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AMATH 503
Homework #3

$\boxed{\text{Question 1}}$ $\boxed{\text{Part a}}$

$$u_t = c^2 \Delta u \qquad -\infty < x, y, z < \infty, \; t > 0$$

$$u(x, y, z, 0) = \delta(x)\delta(y)\delta(z)$$

$$u \to 0 \text{ as } |x| \to \infty, |y| \to \infty, |z| \to \infty$$

(Following lecture 10 35 minute mark for the Drunken sailor Problem)

$$\mathcal{F}_x[u(x,y,z,t)] = \int_{-\infty}^{\infty} u(x,y,z,t)e^{i\omega_1 x}\,dx$$

$$\mathcal{F}_y[u(x,y,z,t)] = \int_{-\infty}^{\infty} u(x,y,z,t)e^{i\omega_2 x}\,dx$$

$$\mathcal{F}_z[u(x,y,z,t)] = \int_{-\infty}^{\infty} u(x,y,z,t)e^{i\omega_3 x}\,dx$$

The Fourier space:

$$\hat{F}_x \hat{F}_y \hat{F}_z [u(x,y,z,t)] = \hat{F}_x [\hat{F}_y [\hat{F}_z [u(x,y,t)]]\}$$

$$\hat{F}_x \hat{F}_y \hat{F}_z [u] = \mathcal{U}(\omega_1, \omega_2, \omega_3, t)$$
$$\hat{F}_x \hat{F}_y \hat{F}_z [u_t] = \mathcal{U}_t(\omega_1, \omega_2, \omega_3, t)$$
$$\hat{F}_x \hat{F}_y \hat{F}_z [u_{xx}] = (i\omega_1)^2 \mathcal{U}(\omega_1, \omega_2, \omega_3, t)$$
$$\hat{F}_x \hat{F}_y \hat{F}_z [u_{yy}] = (i\omega_2)^2 \mathcal{U}(\omega_1, \omega_2, \omega_3, t)$$
$$\hat{F}_x \hat{F}_y \hat{F}_z [u_{zz}] = (i\omega_3)^2 \mathcal{U}(\omega_1, \omega_2, \omega_3, t)$$

The PDE in Fourier space is

$$\mathcal{U}_t(\omega_1, \omega_2, \omega_3, t) = -c^2(\omega_1^2 + \omega_2^2 + \omega_3^2) \cdot \mathcal{U}(\omega_1, \omega_2, \omega_3, t)$$

$$\mathcal{U}(\omega_1, \omega_2, \omega_3, t) = \mathcal{U}(\omega_1, \omega_2, \omega_3, 0) e^{-c^2(\omega_1^2 + \omega_2^2 + \omega_3^2)t}$$
$$\mathcal{U}(\omega_1, \omega_2, \omega_3, 0) = \hat{F}_x \hat{F}_y \hat{F}_z [u(x,y,z,0)] =$$
$$= \hat{F}_x \hat{F}_y \hat{F}_z [\delta(x)\delta(y)\delta(z)] =$$

$$= \tilde{F}_x[\delta(x)]\,\tilde{F}_y[\delta(y)]\,\tilde{F}_z[\delta(z)]$$

$$= 1 \cdot 1 \cdot 1 = 1$$

$$\mathcal{U}(w_1, w_2, w_3, t) = e^{-c^2 w_1^2 t} \cdot e^{-c^2 w_2^2 t} \cdot e^{-c^2 w_3^2 t}$$

$$u(x, y, t) = \tilde{F}_x^{-1} \tilde{F}_y^{-1} \tilde{F}_z^{-1} [\mathcal{U}(w_1, w_2, w_3, t)] =$$

$$= \tilde{F}_x^{-1}\left[\tilde{F}_y^{-1}\left[\tilde{F}_z^{-1}\left[e^{-c^2 w_1^2 t} \cdot e^{-c^2 w_2^2 t} \cdot e^{-c^2 w_3^2 t}\right]\right]\right] =$$

$$= \tilde{F}_x^{-1}\left[e^{-c^2 w_1^2 t} \cdot \tilde{F}_y^{-1}\left[e^{-c^2 w_2^2 t}\, \tilde{F}_z^{-1}\left[e^{-c^2 w_3^2 t}\right]\right]\right]$$

$$= \tilde{F}_x^{-1}\left[e^{-c^2 w_1^2 t}\right] \cdot \tilde{F}_y^{-1}\left[e^{-c^2 w_2^2 t}\right] \cdot \tilde{F}_z^{-1}\left[e^{-c^2 w_3^2 t}\right]$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \cdot e^{\frac{-x^2}{4c^2 t}} \cdot \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-\frac{y^2}{4c^2 t}} \cdot \frac{1}{\sqrt{4\pi c^2 t}} \cdot e^{\frac{-z^2}{4c^2 t}}$$

$$= \frac{1}{\sqrt{4\pi c^2 t}} \cdot e^{\frac{-(x^2+y^2+z^2)}{4c^2 t}} = \frac{1}{(4\pi c^2 t)^{3/2}}\, e^{-\frac{|\vec{x}|^2}{4c^2 t}}$$

$$\vec{x} = (x, y, z)$$

$$\boxed{u(x, y, z) = \frac{1}{8c^3(\pi t)^{3/2}} \cdot e^{-\frac{|\vec{x}|^2}{4c^2 t}}, \quad |\vec{x}| = \sqrt{x^2+y^2+z^2}}$$

$\boxed{\text{Question 1 Part B}}$

N- dimensional drunken sailor problem

$$u_t = c^2 \Delta u$$
$$u(x,0) = \prod_{i=1}^{n} \delta(x_i) \qquad -\infty < x < \infty$$

From part a, we have a solution to a 3-D drunken sailor problem:

$$u(x,y,z) = \frac{1}{(4\pi c^2 t)^{3/2}} \cdot e^{-\frac{|\vec{x}|^2}{4c^2 t}}$$

For n-dimensional drunken sailor problem, my guess is this:

$$\boxed{u(\vec{x},t) = \frac{1}{(4\pi c^2 t)^{n/2}} \, e^{\frac{-|\vec{x}|^2}{4c^2 t}}, \; |\vec{x}| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}$$

Question 2

Prove:

$$\mathcal{F}_x \mathcal{F}_y [u_{xx}] = (i\omega_1)^2 \cdot \mathcal{U}(\omega_1, \omega_2, t)$$

Per lecture 10, 22:52 mark, we know that

$$\mathcal{F}[u_{xx}] = (i\omega_1)^2 \int_{-\infty}^{\infty} u(x,t)\, e^{i\omega_1 x}\, dx =$$

$$= (i\omega_1^2) \cdot \mathcal{U}(\omega_1, \omega_2\, t)$$

And from problem 1, we can conclude:

$$\mathcal{F}_x \hat{\mathcal{F}}_y [u(x,y,t)] = \hat{\mathcal{F}}_x \hat{\mathcal{F}}_y [u(x,y,t)]$$

$$= \int_{\infty}^{\infty} \int_{\infty}^{\infty} u(x,y,t)\, e^{i\omega_1 x} e^{i\omega_2 y}\, dy\, dx$$

1

$$= \mathcal{U}(\omega_1, \omega_2, t)$$

we can start the proof as follows:

$$\hat{F}_x \hat{F}_y [u_{xx}] = \hat{F}_x [\hat{F}_y [u_{xx}]]$$

$$= \hat{F}_x \left[\int_{-\infty}^{\infty} \frac{d^2}{dx^2} u(x,y,t) e^{iw_2 y} dy \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^2}{dx^2} u(x,y,t) e^{iw_2 y} e^{iw_1 x} dy dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{d^2}{dx^2} u(x,y,t) e^{iw_1 x} dx \right] e^{iw_2 y} dy$$

Integration by arts

$$= \int_{-\infty}^{\infty} \Big(\underbrace{(e^{iw_1 x} u_x(x,y,t)) \Big|_{-\infty}^{\infty}}_{\text{this goes to zero based on boundary conditions}} - iw_1 \int_{-\infty}^{\infty} e^{iw_1 x} u_x(x,y,t) dx \Big) e^{iw_2 y} dy$$

$$= \int_{-\infty}^{\infty} \Big(-iw_1 \Big(\int_{-\infty}^{\infty} e^{iw_1 x} u_x(x,y,t) dx \Big) \Big) e^{iw_2 y} dy$$

Integration by parts again

$$= \int_{-\infty}^{\infty} \Big(-iw_1 \Big(\underbrace{(e^{iw_1 x} u(x,y,t)) \Big|_{-\infty}^{\infty}}_{\text{this goes to zero based on boundary conditions}} - iw_1 \int_{-\infty}^{\infty} e^{iw_1 x} u(x,y,t) dx \Big) \Big) e^{iw_2 y} dy$$

$$= (i\omega_1)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t) e^{i\omega_1 x} e^{i\omega_2 y} \, dx\, dy$$

using 1 from above

$$= \boxed{(i\omega_1)^2\, \mathcal{U}(\omega_1, \omega_2, t)}$$

as desired.