EN1060 Signals and Systems: Fourier Transform

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Section 1

Continuous-Time Fourier Transform

Outline

1. Continuous-Time Fourier Transform
   - Introduction
   - Development of the Fourier Transform Representation

2. The Fourier Transform for Periodic Signals

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
  1. Continuous-time Fourier series
  2. Continuous-time Fourier transform
  3. Discrete-time Fourier series
  4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

Fourier Transform

- In the last lecture, we represented a periodic signal as a linear combination of complex exponentials.
- We use Fourier transform to represent aperiodic signals. A larger class of signals, including all signals with finite energy, can be represented through a linear combination of complex exponentials.
- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an integral rather than a sum.
- The resulting spectrum of coefficients in this representation is called the Fourier transform.
- The synthesis integral itself, which uses these coefficients to represent the signal as a linear combination of complex exponentials, is called the inverse Fourier transform.

Outline

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodically repeats with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

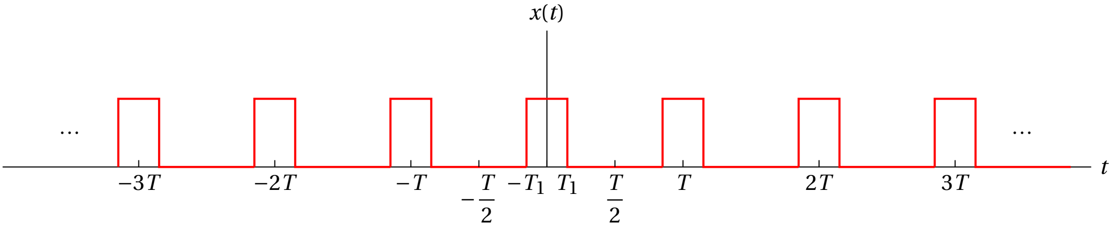


Figure: Periodic square wave

The Fourier series coefficients a_k of this wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \tag{1}$$

We plotted this for a fixed value of T_1 and several values of T (shown in the next slide). An alternative wave of interpreting eq. 1 is as samples of an envelope function:

$$Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0}$$

For fixed T_1, the envelope of Ta_k is independent of T.

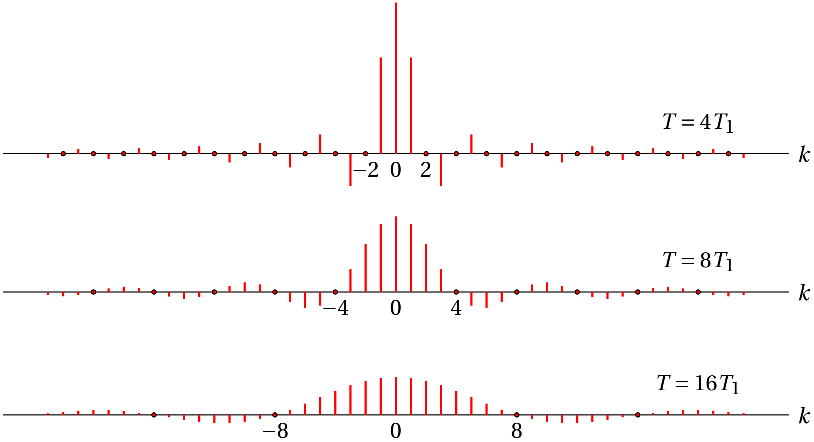
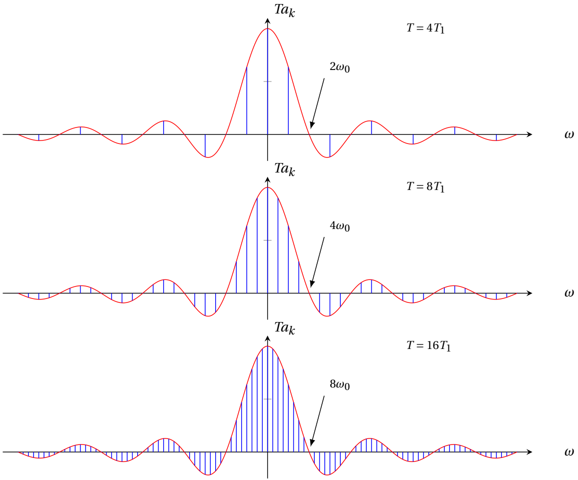


Figure: Plots of scaled Fourier series coefficients Ta_k

Plots of scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: $T = 4T_1$, $T = 8T_1$, $T = 16T_1$.



The Fourier series coefficients and their envelope for periodic square wave for several values of T (with T_1 fixed): $T = 4T_1$, $T = 8T_1$, $T = 16T_1$. The coefficients are regularly-spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Figure: Fourier series coefficients and their envelope for periodic square wave.

Fourier Transform: Synthesis and Analysis Equations

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega. \tag{2}$$

Fourier transform or Fourier integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{3}$$

The transform $X(j\omega)$ of an aperiodic signal $x(t)$ is referred to as the spectrum of $x(t)$.

Relation with a_k

Assume that the Fourier transform of $x(t)$ is $X(j\omega)$.
If we construct a periodic signal $\tilde{x}(t)$ by repeating the aperiodic signals $x(t)$ with period T, its Fourier series coefficients are

$$a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0} \tag{4}$$

Assume that we evaluated $X(j\omega)$ according to eq. 3, and left $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in 2:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

When is $\hat{x}(t)$ a valid representation of the original signal $x(t)$? We define the error between $\hat{x}(t)$ and $x(t)$ as

$$e(t) = \hat{x}(t) - x(t).$$

If $x(t)$ has finite energy (square integrable), i.e.,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty, \tag{5}$$

$X(j\omega)$ is finite, and

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0, \tag{6}$$

If $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t, there is no energy in their difference.

There are alternative conditions sufficient to ensure that $\hat{x}(t)$ is qual to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

1. $x(t)$ is absolutely integrable, i.e.,
$$\int_{-\infty}^{\infty} |x(t)|dt < \infty, \tag{7}$$
2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have finite number of discontinuities have a Fourier transform.

Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

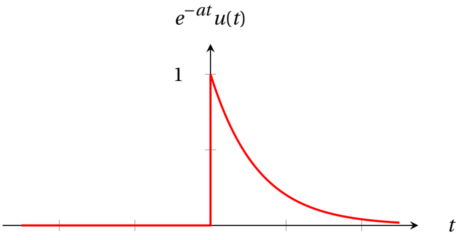


Figure: $e^{-at}u(t), \quad a > 0.$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

$$X(j\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \frac{1}{a+j\omega}\; e^{-(a+j\omega)t}\Big|_0^{\infty}$$

$$X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0.$$

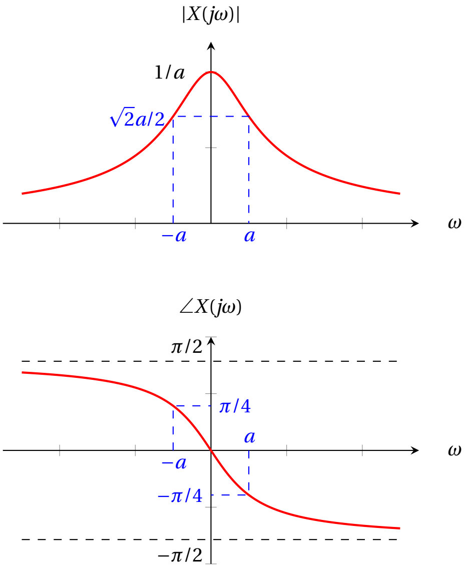


Figure: Fourier transform of the signal $e^{-at}u(t), \quad a > 0$.

Example

Find the Fourier transform of the signal

$$x(t) = e^{-a|t|}, \quad a > 0.$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \\ &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt. \\ X(j\omega) &= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt \\ X(j\omega) &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}, \\ &= \frac{2a}{a^2 + \omega^2}. \end{aligned}$$

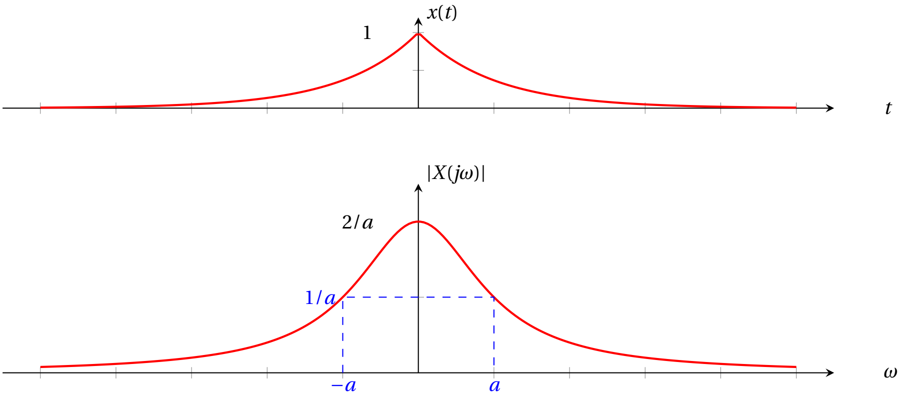


Figure: Fourier transform of the signal $e^{-a|t|}, \quad a > 0.$

Example

Determine the Fourier transform of the unit impulse

$$x(t) = \delta(t).$$

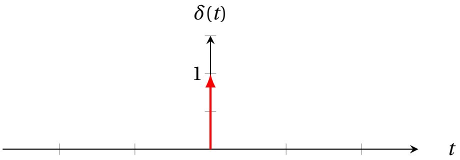


Figure: $\delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1.$$

The unit impulse has a Fourier transform consisting of qual contributions at all frequencies.

Rectangular Pulse

Example

Determine the Fourier transform of the signal

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1. \end{cases}$$

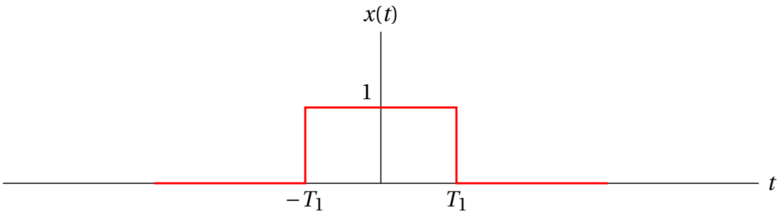


Figure: Rectangular pulse and the Fourier transform.

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt. \\ &= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T_1}^{T_1} \\ &= 2\frac{\sin \omega T_1}{\omega}. \end{aligned}$$

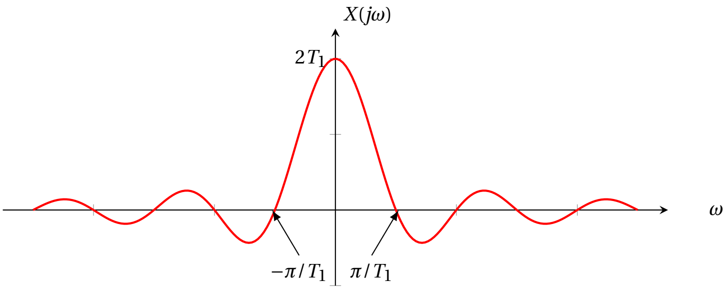


Figure: Fourier transform of the rectangular pulse.

Example

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine $x(t)$.

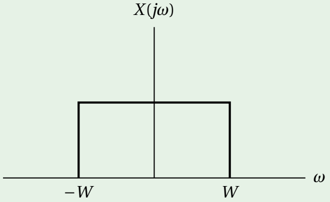


Figure: Fourier transform for $x(t)$.

Using the synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$
$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega.$$
$$= \frac{\sin Wt}{\pi t}.$$

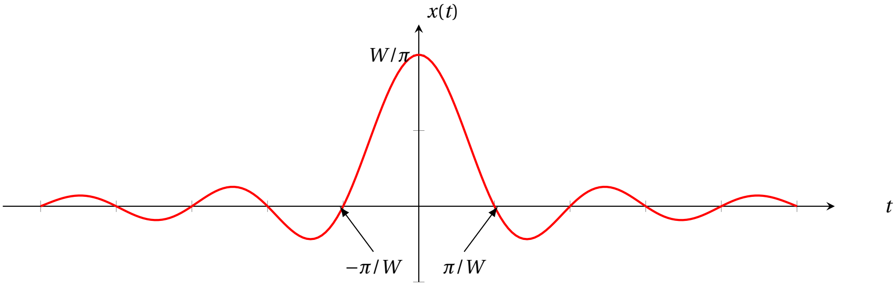


Figure: Time function.

The sinc Function

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{8}$$

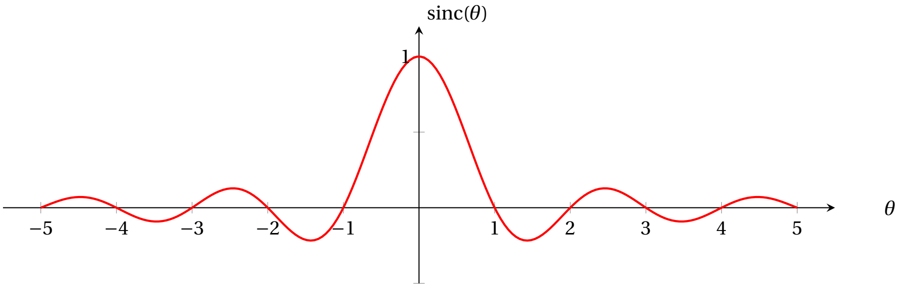


Figure: Fourier transform for $x(t)$.

Express

$$\frac{2\sin\omega T_1}{\omega}$$

and

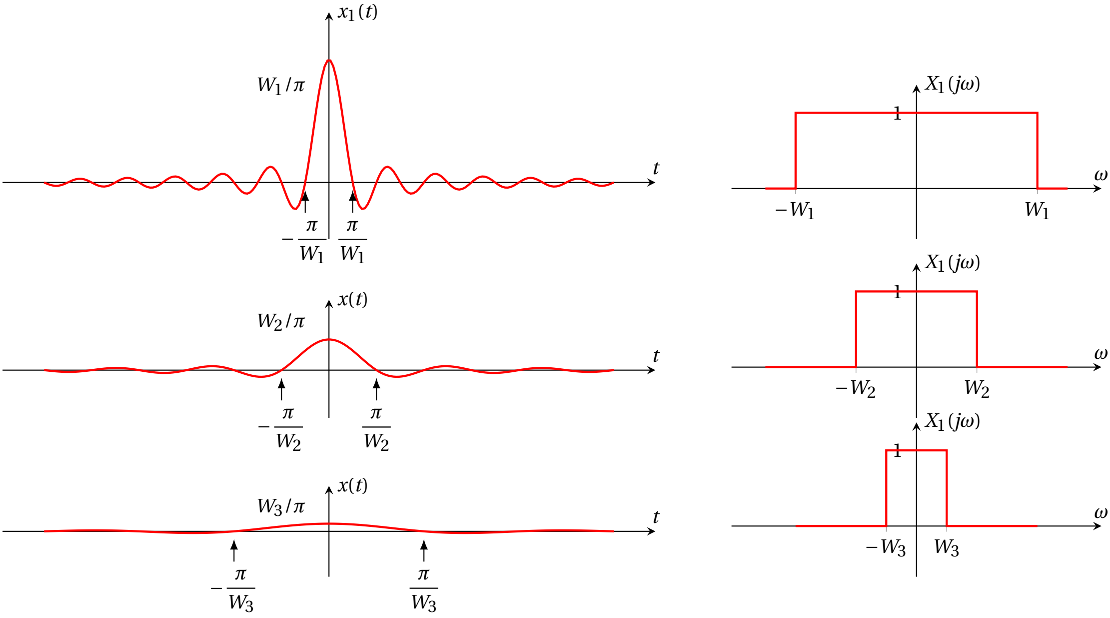
$$\frac{\sin Wt}{\pi t}$$

as sinc functions.

$$\frac{2\sin\omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$
$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

What Happens when W Increases?

- As W increases, $X(j\omega)$ becomes broader, while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of this signal (i.e., the part of the signal for $|t| < \pi/W$) becomes narrower.
- In fact, in the limit as $W \to \infty$, $X(j\omega) = 1$ for all ω, and consequently, we see that $x(t)$ converges to an impulse as $W \to \infty$.
- The behavior is an example of the inverse relationship that exists between the time and frequency domains.


$x_1(t)$
W_1/π
t
$-\frac{\pi}{W_1}$
$\frac{\pi}{W_1}$
$X_1(j\omega)$
1
ω
$-W_1$
W_1
$x(t)$
W_2/π
t
$-\frac{\pi}{W_2}$
$\frac{\pi}{W_2}$
$X_1(j\omega)$
1
ω
$-W_2$
W_2
$x(t)$
W_3/π
t
$-\frac{\pi}{W_3}$
$\frac{\pi}{W_3}$
$X_1(j\omega)$
1
ω
$-W_3$
W_3

Section 2

The Fourier Transform for Periodic Signals

The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transformof a periodic signal directly from its Fourier series representation.
Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$, i.e.,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \tag{9}$$

Let's determine the signal $x(t)$:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega, \\ &= e^{j\omega_0 t}. \end{aligned}$$

More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, i.e.,

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \tag{10}$$

then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{11}$$

which is exactly the Fourier series representation of a periodic signal.
Thus, the Fourier transformof a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

Example

Find the Fourier transformof the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0).$$

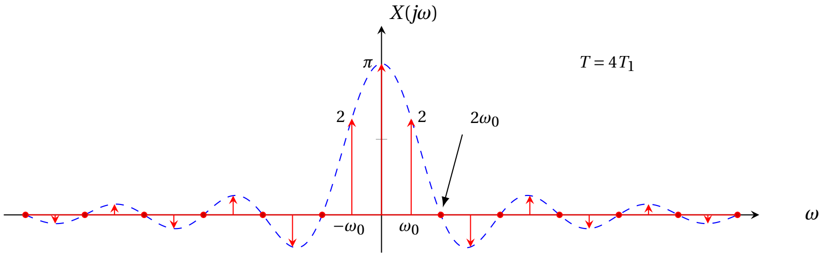


Figure: Fourier transform of a symmetric periodic square wave.

Example

Find the Fourier transformof

$$x(t) = \sin \omega_0 t.$$

and

$$x(t) = \cos \omega_0 t.$$

$$x(t) = \sin \omega_0 t.$$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, k \neq 1 \text{ or } 1$$

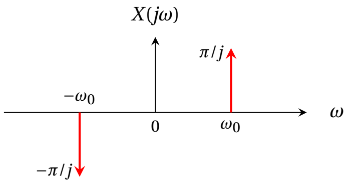


Figure: Fourier transform of the $x(t) = \sin \omega_0 t$.

$$x(t) = \cos \omega_0 t.$$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2}, \qquad a_{-1} = -\frac{1}{2},$$

$$a_k = 0, k \neq 1 \text{ or } 1$$

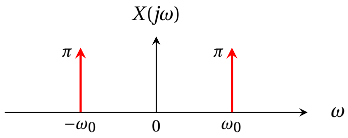


Figure: Fourier transform of the $x(t) = \cos \omega_0 t$.

Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

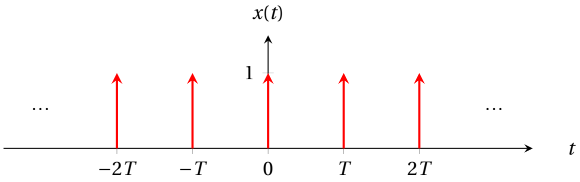


Figure: Pulse train.

The Fourier series coefficients for this signal:

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta t e^{-j\omega_0 t} dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$,

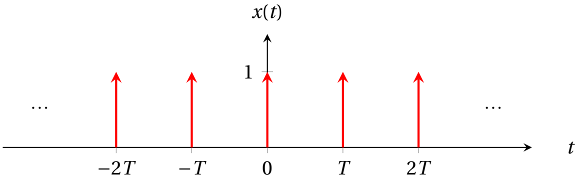


Figure: Periodic impulse train and its Fourier transform.

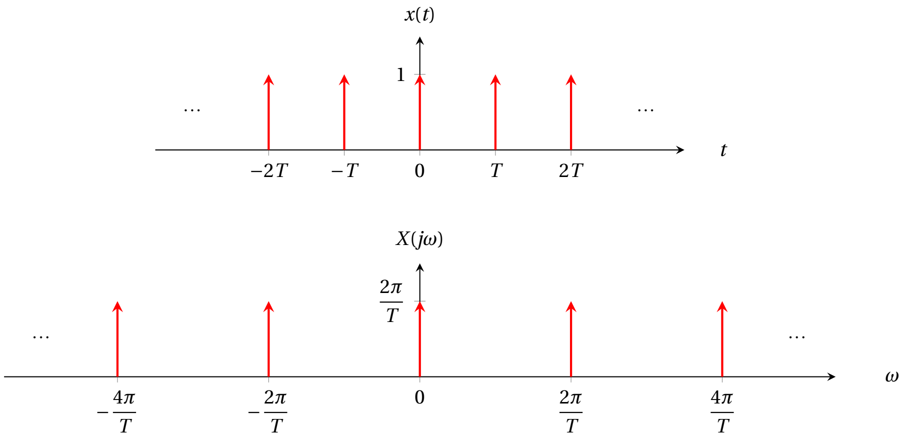


Figure: Periodic impulse train and its Fourier transform.

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Section 1

Fourier Transform Properties

Fourier Transform: Recall

Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega. \tag{1}$$

Analysis equation:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt. \tag{2}$$

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega). \tag{3}$$

Linearity

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

and

$$y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(j\omega).$$

then

$$ax(t) + by(t) \stackrel{\mathscr{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega).$$

Time Shifting

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega).$$

then

$$x(t - t_0) \xleftrightarrow{\mathscr{F}} e^{-j\omega t_0} X(j\omega).$$

Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$$

$$\begin{aligned} x(t-t_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega(t-t_0)}\, d\omega. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right) e^{j\omega t}\, d\omega. \end{aligned}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}d\omega.$$

This is the synthesis equation for $x(t-t_0)$. Therefore,

$$\mathscr{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega).$$

Magnitude of the Fourier transform not altered. Time shift introduces a phase shift $-\omega t_0$, which is a linear function of ω.

Conjugation and Conjugate Symmetry

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x^*(t) \overset{\mathscr{F}}{\longleftrightarrow} X^*(-j\omega).$$

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

Replacing ω by $-\omega$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt$$

If $x(t)$ is real, i.e., $x(t) = x^*(t)$, $X(j\omega)$ has conjugate symmetry.

$$X(-j\omega) = X^*(j\omega) \quad [x(t) \text{ real}]$$

Using Conjugate Symmetry

Use the conjugate property to comment about the symmetry of Fourier transform of a signal $x(t)$ if

1. $x(t)$ is real,
2. $x(t)$ is real and even, and
3. $x(t)$ is real and odd.

Expressing $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathfrak{Re}\left\{X(j\omega)\right\} + j\mathfrak{Im}\left\{X(j\omega)\right\},$$

then if $x(t)$ is real $[x(t) = x^*(t)]$

$$\begin{aligned} \mathfrak{Re}\left\{X(j\omega)\right\} &= \mathfrak{Re}\left\{X(-j\omega)\right\} \quad \text{and} \\ \mathfrak{Im}\left\{X(j\omega)\right\} &= -\mathfrak{Im}\left\{X(-j\omega)\right\} \end{aligned}$$

That is, the real part of the Fourier transform is an even function of frequency, and the imaginary part is an odd function of frequency. Considering

$$X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)},$$

we see that $|X(j\omega)|$ is an even function of frequency, and $\angle X(j\omega)$ is an odd function of frequency.

If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.
Proof:

$$X(-j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$$

With the substitution $\tau = -t$

$$X(-j\omega) = \int_{-\infty}^{\infty} x(-\tau)e^{-j\omega\tau}d\tau$$

Since $x(-\tau) = x(\tau)$ we have

$$X(-j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$X(-j\omega) = X(j\omega)$$

In a similar manner, it can be shown that if $x(t)$ is a real and odd function of time, so that $x(t) = -x(-t)$, then $X(j\omega)$ is purely imaginary and odd.

Fourier Transforms of Odd and Even Parts

A real function $x(t)$ can be expressed as

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t) = \mathfrak{Ev}\{x(t)\}$ is the even part of $x(t)$ and $x_o(t) = \mathfrak{Od}\{x(t)\}$ is the odd part of $x(t)$. Express Fourier transforms of

1. $x_e(t) = \mathfrak{Ev}\{x(t)\}$, and
2. $x_o(t) = \mathfrak{Od}\{x(t)\}$.

in terms of $X(j\omega)$.

From the linearity of the Fourier transform,

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},$$

and from the preceding discussion, $\mathfrak{F}\{x_e(t)\}$ is a real function and $\mathfrak{F}\{x_o(t)\}$ is purely imaginary. Thus, we can conclude that, with $x(t)$ real,

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega),$$

$$\mathfrak{Ev}\{x(t)\} \overset{\mathscr{F}}{\longleftrightarrow} \mathfrak{Re}\left\{X(j\omega)\right\},$$

$$\mathfrak{Od}\{x(t)\} \overset{\mathscr{F}}{\longleftrightarrow} j\mathfrak{Im}\left\{X(j\omega)\right\}.$$

Example

Use the symmetry properties of the Fourier transform to evaluate the Fourier transform of

$$x(t) = e^{-a|t|}, \quad a > 0.$$

We have already found that

$$e^{-at} \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{a + j\omega}.$$

$$\begin{aligned} x(t) = e^{-a|t|} &= e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathfrak{Ev}\left\{e^{-at}u(t)\right\}. \end{aligned}$$

Since $e^{-at}u(t)$is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathfrak{Ev}\left\{e^{-at}u(t)\right\} \stackrel{\mathscr{F}}{\longleftrightarrow} \mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\}.$$

$$X(j\omega) = 2\mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\} = \frac{2a}{a^2 + \omega^2}.$$

Differentiation and Integration

Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Differentiating both sides of the equation

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega X(j\omega).$$

Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).$$

Example

Determine the Fourier transform of the unit step $x(t) = u(t)$ making use of the knowledge that

$$g(t) = \delta(t) \stackrel{\mathscr{F}}{\longleftrightarrow} G(j\omega) = 1.$$

Noting that

$$x(t) = \int_{-\infty}^{t} g(\tau)\, d\tau$$

we obtain that

$$X(j\omega) = \frac{1}{j\omega} G(j\omega) + \pi G(0)\delta(\omega).$$

Since $G(j\omega) = 1$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

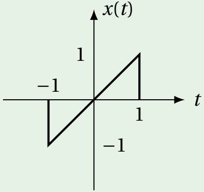
Observe that, we can apply the differentiation property to recover the transform of the impulse:

$$\delta(t) = \frac{du(t)}{dt} \stackrel{\mathscr{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1.$$

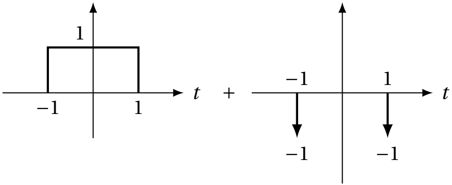
Note: $\omega\delta(\omega) = 0$

Example

Determine the Fourier transform of the signal $x(t)$ shown below:



$g(t) = \frac{dx(t)}{dt} =$



$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

$$X(j\omega) = \frac{1}{j\omega} G(j\omega) + \pi G(0)\delta(\omega).$$

As $G(0) = 0$

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Note: $X(j\omega)$ is purely imaginary and odd.

Time and Frequency Scaling

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x(at) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

where a is a real constant.
Letting $a = -1$

$$x(-t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(-j\omega).$$

The scaling property is another example of the inverse relationship between time and frequency.

Duality

Because of the similarity between the synthesis equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega. \tag{4}$$

and the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{5}$$

for any transform pair, there is a dual pair with the time and frequency variables interchanged.

We determined the Fourier transform of the square pulse as

$$x_1(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1, \end{cases} \overset{\mathscr{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

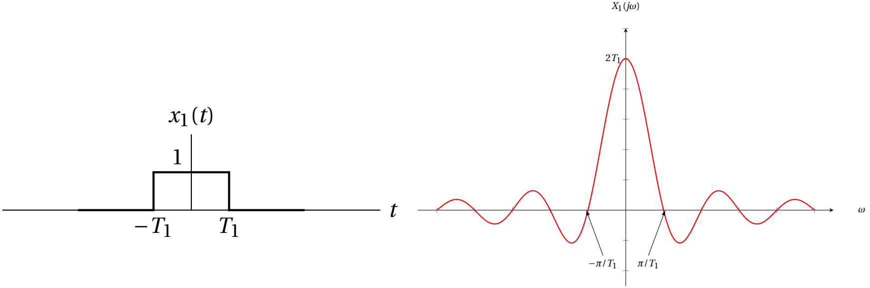


Figure: Rectangular pulse and the Fourier transform.

We also determined that for a time-domain signal that is similar in shape to the $X_1(j\omega)$ as

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{\mathscr{F}} X_2(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

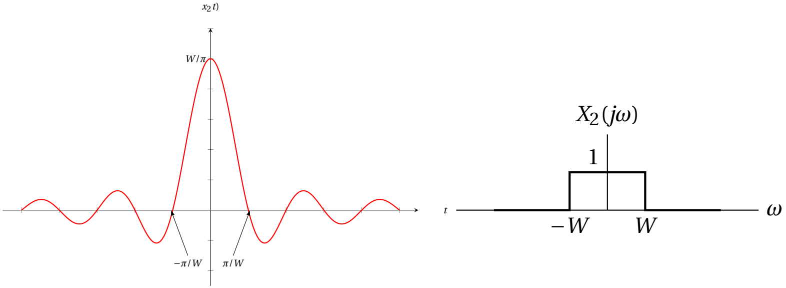


Figure: Fourier transform for $x(t)$.

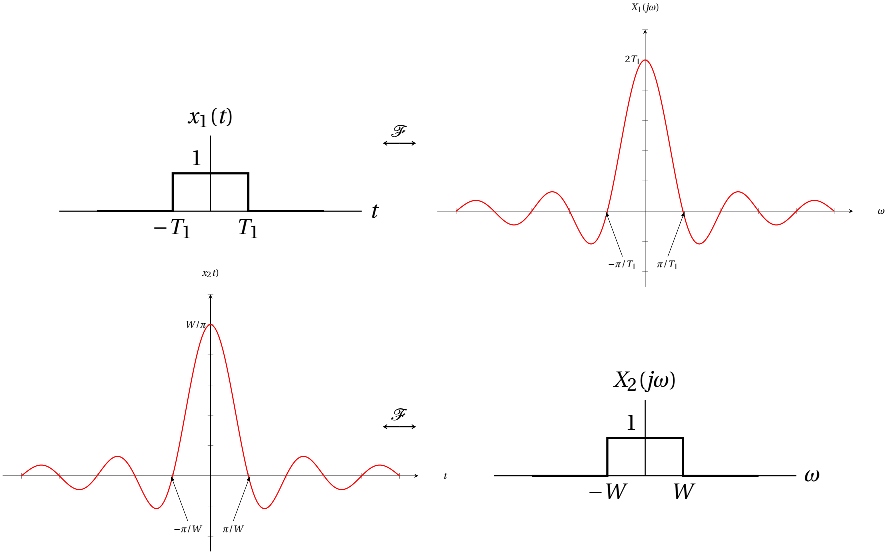


Figure: Duality.

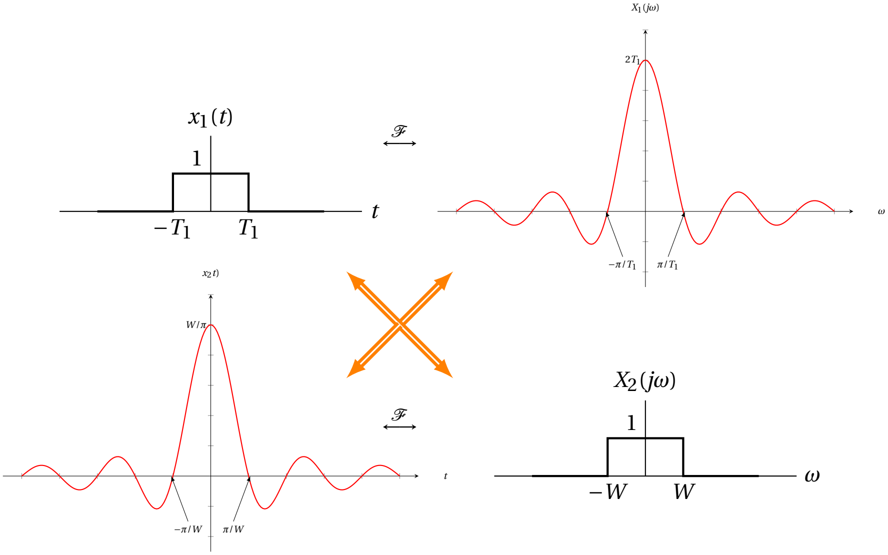


Figure: Duality.

Example

Use the duality property to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

$$x(t) = e^{-|t|} \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}\,d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}\,d\omega.$$

Now interchanging the names of variables t and ω

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}\,dt.$$

The right-hand side of this expression is the Fourier transform analysis equation for $2/(1+t^2)$. Thus

$$\mathscr{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}.$$

More Properties Using Duality

$$-jtx(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.$$

$$e^{j\omega_0 t}x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)\,d\eta.$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

Section 2

The Convolution Property

Convolution Property

$$y(t) = h(t) * x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

This equation is of major importance in signal and system analysis. This says that the Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

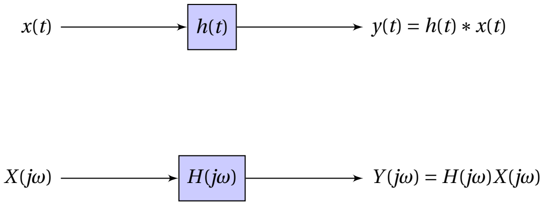


Figure: Convolution property.

Example

An LTI system has the impulse response

$$h(t) = \delta(t - t_0).$$

If the Fourier transform of the input signal $x(t)$ is $X(j\omega)$, what is the Fourier transform of the output?

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\begin{aligned} h(t) &= \delta(t - t_0) \\ H(j\omega) &= e^{-j\omega t_0} \\ Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0} X(j\omega) \end{aligned}$$

Example

What is the frequency response of the differentiator?

The input output relationship of the differentiator is

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property

$$Y(j\omega) = j\omega X(j\omega).$$

Consequently, the frequency response of the differentiator is

$$H(j\omega) = j\omega.$$

Example

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, find $y(t)$ by transfroming the problem into the frequency domain.

$$X(j\omega) = \frac{1}{b + j\omega}$$

$$X(j\omega) = \frac{1}{a + j\omega}$$

Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion.

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}$$

$b \neq a$

$$A = \frac{1}{b - a} = -B,$$

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]$$

By inspection

$$y(t) = \frac{1}{b - a}\left[e^{-at}u(t) - e^{-bt}u(t)\right].$$

For the case $a = b$,

$$Y(j\omega) = \frac{1}{(a + j\omega)^2}.$$

Recognizing this as

$$\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right],$$

we can use the dual of the differentiation property,

$$e^{-at}u(t) \xleftrightarrow{\mathscr{F}} \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \xleftrightarrow{\mathscr{F}} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2},$$

and consequently,

$$y(t) = te^{-at}u(t).$$

Multiplication Property

The convolution property states that convolution in time domain corresponds to multiplication in frequency domain. Because of the duality between time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \stackrel{\mathscr{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)].$$

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other. Consequently, the multiplication of two signals is often referred to as amplitude modulation. For this reason, this equation is sometime referred to as the modulation property.

Example

Let $s(t)$ be a signal whose spectrum is depicted in the figure below. Also consider the signal

$$p(t) = \cos\omega_0 t.$$

Show the spectrum of $r(t) = s(t)p(t)$.

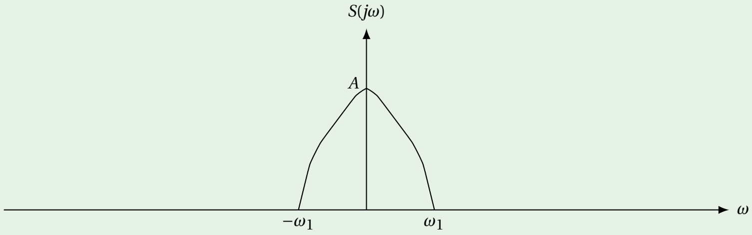


Figure: Spectrum of signal $s(t)$.

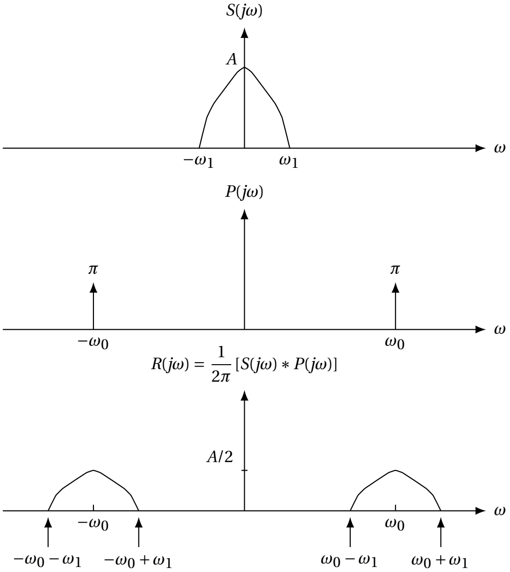


Figure: Fourier transform of $r(t) = s(t)p(t)$.