Thalagala B. P. 180631J | EN4553 - Assignment 01.

$\boxed{Q1}$ 1. $V[X] = E[(X-\mu_x)^2]$; By definition

$$= \int_{-\infty}^{\infty} (X-\mu_x)^2 f(x)\, dx \quad ; \text{By def}^n \text{ of } E[g(x)]$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu_x X + \mu_x^2) f(x)\, dx \quad ; \text{Linear Transformation properties.}$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\, dx}_{E[X^2]} - 2\mu_x \underbrace{\int_{-\infty}^{\infty} x f(x)\, dx}_{\mu_x} + \mu_x^2 \underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{=1}$$

$$= E[X^2] - 2\mu_x^2 + \mu_x^2 = E[X^2] - \mu_x^2$$

$\therefore$ $V[X] = E[X^2] - E[X]^2 \longrightarrow$ (A)

(Q2.) $E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_x\, dx$

$$= a\underbrace{\int_{-\infty}^{\infty} x f(x)\, dx}_{E[X]} + b\underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{=1}$$

$\cancel{E[aX+b]} =$ $a E[X] + b \times 1$

$\therefore$ $E[aX+b] = a E[X] + b \longrightarrow$ (B.)

3. $V[aX+b] = E[(aX+b)^2] - E[aX+b]^2$; By (A)

$= \cancel{E[a^2X^2 + 2abX + b^2]}$

$= \int_{-\infty}^{\infty} (ax+b)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} (ax+b) f(x)\,dx\right]^2$; By def^n of $E[X]$.

$= a^2 \int_{-\infty}^{\infty} x^2 f(x)\,dx + 2ab \int_{-\infty}^{\infty} x f(x)\,dx + b^2 \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1}$

$- \left[a \int_{-\infty}^{\infty} x f(x)\,dx + b \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1}\right]^2$

$= \cancel{a^2 E[X^2]}$

$= a^2 E[X^2] + 2ab\,E[X] + b^2 - [a E[X] + b]^2$

$= a^2 E[X^2] + 2ab\,E[X] + b^2 - a^2 E[X]^2 - 2ab\,E[X^2] - b^2$

$= a^2 E[X^2] - a^2 E[X]^2$

$\therefore V[aX+b] = a^2 \{\underbrace{E[X^2] - E[X]^2}_{V[X] \text{ by (A)}}\}$

$V[aX+b] = a^2 V[X]$

(4) $E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y).f(x,y)\,dx\,dy$; By def^n (2)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x.f(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y.f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\left\{\int_{-\infty}^{\infty} f(x,y)\,dy\right\}}_{f(x)} dx + \int_{-\infty}^{\infty} y \underbrace{\left\{\int_{-\infty}^{\infty} f(x,y)\,dx\right\}}_{f(y)} dy$$

$$= \int_{-\infty}^{\infty} x\,f(x)\,dx + \int_{-\infty}^{\infty} y\,f(y)\,dy$$

$\therefore$ $E[X+Y] = E[X] + E[Y] \longrightarrow$ (c)

(5) $V[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$

$$= E[X^2 + 2XY + Y^2] - (E[X+Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - (E[X] + E[Y])^2 \text{ ; from (c)}$$

$$= E[X^2] + E[Y^2] - E[X]^2 - E[Y]^2 + 2E[XY] - 2E[X].E[Y]$$

$$= \underbrace{(E[X^2] - E[X]^2)} + \underbrace{(E[Y^2] - E[Y]^2)} + 2\{E[XY] - E[X].E[Y]\}$$

$$= V[X] + V[Y] + 2\{\underbrace{E[XY] - E[X].E[Y]}\}$$

$$= \cancel{V[X] + V[Y] + 2\{E[XY] - 2E[X].E[Y] + E[X]E[Y]\}}$$

$$= V[X] + V[Y] + 2\,cov(X,Y)$$ Proof is done later

(6) $E[g(x).h(y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x).h(y).f(x,y)\,dx\,dy$

Since X and Y are independant :

$f(x,y) = f(x).f(y)$ ← Product of marginal distributions.

$\therefore \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x).h(y).f(x).f(y)\,dx\,dy$

$= \underbrace{\int_{-\infty}^{\infty} g(x).f(x)}\times\int_{-\infty}^{\infty} h(y).f(y)\,dy$

$= E[g(x)] \times E[h(y)]$

$\therefore$ $\underline{E[g(x).h(y)] = E[g(x)].E[h(y)]}$ if X and Y are independant.

7. $cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$; By defⁿ

$= E[XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y]$

$= E[XY] - \mu_Y \underbrace{E[X]}_{\mu_X} - \mu_X \underbrace{E[Y]}_{\mu_Y} + \mu_X \mu_Y$; By 4

$= E[XY] - \mu_X \mu_Y$

$\therefore cov(X,Y) = E[XY] - E[X].E[Y]$

8. $cov(X,Y) = E[XY] - E[X].E[Y]$ from 7

if X and Y are independant, $\Rightarrow$ from 6

$E[XY] = E[X].E[Y]$

$\therefore cov(X,Y) = E[X]E[Y] - E[X].E[Y]$

$cov(X,Y) = 0$

$\boxed{Q2}$

$$f(x_1; x_2) = \begin{cases} K(x_1 + x_2), & (0 \le x_1 \le 1) \wedge (0 \le x_2 \le 1) \\ 0, & \text{elsewhere} \end{cases}$$

(1a) $$\int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} f(x_1, x_2)\, dx_2\, dx_1 = 1$$

$$\int_0^1 \int_0^1 K(x_1 + x_2)\, \underline{dx_2}\, dx_1 = 1$$

$$K \cdot \int_0^1 \left[x_1 x_2 + \frac{x^2}{2} \right]_0^1 dx_1 = 1 \quad ; (\text{integrate w.r.t } x_2)$$

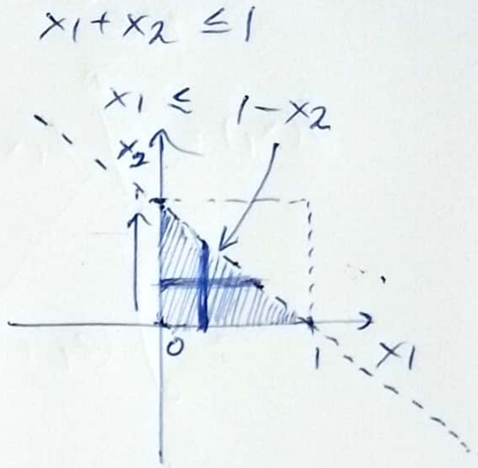
$$\int_0^1 \left(x_1 + \frac{1}{2} \right) dx_1 = \frac{1}{K}$$

$$\left[\frac{x_1^2}{2} + \frac{1}{2} x_1 \right]_0^1 = \frac{1}{K}$$

$$\left(\frac{1}{2} + \frac{1}{2} \right) = \frac{1}{K} \Rightarrow \underline{\underline{K = 1}}$$

$\therefore$ Joint distribution = $$f(x_1, x_2) = \begin{cases} (x_1 + x_2), & (0 \le x_1 \le 1) \wedge (0 \le x_2 \le 1) \\ 0, & \text{elsewhere.} \end{cases}$$

(2) $$Pr(x_1 + x_2 \le 1) = \iint\limits_{(x_1+x_2\le 1)} f(x,y)\,dx\,dy$$



$$= \int_{x_2=0}^{1} \int_{x_1=0}^{1-x_2} (x_1 + x_2)\, dx_1\, dx_2$$

$$= \int_{x_2=0}^{1} \left[\frac{x_1^2}{2} + x_2 x_1 \right]_0^{1-x_2} dx_2$$

$$= \int_0^1 \frac{(1-x_2)^2}{2} + x_2(1-x_2)\, dx_2$$

$$= \int_0^1 \frac{1}{2}\left[1 - \cancel{2x_2} + x_2^2 + \cancel{2x_2} - 2x_2^2\right] dx_2$$

$$= \frac{1}{2}\int_0^1 (1 - 2x_2^2)\, dx_2 = \frac{1}{2}\left[x_2 - \frac{2}{3}x_2^3\right]_0^1$$

$$= \cancel{\frac{1}{2}\left[x_2 - \frac{2}{3}x_2^3\right]_0^1} = \frac{1}{2}\left[1 - \frac{2}{3}\times 1\right] = \frac{1}{2}\times\frac{1}{3}$$

$$\therefore\ \underline{Pr(x_1 + x_2 \le 1) = \frac{1}{6}}$$

(3) Marginal Density f^n:

$$f_{x_1}(x_1) = \int_{-\alpha}^{\alpha} f(x_1, x_2)\, dx_2$$

$$= \int_0^1 (x_1 + x_2)\, dx_2$$

$$= \left[x_1 x_2 + \frac{x_2^2}{2} \right]_0^1$$

$$f(x_1) = x_1 + \frac{1}{2}$$

$$\therefore f(x_1) = \begin{cases} x_1 + 1/2, & 0 \le x_1 \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Similarly: $$f(x_2) = \begin{cases} x_2 + 1/2, & 0 \le x_2 \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

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(4) Using the product rule: $P(A|B) = \frac{P(A \cap B)}{P(B)}$, for $P(B) > 0$.

$$\therefore f(x_1 | x_2) = \frac{f(x_1, x_2)}{f(x_2)}$$

$$= \frac{x_1 + x_2}{x_2 + \frac{1}{2}} \quad ; (\text{for } 0 \le x_2 \le 1)$$

$$\therefore f(x_1 | x_2)\Big|_{x_2 = x_2'} = \begin{cases} \dfrac{x_1 + x_2'}{x_2' + \frac{1}{2}}, & 0 \le x_1 \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

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(5) $Pr(x_1 > 0.75) =$

$$= \int_{x_1=0.75}^{1} f(x_1)\, dx_1$$

$$= \int_{0.75}^{1} \left(x_1 + \frac{1}{2}\right) dx_1$$

$$= \left[\frac{x_1^2}{2} + \frac{x_1}{2}\right]_{0.75}^{1}$$

$$\therefore \; \underline{\underline{Pr(x_1 > 0.75) = 0.344}}$$

Probability that car's fuel tank is more than 75%

(6) $$Pr(x_1 > 0.75 \mid x_2 = 0.5) = \int_{0.75}^{1} f(x_1 \mid x_2 = 0.5)\, dx_1$$

$$= \int_{0.75}^{1} \frac{x_1 + 0.5}{(0.5 + 0.5)}\, dx_1$$

$$= \int_{0.75}^{1} \left(x_1 + \frac{1}{2}\right) dx_1$$

Probability that car's fuel tank is more than 75% given van's fuel tank is exactly 50% full.

$$\therefore \; \underline{\underline{Pr(x_1 > 0.75 \mid x_2 = 0.5) = 0.344}}$$

Q3. $x_1 \sim N(\mu_1, \sigma_1) \longrightarrow$ Sample with Prob. $- p$

$x_2 \sim N(\mu_2, \sigma_2) \longrightarrow$ Sample with Prob. $- (1-p)$

① Let's define a Bernoulli - distributed random variable I s.t.

$$Y = IX_1 + (1-I)x_2$$ ← I is idependant of both x_1 and x_2.

$$E[Y] = E[IX_1 + (1-I)x_2]$$

$$= E[IX_1] + E[(1-I)x_2]$$

$$= E[I].E[X_1] + E[X_2] - E[I].E[X_2]$$; Since I independant of both X_1 and X_2.

$$= E[I].\mu_1 + \mu_2(1 - E[I]) \rightarrow (A)$$

Finding $E[I]$: Probability Mass Function of I $= \begin{cases} q = 1-p, & \text{if } k=0 \\ p, & \text{if } k=1 \end{cases}$

$$\therefore E[I] = \sum_{k=0}^{1} k.Pr(k) = 0 \times Pr(0) + 1 \times Pr(1)$$

$$= 0 \times (1-p) + 1 \times p$$

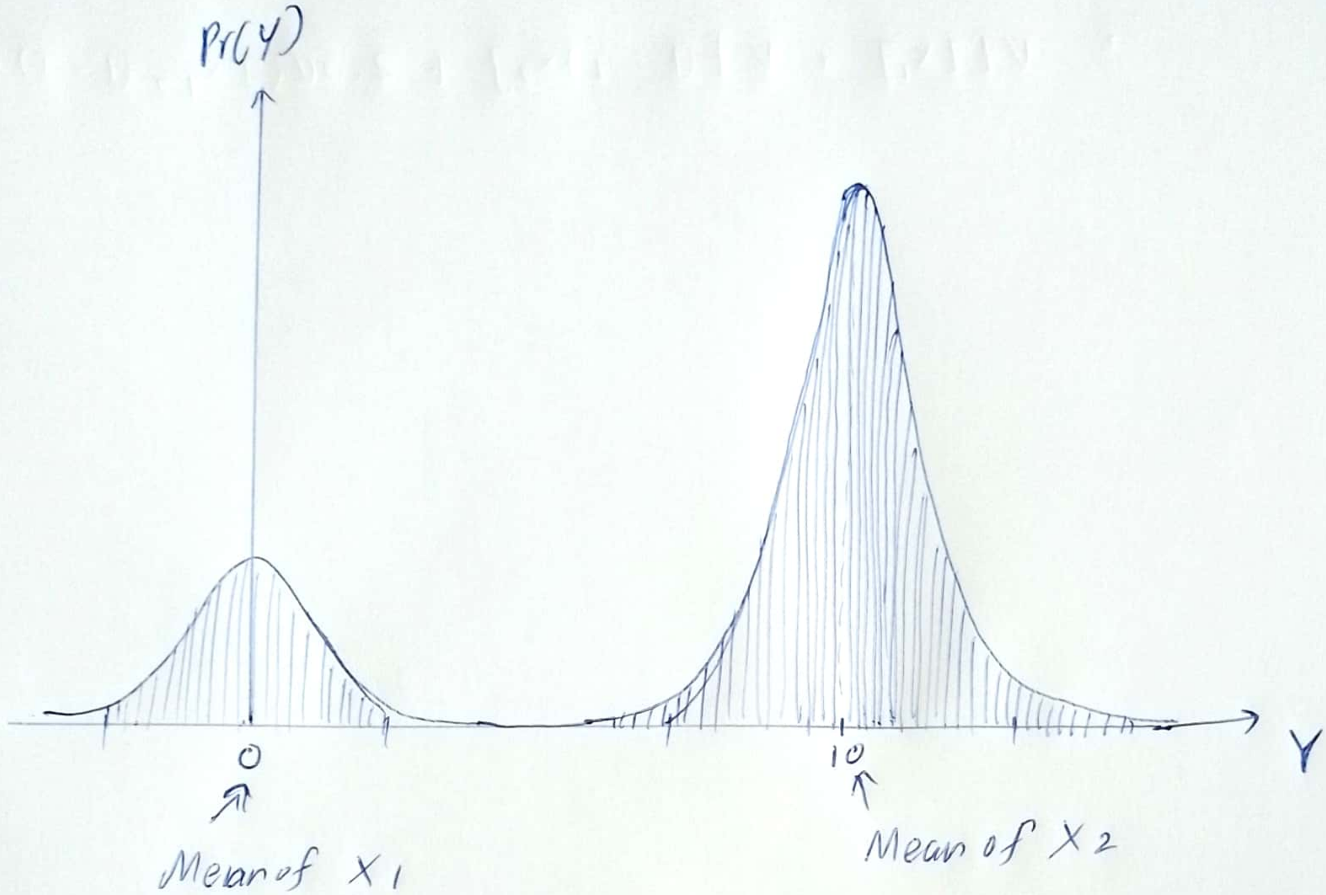
$$E[I] = p \leftarrow (B)$$

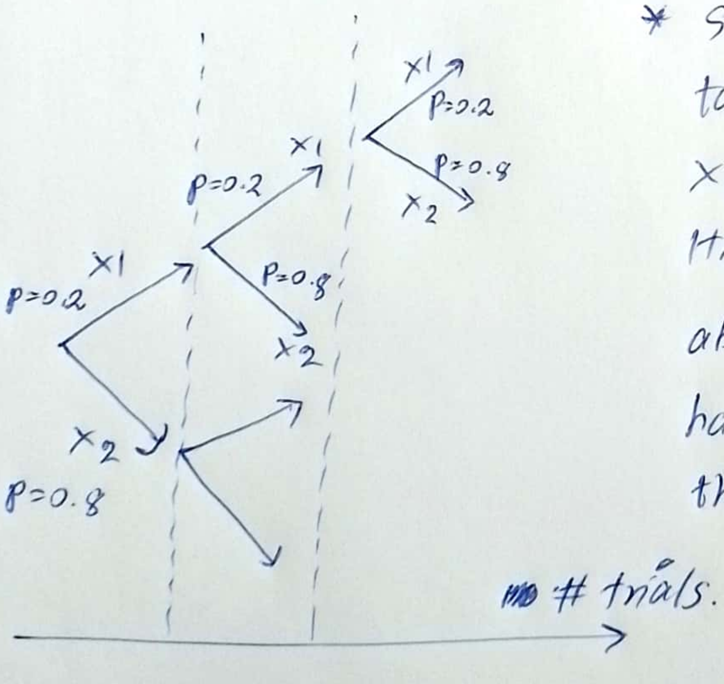
from (A) and (B),

$$E[Y) = E[I].\mu_1 + \mu_2(1 - E[I])$$

$$\underline{\underline{E[Y] = p\mu_1 + (1-p)\mu_2}}$$

3.





* since probability of taking a sample from X_2 distribution is much Higher than that of X_1, above probability distribution has a higher density around the mean of X_2.