

Department of Electronic and Telecommunication Engineering

University of Moratuwa, Sri Lanka

EN3053 - Digital Communications - I



# Assignment No. 1

Submitted by

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$$S(t) = \sum_{k=0}^{N-1} x[k] \phi_k(t) \quad 0 \leq t < NT$$

where  $x[k] \in \mathbb{C}$ ,  $N \in \mathbb{Z}^+$ ,  $T > 0$  and

$$\phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt}$$

consider,  $\langle \phi_{k(t)}, \phi_{l(t)} \rangle$ : the Hermitian Inner Product.

$$\begin{aligned} &= \int_0^{NT} \phi_{k(t)} \cdot \phi_{l(t)}^* dt \\ &= \int_0^{NT} \left( \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt} \right) \cdot \left( \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} lt} \right)^* dt \\ &= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} kt} \cdot e^{-j \frac{2\pi}{NT} lt} dt \\ &= \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (k-l)t} dt \end{aligned}$$

Case 1 if  $k=l$ :

$$\begin{aligned} &\Rightarrow \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} (0)t} dt \\ &= \frac{1}{NT} \int_0^{NT} 1 dt = \frac{1}{NT} [t]_0^{NT} = \frac{1}{NT} [NT - 0] \\ &= \underline{\underline{1}} \longrightarrow \textcircled{1} \end{aligned}$$

Case 2 :- if  $k \neq \ell$

Since  $k, \ell \in \mathbb{Z}_+^* \Rightarrow (k-\ell) \in \mathbb{Z}$

∴ Let  $(k-\ell) = m \in \mathbb{Z}$

$$\begin{aligned}
 & \frac{1}{NT} \int_0^{NT} e^{j \frac{2\pi}{NT} mt} dt \\
 &= \frac{1}{NT} \left[ \frac{e^{j \frac{2\pi}{NT} m \cdot t}}{j \frac{2\pi}{NT} m} \right] \Big|_0^{NT} \\
 &= \frac{1}{j^{2\pi m}} \left[ e^{j \frac{2\pi}{NT} m \cdot NT} - 1 \right] \\
 &\stackrel{2}{=} \frac{1}{j^{2\pi m}} \left[ e^{j(2\pi)m} - 1 \right] \text{ where } m \in \mathbb{Z} \\
 &= \frac{1}{j^{2\pi m}} \left[ \underbrace{\cos(2\pi m)}_{(Always=1)} + j \overbrace{\sin(2\pi m)}^0 - 1 \right] \\
 &= \frac{1}{j^{2\pi m}} [1 - 1] \Rightarrow 0 \longrightarrow \textcircled{2}
 \end{aligned}$$

from ① and ②

$$\text{∴ } \int_0^{NT} \phi_k(t) \cdot \phi_\ell^*(t) dt = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

Q1

b.) Total Energy  $\gamma = \langle s(t), s(t) \rangle$   
Content of  $s(t)$

$$= \int_0^{NT} s(t) (s(t))^* dt \quad ; (0 < t \leq NT)$$

$$= \int_0^{NT} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[ \sum_{k=0}^{N-1} x[k] \phi_k^*(t) \right]^* dt$$

↓ complex conjugate

Since  $(z_1 + z_2)^* = z_1^* + z_2^*$

&  $z_1, z_2 \in \mathbb{C}$  we can apply the conjugate to above summation, separately.

$$= \int_0^{NT} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(t) \right] \left[ \sum_{k=0}^{N-1} (x[k])^* \phi_k^*(t) \right] dt$$

= from part a ~~as~~ terms are vanished when the  $k$  values are different.

$$\Rightarrow \sum_{k=0}^{N-1} \left\{ x[k] (x[k])^* \right\} \underbrace{\int_0^{NT} \phi_k(t) \cdot \phi_k^*(t) dt}_{=1} \quad , \text{from } \textcircled{a}$$

$$\Rightarrow \epsilon = \sum_{k=0}^{N-1} |x[k]|^2$$

**Q1**

$$C.) \sum_{m=0}^{N-1} \phi_k(mT) \phi_k^*(mT) \quad ; \quad \phi_k(t) = \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} kt}$$

$$\Rightarrow \sum_{m=0}^{N-1} \left[ \frac{1}{\sqrt{NT}} e^{j \frac{2\pi}{NT} k(mT)} \right] \left[ \frac{1}{\sqrt{NT}} e^{-j \frac{2\pi}{NT} k(mT)} \right]^*$$

$$= \sum_{m=0}^{N-1} \frac{1}{NT} \left[ e^{j \frac{2\pi}{N} k \cdot m} \right] \left[ e^{-j \frac{2\pi}{N} k \cdot m} \right]$$

$$= \frac{1}{NT} \sum_{m=0}^{N-1} e^{j \frac{2\pi}{N} m(k-l)}$$

$$= \frac{1}{NT} \cdot \underbrace{\sum_{m=0}^{N-1} \left[ e^{j \frac{2\pi}{N} (k-l)} \right]^m}_2$$

$$= \frac{1}{NT} \cdot \frac{1 - \left[ e^{j \frac{2\pi}{N} (k-l)} \right]^N}{1 - e^{j \frac{2\pi}{N} (k-l)}} \quad \text{using the given hint.}$$

$$= \frac{1}{NT} \cdot \frac{1 - e^{j 2\pi (k-l)}}{1 - e^{j \frac{2\pi}{N} (k-l)}} \quad \rightarrow \textcircled{1}$$

**case I** when  $(k-l) \neq 0 \Rightarrow k \neq l$

since  $k, l \in \mathbb{Z}_0^+ \Rightarrow (k-l) \in \mathbb{Z}$

$$\therefore e^{j\frac{2\pi}{N}(k-l)} = 1 \quad \forall (k-l) \in \mathbb{Z}$$

$$\therefore \frac{\frac{1}{NT} \left[ 1 - e^{j\frac{2\pi}{N}(k-l)} \right]}{1 - e^{j\frac{2\pi}{N}(k-l)}} = \frac{\frac{1}{NT} (1 - 1)}{(1 - e^{j\frac{2\pi}{N}(k-l)})}$$

$$= 0 \rightarrow ②.$$

**case 2**

when  $k = l$ , using L'hospital's rule.

$$\begin{aligned} &= \lim_{(k-l) \rightarrow 0} \frac{\frac{1}{NT} \left[ 1 - e^{j\frac{2\pi}{N}(k-l)} \right]}{1 - e^{j\frac{2\pi}{N}(k-l)}} \\ &= \lim_{(k-l) \rightarrow 0} \frac{\frac{1}{NT} \left[ \cancel{1} - j\frac{2\pi}{N} \cdot e^{j\frac{2\pi}{N}(k-l)} \right]}{-j\frac{2\pi}{N} e^{j\frac{2\pi}{N}(k-l)}} \quad \text{differentiate w.r.t } (k-l) \end{aligned}$$

$$= \lim_{(k-l) \rightarrow 0} \frac{\frac{1}{NT} \times N \times \frac{e^{j\frac{2\pi}{N}(k-l)}}{e^{j\frac{2\pi}{N}(k-l)}}}{-j\frac{2\pi}{N}}$$

$$= \frac{1}{NT} \times N \times \frac{1}{-j\frac{2\pi}{N}}$$

$$= \frac{1}{T} \rightarrow ③$$

$$\therefore \text{from } ② \text{ and } ③ \Rightarrow \left\{ \sum_{m=0}^{N-1} \phi_k(mT) \phi_l^*(mT) = \begin{cases} \frac{1}{T} & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases} \right.$$

**Q1**

d.)  $s[m] = s(mT)$ ,  $m = 0, 1, 2, \dots, N-1$

$$\begin{aligned}
 \text{Consider RHS} &= T \times \sum_{m=0}^{N-1} s[m] \phi_e^*(mT) \\
 &= T \times \sum_{m=0}^{N-1} s(mT) \cdot \phi_e^*(mT) \\
 &= \pi x \left[ \sum_{m=0}^{N-1} s(mT) \right] \phi_e^*(t) \\
 &= T \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] \phi_k(t) \right] \phi_e^*(t) \Big|_{t=mT} \\
 &= T \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] \cdot \phi_k(mT) \cdot \phi_e^*(mT) \right] \\
 &= \sum_{k=0}^{N-1} x[k] \left\{ (T) \underbrace{\sum_{m=0}^{N-1} \phi_k(mT) \phi_e^*(mT)}_{\text{from part C; this part is } \frac{1}{T} \text{ when } k=e \text{ and vanishes otherwise.}} \right\} \\
 \text{so when } k=e &\quad (\downarrow) \\
 &\Rightarrow x[e] \times T \times \frac{1}{T}
 \end{aligned}$$

$$\therefore x[e] = T \sum_{m=0}^{N-1} s[m] \phi_e^*(mT)$$

for  $e = 0, 1, 2, \dots, N-1$

Q2. a)

consider  $\left| \sum_{k=1}^M \phi_k \right|^2 \geq 0 \rightarrow \textcircled{1}$

$$\begin{aligned}\left| \sum_{k=1}^M \phi_k \right|^2 &= \left( \sum_{k=1}^M \phi_k \right)^T \left( \sum_{k=1}^M \phi_k \right) \\ &= (\phi_1 + \phi_2 + \dots + \phi_M)^T (\phi_1 + \phi_2 + \dots + \phi_M) \\ &= (\phi_1^T + \phi_2^T + \dots + \phi_M^T)(\phi_1 + \phi_2 + \dots + \phi_M)\end{aligned}$$

In this expression there will be,

- (i)  $M$  terms, where  $i=j$  for  $\phi_i^T \phi_j$
- (ii)  $M(M-1)$  terms, where  $i \neq j$  for  $\phi_i^T \phi_j$   
(for each  $\phi_i \in \mathbb{R}^{(M-1) \times 1}$   $\phi_j$ 's such that  $i \neq j$ ).

so  $\left| \sum_{k=1}^M \phi_k \right|^2 = \underbrace{(M \times 1)}_{\text{from (i)}} + \underbrace{M(M-1)\rho}_{\text{from (ii)}}$

$$\left| \sum_{k=1}^M \phi_k \right|^2 \geq 0 \rightarrow \text{from } \textcircled{1}$$

$$M + M(M-1)\rho \geq 0$$

$$1 + (M-1)\rho \geq 0 \quad (\because M > 0)$$

$$\frac{-1}{M-1} \leq \rho \quad \text{--- } \textcircled{2} \quad (\because (M-1) > 0)$$


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$$\rho \leq 1 \longrightarrow \textcircled{3}$$

∴ From ② and ③,

$$\frac{-1}{M-1} \leq \rho \leq 1$$

Q2. b.)

Assume  $\{\phi_i\}$  is a simplex set. Then following expression must equal to zero, as  $\{\phi_i\}$ 's are already in the minimum energy constellation.

$$\cancel{\sum_{i=1}^M} \quad \sum_{i=1}^M p_i \phi_i = 0$$

Since  $\{\phi_i\}$  are equally likely,  $p_i = \frac{1}{M}$

$$\therefore \frac{1}{M} \sum_{i=1}^M \phi_i = 0$$

$$\Rightarrow \sum_{i=1}^M \phi_i = 0$$

consider,  $\left\| \left( \sum_{i=1}^M \phi_i \right) \right\|^2 = \|\phi\|^2$

$$\underbrace{\left( \sum_{i=1}^M \phi_i \right)^T}_{\text{from part (a)}} \underbrace{\left( \sum_{i=1}^M \phi_i \right)}_{\text{is simplified into}} = 0$$

from part (a) this part is simplified into,

$$M + M(M-1)\rho = 0$$

$$\therefore \rho = \frac{-1}{M-1} \quad //$$

[Q2] c.)

① Determining the minimum energy set corresponding to the set  $\{\phi_i\}$ .

\* Let  $\{\beta_i\}$  be the minimum energy set of the signal set  $\{\phi_i\}$ .

$$\Rightarrow \{\beta_i\} = \{\phi_i - \underline{a}\} \text{ where, } \underline{a} = \frac{1}{M} \sum_{i=1}^M \phi_i$$

$$\begin{aligned}
 & \text{consider } \beta_i^T \beta_e = (\phi_i - \underline{a})^T (\phi_e - \underline{a}) \\
 &= (\phi_i^T - \underline{a}^T)(\phi_e - \underline{a}) \\
 &= \phi_i^T \phi_e - \phi_i^T \underline{a} - \underline{a}^T \phi_e + \underline{a}^T \underline{a} \\
 &= \phi_i^T \phi_e - \phi_i^T \left( \frac{1}{M} \sum_{i=1}^M \phi_i \right) - \left( \frac{1}{M} \sum_{i=1}^M \phi_i \right)^T \phi_e + \underline{a}^T \underline{a} \\
 &= \phi_i^T \phi_e - \frac{1}{M} [1 + (M-1)\rho] - \underbrace{\frac{1}{M} [1 + (M-1)\rho]}_{\text{from part (a)}} + \|\underline{a}\|^2 \\
 &= \phi_i^T \phi_e - \frac{2}{M} [1 + (M-1)\rho] + \frac{1}{M^2} [M + (M-1)\rho] \\
 &= \phi_i^T \phi_e - \frac{1}{M} [1 + (M-1)\rho] \quad \longrightarrow ①
 \end{aligned}$$

from def<sup>h</sup>:  $Q_i^T Q = \begin{cases} 1 & i = \ell \\ \rho & i \neq \ell \end{cases}$  Q2 C) contn:

$\therefore (\beta_i^T \beta_\ell) = \begin{cases} 1 - \frac{1}{M} [1 + (M-1)\rho] & ; i = \ell \\ \rho - \frac{1}{M} [1 + (M-1)\rho] & ; i \neq \ell \end{cases}$

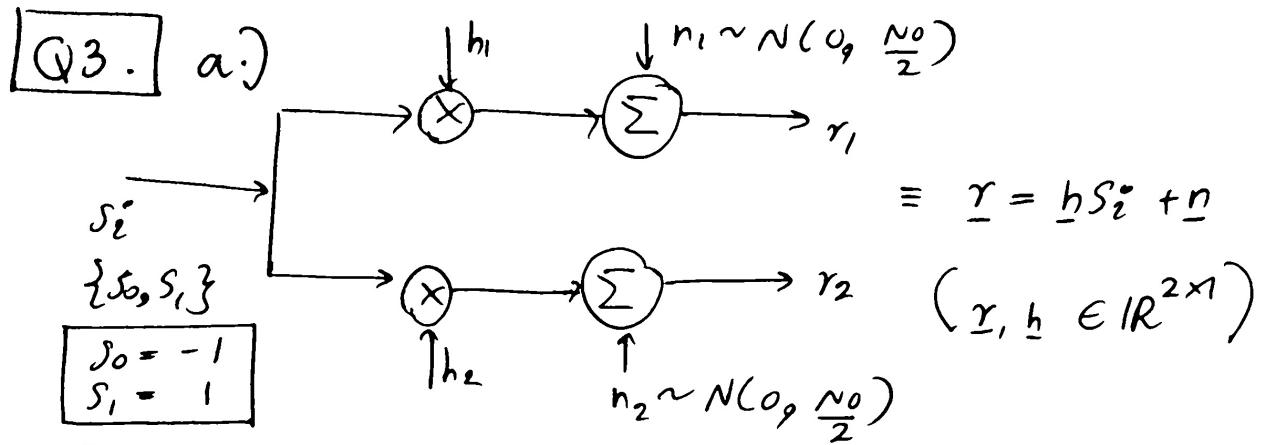
"  $\beta_i^T \beta_\ell = \begin{cases} \frac{(M-1)(1-\rho)}{M} & ; i = \ell \\ \frac{\rho-1}{M} & ; i \neq \ell \end{cases}$  " - ②

from part (b)

For the unit average  $\rightarrow \alpha_i^T \alpha_\ell = \begin{cases} 1 & ; i = \ell \\ \rho = \frac{1}{M-1} & ; i \neq \ell \end{cases}$   
Simplex set

By rearranging ②,  
 $\Rightarrow \beta_i^T \beta_\ell = \begin{cases} 1 \times \left[ \frac{(M-1)(1-\rho)}{M} \right] & ; i = \ell \\ -\frac{1}{M-1} \times \left[ \frac{(M-1)(1-\rho)}{M} \right] & ; i \neq \ell \end{cases}$

$\therefore$  we can observe that minimum energy set  $\{\beta_i\}$  is a scaled constellation of the simplex set  $\{\alpha_i\}$ .



a.) Consider the following a posteriori probabilities.

① For the receiver to classify the observed signal as  $s_0$   $\Rightarrow P(s_0 | r) \stackrel{s_0}{>} P(s_1 | r) \rightarrow \textcircled{1}$

② For the receiver to classify the observed signal as  $s_1$   $\Rightarrow P(s_0 | r) \stackrel{s_1}{<} P(s_1 | r) \rightarrow \textcircled{2}$

By combining ① and ②, for

$$P(s_0 | r) \stackrel{s_0}{>} P(s_1 | r) \leftarrow \text{This is what demanded by the receiver!} \quad \textcircled{3}$$

But what is known, to us, are the following probabilities.

$$P(r | s_0), P(r | s_1)$$

∴ By using Bayes theorem, (3) can be rewritten as follows.

$$\frac{P(\underline{s}_0 | \underline{z})}{P(\underline{s}_1 | \underline{z})} \stackrel{\underline{s}_0}{\gtrless} \frac{P(\underline{s}_1 | \underline{z})}{P(\underline{s}_0 | \underline{z})}$$

$$\frac{P(\underline{r} | \underline{s}_0) \times P(\underline{s}_0)}{P(\underline{r})} \stackrel{\underline{s}_0}{\gtrless} \frac{P(\underline{r} | \underline{s}_1) \times P(\underline{s}_1)}{P(\underline{r})}$$

Since  $\underline{s}_0$  and  $\underline{s}_1$  are equally likely signals,

$$P(\underline{s}_0) = P(\underline{s}_1) = \frac{1}{2}$$

∴  $P(\underline{z} | \underline{s}_0) \stackrel{\underline{s}_0}{\gtrless} P(\underline{z} | \underline{s}_1)$

Since  $\underline{r} = \underline{h}\underline{s}_i + \underline{n}$

$$\begin{cases} \underline{s}_i = \underline{s}_0 \rightarrow \underline{r} = \underbrace{\underline{h}(-1)}_{\underline{s}_0} + \underline{n} \\ \underline{s}_i = \underline{s}_1 \rightarrow \underline{r} = \underbrace{\underline{h}(1)}_{\underline{s}_1} + \underline{n} \end{cases}$$

∴  $P(\underline{z} | -\underline{h}) \stackrel{\underline{s}_0}{\gtrless} P(\underline{z} | \underline{h})$

$$N_2(-\underline{h}, \frac{N_0}{2} I_2) \stackrel{\underline{s}_0}{\gtrless} N_2(\underline{h}, \frac{N_0}{2} I_2)$$

$$e^{-\|\underline{r} - (-\underline{h})\|^2} \stackrel{\underline{s}_0}{\gtrless} e^{-\|\underline{r} - (\underline{h})\|^2}$$

$$\|\underline{r} + \underline{h}\|^2 \stackrel{\underline{s}_1}{\gtrless} \|\underline{r} - \underline{h}\|^2$$

$$\|\underline{r} + \underline{h}\|^2 \stackrel{S_1}{\underset{S_0}{\gtrless}} \|\underline{r} - \underline{h}\|^2$$

$$(\underline{r} + \underline{h})^T (\underline{r} + \underline{h}) \stackrel{S_1}{\underset{S_0}{\gtrless}} (\underline{r} - \underline{h})^T (\underline{r} - \underline{h})$$

$$(\underline{r}^T + \underline{h}^T)(\underline{r} + \underline{h}) \stackrel{S_1}{\underset{S_0}{\gtrless}} (\underline{r}^T - \underline{h}^T)(\underline{r} - \underline{h})$$

$$\begin{aligned} \underline{r}^T \underline{r} + \underline{r}^T \underline{h} & \stackrel{S_1}{\underset{S_0}{\gtrless}} \underline{r}^T \underline{r} - \underline{r}^T \underline{h} \\ + \underline{h}^T \underline{r} + \underline{h}^T \underline{h} & \stackrel{S_1}{\underset{S_0}{\gtrless}} - \underline{h}^T \underline{r} + \underline{h}^T \underline{h} \end{aligned}$$

Since  $\underline{h}, \underline{r} \in \mathbb{R}^{2 \times 1} \Rightarrow \underline{r}^T \underline{h} = \underline{h}^T \underline{r}$

$$\therefore 2 \underline{r}^T \underline{h} \stackrel{S_1}{\underset{S_0}{\gtrless}} -2 \underline{r}^T \underline{h}$$

$$4 \underline{r}^T \underline{h} \stackrel{S_1}{\underset{S_0}{\gtrless}} 0$$

$$\boxed{\underline{r}^T \underline{h} \stackrel{S_1}{\underset{S_0}{\gtrless}} 0} \leftarrow \text{Optimum MAP decision rule.}$$

$$\underline{r}^T h \geq_{S_0} 0$$

$$[r_1 \ r_2] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \geq_{S_0} 0$$

$$r_1 h_1 + r_2 h_2 \geq_{S_0} 0$$

$$r_1 \geq -\left(\frac{h_2}{h_1}\right) r_2$$

$h_1 < 0$  inequality changes

$h_1 > 0$  inequality remains the same.

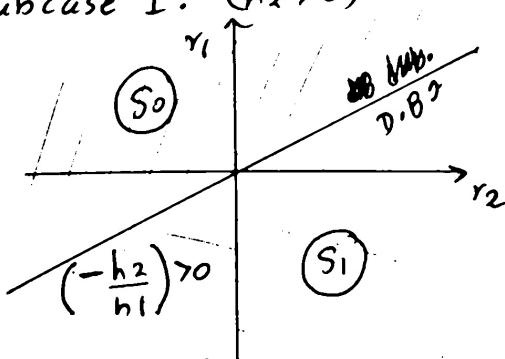
# case I

$$(h_1 < 0)$$

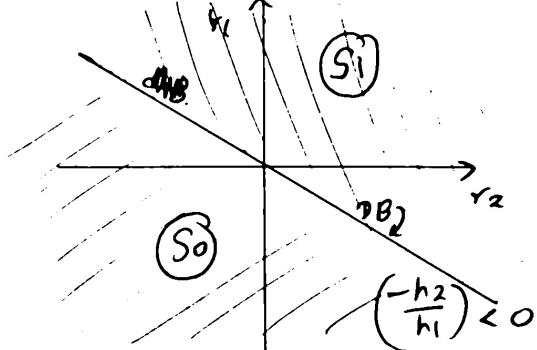
# case II

$$(h_1 > 0)$$

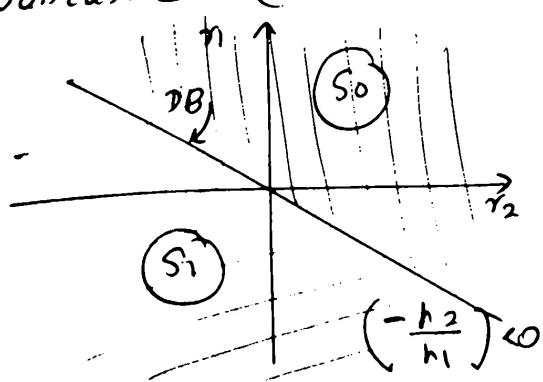
Subcase I. ( $h_2 > 0$ )



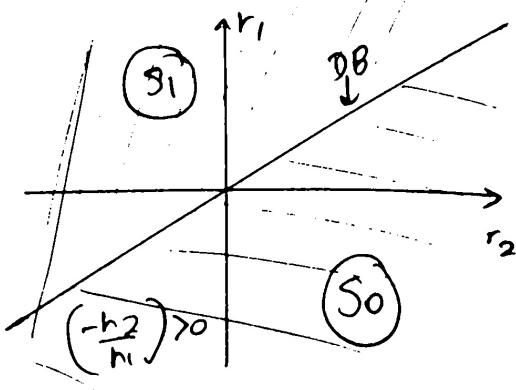
Subcase I. ( $h_2 > 0$ )



Subcase II. ( $h_2 < 0$ )



Subcase II.  $h_2 < 0$



Q3 b.)

Under the hypothesis ( $s_1 = 1$ )

$$\underline{r} = \underline{h} \underline{s}_1 + \underline{n}$$

$$\underline{r} = \underline{h}(1) + \underline{n}$$

$$\underline{r} = \underline{h} + \underline{n}$$

$$\underline{r}^T \underline{h} = (\underline{h} + \underline{n})^T \underline{h}$$

$$\text{consider } E\{\underline{r}^T \underline{h}\} = \mu_h$$

$$= E\{(\underline{h} + \underline{n})^T \underline{h}\}$$

$$= E\{(\underline{h}^T + \underline{n}^T) \cdot \underline{h}\}$$

Here  $\underline{h}^T$  is a vector of constants and

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \underline{n} \sim N_2(0, \frac{N_0}{2} I_2) \Rightarrow E\{\underline{n}^T\} = E\{\underline{n}\} = \underline{0}$$

$$\Rightarrow E\{\underline{h}^T \underline{n} + \underline{n}^T \underline{h}\} = \cancel{\underline{h}^T \underline{n}} + \cancel{\underline{n}^T \underline{h}}$$

$$= E\{\underline{h}^T \underline{h}\} + \underbrace{E\{\underline{n}^T \underline{h}\}}_0$$

$$\therefore \mu_h = E\{\underbrace{\|\underline{h}\|^2}_{\text{constant}}\} = \|\underline{h}\|^2 \rightarrow ①$$

$$\text{consider } \sigma_h^2 = \text{Var}(r^T h) = E\{(r^T h - \|h\|^2)^2\}$$

$$= E\{(r^T h - \|h\|^2)(r^T h - \|h\|^2)\}$$

$$= E\{(r^T h)^2\} - 2\|h\|^2 E\{r^T h\} + E\{\|h\|^4\}$$

$$= E\{(r^T h)^2\} - 2\|h\|^4 + \|h\|^4 \quad \text{from ①}$$

$$= E\{(r^T h)^2\} - \|h\|^4$$

$$r^T h = \|h\|^2 + n^T h.$$

$$= E\{(\|h\|^2 + n^T h)^2\} - \|h\|^4$$

$$= E\{\|h\|^4 + 2\|h\|^2 n^T h + (n^T h)^2\} - \|h\|^4$$

$$= E\{\|h\|^4\} + \underbrace{2\|h\|^2 E\{n^T h\}}_0 + E\{(n^T h)^2\} - \|h\|^4$$

$$\Rightarrow \text{Var}(r^T h) = E\{(n^T h)^2\} \quad \left| \begin{array}{l} n^T h = [n_1 \ n_2] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ = (n_1 h_1 + n_2 h_2) \end{array} \right.$$

$$= E\{(n_1 h_1 + n_2 h_2)^2\} = E\{n_1^2 h_1^2\} + E\{n_2^2 h_2^2\} + E\{n_1 n_2 (h_1 h_2)\}$$

$$= h_1^2 \underbrace{E\{n_1^2\}}_{\frac{N_0}{2}} + h_2^2 \underbrace{E\{n_2^2\}}_{\frac{N_0}{2}} + \underbrace{h_1 h_2 E\{n_1 n_2\}}_{\underbrace{\overbrace{E[n_1] E[n_2]}_{0 \times 0}}_{(-0)}}$$

$\frac{N_0}{2}$

$\frac{N_0}{2}$

$\underbrace{\overbrace{E[n_1] E[n_2]}_{0 \times 0}}_{(-0)}$

$$\therefore \text{Var}(\underline{r^T h}) = h_1^2 \frac{N_0}{2} + h_2^2 \frac{N_0}{2}$$

$$= \underbrace{(h_1^2 + h_2^2)}_{\frac{N_0}{2}}$$

$$\text{Var}(r^T h) = \|h\|^2 \frac{N_0}{2} \rightarrow ③$$

$\therefore$  from ① and ②,

$$E\{\underline{r^T h}\} = \|h\|^2 = \mu_h$$

$$\text{Var}\{\underline{r^T h}\} = \|h\|^2 \frac{N_0}{2} = \sigma_h^2$$

$$\therefore \underline{r^T h} \sim \mathcal{N}(\mu_h, \sigma_h^2)$$

Q3 c.)

$$P(\varepsilon) = P(\varepsilon | s_0) \underbrace{P(s_0)}_{\frac{1}{2}} + P(\varepsilon | s_1) \underbrace{P(s_1)}_{\frac{1}{2}}$$

$$P(\varepsilon) = \frac{1}{2} [P(\varepsilon | s_0) + P(\varepsilon | s_1)] \longrightarrow \textcircled{1}$$

from part (a) the optimum MAP decision is

$$\text{correct } s_1 : \underline{r^T h} > 0 \rightarrow P(\varepsilon | s_1) = P(\underline{r^T h} < 0)$$

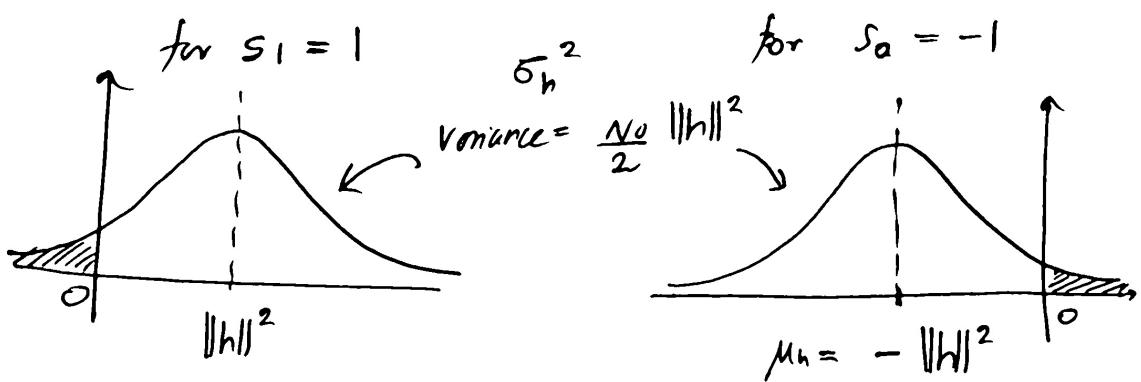
$$\text{correct } s_0 : \underline{r^T h} < 0 \rightarrow P(\varepsilon | s_0) = P(\underline{r^T h} > 0)$$

from Part b.: under hypothesis  $s_1 = 1$

$$s_1 : \underline{r^T h} \sim N(-\|h\|^2, \frac{N_0}{2}\|h\|^2)$$

Some way under the Hypothesis  $s_0 = -1$

$$\underline{r^T h} = (\underline{h} + \underline{n}) \cdot \underline{h} \Rightarrow \underline{r^T h} \sim N(-\|h\|^2, \frac{N_0}{2}\|h\|^2)$$



∴ ① can be rewritten as, follows,

$$\begin{aligned}
 P(a) &= \frac{1}{2} [P(a|s_0) + P(a|s_1)] \\
 &= \frac{1}{2} [P(\underline{r}^T h > 0 | s_0) + P(\underline{r}^T h < 0 | s_1)] \\
 &= \frac{1}{2} \left\{ \int_{-\infty}^{\infty} N(-||h||^2, \frac{N_0}{2} ||h||^2) + \int_{-\infty}^{\infty} N(+||h||^2, \frac{N_0}{2} ||h||^2) \right.
 \end{aligned}$$

Probability distributions are mirror images of each other!  
therefore we can use either of the distributions to  
solve above expression.

$$\text{∴ } P(a) = \frac{1}{2} \times 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi \sigma_h^2}} e^{-\frac{[(\underline{r}^T h - (-||h||^2))^2]}{2 \sigma_h^2}} d(\underline{r}^T h)$$

$$P(a) = \frac{1}{\sqrt{2\pi \sigma_h^2}} \int_0^{\infty} \exp \left[ -\frac{[(\underline{r}_0^T h + ||h||^2)^2]}{2 \sigma_h^2} \right] d(\underline{r}^T h)$$

$$\text{Substitute : } \frac{\underline{r}^T h + ||h||^2}{\sigma_h} = t$$

Limits.

$$\text{When } r^T h = 0 \rightarrow t = \frac{\|h\|^2}{\sigma_h}$$

$$r^T h \rightarrow \infty \rightarrow t \rightarrow \infty$$

$$\sigma_h dt = d(r^T h)$$

$$\therefore P(\epsilon) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{\|h\|^2}{\sigma_h}}^{\infty} e^{-t^2/2} \cdot \sigma_h dt$$

$$\therefore P(\epsilon) = Q\left(\frac{\|h\|^2}{\sigma_h}\right)$$

$$= Q\left(\frac{\|h\|^2}{\sqrt{\frac{No}{2}\|h\|^2}}\right)$$

$$\therefore P(\epsilon) = Q\left[\sqrt{\frac{2\|h\|^2}{No}}\right]$$

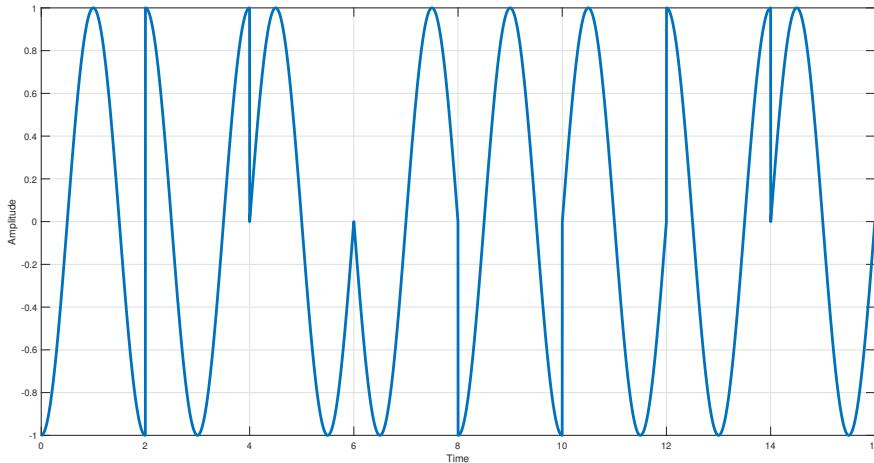
$$\text{Where, } Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-t^2/2} dt$$

## Question 4

(a) The sequence of signals (waveforms) corresponding to the binary sequence

Please note that when plotting the below figure following values were assumed for the parameters.

Bit interval ( $T_b$ )      1 s  
 Symbol interval ( $T$ )     $2 \times T_b = 2$  s



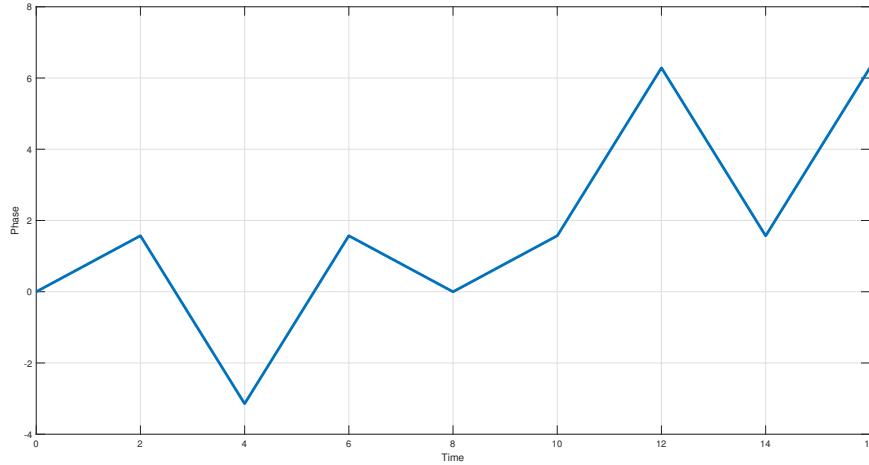
```

1 T_b = 1; % bit interval
2 h = 2; % number of bits per symbol
3 T = h*T_b; % symbol interval
4 binary_seq = [1 1 0 0 1 0 0 1 1 1 1 0 0 0 1 0];
5 signals = length(binary_seq)/h;
6 resolution = 1000; % number of data points for a smooth plot
7 t = linspace(0,T,resolution);
8
9 % generating symbols according to the mapping rule
10 s_0_t = sqrt(2/T)*cos(2*pi*t/T + pi*0/2); % i = 0
11 s_1_t = sqrt(2/T)*cos(2*pi*t/T + pi*1/2); % i = 1
12 s_2_t = sqrt(2/T)*cos(2*pi*t/T + pi*2/2); % i = 2
13 s_3_t = sqrt(2/T)*cos(2*pi*t/T + pi*3/2); % i = 3
14
15 % constructing the signal
16 tx_signal = zeros(signals, resolution);
17 signal = 1;
18 for index = 1:h:length(binary_seq) %initVal:step:endVal
19     if binary_seq(index) == 0 && binary_seq(index +1) == 0
20         tx_signal(signal,:) = s_0_t;
21     elseif binary_seq(index) == 0 && binary_seq(index +1) == 1
22         tx_signal(signal,:) = s_1_t;
23     elseif binary_seq(index) == 1 && binary_seq(index +1) == 1
24         tx_signal(signal,:) = s_2_t;
25     elseif binary_seq(index) == 1 && binary_seq(index +1) == 0
26         tx_signal(signal,:) = s_3_t;
27     end
28     signal = signal +1;
29 end
30 tx_signal = reshape(tx_signal',1,[ ]); % pre-processing for plotting
31 plot(linspace(0,signals*T, length(tx_signal)), tx_signal, 'LineWidth',3);
32 grid on; xlabel('Time'); ylabel('Amplitude')
```

## (b) The Quaternary MSK phase trajectory

Please note that when plotting the below figure following values were assumed for the parameters.

Bit interval ( $T_b$ )      1 s  
 Symbol interval (T)     $2 \times T_b = 2$  s



```

1 T_b = 1; % bit interval
2 h = 2; % number of bits per symbol
3 T = h*T_b; % symbol interval
4 binary_seq = [1 1 0 0 1 0 0 1 1 1 0 0 0 1 0];
5 num_of_Xks = length(binary_seq)/h;
6 resolution = 1000; % number of data points for a smooth plot
7 t = linspace(0,T,resolution);

8 % generating X_k s
9 X_ks = zeros(1,num_of_Xks);
10 k = 1;
11 for index = 1:h:length(binary_seq) %initVal:step:endVal
12     if binary_seq(index) == 0 && binary_seq(index +1) == 0
13         X_ks(k) = -3;
14     elseif binary_seq(index) == 0 && binary_seq(index +1) == 1
15         X_ks(k) = -1;
16     elseif binary_seq(index) == 1 && binary_seq(index +1) == 1
17         X_ks(k) = +1;
18     elseif binary_seq(index) == 1 && binary_seq(index +1) == 0
19         X_ks(k) = +3;
20     end
21     k = k +1;
22 end

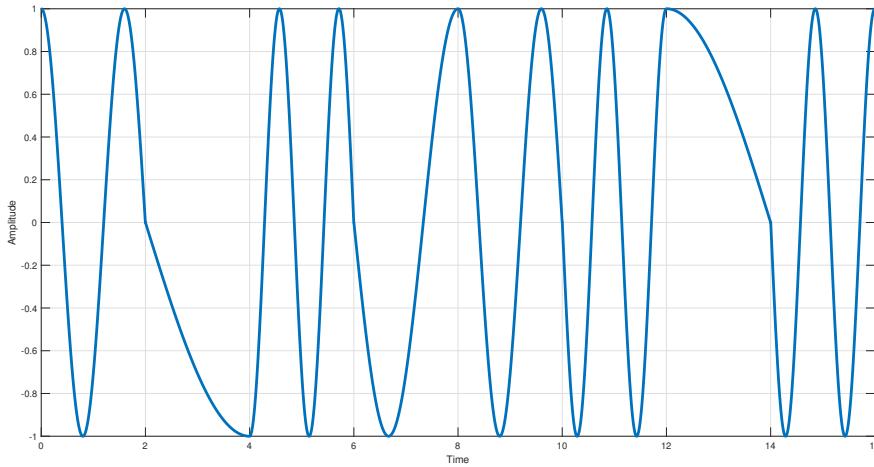
23 phase = zeros(num_of_Xks, resolution);
24 for n = 0:(num_of_Xks-1)
25     t = linspace(n*T, (n+1)*T, resolution);
26     phase(n+1,:) = (pi/2)*sum(X_ks(1:n)) + (pi/2)*X_ks(n+1)*(t/T -n);
27 end
28 phase = reshape(phase',1,[]);
29 plot(linspace(0,num_of_Xks*T, length(phase)), phase, 'LineWidth',3);
30 grid on; xlabel('Time'); ylabel('Phase')
31

```

### (c) The Quaternary MSK waveform corresponding to the binary sequence

Please note that when plotting the below figure following values were assumed for the parameters.

Bit interval ( $T_b$ )      1 s  
 Symbol interval (T)     $2 \times T_b = 2$  s



```

1 T_b = 1; % bit interval
2 h = 2; % number of bits per symbol
3 T = h*T_b; % symbol interval
4 binary_seq = [1 1 0 0 1 0 0 1 1 1 1 0 0 0 1 0];
5 num_of_Xks = length(binary_seq)/h;
6 resolution = 1000; % number of data points for a smooth plot
7 t = linspace(0,T,resolution);

8
9 % generating X_k s
10 X_ks = zeros(1,num_of_Xks);
11 k = 1;
12 for index = 1:h:length(binary_seq) %initVal:step:endVal
13     if binary_seq(index) == 0 && binary_seq(index +1) == 0
14         X_ks(k) = -3;
15     elseif binary_seq(index) == 0 && binary_seq(index +1) == 1
16         X_ks(k) = -1;
17     elseif binary_seq(index) == 1 && binary_seq(index +1) == 1
18         X_ks(k) = +1;
19     elseif binary_seq(index) == 1 && binary_seq(index +1) == 0
20         X_ks(k) = +3;
21     end
22     k = k +1;
23 end

24
25 QMSK = zeros(num_of_Xks , resolution);
26 % generating quaternary MSK waveform
27 for n = 0:(num_of_Xks-1)
28     t = linspace(n*T, (n+1)*T, resolution);
29     phase = (pi/2)*sum(X_ks(1:n)) + (pi/2)*X_ks(n+1)*(t/T -n);
30     QMSK(n+1,:) = sqrt(2/T)*cos(2*pi*t/T + phase);
31 end
32 QMSK = reshape(QMSK ',1,[]); % pre-processing for plotting
33 plot(linspace(0,num_of_Xks*T, length(QMSK)), QMSK, 'LineWidth',3);
34 grid on; xlabel('Time'); ylabel('Amplitude')
```