Department of Electronic and Telecommunication Engineering

University of Moratuwa, Sri Lanka

EN3053 - Digital Communications - I

Assignment No. 1

Submitted by

Thalagala B.P. 180631J

Submitted on

December 7, 2021

Contents

**** PDF is clickable***

Note:

*All the executable codes, a simulink model and a voice sample are included in a separate folder named as **EN2053-Assignment 2-180574K-180631J.rar** which was submitted along with this report.*

*Before run the **VoiceTransmissionBaseband.slx** simulink simulation related to the part 5 of the Task 1 ,(included in a separate folder named as **Task1-RFPM** inside the above folder) change the directories related to the voice input and output. Voice sample named as **voice.wav** is also included in the same folder.*

Additionally all the materials related to Task 1 can also be found at https://github.com/bimalka98/RF-Propagation-Model

$$S(t) = \sum_{k=0}^{N-1} x[k]\, \phi_k(t) \qquad 0 \leq t < NT$$

where $x[k] \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $T > 0$ and

$$\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} \cdot kt}$$

consider, $\langle \phi_k(t), \phi_\ell(t) \rangle$: the Hermitian Inner Product.

$$= \int_0^{NT} \phi_k(t) \cdot \phi_\ell^*(t)\, dt$$

$$= \int_0^{NT} \left(\frac{1}{\sqrt{NT}} \cdot e^{j \frac{2\pi}{NT} kt} \right) \cdot \left(\frac{1}{\sqrt{NT}} \cdot e^{j \frac{2\pi}{NT} \ell t} \right)^* dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} kt} \cdot e^{-j \frac{2\pi}{NT} \ell t}\, dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (k-\ell) t}\, dt$$

$\boxed{\text{Case } 1 :}$ if $k = \ell$:

$$\Rightarrow \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (0) \cdot t}\, dt$$

$$= \frac{1}{NT} \int_0^{NT} 1\, dt = \frac{1}{NT} [t]_0^{NT} = \frac{1}{NT} [NT - 0]$$

$$= \underline{\underline{1}} \longrightarrow (1)$$

$\boxed{\text{Case 2 :}}$ if $k \neq \ell$

Since $k, \ell \in \mathbb{Z}_0^+ \Rightarrow (k-\ell) \in \mathbb{Z}$

$\therefore$ Let $(k-\ell) = m \in \mathbb{Z}$

$$\therefore \frac{1}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT} mt} \, dt$$

$$= \frac{1}{\cancel{NT}} \left. \frac{e^{j\frac{2\pi}{NT} \cdot m \cdot t}}{j\frac{2\pi}{\cancel{NT}} \cdot m} \right|_0^{NT}$$

$$= \frac{1}{j2\pi m} \cdot \left[e^{j\frac{2\pi}{\cancel{NT}} \cdot m \cdot \cancel{NT}} - 1 \right]$$

$$= \frac{1}{j2\pi m} \left[e^{j(2\pi)m} - 1 \right] \quad \text{where } m \in \mathbb{Z}$$

$$= \frac{1}{j2\pi m} \left[\underbrace{\cos(2\pi m)}_{(\text{always } = 1)} + j \cancel{\sin(2\pi m)}^{\,0} - 1 \right]$$

$$= \frac{1}{j2\pi m} [1 - 1] \Rightarrow 0 \longrightarrow (2)$$

from (1) and (2)

$$\therefore \int_0^{NT} \phi_k(t) \cdot \phi_\ell^*(t) \, dt = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

$\boxed{\text{Q1}}$

b.) Total Energy Content of $s(t)$ $\Big\} = \langle s(t), s(t) \rangle$

$$= \int_0^{NT} s(t) \cdot (s(t))^* \, dt \quad ; (0 < t \leq NT)$$

complex conjugate

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[\sum_{k=0}^{N-1} x[k] \, \phi_k(t) \right]^* dt$$

Since $(z_1 + z_2)^* = z_1^* + z_2^*$

$\forall \, z_1, z_2 \in \mathbb{C}$ we can apply the conjugate to above summation. seperately.

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[\sum_{k=0}^{N-1} (x[k])^* \, \phi_k^*(t) \right] dt$$

= from part ⓐ terms are vanished when the k values are different.

$$\Rightarrow \sum_{k=0}^{N-1} \left\{ [x[k] \cdot (x[k])^*] \underbrace{\int_0^{NT} \phi_k(t) \cdot \phi_k^*(t) \, dt}_{=1 \longrightarrow \text{from ⓐ}} \right\}$$

$$\Rightarrow \mathcal{E} = \sum_{k=0}^{N-1} |x[k]|^2$$

$\boxed{Q1}$

c.) $\sum_{m=0}^{N-1} \phi_k(mT)\, \phi_l^*(mT)$; $\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}kt}$

$$\Rightarrow \sum_{m=0}^{N-1} \left[\frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}k(mT)}\right]\left[\frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}l(mT)}\right]^*$$

$$= \sum_{m=0}^{N-1} \frac{1}{NT}\left[e^{j\frac{2\pi}{N}k.m}\right]\left[e^{-j\frac{2\pi}{N}l.m}\right]$$

$$= \frac{1}{NT} \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}m(k-l)}$$

$$= \frac{1}{NT} \cdot \underbrace{\sum_{m=0}^{N-1} \left[e^{j\frac{2\pi}{N}(k-l)}\right]^m}$$

$$= \frac{1}{NT} \quad \frac{1 - \left[e^{j\frac{2\pi}{N}(k-l)}\right]^N}{1 - e^{j\frac{2\pi}{N}(k-l)}}$$ using the given hint.

$$= \frac{1}{NT} \cdot \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} \longrightarrow ①$$

$\boxed{\text{Case I}}$ when $(k-l) \neq 0 \Rightarrow k \neq l$

since $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$$\therefore \quad e^{j2\pi(k-l)} = 1 \quad \forall \ (k-l) \in \mathbb{Z}$$

$$\therefore \quad \frac{1}{NT} \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} = \frac{1}{NT} \frac{(1-1)}{(1 - e^{j\frac{2\pi}{N}(k-l)})}$$

$$= 0 \longrightarrow ②.$$

$\boxed{\text{case 2}}$ when $k = l$, using L'hopital's rule.

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \frac{-j2\pi \cdot e^{j2\pi(k-l)}}{-j\frac{2\pi}{N} e^{j\frac{2\pi}{N}(k-l)}} \qquad \text{differentiate w.r.t } (k-l)$$

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \times N \times \frac{e^{j2\pi(k-l)}}{e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \frac{1}{NT} \times N \times \frac{1}{1}$$

$$= \frac{1}{T} \longrightarrow ③$$

$$\therefore \text{ from ② and ③} \Rightarrow \left\{ \sum_{m=0}^{N-1} \phi_k(mT) \, \phi_l^*(mT) = \begin{cases} \frac{1}{T} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \right.$$

$\boxed{Q1}$

d.) $s[m] = s(mT)$; $m = 0, 1, 2, \dots, N-1$

Consider $RHS = T \times \sum_{m=0}^{N-1} s[m]\,\phi_l^*(mT)$

$$= T \times \sum_{m=0}^{N-1} s(mT).\,\phi_l^*(mT)$$

$$= \cancel{T \times \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k] \right.}$$

$$= T \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k]\,\phi_k(t) \right] \phi_l^*(t) \Bigg|_{t=mT}$$

$$= T \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k].\,\phi_k(mT).\,\phi_l^*(mT) \right]$$

$$= T \sum_{k=0}^{N-1} x[k] \left\{ (T) \underbrace{\sum_{m=0}^{N-1} \phi_k(mT)\,\phi_l^*(mT)}_{} \right\}$$

from part (c); this part is $\frac{1}{T}$ when $k=l$ and vanishes otherwise.

so when $k=l$ $(k=l)$

$$= \cancel{\sum_{k=0}^{N-1} x[k]}$$

$$\Rightarrow \quad x[l] \times T \times \frac{1}{T}$$

$$\therefore \quad x[l] = T.\sum_{m=0}^{N-1} s[m]\,\phi_l^*(mT)$$

for $l = 0, 1, 2, \dots, N-1$

$\boxed{Q2.}$ a)

consider $\left|\sum_{k=1}^{M} \theta_k\right|^2 \geq 0 \longrightarrow$ ①

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 = \left(\sum_{k=1}^{M} \theta_k\right)^T \left(\sum_{k=1}^{M} \theta_k\right)$$

$$= (\theta_1 + \theta_2 + \dots + \theta_M)^T (\theta_1 + \theta_2 + \dots + \theta_M)$$

$$= (\theta_1^T + \theta_2^T + \dots + \theta_M^T)(\theta_1 + \theta_2 + \dots + \theta_M)$$

In this expression there will be,

(I) M terms, where $i=j$ for $\theta_i^T \theta_j$

(II) $M(M-1)$ terms, wher $i \neq j$ for $\theta_i^T \theta_j$
(for each θ_i $\exists$, $(M-1)$ θ_j's such that $i \neq j$.

$\therefore$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 = \underbrace{(M \times 1)}_{\text{from (I)}} + \underbrace{M(M-1)\rho}_{\text{from (II)}}$$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 \geq 0 \longrightarrow \text{from ①}$$

$$M + M(M-1)\rho \geq 0$$

$$1 + (M-1)\rho \geq 0 \quad (\because \text{ since } M > 0)$$

$$\frac{-1}{M-1} \leq \rho \longrightarrow ② \quad (\because \text{ since } (M-1) > 0)$$

$\rho \leq 1 \longrightarrow 3$

$\therefore$ From 2 and 3,

$$\frac{-1}{M-1} \leq \rho \leq 1$$

$\boxed{Q2.}$ b.)

Assume $\{\theta_i\}$ is a simplex set. Then following expression must equal to zero, as $\{\theta_i\}$s are already in the minimum energy constellation.

$$\sum_{i=1}^{M} P_i \theta_i = \underline{0}$$

Since $\{\theta_i\}$ are equally likely, $P_i = \frac{1}{M}$

$$\therefore \frac{1}{M} \sum_{i=1}^{M} \theta_i = \underline{0}$$

$$\Rightarrow \sum_{i=1}^{M} \theta_i = \underline{0}$$

consider, $$\left\|\left(\sum_{i=1}^{M} \theta_i\right)\right\|^2 = \|\underline{0}\|^2$$

$$\underbrace{\left(\sum_{i=1}^{M} \theta_i\right)^T \left(\sum_{i=1}^{M} \theta_i\right)} = 0$$

from part (a) this part is simplified into,

$$M + M(M-1)\rho = 0$$

$$\therefore \rho = \frac{-1}{M-1} \quad //$$

$\boxed{Q.2}$ c.)

① Determining the minimum energy set corresponding to the set $\{\theta_i\}$.

* Let $\{\beta_i\}$ be the minimum energy set of the signal set $\{\theta_i\}$.

$$\Rightarrow \{\beta_i\} = \{\theta_i - \underline{a}\} \text{ where, } \underline{a} = \frac{1}{M}\sum_{i=1}^{M} \theta_i$$

consider $\beta_i^T \beta_\ell = (\theta_i - \underline{a})^T(\theta_\ell - \underline{a})$

$$= (\theta_i^T - \underline{a}^T)(\theta_\ell - \underline{a})$$

$$= \theta_i^T\theta_\ell - \theta_i^T\underline{a} - \underline{a}^T\theta_\ell + \underline{a}^T\underline{a}$$

$$= \theta_i^T\theta_\ell - \theta_i^T\left(\frac{1}{M}\sum_{i=1}^{M}\theta_i\right) - \left(\frac{1}{M}\sum_{i=1}^{M}\theta_i\right)^T\theta_\ell + \underline{a}^T\underline{a}$$

$$= \theta_i^T\theta_\ell - \frac{1}{M}[1+(M-1)\rho] - \frac{1}{M}[1+(M-1)\rho] + \|\underline{a}\|^2$$

from part (a.)

$$= \theta_i^T\theta_\ell - \frac{2}{M}[1+(M-1)\rho] + \frac{1}{M^2}[M+(M-1)\rho]$$

$$= \theta_i^T\theta_\ell - \frac{1}{M}[1+(M-1)\rho] \longrightarrow ①$$

from def^n: $\theta_i^T \theta = \begin{cases} 1 & i = l \\ \rho & i \neq l \end{cases}$ Q2 c) contin:

$$\therefore (\beta_i)^T(\beta_l) = \begin{cases} 1 - \frac{1}{M}[1 + (M-1)\rho] \; ; & i = l \\ \rho - \frac{1}{M}[1 + (M-1)\rho] \; ; & i \neq l \end{cases}$$

$$\therefore \beta_i^T \beta_l = \left\{ \begin{array}{ll} \frac{(M-1)(1-\rho)}{M} \; ; & i = l \\ \frac{(\rho-1)}{M} \; ; & i \neq l \end{array} \right\} - (2)$$

from part (b)

For the unit average Simplex set $\rightarrow$ $\alpha_i^T \alpha_l = \begin{cases} 1 & ; i = l \\ \rho = \frac{-1}{M-1} & ; i \neq l \end{cases}$

By rearranging (2),

$$\Rightarrow \beta_i^T \beta_l = \begin{cases} 1 \times \left[\frac{(M-1)(1-\rho)}{M}\right] & i = l \\ \frac{-1}{M-1} \times \left[\frac{(M-1)(1-\rho)}{M}\right] ; & i \neq l \end{cases}$$

$\therefore$ We can observe that minimum energy set $\{\beta_i\}$ is a scaled constellation of the simplex set $\{\alpha_i\}$.