$$S(t) = \sum_{k=0}^{N-1} x[k]\, \phi_k(t) \qquad 0 \leq t < NT$$

where $x[k] \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $T > 0$ and

$$\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} \cdot kt}$$

consider, $< \phi_k(t), \phi_\ell(t) >$ : the Hermitian Inner Product.

$$= \int_0^{NT} \phi_k(t) \cdot \phi_\ell^*(t)\, dt$$

$$= \int_0^{NT} \left( \frac{1}{\sqrt{NT}} \cdot e^{j \frac{2\pi}{NT} kt} \right) \cdot \left( \frac{1}{\sqrt{NT}} \cdot e^{j \frac{2\pi}{NT} \ell t} \right)^* dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} kt} \cdot e^{-j \frac{2\pi}{NT} \ell t}\, dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (k-\ell) t}\, dt$$

$\boxed{\text{Case } 1:}$ if $k = \ell$:

$$\Rightarrow \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (0).t}\, dt$$

$$= \frac{1}{NT} \int_0^{NT} 1\, dt = \frac{1}{NT} [t]_0^{NT} = \frac{1}{NT} [NT - 0]$$

$$= \underline{\underline{1}} \longrightarrow ①$$

$\boxed{\text{Case 2 :}}$ if $k \neq l$

Since $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$\therefore$ Let $(k-l) = m \in \mathbb{Z}$

$$\therefore \frac{1}{NT} \cdot \int_0^{NT} e^{j\frac{2\pi}{NT} mt} \, dt$$

$$= \frac{1}{\cancel{NT}} \left. \frac{e^{j\frac{2\pi}{NT} \cdot m \cdot t}}{j\frac{2\pi}{\cancel{NT}} \cdot m} \right|_0^{NT}$$

$$= \frac{1}{j2\pi m} \cdot \left[ e^{j\frac{2\pi}{\cancel{NT}} \cdot m \cdot \cancel{NT}} - 1 \right]$$

$$= \frac{1}{j2\pi m} \left[ e^{j(2\pi)m} - 1 \right] \quad \text{where } m \in \mathbb{Z}$$

$$= \frac{1}{j2\pi m} \left[ \underbrace{\cos(2\pi m)}_{(\text{always} = 1)} + j \overbrace{\cancel{\sin(2\pi m)}}^{0} - 1 \right]$$

$$= \frac{1}{j2\pi m} [1 - 1] \Rightarrow 0 \longrightarrow ②$$

from ① and ②

$$\therefore \int_0^{NT} \phi_k(t) \cdot \phi_l^*(t) \, dt = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$\boxed{\text{Q1}}$

b.) $\left.\begin{array}{l}\text{Total Energy}\\ \text{Content of } S(t)\end{array}\right\} = \langle S(t), S(t) \rangle$

$$= \int_0^{NT} S(t)\,(S(t))^* \, dt \quad ; (0 < t \leq NT)$$

complex conjugate

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k].\,\phi_k(t)\right]\left[\sum_{k=0}^{N-1} x[k]\,\phi_k(t)\right]^* dt$$

Since $(z_1 + z_2)^* = z_1^* + z_2^*$

$\forall\ z_1, z_2 \in \mathbb{C}$ we can apply the conjugate to above summation, seperately.

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k].\,\phi_k(t)\right]\left[\sum_{k=0}^{N-1} (x[k])^*\,\phi_k^*(t)\right] dt$$

= from part ⓐ terms are vanished when the k values are different.

$$\Rightarrow \sum_{k=0}^{N-1}\left\{\left[x[k].(x[k])^*\right] \underbrace{\int_0^{NT} \phi_k(t).\,\phi_k^*(t)\,dt}_{=1}\right\}$$

$\longrightarrow$ from ⓐ

$$\Rightarrow \mathcal{E} = \sum_{k=0}^{N-1} |x[k]|^2$$

Q1

c.) $\sum_{m=0}^{N-1} \phi_k(mT)\, \phi_l^*(mT)$ ; $\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}kt}$

$$\Rightarrow \sum_{m=0}^{N-1} \left[\frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}k(mT)}\right]\left[\frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}l(mT)}\right]^*$$

$$= \sum_{m=0}^{N-1} \frac{1}{NT}\left[e^{j\frac{2\pi}{N}k.m.}\right]\left[e^{-j\frac{2\pi}{N}l.m}\right]$$

$$= \frac{1}{NT} \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}m(k-l)}$$

$$= \frac{1}{NT} \cdot \underbrace{\sum_{m=0}^{N-1} \left[e^{j\frac{2\pi}{N}(k-l)}\right]^m}$$

$$= \frac{1}{NT} \frac{1 - \left[e^{j\frac{2\pi}{N}(k-l)}\right]^N}{1 - e^{j\frac{2\pi}{N}(k-l)}}$$ using the given hint.

$$= \frac{1}{NT} \cdot \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} \longrightarrow ①$$

$\boxed{\text{Case I}}$ when $(k-l) \neq 0 \Rightarrow k \neq l$

since $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$$\therefore \quad e^{j2\pi(k-l)} = 1 \quad \forall \ (k-l) \in \mathbb{Z}$$

$$\therefore \quad \frac{1}{NT} \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} = \frac{1}{NT} \frac{(1-1)}{\left(1 - e^{j\frac{2\pi}{N}(k-l)}\right)}$$

$$= 0 \longrightarrow ②.$$

$\boxed{\text{case 2}}$ when $k = l$, using L'hopital's rule.

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \frac{-j2\pi \cdot e^{j2\pi(k-l)}}{-j\frac{2\pi}{N} e^{j\frac{2\pi}{N}(k-l)}} \qquad \text{differentiate w.r.t } (k-l)$$

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \times N \times \frac{e^{j2\pi(k-l)}}{e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \frac{1}{NT} \times N \times \frac{1}{1}$$

$$= \frac{1}{T} \longrightarrow ③$$

$$\therefore \text{ from ② and ③} \Rightarrow \left\{ \sum_{m=0}^{N-1} \phi_k(mT) \phi_l^*(mT) = \begin{cases} \frac{1}{T} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \right.$$

Q1

d.) $s[m] = s(mT)$; $m = 0, 1, 2, \cdots, N-1$

Consider $RHS = T \times \sum_{m=0}^{N-1} s[m]\, \phi_l^*(mT)$

$$= T \times \sum_{m=0}^{N-1} s(mT) \cdot \phi_l^*(mT)$$

$$= \cancel{T \times \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] \right.}$$

$$= T \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k]\, \phi_k(t) \right] \phi_l^*(t) \Bigg|_{t=mT}$$

$$= T \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(mT) \cdot \phi_l^*(mT) \right]$$

$$= \sum_{k=0}^{N-1} x[k] \left\{ (T) \underbrace{\sum_{m=0}^{N-1} \phi_k(mT)\, \phi_l^*(mT)}_{} \right\}$$

from part (c); this part is $\frac{1}{T}$ when $k = l$ and vanishes otherwise.

so when $k = l$ $\quad (k = l)$

$$\Rightarrow x[l] \times T \times \frac{1}{T}$$

$$\therefore \quad x[l] = T \cdot \sum_{m=0}^{N-1} s[m]\, \phi_l^*(mT)$$

for $l = 0, 1, 2, \cdots, N-1$

$\boxed{Q2.}$ a.)

consider $\left|\sum_{k=1}^{M} \theta_k\right|^2 \geq 0 \longrightarrow ①$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 = \left(\sum_{k=1}^{M} \theta_k\right)^T\left(\sum_{k=1}^{M} \theta_k\right)$$

$$= (\theta_1 + \theta_2 + \dots + \theta_M)^T(\theta_1 + \theta_2 + \dots + \theta_M)$$

$$= (\theta_1^T + \theta_2^T + \dots + \theta_M^T)(\theta_1 + \theta_2 + \dots + \theta_M)$$

In this expression there will be,

(I) $M$ terms, where $i=j$ for $\theta_i^T \theta_j$

(II) $M(M-1)$ terms, wher $i \neq j$ for $\theta_i^T \theta_j$
( for each $\theta_i$ $\exists$, $(M-1)$ $\theta_j$'s such that $i \neq j$.

$\therefore$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 = \underbrace{(M \times 1)}_{\text{from (I)}} + \underbrace{M(M-1)\rho}_{\text{from (II)}}$$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 \geq 0 \longrightarrow \text{from } ①$$

$$M + M(M-1)\rho \geq 0$$

$$1 + (M-1)\rho \geq 0 \quad (\because \text{Since } M>0)$$

$$\frac{-1}{M-1} \leq \rho \longrightarrow ② \quad (\because \text{Since } (M-1)>0)$$

$\rho \leq 1 \longrightarrow$ ③

$\therefore$ From ② and ③,

$$\frac{-1}{M-1} \leq \rho \leq 1$$

$\boxed{Q2.}$ b.)

Assume $\{\mathcal{O}_i\}$ is a Simplex set. Then following expression must equal to zero, as $\{\mathcal{O}_i\}$s are already in the minimum energy constellation.

$$\sum_{i=1}^{M} p_i \mathcal{O}_i = \underline{0}$$

Since $\{\mathcal{O}_i\}$ are equally likely, $p_i = \frac{1}{M}$

$$\therefore \quad \frac{1}{M} \sum_{i=1}^{M} \mathcal{O}_i = \underline{0}$$

$$\Rightarrow \quad \sum_{i=1}^{M} \mathcal{O}_i = \underline{0}$$

consider, $$\left\| \left( \sum_{i=1}^{M} \mathcal{O}_i \right) \right\|^2 = \|\underline{0}\|^2$$

$$\underbrace{\left( \sum_{i=1}^{M} \mathcal{O}_i \right)^T \left( \sum_{i=1}^{M} \mathcal{O}_i \right)} = 0$$

from part (a) this part is simplified into,

$$M + M(M-1)\rho = 0$$

$$\therefore \quad \rho = \frac{-1}{M-1} \quad //$$

Q.2 c.)

(I) Determining the minimum energy set corresponding to the set $\{\theta_i\}$.

* Let $\{\beta_i\}$ be the minimum energy set of the signal set $\{\theta_i\}$.

$$\Rightarrow \{\beta_i\} = \{\theta_i - \underline{a}\} \text{ where, } \underline{a} = \frac{1}{M}\sum_{i=1}^{M}\theta_i$$

consider $\beta_i^T \beta_\ell = (\theta_i - \underline{a})^T(\theta_\ell - \underline{a})$

$$= (\theta_i^T - \underline{a}^T)(\theta_\ell - \underline{a})$$

$$= \theta_i^T\theta_\ell - \theta_i^T\underline{a} - \underline{a}^T\theta_\ell + \underline{a}^T\underline{a}$$

$$= \theta_i^T\theta_\ell - \theta_i^T\left(\frac{1}{M}\sum_{i=1}^{M}\theta_i\right) - \left(\frac{1}{M}\sum_{i=1}^{M}\theta_i\right)^T\theta_\ell + \underline{a}^T\underline{a}$$

$$= \theta_i^T\theta_\ell - \frac{1}{M}[1+(M-1)\rho] - \frac{1}{M}[1+(M-1)\rho] + \|\underline{a}\|^2$$

from part (a.)

$$= \theta_i^T\theta_\ell - \frac{2}{M}[1+(M-1)\rho] + \frac{1}{M^2}[M+(M-1)\rho]$$

$$= \theta_i^T\theta_\ell - \frac{1}{M}[1+(M-1)\rho] \longrightarrow (1)$$

Q2 c) contin:

from def$^n$: $\quad \theta_i^T \theta = \begin{cases} 1 & i = l \\ \rho & i \neq l \end{cases}$

$$\therefore \; (\beta_i)^T(\beta_l) = \begin{cases} 1 - \frac{1}{M}[1 + (M-1)\rho] \; ; & i = l \\ \rho - \frac{1}{M}[1 + (M-1)\rho] \; ; & i \neq l \end{cases}$$

$$\therefore \; \beta_i^T \beta_l = \left\{ \begin{array}{ll} \frac{(M-1)(1-\rho)}{M} \; ; & i = l \\ \frac{(\rho-1)}{M} \; ; & i \neq l \end{array} \right\} - (2)$$

from part (b)

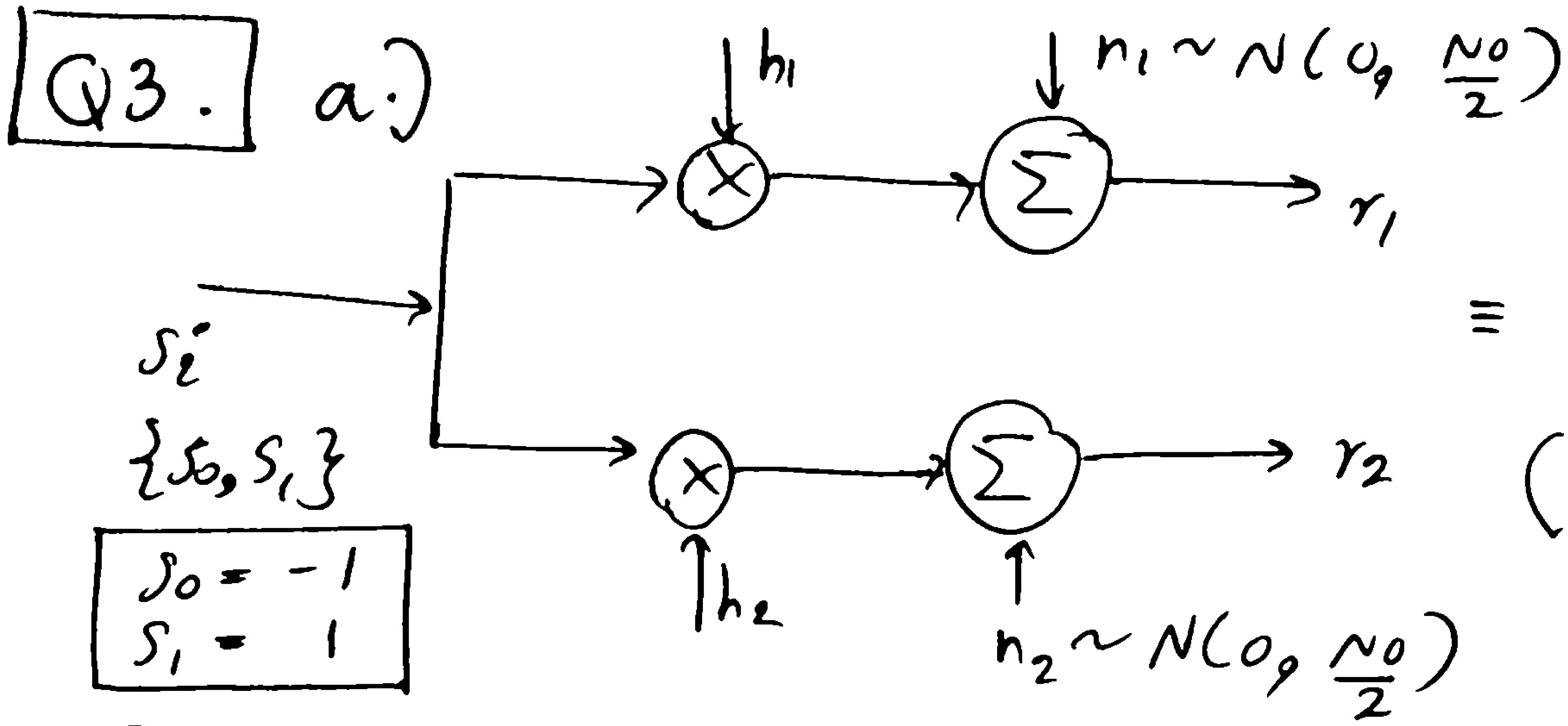
For the unit average Simplex set $\rightarrow$ $\quad \alpha_i^T \alpha_l = \begin{cases} 1 & ; i = l \\ \rho = \frac{-1}{M-1} & ; i \neq l \end{cases}$

By rearranging (2),

$$\Rightarrow \beta_i^T \beta_l = \begin{cases} 1 \times \left[\frac{(M-1)(1-\rho)}{M}\right] & i = l \\ -\frac{1}{M-1} \times \left[\frac{(M-1)(1-\rho)}{M}\right] ; & i \neq l \end{cases}$$

$\therefore$ We can observe that minimum energy set $\{\beta_i\}$ is a scaled constellation of the simplex set $\{\alpha_i\}$.

Q3. a:)




$\equiv \quad \underline{r} = \underline{h} S_i + \underline{n}$

$(\underline{r}, \underline{h} \in \mathbb{R}^{2\times 1})$

a.) Consider the following a posteriori probabilities.

(I) For the receiver to classify the observed signal as $S_0$ $\Big\} \Rightarrow P(S_0 | \underline{r}) \overset{S_0}{>} P(S_1 | \underline{r}) \rightarrow (1)$

(II) For the receiver to classify the observed signal as $S_1$ $\Big\} \Rightarrow P(S_0 | \underline{r}) \underset{S_1}{<} P(S_1 | \underline{r}) \rightarrow (2)$

By combining (1) and (2), ~~for~~

$$P(S_0 | \underline{r}) \underset{S_1}{\overset{S_0}{\gtrless}} P(S_1 | \underline{r}) \quad (3)$$

$\leftarrow$ This is what demanded by the receiver!

But what is known, to us, are the following probabilities.

$$P(\underline{r} | S_0) \; , \; P(\underline{r} | S_1)$$

∴ By using Bayes theorem, (3) can be rewritten as follows.

$$P(\underline{s_0}|\underline{r}) \underset{s_1}{\overset{s_0}{\gtrless}} P(\underline{s_1}|\underline{r})$$

$$\frac{P(\underline{r}|\underline{s_0}) \times P(\underline{s_0})}{P(\underline{r})} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{P(\underline{r}|\underline{s_1})\, P(\underline{s_1})}{P(\underline{r})}$$

Since $s_0$ and $s_1$ are equally likely signals,

$$P(\underline{s_0}) = P(\underline{s_1}) = \frac{1}{2}$$

∴

$$P(\underline{r}|\underline{s_0}) \underset{s_1}{\overset{s_0}{\gtrless}} P(\underline{r}|\underline{s_1})$$

Since $\underline{r} = \underline{h} s_i + \underline{n}$

$s_i = s_0 \rightarrow \underline{r} = \underbrace{\underline{h}(-1)}_{\underline{s_0}} + \underline{n}$

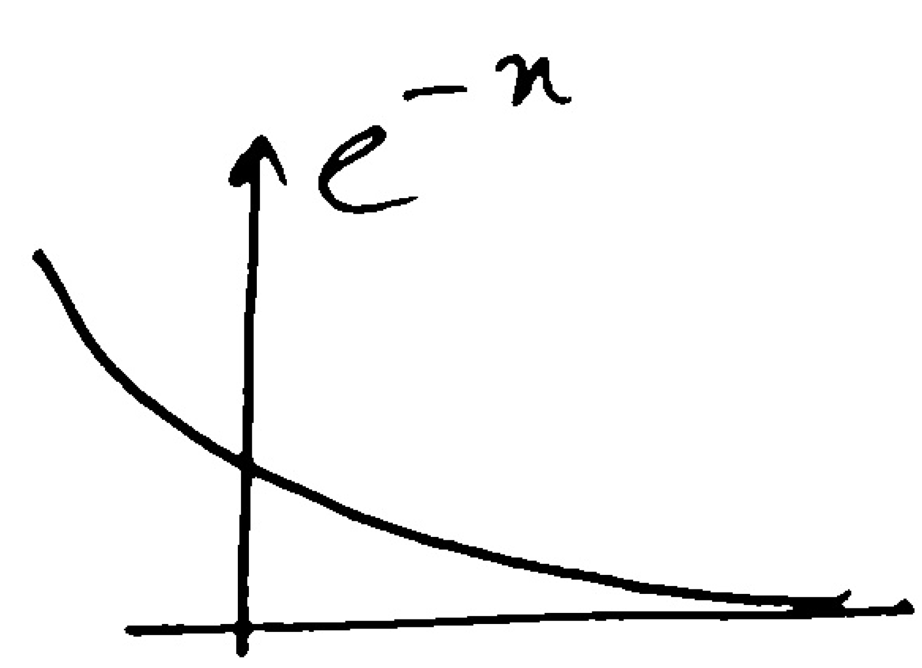
$s_i = s_1 \rightarrow \underline{r} = \underbrace{\underline{h}(1)}_{\underline{s_1}} + \underline{n}$

So

$$P(\underline{r}|-\underline{h}) \underset{s_1}{\overset{s_0}{\gtrless}} P(\underline{r}|\underline{h})$$

$$N_2\left(-\underline{h}, \frac{N_0}{2} I_2\right) \underset{s_1}{\overset{s_0}{\gtrless}} N_2\left(\underline{h}, \frac{N_0}{2} I_2\right)$$

$$e^{-\|\underline{r}-(-\underline{h})\|^2} \underset{s_1}{\overset{s_0}{\gtrless}} e^{-\|\underline{r}-(\underline{h})\|^2}$$




$$\|\underline{r}+\underline{h}\|^2 \underset{s_0}{\overset{s_1}{\gtrless}} \|\underline{r}-\underline{h}\|^2$$

$$\|\underline{r} + \underline{h}\|^2 \underset{S_0}{\overset{S_1}{\gtrless}} \|\underline{r} - \underline{h}\|^2$$

$$(\underline{r} + \underline{h})^T (\underline{r} + h) \underset{S_0}{\overset{S_1}{\gtrless}} (\underline{r} - \underline{h})^T (\underline{r} - \underline{h})$$

$$(\underline{r}^T + \underline{h}^T)(\underline{r} + \underline{h}) \underset{S_0}{\overset{S_1}{\gtrless}} (r^T - h^T)(\underline{r} - \underline{h})$$

$$\begin{aligned} &\underline{r}^T \underline{r} + \underline{r}^T \underline{h} & & \underline{r}^T . \underline{r} - r^T . \underline{h} \\ &+ h^T . \underline{r} + \underline{h}^T . \underline{h} & \underset{S_1}{\overset{S_1}{\gtrless}} & - \underline{h}^T \underline{r} + h^T . h \end{aligned}$$

Since $h, r \in \mathbb{R}^{2\times 1} \Rightarrow \underline{r}^T \underline{h} = \underline{h}^T \underline{r}$

$$\therefore \quad 2 r^T h \underset{S_0}{\overset{S_1}{\gtrless}} -2 \underline{r}^T \underline{h}$$

$$4 \underline{r}^T \underline{h} \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$\Downarrow$$

$$\boxed{\underline{r}^T \underline{h} \underset{S_0}{\overset{S_1}{\gtrless}} 0}$$

$\leftarrow$ Optimum MAP decision rule.

$$\underline{r}^T \underline{h} \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$r_1 h_1 + r_2 h_2 \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$\boxed{r_1 \gtrless -\left(\frac{h_2}{h_1}\right) r_2}$$

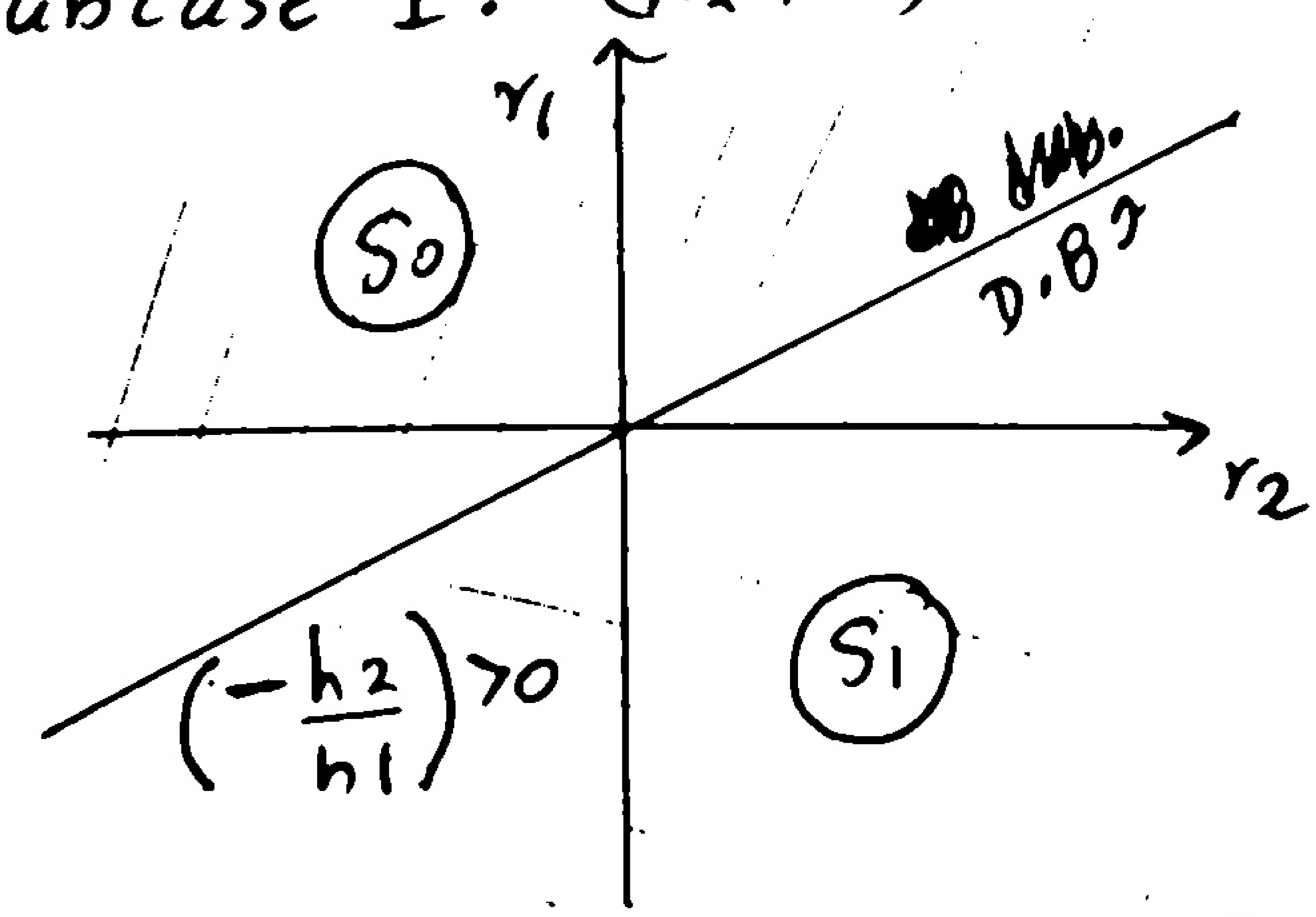
$h_1 < 0$ inequality changes

$h_1 > 0$ inequality remains the same.

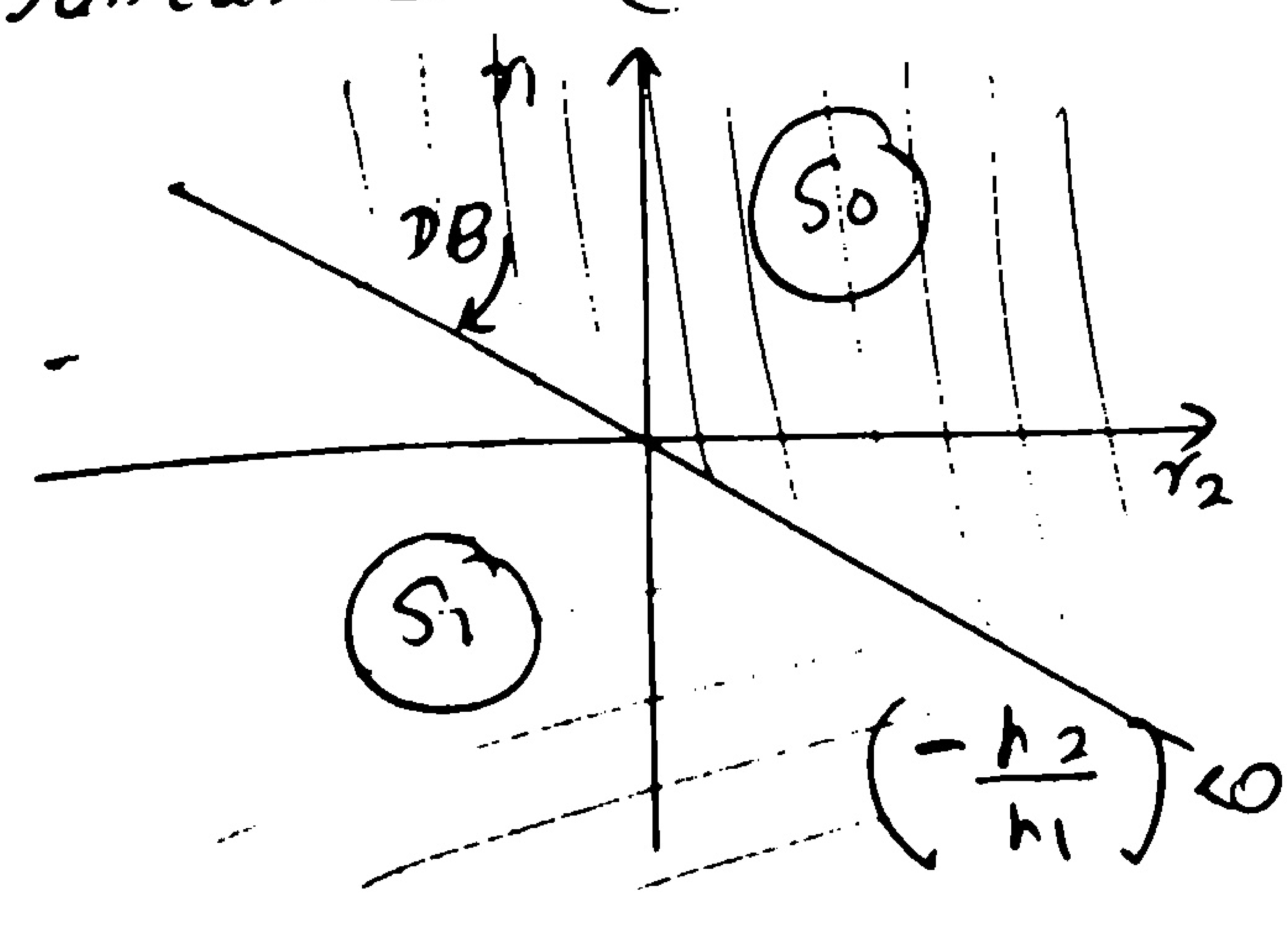
# case (I) ($h_1 < 0$)

$$\boxed{r_1 \underset{S_1}{\overset{S_0}{\gtrless}} -\left(\frac{h_2}{h_1}\right) r_2}$$

Subcase I. ($h_2 > 0$)



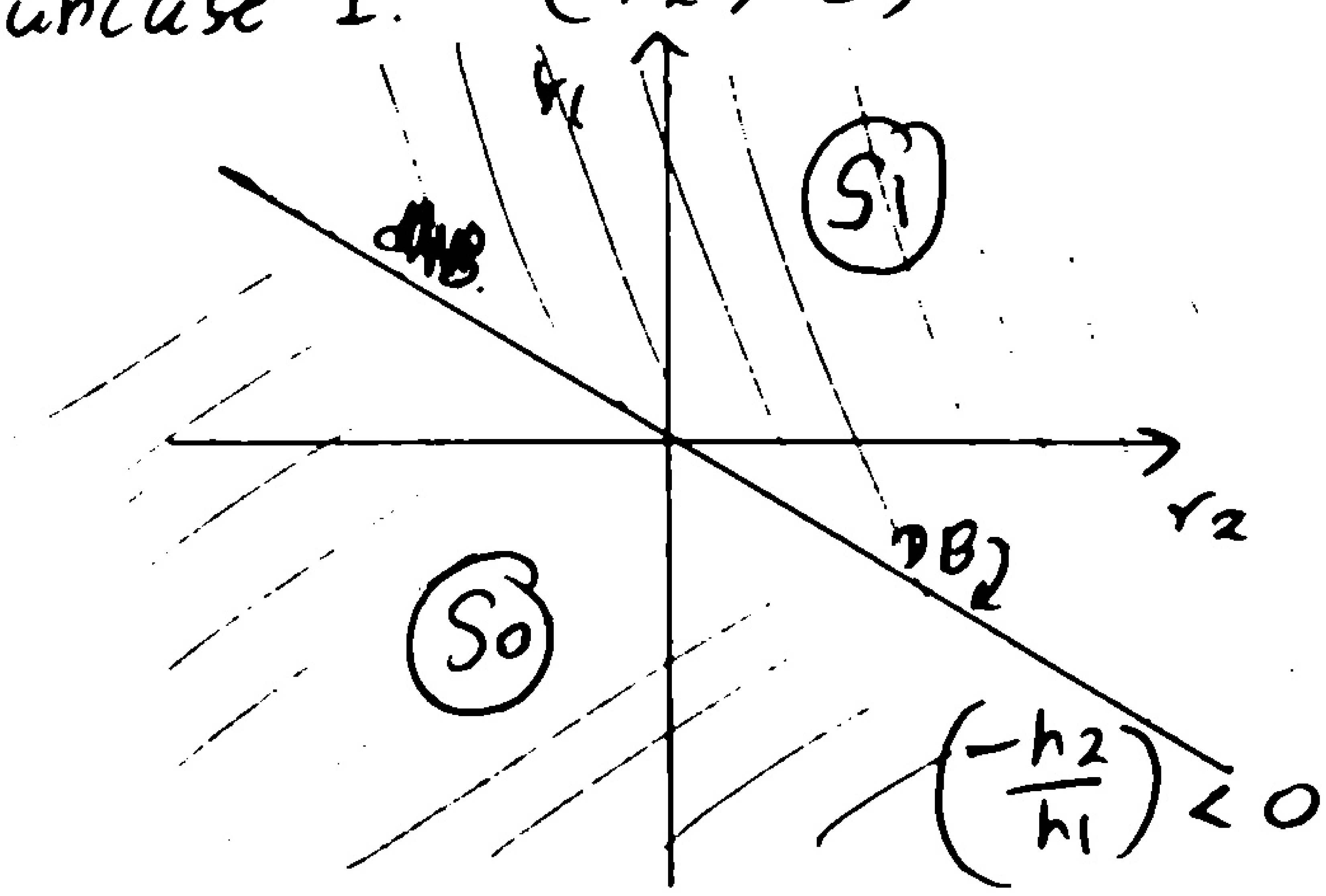

Subcase II. ($h_2 < 0$)



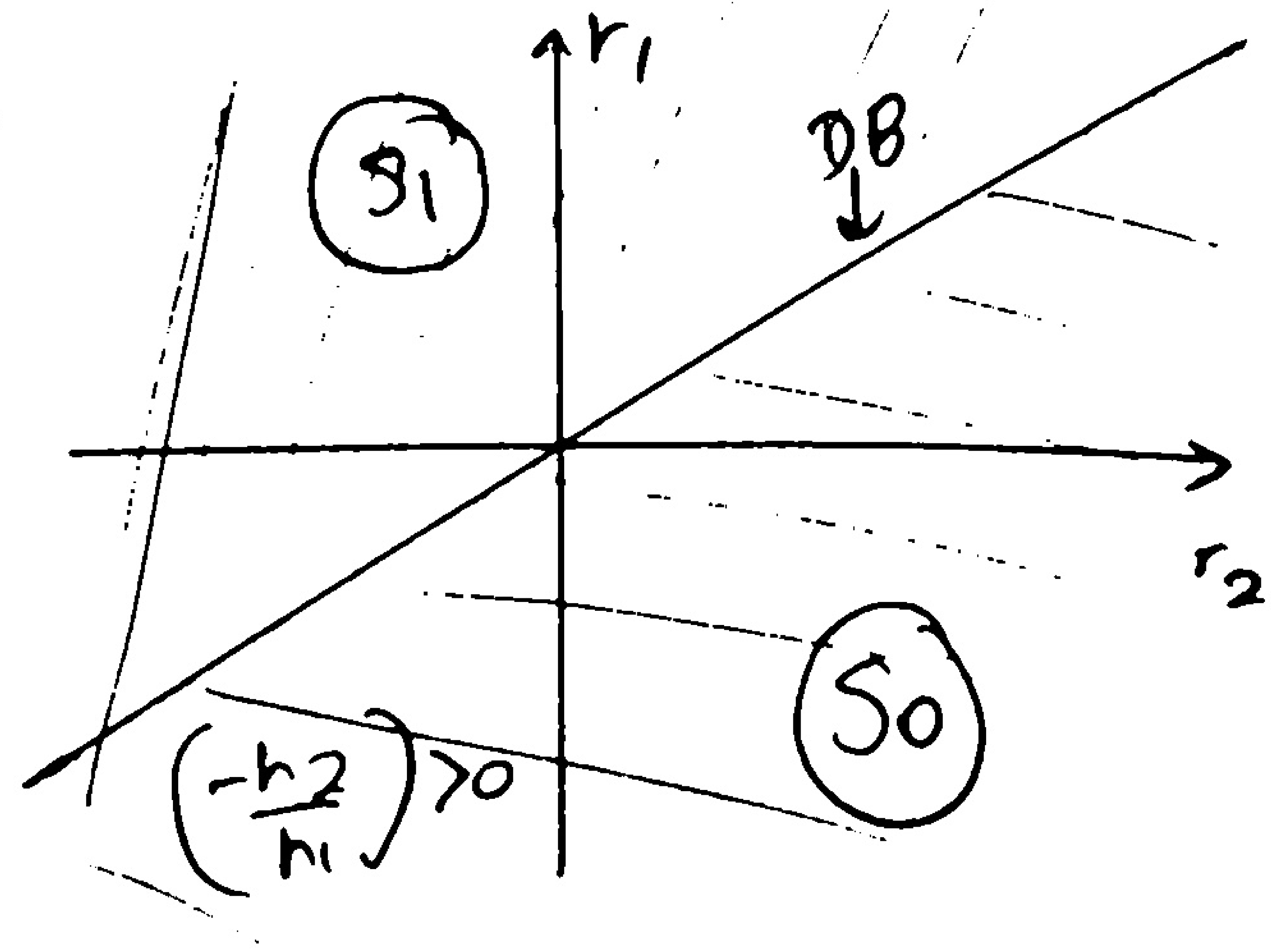

# case (II) ($h_1 > 0$)

$$\boxed{r_1 \underset{S_0}{\overset{S_1}{\gtrless}} -\left(\frac{h_2}{h_1}\right) r_2}$$

Subcase I. ($h_2 > 0$)




Subcase II. $h_2 < 0$



$\boxed{Q3}$ b.)

Under the hypothesis $(s_1 = 1)$.

$$\underline{r} = \underline{h} s_i + \underline{n}$$

$$\underline{r} = h(1) + \underline{n}$$

$$\underline{r} = \underline{h} + \underline{n}$$

$$\underline{r}^T \cdot \underline{h} = (\underline{h} + \underline{n})^T \cdot \underline{h}$$

consider $E\{r^T h\} = \mu_h$

$$= E\{(\underline{h} + \underline{n})^T \cdot \underline{h}\}$$

$$= E\{(\underline{h}^T + \underline{n}^T) \cdot \underline{h}\}$$

Here $h^T$ is a vector of constants and

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \underline{n} \sim N_2\left(0, \frac{N_0}{2} I_2\right) \Rightarrow E\{\underset{\sim}{n}^T\} = E\{\underset{\sim}{n}\} = \underset{\sim}{0}$$

$$\Rightarrow \cancel{E\{h^T n + \|h\|^2\}} = \cancel{h^T E\{n\}} + \cancel{E\{\|h\|^2\}}$$

$$= E\{\underline{h}^T \cdot \underline{h}\} + \underbrace{E\{n^T \cdot h\}}_{0}$$

$$\therefore \mu_h = E\{\underset{\text{constant}}{\|h\|^2}\} = \|h\|^2 \longrightarrow (1)$$

consider $\sigma_h^2 = Var(r^T h) = E\{(r^T h - \|h\|^2)^2\}$

$= E\{(\underline{r}^T \underline{h} - \|h\|^2)(\underline{r}^T \underline{h} - \|h\|^2)\}$

$= E\{(\underline{r}^T \underline{h})^2\} - 2\|h\|^2 \underbrace{E\{r^T h\}}_{\text{from } ①} + E\{\|h\|^4\}$

$= E\{(\underline{r}^T \underline{h})^2\} - 2\|h\|^4 + \|h\|^4$

$= E\{(\underline{r}^T h)^2\} - \|h\|^4$

$$\underline{r}^T h = \|h\|^2 + \underline{n}^T \cdot \underline{h}.$$

$= E\{(\|h\|^2 + n^T h)^2\} - \|h\|^4$

$= E\{\|h\|^4 + 2\|h\|^2 \underline{n}^T \cdot \underline{h} + (n^T h)^2\} - \|h\|^4$

$= E\{\cancel{\|h\|^4}\} + \underbrace{2\|h\|^2 E\{n^T h\}}_{0} + E\{(n^T h)^2\} - \cancel{\|h\|^4}$

$\Rightarrow Var(r^T h_1) = E\{(n^T h)^2\}$

$$\boxed{n^T h = [n_1 \; n_2]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = (n_1 h_1 + n_2 h_2)}$$

$= E\{(n_1 h_1 + n_2 h_2)^2\}$

$= E\{n_1^2 (h_1^2)\} + E\{n_2^2 h_2^2\} + E\{n_1 n_2 (h_1 h_2)\}$

$= h_1^2 \cdot \underbrace{E\{n_1^2\}}_{\frac{N_0}{2}} + h_2^2 \underbrace{E\{n_2^2\}}_{\frac{N_0}{2}} + h_1 h_2 \underbrace{E\{n_1 n_2\}}_{\underbrace{E[n_1]\,E[n_2]}_{\substack{0 \times 0 \\ (=0)}}}$

$$\therefore \ Var(\underline{r}^T\underline{h}) = h_1^2 \frac{N_0}{2} + h_2^2 \frac{N_0}{2}$$

$$= \underbrace{(h_1^2 + h_2^2)} \frac{N_0}{2}$$

$$var(r^T h) = \|h\|^2 \frac{N_0}{2} \longrightarrow (2)$$

$\therefore$ from (1) and (2),

$$E\{\underline{r}^T\underline{h}\} = \|h\|^2 = \mu_h$$

$$Var\{\underline{r}^T\underline{h}\} = \|h\|^2 \frac{N_0}{2} = \sigma_h^2$$

$$\therefore \ \underline{\underline{\underline{r}^T\underline{h} \sim N(\mu_h, \sigma_h^2)}}$$

$\boxed{Q3}$ c.)

$$P(e) = P(e|s_0) \times \underbrace{P(s_0)}_{\frac{1}{2}} + P(e|s_1)\,\underbrace{P(s_1)}_{1/2}$$

$$P(e) = \frac{1}{2}[P(e|s_0) + P(e|s_1)] \longrightarrow ①$$

from part (a.) the optimum MAP decision is

correct $s_1$: $\quad \underline{r}^T\underline{h} > 0 \rightarrow P(e|s_1) = P(\underline{r}^T\underline{h} < 0)$

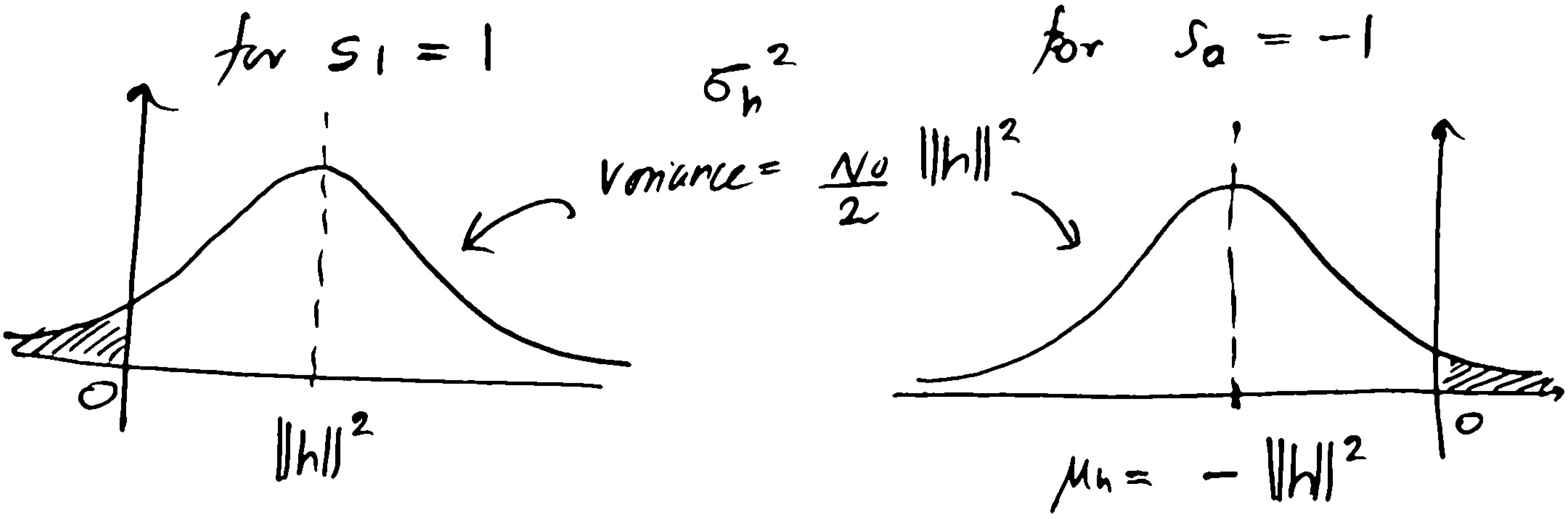
correct $s_0$: $\quad \underline{r}^T\underline{h} < 0 \rightarrow P(e|s_0) = P(\underline{r}^T\underline{h} > 0)$

from part (b.): under hypothesis $s_1 = 1$

$s_1$: $\quad r^Th \sim N(\|h\|^2, \frac{N_0}{2}\|h\|^2)$

Same way under the Hypothesis $s_0 = -1$

$$\underline{r}^T\cdot\underline{h} = (-\underline{h} + \underline{n})\cdot\underline{h} \Rightarrow \underline{r}^T\cdot\underline{h} \sim N(-\|h\|^2, \frac{N_0}{2}\|h\|^2)$$




$\therefore$ (1) can be rewritten as, follows.

$$P(e) = \frac{1}{2}\left[P(e|s_0) + P(e|s_1)\right]$$

$$= \frac{1}{2}\left[P(\underline{r}^T\underline{h} > 0 | s_0) + P(r^T h < 0 | s_1)\right]$$

$$= \frac{1}{2}\left\{ \int_0^{\infty} N\left(-\|h\|^2, \frac{N_0}{2}\|h\|^2\right) + \int_{-\infty}^{0} N\left(+\|h\|^2; \frac{N_0}{2}\|h\|^2\right)\right.$$

Probability distributions are mirror images of each other! therefore we can use either of the distributions to solve above expression..

$$\therefore \; P(e) = \frac{1}{2} \times 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma_h^2}} e^{-\frac{[\underline{r}^T\underline{h} - (-\|h\|^2)]}{2\sigma_h^2}} d(r^T h)$$

$$P(e) = \frac{1}{\sqrt{2\pi}\,\sigma_h} \int_0^{\infty} \exp\left[-\frac{(\underline{r}_0^T\underline{h} + \|h\|^2)^2}{2\sigma_h^2}\right] d(\underline{r}^T\underline{h})$$

Substitute : $\dfrac{\underline{r}^T\underline{h} + \|h\|^2}{\sigma_h} = t$

limits.

when $r^T h = 0 \longrightarrow t = \frac{\|h\|^2}{\sigma_h}$

$r^T h \to \infty \longrightarrow t \to \infty$

$$\sigma_h \, dt = d(r^T h)$$

$$\therefore P(\varepsilon) = \frac{1}{\sqrt{2\pi}\,\cancel{\sigma_h}} \int_{\frac{\|h\|^2}{\sigma_h}}^{\infty} \exp(-t^2/2) \cdot \cancel{\sigma_h}\, dt$$

$$\therefore P(\varepsilon) = Q\left(\frac{\|h\|^2}{\sigma_h}\right)$$

$$= Q\left(\frac{\|h\|^2}{\sqrt{\frac{N_0}{2}\|h\|^2}}\right)$$

$$\therefore P(e) = Q\left[\frac{2\,\|h\|^2}{N_0}\right]$$

Where, $Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-t^2/2}\, dt$