Problem Sheet 2

Numerical Methods for CSE

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1. Lyapunov Equation

1.a. Linear Mapping

To prove (for $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n,n}, \alpha \in \mathbb{R}$ and a fixed $\mathbf{A}$):

- $L(\mathbf{U} + \mathbf{V}) = L(\mathbf{U}) + L(\mathbf{V})$
- $L(\alpha \cdot \mathbf{U}) = \alpha \cdot L(\mathbf{U})$

The first property:

$$
\begin{aligned}
L(\mathbf{U} + \mathbf{V}) &= \mathbf{A}(\mathbf{U} + \mathbf{V}) + (\mathbf{U} + \mathbf{V})\mathbf{A}^T \\
&= \mathbf{AU} + \mathbf{AV} + \mathbf{UA}^T + \mathbf{VA}^T \\
&= \mathbf{AU} + \mathbf{UA}^T + \mathbf{AV} + \mathbf{VA}^T \\
&= L(\mathbf{U}) + L(\mathbf{V})
\end{aligned}
$$

And the second property:

$$
\begin{aligned}
L(\alpha \cdot \mathbf{U}) &= \mathbf{A}(\alpha \cdot \mathbf{U}) + (\alpha \cdot \mathbf{U})\mathbf{A}^T \\
&= \alpha \cdot (\mathbf{AU}) + \alpha \cdot (\mathbf{UA}^T) \\
&= \alpha \cdot (L(\mathbf{U}))
\end{aligned}
$$

1.c. Linear Equations

We have

$$
\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \mathbf{X} + \mathbf{X} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \mathbf{I}
$$

This gives us the following equations:

$$
\begin{aligned}
1 &= 2\mathbf{X}_{11} + \mathbf{X}_{21} + 2\mathbf{X}_{11} + \mathbf{X}_{11} = 4\mathbf{X}_{11} + \mathbf{X}_{21} + \mathbf{X}_{12} \\
0 &= -\mathbf{X}_{11} + 5\mathbf{X}_{21} + \mathbf{X}_{22} \\
0 &= -\mathbf{X}_{11} + 5\mathbf{X}_{12} + \mathbf{X}_{22} \\
1 &= -\mathbf{X}_{21} - \mathbf{X}_{12} + 6\mathbf{X}_{21}
\end{aligned}
$$