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Numerical Methods for CSE

ETH Zürich
D-MATH

Problem Sheet 9

Problem 1. Chebychev interpolation of analytic functions (core problem)

This problem concerns Chebychev interpolation (cf. [1, Section 4.1.3]). Using techniques from complex analysis, notably the residue theorem [1, Thm. 4.1.56], in class we derived an expression for the interpolation error [1, Eq. (4.1.62)] and from it an error bound [1, Eq. (4.1.63)], as much sharper alternative to [1, Thm. 4.1.37] and [1, Lemma 4.1.41] for *analytic* interpolands. The bound tells us that for all $t \in [a,b]$

$$|f(t) - \mathsf{L}_{\mathcal{T}} f(t)| \le \left| \frac{w(x)}{2\pi i} \int_\gamma \frac{f(z)}{(z-t)w(z)} dz \right| \le \frac{|\gamma|}{2\pi} \frac{\max_{a\le\tau\le b}|w(\tau)|}{\min_{z\in\gamma}|w(z)|} \frac{\max_{z\in\gamma}|f(z)|}{d([a,b],\gamma)} ,$$

where $d([a,b],\gamma)$ is the geometric distance of the integration contour $\gamma \subset \mathbb{C}$ from the interval $[a,b] \subset \mathbb{C}$ in the complex plane. The contour γ must be contractible in the domain D of analyticity of f and must wind around $[a,b]$ exactly once, see [1, Fig. 124].

Now we consider the interval $[-1,1]$. Following [1, Rem. 4.1.87], our task is to find an upper bound for this expression, in the case where f possesses an analytical extension to a complex neighbourhood of $[-1,1]$.

For the analysis of the Chebychev interpolation of analytic functions we used the elliptical contours, see [1, Fig. 138],

$$\gamma_\rho(\theta) := \cos(\theta - i\log(\rho)) , \quad \forall 0 \le \theta \le 2\pi , \quad \rho > 1 . \tag{23}$$

(1a) ⚁ Find an upper bound for the length $|\gamma_\rho|$ of the contour γ_ρ.

HINT: You may use the arc-length formula for a curve $\gamma : I \to \mathbb{R}^2$:

$$|\gamma| = \int_I \|\dot\gamma(\tau)\| d\tau, \tag{24}$$

where $\dot{\gamma}$ is the derivative of γ w.r.t the parameter τ. Recall that the "length" of a complex number z viewed as a vector in $\mathbb{R}^2$ is just its modulus.

Now consider the S-curve function (the logistic function):

$$f(t) := \frac{1}{1 + e^{-3t}}\ , \quad t \in \mathbb{R}\ .$$

(1b) ⚁ Determine the maximal domain of analyticity of the extension of f to the complex plane $\mathbb{C}$.

HINT: Consult [1, Rem. 4.1.64].

(1c) ⚁ Write a MATLAB function that computes an approximation M of:

$$\min_{\rho>1} \frac{\max_{z\in\gamma_\rho} |f(z)|}{d([-1,1],\gamma_\rho)}, \tag{25}$$

by sampling, where the distance of $[a,b]$ from γ_ρ is formally defined as

$$d(\gamma,[a,b]) := \inf\{|z-t| \mid z \in \gamma, t \in [a,b]\}. \tag{26}$$

HINT: The result of (1b), together with the knowledge that γ_ρ describes an ellipsis, tells you the maximal range $(1, \rho_{max})$ of ρ. Sample this interval with 1000 equidistant steps.

HINT: Apply geometric reasoning to establish that the distance of γ_ρ and $[-1,1]$ is $\frac{1}{2}(\rho + \rho^{-1}) - 1$.

HINT: If you cannot find ρ_{max} use $\rho_{max} = 2.4$.

HINT: You can exploit the properties of $\cos$ and the hyperbolic trigonometric functions $\cosh$ and $\sinh$.

(1d) ⚁ Based on the result of (1c), and [1, Eq. (4.1.89)], give an "optimal" bound for

$$\|f - L_n f\|_{L^\infty([-1,1])},$$

where L_n is the operator of Chebychev interpolation on $[-1,1]$ into the space of polynomials of degree $\leq n$.

(1e) ⚀ Graphically compare your result from (1d) with the measured supremum norm of the approximation error of Chebychev interpolation of f on $[-1,1]$ for polynomial degree $n = 1, \ldots, 20$. To that end, write a MATLAB-code and rely on the provided function `intpolyval` (cf. [1, Code 4.4.8]).

HINT: Use semi-logarithmic scale for your plot `semilogy`.

(1f) ⚃ Rely on pullback to $[-1,1]$ to discuss how the error bounds in [1, Eq. (4.1.89)] will change when we consider Chebychev interpolation on $[-a,a], a > 0$, instead of $[-1,1]$, whilst keeping the function f fixed.

Problem 2. Piecewise linear approximation on graded meshes (core problem)

One of the messages given by [1, Section 4.1.3] is that the quality of an interpolant depends heavily on the choice of the interpolation nodes. If the function to be interpolated has a "bad behavior" in a small part of the domain, for instance it has very large derivatives of high order, more interpolation points are required in that area of the domain. Commonly used tools to cope with this task, are *graded meshes*, which will be the topic of this problem.

Given a mesh $\mathcal{T} = \{0 \le t_0 < t_1 < \cdots < t_n \le 1\}$ on the unit interval $I = [0,1]$, $n \in \mathbb{N}$, we define the *piecewise linear* interpolant

$$\mathsf{I}_{\mathcal{T}} : C^0(I) \to \mathcal{P}_{1,\mathcal{T}} = \{s \in C^0(I),\ s_{|[t_{j-1},t_j]} \in \mathcal{P}_1\ \forall\, j\}, \quad \text{s.t.} \quad \big(\mathsf{I}_{\mathcal{T}} f\big)(t_j) = f(t_j), \quad j = 0,\dots,n;$$

(see also [1, Section 3.3.2]).

(2a) ⚁ If we choose the uniform mesh $\mathcal{T} = \{t_j\}_{j=0}^{n}$ with $t_j = j/n$, given a function $f \in C^2(I)$, what is the asymptotic behavior of the error

$$\|f - \mathsf{I}_{\mathcal{T}} f\|_{L^\infty(I)},$$

when $n \to \infty$?

HINT:: look for a suitable estimate in [1, Section 4.5.1].

(2b) ⚁ What is the regularity of the function

$$f : I \to \mathbb{R}, \qquad f(t) = t^\alpha, \qquad 0 < \alpha < 2\,?$$

In other words, for which $k \in \mathbb{N}$ do we have $f \in C^k(I)$?

HINT: Notice that I is a closed interval and check the continuity of the derivatives in the endpoints of I.

(2c) ⚂ Study numerically the h-convergence of the piecewise linear approximation of $f(t) = t^\alpha$ ($0 < \alpha < 2$) on uniform meshes; determine the order of convergence using linear regression based on MATLAB's `polyfit`, see [1, Section 4.5.1].

HINT: Linear regression and `polyfit` have not been introduced yet. Please give a quick look at the examples in http://ch.mathworks.com/help/matlab/ref/polyfit.html#examples to see `polyfit` in action. For instance, the code to determine the slope of a line approximating a sequence of points $(x_i, y_i)_i$ in doubly logarithmic scale is

```
P = polyfit(log(x),log(y),1);
slope = P(1);
```

(2d) ⚁ In which mesh interval do you expect $|f - \mathsf{I}_\mathcal{T} f|$ to attain its maximum?

HINT:: you may use the code from the previous subtask to get an idea.

HINT: 2: what's the meaning of [1, Thm. 4.1.37] in the piecewise linear setting?

(2e) ⚁ Compute by hand the exact value of $\|f - \mathsf{I}_\mathcal{T} f\|_{L^\infty(I)}$.

Compare the order of convergence obtained with the one observed numerically in (2b).

HINT:: use the result of (2d) to simplify the problem.

(2f) ⚁ Since the interpolation error is concentrated in the left part of the domain, it seems reasonable to use a finer mesh only in this part. A common choice is an **algebraically graded mesh**, defined as

$$\mathcal{G} = \left\{ t_j = \left(\frac{j}{n}\right)^\beta, \quad j = 0, \dots, n \right\},$$

for a parameter $\beta > 1$. An example is depicted in Figure 5 for $\beta = 2$.

For a fixed parameter α in the definition of f, numerically determine the rate of convergence of the piecewise linear interpolant $\mathsf{I}_\mathcal{G}$ on the graded mesh $\mathcal{G}$ as a function of the parameter β. Try for instance $\alpha = 1/2$, $\alpha = 3/4$ or $\alpha = 4/3$.

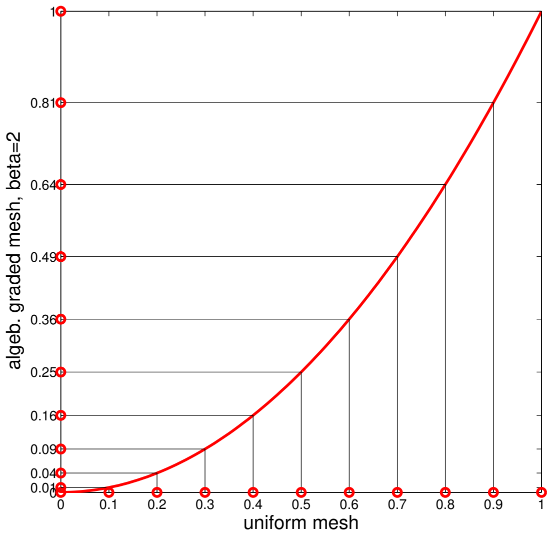
How do you have to choose β in order to recover the optimal rate $\mathcal{O}(n^{-2})$ (if possible)?

Problem 3. Chebyshev polynomials and their properties

Let $T_n \in \mathcal{P}_n$ be the n-th Chebyshev polynomial, as defined in [1, Def. 4.1.67] and $\xi_0^{(n)}, \dots, \xi_{n-1}^{(n)}$ be the n zeros of T_n. According to [1, Eq. (4.1.75)], these are given by

$$\xi_j^{(n)} = \cos\left(\frac{2j+1}{2n}\pi\right), \quad j = 0, \dots, n-1. \tag{27}$$

Figure 5: Graded mesh $x_j = (j/n)^2$, $j = 0, \ldots, 10$.



We define the family of discrete L^2 semi inner products, *cf.* [1, Eq. (4.2.21)],

$$(f, g)_n := \sum_{j=0}^{n-1} f(\xi_j^{(n)}) g(\xi_j^{(n)}), \quad f, g \in C^0([-1, 1]) \tag{28}$$

and the special weighted L^2 inner product

$$(f, g)_w := \int_{-1}^{1} \frac{1}{\sqrt{1 - t^2}} f(t) g(t) \, dt \quad f, g \in C^0([-1, 1]) \tag{29}$$

(3a) Show that the Chebyshev polynomials are an orthogonal family of polynomials with respect to the inner product defined in (29) according to [1, Def. 4.2.24], namely $(T_k, T_l)_w = 0$ for every $k \neq l$.

HINT: Recall the trigonometric identity $2\cos(x)\cos(y) = \cos(x + y) + \cos(x - y)$.

Consider the following statement.

> **Theorem.** The family of polynomials $\{T_0, \ldots, T_n\}$ is an orthogonal basis ($\rightarrow$ [1, Def. 4.2.13]) of $\mathcal{P}_n$ with respect to the inner product $(\ ,\ )_{n+1}$ defined in (28).

(3b) Write a C++ code to test the assertion of the theorem.

HINT: [1, Code 4.1.70] demonstrates the efficient evaluation of Chebychev polynomials based on their 3-term recurrence formula from [1, Thm. 4.1.68].

(3c) Prove the theorem.

HINT: Use the relationship of trigonometric functions and the complex exponential together with the summation formula for geometric sums.

(3d) Given a function $f \in C^0([-1,1])$, find an expression for the best approximant $q_n \in \mathcal{P}_n$ of f in the discrete L^2-norm:

$$q_n = \operatorname*{argmin}_{p \in \mathcal{P}_n} \|f - p\|_{n+1} ,$$

where $\| \; \|_{n+1}$ is the norm induced by the scalar product $(\; , \;)_{n+1}$. You should express q_n through an expansion in Chebychev polynomials of the form

$$q_n = \sum_{j=0}^{n} \alpha_j T_j \tag{30}$$

for suitable coefficients $\alpha_j \in \mathbb{R}$.

HINT: The task boils down to determining the coefficients α_j. Use the theorem you have just proven and a slight extension of [1, Cor. 4.2.14].

(3e) Write a C++ function

```
template <typename Function>
void bestpolchebnodes(const Function &f, Eigen::VectorXd &alpha)
```

that returns the vector of coefficients $(\alpha_j)_j$ in (30) given a function f. Note that the degree of the polynomial is indirectly passed with the length of the output `alpha`. The input `f` is a lambda-function, e.g.

```
auto f = [] (double & x) {return 1/(pow(5*x,2)+1);};
```

(3f) Test `bestpolchebnodes` with the function $f(x) = \frac{1}{(5x)^2+1}$ and $n = 20$. Approximate the supremum norm of the approximation error by sampling on an equidistant grid with 10^6 points.

HINT: Again, [1, Code 4.1.70] is useful for evaluating Chebychev polynomials.

(3g) Let L_j, $j = 0, \dots, n$, be the Lagrange polynomials associated with the nodes $t_j = \xi_j^{(n+1)}$ of Chebyshev interpolation with $n + 1$ nodes on $[-1, 1]$, see [1, Eq. (4.1.75)]. Show that

$$L_j = \frac{1}{n+1} + \frac{2}{n+1} \sum_{l=1}^{n} T_l(\xi_j^{(n+1)}) T_l.$$

HINT: Again use the above theorem to express the coefficients of a Chebychev expansion of L_j.

Problem 4. Piecewise cubic Hermite interpolation

Piecewise cubic Hermite interpolation with exact slopes on a mesh

$$\mathcal{M} := \{a = x_0 < x_1 < \cdots < x_n = b\}$$

was defined in [1, Section 3.4]. For $f \in C^4([a, b])$ it enjoys h-convergence with rate 4 as we have seen in [1, Exp. 4.5.15].

Now we consider cases, where perturbed or reconstructed slopes are used. For instance, this was done in the context of monotonicity preserving piecewise cubic Hermite interpolation as discussed in [1, Section 3.4.2].

(4a) Assume that piecewise cubic Hermite interpolation is based on perturbed slopes, that is, the piecewise cubic function s on $\mathcal{M}$ satisfies:

$$s(x_j) = f(x_j) \quad , \quad s'(x_j) = f'(x_j) + \delta_j,$$

where the δ_j may depends on $\mathcal{M}$, too.

Which rate of asymptotic h-convergence of the sup-norm of the approximation error can be expected, if we know that for all j

$$|\delta_j| = O(h^{\beta}) \; , \quad \beta \in \mathbb{N}_0 \; ,$$

for mesh-width $h \to 0$.

HINT: Use a local generalized cardinal basis functions, cf. [1, § 3.4.3].

(4b) ⚃ Implement a strange piecewise cubic interpolation scheme in C++ that satisfies:

$$s(x_j) = f(x_j) \quad , \quad s'(x_j) = 0$$

and empirically determine its convergence on a sequence of equidistant meshes of $[-5, 5]$ with mesh-widths $h = 2^{-l}, l = 0, \ldots, 8$ and for the interpoland $f(t) := \frac{1}{1+t^2}$.

As a possibly useful guideline, you can use the provided C++ template, see the file `piecewise_hermite_interpolation_template.cpp`.

Compare with the insight gained in (4a).

(4c) ⚀ Assume equidistant meshes and reconstruction of slopes by a particular averaging. More precisely, the $\mathcal{M}$-piecewise cubic function s is to satisfy the generalized interpolation conditions

$$s(x_j) = f(x_j),$$
$$s'(x_j) = \begin{cases} \frac{-f(x_2)+4f(x_1)-3f(x_0)}{2h} & \text{for } j = 0 \;, \\ \frac{f(x_{j+1})-f(x_{j-1})}{2h} & \text{for } j = 1, \ldots, n-1 \;, \\ \frac{3f(x_n)-4f(x_{n-1})+f(x_{n-2})}{2h} & \text{for } j = n \;. \end{cases}$$

What will be the rate of h-convergence of this scheme (in sup-norm)?

(You can solve this exercise either theoretically or determine an empiric convergence rate in a numerical experiment.)

HINT: If you opt for the theoretical approach, you can use what you have found in subsubsection (4a). To find perturbation bounds, rely on the Taylor expansion formula with remainder, see [1, Ex. 1.5.58].

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References

[1] R. Hiptmair. *Lecture slides for course "Numerical Methods for CSE"*. http://www.sam.math.ethz.ch/~hiptmair/tmp/NumCSE/NumCSE15.pdf. 2015.