Lecture 8: Jet Mass with Grooming

In this lecture, we still focus on QCD jets. We want to see how different grooming algorithms clean up the jets.

1. Soft Drop Mass

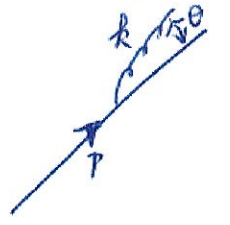
In the soft drop, one uses the soft drop condition:

$$\min(p_{T,i}, p_{T,j}) > z_{cut}(p_{T,i} + p_{T,j})\left(\frac{\theta}{R}\right)^\beta \qquad (1)$$

Here, as before, we only calculate the LL result.

1.1 LO SD mass:

At lowest order, we only need to consider collinear & soft radiation:



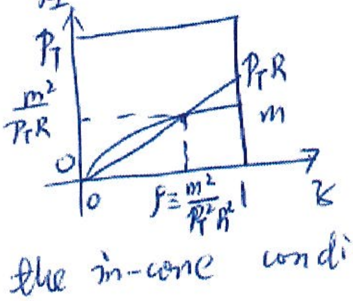
The SD condition becomes

$$z p_T = k_T > z_{cut}(p_T + k_T)\left(\frac{\theta}{R}\right)^\beta = z_{cut}\left(\frac{k_\perp}{z p_T R}\right)^\beta p_T \qquad (2)$$

Using the variables $k_\perp$ & z, one has

$$m^2 = 2E\, n\cdot k = \frac{k_\perp^2}{z} \qquad (3)$$

and the phase-space for k is



Recall that the in-cone condition is (4)

$$\frac{k_\perp}{z p_T} < R$$

We only need to consider the in-cone radiation.

The SD condition (2) can be written in the form

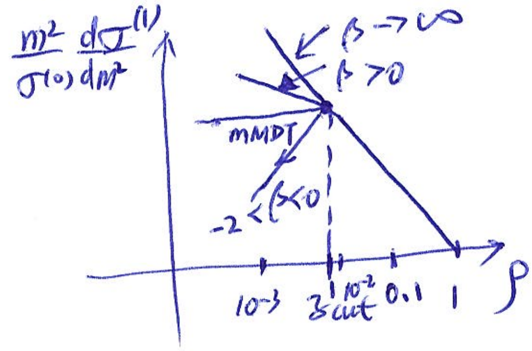
$$z > z_{cut}^{\frac{2}{2+\beta}}\, \rho^{\frac{\beta}{2+\beta}}, \qquad (5)$$

where we have used the relation between m and $k_\perp, z$ in (3). Recall the calculation for the jet mass, and we have

$$\frac{1}{\sigma^{(0)}}\frac{d}{dm^2}\sigma^{(1)} = \frac{\alpha_s C_F}{\pi}\frac{1}{m^2}\int_0^1 \frac{dz}{z}\,\theta(z-\rho)\,\theta\left(z - z_{cut}^{\frac{2}{2+\beta}}\rho^{\frac{\beta}{2+\beta}}\right)$$

$$= \begin{cases} \dfrac{\alpha_s C_F}{\pi}\dfrac{1}{m^2}\log\dfrac{1}{\rho} & \text{if } \rho > z_{cut} \\[2ex] \dfrac{\alpha_s C_F}{\pi}\dfrac{1}{m^2}\log\dfrac{1}{z_{cut}^{\frac{2}{2+\beta}}\rho^{\frac{\beta}{2+\beta}}} = \dfrac{\alpha_s C_F}{\pi}\dfrac{1}{m^2}\left(\dfrac{2}{2+\beta}\log\dfrac{1}{z_{cut}} + \dfrac{\beta}{2+\beta}\log\dfrac{1}{\rho}\right) & \\ \quad \text{otherwise.} & \end{cases} \qquad (6)$$

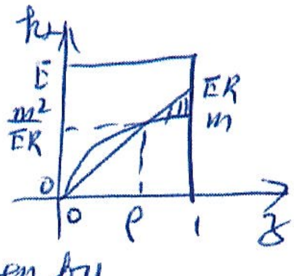
<u>The effect of β</u>:



Now, let us calculate the cumulative distribution. For this task, it is more convenient to use the Lund diagrams in which one uses $\frac{k_\perp}{E}$ and $\frac{1}{\theta}$ in the logarithmic scale. In terms of these two variables, one has

$$z = \frac{k_\perp}{E}\frac{1}{\theta} \quad (7)$$

Recall that in jet mass calculation, we have the phase space of k as follows



with the shaded region given by

$$i)\ 1 > z > \rho \qquad ii)\ zER > k_\perp > z^{\frac{1}{2}} m \quad (8)$$

In terms of $\frac{k_\perp}{E}$ and $\frac{1}{\theta}$, one accordingly has

$$i)\ -\log\frac{1}{\theta} > \log\frac{k_\perp}{E} > -\log\frac{1}{\theta} - \log\frac{1}{\rho} \quad (9a)$$

$$ii)\ \log\frac{1}{\theta} > \log\frac{1}{R} \ \text{(out cone)} \quad (9b) \qquad (9)$$

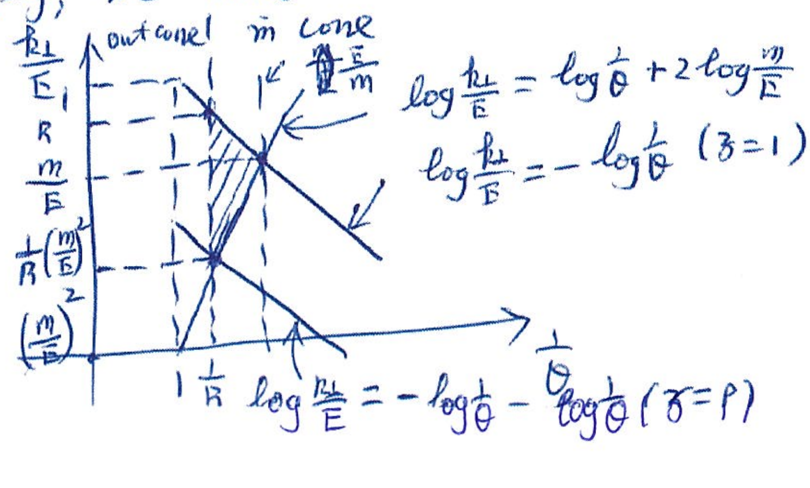
$$\text{and} \quad \log\frac{k_\perp}{E} > \log\frac{1}{\theta} + 2\log\frac{m}{E} \ \text{(mass)} \quad (9c)$$

Here, one can easily get the crossing points of each boundaries

$$-\log\frac{1}{\theta} = \log\frac{1}{\theta} + 2\log\frac{m}{E} \Rightarrow \log\frac{1}{\theta} = -\log\frac{m}{E}$$

$$-\log\frac{1}{\theta} - \log\frac{1}{\rho} = \log\frac{1}{\theta} + 2\log\frac{m}{E} \Rightarrow \log\frac{1}{\theta} = \log\frac{E}{m} + \log\frac{m}{ER}$$
$$= \log\frac{1}{R} \quad (10)$$

Accordingly, we have



(3)

Now let us define

$$\ell \equiv \log\frac{k_\perp}{E}, \qquad \eta \equiv \log\frac{1}{\theta} \qquad (11)$$

In terms of these variables, we have

$$\Sigma^{(1)}_{\text{ungroom}} = -\frac{2\alpha_s C_F}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{E}{m}} d\eta \int_{\eta+2\log\frac{m}{E}}^{-\eta} d\ell$$

$$= -\frac{2\alpha_s C_F}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{E}{m}} d\eta \left(-2\eta - 2\log\frac{m}{E}\right)$$

$$= -\frac{2\alpha_s C_F}{\pi}\left[-\left(\log^2\frac{E}{m} - \log^2\frac{1}{R}\right) - 2\log\frac{RE}{m}\log\frac{m}{E}\right]$$

$$= -\frac{2\alpha_s C_F}{\pi}\left[-\log\frac{ER}{m}\log\frac{E}{mR} + 2\log\frac{ER}{m}\log\frac{E}{m}\right]$$

$$= -\frac{2\alpha_s C_F}{\pi}\left[\log\frac{ER}{m}\left(2\log\frac{E}{m} - \log\frac{E}{mR}\right)\right] \qquad (12)$$

$$= -\frac{2\alpha_s C_F}{\pi}\log^2\frac{ER}{m} = -\frac{\alpha_s C_F}{2\pi}\log^2\frac{1}{\rho}. \quad \square$$

We hence reproduced the ungroomed mass distribution.

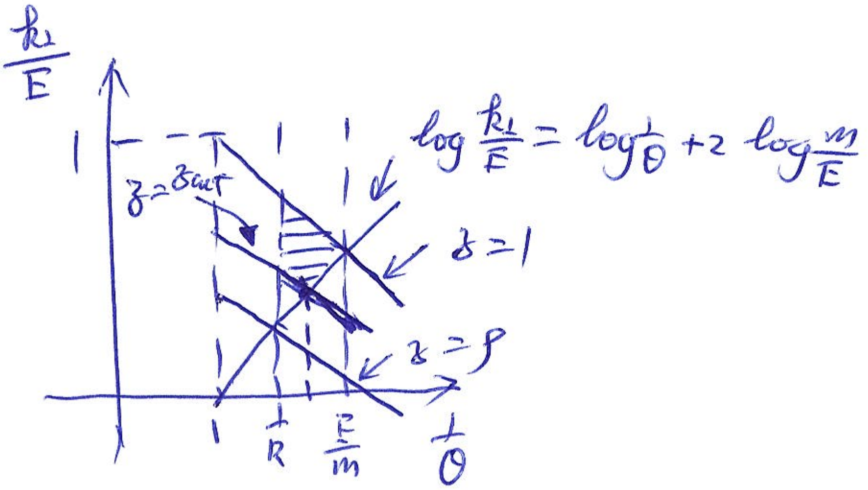
The SD condition now takes the form

$$\log\frac{k_\perp}{E} \geq -(1+\beta)\log\frac{1}{\theta} - \log\frac{1}{z_{cut}} + \beta\log\frac{1}{R} \qquad (13)$$

Let us first take $\beta = 0$ and have

$$\log\frac{k_\perp}{E} \geq -\log\frac{1}{\theta} - \log\frac{1}{z_{cut}} \qquad (14)$$

In this case, only if $\frac{1}{\rho} > \frac{1}{z_{cut}}$ or $\rho < z_{cut}$ $\Sigma^{(1)}$ is modified. In the Lund jet plane, one has



Here we need the crossing point of $z = z_{cut}$ and $\log\frac{k_\perp}{E} = \log\frac{1}{\theta} + 2\log\frac{m}{E}$:

$$\log\frac{k_\perp}{E} = -\log\frac{1}{\theta} - \log\frac{1}{z_{cut}}$$

$$= \log\frac{1}{\theta} + 2\log\frac{m}{E}$$

$$\Rightarrow \log\frac{1}{\theta} = -\log\frac{m}{E} - \frac{1}{2}\log\frac{1}{z_{cut}}$$

$$= -\log\left(\frac{m}{E\, z_{cut}^{1/2}}\right)$$

$$\theta = \frac{m}{E\, z_{cut}^{1/2}}$$

Now for $\beta = 0$, i.e., mMDT, one has

$$-\Sigma^{(1)} = \frac{\alpha_s C_i}{2\pi}\log^2\frac{1}{\rho} - \frac{2\alpha_s C_i}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{E z_{cut}^{1/2}}{m}} d\eta \int_{\eta + \log\frac{m^2}{E^2}}^{-\eta - \log\frac{1}{z_{cut}}} d\ell$$

$$= \frac{\alpha_s C_i}{2\pi}\log^2\frac{1}{\rho} - \frac{2\alpha_s C_i}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{E\delta_{cut}^{1/2}}{m}} d\eta\left[-2\eta + \log\frac{\delta_{cut}E^2}{m^2}\right]$$

$$= \frac{\alpha_s C_i}{2\pi}\log^2\frac{1}{\rho} - \frac{2\alpha_s C_i}{\pi}\Big[-\Big(\log^2\frac{E\delta_{cut}^{1/2}}{m} - \log\frac{1}{R}\Big)$$

$$+ \log\frac{ER\delta_{cut}^{1/2}}{m}\log\frac{\delta_{cut}E^2}{m^2}\Big]$$

$$= \frac{\alpha_s C_i}{2\pi}\log^2\frac{1}{\rho} - \frac{2\alpha_s C_i}{\pi}\Big[\log\frac{ER\delta_{cut}^{1/2}}{m}\Big(-\log\frac{E\delta_{cut}^{1/2}}{mR}$$

$$+ \log\frac{\delta_{cut}E^2}{m^2}\Big]$$

$$= \frac{\alpha_s C_i}{2\pi}\log^2\frac{1}{\rho} - \frac{2\alpha_s C_i}{\pi}\Big[\log\frac{ER\delta_{cut}^{1/2}}{m}\log\frac{\delta_{cut}^{1/2}ER}{m}\Big]$$

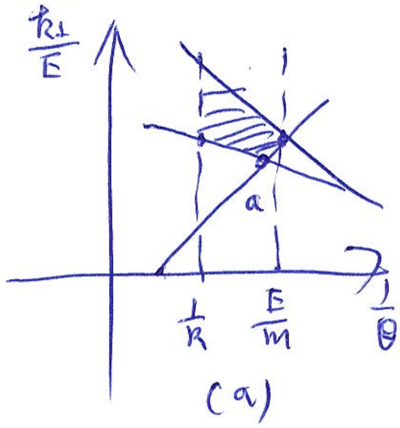
$$= \frac{\alpha_s C_i}{2\pi}\Big[\log^2\frac{1}{\rho} - \log^2\frac{\delta_{cut}}{\rho}\Big]. \quad \text{II}$$

This calculation can be easily generalized to any values of β. Note that at $\theta = R$, the SD condition line cross $\frac{1}{\theta} = \frac{1}{R}$ at a point independent of β, that is, $\log\frac{k_\perp}{E} = -\log\frac{1}{R} - \log\frac{1}{\delta_{cut}}$

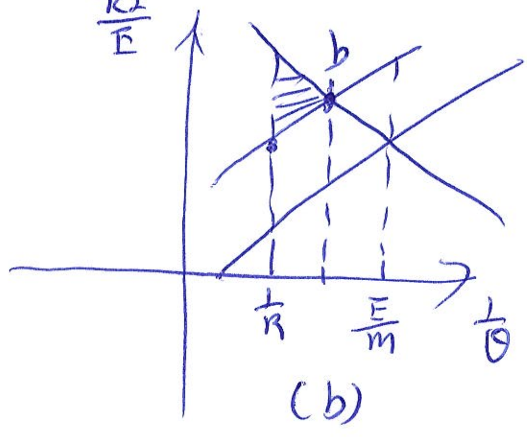
As ~~log as~~ long as $-\log\frac{1}{R} > -\log\frac{1}{R} - \log\frac{1}{\delta_{cut}} > -\log\frac{1}{R} + \log\rho$

$\delta_{cut} > \rho$, one has a different phase space.

we have two cases



(a)



(b)

For (a), we need to know the crossing point a in the figure:

$$-(1+\beta)\log\frac{1}{\theta}-\log\frac{1}{z_{cut}}+\beta\log\frac{1}{R}=\log\frac{1}{\theta}+2\log\frac{m}{E}$$

$$\Rightarrow \log\frac{1}{\theta}=\frac{1}{2+\beta}\left[\beta\log\frac{1}{R}-\log\frac{1}{z_{cut}}-2\log\frac{m}{E}\right]$$

$$=\frac{1}{2+\beta}\left[\beta\log\frac{1}{R}+\log\frac{z_{cut}}{\rho}+\log\frac{1}{R^2}\right]$$

$$=\frac{1}{2+\beta}\left[(2+\beta)\log\frac{1}{R}+\log\frac{z_{cut}}{\rho}\right]$$

$$=\log\frac{1}{R}+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}.$$

Accordingly,

$$(a)=\Sigma^{(1)}_{ungroom}+\frac{2\alpha_s C_i}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{1}{R}+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}}d\eta\int_{\eta+2\log\frac{m}{E}}^{-(1+\beta)\eta-\log\frac{1}{z_{cut}}+\beta\log\frac{1}{R}}d\ell$$

$$=\Sigma^{(1)}_{ungroom}+\frac{2\alpha_s C_i}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{1}{R}+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}}d\eta\left[-(2+\beta)\eta-\log\frac{1}{z_{cut}}+\beta\log\frac{1}{R}-2\log\frac{m}{E}\right]$$

$$=\Sigma^{(1)}_{ungroom}+\frac{2\alpha_s C_i}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{1}{R}+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}}d\eta\left[-(2+\beta)\eta+\log\frac{z_{cut}}{\rho}+2\log\frac{1}{R}+\beta\right.$$

$$=\Sigma^{(1)}_{ungroom}+\frac{2\alpha_s C_i}{\pi}\int_{\log\frac{1}{R}}^{\log\frac{1}{R}+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}}d\eta\left[-(2+\beta)\eta+\log\frac{z_{cut}}{\rho}+(2+\beta)\log\frac{1}{R}\right]$$

$$=\Sigma^{(1)}_{ungroom}+\frac{2\alpha_s C_i}{\pi}\left[-\frac{2+\beta}{2}\,\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}\left(\log\frac{1}{R^2}+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}\right)\right.$$

$$\left.+\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}\left(\log\frac{z_{cut}}{\rho}+(2+\beta)\log\frac{1}{R}\right)\right]$$

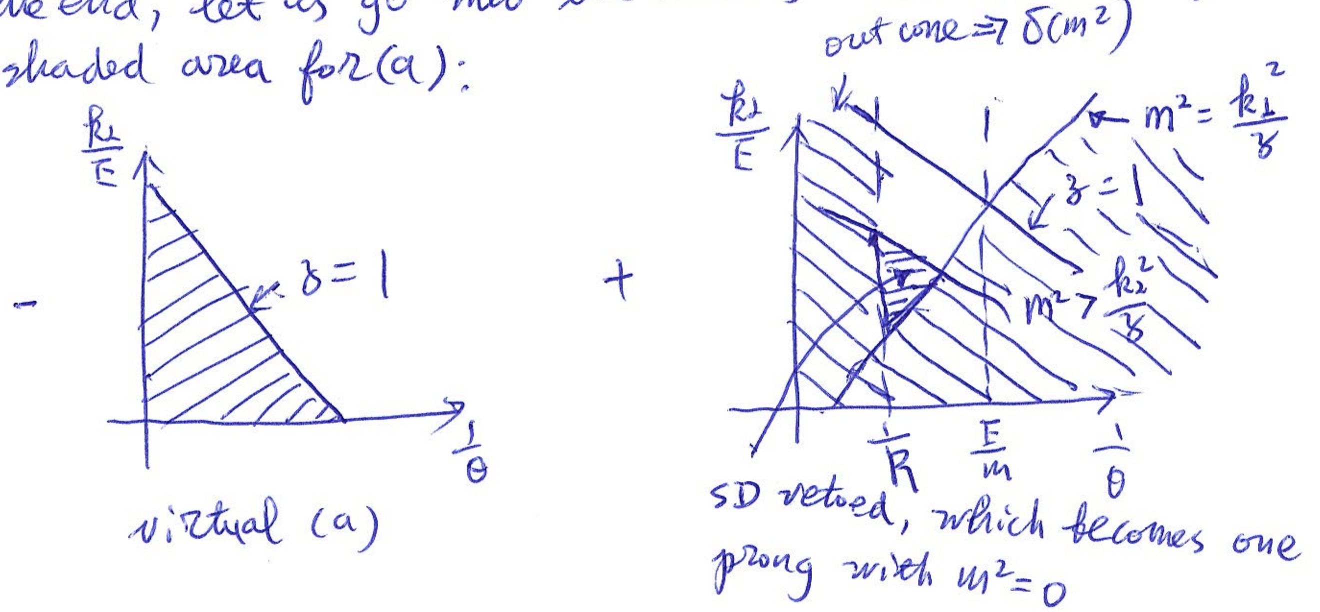
$$=\Sigma^{(1)}_{ungroom}+\frac{2\alpha_s C_i}{\pi}\left[\frac{1}{2+\beta}\log\frac{z_{cut}}{\rho}\right]\left[-\frac{1}{2}\log\frac{z_{cut}}{\rho}-\frac{2+\beta}{2}\log\frac{1}{R}\right.$$

$$\left.+\log\frac{z_{cut}}{\rho}+(2+\beta)\log\frac{1}{R}\right]$$

$$= \Sigma^{(1)}_{\text{mgroom}} + \frac{\alpha_s C_i}{\pi} \frac{1}{2+\beta} \log^2 \frac{z_{cut}}{\rho}$$

$$= -\frac{\alpha_s C_i}{\pi} \left[\frac{1}{2} \log^2 \frac{1}{\rho} - \frac{1}{2+\beta} \log^2 \frac{z_{cut}}{\rho} \right].$$

At the end, let us go into the details about how one gets the shaded area for (a):



$= (a).$ □

Note that in this lecture we still count $\log z_{cut} \sim \log \rho$ as big logs. In principle, one can also get single $\log \rho$ terms correct, which we will not touch on in this lecture.

2. Signal v.s. background jet mass distribution.

At the end of this lecture, let us take a look at the effects of grooming for signal jets. Recall that for a particle X with $m_X \gg \Gamma$, we have

$$\frac{d}{dm^2}\Sigma_X = \frac{1}{2\pi}\frac{2m_X\Gamma(X\to q\bar{q})}{(m^2-m_X^2)^2+m_X^2\Gamma^2}.$$

Now, we are focusing on highly boosted case and clustering $q\bar{q}$ into one fat jet with jet radius R. In this case, we have

$$\Delta R_{q\bar{q}} = \frac{m}{p_T}\frac{1}{\sqrt{z(1-z)}} \Longleftrightarrow \left(\frac{\Delta R_{q\bar{q}}}{R}\right)^2 = \frac{\rho}{z(1-z)}.$$

For simplicity we assume $\Gamma(X\to q\bar{q})$ is independent of z, the fraction of the transverse momentum of X carried by q. The in-cone condition is hence given by:

$$1 > \frac{\rho}{z(1-z)}.$$

Accordingly, we have

$$\frac{d}{dm^2}\Sigma_{X\to J} = \frac{1}{2\pi}\frac{2m_X\Gamma(X\to q\bar{q})}{(m^2-m_X^2)^2+m_X^2\Gamma^2}\int_0^1 dz\,\Theta\left(1-\frac{\rho}{z(1-z)}\right)$$

Since $\rho \ll 1$, we have

$$\frac{d}{dm^2}\Sigma_{X\to J} = \frac{1}{\pi}\frac{m_X\Gamma(X\to q\bar{q})}{(m^2-m_X^2)^2+m_X^2\Gamma^2}(1-2\rho).$$

Near the threshold $m\sim m_X$, one has

$$\frac{d}{dm^2}\Sigma_{X\to J} = \frac{1}{\pi}\frac{1}{m_X\Gamma}\left(\frac{\Gamma(X\to q\bar{q})}{\Gamma}\right) = \frac{1}{\pi}\frac{\Gamma(X\to q\bar{q})}{\Gamma}\frac{1}{m_X^2}\left(\frac{m_X}{\Gamma}\right)$$

$$\sim \frac{10}{m_X^2} \quad \text{for } W/Z$$

with $m_X\sim 100$ GeV, $\Gamma\sim 2$ GeV, $\frac{\Gamma(X\to q\bar{q})}{\Gamma}\sim(60-70)\%\sim 1$.

In comparison with QCD jets:

$$\frac{d}{dm^2}\Sigma_{QCD} = \frac{\alpha_s C_i}{\pi}\frac{1}{m^2}\log\frac{1}{\rho}.$$

For $m_X \sim 100\,\text{GeV}$ and $p_T \sim 1\,\text{TeV}$, we have

$$\frac{d}{dm^2}\Sigma_{QCD} \sim \frac{0.1}{m^2}$$

This, however, will be, in most cases, compensated by the big σ's for QCD jets than X-jets in the final cross-section. For example, we could have

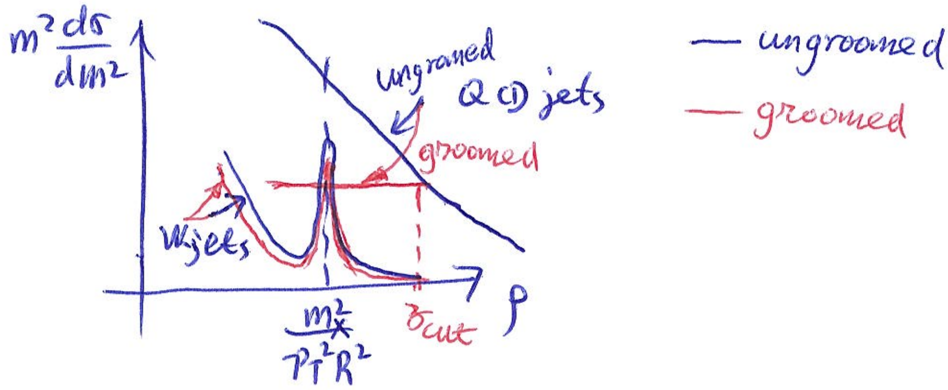


Fig. ungroomed jet mass $\rightarrow$ groomed jet mass

Now let us calculate the SD mass distribution of X-jets:

$$\frac{d}{dm^2}\Sigma_X^{SD} = \frac{1}{\pi}\frac{m_X\Gamma(X\to q\bar{q})}{(m^2-m_X^2)^2+m_X^2\Gamma^2}\, 2\int_0^{\frac{1}{2}} dz\, \Theta(z-\rho)$$

$$\times\,\Theta\left(z - z_{cut}\left(\frac{\rho}{z}\right)^{\frac{\beta}{2}}\right) \text{ for } \rho \ll 1.$$

$$= \frac{1}{\pi}\frac{m_X\Gamma(X\to q\bar{q})}{(m^2-m_X^2)^2+m_X^2\Gamma^2}\left[1-\max\left(\rho,\, \rho\left(\frac{z_{cut}}{\rho}\right)^{\frac{2}{2+\beta}}\right)\right]$$

we have $\rho \sim 10^{-2}$. Now let us take mMDT ($\beta=0$) as an example

$$\frac{d}{dm^2}\Sigma_{QCD}^{SD(\beta=0)} = \frac{\alpha_s C_i}{\pi}\frac{1}{m^2}\log\frac{1}{z_{cut}}$$

For $z_{cut}=0.1$, it decreases by a factor of $\frac{\log\frac{1}{z_{cut}}}{\log\frac{1}{\rho}} = 0.5$, which the cumulative distribution only decreases about 10%.