EN1060 Signals and Systems: Introduction

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Section 1

Introduction to Signals and Systems

Outline

- Signals and systems find many application in communications, automatic control, and form the basis for signal processing, machine vision, and pattern recognition.
- Electrical signals (voltages and currents in circuits, electromagnetic communication signals), acoustic signals, image and video signals, and biological signals are all example of signals that we encounter.
- They are functions of independent variables and carry information.

- We define a system as a mathematical relationship between an input signal and an output signal.
- We can use systems to analyze and modify signals.
- Signals and systems have brought about revolutionary changes.
- In this course we will study the fundamentals of signals and systems.
- Types of signals in continuous time and discrete time, linear time-invariant (LTI) systems, Fourier analysis, sampling, Laplace transform, z-transform, and stability of systems are the core components of the course.

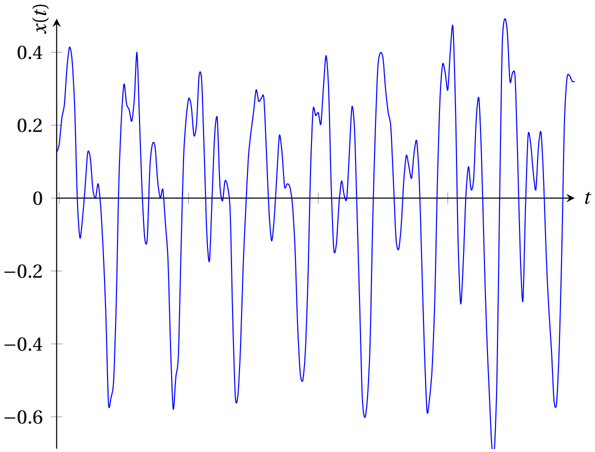
After completing this course you will be able to do the following:

- Differentiate between continuous-time, discrete-time, and digital signals, and techniques applicable to the analysis of each type.
- Apply appropriate theoretical principles to characterize the behavior of linear time-invariant (LTI) Systems.
- Use Fourier techniques to understand frequency-domain characteristics of signals.
- Use appropriate theoretical principles for sampling and reconstruction of analog signals.
- Use the Laplace transform and the z-transform to treat a class of signals and systems broader than what Fourier techniques can handle.

- In this course we study signals and systems that process these signals.
- Categories of signals:
  - Continuous-time signals: independent variable is continuous, $x(t)$
  - Discrete-time signals:independent variable is an integer, $x[n]$
- There are some very strong similarities and also some very important differences between discrete-time signals and systems and continuous-time signals and systems.

Continuous-Time Signals $x(t)$

- The independent variable is continuous.
- E.g., sound pressure at a microphone as a function of time (one-dimensional signal).
- E.g., image brightness as a function of two spatial variables (two-dimensional signal).
- Con convenience, we refer to the independent variable as time.



A function of a continuous variable

A speech signal: a continuous-time, one-dimensional signal

An image on a film: a continuous-time, two-dimensional signal

- Function of an integer variable.
- Takes on values at integer values of the argument of $x[n]$.

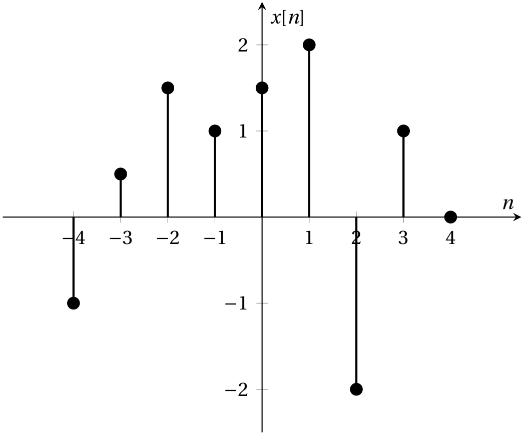


Figure: DT Signal

- What is a digital signal?
  - A quantized discrete-time signal. I.e., $x[n,m]$ can take only a value from a finite set of values.
- What is a digital image?
  - A two-dimensional, quantized, discrete-time signal.
  - A 600×800 image: $n \in [0,599]$, $m \in [0,799]$, $x[n,m] \in [0,255]$. 8-bit image.

Systems

- A system processes signals.
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm for post-flight analysis of a space launch.
  - An edge detection algorithm for medical images.

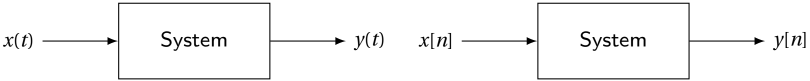


Figure: CT and DT Systems.

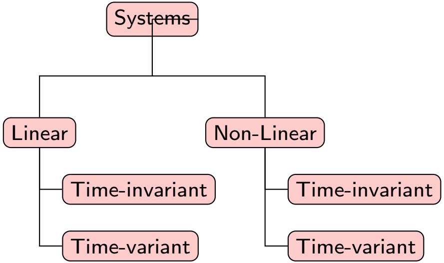
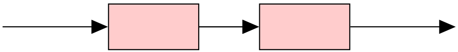


Figure: System types.

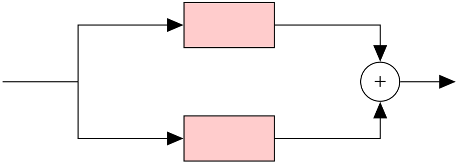
This course is focused on the class of linear, time-invariant (LTI) systems.

- Dynamics of an aircraft.
- An algorithm for analyzing financial and economic factors to predict bond prices.
- An algorithm for post-flight analysis of a space launch.
- An edge detection algorithm for medical images.

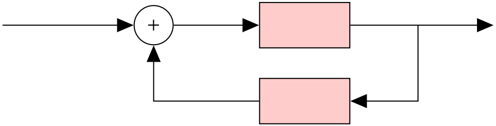
- To build more complex systems by interconnecting simpler subsystems.
- To modify the response of a system.
- E.g.: amplifier design, stabilizing unstable systems.



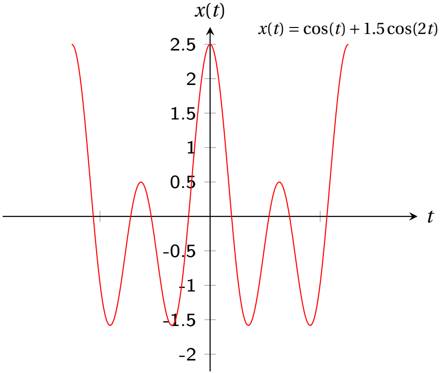
Series (Cascade)



Parallel



Feedback



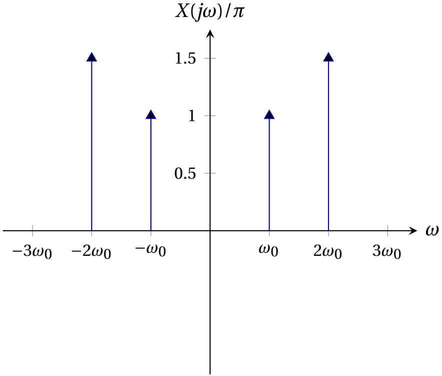


Figure: Domains.

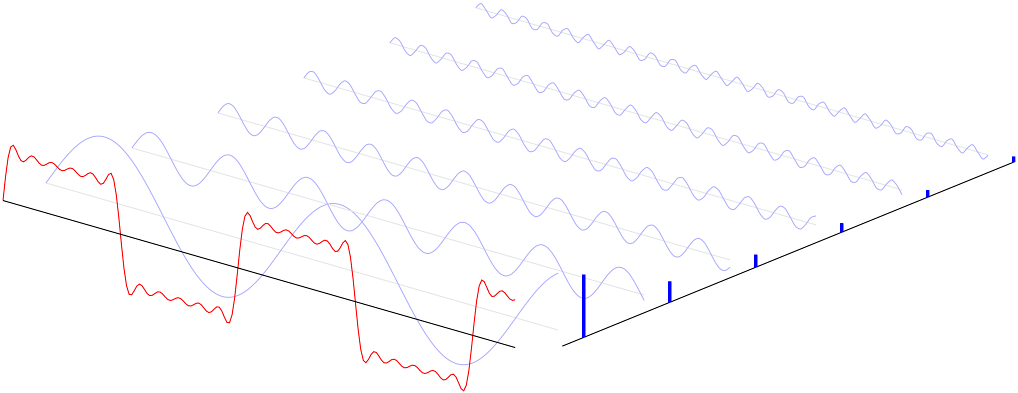


Figure: Square wave: time and frequency domains.

EN1060 Signals and Systems: Signals

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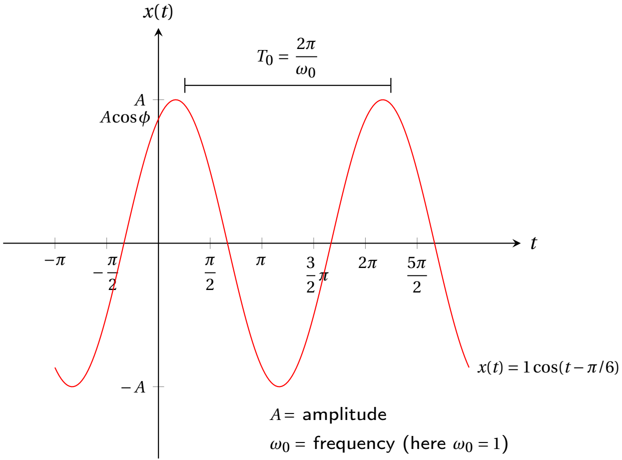
Section 1

Signals

Outline

$$x(t) = A\cos(\omega_0 t + \phi). \tag{1}$$



$A =$ amplitude

$\omega_0 =$ frequency (here $\omega_0 = 1$)

$\phi =$ phase (here $\phi = -\pi/6$)

Sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t+T) \tag{2}$$

for all values of t. Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t + \phi)$

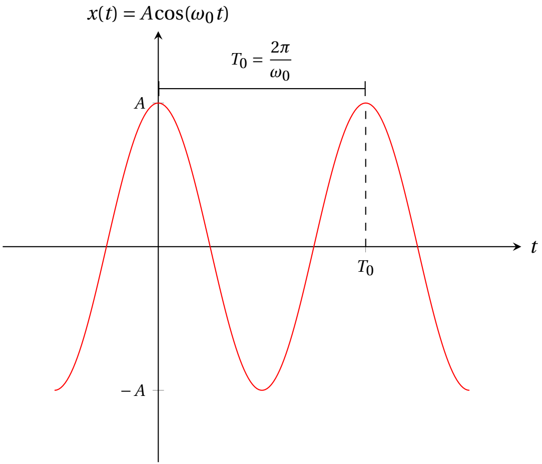
$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0 (t + T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

$T = \dfrac{2\pi m}{\omega_0} \Rightarrow$ fundamental period $T_0 = \dfrac{2\pi}{\omega_0}$.

A time-shift in a CT sinusoid is equivalent to a phase shift.
E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

$$A\cos[\omega_0(t+t_0)] = A\cos(\omega_0 t + \omega_0 t_0) = A\cos(\omega_0 t + \Delta\phi), \quad \Delta\phi \text{ is a change in phase.}$$

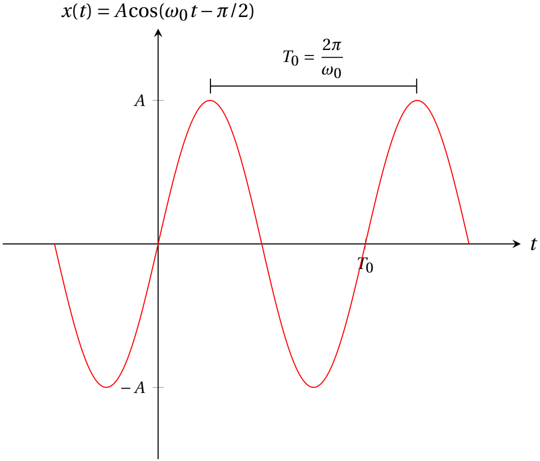
$$A\cos[\omega_0(t+t_0)+\phi] = A\cos(\omega_0 t + \omega_0 t_0 + \phi) = A\cos(\omega_0(t+t_1)), \quad t_1 = t_0 + \phi/\omega_0.$$



This signal is even. If we mirror an even signal about the time origin, it would look exactly the same.

Periodic: $x(t) = x(t + T)$.
Even: $x(t) = x(-t)$.

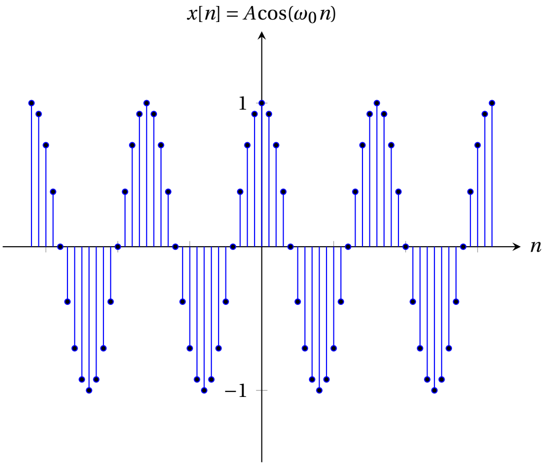
Phase of a Sinusoidal: $\phi = -\pi/2$



This signal is odd. If we flip an odd signal about the time origin, we also multiply it by a $(-)$ sign to get the original signal.

Periodic: $x(t) = x(t+T)$.
Odd: $x(t) = -x(-t)$.

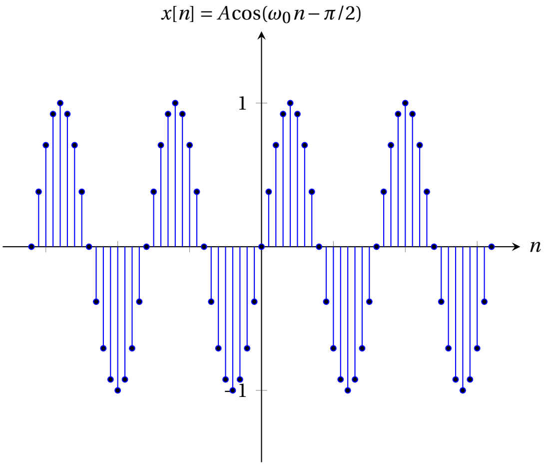
$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = 0$



The independent variable is an integer.
The sequence takes values only at integer values of he argument.
This signal is even.

Even: $x[n] = x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$
$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}$.

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = -\pi/2$



The independent variable is an integer.
The sequence takes values only at inter values of he argument.
This signal is odd.

Odd: $x[n] = -x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$
$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}$. $\phi = -\pi/2$, $x[n] = A\cos(\omega_0 n + \phi) = A\cos(\omega_0(n + n_0))$. n_0 must be an integer.
$n_0 = \frac{\phi}{\omega_0} = \frac{\pi/2}{\pi/8} = 4$.

Q

Does a phase change always correspond to a time shift in discrete-time signals?

Answer: No.

$$A\cos[\omega_0 n+\phi)] = A\cos[\omega_0(n+n_0)]$$
$$\omega_0 n+\omega_0 n_0 = \omega_0 n+\phi$$
$$\omega_0 n_0 = \phi \quad n_0 \text{ is an integer.}$$

- Depending on ϕ and ω_0, n_0 many not come out to be an integer.
- In discrete time, the amount of time shift must be an integer.

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

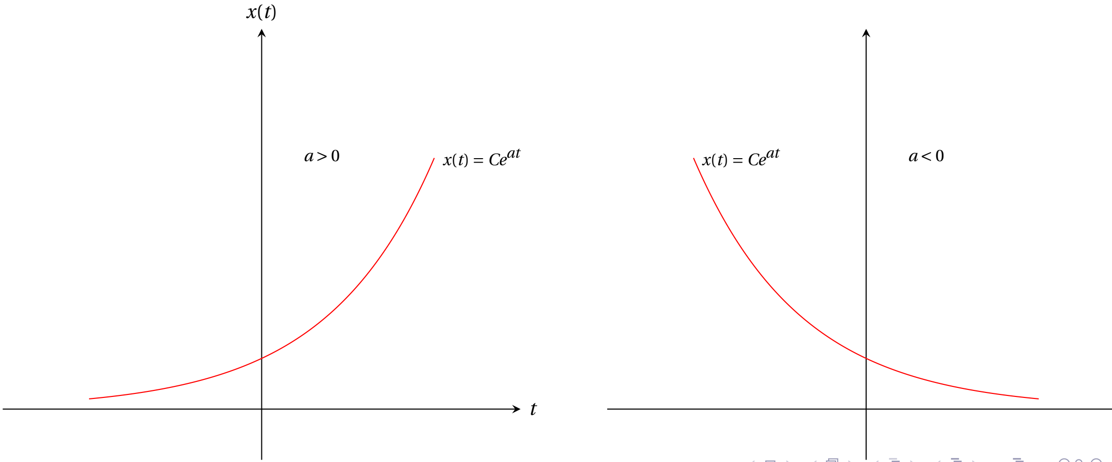
$$A\cos[\omega_0(n+N)+\phi] = A\cos[\omega_0 n + \omega_0 N + \phi]$$

$\omega_0 N$ must be an integer multiple of 2π.
Periodic $\Rightarrow$ $\omega_0 N = 2\pi m$

$$N = \frac{2\pi m}{\omega_0} \tag{4}$$

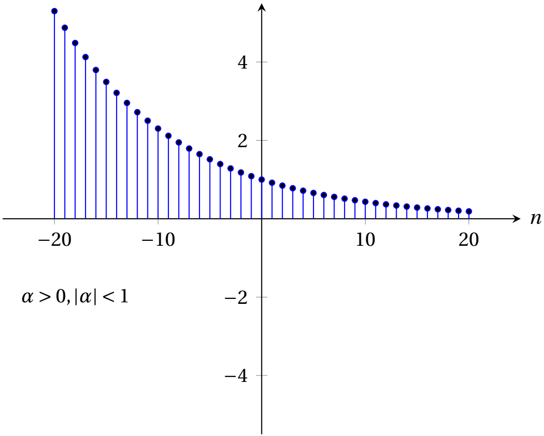
N and m must be integers.
Smallest N, if any, is the fundamental period.
N may not be an integer. In this case, the signal is not periodic.

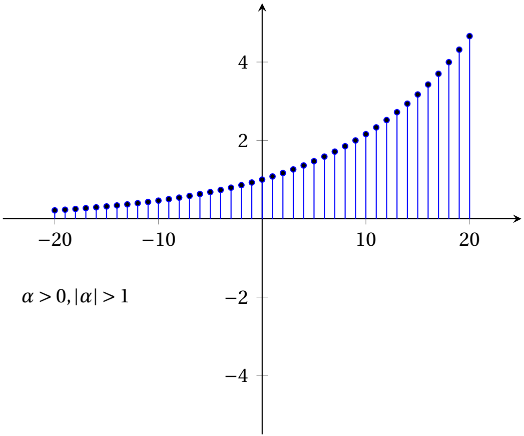
$$x(t) = Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers}$$
$$= Ce^{at_0} e^{at}.$$



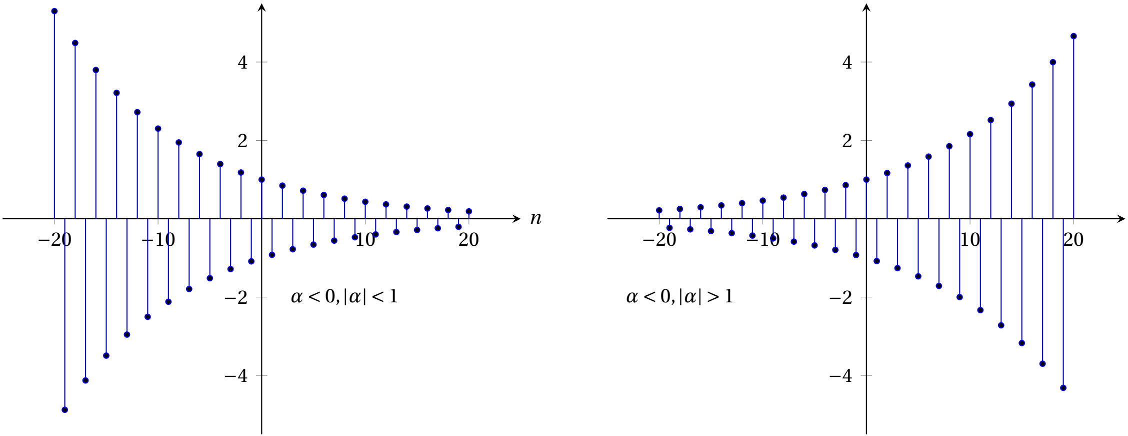
$x[n] = Ce^{\beta n} = C\alpha^n$, $\quad C$ and α are real numbers

$x[n] = C\alpha^n, \quad \alpha = 0.92$





$x[n] = C\alpha^n, \quad \alpha = -0.92$



$$x(t) = Ce^{at} \quad C \text{ and } a \text{ are complex numbers.}$$

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

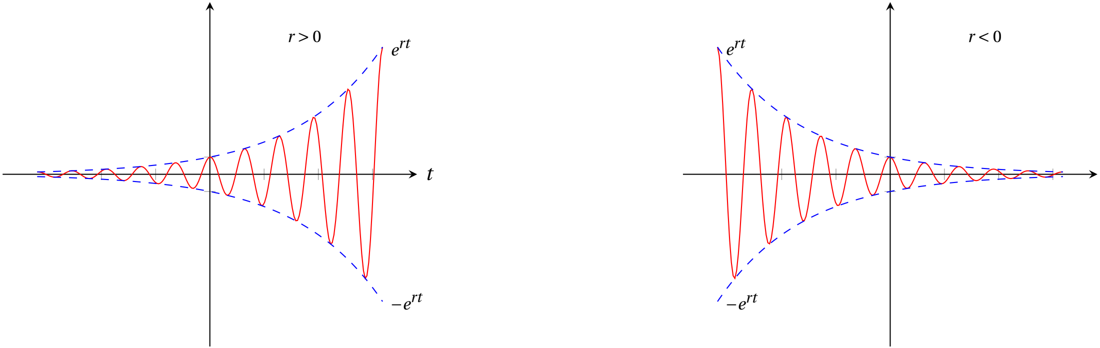
$$x(t) = |C|e^{j\theta} e^{(r+j\omega_0)t}$$

$$= |C|e^{rt} \boxed{e^{j(\omega_0 t+\theta)}}$$

$$= \boxed{|C|e^{rt}} [\cos(\omega_0 t + \theta) + \boxed{j} \sin(\omega_0 t + \theta)]$$

- $e^{j(\omega_0 t+\theta)} = \cos(\omega_0 t + \theta) + j\sin(\omega_0 t + \theta)$
- Real
- 90° out of phase

$$x(t) = |C|e^{rt}\cos(\omega_0 t + \phi)$$



$$x[n] = C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \tag{5}$$

$$C = |C|e^{j\theta} \tag{6}$$

$$\alpha = |\alpha|e^{j\omega_0} \tag{7}$$

$$x[n] = |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \tag{8}$$

$$= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta) \tag{9}$$

$$\tag{10}$$

Comments:

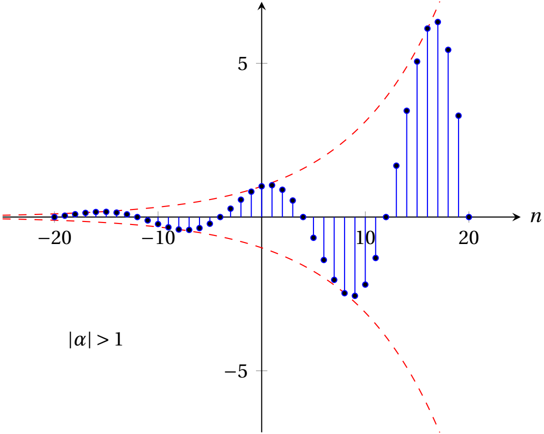
- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

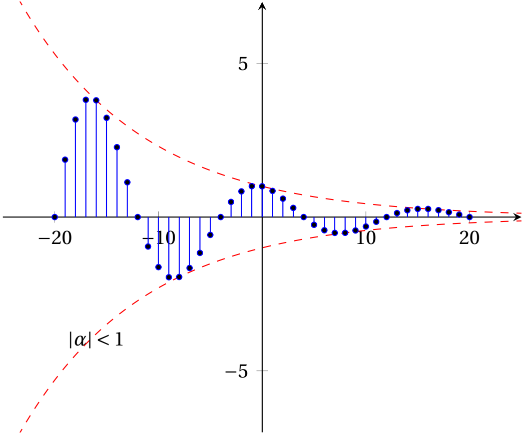
Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

- For the CT counterpart $e^{j\omega_0 t}$,
  1. The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal.
  2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.
- In DT, as
$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$
  the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0 .
- Although in CT $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In DT, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 + 2\pi$, $\omega_0 + 4\pi$, and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0.
- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

DT Complex Exponentials Plot

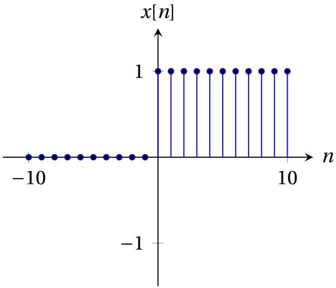
$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta), \quad |\alpha| = 1.12, \theta = 0$





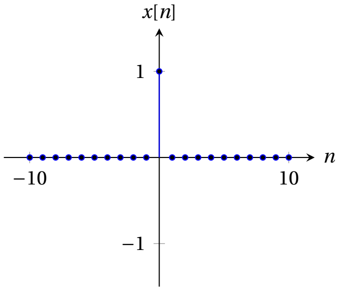
Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{11}$$



Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

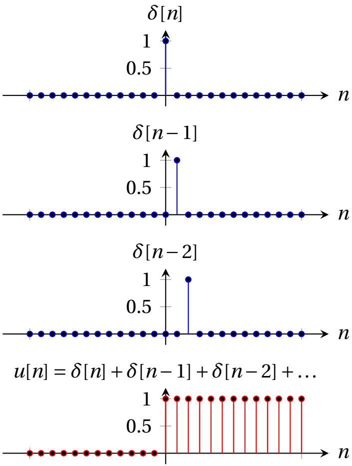
$$u[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{12}$$



DT Step and Impulse

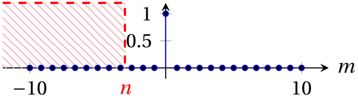
Unit impulse is the first backward difference of the unit step sequence.

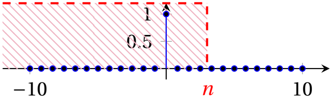
$$\delta[n] = u[n] - u[n-1]. \tag{13}$$



The unit step sequence is the running sum of the unit impulse.

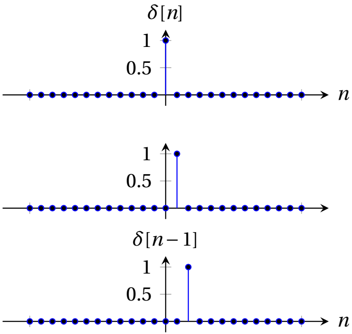
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{14}$$





The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{15}$$



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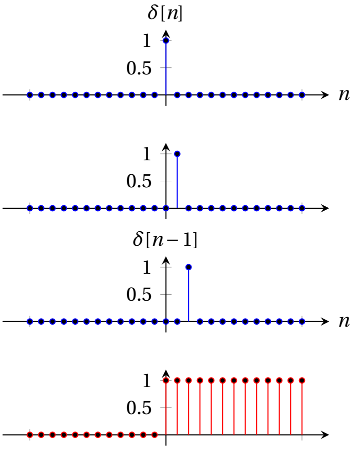
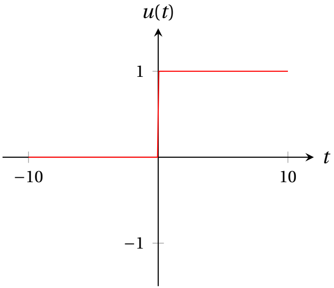
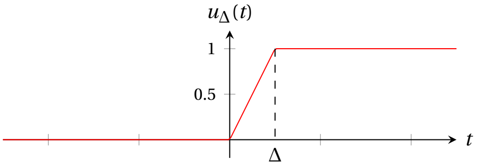


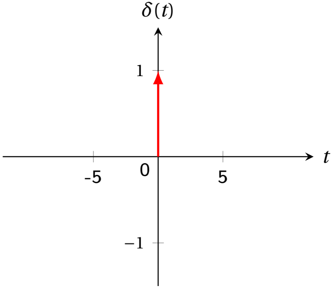
$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{16}$$

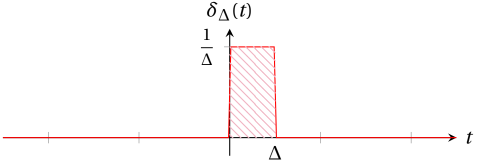




$u_\Delta(t) \rightarrow u(t)$ as $\Delta \rightarrow 0.$

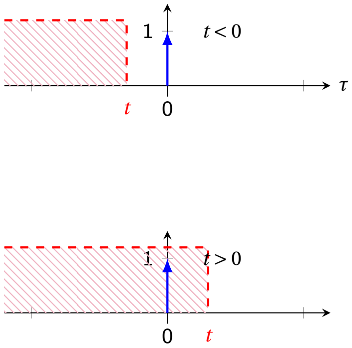
$$\delta(t) = \frac{du(t)}{dt}. \qquad (17)$$





$\delta_\Delta(t) \rightarrow \delta(t)$ as $\Delta \rightarrow 0.$
area $= 1$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \tag{18}$$



The total energy over a time interval $t_1 \le t \le t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \le n \le n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{19}$$

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \int_{n=-\infty}^{+\infty} |x[n]|^2. \tag{20}$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt. \tag{21}$$

Total energy in a DT signal:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{22}$$

With these definitions, we can identify three important classes of signals:

1. Energy signals: Signals with finite total energy $E_\infty < \infty$. These have zero average power.
2. Power signals: Signals with finite average power $0 < P_\infty < \infty$. As $P_\infty > 0$, $E_\infty = \infty$.
3. Signals with neither E_∞ nor P_∞ are finite.

EN1060 Signals and Systems: Fourier Series

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Section 1

Continuous-Time Fourier Series

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectra, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

Outline

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned} \tag{1}
$$

The set of coefficients $\{a_k\}$ is called the Fourier series coefficients or the spectral coefficients of $x(t)$. The coefficient a_0 is the dc or constant component of $x(t)$, given by Equation 1 with $k=0$:

$$
a_0 = \frac{1}{T}\int_T x(t)dt, \tag{2}
$$

which is simply the average of $x(t)$ over one period.

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing the right-hand side of this equation and Equation 1, we obtain

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$
$$a_k = 0, \qquad k \neq \pm 1.$$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a liner combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

Collecting terms,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

The Fourier coefficients are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = \left(1 - \frac{j}{2}\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = \left(1 + \frac{j}{2}\right),$$

$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1 + j),$$
$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, |k| > 2.$$

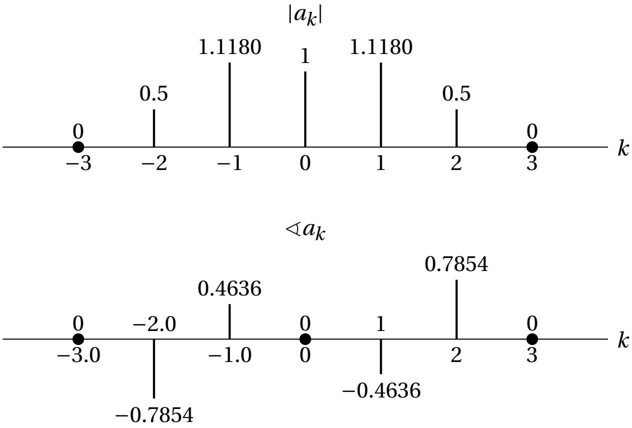

Figure: $|a_k|$, $\sphericalangle a_k$

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

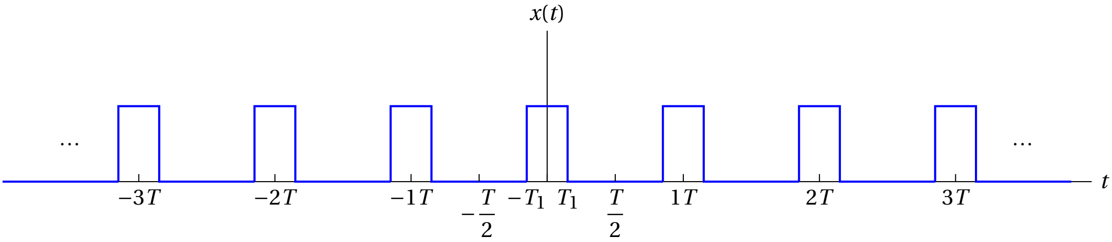

Figure: Periodic square wave

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)\,dt,\\
&= \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt,\\
&= \frac{2T_1}{T}.
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt,\\
&= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt,\\
&= -\,\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}
\end{aligned}
$$

$$
a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j}\right]
$$

$$
a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{2\sin(k\omega_0 T_1)}{k\pi},\, k \neq 0.
$$

For $T = 4T_1$

$$a_k = 0, \quad k \text{ even.}$$

$$
\begin{aligned}
a_0 &= \frac{1}{2} \\
a_1 = a_{-1} &= \frac{1}{\pi} \\
a_3 = a_{-3} &= \frac{1}{3\pi} \\
a_5 = a_{-5} &= \frac{1}{5\pi}
\end{aligned}
$$

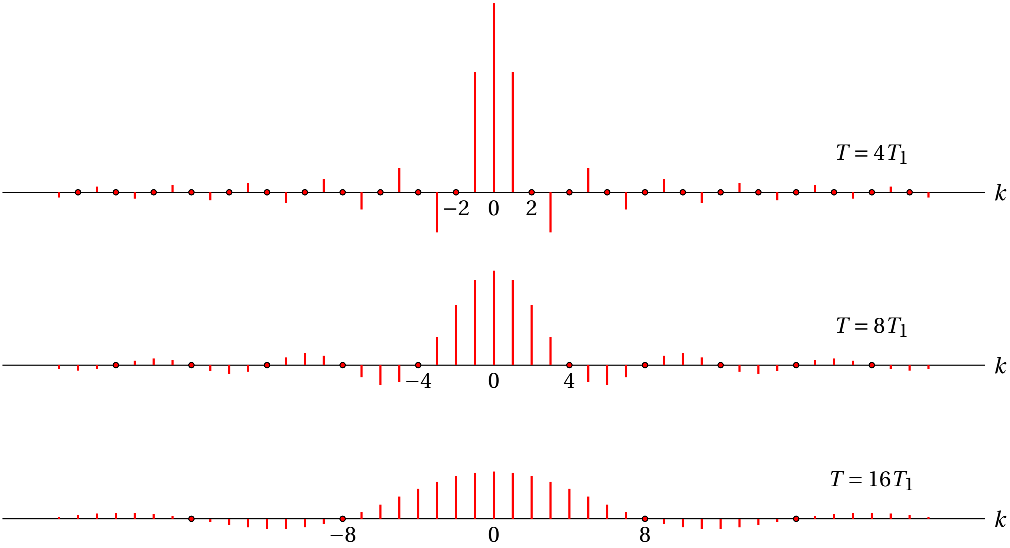

Figure: Plots of scaled Fourier series coefficients Ta_k

Section 2

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients are denoted by a_k, then

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \tag{3}$$

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathscr{FS}} a_k,$$
$$y(t) \xleftrightarrow{\mathscr{FS}} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $X(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathscr{FS}} c_k = Aa_k + Bb_k. \tag{4}$$

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

$$\begin{aligned}
x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, &\quad x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} b_k, \\
b_k &= \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt, \\
&= \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} dt, \\
&= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau, \\
&= e^{-jk\omega_0 t_0} a_k.
\end{aligned}$$

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k.$$

Note: $|a_k| = |b_k|$

If

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} a_k \tag{6}$$

then

$$x(-t) \overset{\mathscr{FS}}{\longleftrightarrow} a_{-k}. \tag{7}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{8}$$

Substitution: $k = -m$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{-jm2\pi t/T}.$$

- Time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients.
- If $x(t)$ is even—$x(-t) = x(t)$—then its Fourier series coefficients are also even, i.e., $a_{-k} = a_k$.
- If $x(t)$ is odd—$x(-t) = -x(t)$—then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

Time scaling, in general, changes the period.
If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{9}$$

While Fourier coefficients have not changes, the Fourier series representation has changed because of the change in the fundamental frequency.

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathscr{FS}} a_k,$$
$$y(t) \xleftrightarrow{\mathscr{FS}} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, its Fourier series coefficients h_k are

$$x(t)y(t) \xleftrightarrow{\mathscr{FS}} \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{10}$$

Conjugation and Conjugate Symmetry

- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real—$x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric,i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.
- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{11}$$

Note: Left-hand side of equation 11 is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$.

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2\, dt = |a_k|^2. \tag{12}$$

So, $|a_k|^2$ is the average power in the kth harmonic component of $x(k)$.
Thus, what Parseval's relation states is that the total power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 5.

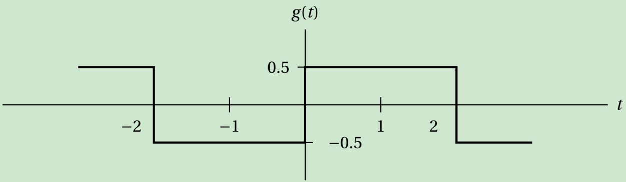

Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

We notice that

$$g(t) = x(t-1) - 1/2,$$

with $T = 4$ and $T_1 = 1$. If FS coefficients of $x(t)$ are denoted by a_k, the FS coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2},$$

The FS coefficients of the dc offset $-1/2$

$$c_k = \begin{cases} 0, & \text{for } k \neq 0, \\ -\dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

Applying the linearity property, the FS coefficients of $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0, \\ a_0 - \dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

yielding

$$d_k = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0, \\ 0 & \text{for } k = 0. \end{cases}$$

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 6. The derivative signal is the signal $g(t)$ in Figure 5. Using this information, find the Fourier series coefficients of $x(t)$.

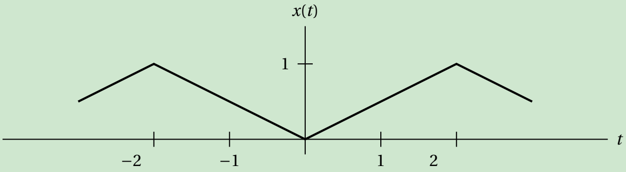

Figure: Figure for example

The derivative of this signal is the signal $g(t)$ in the previous example. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k we see that the differentiation property indicates that

$$d_k = jk(\pi/2)\,e_k.$$

This equation can be used to express e_k in terms of d_k except when $k=0$. Specifically,

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0.$$

For $k=0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{13}$$

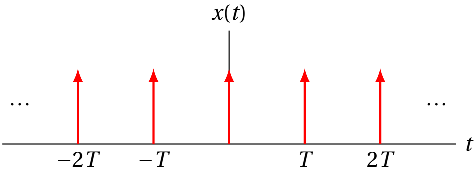

Figure: Impulse train

To determine the Fourier series coefficients a_k, we select the interval of integration to be $-T/2 \leq t \leq T/2$. Within this interval, $x(t)$ is the same as $\delta(t)$.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi/T} dt = \frac{1}{T}.$$

In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index k).

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

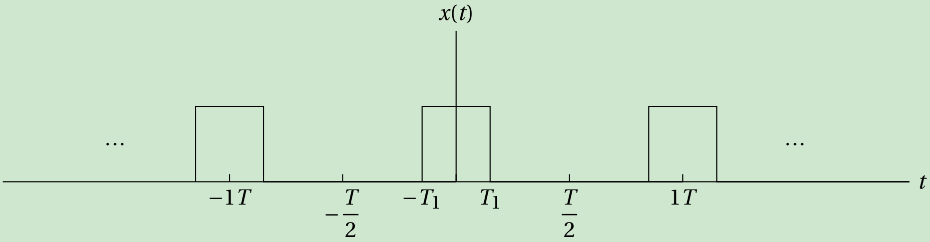

Figure: Figure for example

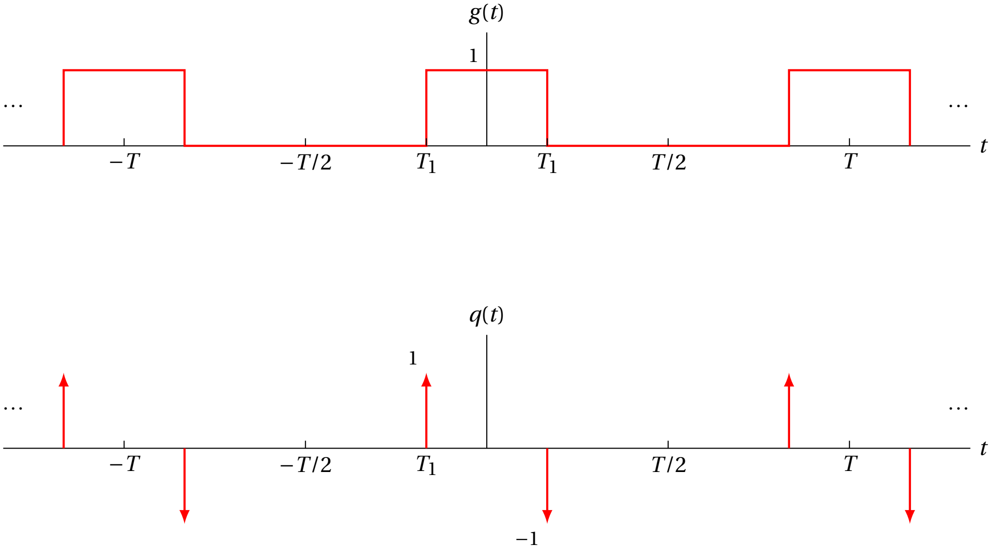

Figure: Periodic square wave and its derivative.

We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t+T_1) - x(t-T_1).$$

Fourier series coefficients b_k of $q(t)$ may be expressed in terms of the Fourier series coefficients a_k of $x(t)$; that is,

$$\begin{aligned} b_k &= e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k, \quad \omega_0 = 2\pi/T, \\ &= \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] = \frac{2j\sin(k\omega_0 T_1}{T}). \end{aligned}$$

Since $q(t)$ is the derivative of $g(t)$, we can use the differentiation property:

$$b_k = jk\omega_0 c_k,$$

where the c_k are the Fourier series coefficients of $g(t)$. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

Since c_0 is just the average value of $g(t)$ over one period,

$$c_0 = \frac{2T_1}{T}.$$

Complex Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \tag{14}$$

Trigonometric Fourier Series

$$x(t) = A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$
$$A_k = \frac{1}{T}\int_T x(t) A_k \cos k\omega_0 t dt \tag{15}$$
$$B_k = \frac{1}{T}\int_T x(t) A_k \sin k\omega_0 t dt$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$x(t) = C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k)$$
$$C_0 = A_0 \tag{16}$$
$$C_k = \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)$$

Relationship

$$A_0 = a_0$$
$$A_k = \frac{a_k + a_{-k}}{2}$$
$$B_k = j\frac{a_k - a_{-k}}{2} \tag{17}$$
$$\omega_0 = \frac{2\pi}{T}$$

Section 3

Convergence of Fourier Series

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2\,dt$$

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.
- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |e_N(t)|^2\, dt < \infty \tag{18}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.
- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Alternative Conditions (Dirichlet Conditions)

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1
Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\, dt < \infty. \tag{19}$$

This guarantees that a_ks are finite.

Condition 2
In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3
In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Functions that Violate Dirichlet Conditions

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)|\, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period T = 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.