Department of Electronic and Telecommunication Engineering

University of Moratuwa, Sri Lanka

EN3053 - Digital Communications - I

Assignment No. 1

Submitted by

Thalagala B.P. 180631J

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$$S(t) = \sum_{k=0}^{N-1} x[k]\,\phi_k(t) \qquad 0 \leq t < NT$$

where $x[k] \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $T > 0$ and

$$\phi_k(t) = \frac{1}{\sqrt{NT}}\, e^{j\frac{2\pi}{NT}.kt}$$

consider, $\langle \phi_k(t), \phi_\ell(t) \rangle$: the Hermitian Inner Product.

$$= \int_0^{NT} \phi_k(t) . \phi_\ell^*(t)\, dt$$

$$= \int_0^{NT} \left(\frac{1}{\sqrt{NT}} . e^{j\frac{2\pi}{NT}kt} \right) . \left(\frac{1}{\sqrt{NT}} . e^{j\frac{2\pi}{NT}\ell t} \right)^* dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT}kt} . e^{-j\frac{2\pi}{NT}\ell t}\, dt$$

$$= \frac{1}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT}(k-\ell)t}\, dt$$

$\boxed{\text{Case 1 :}}$ if $k = \ell$:

$$\Rightarrow \frac{1}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT}(0).t}\, dt$$

$$= \frac{1}{NT} \int_0^{NT} 1\, dt = \frac{1}{NT} [t]_0^{NT} = \frac{1}{NT} [NT - 0]$$

$$= \underline{1} \longrightarrow (1)$$

$\boxed{\text{Case 2 :}}$ if $k \neq l$

Since $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$\therefore$ Let $(k-l) = m \in \mathbb{Z}$

$$\therefore \frac{1}{NT} \int_0^{NT} e^{j\frac{2\pi}{NT} mt} \, dt$$

$$= \frac{1}{\cancel{NT}} \left. \frac{e^{j\frac{2\pi}{NT} \cdot m \cdot t}}{j\frac{2\pi}{\cancel{NT}} \cdot m} \right|_0^{NT}$$

$$= \frac{1}{j2\pi m} \cdot \left[e^{j\frac{2\pi}{\cancel{NT}} \cdot m \cdot \cancel{NT}} - 1 \right]$$

$$= \frac{1}{j2\pi m} \left[e^{j(2\pi)m} - 1 \right] \quad \text{where } m \in \mathbb{Z}$$

$$= \frac{1}{j2\pi m} \left[\underbrace{\cos(2\pi m)}_{(\text{always} = 1)} + j \cancel{\sin(2\pi m)}^{0} - 1 \right]$$

$$= \frac{1}{j2\pi m} \left[1 - 1 \right] \Rightarrow 0 \longrightarrow (2)$$

from (1) and (2)

$$\therefore \int_0^{NT} \phi_k(t) \cdot \phi_l^*(t) \, dt = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

$\boxed{Q1}$

b.) Total Energy Content of $S(t)$ $\Big\} = \langle S(t), S(t) \rangle$

$$= \int_0^{NT} S(t)\,(S(t))^* \, dt \quad ; (0 < t \leq NT)$$

complex conjugate

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k].\ \phi_k(t)\right]\left[\sum_{k=0}^{N-1} x[k]\ \phi_k(t)\right]^* dt$$

Since $(z_1 + z_2)^* = z_1^* + z_2^*$

$\forall\ z_1, z_2 \in \mathbb{C}$ we can apply the conjugate to above summation, seperately.

$$= \int_0^{NT} \left[\sum_{k=0}^{N-1} x[k].\ \phi_k(t)\right]\left[\sum_{k=0}^{N-1} (x[k])^*\ \phi_k^*(t)\right] dt$$

= from part ⓐ terms are vanished when the k values are different.

$$\Rightarrow \sum_{k=0}^{N-1}\left\{\left[x[k].(x[k])^*\right] \underbrace{\int_0^{NT} \phi_k(t).\ \phi_k^*(t)\ dt}_{=1 \longrightarrow \text{from ⓐ}}\right\}$$

$$\Rightarrow \ \mathcal{E} = \sum_{k=0}^{N-1} |x[k]|^2$$

$\boxed{Q1}$

c.) $\sum_{m=0}^{N-1} \phi_k(mT)\, \phi_l^*(mT)$; $\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}kt}$

$$\Rightarrow \sum_{m=0}^{N-1} \left[\frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}k(mT)}\right]\left[\frac{1}{\sqrt{NT}} e^{j\frac{2\pi}{NT}l(mT)}\right]^*$$

$$= \sum_{m=0}^{N-1} \frac{1}{NT}\left[e^{j\frac{2\pi}{N}k.m.}\right]\left[e^{-j\frac{2\pi}{N}l.m}\right]$$

$$= \frac{1}{NT} \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}m(k-l)}$$

$$= \frac{1}{NT} \cdot \underbrace{\sum_{m=0}^{N-1} \left[e^{j\frac{2\pi}{N}(k-l)}\right]^m}$$

$$= \frac{1}{NT} \quad \frac{1-\left[e^{j\frac{2\pi}{N}(k-l)}\right]^N}{1-e^{j\frac{2\pi}{N}(k-l)}}$$ using the given hint.

$$= \frac{1}{NT} \cdot \frac{1-e^{j2\pi(k-l)}}{1-e^{j\frac{2\pi}{N}(k-l)}} \longrightarrow ①$$

$\boxed{\text{Case I}}$ when $(k-l) \neq 0 \Rightarrow k \neq l$

since $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$$\therefore \quad e^{j2\pi(k-l)} = 1 \quad \forall\ (k-l) \in \mathbb{Z}$$

$$\therefore \quad \frac{1}{NT} \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} = \frac{1}{NT} \frac{(1-1)}{\left(1 - e^{j\frac{2\pi}{N}(k-l)}\right)}$$

$$= 0 \longrightarrow ②.$$

$\boxed{\text{case 2}}$ when $k = l$, using L'hopital's rule.

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \frac{-j2\pi \cdot e^{j2\pi(k-l)}}{-j\frac{2\pi}{N} e^{j\frac{2\pi}{N}(k-l)}} \qquad \text{differentiate w.r.t } (k-l)$$

$$= \lim_{(k-l)\to 0} \frac{1}{NT} \times N \times \frac{e^{j2\pi(k-l)}}{e^{j\frac{2\pi}{N}(k-l)}}$$

$$= \frac{1}{NT} \times N \times \frac{1}{1}$$

$$= \frac{1}{T} \longrightarrow ③$$

$$\therefore \text{ from ② and ③} \Rightarrow \left\{ \sum_{m=0}^{N-1} \phi_k(mT)\, \phi_l^*(mT) = \begin{cases} \frac{1}{T} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \right.$$

$\boxed{Q1}$

d.) $s[m] = s(mT)$; $m = 0, 1, 2, \cdots, N-1$

consider $RHS = T \times \sum_{m=0}^{N-1} s[m]\, \phi_{\ell}^{*}(mT)$

$$= T \times \sum_{m=0}^{N-1} s(mT) . \phi_{\ell}^{*}(mT)$$

$$= T \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k]\, \phi_k(t) \right] \phi_{\ell}^{*}(t) \Big|_{t=mT}$$

$$= T \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x[k] . \phi_k(mT) . \phi_{\ell}^{*}(mT) \right]$$

$$= \sum_{k=0}^{N-1} x[k] \left\{ (T) \underbrace{\sum_{m=0}^{N-1} \phi_k(mT) \phi_{\ell}^{*}(mT)} \right\}$$

from part ⓒ; this part is $\frac{1}{T}$ when $k=\ell$ and vanishes otherwise.

So when $k=\ell$ $(k=\ell)$

$$\Rightarrow \quad x[\ell] \times T \times \frac{1}{T}$$

$\therefore \quad x[\ell] = T . \sum_{m=0}^{N-1} s[m]\, \phi_{\ell}^{*}(mT)$

for $\ell = 0, 1, 2, \cdots, N-1$

$\boxed{Q2.}$ a)

consider $\left|\sum_{k=1}^{M} \theta_k\right|^2 \geq 0 \longrightarrow (1)$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 = \left(\sum_{k=1}^{M} \theta_k\right)^T\left(\sum_{k=1}^{M} \theta_k\right)$$

$$= (\theta_1 + \theta_2 + \dots + \theta_M)^T(\theta_1 + \theta_2 + \dots + \theta_M)$$

$$= (\theta_1^T + \theta_2^T + \dots + \theta_M^T)(\theta_1 + \theta_2 + \dots + \theta_M)$$

In this expression there will be,

(I) M terms, where $i=j$ for $\theta_i^T\theta_j$

(II) $M(M-1)$ terms, wher $i \neq j$ for $\theta_i^T\theta_j$
(for each θ_i $\exists$, $(M-1)$ θ_j's such that $i \neq j$.

$\therefore$
$$\left|\sum_{k=1}^{M} \theta_k\right|^2 = \underbrace{(M \times 1)}_{\text{from (I)}} + \underbrace{M(M-1)\rho}_{\text{from (II)}}$$

$$\left|\sum_{k=1}^{M} \theta_k\right|^2 \geq 0 \longrightarrow \text{from (1)}$$

$$M + M(M-1)\rho \geq 0$$

$$1 + (M-1)\rho \geq 0 \quad (\because \text{since } M > 0)$$

$$\frac{-1}{M-1} \leq \rho \longrightarrow (2) \quad (\because \text{since } (M-1) > 0)$$

$\rho \leq 1 \longrightarrow (3)$

$\therefore$ From (2) and (3),

$$\frac{-1}{M-1} \leq \rho \leq 1$$

$\boxed{Q2.}$ b.)

Assume $\{\mathcal{O}_i\}$ is a Simplex set. Then following expression must equal to zero, as $\{\mathcal{O}_i\}$s are already in the minimum energy constellation.

$$\sum_{i=1}^{M} P_i \mathcal{O}_i = \underline{0}$$

Since $\{\mathcal{O}_i\}$ are equally likely, $P_i = \frac{1}{M}$

$$\therefore \quad \frac{1}{M} \sum_{i=1}^{M} \mathcal{O}_i = \underline{0}$$

$$\Rightarrow \quad \sum_{i=1}^{M} \mathcal{O}_i = \underline{0}$$

consider, $$\left\| \left(\sum_{i=1}^{M} \mathcal{O}_i \right) \right\|^2 = \|\underline{0}\|^2$$

$$\underbrace{\left(\sum_{i=1}^{M} \mathcal{O}_i \right)^T \left(\sum_{i=1}^{M} \mathcal{O}_i \right)} = 0$$

from part (a) this part is simplified into,

$$M + M(M-1)\rho = 0$$

$$\therefore \quad \rho = \frac{-1}{M-1}$$

$\boxed{Q.2}$ c.)

(I) Determining the minimum energy set corresponding to the set $\{\theta_i\}$.

* Let $\{\beta_i\}$ be the minimum energy set of the signal set $\{\theta_i\}$.

$$\Rightarrow \{\beta_i\} = \{\theta_i - \underline{a}\} \text{ where, } \underline{a} = \frac{1}{M}\sum_{i=1}^{M}\theta_i$$

consider $\beta_i^T \beta_l = (\theta_i - \underline{a})^T(\theta_l - \underline{a})$

$$= (\theta_i^T - \underline{a}^T)(\theta_l - \underline{a})$$

$$= \theta_i^T\theta_l - \theta_i^T\underline{a} - \underline{a}^T\theta_l + \underline{a}^T\underline{a}$$

$$= \theta_i^T\theta_l - \theta_i^T\left(\frac{1}{M}\sum_{i=1}^{M}\theta_i\right) - \left(\frac{1}{M}\sum_{i=1}^{M}\theta_i\right)^T\theta_l + \underline{a}^T\underline{a}$$

$$= \theta_i^T\theta_l - \frac{1}{M}[1 + (M-1)\rho] - \frac{1}{M}[1 + (M-1)\rho] + \underbrace{\|\underset{\sim}{a}\|^2}_{\text{from part (a.)}}$$

$$= \theta_i^T\theta_l - \frac{2}{M}[1 + (M-1)\rho] + \frac{1}{M^2}[M + (M-1)\rho]$$

$$= \theta_i^T\theta_l - \frac{1}{M}[1 + (M-1)\rho] \longrightarrow (1)$$

Q2 c) contin:

from def^n: $\theta_i^T \theta = \begin{cases} 1 & i = l \\ \rho & i \neq l \end{cases}$

$\therefore$ $(\beta_i)^T(\beta_l) = \begin{cases} 1 - \frac{1}{M}[1 + (M-1)\rho] \; ; & i = l \\ \rho - \frac{1}{M}[1 + (M-1)\rho] \; ; & i \neq l \end{cases}$

$$\therefore \beta_i^T \beta_l = \left\{ \begin{array}{ll} \frac{(M-1)(1-\rho)}{M} \; ; & i = l \\ \frac{(\rho-1)}{M} \; ; & i \neq l \end{array} \right\} - (2)$$

from part (b)

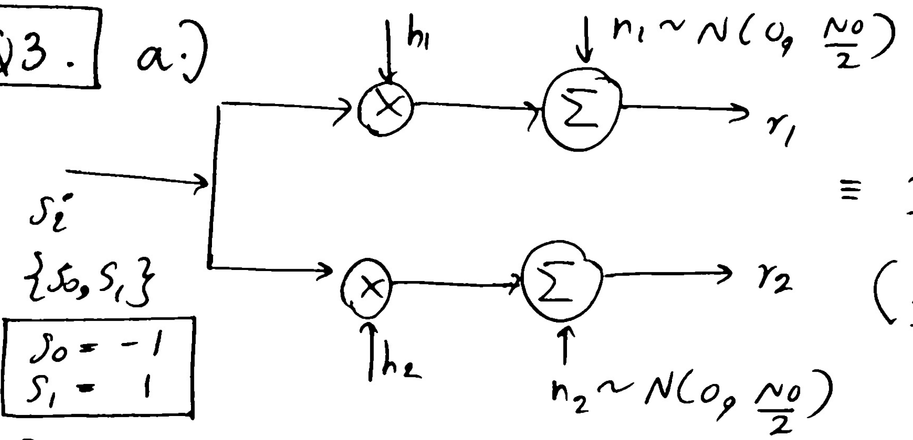
For the unit average Simplex set $\rightarrow$ $\alpha_i^T \alpha_l = \begin{cases} 1 & ; i = l \\ \rho = \frac{-1}{M-1} & ; i \neq l \end{cases}$

By rearranging (2),

$$\Rightarrow \beta_i^T \beta_l = \begin{cases} 1 \times \left[\frac{(M-1)(1-\rho)}{M}\right] & i = l \\ \frac{-1}{M-1} \times \left[\frac{(M-1)(1-\rho)}{M}\right] ; & i \neq l \end{cases}$$

$\therefore$ We can observe that minimum energy set $\{\beta_i\}$ is a scaled constellation of the simplex set $\{\alpha_i\}$.

Q3. a:)



$$\equiv \underline{r} = \underline{h} s_i + \underline{n}$$

$$(\underline{r}, \underline{h} \in \mathbb{R}^{2\times 1})$$

a.) Consider the following a posteriori probabilities.

(I) For the receiver to classify the observed signal as s_0 $\Rightarrow P(s_0|\underline{r}) \overset{s_0}{>} P(s_1|\underline{r}) \rightarrow (1)$

(II) For the receiver to classify the observed signal as s_1 $\Rightarrow P(s_0|\underline{r}) \underset{s_1}{<} P(s_1|\underline{r}) \rightarrow (2)$

By combining (1) and (2), ~~for~~

$$P(s_0|\underline{r}) \underset{s_1}{\overset{s_0}{\gtrless}} P(s_1|\underline{r})$$ ← (3) This is what demanded by the receiver!

But what is known, to us, are the following probabilities.

$$P(\underline{r}|s_0), \quad P(\underline{r}|s_1)$$

∴ By using Bayes theorem, ③ can be rewritten as follows.

$$P(\underline{s}_0 \mid \underline{r}) \underset{s_1}{\overset{s_0}{\gtrless}} P(\underline{s}_1 \mid \underline{r})$$

$$\frac{P(\underline{r} \mid \underline{s}_0) \times P(\underline{s}_0)}{P(\underline{r})} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{P(\underline{r} \mid \underline{s}_1)\, P(\underline{s}_1)}{P(\underline{r})}$$

Since s_0 and s_1 are equally likely signals,

$$P(\underline{s}_0) = P(\underline{s}_1) = \frac{1}{2}$$

$$\therefore \quad P(\underline{r} \mid \underline{s}_0) \underset{s_1}{\overset{s_0}{\gtrless}} P(\underline{r} \mid \underline{s}_1)$$

Since $\underline{r} = \underline{h} s_i + \underline{n}$

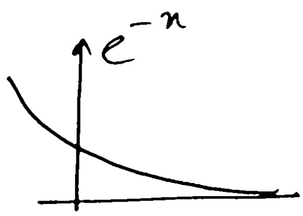
$$s_i = s_0 \rightarrow \underline{r} = \overbrace{\underline{h}(-1)}^{\underline{s}_0} + \underline{n}$$

$$s_i = s_1 \rightarrow \underline{r} = \underbrace{\underline{h}(1)}_{\underline{s}_1} + \underline{h}$$

$$\therefore \quad P(\underline{r} \mid -\underline{h}) \underset{s_1}{\overset{s_0}{\gtrless}} P(\underline{r} \mid \underline{h})$$

$$N_2\left(-\underline{h}, \frac{N_0}{2} I_2\right) \underset{s_1}{\overset{s_0}{\gtrless}} N_2\left(\underline{h}, \frac{N_0}{2} I_2\right)$$

$$e^{-\|\underline{r} - (-\underline{h})\|^2} \underset{s_1}{\overset{s_0}{\gtrless}} e^{-\|\underline{r} - (\underline{h})\|^2}$$



$$\|\underline{r} + \underline{h}\|^2 \underset{s_0}{\overset{s_1}{\gtrless}} \|\underline{r} - \underline{h}\|^2$$

$$\|\underline{r}+\underline{h}\|^2 \underset{S_0}{\overset{S_1}{\gtrless}} \|\underline{r}-\underline{h}\|^2$$

$$(\underline{r}+\underline{h})^T(\underline{r}+\underline{h}) \underset{S_0}{\overset{S_1}{\gtrless}} (\underline{r}-\underline{h})^T(\underline{r}-\underline{h})$$

$$(\underline{r}^T+\underline{h}^T)(\underline{r}+\underline{h}) \underset{S_0}{\overset{S_1}{\gtrless}} (r^T-h^T)(\underline{r}-\underline{h})$$

$$\begin{aligned} &\underline{r}^T\underline{r} + \underline{r}^T\underline{h} && \quad \underline{r}^T.\underline{r} - r^T.\underline{h} \\ &+ h^T.\underline{r} + \underline{h}^T.\underline{h} \underset{S_1}{\overset{S_1}{\gtrless}} && \quad - \underline{h}^T\underline{r} + h^T.h \end{aligned}$$

Since $\quad h, r \in \mathbb{R}^{2\times 1} \Rightarrow \underline{r}^T\underline{h} = \underline{h}^T\underline{r}$

$$\therefore \quad 2r^Th \underset{S_0}{\overset{S_1}{\gtrless}} -2\underline{r}^T\underline{h}$$

$$4\underline{r}^T\underline{h} \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$\Downarrow$$

$$\boxed{\underline{r}^T\underline{h} \underset{S_0}{\overset{S_1}{\gtrless}} 0}$$ ← Optimum MAP decision rule.

$$\underline{r}^T \underline{h} \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$[r_1 \quad r_2]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$r_1 h_1 + r_2 h_2 \underset{S_0}{\overset{S_1}{\gtrless}} 0$$

$$\boxed{r_1 \gtrless -\left(\frac{h_2}{h_1}\right) r_2}$$

$h_1 < 0$ inequality changes

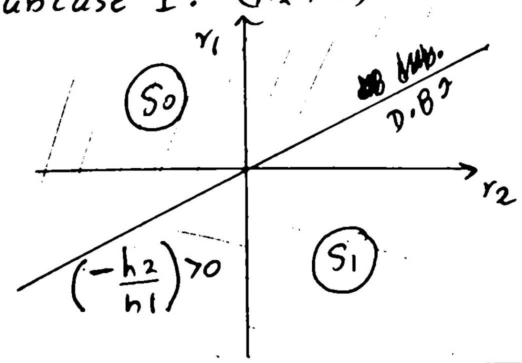
$h_1 > 0$ inequality remains the same.

case (I)

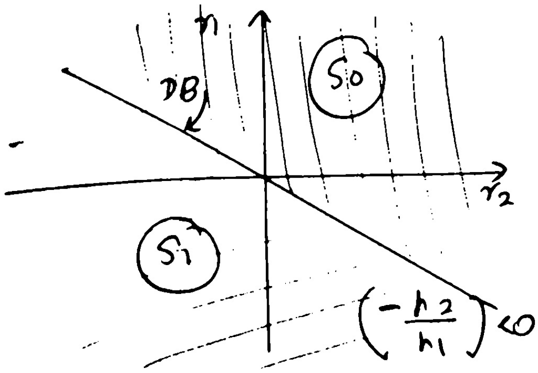
$(h_1 < 0)$

$$\boxed{r_1 \underset{S_1}{\overset{S_0}{\gtrless}} -\left(\frac{h_2}{h_1}\right) r_2}$$

Subcase I. $(h_2 > 0)$



Subcase II. $(h_2 < 0)$

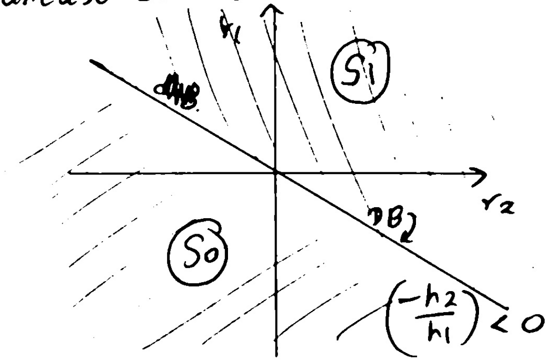


case (II)

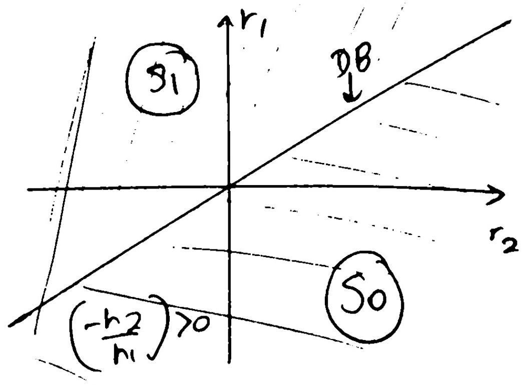
$(h_1 > 0)$

$$\boxed{r_1 \underset{S_0}{\overset{S_1}{\gtrless}} -\left(\frac{h_2}{h_1}\right) r_2}$$

Subcase I. $(h_2 > 0)$



Subcase II. $h_2 < 0$



$\boxed{Q3}$ b.)

Under the hypothesis $(s_1 = 1)$.

$$\underline{r} = \underline{h} s_i + \underline{n}$$

$$\underline{r} = h(1) + \underline{n}$$

$$\underline{r} = \underline{h} + \underline{n}$$

$$\underline{r}^T \cdot \underline{h} = (\underline{h} + \underline{n})^T \cdot \underline{h}$$

consider $E\{r^T h\} = \mu_h$

$$= E\{(\underline{h} + \underline{n})^T \cdot \underline{h}\}$$

$$= E\{(\underline{h}^T + \underline{n}^T) \cdot \underline{h}\}$$

Here h^T is a vector of constants and

$$\begin{bmatrix} n1 \\ n2 \end{bmatrix} = \underline{n} \sim \mathcal{N}_2\left(0, \frac{N_0}{2} I_2\right) \Rightarrow E\{\underset{\sim}{n}^T\} = E\{\underset{\sim}{n}\} = \underset{\sim}{0}$$

$$\Rightarrow \cancel{E\{h^T n + \|h\|^2\}} = \cancel{h^T E\{n\} + E\{\|h\|^2\}}$$

$$= E\{\underline{h}^T \cdot \underline{h}\} + \underbrace{E\{n^T h\}}_{0}$$

$$\therefore \quad \mu_h = E\{\|h\|^2\} = \|h\|^2 \longrightarrow (1)$$

constant

consider $\sigma_h^2 = Var(r^T h) = E\{(r^T h - \|h\|^2)^2\}$

$$= E\{(\underline{r}^T\underline{h} - \|h\|^2)(\underline{r}^T\underline{h} - \|h\|^2)\}$$

$$= E\{(\underline{r}^T\underline{h})^2\} - 2\|h\|^2 \underbrace{E\{r^T h\}}_{\text{from ①}} + E\{\|h\|^4\}$$

$$= E\{(\underline{r}^T\underline{h})^2\} - 2\|h\|^4 + \|h\|^4$$

$$= E\{(\underline{r}^T\underline{h})^2\} - \|h\|^4$$

$$\underline{r}^T\underline{h} = \|h\|^2 + \underline{n}^T\underline{h}.$$

$$= E\{(\|h\|^2 + n^T h)^2\} - \|h\|^4$$

$$= E\{\|h\|^4 + 2\|h\|^2 \underline{n}^T\underline{h} + (n^T h)^2\} - \|h\|^4$$

$$= E\{\cancel{\|h\|^4}\} + \underbrace{2\|h\|^2 E\{n^T h\}}_{0} + E\{(n^T h)^2\} - \cancel{\|h\|^4}$$

$$\Rightarrow Var(r^T h_1) = E\{(n^T h)^2\}$$

$$\boxed{n^T h = [n_1 \; n_2]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = (n_1 h_1 + n_2 h_2)}$$

$$= E\{(n_1 h_1 + n_2 h_2)^2\}$$

$$= E\{n_1^2 (h_1^2)\} + E\{n_2^2 h_2^2\} + E\{n_1 n_2 (h_1 h_2)\}$$

$$= h_1^2 \cdot \underbrace{E\{n_1^2\}}_{\frac{N_0}{2}} + h_2^2 \underbrace{E\{n_2^2\}}_{\frac{N_0}{2}} + h_1 h_2 \underbrace{E\{n_1 n_2\}}_{\underbrace{E[n_1]\,E[n_2]}_{0 \times 0}}$$

$(= 0)$

$$\therefore Var(\underline{r}^T\underline{h}) = h_1^2 \frac{N_0}{2} + h_2^2 \frac{N_0}{2}$$

$$= \underbrace{(h_1^2 + h_2^2)} \frac{N_0}{2}$$

$$Var(r^T h) = \|h\|^2 \frac{N_0}{2} \longrightarrow (2)$$

$\therefore$ from (1) and (2),

$$E\{\underline{r}^T\underline{h}\} = \|h\|^2 = \mu_h$$

$$Var\{\underline{r}^T\underline{h}\} = \|h\|^2 \frac{N_0}{2} = \sigma_h^2$$

$$\therefore \underline{\underline{\underline{r}^T\underline{h} \sim N(\mu_h, \sigma_h^2)}}$$

$\boxed{Q3}$ c.)

$$P(\varepsilon) = P(\varepsilon \mid S_0) \times \underbrace{P(S_0)}_{\frac{1}{2}} + P(\varepsilon \mid S_1)\underbrace{P(S_1)}_{1/2}$$

$$P(\varepsilon) = \frac{1}{2}\left[P(\varepsilon \mid S_0) + P(\varepsilon \mid S_1)\right) \longrightarrow (1)$$

from part (a.) the optimum MAP decision is

correct S_1: $\quad \underline{r}^T\underline{h} > 0 \rightarrow P(\varepsilon \mid S_1) = P(\underline{r}^T\underline{h} < 0)$

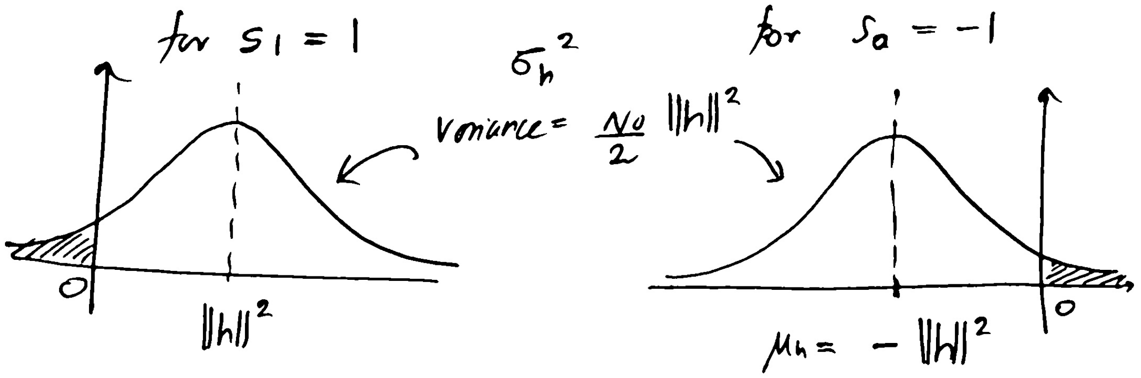
correct S_0: $\quad \underline{r}^T\underline{h} < 0 \rightarrow P(\varepsilon \mid S_0) = P(\underline{r}^T\underline{h} > 0)$

from part ~~(~~(b.): under hypothesis $S_1 = 1$

$$S_1: \quad r^T h \sim N\left(\|h\|^2, \frac{N_0}{2}\|h\|^2\right)$$

Same way under the Hypothesis $S_0 = -1$

$$\underline{r}^T\cdot\underline{h} = (-\underline{h} + \underline{n})\cdot\underline{h} \Rightarrow \underline{r}^T\cdot\underline{h} \sim N\left(-\|h\|^2, \frac{N_0}{2}\|h\|^2\right)$$



∴ ① can be rewritten as, follows,

$$P(e) = \frac{1}{2}\left[P(e|s_0) + P(e|s_1)\right]$$

$$= \frac{1}{2}\left[P(\underline{r}^T\underline{h} > 0 | s_0) + P(r^T h < 0 | s_1)\right]$$

$$= \frac{1}{2}\left\{ \int_0^{\infty} N\left(-\|h\|^2, \frac{N_0}{2}\|h\|^2\right) + \int_{-\infty}^{0} N\left(+\|h\|^2, \frac{N_0}{2}\|h\|^2\right)\right.$$

Probability distributions are mirror images of each other! therefore we can use either of the distributions to solve above expression..

$$\therefore \; P(e) = \frac{1}{2} \times 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma_h^2}} \, e^{-\frac{[\underline{r}^T\underline{h} - (-\|h\|^2)]}{2\sigma_h^2}} \, d(r^T h)$$

$$P(e) = \frac{1}{\sqrt{2\pi}\,\sigma_h} \int_0^{\infty} \exp\left[\frac{-(\underline{r}_0^T\underline{h} + \|h\|^2)^2}{2\sigma_h^2}\right] d(\underline{r}^T\underline{h})$$

Substitute : $\dfrac{\underline{r}^T\underline{h} + \|h\|^2}{\sigma_h} = t$

limits.

when $\gamma^T h = 0 \longrightarrow t = \frac{\|h\|^2}{\sigma_h}$

$\gamma^T h \to \alpha \longrightarrow t \longrightarrow \infty$

$\sigma_h \, dt = d(\gamma^T h)$

$$\therefore P(\varepsilon) = \frac{1}{\sqrt{2\pi}\,\cancel{\sigma_h}} \int_{\frac{\|h\|^2}{\sigma_h}}^{\infty} \exp(-t^2/2) . \cancel{\sigma_h}\, dt$$

$$\therefore P(\varepsilon) = Q\left(\frac{\|h\|^2}{\sigma_h}\right)$$

$$= Q\left(\frac{\|h\|^2}{\sqrt{\frac{N_0}{2}\|h\|^2}}\right)$$

$$\therefore P(e) = Q\left[\frac{2\|h\|^2}{N_0}\right]$$

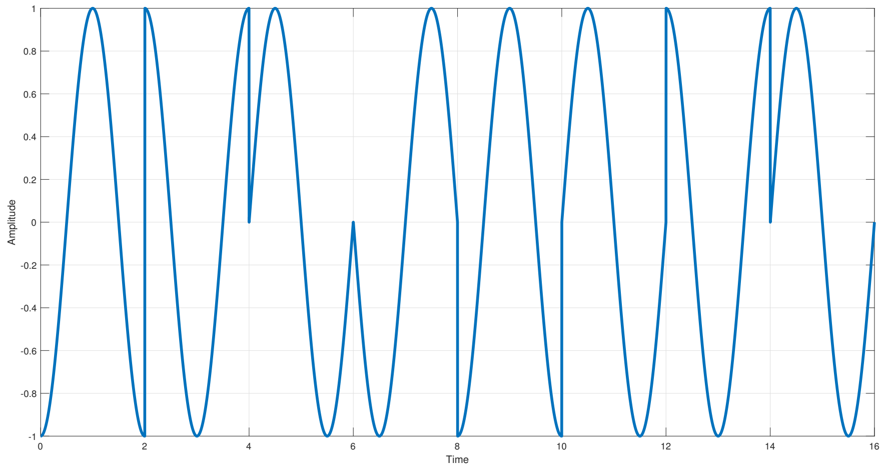
Where, $Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-t^2/2}\, dt$

Question 4

(a) The sequence of signals (waveforms) corresponding to the binary sequence

Please note that when plotting the below figure following values were assumed for the parameters.

Bit interval (T_b)	$1\ s$
Symbol interval (T)	$2 \times T_b = 2\ s$

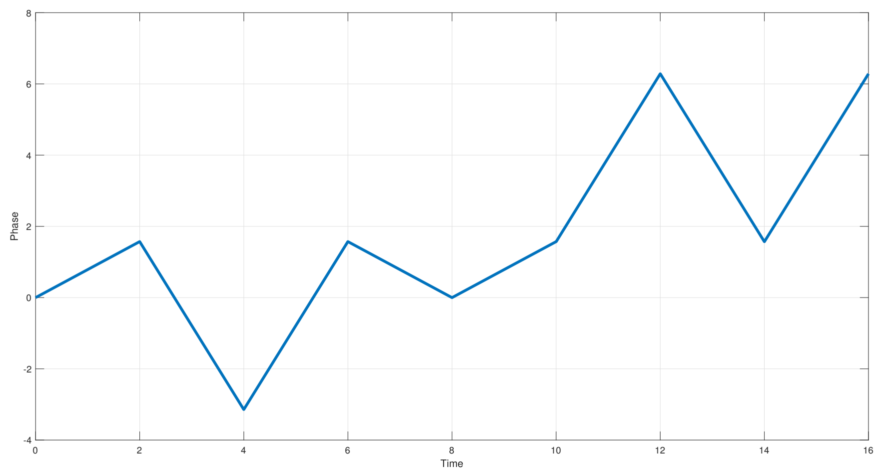



```
T_b = 1;              % bit interval
h = 2;                % number of bits per symbol
T = h*T_b;            % symbol interval
binary_seq = [1 1 0 0 1 0 0 1 1 1 1 0 0 0 1 0];
signals = length(binary_seq)/h;
resolution = 1000;  % number of data points for a smooth plot
t = linspace(0,T,resolution);

% generating symbols according to the mapping rule
s_0_t = sqrt(2/T)*cos(2*pi*t/T + pi*0/2); % i = 0
s_1_t = sqrt(2/T)*cos(2*pi*t/T + pi*1/2); % i = 1
s_2_t = sqrt(2/T)*cos(2*pi*t/T + pi*2/2); % i = 2
s_3_t = sqrt(2/T)*cos(2*pi*t/T + pi*3/2); % i = 3

% constructing the signal
tx_signal = zeros(signals, resolution);
signal = 1;
for index = 1:h:length(binary_seq) %initVal:step:endVal
    if binary_seq(index) == 0 && binary_seq(index +1) == 0
        tx_signal(signal,:) = s_0_t;
    elseif binary_seq(index) == 0 && binary_seq(index +1) == 1
        tx_signal(signal,:) = s_1_t;
    elseif binary_seq(index) == 1 && binary_seq(index +1) == 1
        tx_signal(signal,:) = s_2_t;
    elseif binary_seq(index) == 1 && binary_seq(index +1) == 0
        tx_signal(signal,:) = s_3_t;
    end
    signal = signal +1;
end
tx_signal = reshape(tx_signal',1,[]); % pre-processing for plotting
plot(linspace(0,signals*T, length(tx_signal)), tx_signal, 'LineWidth',3);
grid on; xlabel('Time'); ylabel('Amplitude')
```

(b) The Quaternary MSK phase trajectory

Please note that when plotting the below figure following values were assumed for the parameters.

Bit interval (T_b)	$1\ s$
Symbol interval (T)	$2 \times T_b = 2\ s$

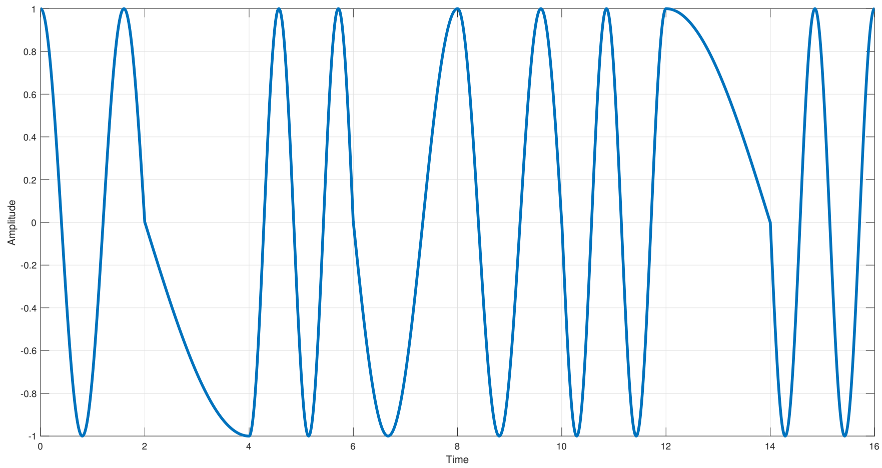



```
T_b = 1;              % bit interval
h = 2;                % number of bits per symbol
T = h*T_b;            % symbol interval
binary_seq = [1 1 0 0 1 0 0 1 1 1 1 0 0 0 1 0];
num_of_Xks = length(binary_seq)/h;
resolution = 1000;  % number of data points for a smooth plot
t = linspace(0,T,resolution);

% genrating X_k s
X_ks = zeros(1,num_of_Xks);
k = 1;
for index = 1:h:length(binary_seq) %initVal:step:endVal
    if binary_seq(index) == 0 && binary_seq(index +1) == 0
        X_ks(k) = -3;
    elseif binary_seq(index) == 0 && binary_seq(index +1) == 1
        X_ks(k) = -1;
    elseif binary_seq(index) == 1 && binary_seq(index +1) == 1
        X_ks(k) = +1;
    elseif binary_seq(index) == 1 && binary_seq(index +1) == 0
        X_ks(k) = +3;
    end
    k = k +1;
end

phase = zeros(num_of_Xks, resolution);
for n = 0:(num_of_Xks-1)
    t = linspace(n*T, (n+1)*T, resolution);
    phase(n+1,:) = (pi/2)*sum(X_ks(1:n)) + (pi/2)*X_ks(n+1)*(t/T -n);
end
phase = reshape(phase',1,[]); % pre-processing for plotting
plot(linspace(0,num_of_Xks*T, length(phase)), phase, 'LineWidth',3);
grid on; xlabel('Time'); ylabel('Phase')
```

(c) The Quaternary MSK waveform corresponding to the binary sequence

Please note that when plotting the below figure following values were assumed for the parameters.

Bit interval (T_b)	1 s
Symbol interval (T)	$2 \times T_b = 2\ s$




```
T_b = 1;              % bit interval
h = 2;                % number of bits per symbol
T = h*T_b;            % symbol interval
binary_seq = [1 1 0 0 1 0 0 1 1 1 1 0 0 0 1 0];
num_of_Xks = length(binary_seq)/h;
resolution = 1000;  % number of data points for a smooth plot
t = linspace(0,T,resolution);

% genrating X_k s
X_ks = zeros(1,num_of_Xks);
k = 1;
for index = 1:h:length(binary_seq) %initVal:step:endVal
    if binary_seq(index) == 0 && binary_seq(index +1) == 0
        X_ks(k) = -3;
    elseif binary_seq(index) == 0 && binary_seq(index +1) == 1
        X_ks(k) = -1;
    elseif binary_seq(index) == 1 && binary_seq(index +1) == 1
        X_ks(k) = +1;
    elseif binary_seq(index) == 1 && binary_seq(index +1) == 0
        X_ks(k) = +3;
    end
    k = k +1;
end

QMSK = zeros(num_of_Xks, resolution);
% generating quaternary MSK waveform
for n = 0:(num_of_Xks-1)
    t = linspace(n*T, (n+1)*T, resolution);
    phase = (pi/2)*sum(X_ks(1:n)) + (pi/2)*X_ks(n+1)*(t/T -n);
    QMSK(n+1,:) = sqrt(2/T)*cos(2*pi*t/T + phase);
end
QMSK = reshape(QMSK',1,[]); % pre-processing for plotting
plot(linspace(0,num_of_Xks*T, length(QMSK)), QMSK, 'LineWidth',3);
grid on; xlabel('Time'); ylabel('Amplitude')
```