EN 3053 Digital Communication-I
Semester 5
Assignment No. 1
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1. A certain signal $s(t)$ assumes the following decomposition

$$s(t) = \sum_{k=0}^{N-1} x[k]\phi_k(t), \quad 0 \leq t < NT,$$

where $x[k] \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $T > 0$, and

$$\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}kt}$$

with $j = \sqrt{-1}$.

(a) Show that

$$\int_0^{NT} \phi_k(t)\phi_\ell^*(t)\,\mathrm{d}t = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

where $(\cdot)^*$ denotes the complex conjugate operator. **[2.5 marks]**

(b) Hence show that the total energy content of $s(t)$ is given by **[2.5 marks]**

$$\mathcal{E} = \sum_{k=0}^{N-1} |x[k]|^2 .$$

(c) Show that **[10 marks]**

$$\sum_{m=0}^{N-1} \phi_k(mT)\phi_\ell^*(mT) = \begin{cases} \frac{1}{T} & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

[**Hint:** *You may use the formula* $\sum_{m=0}^{N-1} z^m = \frac{1 - z^N}{1 - z}$, *where* $z \in \mathbb{C}$ *and* $z \neq 1$.]

(d) Let us now obtain a discrete-time sequence of length N from the continuous-time signal $s(t)$ through sampling. In particular, the discrete-time sequence $s[m]$ can be written as

$$s[m] = s(mT), \quad m = 0, 1, \ldots, N-1.$$

Show that **[10 marks]**

$$x[\ell] = T \sum_{m=0}^{N-1} s[m]\phi_\ell^*(mT), \quad \ell = 0, 1, \ldots, N-1.$$

2. Assume that a set $\{\boldsymbol{\theta_i}\}$ of M equally likely real vectors satisfies the equations

$$\boldsymbol{\theta}_i^T \boldsymbol{\theta}_\ell = \begin{cases} 1 & i = \ell \\ \rho & i \neq \ell. \end{cases}$$

(a) Prove that $-\dfrac{1}{M-1} \leq \rho \leq 1$. [**Hint**: *Consider* $\left|\sum_{k=1}^{M} \boldsymbol{\theta}_k\right|^2$.] **[5 marks]**

(b) Show that the left-hand equality (i.e., $\rho = -\frac{1}{M-1}$) is achieved by the unit average energy simplex set.[1] **[10 marks]**

(c) Prove for any allowable ρ that the signal set $\{\mathbf{s}_i\}$, with $\mathbf{s}_i = \sqrt{E_\rho}\boldsymbol{\theta}_i$ for all i, has the same error probability as the simplex signal set with energy **[10 marks]**

$$E_s = E_\rho \left(1 - \frac{1}{M}\right)(1 - \rho).$$

[**Hint**: *First, determine the minimum energy set corresponding to the set* $\{\boldsymbol{\theta}_i\}$. *Then show that up to a scaling factor the minimum energy set and the simplex set have the same geometry.*]

3. One of the two equally likely signals $s_0 = -1$, $s_1 = 1$ is transmitted over a certain AWGN channel as shown in Fig. 1. The two noise random variables n_1 and n_2 are statistically independent of the transmitted signal and of each other. Consequently, we can write the received signals, for $i = 0, 1$, as

$$r_1 = h_1 s_i + n_1$$
$$r_2 = h_2 s_i + n_2.$$

Now it is convenient to rewrite the above equations in vector form as

$$\underbrace{\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}}_{\boldsymbol{r}} = \underbrace{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}_{\boldsymbol{h}} s_i + \underbrace{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}}_{\boldsymbol{n}},$$

which in turn gives

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[1] Given a set of M equally likely *orthogonal* vectors $\{\boldsymbol{\beta}_i\}$, the set $\{\boldsymbol{\beta}_i - \mathbf{a}\}$ with $\mathbf{a} = \frac{1}{M}\sum_{k=1}^{M} \boldsymbol{\beta}_i$ is known as the *simplex* set.

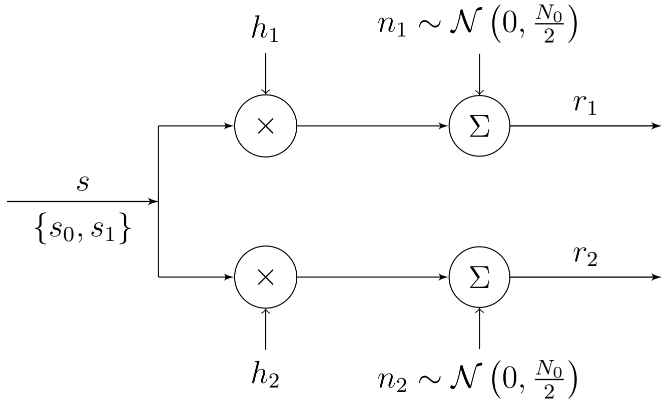

Figure 1: A diversity communication system.

$$\boldsymbol{r} = \boldsymbol{h} s_i + \boldsymbol{n}, \quad i = 0, 1,$$

where $\boldsymbol{r}, \boldsymbol{h} \in \mathbb{R}^{2\times 1}$, and $\boldsymbol{n} \sim \mathcal{N}_2\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}_2\right)$ with $\mathbf{I}_2$ denoting the 2×2 identity matrix.

(a) Show that the optimum MAP decision rule is given by

$$\boldsymbol{r}^T \boldsymbol{h} \underset{s_0}{\overset{s_1}{\gtrless}} 0$$

and draw the corresponding decision boundary on (r_1, r_2) plane. **[5 marks]**

(b) Show that

$$\boldsymbol{r}^T \boldsymbol{h} \sim \mathcal{N}\left(\mu_{\boldsymbol{h}}, \sigma_{\boldsymbol{h}}^2\right)$$

where

$$\begin{aligned} \mu_{\boldsymbol{h}} &= \mathbb{E}\left\{\boldsymbol{r}^T \boldsymbol{h}\right\} = ||\boldsymbol{h}||^2 \\ \sigma_{\boldsymbol{h}}^2 &= \mathrm{Var}\left(\boldsymbol{r}^T \boldsymbol{h}\right) = \mathbb{E}\left\{\left(\boldsymbol{r}^T \boldsymbol{h} - ||\boldsymbol{h}||^2\right)^2\right\} = \frac{N_0}{2}||\boldsymbol{h}||^2 \end{aligned}$$

with $||\cdot||$ denoting the Euclidean norm. **[10 marks]**

(c) Hence show that the BER is given by

$$\Pr(\varepsilon) = \mathcal{Q}\left(\sqrt{2\frac{||\boldsymbol{h}||^2}{N_0}}\right)$$

where $\mathcal{Q}(z)$ denotes the Gaussian-Q function.[2] **[5 marks]**

(d) Explain how the coherent BPSK BER formula is related to the above BER result. **[5 marks]**

[2] $\mathcal{Q}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{t^2}{2}} \mathrm{d}t$.

4. (a) A transmitter uses the signals $\{s_i(t)\}$ to communicate one of 4 equally likely messages over an AWGN channel, where for $i = 0, 1, 2, 3$

$$s_i(t) = \begin{cases} \sqrt{\frac{2}{T}} \cos\left(2\pi \frac{t}{T} + \frac{\pi i}{2}\right), & 0 \le t < T \\ 0, & \text{elsewhere.} \end{cases}$$

Moreover, we adopt the following mapping to map bits to message waveforms:

$$(0,0) \mapsto s_0(t) \quad (0,1) \mapsto s_1(t) \quad (1,1) \mapsto s_2(t) \quad (1,0) \mapsto s_3(t)$$

where the bit interval T_b is related to the symbol interval via $T = 2T_b$. Now consider the binary sequence $\mathcal{S} = \{1,1,0,0,1,0,0,1,1,1,1,0,0,0,1,0\}$. Plot the sequence of signals (waveforms) corresponding to the binary sequence $\mathcal{S}$ for $0 \le t < 8T$. **[5 marks]**

(b) A certain quaternary minimum-shift keying (MSK) waveform can be written as

$$s(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi \frac{t}{T} + \Phi_n(t)\right), \quad nT \le t \le (n+1)T$$

where

$$\begin{aligned} \Phi_n(t) &= \underbrace{\frac{\pi}{2} \sum_{k=-\infty}^{n-1} X_k}_{\theta_n} + \frac{\pi}{2} X_n \left(\frac{t}{T} - 1\right) \\ &= \theta_n + \frac{\pi}{2} X_n \left(\frac{t}{T} - 1\right) \end{aligned}$$

and each $X_k \in \{-3, -1, +1, +3\}$ is chosen according to the mapping

$$(0,0) \mapsto -3 \quad (0,1) \mapsto -1 \quad (1,1) \mapsto +1 \quad (1,0) \mapsto +3.$$

Draw the quaternary MSK phase trajectory (i.e., $\Phi_n(t)$ vs. t) corresponding to the binary sequence $\mathcal{S}$ for $0 \le t \le 8T$. **[10 marks]**

(c) Plot the quaternary MSK waveform corresponding to the binary sequence $\mathcal{S}$ for $0 \le t \le 8T$. **[10 marks]**