

Department of Electronic and Telecommunication Engineering

University of Moratuwa, Sri Lanka

EN3053 - Digital Communications - I



# Assignment No. 1

Submitted by

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## Contents

\* *PDF is clickable*

*Note:*

All the executable codes, a simulink model and a voice sample are included in a separate folder named as **EN2053-Assignment 2-180574K-180631J.rar** which was submitted along with this report.

Before run the **VoiceTransmissionBaseband.slx** simulink simulation related to the part 5 of the Task 1 ,(included in a separate folder named as **Task1-RFPM** inside the above folder) change the directories related to the voice input and output. Voice sample named as **voice.wav** is also included in the same folder.

Additionally all the materials related to Task 1 can also be found at <https://github.com/bimalka98/RF-Propagation-Model>

$$S(t) = \sum_{k=0}^{N-1} x[k] \phi_k(t) \quad 0 \leq t < NT$$

where  $x[k] \in \mathbb{C}$ ,  $N \in \mathbb{Z}^+$ ,  $T > 0$  and

$$\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt}$$

Consider,  $\langle \phi_k(t), \phi_l(t) \rangle$ : the Hermitian Inner Product.

$$= \int_0^{NT} \phi_k(t) \cdot \phi_l^*(t) dt$$

$$= \int_0^{NT} \left( \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt} \right) \cdot \left( \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} lt} \right)^* dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} kt} \cdot e^{-j \frac{2\pi}{NT} lt} dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (k-l)t} dt$$

Case 1 :- if  $k=l$ :

$$\Rightarrow \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (0)t} dt$$

$$= \frac{1}{NT} \int_0^{NT} 1 dt = \frac{1}{NT} [t]_0^{NT} = \frac{1}{NT} [NT - 0]$$

$$= \underline{\underline{1}} \longrightarrow \textcircled{1}$$

Case 2 : if  $k \neq l$

Since  $k, l \in \mathbb{Z}^+ \Rightarrow (k-l) \in \mathbb{Z}$

&  $\therefore$  Let  $(k-l) = m \in \mathbb{Z}$

$$\frac{v_0}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT} mt} dt$$

$$= \frac{1}{NT} \left[ \frac{e^{j\frac{2\pi}{NT} m \cdot t}}{j\frac{2\pi}{NT} \cdot m} \right] \Big|_0^{NT}$$

$$= \frac{1}{j2\pi m} \left[ e^{j\frac{2\pi}{NT} m \cdot NT} - 1 \right]$$

$$= \frac{1}{j2\pi m} \left[ e^{j(2\pi)m} - 1 \right] \text{ where } m \in \mathbb{Z}$$

$$= \frac{1}{j2\pi m} \left[ \underbrace{\cos(2\pi m)}_{(\text{Always } = 1)} + j \sin(\cancel{2\pi m}) - 1 \right]$$

$$= \frac{1}{j2\pi m} [1 - 1] \Rightarrow 0 \longrightarrow \textcircled{2}$$

from ① and ②

$$\therefore \int_0^{NT} \phi_k(t) \cdot \phi_l^*(t) dt = \begin{cases} 1 & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases}$$

Q1

b.) Total Energy  $\gamma = \langle s(t), s(t) \rangle$   
 Content of  $s(t)$

$$= \int_0^{NT} s(t) (s(t))^* dt \quad ; (0 < t < NT)$$

$$= \int_0^{NT} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[ \sum_{k=0}^{N-1} x[k] \phi_k^*(t) \right]^* dt \quad \begin{matrix} \text{complex} \\ \text{conjugate} \end{matrix}$$

Since  $(z_1 + z_2)^* = z_1^* + z_2^*$

&  $z_1, z_2 \in \mathbb{C}$  we can apply the conjugate to above summation. separately.

$$= \int_0^{NT} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[ \sum_{k=0}^{N-1} (x[k])^* \phi_k^*(t) \right] dt$$

= from part ② terms are vanished when the  $k$  values are different.

$$\Rightarrow \sum_{k=0}^{N-1} \left\{ x[k] (x[k])^* \right\} \underbrace{\int_0^{NT} \phi_k(t) \cdot \phi_k^*(t) dt}_{=1} \xrightarrow{\text{from ②}}$$

$$\Rightarrow \gamma = \sum_{k=0}^{N-1} |x[k]|^2$$

**Q1**

$$C.) \sum_{m=0}^{N-1} \phi_k(mT) \phi_\ell^*(mT) \quad ; \quad \phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt}$$

$$\Rightarrow \sum_{m=0}^{N-1} \left[ \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} k(mT)} \right] \left[ \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} \ell(mT)} \right]^*$$

$$= \sum_{m=0}^{N-1} \frac{1}{NT} \left[ e^{j \frac{2\pi}{N} k \cdot m} \right] \left[ e^{-j \frac{2\pi}{N} \ell \cdot m} \right]$$

$$= \frac{1}{NT} \sum_{m=0}^{N-1} e^{j \frac{2\pi}{N} m(k-\ell)}$$

$$= \frac{1}{NT} \cdot \sum_{m=0}^{N-1} \underbrace{\left[ e^{j \frac{2\pi}{N} (k-\ell)} \right]^m}$$

$$= \frac{1}{NT} \frac{1 - \left[ e^{j \frac{2\pi}{N} (k-\ell)} \right]^N}{1 - e^{j \frac{2\pi}{N} (k-\ell)}} \quad \text{using the given hint.}$$

$$= \frac{1}{NT} \cdot \frac{1 - e^{j 2\pi (k-\ell)}}{1 - e^{j \frac{2\pi}{N} (k-\ell)}} \rightarrow \textcircled{1}$$

Case I When  $(k-l) \neq 0 \Rightarrow k \neq l$

Since  $k, l \in \mathbb{Z}_+^+ \Rightarrow (k-l) \in \mathbb{Z}$

$$\therefore e^{j\frac{2\pi}{N}(k-l)} = 1 \quad \text{if } (k-l) \in \mathbb{Z}$$

$$\begin{aligned} \therefore \frac{1}{NT} \left( \frac{1 - e^{j\frac{2\pi}{N}(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} \right) &= \frac{1}{NT} \frac{(1 - 1)}{(1 - e^{j\frac{2\pi}{N}(k-l)})} \\ &= 0 \quad \rightarrow \textcircled{2}. \end{aligned}$$

Case 2 When  $k = l$ , using L'Hopital's rule.

$$\begin{aligned} &= \lim_{(k-l) \rightarrow 0} \frac{1}{NT} \frac{1 - e^{j\frac{2\pi}{N}(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} \\ &= \lim_{(k-l) \rightarrow 0} \frac{1}{NT} \frac{1 \cancel{(k-l) \rightarrow 0} - j\frac{2\pi}{N} \cdot e^{j\frac{2\pi}{N}(k-l)}}{-j\frac{2\pi}{N} e^{j\frac{2\pi}{N}(k-l)}} \quad \begin{matrix} \text{differentiate} \\ \text{w.r.t } (k-l) \end{matrix} \\ &= \lim_{(k-l) \rightarrow 0} \frac{1}{NT} \times N \times \frac{e^{j\frac{2\pi}{N}(k-l)}}{e^{j\frac{2\pi}{N}(k-l)}} \end{aligned}$$

$$= \frac{1}{NT} \times N \times \frac{1}{1}$$

$$= \frac{1}{T} \rightarrow \textcircled{3}$$

$$\therefore \text{from } \textcircled{2} \text{ and } \textcircled{3} \Rightarrow \left\{ \sum_{m=0}^{N-1} \phi_k^{(mT)} \phi_l^{*(mT)} \right\} = \begin{cases} \frac{1}{T} & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases}$$

Q1

d.)  $s[m] = s(mT)$ ,  $m = 0, 1, 2, \dots, N-1$

Consider  $RHS = T \times \sum_{m=0}^{N-1} s[m] \phi_e^*(mT)$

$$= T \times \sum_{m=0}^{N-1} s(mT) \cdot \phi_e^*(mT)$$

$$= T \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] \phi_k(t) \right] \phi_e^*(t) \Big|_{t=mT}$$

$$= T \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(mT) \cdot \phi_e^*(mT) \right]$$

$$= T \underbrace{\sum_{k=0}^{N-1} x[k] \left\{ \underbrace{(T) \sum_{m=0}^{N-1} \phi_k(mT) \phi_e^*(mT)}_{\text{from part C; this part is } \frac{1}{T} \text{ when } k=l \text{ and vanishes otherwise.}} \right\}}$$

so when  $k=l$  ( $k=l$ )

$$= \sum_{k=0}^{N-1} x[l]$$

$$\Rightarrow x[l] \times T \times \frac{1}{T}$$

$\therefore x[l] = T \sum_{m=0}^{N-1} s[m] \phi_e^*(mT)$   
for  $l = 0, 1, 2, \dots, N-1$

Q 2. a)

① consider  $\left| \sum_{k=1}^M \phi_k \right|^2 \geq 0 \rightarrow \textcircled{1}$

$$\begin{aligned}\left| \sum_{k=1}^M \phi_k \right|^2 &= \left( \sum_{k=1}^M \phi_k \right)^T \left( \sum_{k=1}^M \phi_k \right) \\ &= (\phi_1 + \phi_2 + \dots + \phi_M)^T (\phi_1 + \phi_2 + \dots + \phi_M) \\ &= (\phi_1^T + \phi_2^T + \dots + \phi_M^T)(\phi_1 + \phi_2 + \dots + \phi_M)\end{aligned}$$

In this expression there will be,

③  $M$  terms, where  $i=j$  for  $\phi_i^T \phi_j$

④  $M(M-1)$  terms, where  $i \neq j$  for  $\phi_i^T \phi_j$   
(for each  $\phi_i \exists (M-1) \phi_j$ 's such that  $i \neq j$ .)

$$\left| \sum_{k=1}^M \phi_k \right|^2 = \underbrace{(M \times 1)}_{\text{from } \textcircled{1}} + \underbrace{M(M-1)\rho}_{\text{from } \textcircled{2}}$$

$$\left| \sum_{k=1}^M \phi_k \right|^2 \geq 0 \rightarrow \text{from } \textcircled{1}$$

$$M + M(M-1)\rho \geq 0$$

$$1 + (M-1)\rho \geq 0 \quad (\because M > 0)$$

$$\frac{-1}{M-1} \leq \rho \quad \text{--- } \textcircled{2} \quad (\because (M-1) > 0)$$

$$\rho \leq 1 \longrightarrow \textcircled{3}$$

∴ From ② and ③,

$$\frac{-1}{M-1} \leq \rho \leq 1$$

Q2. b.)

Assume  $\{\phi_i\}$  is a simplex set. Then following expression must equal to zero, as  $\{\phi_i\}$ s are already in the minimum energy constellation.

$$\sum_{i=1}^M p_i \phi_i = 0$$

Since  $\{\phi_i\}$  are equally likely,  $p_i = \frac{1}{M}$

$$\therefore \frac{1}{M} \sum_{i=1}^M \phi_i = 0$$

$$\Rightarrow \sum_{i=1}^M \phi_i = 0$$

consider,  $\left\| \left( \sum_{i=1}^M \phi_i \right) \right\|^2 = \|\phi\|^2$

$$\underbrace{\left( \sum_{i=1}^M \phi_i \right)^T \left( \sum_{i=1}^M \phi_i \right)}_{} = 0$$

from part (a) this part is simplified into,

$$M + M(M-1)\rho = 0$$

$$\therefore \rho = \frac{-1}{M-1}$$

Q.2 c.)

② Determining the minimum energy set corresponding to the set  $\{\phi_i^o\}$ .

\* Let  $\{\beta_i^o\}$  be the minimum energy set of the signal set  $\{\phi_i^o\}$ .

$$\Rightarrow \{\beta_i^o\} = \{\phi_i^o - \underline{a}\} \text{ where, } \underline{a} = \frac{1}{M} \sum_{i=1}^M \phi_i^o$$

$$\text{consider } \beta_i^o \top \beta_e = (\phi_i^o - \underline{a}) \top (\phi_e - \underline{a})$$

$$= (\phi_i^o \top - \underline{a} \top)(\phi_e - \underline{a})$$

$$= \phi_i^o \top \phi_e - \phi_i^o \top \underline{a} - \underline{a} \top \phi_e + \underline{a} \top \underline{a}$$

$$= \phi_i^o \top \phi_e - \phi_i^o \left( \frac{1}{M} \sum_{i=1}^M \phi_i^o \right) - \left( \frac{1}{M} \sum_{i=1}^M \phi_i^o \right) \top \phi_e + \underline{a} \top \underline{a}$$

$$= \phi_i^o \top \phi_e - \frac{1}{M} [1 + (M-1)\rho] - \frac{1}{M} [1 + (M-1)\rho] + \underbrace{\|\underline{a}\|^2}_{\text{from part(a)}}$$

$$= \phi_i^o \top \phi_e - \frac{2}{M} [1 + (M-1)\rho] + \frac{1}{M^2} [M + (M-1)\rho]$$

$$= \phi_i^o \top \phi_e - \frac{1}{M} [1 + (M-1)\rho] \rightarrow ①$$

from def:  $\beta_i^T \beta_i = \begin{cases} 1 & i = \ell \\ \rho & i \neq \ell \end{cases}$  Q2 c) contin:

$$\therefore (\beta_i^T \beta_\ell) = \begin{cases} 1 - \frac{1}{M} [1 + (M-1)\rho] ; & i = \ell \\ \rho - \frac{1}{M} [1 + (M-1)\rho] ; & i \neq \ell \end{cases}$$

" "  $\beta_i^T \beta_\ell = \begin{cases} \frac{(M-1)(1-\rho)}{M} ; & i = \ell \\ \frac{\rho-1}{M} ; & i \neq \ell \end{cases}$  - (2)

from part (b)

For the unit average  $\rightarrow \alpha_i^T \alpha_\ell = \begin{cases} 1 ; & i = \ell \\ \rho = \frac{1}{M-1} ; & i \neq \ell \end{cases}$

By rearranging (2),

$$\Rightarrow \beta_i^T \beta_\ell = \begin{cases} 1 \times \left[ \frac{(M-1)(1-\rho)}{M} \right] ; & i = \ell \\ -\frac{1}{M-1} \times \left[ \frac{(M-1)(1-\rho)}{M} \right] ; & i \neq \ell \end{cases}$$

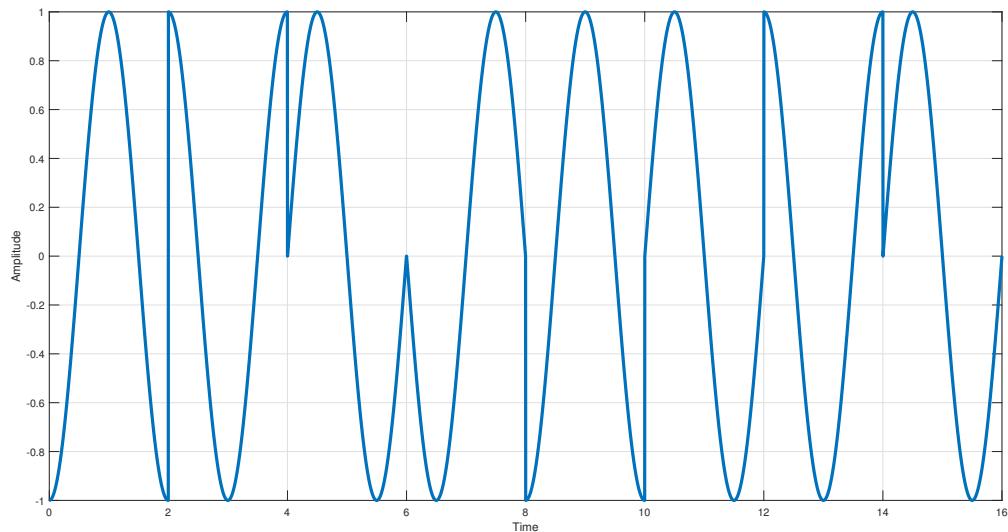
" " we can observe that minimum energy set  $\{\beta_i\}$  is a scaled constellation of the simplex set  $\{\alpha_i\}$ .

## Q4

(a)

Please note that when plotting the below figure following values were assumed for the parameters.

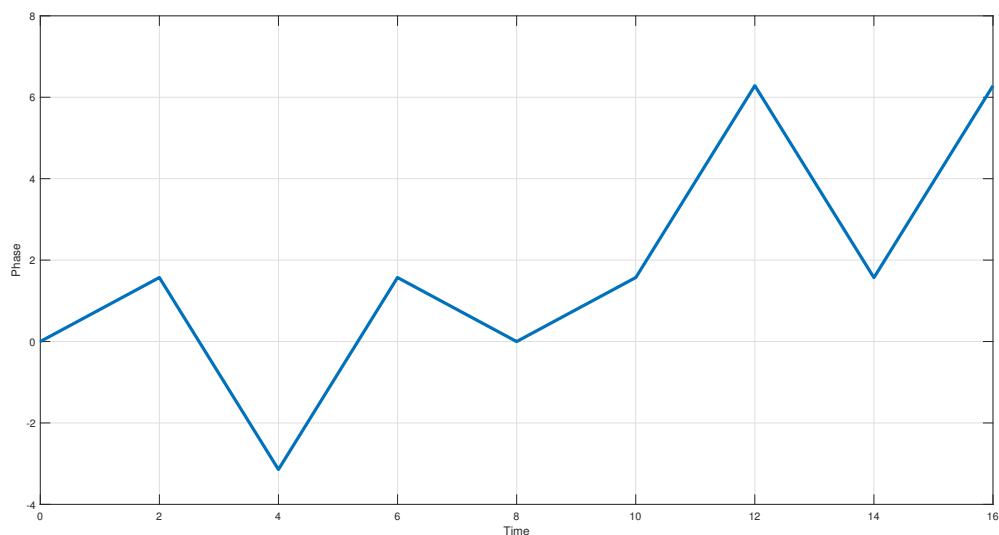
$$\begin{array}{ll} \text{Bit interval } (T_b) & 1 \text{ s} \\ \text{Symbol interval via } (T) & 2 \times T_b = 2 \text{ s} \end{array}$$



(b)

Please note that when plotting the below figure following values were assumed for the parameters.

$$\begin{array}{ll} \text{Bit interval } (T_b) & 1 \text{ s} \\ \text{Symbol interval via (T)} & 2 \times T_b = 2 \text{ s} \end{array}$$



(c)

Please note that when plotting the below figure following values were assumed for the parameters.

$$\begin{array}{ll} \text{Bit interval } (T_b) & 1 \text{ s} \\ \text{Symbol interval via } (T) & 2 \times T_b = 2 \text{ s} \end{array}$$

