EN4553 - Assignment 1

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The following definitions (along with other basic theorems of probability theory) will be useful in completing this assignment. Note that, although we consider only continuous random variables below, analogous definitions exist for discrete random variables as well.

Definition 1. *(Expected value and variance): Let X be a continuous random variable with the probability density function $f(x)$. Then the expected value (or the mean) of X is*[1]*,*

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x f(x) dx.$$

The expected value of a function $g(X)$ of X is,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x) dx.$$

The variance of X is,

$$V[X] = \sigma_X^2 = E[(X - \mu_X)^2].$$

Definition 2. *(Expected value - multivariate case): Let $g(X_1, X_2, \ldots, X_k)$ be a function of continuous random variables, $X_1, X_2, \ldots, X_k$, which have the joint probability density function $f(x_1, x_2, \ldots, x_k)$. Then the expected value of $g(X_1, X_2, \ldots, X_k)$ is,*

$$E[g(X_1, X_2, \ldots, X_k)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \ldots, x_k) f(x_1, x_2, \ldots, x_k) dx_1 dx_2 \ldots dx_k.$$

Definition 3. *(Covariance and correlation) Let X_1 and X_2 be random variables with means μ_1 and μ_2, respectively. Then the covariance of X_1 and X_2 is,*

$$\mathrm{cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)].$$

The correlation coefficient of X_1 and X_2 is,

$$\rho_{X_1, X_2} = \frac{\mathrm{cov}(X_1, X_2)}{\sigma_1 \sigma_2},$$

where σ_1 and σ_2 are the standard deviations of X_1 and X_2, respectively.

[1] Here and everywhere else, without explicitly stating it, we assume that the integral converges absolutely.

Q1: Prove the following, where X, Y are random variables, a, b are constants, and g, h are functions. It is sufficient to prove the results for continuous X and Y although they hold for the discrete case as well (2 marks each).

1. $V[X] = E[X^2] - E[X]^2$.
2. $E[aX + b] = aE[X] + b$.
3. $V[aX + b] = a^2 V[X]$.
4. $E[X + Y] = E[X] + E[Y]$.
5. $V[X + Y] = V[X] + V[Y] + 2\,\mathrm{cov}(X, Y)$.
6. $E[g(X)\,h(Y)] = E[g(X)]\,E[h(Y)]$, when X and Y are independent.
7. $\mathrm{cov}(X, Y) = E[XY] - E[X]E[Y]$.
8. cov(X, Y) vanishes when X and Y are independent (*i.e.*, independent random variables are uncorrelated).

Q2: Nimal has a car and a van. Let X_1 and X_2 denote the filled proportion of the capacity of the fuel tank at the start of a day, for the car and the van, respectively. Nimal finds out that the joint probability distribution of X_1 and X_2 can be modeled by:

$$f(x_1, x_2) = \begin{cases} k(x_1 + x_2), & 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

1. Find the value of the constant k (2 marks).
2. Find $P(X_1 + X_2 \leq 1)$ (2 marks).
3. Find the marginal density functions for X_1 and X_2 (4 marks).
4. Find the conditional density function for X_1 given that X_2 is x_2' (2 marks).
5. Find the probability that the car's fuel tank is more than 75% full on a random morning (2 marks).
6. The van's fuel tank is exactly 50% full on a given morning. Find the probability that the car's fuel tank is more than 75% full (4 marks).

Q3: There are two random variables $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$. Define a new random variable Y. In each realization of Y, we sample from X_1's distribution with a probability p, and X_2's distribution with a probability $(1 - p)$.

1. Show that the mean of Y is given by (2 marks):
$$E[Y] = p\mu_1 + (1 - p)\mu_2.$$
2. Show that the variance of Y is given by (4 marks):
$$V[Y] = p\,\sigma_1^2 + (1 - p)\,\sigma_2^2 + p(1 - p)\,(\mu_1 - \mu_2)^2.$$
3. Roughly sketch Y's probability density function for $\mu_1 = 0, \sigma_1 = 1, \mu_2 = 10, \sigma_2 = 1, p = 0.2$ (2 marks).
4. Using the library functions `numpy.random.randn()`, and `numpy.random.uniform()` write code to simulate $10,000$ realizations of Y. Then use `seaborn.distplot()` to plot the distribution to verify that your sketch above is accurate. Include your code and the plot in the answer sheet. (4 marks)

   Hint: For the first two parts, you could define a Bernoulli-distributed random variable I such that $Y = IX_1 + (1 - I)X_2$, while noting that I is independent of both X_1 and X_2. Alternatively, you could work out the expected values of functions of Y with the help of a tree diagram.