

$$S(t) = \sum_{k=0}^{N-1} x[k] \phi_k(t) \quad 0 \leq t < NT$$

where $x[k] \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $T > 0$ and

$$\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} k t}$$

Consider, $\langle \phi_n(t), \phi_\ell(t) \rangle$: the Hermitian Inner Product.

$$= \int_0^{NT} \phi_n(t) \cdot \phi_\ell^*(t) dt$$

$$= \int_0^{NT} \left(\frac{1}{\sqrt{NT}} \cdot e^{j \frac{2\pi}{NT} kt} \right) \cdot \left(\frac{1}{\sqrt{NT}} \cdot e^{j \frac{2\pi}{NT} \ell t} \right)^* dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} kt} \cdot e^{-j \frac{2\pi}{NT} \ell t} dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (k-\ell)t} dt$$

Case 1 : if $k=\ell$:

$$\Rightarrow \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (0)t} dt$$

$$= \frac{1}{NT} \int_0^{NT} 1 dt = \frac{1}{NT} [t]_0^{NT} = \frac{1}{NT} [NT - 0]$$

= 1 \longrightarrow ①

Case 2 : if $k \neq l$

Since $k, l \in \mathbb{Z}^+ \Rightarrow (k-l) \in \mathbb{Z}$

∴ Let $(k-l) = m \in \mathbb{Z}$

$$\frac{1}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT} mt} dt$$

$$= \frac{1}{NT} \left[\frac{e^{j\frac{2\pi}{NT} \cdot m \cdot t}}{j\frac{2\pi}{NT} \cdot m} \right] \Big|_0^{NT}$$

$$= \frac{1}{j^{2\pi m}} \left[e^{j\frac{2\pi}{NT} \cdot m \cdot NT} - 1 \right]$$

$$= \frac{1}{j^{2\pi m}} \left[e^{j(2\pi)m} - 1 \right] \text{ where } m \in \mathbb{Z}$$

$$= \frac{1}{j^{2\pi m}} \left[\underbrace{\cos(2\pi m)}_{(\text{Always } = 1)} + j \overbrace{\sin(2\pi m)}^0 - 1 \right]$$

$$= \frac{1}{j^{2\pi m}} [1 - 1] \Rightarrow 0 \rightarrow \textcircled{2}$$

from ① and ②

$$\therefore \int_0^{NT} \phi_k(t) \cdot \phi_l^*(t) dt = \begin{cases} 1 & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases}$$

Q1

b.) Total Energy $\gamma = \langle s(t), s(t) \rangle$
 Content of $s(t)$

$$= \int_0^{NT} s(t) (s(t))^* dt ; (0 < t < NT)$$

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[\sum_{k=0}^{N-1} x[k] \phi_k^*(t) \right]^* dt$$

complex conjugate

Since $(z_1 + z_2)^* = z_1^* + z_2^*$

& $z_1, z_2 \in \mathbb{C}$ we can apply the conjugate to above summation. separately.

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[\sum_{k=0}^{N-1} (x[k])^* \phi_k^*(t) \right] dt$$

= from part ① terms are vanished when the k values are different.

$$\Rightarrow \sum_{k=0}^{N-1} \{x[k] (x[k])^*\} \int_0^{NT} \phi_k(t) \cdot \phi_k^*(t) dt$$

$\underbrace{= 1}_{\text{from ①}}$

$$\Rightarrow E = \sum_{k=0}^{N-1} |x[k]|^2$$

Q 1

$$C) \sum_{m=0}^{N-1} \phi_k(mT) \phi_l^*(mT) \quad ; \quad \phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt}$$

$$\Rightarrow \sum_{m=0}^{N-1} \left[\frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} k(mT)} \right] \left[\frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} l(mT)} \right]^*$$

$$= \sum_{m=0}^{N-1} \frac{1}{NT} \left[e^{j \frac{2\pi}{N} k \cdot m} \right] \left[e^{-j \frac{2\pi}{N} l \cdot m} \right]$$

$$= \frac{1}{NT} \sum_{m=0}^{N-1} e^{j \frac{2\pi}{N} m(k-l)}$$

$$= \frac{1}{NT} \cdot \sum_{m=0}^{N-1} \left[e^{j \frac{2\pi}{N} (k-l)} \right]^m$$

(brace)

$$= \frac{1}{NT} \frac{1 - \left[e^{j \frac{2\pi}{N} (k-l)} \right]^N}{1 - e^{j \frac{2\pi}{N} (k-l)}}$$

using the given
hint.

$$= \frac{1}{NT} \cdot \frac{1 - e^{j 2\pi (k-l)}}{1 - e^{j \frac{2\pi}{N} (k-l)}} \rightarrow \textcircled{1}$$

Case I. When $(k-l) \neq 0 \Rightarrow k \neq l$

Since $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$$\therefore e^{j2\pi(k-l)} = 1 \quad \text{if } (k-l) \in \mathbb{Z}$$

$$\begin{aligned} \text{so } \frac{1}{NT} e^{\frac{1-e^{j2\pi(k-l)}}{1-e^{j\frac{2\pi}{N}(k-l)}}} &= \frac{1}{NT} \frac{(1-1)}{(1-e^{j\frac{2\pi}{N}(k-l)})} \\ &= 0 \rightarrow \textcircled{2}. \end{aligned}$$

Case 2 When $k=l$, using L'Hopital's rule.

$$\begin{aligned} &= \lim_{(k-l) \rightarrow 0} \frac{1}{NT} \frac{1-e^{j2\pi(k-l)}}{1-e^{j\frac{2\pi}{N}(k-l)}} \\ &= \lim_{(k-l) \rightarrow 0} \frac{1}{NT} \cdot \frac{1-j2\pi \cdot e^{j2\pi(k-l)}}{-j\frac{2\pi}{N} e^{j\frac{2\pi}{N}(k-l)}} \quad \begin{matrix} \text{differentiate} \\ \text{w.r.t } (k-l) \end{matrix} \end{aligned}$$

$$= \lim_{(k-l) \rightarrow 0} \frac{1}{NT} \times N \times \frac{e^{j2\pi(k-l)}}{e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \frac{1}{NT} \times N \times \frac{1}{1}$$

$$= \frac{1}{T} \rightarrow \textcircled{3}$$

$$\therefore \text{from } \textcircled{2} \text{ and } \textcircled{3} \Rightarrow \left\{ \sum_{m=0}^{N-1} \phi_k(mT) \phi_l^*(mT) = \begin{cases} \frac{1}{T} & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases} \right.$$

Q4

d.) $s[m] = s(mT)$, $m = 0, 1, 2, \dots, N-1$

Consider $RHS = T \times \sum_{m=0}^{N-1} s[m] \phi_e^*(mT)$

$$= T \times \sum_{m=0}^{N-1} s(mT) \cdot \phi_e^*(mT)$$

$$= \pi x \left[\sum_{k=0}^{N-1} x[k] \phi_k(t) \right] \phi_e^*(t)$$

$$= T \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k] \phi_k(mT) \right] \phi_e^*(t) \Big|_{t=mT}$$

$$= T \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k] \phi_k(mT) \cdot \phi_e^*(mT) \right]$$

$$= \sum_{k=0}^{N-1} x[k] \left\{ (T) \underbrace{\sum_{m=0}^{N-1} \phi_k(mT) \phi_e^*(mT)}_{(k=l)}$$

so when $k=l$

from part C; this part is $\frac{1}{T}$ when $k=l$ and vanishes otherwise.

$$\Rightarrow x[l] \times T \times \frac{1}{T}$$

$\therefore x[l] = T \sum_{m=0}^{N-1} s[m] \phi_e^*(mT)$

for $l = 0, 1, 2, \dots, N-1$

Q 2. a)

consider $\left| \sum_{k=1}^M \theta_k \right|^2 \geq 0 \rightarrow \textcircled{1}$

$$\begin{aligned} \left| \sum_{k=1}^M \theta_k \right|^2 &= \left(\sum_{k=1}^M \theta_k \right)^T \left(\sum_{k=1}^M \theta_k \right) \\ &= (\theta_1 + \theta_2 + \dots + \theta_M)^T (\theta_1 + \theta_2 + \dots + \theta_M) \\ &= (\theta_1^T + \theta_2^T + \dots + \theta_M^T)(\theta_1 + \theta_2 + \dots + \theta_M) \end{aligned}$$

In this expression there will be,

- (i) M terms, where $i=j$ for $\theta_i^T \theta_j$
- (ii) $M(M-1)$ terms, where $i \neq j$ for $\theta_i^T \theta_j$
(for each $\theta_i^T \exists (M-1) \theta_j$'s such that $i \neq j$.)

$$\left| \sum_{k=1}^M \theta_k \right|^2 = \underbrace{(M \times 1)}_{\text{from } \textcircled{1}} + \underbrace{M(M-1)\rho}_{\text{from } \textcircled{2}}$$

$$\left| \sum_{k=1}^M \theta_k \right|^2 \geq 0 \rightarrow \text{from } \textcircled{1}$$

$$M + M(M-1)\rho \geq 0$$

$$1 + (M-1)\rho \geq 0 \quad (\because M > 0)$$

$$\frac{-1}{M-1} \leq \rho \rightarrow \textcircled{2} \quad (\because (M-1) > 0)$$

$$\rho \leq 1 \longrightarrow \textcircled{3}$$

∴ From ② and ③,

$$\frac{-1}{M-1} \leq \rho \leq 1$$

Q2. b.)

Assume $\{\phi_i\}$ is a simplex set. Then following expression must equal to zero, as $\{\phi_i\}$ s are already in the minimum energy constellation.

$$\sum_{i=1}^M p_i \phi_i = 0$$

Since $\{\phi_i\}$ are equally likely, $p_i = \frac{1}{M}$

$$\therefore \frac{1}{M} \sum_{i=1}^M \phi_i = 0$$

$$\Rightarrow \sum_{i=1}^M \phi_i = 0$$

consider, $\left\| \left(\sum_{i=1}^M \phi_i \right) \right\|^2 = \|0\|^2$

$$\underbrace{\left(\sum_{i=1}^M \phi_i \right)^T \left(\sum_{i=1}^M \phi_i \right)}_{} = 0$$

from part (a) this part is simplified into,

$$M + M(M-1)\rho = 0$$

$$\therefore \rho = \frac{-1}{M-1}$$

Q.2

c.)

② Determining the minimum energy set corresponding to the set $\{\phi_i^o\}$.

* Let $\{\beta_i^o\}$ be the minimum energy set of the signal set $\{\phi_i^o\}$.

$$\Rightarrow \{\beta_i^o\} = \{\phi_i^o - \underline{a}\} \text{ where, } \underline{a} = \frac{1}{M} \sum_{i=1}^M \phi_i^o$$

Consider

$$\beta_i^o \top \beta_e = (\phi_i^o - \underline{a}) \top (\phi_e - \underline{a})$$

$$= (\phi_i^o \top - \underline{a} \top)(\phi_e - \underline{a})$$

$$= \phi_i^o \top \phi_e - \phi_i^o \top \underline{a} - \underline{a} \top \phi_e + \underline{a} \top \underline{a}$$

$$= \phi_i^o \top \phi_e - \phi_i^o \left(\frac{1}{M} \sum_{i=1}^M \phi_i^o \right) - \left(\frac{1}{M} \sum_{i=1}^M \phi_i^o \right) \top \phi_e + \underline{a} \top \underline{a}$$

$$= \phi_i^o \top \phi_e - \frac{1}{M} [1 + (M-1)\rho] - \underbrace{\frac{1}{M} [1 + (M-1)\rho]}_{\text{from part(a)}} + \|\underline{a}\|^2$$

$$= \phi_i^o \top \phi_e - \frac{2}{M} [1 + (M-1)\rho] + \frac{1}{M^2} [M + (M-1)\rho]$$

$$= \phi_i^o \top \phi_e - \frac{1}{M} [1 + (M-1)\rho] \rightarrow ①$$

from def^h: $\beta_i^T \beta_l = \begin{cases} 1 & i = l \\ \rho & i \neq l \end{cases}$ Q2 c.) contin:

$\therefore (\beta_i^T \beta_l) = \begin{cases} 1 - \frac{1}{M} [1 + (M-1)\rho] ; & i = l \\ \rho - \frac{1}{M} [1 + (M-1)\rho] ; & i \neq l \end{cases}$

$\therefore \beta_i^T \beta_l = \begin{cases} \frac{(M-1)(1-\rho)}{M} ; & i = l \\ \frac{\rho-1}{M} ; & i \neq l \end{cases} \quad \textcircled{2}$

from part (b)

For the unit average $\Rightarrow \alpha_i^T \alpha_l = \begin{cases} 1 ; & i = l \\ \rho = \frac{1}{M-1} ; & i \neq l \end{cases}$

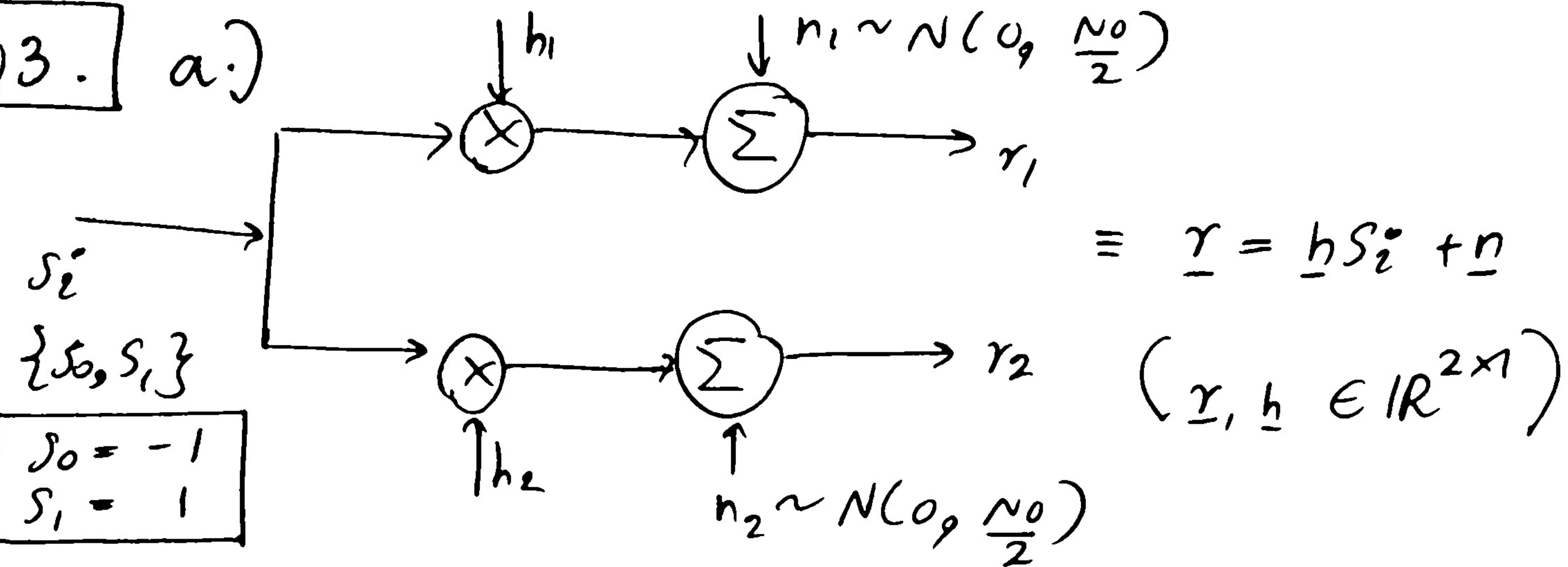
Simplex set

By rearranging $\textcircled{2}$,

$$\Rightarrow \beta_i^T \beta_l = \begin{cases} 1 \times \left[\frac{(M-1)(1-\rho)}{M} \right] ; & i = l \\ -\frac{1}{M-1} \times \left[\frac{(M-1)(1-\rho)}{M} \right] ; & i \neq l \end{cases}$$

\therefore we can observe that minimum energy set $\{\beta_i\}$ is a scaled constellation of the simplex set $\{\alpha_i\}$.

Q3. a.)



a.) Consider the following a posteriori probabilities.

① For the receiver to classify the observed signal as s_0

$$\Rightarrow P(s_0 | r) > P(s_1 | r) \rightarrow ①$$

② For the receiver to classify the observed signal as s_1

$$\Rightarrow P(s_0 | r) < P(s_1 | r) \rightarrow ②$$

By combining ① and ②, for

$$P(s_0 | r) \stackrel{s_0}{\geq} P(s_1 | r)$$

③ This is what demanded by the receiver!

But what is known, to us, are the following probabilities.

$$P(r | s_0), P(r | s_1)$$

∴ By using Bayes theorem, ③ can be re written as follows.

$$P(\underline{s}_0 | \underline{r}) \underset{\underline{s}_1}{\gtrless} P(\underline{s}_1 | \underline{r})$$

$$\frac{P(\underline{r} | \underline{s}_0) \times P(\underline{s}_0)}{P(\underline{r})} \underset{\underline{s}_1}{\gtrless} \frac{P(\underline{r} | \underline{s}_1) P(\underline{s}_1)}{P(\underline{r})}$$

Since \underline{s}_0 and \underline{s}_1 are equally likely signals,

$$P(\underline{s}_0) = P(\underline{s}_1) = \frac{1}{2}$$

$$\therefore P(\underline{r} | \underline{s}_0) \underset{\underline{s}_1}{\gtrless} P(\underline{r} | \underline{s}_1)$$

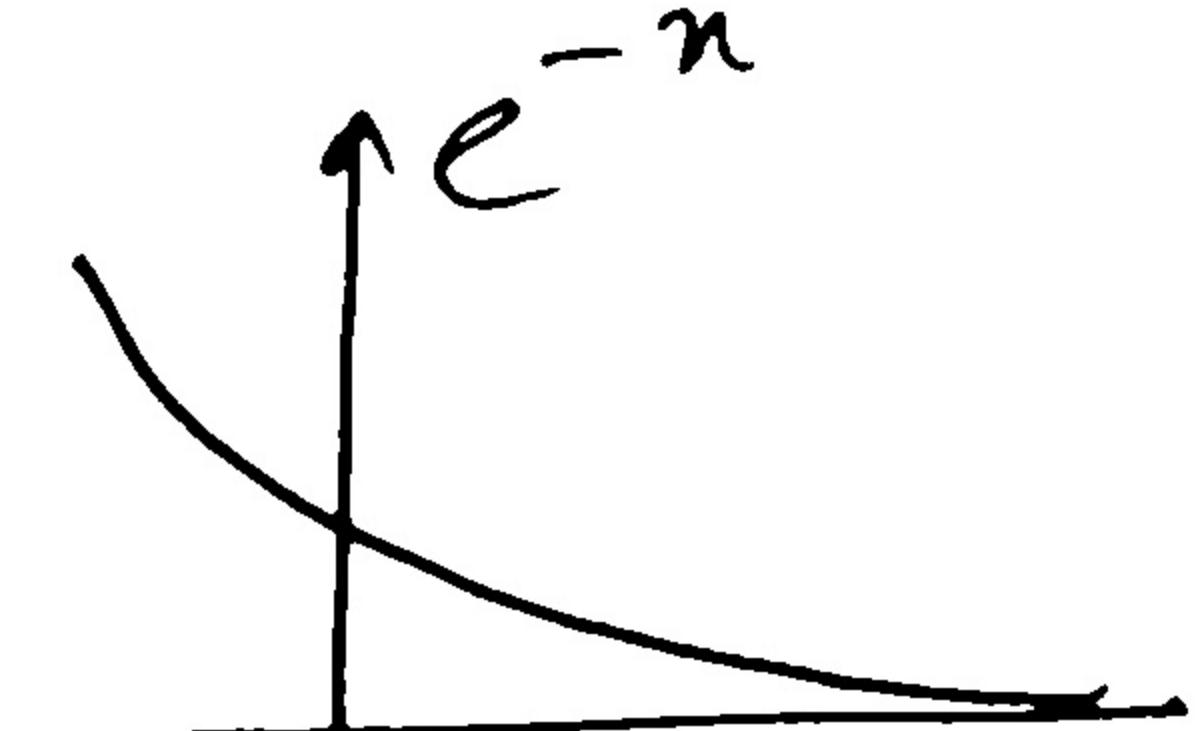
Since $\underline{r} = \underline{h}\underline{s}_i + \underline{n}$

$$\begin{aligned} \underline{s}_i &= \underline{s}_0 \rightarrow \underline{r} = \underbrace{\underline{h}(-1)}_{\underline{s}_0} + \underline{n} \\ \underline{s}_i &= \underline{s}_1 \rightarrow \underline{r} = \underbrace{\underline{h}(1)}_{\underline{s}_1} + \underline{n} \end{aligned}$$

$$\therefore P(\underline{r} | -\underline{h}) \underset{\underline{s}_1}{\gtrless} P(\underline{r} | \underline{h})$$

$$N_2(-\underline{h}, \frac{N_0}{2} I_2) \underset{\underline{s}_1}{\gtrless} N_2(\underline{h}, \frac{N_0}{2} I_2)$$

$$e^{-\|\underline{r} - (-\underline{h})\|^2} \underset{\underline{s}_1}{\gtrless} e^{-\|\underline{r} - (\underline{h})\|^2}$$



$$\|\underline{r} + \underline{h}\|^2 \underset{\underline{s}_0}{\gtrless} \|\underline{r} - \underline{h}\|^2$$

$$\|\underline{r} + \underline{h}\|^2 \stackrel{s_1}{\underset{s_0}{\gtrless}} \|\underline{r} - \underline{h}\|^2$$

$$(\underline{r} + \underline{h})^T (\underline{r} + \underline{h}) \stackrel{s_1}{\underset{s_0}{\gtrless}} (\underline{r} - \underline{h})^T (\underline{r} - \underline{h})$$

$$(\underline{r}^T + \underline{h}^T) (\underline{r} + \underline{h}) \stackrel{s_1}{\underset{s_0}{\gtrless}} (\underline{r}^T - \underline{h}^T) (\underline{r} - \underline{h})$$

$$\begin{aligned} \underline{r}^T \underline{r} + \underline{r}^T \underline{h} & \stackrel{s_1}{\gtrless} \underline{r}^T \underline{r} - \underline{r}^T \underline{h} \\ + \underline{h}^T \underline{r} + \underline{h}^T \underline{h} & \stackrel{s_1}{\gtrless} - \underline{h}^T \underline{r} + \underline{h}^T \underline{h} \end{aligned}$$

Since $\underline{h}, \underline{r} \in \mathbb{R}^{2 \times 1} \Rightarrow \underline{r}^T \underline{h} = \underline{h}^T \underline{r}$

$$\therefore 2 \underline{r}^T \underline{h} \stackrel{s_1}{\underset{s_0}{\gtrless}} - 2 \underline{r}^T \underline{h}$$

$$4 \underline{r}^T \underline{h} \stackrel{s_1}{\underset{s_0}{\gtrless}} 0$$

$$\boxed{\underline{r}^T \underline{h} \stackrel{s_1}{\underset{s_0}{\gtrless}} 0}$$

\leftarrow Optimum MAP decision rule.

$$\underline{r}^T h \geq_{S_0} 0$$

$$[r_1 \ r_2] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \geq_{S_0} 0$$

$$r_1 h_1 + r_2 h_2 \geq_{S_0} 0$$

$$r_1 \geq_{S_0} -\left(\frac{h_2}{h_1}\right) r_2$$

$h_1 < 0$ inequality changes

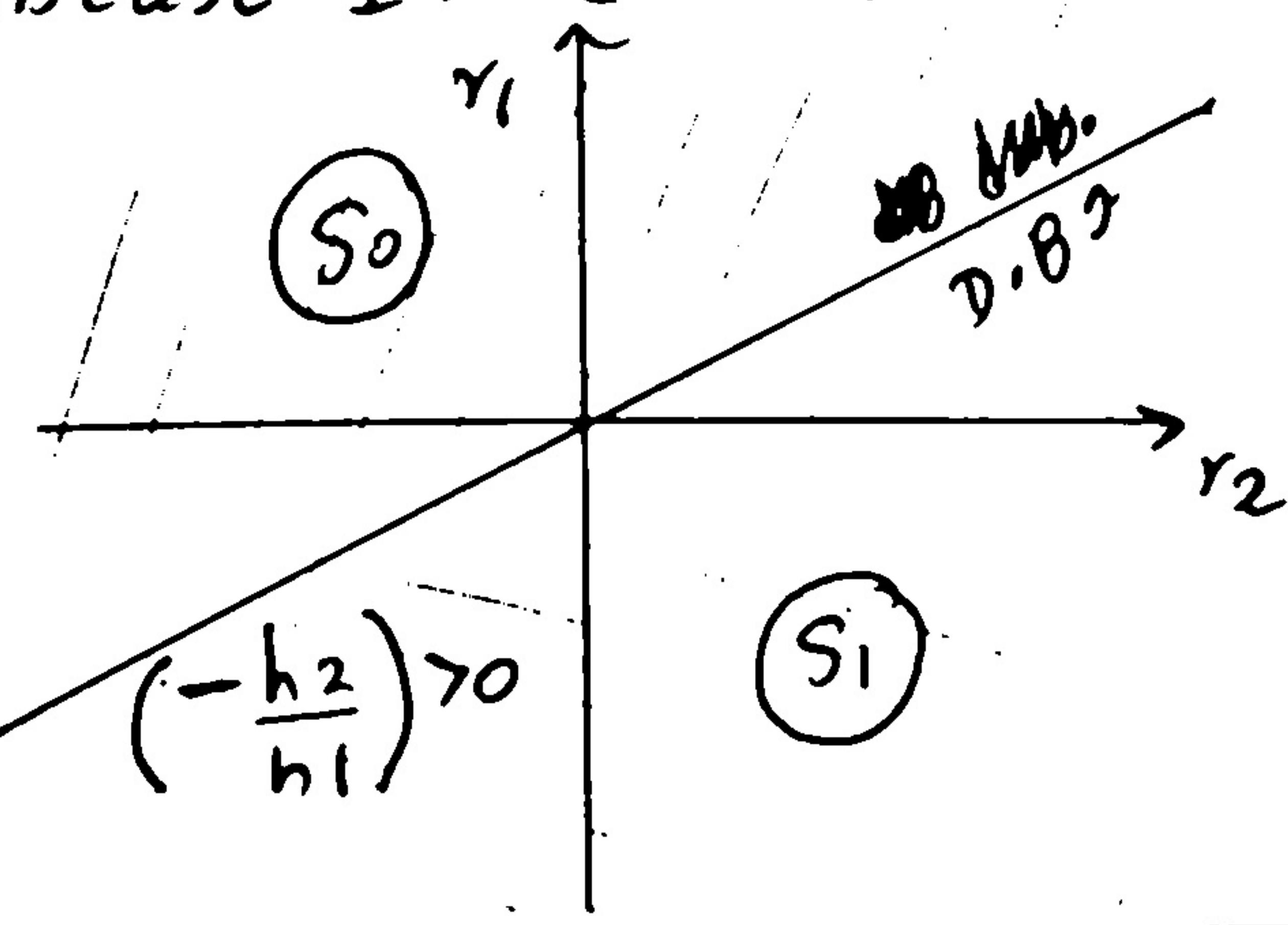
$h_1 > 0$ inequality remains the same.

case I

$$(h_1 < 0)$$

$$r_1 \geq_{S_1} -\left(\frac{h_2}{h_1}\right) r_2$$

Subcase I. ($h_2 > 0$)

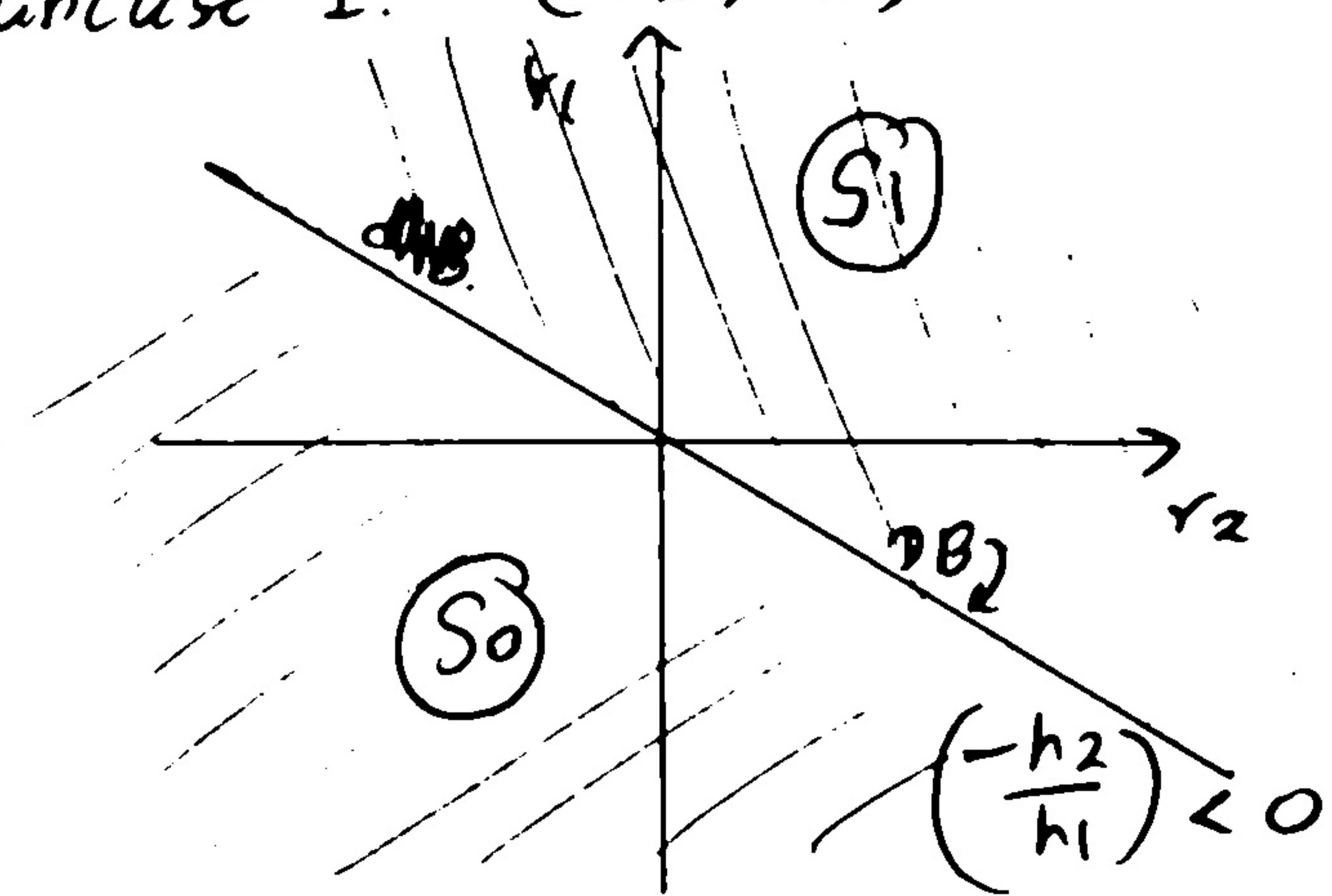


case II

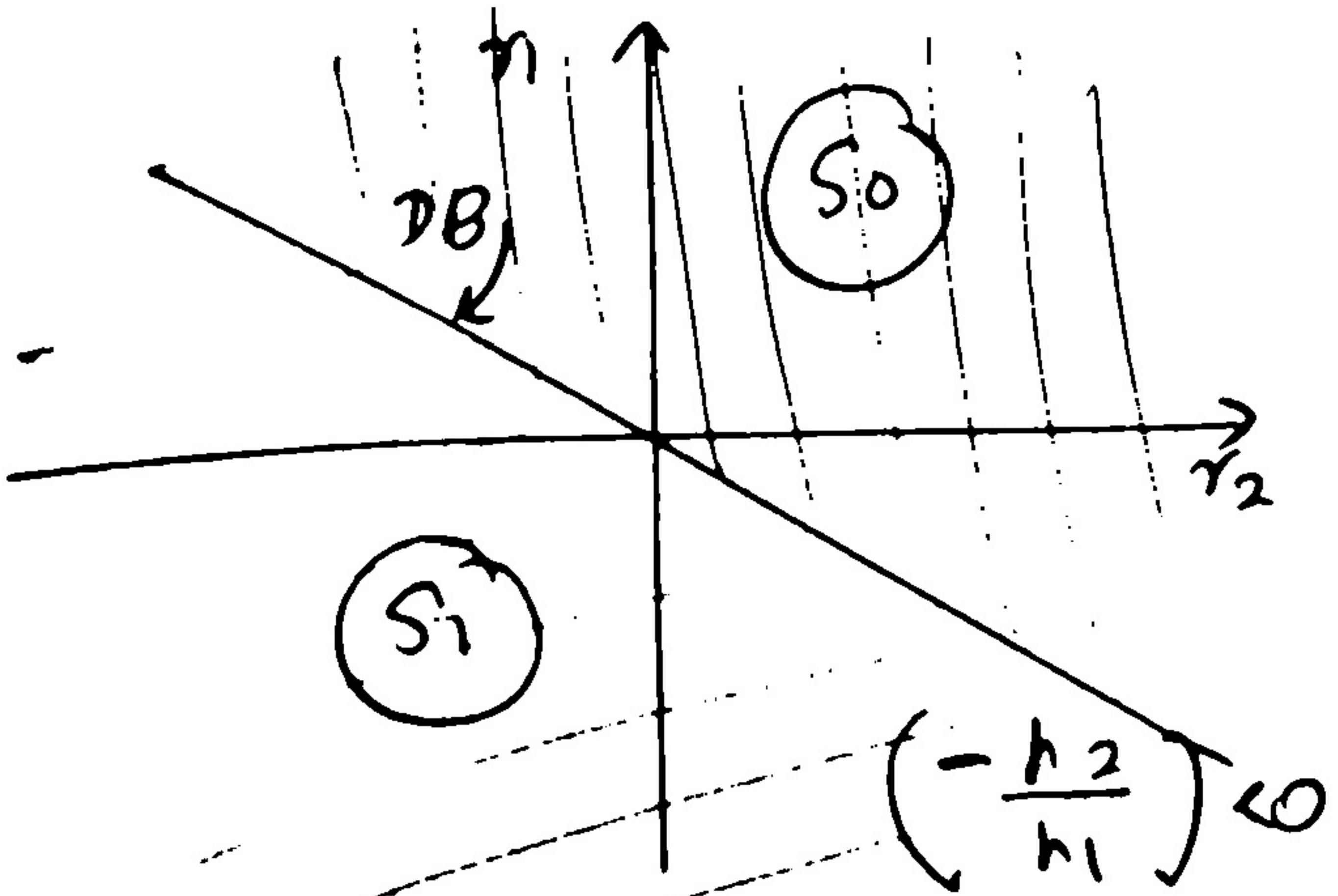
$$h_2 (h_1 > 0)$$

$$r_1 \geq_{S_0} -\left(\frac{h_2}{h_1}\right) r_2$$

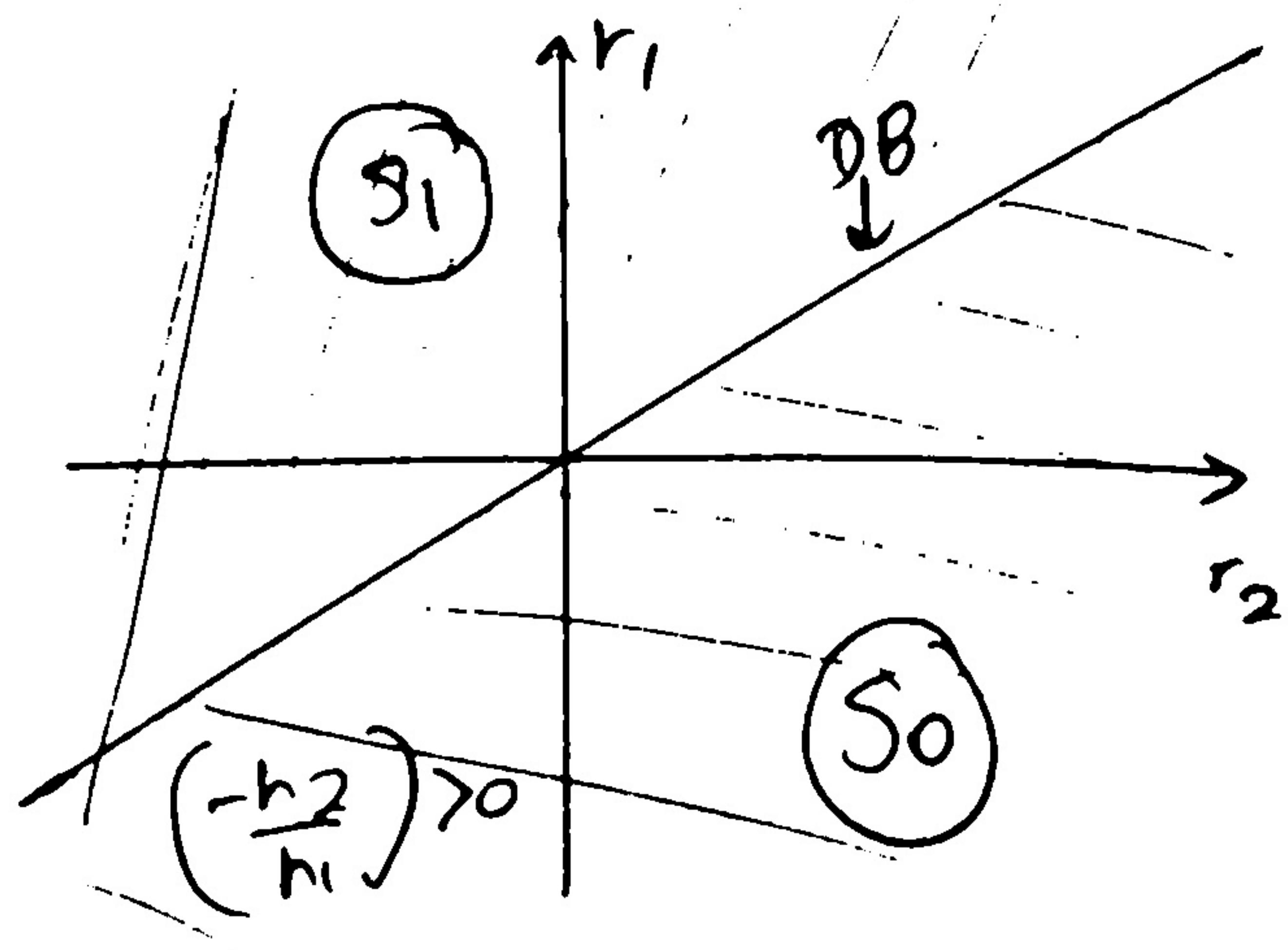
Subcase I. ($h_2 > 0$)



Subcase II. ($h_2 < 0$)



Subcase II. $h_2 < 0$



Q3

b.)

Under the hypothesis ($s_1 = 1$).

$$\underline{r} = \underline{h} \underline{s}_1 + \underline{n}$$

$$\underline{r} = \underline{h}(1) + \underline{n}$$

$$\underline{r} = \underline{h} + \underline{n}$$

$$\underline{r}^T \underline{h} = (\underline{h} + \underline{n})^T \underline{h}$$

$$\text{Consider } E\{\underline{r}^T \underline{h}\} = \mu_h$$

$$= E\{(\underline{h} + \underline{n})^T \underline{h}\}$$

$$= E\{(\underline{h}^T + \underline{n}^T) \cdot \underline{h}\}$$

Here \underline{h}^T is a vector of constants and

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \underline{n} \sim N_2(0, \frac{N_0}{2} I_2) \Rightarrow E\{\underline{n}^T\} = E\{\underline{n}\} = 0$$

$$\Rightarrow E\{\underline{n}^T \underline{h} + \underline{h}^T \underline{n}\} = E\{\underline{n}^T \underline{h}\} + E\{\underline{h}^T \underline{n}\}$$

$$= E\{\underline{h}^T \underline{h}\} + \underbrace{E\{\underline{n}^T \underline{h}\}}_0$$

$$\therefore \mu_h = E\{\underbrace{\|\underline{h}\|^2}_{\text{constant}}\} = \|\underline{h}\|^2 \rightarrow ①$$

$$\text{consider } \sigma_h^2 = \text{Var}(r^T h) = E\{(r^T h - \|h\|^2)^2\}$$

$$= E\{(r^T h - \|h\|^2)(r^T h - \|h\|^2)\}$$

$$= E\{(r^T h)^2\} - 2\|h\|^2 \underbrace{E\{r^T h\}}_{\text{from D}} + E\{\|h\|^4\}$$

$$= E\{(r^T h)^2\} - 2\|h\|^4 + \|h\|^4$$

$$^2 E\{(r^T h)^2\} - \|h\|^4$$

$$r^T h = \|h\|^2 + n^T h.$$

$$= E\{(\|h\|^2 + n^T h)^2\} - \|h\|^4$$

$$= E\{\|h\|^4 + 2\|h\|^2 n^T h + (n^T h)^2\} - \|h\|^4$$

$$= E\{\|h\|^4\} + \underbrace{2\|h\|^2 E\{n^T h\}}_0 + E\{(n^T h)^2\} - \|h\|^4$$

$$\Rightarrow \text{Var}(r^T h) = E\{(n^T h)^2\} \quad | \quad \begin{aligned} n^T h &= [n_1 \ n_2] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= (n_1 h_1 + n_2 h_2) \end{aligned}$$

$$= E\{(n_1 h_1 + n_2 h_2)^2\}$$

$$= E\{n_1^2(h_1^2)\} + E\{n_2^2(h_2^2)\} + E\{n_1 n_2 (h_1 h_2)\}$$

$$= h_1^2 \cdot \underbrace{E\{n_1^2\}}_{\frac{N_0}{2}} + h_2^2 \cdot \underbrace{E\{n_2^2\}}_{\frac{N_0}{2}} + \underbrace{n_1 n_2 E\{n_1 n_2\}}_{\underbrace{E[n_1] E[n_2]}_{0 \times 0} (= 0)}$$

$$\therefore \text{Var}(\underline{r}^T \underline{h}) = h_1^2 \frac{N_0}{2} + h_2^2 \frac{N_0}{2}$$

$$= \underbrace{(h_1^2 + h_2^2)}_{\frac{N_0}{2}}$$

$$\text{Var}(\underline{r}^T \underline{h}) = \|\underline{h}\|^2 \frac{N_0}{2} \rightarrow \textcircled{2}$$

\therefore from ① and ②,

$$E\{\underline{r}^T \underline{h}\} = \|\underline{h}\|^2 = \mu_h$$

$$\text{Var}\{\underline{r}^T \underline{h}\} = \|\underline{h}\|^2 \frac{N_0}{2} = \sigma_h^2$$

$$\therefore \underline{r}^T \underline{h} \sim \mathcal{N}(\mu_h, \sigma_h^2)$$

Q3 c.)

$$P(\varepsilon) = P(\varepsilon | S_0) \underbrace{P(S_0)}_{\frac{1}{2}} + P(\varepsilon | S_1) \underbrace{P(S_1)}_{\frac{1}{2}}$$

$$P(\varepsilon) = \frac{1}{2}[P(\varepsilon | S_0) + P(\varepsilon | S_1)] \longrightarrow \textcircled{1}$$

from part (a) the optimum MAP decision is

$$\text{correct } S_1 : \underline{r^T h} > 0 \rightarrow P(\varepsilon | S_1) = P(\underline{r^T h} < 0)$$

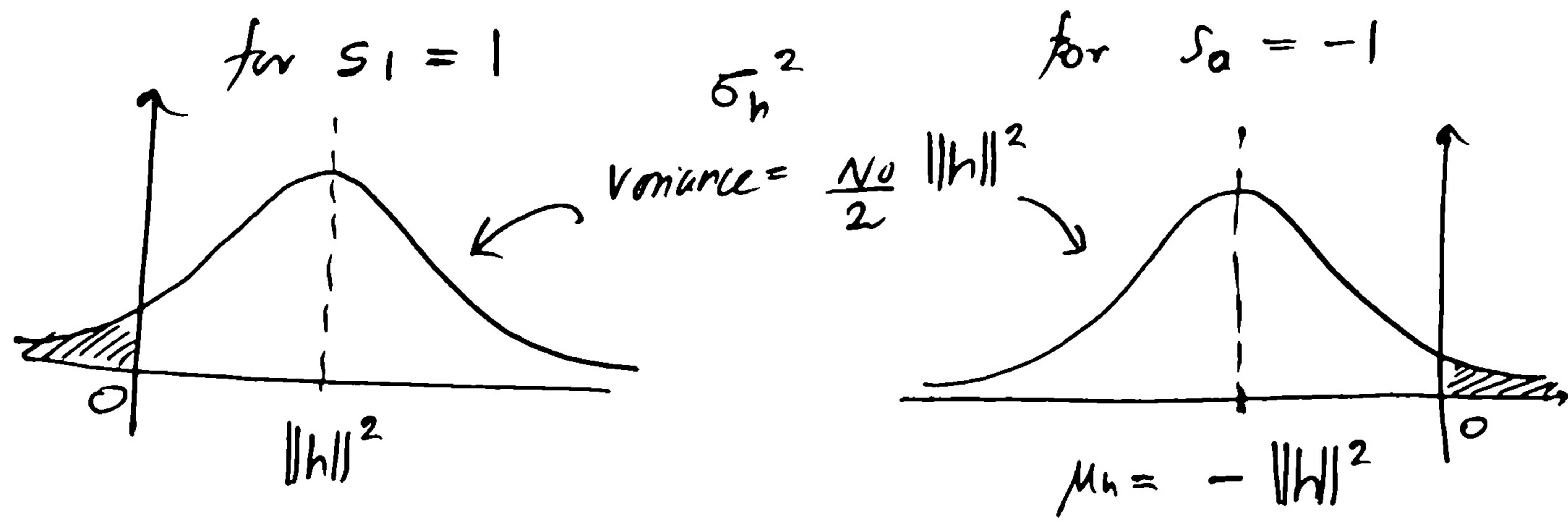
$$\text{correct } S_0 : \underline{r^T h} < 0 \rightarrow P(\varepsilon | S_0) = P(\underline{r^T h} > 0)$$

from Part b.: under hypothesis $S_1 = 1$

$$S_1 : \underline{r^T h} \sim N(-\|h\|^2, \frac{N_0}{2}\|h\|^2)$$

Same way under the hypothesis $S_0 = -1$

$$\underline{r^T h} = (\underline{h} + \underline{n}) \cdot \underline{h} \Rightarrow \underline{r^T h} \sim N(-\|h\|^2, \frac{N_0}{2}\|h\|^2)$$



∴ ① can be rewritten as, follows,

$$\begin{aligned}
 P(a) &= \frac{1}{2} [P(a|s_0) + P(a|s_1)] \\
 &= \frac{1}{2} [P(\underline{r}^T \underline{h} > 0 | s_0) + P(\underline{r}^T \underline{h} < 0 | s_1)] \\
 &= \frac{1}{2} \left\{ \int_{-\infty}^{\infty} N(-\|\mathbf{h}\|^2, \frac{N_0}{2} \|\mathbf{h}\|^2) + \int_{-\infty}^{0} N(+\|\mathbf{h}\|^2, \frac{N_0}{2} \|\mathbf{h}\|^2) \right.
 \end{aligned}$$

Probability distributions are mirror images of each other!
therefore we can use either of the distributions to
solve above expression.

$$-\frac{[\underline{r}^T \underline{h} - (-\|\mathbf{h}\|^2)]}{2 \sigma_h^2}$$

$$\therefore P(a) = \frac{1}{2} \times 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi} \sigma_h^2} e^{-\frac{[\underline{r}^T \underline{h} - (-\|\mathbf{h}\|^2)]^2}{2 \sigma_h^2}} d(\underline{r}^T \underline{h})$$

$$P(a) = \frac{1}{\sqrt{2\pi} \sigma_h} \int_0^{\infty} \exp \left[-\frac{(\underline{r}^T \underline{h} + \|\mathbf{h}\|^2)^2}{2 \sigma_h^2} \right] d(\underline{r}^T \underline{h})$$

Substitute : $\frac{\underline{r}^T \underline{h} + \|\mathbf{h}\|^2}{\sigma_h} = t$

Limits.

$$\text{When } \gamma^T h = 0 \rightarrow t = \frac{\|h\|^2}{\sigma_h}$$

$$\gamma^T h \rightarrow \infty \rightarrow t \rightarrow \infty$$

$$\sigma_h \cdot dt = d(\gamma^T h)$$

$$\therefore P(q) = \frac{1}{\sqrt{2\pi}\sigma_h} \int_{\frac{\|h\|^2}{\sigma_h}}^{\infty} e^{-t^2/2} \cdot \sigma_h dt$$

$$\therefore P(q) = Q\left(\frac{\|h\|^2}{\sigma_h}\right)$$

$$= Q\left(\frac{\|h\|^2}{\sqrt{\frac{N_0 \|h\|^2}{2}}}\right)$$

$$\therefore P(q) = Q\left[\sqrt{\frac{2\|h\|^2}{N_0}}\right]$$

$$\text{Where, } Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-t^2/2} dt$$