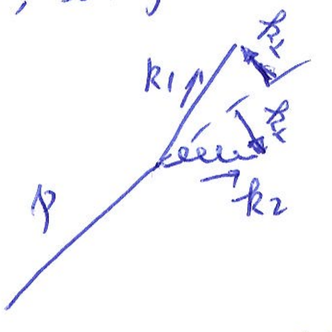
Lectures 4&5: Jet mass

Jets are massive with mass
$$m^2 = \Big(\sum_{i\in \text{jet}} k_i\Big)^2. \qquad (1)$$

Here we focus on QCD jets, which are initiated by a single hard parton. In the collinear limit, one has
$$p_T \Delta R \simeq E\sqrt{\Delta\Omega} \simeq E\,\Delta\theta \quad \text{with } \theta \text{ the opening angle} \qquad (2)$$
we will work in (E,θ) coordinates but our result is simply given by the replacement: $E \to p_T$ for hadron collisions.

We only calculate the leading logarithmic (LL) results. For such a calculation, we only need the formula for soft gluon emission:
$$\frac{\sigma_r^{(1)}}{\sigma^{(0)}} = \frac{2\alpha_s C_F}{\pi}\int_0^E \frac{dk_\perp}{k_\perp}\int_0^1 \frac{dz}{z} \qquad (3)$$
Here, "r" ~~denotes~~ means it is from real gluon emission and "⊥" denotes the transverse direction with respect to the momentum of the hard parton, that is, the jet momentum (in the soft limit):



Here we ignore hadronization and take the two daughters as massless on-shell particles. In the collinear limit, one has
$$m^2 = 2k_1 k_2 = 2k_\perp^2 + \frac{1-z}{z}k_\perp^2 + \frac{z}{1-z}k_\perp^2$$
$$= \frac{k_\perp^2}{(1-z)z} \longrightarrow \frac{k_\perp^2}{z} \qquad (4)$$

Here both k_1 & k_2 can be written in the form

$$k_1^\mu = zE\, n^\mu + k_\perp^\mu + \frac{|k_\perp|^2}{4zE}\bar{n}^\mu , \quad k_2^\mu = (1-z)E\, n^\mu - k_\perp^\mu + \frac{|k_\perp|^2}{4(1-z)E}\bar{n}^\mu \qquad (5)$$

with $$n^\mu = \left(1, \frac{\vec{p}_J}{|\vec{p}_J|}\right) \text{ and } \bar{n}^\mu = \left(1, -\frac{\vec{p}_J}{|\vec{p}_J|}\right). \qquad (6)$$

and $$k_\perp^\mu = (0, \vec{k}_\perp, 0). \qquad (7)$$

Now, we work in both collinear and soft limit and have

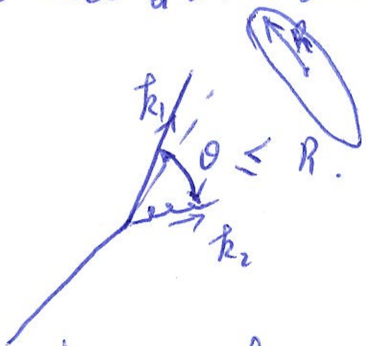
$$m^2 \simeq \frac{k_\perp^2}{z} \qquad (8).$$

4.1 Jet mass cross section at one-loop

In soft and collinear limit, we have

$$\frac{1}{\sigma^{(0)}}\frac{d\sigma^{(1)}}{dm^2} = \frac{2\alpha_s C_F}{\pi}\int_0^E \frac{dk_\perp}{k_\perp}\int_0^1 \frac{dz}{z}\,\delta\!\left(m^2 - \frac{k_\perp^2}{z}\right)\Theta_{in}(k_\perp, z) \qquad (9)$$

Here the in-cone condition Θ_{in} is to constrain the phase-space of $k_\perp$ and z in the jet cone with jet radius R:



which gives $$\theta = \frac{k_\perp}{zE} + \frac{k_\perp}{(1-z)E} = \frac{k_\perp}{z(1-z)E} \qquad (10)$$

$$\simeq \frac{1}{z}\frac{k_\perp}{E} < R.$$

As a result, we have

$$\frac{1}{\sigma^{(0)}}\frac{d\sigma^{(1)}}{dm^2} = \frac{2\alpha_s C_F}{\pi}\int_0^E \frac{dk_\perp}{k_\perp}\,\frac{1}{z}\,\frac{1}{\frac{k_\perp^2}{z^2}}\,\theta\!\left(R - \frac{1}{z}\frac{k_\perp}{E}\right) \qquad (11)$$

with $$1 \geq z = \frac{k_\perp^2}{m^2} > 0. \qquad (12)$$

Accordingly, we have

$$\frac{1}{\sigma^{(0)}}\frac{d\sigma^{(1)}}{dm^2} = \frac{2\alpha_s C_F}{\pi}\frac{1}{m^2}\int_0^m \frac{dk_\perp}{k_\perp}\,\theta\left(R - \frac{m^2}{E k_\perp}\right)$$

$$= \frac{2\alpha_s C_F}{\pi}\frac{1}{m^2}\int_{\frac{m^2}{RE}}^{m} \frac{dk_\perp}{k_\perp} = \frac{2\alpha_s C_F}{\pi}\frac{1}{m^2}\log\left(\frac{RE}{m}\right)$$

$$\times\,\theta\left(RE - m\right) \tag{13}$$

That is, this log ~~is~~ becomes relevant and large at high energies at fixed R.

4.2 The cumulative distribution at one loop

The cumulative distribution is defined as the normalized cross section for measuring a value of the jet mass below a certain m^2:

$$\Sigma(m^2) \equiv \frac{1}{\sigma^{(0)}} \int_0^{m^2} dm'^2 \frac{d\sigma}{dm'^2} = 1 + \alpha_s \Sigma^{(1)} + \mathcal{O}(\alpha_s^2). \quad (14)$$

Again, we focus on LL result. The difference between $\Sigma(m^2)$ & $\frac{d\sigma}{dm^2}$ is that the virtual diagrams also contribute here in the limit $m' \to 0$ in (4), which cancel IR divergence in the real contribution in this limit.

For LL result, we only need to figure out the ~~phas~~ phase space for real and virtual contributions. The LL phase-space for the virtual contribution is the full space

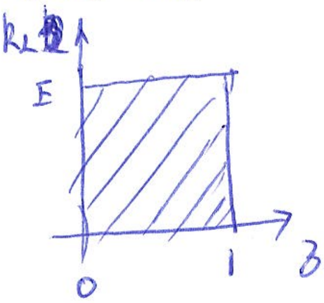


Fig 1. phase-space for the virtual contribution.

Now, there are two contraints for the real gluons:

$$\begin{cases} m^2 > \dfrac{k_\perp^2}{z} & \text{mass} \\ R < \dfrac{1}{z}\dfrac{k_\perp}{E} & \text{out cone} \end{cases} \quad (15)$$

which gives

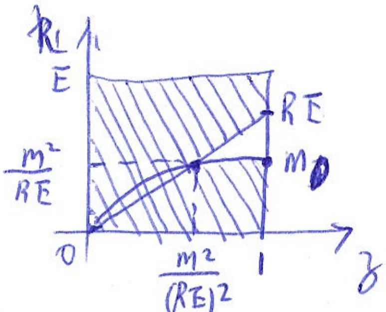


Fig 2. phase-space for the real gluon.

Here, the crossing point is given by $\frac{k_\perp}{RE} = \frac{k_\perp^2}{m^2} \iff \begin{cases} k_\perp = \frac{m^2}{RE} \\ z = \frac{m^2}{(RE)^2} \end{cases}$ (16)

By combining Figs. 1 & 2, we obtain the phase-space for $\Sigma^{(1)}$

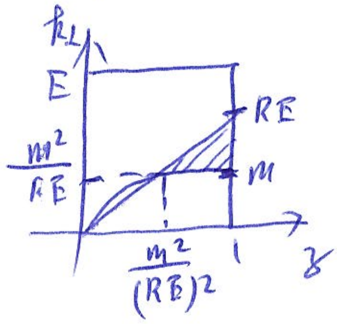


Fig3. phase-space for $\alpha_s\Sigma^{(1)}$.

Accordingly, one has

$$\alpha_s\Sigma^{(1)} = -\frac{2\alpha_s C_F}{\pi}\int_{\frac{m^2}{(RE)^2}}^{1}\frac{dz}{z}\int_{z^{\frac{1}{2}}m}^{zRE}\frac{dk_\perp}{k_\perp}$$

$$= -\frac{2\alpha_s C_F}{\pi}\int_{\frac{m^2}{(RE)^2}}^{1}\frac{dz}{z}\log\left(\frac{z^{\frac{1}{2}}RE}{m}\right)$$

$$= -\frac{2\alpha_s C_F}{\pi}\left[-\frac{1}{4}\log^2\left(\frac{m^2}{(RE)^2}\right)-\log\left(\frac{m^2}{(RE)^2}\right)\log\left(\frac{RE}{m}\right)\right.$$

$$= -\frac{\alpha_s C_F}{2\pi}\log^2\left(\frac{(RE)^2}{m^2}\right) \qquad (17)$$

In the literature, one usually defines $\rho \equiv \frac{4m^2}{R^2E^2}$ and write

$$\alpha_s\Sigma^{(1)} = -\frac{\alpha_s C_F}{2\pi}\log^2\left(\frac{1}{\rho}\right) \qquad (18)$$

At the end, we comment on the relevance of this ~~cell~~ resul to pp collisons. In pp collisions, the jet radius R is defi in the y-φ space. As we discussed in Lecture 3, we hav

$$\Delta\Omega = \frac{R^2}{\cosh^2\eta}$$

Here, in the above derivation, we took $R \sim \Delta\Omega^{\frac{1}{2}}$. Therefore, for pp collisions, we have (see Eq(2))

$$\rho = \frac{4m^2}{R^2\frac{E^2}{\cosh^2\eta}} = \frac{4m^2}{R^2p_T^2}.$$

4.3 Leading Log Resummation

In perturbative calculations, one often finds large logarithms. In this case, a sensible theoretical prediction entails summing over such logarithmic terms up to all orders in α_s. This procedure is called <u>resummation</u>. In order to do power counting for logarithmic resummation, one usually takes $\alpha_s L \sim 1$ with L standing for the logarithm in the problem. Many observables like Σ schematically take an exponential form:

$$\sigma^{resum}(\rho) \equiv \sigma_0 g_0 \exp\Big\{ \underbrace{L g_1 \alpha_s L}_{LL} + \underbrace{g_2 \alpha_s L}_{NLL} + \underbrace{\alpha_s g_3 \alpha_s L}_{NNLL} + \cdots + \alpha_s^i g_{i+1} \alpha_s L + \cdots \Big\} . \quad (19)$$

In order to make predictions for a broader range of ρ, the above resummed result needs to be matched to the fixed order calculations:

$$\sigma(\rho) = \sigma^{fixed}(\rho) + \sigma^{resum}(\rho) - (\text{double counting}) \quad (20)$$

To be more specific, $\sigma(\rho)$ can be taken as σ^{resum} with g_0 matched to fixed order result. For example, in order to get $\sigma(\rho)$:

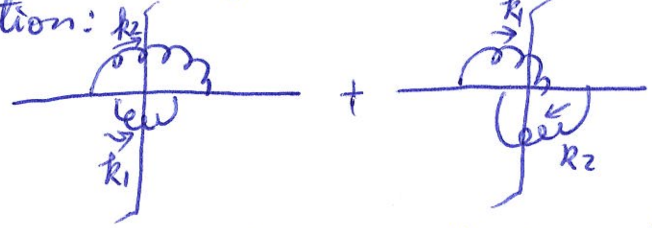
at <u>LL</u> and <u>NLL</u>, $\sigma_0 g_0$ needs only to be matched to tree-level calculation

at NNLL, $\sigma_0 g_0$ needs to be matched to one-loop resul

Here, to illustrate the effect of resummation, we carry out the LL resummation.

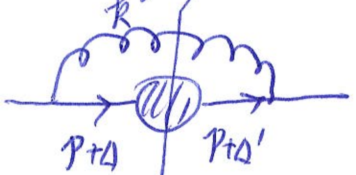
We use anti-k_T algorithm in which the jet is a perfect cone in ~~the~~ soft limit, All soft particles are combined with the hard parton,

Let us start with two ~~gluon~~ real gluon emission. Here, we have the contribution:

$\Sigma_r^{(2)} =$  $+ (k_1 \leftrightarrow k_2)$ (20)

Recall that we have $\sum_\lambda n\cdot\epsilon_\lambda(k)\, n\cdot\epsilon^*_\lambda(k) = \frac{4\, n\cdot k}{\bar n\cdot k}$ (21)

we need to generalize  $= -g\,\frac{n\cdot\epsilon^*_\lambda(k)}{n\cdot k}$ (22)

to  $= 4g^2\,\frac{n\cdot k}{\bar n\cdot k}\,\frac{1}{n\cdot(k+\Delta)}\,\frac{1}{n\cdot(k+\Delta')}$ (23)

which Δ, Δ' some soft momenta.

We can first work out the color factor:

(a): C_F^2 (b): $T^aT^bT^aT^b = i f^{abc}T^c\,\frac{i}{2} f^{abd}T^d + C_F^2$

$$= C_F\left(C_F - \frac{N_c}{2}\right) = -\frac{C_F}{2N_c} \quad (24)$$

using (23), one can easily see that

$$(a) = \frac{4g^2 C_F}{|\vec{k}_{\perp,1}|^2}\;\frac{4g^2 C_F\, n\cdot k_2}{\bar n\cdot k_2}\;\frac{1}{[n\cdot(k_1+k_2)]^2} \quad (25)$$

The collinear ~~region~~ logarithmic region is given by $n\cdot k_1 \ll n\cdot k_2$ (26)

In this case, we have $(a) = \frac{4g^2C_F}{|\vec{k}_{\perp,1}|^2}\,\frac{4g^2C_F}{|\vec{k}_{\perp,2}|^2}$ (27)

In LL approximation, we ~~put~~ relax the condition (26) and take

$$n\cdot k_1 \lesssim n\cdot k_2 \quad (26')$$

$$(b) = -\frac{1}{2N_c}C_F\left(4g^2\frac{n\cdot k_1}{\bar n\cdot k_1}\right)\left(\frac{4g^2\, n\cdot k_2}{\bar n\cdot k_2}\right)\frac{1}{n\cdot k_2\; n\cdot(k_1+k_2)\; n\cdot k_1\; n\cdot(k_1+k_2)}$$

which does not contribute the LL since ~~both~~ neither $n\cdot k_2 \gg n\cdot k_1$ nor $n\cdot k_1 \gg n\cdot k_2$ could ~~not~~ give the collinear singularities we need.

We shall not evaluate explicitly the corresponding virtual correcti

The virtual correction is simply given by probability conservation.

Since $n\cdot k_2 \gg n\cdot k_1$, one has

$$m^2 = \sum_i 2E\, n\cdot k_i \approx 2E\, n\cdot k_2 \qquad (28)$$

Given the constaint in (26'), the phase space for k_1 is given by

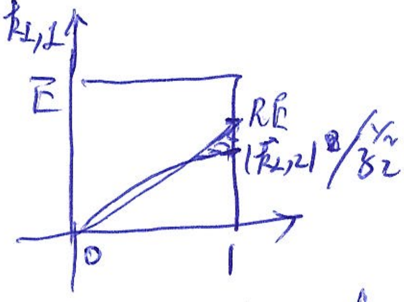


Fig 4 phase-space for k_1

The phase space for k_2 is given by Fig. 3.

Accordingly, at LL

$$\Sigma_r^{(2)} = \left(-\frac{2\alpha_s C_F}{\pi}\right)\int_{\frac{m^2}{(RE)^2}}^{1}\frac{dz_2}{z_2}\int_{z_2^{1/2} m}^{z_2 RE}\frac{dk_{\perp,2}}{k_{\perp,2}}\left(-\frac{\alpha_s C_F}{2\pi}\log^2\overset{\equiv\frac{1}{\xi}}{\left(\frac{z_2 R^2E^2}{|k_{\perp,2}|^2}\right)}\right]$$

$$= \left(-\frac{2\alpha_s C_F}{\pi}\right)\int_{\frac{m^2}{(RE)^2}}^{1}\frac{dz_2}{z_2}\int_{\frac{m^2}{(RE)^2}}^{z_2}\frac{d\xi}{\xi}\left(-\frac{\alpha_s C_F}{2\pi}\log^2\frac{1}{\xi}\right]$$

$$= \left(-\frac{2\alpha_s C_F}{\pi}\right)\int_{\rho}^{1}\frac{dz_2}{z_2}\int_{\rho}^{z_2} d\log\xi\left[-\frac{\alpha_s C_F}{2\pi}\log^2\frac{1}{\xi}\right)$$

$$= -\frac{2\alpha_s C_F}{\pi}\int_{\rho}^{1}\frac{dz_2}{z_2}\left(-\frac{\alpha_s C_F}{2\pi}\right)\left[\frac{1}{3}\log^3 z_2 - \frac{1}{3}\log^3\rho\right]$$

$$= \left(-\frac{2\alpha_s C_F}{\pi}\right)\left(-\frac{\alpha_s C_F}{2\pi}\right)\left[-\frac{1}{4}\frac{1}{3}\log^4\rho + \frac{1}{3}\log^4\rho\right]$$

$$= \frac{1}{2!}\left[-\frac{\alpha_s C_F}{2\pi}\right]^2\log^4\rho \qquad (29).$$

The above calculation can be generalized to arbitrary number soft gluon emission. This gives us the final resummed result

$$\Sigma_{LL} = e^{\underbrace{-\frac{\alpha_s C_F}{2\pi}\log^2\frac{1}{\rho}}_{\text{Sudakov exponent}}} \qquad (30)$$

Accordingly, one has

$$\frac{1}{\sigma^{(0)}}\frac{d\sigma}{dm^2} = \frac{d}{dm^2}\Sigma = \frac{\alpha_s C_F}{\pi}\frac{1}{m^2}\log\frac{1}{\rho}\, e^{-\frac{\alpha_s C_F}{2\pi}\log^2\frac{1}{\rho}} \qquad (31)$$