Superintegrable Systems. III.

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Part I. Classical & Quantum superintegrability, Kepler system (classical), Hydrogen atom (quantum).

Part II. 1) Spin Calogero-Moser systems

Classical:

$$\begin{tikzcd}
T^*G \arrow[r, "\mu_L\times\mu_R"] \arrow[d, "\text{sympl.}"'] & \mathfrak{g}^*\tilde{\times}_{\mathfrak{g}^*/G}\,\mathfrak{g}^* \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] & \mathfrak{g}^* \quad \mathfrak{g}^* \\
\text{Poiss } T^*G/G \arrow[r] & (\mathfrak{g}^*\tilde{\times}_{\mathfrak{g}^*/G}\,\mathfrak{g}^*)/G \arrow[r] & \mathfrak{g}^*/G \\
S(\mathcal{O}) \arrow[u, phantom, sloped, "\subset"] \arrow[r] & P(\mathcal{O}) \arrow[u, phantom, sloped, "\subset"] \arrow[r] & B(\mathcal{O}) \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

$U\mathfrak{g}$ = quant. $C(\mathfrak{g}^*)$

sympl. leaf

$\dim(\mathcal{O})$ $\quad$ $\dim(\mathcal{O})-r$ $\quad$ r

classical superint. system spin CM

2) Quantum spin CM

$$\mathrm{Diff}(G) \longleftarrow U\mathfrak{g}\,\tilde{\otimes}_{Z(\mathfrak{g})}\,U\mathfrak{g} \;\leftrightarrows\; \begin{matrix} U\mathfrak{g} \\ U\mathfrak{g} \end{matrix}$$

$\cup \qquad \cup \qquad \cup$

$$\begin{tikzcd}
\mathrm{Diff}(G)^G \arrow[d, "ev_\mu"] & (Ug \otimes_{Z(g)} Ug)^G \arrow[l, hook'] \arrow[d, "ev_\mu"] & (Ug)^G \arrow[l, hook'] \arrow[d, "ev_\mu"] \arrow[dd, bend left] \\
A_\mu & J_\mu \arrow[l, hook'] & I_\mu \arrow[l, hook'] \\
 & & Z(g)
\end{tikzcd}$$

ev_μ is the evaluation on the representation

G = compact simple

$$\mathcal{H}_\mu = L_2\{ G \to V_\mu \mid f(h) \in V_\mu[0],\ h \in H \text{ and } f(g_1 g g_1^{-1}) = \pi_\mu(g_1) f(g) \}$$

$f(h_1 h h_1^{-1}) = f(h) = \pi_\mu(h_1) f(h)$, $\pi_\mu(h) = 0$, $\forall h \Rightarrow f(h) \in V_\mu[0]$

$$H_\mu \simeq L_2(H \to V_\mu[0])^W$$

- $g = g_1 h g_1^{-1}$, $\quad f(g) = \pi_\mu(g_1) f(h)$
- $f(h_1 h h_1^{-1}) = \pi_\mu(h_1) f(h) = f(h)$

$f(h'')$

For $G = SL_n$, $\mathfrak{g} = \{ e_{ij} \mid \sum_{i=1}^{n} e_{ii} = 0 \}$

Casimirs : $\begin{cases} c_2 = \sum_{ij} e_{ij} e_{ji}, \\ c_3 = \sum_{ijk} e_{ij} e_{jk} e_{ki} \\ \ldots\ldots \end{cases}$

$Z(\mathfrak{g}) = $
$= \mathrm{Pol}(c_2, \ldots c_n)$

Let us compute how Casimirs act

on $\mathcal{H}_\mu = L_2(H \to V_\mu[0])^W$

Natural embedding:

$$\mathfrak{g} \hookrightarrow U(\mathfrak{g}) \subset \mathrm{Diff}(G)$$

$e_{ij} \mapsto -g_{ki} \frac{\partial}{\partial g_{kj}}$, $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$

In general, left and right actions

$X \in \mathfrak{g}$

$$\boxed{(X_L f)(g) = \frac{d}{dt} f(e^{-tX} g) \Big|_{t=0}}$$

$$(X_R f)(g) = \frac{d}{dt} f(g e^{tX})\Big|_{t=0}$$

For $f \in \mathcal{H}_\mu$, by definition:

$$(X_L + X_R) f(g) = \pi_\mu(X) f(g)$$

For sl_n, $\{e_{ij}\}$ - basis $\left(\sum_{i=1}^{n} e_{ii} = 0\right)$

$$(e^R_{ij} f)(h) = \frac{d}{dt} f(h e^{t e_{ij}})\Big|_{t=0} = -h_i h_j^{-1} e^L_{ij} f(h)$$

$$(e^L_{ij} - h_i h_j^{-1} e^L_{ij}) f(h) = \pi_\mu(e_{ij}) f(h)$$

$$e^L_{ij} f(h) = \frac{1}{1 - h_i h_j^{-1}} \pi_\mu(e_{ij}) f(h),$$

$$h_j = e^{i q_j},$$

$$e^L_{jj} f(h) = \frac{d}{dt} f(e^{-t e_{jj}} h) = -h_j \frac{\partial f}{\partial h_j}(h) = -i \frac{\partial f}{\partial q_j}$$

Corollary For the Casimirs we have:

$$= L_2(H \to V_\mu[0]) \quad \sim H_\mu$$

$$\underset{=}{H_2^\mu} = C_2 = \underbrace{-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial q_i^2} + \frac{1}{4}\sum_{i\neq j}\frac{\pi_\mu(e_{ij}e_{ji})}{\sinh^2\left(\frac{q_i-q_j}{2}\right)}}$$

$$H_k^\mu = C_k = \frac{1}{k}\sum_{i_1\cdots i_k} e^L_{i_1 i_2}\, e^L_{i_2 i_3}\cdots e^L_{i_k i_1}$$

Hamilt. of quantum CM

Joint spectrum of $\langle H_2, H_3, \dots\rangle$

$$\mathcal{H}_\mu = \bigoplus_{\lambda\in D_\mu} \mathcal{H}_\mu[\lambda],$$

Eigensubspaces $\quad \mathcal{H}_\mu[\lambda] \simeq \mathrm{Hom}_G(V_\lambda \otimes V_\lambda^*, V_\mu)$

$$\Downarrow$$

$$a$$

$$D_\mu = \{\lambda \mid \mathrm{Hom}_G(V_\lambda\otimes V_\lambda^*, V_\mu) \neq \emptyset\}$$

Eigenvectors in $\mathcal{H}_\mu[\lambda]$:

$$\boxed{\Psi_a(g) = \mathrm{Tr}_{V_\lambda}(\pi_\lambda(g)\, a)},$$

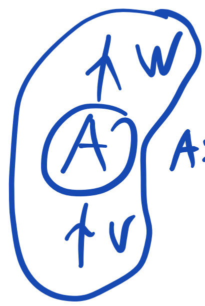
$a : V_\lambda \to V_\lambda \otimes V_\mu$, $\boxed{a \in \mathrm{Hom}_G(V_\lambda \otimes V_\lambda^*, V_\mu)}$

$\boxed{\hat{H}_k \psi_a(g) = c_k(\lambda)\, \psi_a(g)}$ $\boxed{\dim(\mathrm{Hom}_G(\dots)) = \mathrm{mult}(\mu \subset \lambda \otimes \lambda^*)}$

Proposition $\mathcal{H}_\mu(\lambda)$ is irreducible for J_μ.

Thus, we have a quantum superint. system which quantizes classical spin. CM models.

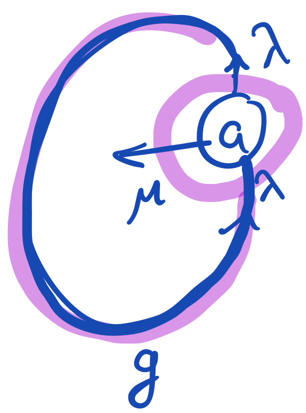
Graphical interpretation:



$A: V \to W$

$$\psi^{\mu}_{a,\lambda}(g) \rightsquigarrow$$

$$= \mathrm{tr}_{V_\lambda}(a\, \pi_\lambda(g))$$



$a : V_\lambda \to V_\mu \otimes V_\lambda$

Think of g as the holonomy along the edge.

$$U^{\mu}_{A} = \exp(-A\, \hat{H}_2) : \mathcal{H}^{\mu} \circlearrowleft$$

$$(U^\mu_A f)(g) = \int_G U^\mu_A(g,g')\, f(g')\, dg'$$

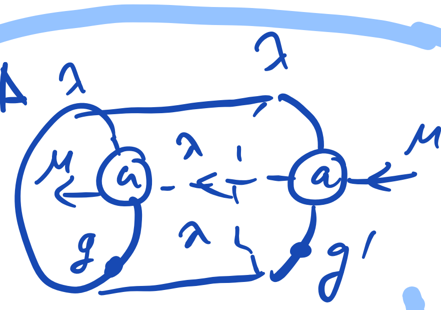
$$U^\mu_A(g,g') = \sum_{\lambda,a} \overline{\psi}^{\mu}_{a,\lambda}(g)\, e^{-Ac_2(\lambda)} \cdots\cdots \psi^{\mu}_{a,\lambda}(g')$$

Clearly:

$$\int_G U^\mu_{A_1}(g,g'')\, U^\mu_{A_2}(g'',g')\, dg'' = U_{A_1+A_2}(g,g')$$

Graphically:

$$U^\mu_A(g,g') \sim \sum_{a,\lambda} e^{-c_2(\lambda)A} \cdots$$



$: V_\mu \circlearrowleft$

$\lambda\otimes\lambda^* \ni \mu$

2D QCD presentation:

Thus

$$U^\mu_A(g,g') = \int_G$$

dg''

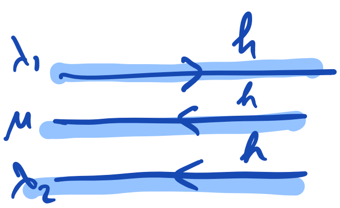
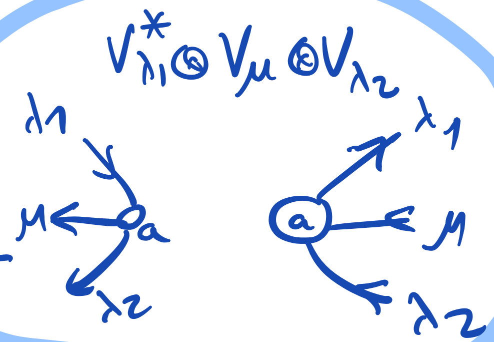
$$= \int_G U_A(g g'' g' g''^{-1})\, \pi^{\mu}(g'')\, dg''$$

$$U_A(g) = \sum_\lambda \chi_\lambda(g) \dim(\lambda)\, e^{-AC_2(\lambda)}$$

(Migdal, Witten)

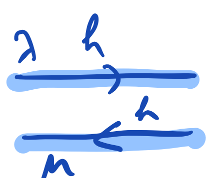
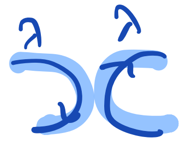
Proof.

Lemma

a) $\int_G$  $dh = \sum_a$ 

$\widehat{|G|}$

or $$\int_G \pi_1^{\lambda_1}(h^{-1})\, \pi_2^{\mu}(h)\, \pi_3^{\lambda_2}(h)\, dh = \sum_a K_a \cdot K_a^t \quad \widehat{|G|}$$

b) $\int_G$  $dh = \frac{|G|}{\dim(\lambda)} \delta_{\lambda,\mu}$ 

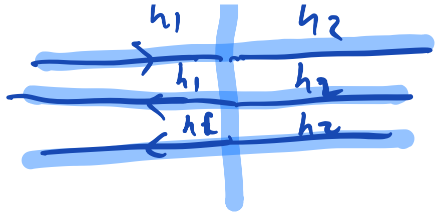
... $\epsilon : \mathbb{C} \to V^* \otimes V$

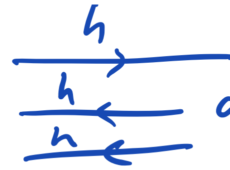
or

$$\int_G \pi_1^{\lambda}(h^{-1})\, \pi_2^{\lambda}(h)\, dh = \frac{|G|}{\dim(\lambda)} \delta_{\lambda,\mu} K_{12}^{0} K_{12}^{0\,t}$$

Proof.

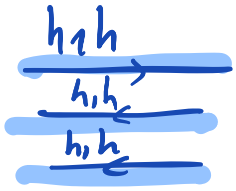
$h_1 h_2$

a) $\displaystyle\iint_{GG}$ 

$dh_1 dh_2 = |G| \displaystyle\int_G$ 

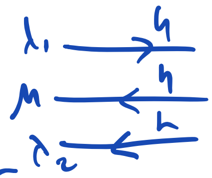
dh

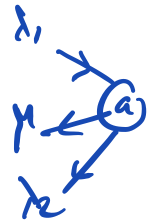
$\displaystyle\int_G$ 

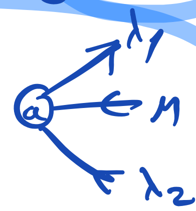
$dh_1 = \displaystyle\int_G$ 

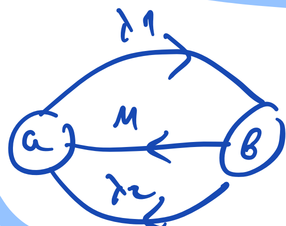
$dh_1 = \displaystyle\int_G$ 

dh_1

$\Rightarrow \displaystyle\int_G$ 

$dh = C \displaystyle\sum_n$ 





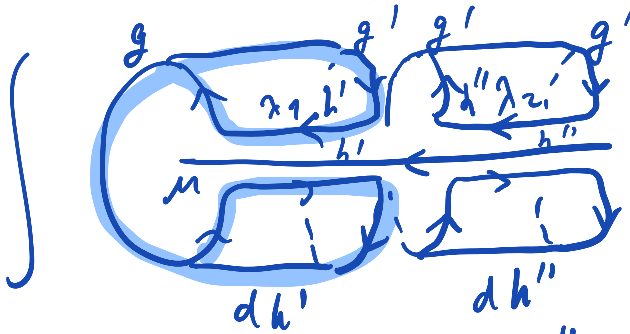
$= \delta_{a,b}$, $\Rightarrow$ $C = |G|$,

b) Similarly ☒

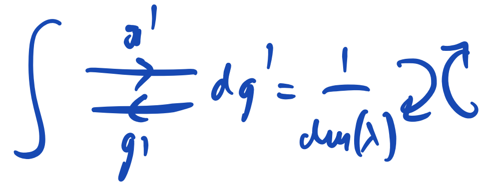
$$\sum_\lambda \int_G \chi_\lambda(g\, g''\, g'\, g''^{-1})\, \pi^\mu(g'')\, e^{-Ac_2(\lambda)}\, dg''$$

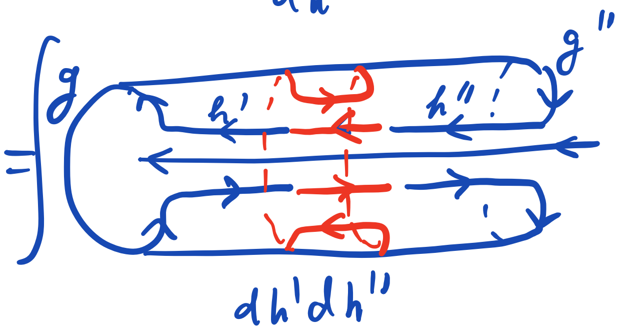
$$= \sum_{\lambda, a} \mathrm{tr}_\lambda(a g)\, e^{-Ac_2(\lambda)}\, \mathrm{tr}_\lambda(g' a)$$ ☒

Semigroup property

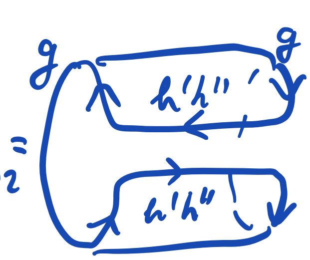


$dg' =$





$\delta_{\lambda_1\lambda_2} =$

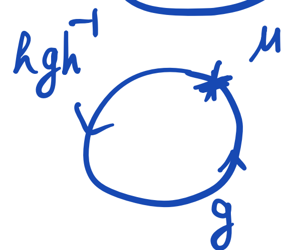


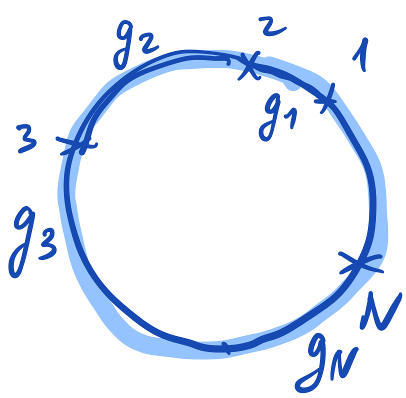
$\dim(\lambda_1)\,\delta_{\lambda_1\lambda_2}$

h/w

N-spin CM model

hgh^{-1}





Think of g_i as the holonomy from i to $i+1$.

$G^{\times N}$

Gauge group

$G^{\times N} = \{h_1, \dots, h_N\}$

vertices

acts on holonomies

$(h_1, \dots, h_N)(g_1, \dots, g_N) = (h_N g_1 h_1^{-1}, h_1 g_2 h_2^{-1}, \dots$

$\ldots, h_{N-1} g_N h_N^{-1})$

$$\theta : G \to G \qquad h\, g\, \theta(h)^{-1}$$

$G^{\times N} : G^{\times N} \circlearrowleft$, cosets: G^N / G^N

Sections:

$$(1, .., h, \ldots 1), \quad \begin{cases} g_j = 1, & j \neq i \\ g_i \in H \end{cases}$$

Thus $G^N / G^N \simeq H/W$ $\qquad T^*(G^{\times N})/G^N \quad (\mathfrak{g}^*)^{\times N}$

$$\mathrm{Diff}(G^N) \circlearrowleft Ug^N \tilde{\otimes}_{Z(\mathfrak{g})^N} Ug^N \qquad \begin{matrix} L \leftarrow Ug^N \\ R \leftarrow Ug^N \end{matrix}$$

left action on $G^{\times N}$ $\qquad$ right action on $G^{\times N}$

$\tilde{\otimes}_{Z(\mathfrak{g})^N}$: $(a_1 \otimes \cdots \otimes a_N) \otimes (z_1 b_1 \otimes \cdots \otimes z_N b_N) =$

$$= (a_1 S(z_2) \otimes \cdots \otimes a_N S(z_1)) \otimes (b_1 \otimes \cdots \otimes b_N)$$

$$\cup \qquad \cup \qquad \cup$$

$$\mathrm{Diff}(G^{\times N})^{G^N} \circlearrowleft \left(Ug^N \tilde{\otimes}_{Z(\mathfrak{g})^N} Ug^N\right)^{G^N} \circlearrowleft Z(\mathfrak{g})^{\otimes N}$$

$(\mathcal{Z}(\mathfrak{g}) = U\mathfrak{g}$ $)$

All of them act on (generalization of $N=1$ case to $N>1$)

$$H^{\mu_1 \ldots \mu_N} = L_2\left(G^{\times N} \to V^{\mu_1} \otimes \cdots \otimes V^{\mu_N} \,\middle|\, \right.$$

$$f(1, \ldots, \underset{i}{h}, \ldots, 1) \in \left(V^{\mu_1} \otimes \cdots \otimes V^{\mu_N}\right)[0], \ \forall i = 1, \ldots, N, \ h \in H$$

$$f(h_N g_1 h_1^{-1}, h_1 g_2 h_2^{-1}, \ldots, h_{N-1} g_N h_N^{-1}) = \left(\pi^{\mu_1}(h_1) \otimes \pi^{\mu_2}(h_2) \otimes \cdots \otimes \pi^{\mu_N}(h_N)\right) \cdot f(g_1, \ldots, g_N) \left. \right)$$

$$H^{\mu_1 \ldots \mu_N} \simeq H_i^{\{\mu\}} = L_2\left(H \to (V^{\mu_1} \otimes \cdots \otimes V^{\mu_N})[0]\right)$$

i natural isomorphisms $i = 1, \ldots, N$

Evaluating the above algebras on $H_i^{\{\mu\}}$ gives

$$\boxed{A^i_{\{\mu\}} \supset J^i_{\{\mu\}} \supset I^i_{\{\mu\}}}$$

Commuting Hamiltonians =

= Images of Casimirs in $I^i_{\{\mu\}}$.

For sl_n: $(n-1)N$ Casimirs

rN

$c_2^{(a)}, \dots, c_n^{(a)}, \quad a = 1, \dots, N$

$$c_k^{(a)} = \sum_{\{i\}} e^{(a)}_{i_1 i_2} e^{(a)}_{i_2 i_3} \cdots e^{(a)}_{i_k i_1}$$

Thm. $c_2^{(i)}$ acting on $\mathcal{H}_i^{\{\mu\}}$ gives

$$H_2^{(i)} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial q_i^2} + \sum_{i<j} \frac{\pi_\mu(e_{ij} e_{ji})}{\sin^2\left(\frac{q_i - q_j}{2}\right)}$$

$\pi_\mu(x) = (\pi_{\mu_1} \otimes \cdots \otimes \pi_{\mu_n})(\Delta(e_{ij} e_{ji})) : (V_{\mu_1} \otimes \cdots \otimes V_{\mu_n})[0]$

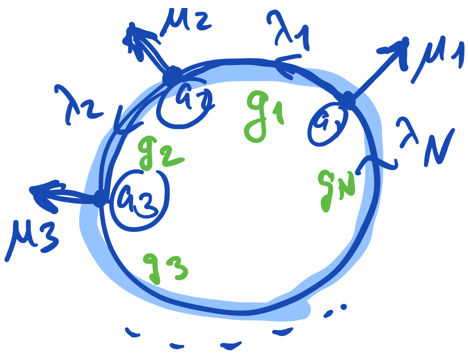
Proof (hw)

Joint spectrum of $I_{\{\mu\}}$

$$\mathcal{H}_{\{\mu\}} \simeq \bigoplus_{\lambda_1, \dots, \lambda_N} \mathcal{H}_{\{\mu\}}[\lambda_1, \dots, \lambda_N]$$

$G^{\times N}$ irreps

$\mathcal{H}_{\{\mu\}}[\lambda_1, \dots, \lambda_N] = \mathrm{Hom}_G(V_{\lambda_N}, V_{\mu_1} \otimes V_{\lambda_1}) \otimes \cdots$



$\otimes \mathrm{Hom}_G(V_{\lambda_{N-1}}, V_{\mu_N} \otimes V_{\lambda_N})$,

$$\Psi^{\{\mu\}}_{\{a\}\{\lambda\}}(g_1,\dots,g_N) = \mathrm{Tr}_{V_{\lambda_N}}\big(a_1 \pi_{\lambda_1}(g_1)\, a_2 \pi_{\lambda_2}(g_2) \dots a_N \pi_{\lambda_N}(g_N)\big)$$

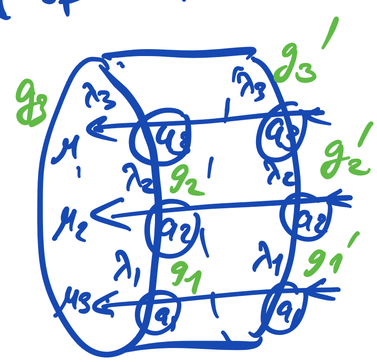
$$a_i : V_{\lambda_{i-1}} \to V_{\mu_i} \otimes V_{\lambda_i}$$

Thm. $J^i_{\{\mu\}}$ acts irreducibly on $\mathcal{H}^{\mu}[\lambda]$

Quantum superintegrable system.

The propagator integral kernel of $\exp(-A\hat{H}_2)$

$$U_A^{\{\mu\}}(\{g\}|\{g'\}) = \sum_{\{a\}\{\lambda\}}$$



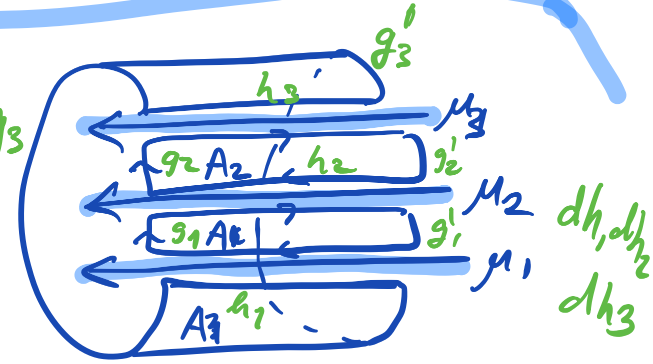
$\cdot \exp\big(-\sum_a c_2(\lambda_a) A_a\big)$

$\exp\big(-\sum_a \sum_{i=1}^{r} c_i(\lambda_a) \cdot A^i_a\big)$

$\cdot A_a)$

In terms of 2D QCD:

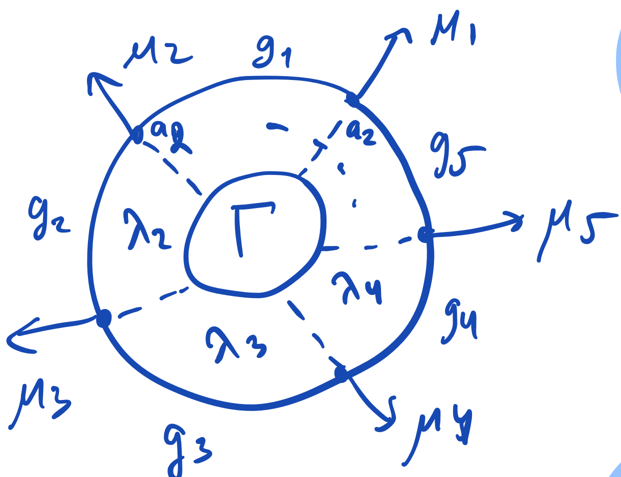
$$U_A^{\{\mu\}}(\{g\}|\{g'\}) = \int_{G^{\times 3}}$$

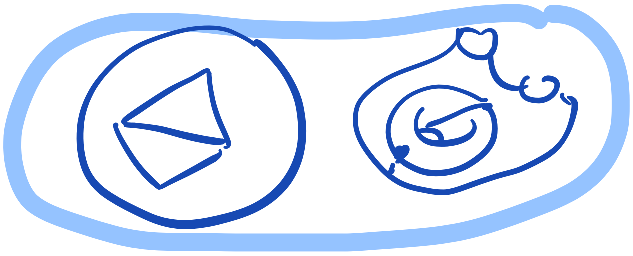


$dh_1 dh_2 dh_3$

$N=3$

2D QCD with nonremovable corners



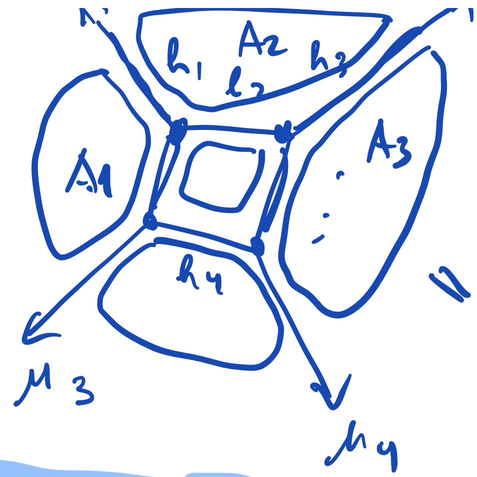


$$= \sum_{\lambda} \mathrm{tr}(a_1 \pi_{\lambda_1}(g_1)$$

colors of internal domains

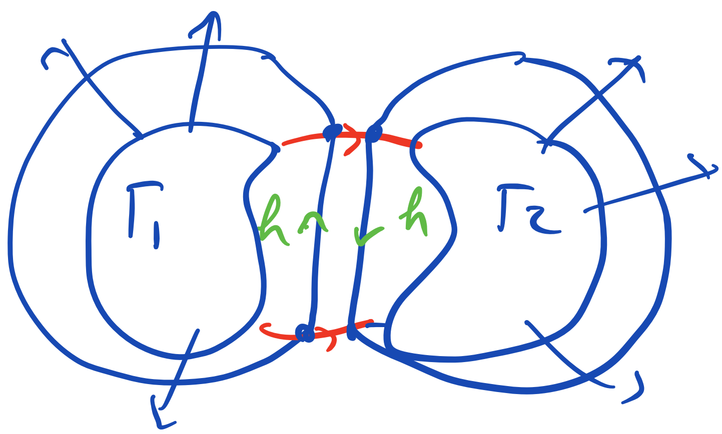
$$\cdots a_N \pi_{\lambda_N}(g_N)) \exp\Big(-\sum_p c_2(\lambda_p) A_p\Big).$$

$\mathrm{weight}(\{\lambda\})_{\{\mu_{int}\}} =$

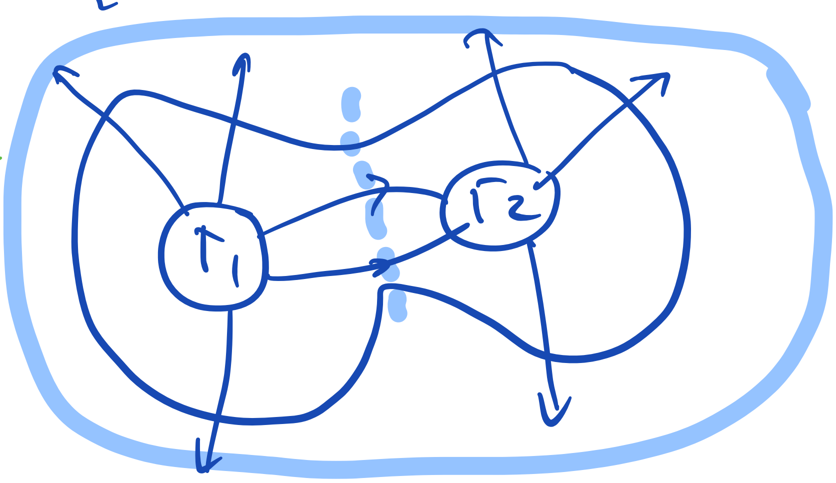
$= \int$ 

$dh_{\text{internal}} \; \prod_p U_{A_p}(\text{hol}(\partial p)) \times \times \Gamma(h, g)$

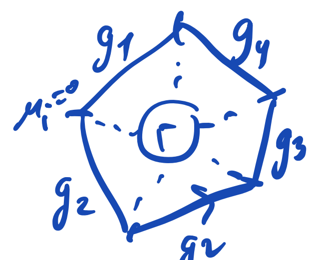
Gluing through an interval:

$\int$ 

$dh =$

$=$ 

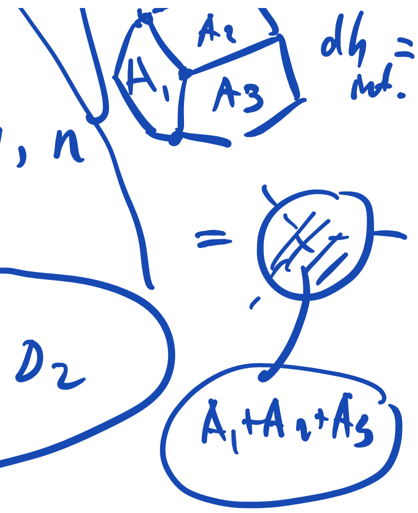
if $\mu_i = 0$



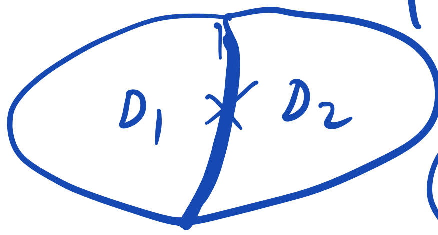
2D QCD with Wilson loops going " 

into the boundary.

Can define it for $\Sigma_{g,n}$



Gluing along segments



$$S_{D_1\cup D_2} = S_{D_1} + S_{D_2}$$

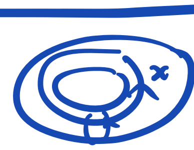
Matitsyn ; Gross + M.

· $U_A(g) = \sum_{\lambda} \mathrm{tr}_\lambda(g) \dim(\lambda)\, e^{-c_2(\lambda)A}$

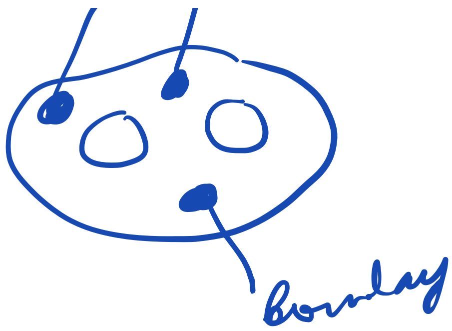
U_n , $n \to \infty$

· Do the same for $\mathrm{tr}(a_1 A_1(g_1)\cdots a_N A_N(g_N))$
not for $\mathrm{tr}_\lambda(g)$

$e^{p} \to 1 + p + \frac{p^2}{2} \cdots$

Rel. spin CM $\rightsquigarrow$  $G \times G \to T^*G$

$\mathfrak{g}^* \simeq \mathfrak{g}$, $\mathfrak{g}$ - simple



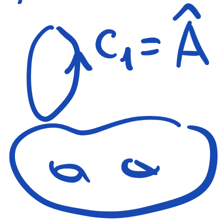
Artamonov, R. 2019

$$G\times G \xrightarrow{\varepsilon\to 0} TG$$

TQFT; RT; $\varepsilon = \frac{1}{c}$

Polyak, R., 2D QCD, $A \in i\mathbb{R}$,

$\uparrow$ semiclass. limit $r\to\infty$

$RT_r\Big($  $\Big)$ $c_1 = \hat{A}$

$q^r = 1$,

$q = e^{\frac{i\pi}{r}}$, $r\to\infty$, $\hat{A}\to\infty$

$$\frac{\hat{A}}{r} = A$$

$K\backslash G/K$ $\underline{\underline{K\backslash G_{\mathbb{R}}/K}}$

L. Feher BC cosets.