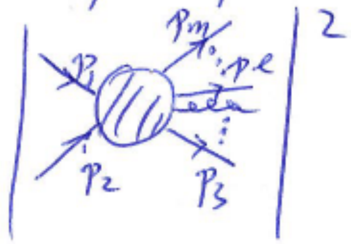
Lecture 2

Here, I would like to sketch a derivation of the formula

$$\sigma_{m+1}(p_1, \cdots, p_\ell, \cdots, p_m, k) \equiv$$ 

(1)

$$\simeq \frac{4g^2 C_\ell}{|\vec{k}_{\perp\ell}|^2}\, \sigma_m(p_1, \cdots, p_\ell + k, \cdots, p_m)$$

when k^μ is soft and collinear to p_ℓ. This is a gauge invariant statement and we use the light-cone gauge $\bar{n}\cdot A = 0$, where we define

$$n^\mu \equiv \frac{p_\ell^\mu}{E_\ell} = \left(1, \frac{\vec{p}_\ell}{|\vec{p}_\ell|}\right) \equiv (1, \hat{p}_\ell)$$
$$\bar{n}^\mu \equiv (1, -\hat{p}_\ell). \qquad (2)$$

It is easy to see that $n^2 = 0 = \bar{n}^2$ and $n\cdot\bar{n} = 2$.

Let us write $k^\mu = (\omega, \vec{k})$ and soft means

$$\omega \ll E_\ell \qquad (3)$$

Collinear means

$$\bar{n}\cdot k \gg |\vec{k}_{\perp\ell}| \gg n\cdot k. \qquad (4)$$

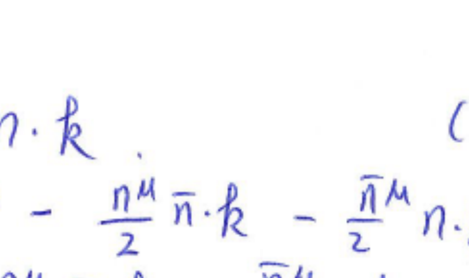
In terms of n & n^μ, $k^\mu_{\perp\ell} \equiv k^\mu - \frac{n^\mu}{2}\bar{n}\cdot k - \frac{\bar{n}^\mu}{2} n\cdot k$ (5)

or equivalently $k^\mu = \frac{n^\mu}{2}\bar{n}\cdot k + \frac{\bar{n}^\mu}{2} n\cdot k + k^\mu_{\perp\ell}$ (6

we introduce some scaling parameter $\lambda \equiv \frac{|\vec{k}_{\perp\ell}|}{\bar{n}\cdot k} \ll 1$ (7

Accordingly, we have $\bar{n}\cdot k \sim 1$, $\vec{k}_{\perp\ell} \sim \lambda$, $n\cdot k \sim \lambda^2$ (8

We need to show that using the above scaling, in the light-con gauge, we have

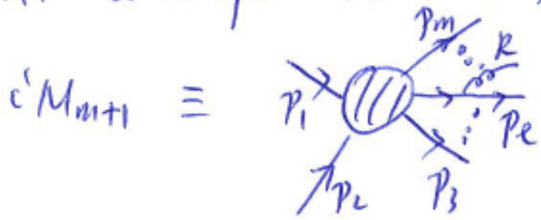
$$\sigma_{m+1}(p_1, \cdots, p_\ell, \cdots, p_m, k) \simeq$$ 

$\sim \lambda^{-}$

(9

and other diagrams are suppressed in λ.

At the amplitude level, one has

$$i\mathcal{M}_{m+1} \equiv$$ 

$$= \bar{u}(p_\ell)\, i g \not\epsilon_\lambda^*(k) \frac{i(\not p_\ell + \not k)}{(p_\ell + k)^2} \underbrace{\sum}_{m}$$

the rest part

$$\simeq \bar{u}(p_\ell)\, i g \not\epsilon_\lambda^*(k) \frac{i(\not p_\ell + \omega \not n)}{2 p_\ell \cdot k} \Sigma_m$$

$$= -\frac{\bar{u}(p_\ell)\, g \not\epsilon_\lambda^*(k)\, u(p_\ell + \omega n)}{2 p_\ell \cdot k}\, i\mathcal{M}(p_1, \cdots, p_\ell + \omega n, \cdots, p_m) \quad (1$$

Since we are focusing the soft limit, one can show that

$$i\mathcal{M}_{m+1} \simeq -g \frac{2 p_\ell \cdot \epsilon_\lambda^*(k)}{2 p_\ell \cdot k}\, i\mathcal{M}(p_1, \cdots, p_\ell + \omega n, \cdots, p_m)$$

$$= -g \frac{n \cdot \epsilon_\lambda^*(k)}{n \cdot k}\, i\mathcal{M}(p_1, \cdots, p_\ell + \omega n, \cdots, p_m) \quad (11$$

From this, we have, including a casimir factor $C_\ell = \begin{cases} C_A & \text{for glu} \\ C_F & \text{for qua} \end{cases}$

$$\sigma_{m+1} \simeq g^2 C_\ell \sum_\lambda \frac{|n \cdot \epsilon_\lambda(k)|^2}{(n \cdot k)^2} \sigma_m(p_1, \cdots, p_\ell + \omega n, \cdots, p \quad (12$$

Using $$P^{\mu\nu} \equiv \sum_\lambda \epsilon_\lambda^\mu(k)\, \epsilon_\lambda^{*\nu}(k) = -g^{\mu\nu} + \frac{\bar{n}^\mu k^\nu + \bar{n}^\nu k^\mu}{\bar{n} \cdot k} \quad (13$$

one has

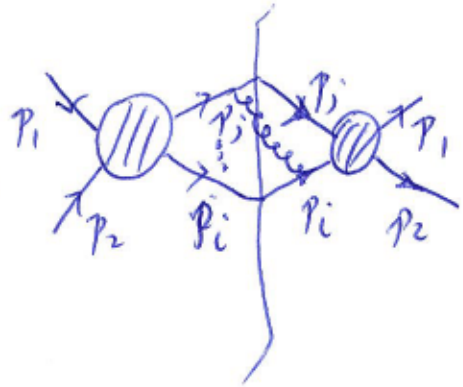
$$\sigma_{m+1} \simeq g^2 C_\ell \frac{1}{(n \cdot k)^2} \frac{4 n \cdot k}{\bar{n} \cdot k} \sigma_m(p_1, \cdots, p_\ell + \omega n, \cdots, p_m)$$

$$= 4 g^2 C_F \frac{1}{|\vec{k}_{\perp\ell}|^2} \sigma_m(p_1, \cdots, p_\ell + k, \cdots p_m)$$

Here, we write $p_\ell + k \simeq p_\ell + \omega n$ by keeping only $\bar{n} \cdot k$ component.

Similarly, we define $n_i^\mu \equiv \frac{p_i^\mu}{E_i}$ and $n_i \cdot n_j \sim 1$ if $i \neq j$

Now, we need to make sure other terms are suppressed. In general, the color factor is complicated but we only need to see how the contribution scales in λ. In general one has

$\sigma^{ij}_{m+1} \equiv$ 

$$\propto \frac{1}{n_i\cdot k\, n_j\cdot k}\, n_i^\mu n_j^\nu P_{\mu\nu}(k) \qquad (15)$$

Inserting (5) into (15), one has, if $n_i \neq n \neq n_j$,

$$\sigma^{ij}_{m+1} \propto \frac{1}{n_i\cdot k\, n_j\cdot k}\left(-n_i\cdot n_j + \frac{n_i\cdot\bar{n}\, n_j\cdot n + n_j\cdot\bar{n}\, n_i\cdot n}{2} + \mathcal{O}(\lambda)\right) \quad (1$$

Here $n_i\cdot k \approx \frac{1}{2} n_i\cdot n\ \bar{n}\cdot k \sim 1 \sim n_j\cdot k$ $\qquad (18)$

Therefore $\sigma^{ij}_{m+1} \sim \lambda$. $\qquad (19)$

This is suppressed compared to σ_{m+1} in (14), which scales as λ^{-2}.

Similarly, if $n_i \neq n$ but $n_j = n$, one has

$$\sigma^{i}_{m+1} \propto \frac{1}{n_i\cdot k\, n\cdot k}\left(-n_i\cdot n + \frac{n_i\cdot\bar{n}\ n\cdot k + 2 n_i\cdot k}{\bar{n}\cdot k}\right)$$

$$= \frac{1}{n_i\cdot k\, n\cdot k}\left(-n_i\cdot n + \frac{n_i\cdot\bar{n}\ n\cdot k + n_i\cdot n\ \bar{n}\cdot k - 2\vec{n}_i}{\bar{n}\cdot k}\right.$$

$$\left. + \frac{n_i\cdot\bar{n}\ n\cdot k}{\bar{n}\cdot k}\right) = \frac{1}{n_i\cdot k\, n\cdot k}\ \frac{2 n_i\cdot\bar{n}\ n\cdot k - 2\vec{n}_i\cdot\vec{k}_\perp}{\bar{n}\cdot k}$$

$$\sim \lambda^{-1} \qquad (20)$$

Therefore, (9) gives the most important contribution $\sim \lambda^{-2}$ while all the other terms are power suppressed. Here, one may wonder whether the soft gluon is emitted in the blob, that is, not on external legs? The answer is No because one does not expect any IR divergence in such diagrams.

According to Eq. (1), one has

$$\sigma^{r}_{m+1}(p_1, \cdots, p_\ell, \cdots, p_m) = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2)\theta(k^0)$$

$$\times\, \sigma_{m+1}(p_1, \cdots, p_\ell, \cdots, p_m, k)$$

$$= \frac{1}{2(2\pi)^3}\int \frac{d\vec{k}^{2}_{\perp\ell}}{|\vec{k}_{\perp\ell}|} \quad \frac{d\bar{n}\cdot k}{\bar{n}\cdot k} \qquad \sigma_m \; 4g^2 C_\ell$$

$$= \frac{\alpha_s C_\ell}{\pi^2} \int \frac{d\bar{n}\cdot k}{\bar{n}\cdot k}\int \frac{d^2\vec{k}_{\perp\ell}}{|\vec{k}_{\perp\ell}|^2}\; \sigma_m\Big(p_1, \cdots, \overbrace{p_\ell + \tfrac{n}{2}\bar{n}\cdot k}^{\equiv p}, \cdots, p_m\Big)$$

$$\equiv \frac{\alpha_s C_\ell}{\pi^2}\int \frac{dz}{z}\int \frac{d^2 k_{\perp\ell}}{|\vec{k}_{\perp\ell}|^2}\,\sigma_m(p_1, \cdots, p, \cdots, p_m) \tag{21}$$

with $z \equiv \dfrac{\bar{n}\cdot k}{n\cdot p}$. (22)

In leading logarithmic approximation, we can set the limits of integration as follows:

$$\sigma^{r}_{m+1} = \frac{2\alpha_s C_\ell}{\pi}\int_0^1 \frac{dz}{z}\int_0^Q \frac{dk_{\perp\ell}}{k_{\perp\ell}}\,\sigma_m \tag{23}$$

It contains both soft and collinear divergences.