

# A Theory of Three-Dimensional Parachute Dynamic Stability

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The three-dimensional motion of a freely descending parachute is studied with a five-degree-of-freedom analysis (the roll motion is neglected). The equations of motion are nondimensionalized and the resulting parameters discussed. Exact expressions are given for the longitudinal and lateral small-disturbance stability of the familiar gliding motion of parachutes. The breakdown of these small-disturbance expressions is illustrated by exact large-disturbance studies. A large longitudinal disturbance of most parachutes will result in a large pitching motion, whereas a large lateral (out of the glide plane) disturbance will usually cause a large angle vertical coning motion. Exact algebraic expressions are given for the coning mode, which is a stable rotation, and a small amount of available coning data is included for comparison. Some parametric computer studies of the various motions are also shown.

## Nomenclature

$A$	= canopy nominal area
$B_m, B_i$	= dimensionless canopy inertia, see Eqs. (25) and (26)
$C_N, C_T$	= canopy normal and tangent force coefficients, Eq. (7)
$C_{N\alpha}, C_{T\alpha}$	= force coefficient slopes at the glide point, Fig. 2
$D_0$	= canopy nominal diam
$E_i, C_i, K_i$	= consts in the small-disturbance analysis
$g$	= acceleration of gravity
$I, i$	= system moment of inertia, Eq. (6)
$L$	= length between canopy and load centroids, Fig. 1
$m, k$	= system mass, Eq. (6)
$M, N$	= aerodynamic moment components
$P, p, Q, q$	= angular velocity components, Fig. 1 and Eq. (4)
$r$	= dimensionless mass ratio, Eq. (6)
$t, t^*$	= time, Eq. (4)
$T$	= period of oscillation
$U, u, V, v, W, w$	= velocity components, Fig. 1 and Eq. (4)
$x, y, z$	= Cartesian coordinates
$X, Y, Z$	= aerodynamic force components
$\alpha$	= canopy angle of attack
$\beta, \gamma$	= coning angles, Fig. 10
$\lambda$	= characteristic root in small-disturbance theory
$\psi, \theta, \phi$	= Euler angles between earth and body axes
$\rho$	= fluid density

## Subscripts

$c$	= canopy
$ch$	= canopy hydrodynamic
$1$	= earth axes
$0$	= at the glide point $\alpha = \alpha_0$
$p$	= payload

## Superscripts

$( )^*$	= dimensionless quantity
$( )'$	= small-disturbance variable
$( \dot{\phantom{a}} )$	= derivative with respect to $t^*$

## Introduction

THE parachute is a well-known method of achieving aerodynamic retardation, and many different types of parachutes have been designed and tested.<sup>1</sup> Some of the engineering properties are now available, such as the static aerodynamic coefficients<sup>2</sup> and opening shock and strength data.<sup>3</sup> However, the reasons for the various dynamic instabilities of a parachute have not yet been fully explained. One problem is that dynamic stability testing of parachutes is difficult, and usually gives crude results.<sup>4</sup> Also, to theorize on parachute instability requires a nonlinear three-dimensional analysis with several assumptions that are difficult to justify.

The general motion of a parachute can be separated into four effects: 1) steady vertical descent, 2) steady gliding, 3) large angle pitching oscillation, and 4) large angle coning motion. The first type of motion, vertical descent, was analyzed by linear theory in early work by Henn<sup>5</sup> and Brown.<sup>6</sup> These papers contained certain errors in their equations of motion, as discussed by Lester,<sup>7</sup> who gave a careful derivation of the equations but did not attempt any solutions. A recent study by Heinrich and Rust<sup>8</sup> gives a conservative criterion for linearized dynamic stability of vertical plane motion.

The second motion, steady gliding, has been discussed under static conditions by Heinrich.<sup>9</sup> The present paper apparently gives the first criterion for longitudinal and lateral dynamic stability of a steady gliding motion.

The third motion, pitching oscillation, has been analyzed in computer solutions by Ludwig and Heins,<sup>10-12</sup> with application to personnel guide surface parachutes. No general criteria were given by these authors for stability.

The final motion, coning, has been noted in tests,<sup>9,13</sup> but the present paper is apparently the first to analyze coning theoretically.

## Equations of Motion

### Simplifying Assumptions

To reduce the problem to one which will yield numerical results, the following assumptions have been made:

- 1) The system consists of a symmetric parachute rigidly connected to a neutral payload.
- 2) The aerodynamic force and hydrodynamic inertia of the payload are negligible.
- 3) There are five degrees of freedom, with the roll of the parachute about its axis of symmetry being ignored.
- 4) The hydrodynamic mass and moment of inertia tensors of the canopy are approximated by single scalar values.

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5) The aerodynamic forces are assumed quasistatic as based on the instantaneous angle of attack of the canopy.

6) The canopy center of pressure is taken at the canopy centroid. Apparently, the actual center of pressure of a parachute canopy has not been tested.

7) A flat earth is assumed with no winds.

These assumptions, although restrictive, still allow one to obtain considerable information about parachute stability.

### Equations of Motion

The equations of motion use the previous assumptions to simplify the general derivation given in Chap. 4 of the text by Etkin.<sup>14</sup> It is convenient to use body axes ( $x, y, z$ ) for the stability analysis and earth-fixed axes ( $x_1, y_1, z_1$ ) to compute the trajectory. These two right-handed systems are illustrated in Fig. 1. In the body axis system, there are linear velocities ( $U, V, W$ ), angular velocities ( $P, Q$ ), aerodynamic forces ( $X, Y, Z$ ), and aerodynamic moments ( $M, N$ ). The  $z$  axis is the axis of symmetry of the parachute. The earth-fixed axes are aligned with the body axes by a series of three Euler angle rotations in the following order:

$\psi$  = a rotation about  $z_1$

$\theta$  = a rotation about  $y_2$  (position of  $y_1$  after  $\psi$ )

$\phi$  = a rotation about  $x_3$  (position of  $x_1$  after  $\psi$  and  $\theta$ )

The five force and moment equations in the body axis system then become

$$m(dU/dt + QW) = X - (m_p + m_c)g \sin \theta$$

$$m(dV/dt - PW) = Y + (m_p + m_c)g \cos \theta \sin \phi$$

$$m(dW/dt + PV - QU) = Z + (m_p + m_c)g \cos \theta \cos \phi \quad (1)$$

$$I(dP/dt) = M - m_p g L_p \cos \theta \sin \phi + m_c g L_c \cos \theta \sin \phi$$

$$I(dQ/dt) = N + \sin \theta (m_c g L_c - m_p g L_p)$$

where  $m = (m_c + m_p + m_{ch})$  and the moment of inertia  $I = (I_c + I_p + I_{ch})$  include the hydrodynamic or apparent inertia of the canopy. The angular velocities are related to the Euler angles by the expressions

$$d\theta/dt = Q \cos \phi \quad d\phi/dt = P + Q \sin \phi \tan \theta \quad (2)$$

Equations (1) and (2) are seven equations in the seven unknowns ( $U, V, W, P, Q, \theta, \phi$ ). The lengths  $L_c$  and  $L_p$  relate the centroids of the canopy and payload to the system centroid, as shown in Fig. 1. Since the canopy is assumed to provide all the aerodynamic forces and moments, the angle of attack is taken to be the instantaneous angle  $\alpha$  between the velocity  $V_c$  of the canopy centroid and the axis of symmetry  $z$ . This angle  $\alpha$  is also shown in Fig. 1.

If desired, the system trajectory in earth-fixed coordinates can be calculated from the following relations:

$$d\psi/dt = Q \sin \phi \sec \theta$$

$$dx_1/dt = U \cos \theta \cos \psi + V(\sin \theta \sin \theta \cos \psi - \cos \theta \sin \psi) + W(\cos \theta \sin \theta \cos \psi + \sin \theta \sin \psi) \quad (3)$$

$$dy_1/dt = U \cos \theta \sin \psi + V(\sin \theta \sin \theta \sin \psi + \cos \theta \cos \psi) + W(\cos \theta \sin \theta \sin \psi - \sin \theta \cos \psi)$$

$$dz_1/dt = -U \sin \theta + V \sin \phi \cos \theta + W \cos \phi \cos \theta$$

However, only Eqs. (1) and (2) are needed for a stability study.

### Nondimensional Equations

It is convenient to define dimensionless variables based upon the system length  $L$  and the steady glide speed  $V_0$ , as

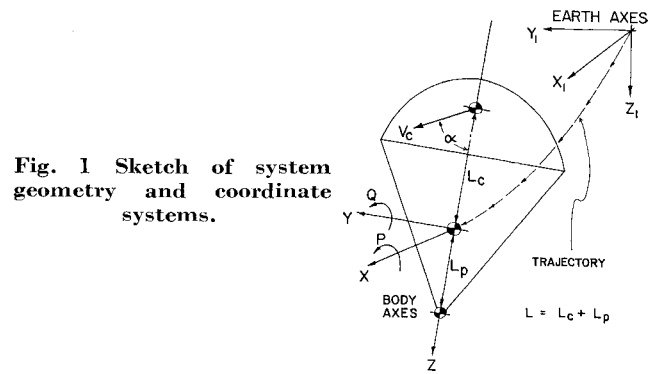


Fig. 1 Sketch of system geometry and coordinate systems.

follows:

$$u = \frac{U}{V_0} \quad v = \frac{V}{V_0} \quad w = \frac{W}{V_0} \quad (4)$$

$$p = \frac{LP}{V_0} \quad q = \frac{LQ}{V_0} \quad t^* = \frac{V_0 t}{L}$$

Substitution into Eqs. (1) gives the following dimensionless equations of motion, whose analysis constitutes the remainder of this paper.

$$\frac{du}{dt^*} = - \frac{C_N V_c^{*2} \left( u - \frac{q}{1+r_c} \right)}{2k(V_c^{*2} - w^2)^{1/2}} - \frac{C_{T_0}}{2k} \sin \theta - qw$$

$$\frac{dv}{dt^*} = - \frac{C_N V_c^{*2} \left( v + \frac{p}{1+r_c} \right)}{2k(V_c^{*2} - w^2)^{1/2}} + \frac{C_{T_0}}{2k} \cos \theta \sin \phi + pw$$

$$\frac{dw}{dt^*} = - \frac{C_T V_c^{*2}}{2k} + \frac{C_{T_0}}{2k} \cos \theta \cos \phi - pv + qu \quad (5)$$

$$\frac{dp}{dt^*} = - \frac{C_N V_c^{*2} \left( v + \frac{p}{1+r_c} \right)}{2i(1+r_c)(V_c^{*2} - w^2)^{1/2}} - \frac{C_{T_0} r_{ch}}{2i(1+r_c)} \cos \theta \sin \phi$$

$$\frac{dq}{dt^*} = \frac{C_N V_c^{*2} \left( u - \frac{q}{1+r_c} \right)}{2i(1+r_c)(V_c^{*2} - w^2)^{1/2}} - \frac{C_{T_0} r_{ch}}{2i(1+r_c)} \sin \theta$$

The dimensionless parameters in Eqs. (5) are defined as

$$r_c = (m_c + m_{ch})/m_p \quad r_{ch} = m_{ch}/(m_c + m_p)$$

$$k = m/\rho A L \quad i = I/\rho A L^3 \quad (6)$$

$$C_{T_0} = (m_c + m_p)g/\frac{1}{2}\rho V_0^2 A$$

where  $\rho$  is the fluid density and  $A$  is a reference area, usually the nominal cloth area of the parachute. The quantities  $C_N$  and  $C_T$  are the normal and tangential aerodynamic force coefficients of the canopy,

$$C_N = (X^2 + Y^2)^{1/2}/\frac{1}{2}\rho V_c^2 A \quad C_T = -Z/\frac{1}{2}\rho V_c^2 A \quad (7)$$

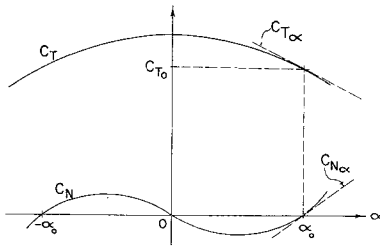
which, from assumption 5, are taken to depend quasi-statically upon the instantaneous angle-of-attack  $\alpha$  of the canopy (Fig. 1),

$$\alpha = \arctan \left\{ 1/w \left[ (u - q/1 + r_c)^2 + (v + p/1 + r_c)^2 \right]^{1/2} \right\} \quad (8)$$

Finally, the quantity  $V_c^*$  is the dimensionless canopy velocity,

$$V_c^{*2} = w^2 + [u - (q/1 + r_c)]^2 + [v + (p/1 + r_c)]^2 \quad (9)$$

Equations (2) and (3) will not be rewritten because their



**Fig. 2 Typical force coefficient curves for a parachute, showing the linear slope approximation.**

dimensionless form is exactly equivalent. These equations are nonlinear but may easily be solved for particular cases on a digital computer. Before discussing any computer solutions, let us derive a small disturbance stability criterion.

### Glide-Point Small-Disturbance Analysis

Because of the unusual shape of the force coefficient curves, most parachutes do not fall vertically, even if they are stable. Figure 2 is a sketch of typical canopy coefficients as a function of the angle of attack. The  $C_N$  curve is antisymmetric, because the normal force changes direction, acting away from the centerline if  $\alpha$  is between  $-\alpha_0$  and  $+\alpha_0$ . A parachute cannot remain at the point ( $\alpha = 0$ ), even though the normal force is zero there, because any slight disturbance will drive the chute away from the origin because of the induced outward normal force. This condition is termed static instability and is derived rigorously in this section. However, as discussed by Heinrich,<sup>9</sup> the point ( $\alpha = \alpha_0$ ) is statically stable, since a slight disturbance causes a normal force which returns the chute back toward  $\alpha_0$ . Heinrich calls  $\alpha_0$  the "stable glide point," because the chute will glide with its tangent force exactly balancing the system weight. Hence the value of  $C_T$  is  $C_{T_0}$ , as defined in Eq. (6). Unfortunately, as we shall see, the glide point, while statically stable, may be dynamically unstable, so that a given parachute may be unable to continue gliding, and instead may oscillate or perform a coning motion.

To investigate linearized stability of the glide point, we assume small disturbances from a steady glide in the  $x$ - $z$  plane. Then the reference state is

$$\begin{aligned} \alpha &= \alpha_0 & C_T &= C_{T_0} & C_N &= 0 \\ u_0 &= \sin \alpha_0 & v &= 0 & w_0 &= \cos \alpha_0 \\ p &= q = \theta = \phi = 0 \end{aligned} \quad (10)$$

Now, define disturbance variables denoted by primes,

$$\begin{aligned} u' &= u - u_0 & v' &= v & w' &= w - w_0 \\ p' &= p & q' &= q & \theta' &= \theta & \phi' &= \phi \end{aligned} \quad (11)$$

Further, assume that the force coefficients are approximated by straight lines in the neighborhood of the glide point, as shown by the dotted lines in Fig. 2,

$$\alpha' = \alpha - \alpha_0 \quad C_N = C_{N\alpha} \alpha' \quad C_T = C_{T_0} + C_{T\alpha} \alpha' \quad (12)$$

If we substitute Eqs. (11) and (12) into Eqs. (5) and neglect squares and higher powers of disturbance variables, we arrive at the following linearized equations of motion about the glide point:

$$\begin{aligned} \left[ D + C_{N\alpha} \frac{\cos \alpha_0}{(2k)} \right] u' - \left[ C_{N\alpha} \frac{\sin \alpha_0}{(2k)} \right] w' + \\ \left[ \cos \alpha_0 \left( 1 - \frac{C_{N\alpha}}{2k(1+r_c)} \right) D + \frac{C_{T_0}}{(2k)} \right] \theta' = 0 \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{1}{2k} (C_{T\alpha} \cos \alpha_0 + 2C_{T_0} \sin \alpha_0) u' + \\ \left[ D + \frac{1}{2k} (2C_{T_0} \cos \alpha_0 - C_{T\alpha} \sin \alpha_0) \right] w' + \\ \left[ \sin \alpha_0 - \frac{C_{T\alpha} \cos \alpha_0 + 2C_{T_0} \sin \alpha_0}{2k(1+r_c)} D \right] \theta' = 0 \end{aligned} \quad (14)$$

$$\begin{aligned} - \frac{C_{N\alpha} \cos \alpha_0}{2i(1+r_c)} u' + \frac{C_{N\alpha} \sin \alpha_0}{2i(1+r_c)} w' + \\ \left[ D^2 + \frac{C_{N\alpha} \cos \alpha_0}{2i(1+r_c)^2} D + \frac{C_{T_0} r_{ch}}{2i(1+r_c)} \right] \theta' = 0 \end{aligned} \quad (15)$$

$$Dv' - [\cos \alpha_0 D + C_{T_0}/(2k)] \phi' = 0 \quad (16)$$

$$[D^2 + C_{T_0} r_{ch}/2i(1+r_c)] \phi' = 0 \quad (17)$$

where the symbol  $D$  is the operator  $d/dt^*$ . For a detailed derivation of these five equations, see Wolf.<sup>15</sup>

These linearized relations have the valuable property that the longitudinal and lateral motions are uncoupled. That is, Eqs. (13–15) describe the longitudinal or pitching motion in the glide plane and contain only the in-plane variables ( $u'$ ,  $w'$ ,  $\theta'$ ), whereas Eqs. (16) and (17) describe the lateral motion normal to the glide plane. Thus the longitudinal and lateral stability of a symmetric parachute may be studied separately, similar to the study of aircraft linearized stability.<sup>14</sup>

### Longitudinal Stability

Since Eqs. (13–15) are linear with constant coefficients, the solutions must be of the exponential form  $e^{\lambda t^*}$ , where  $\lambda$  is a constant. Setting  $u'$ ,  $w'$ , and  $\theta'$  in this exponential form and substituting, we obtain, by requiring the determinant of the coefficients to be zero, a fourth-order algebraic equation for the constant  $\lambda$ ,

$$\lambda^4 + C_1 \lambda^3 + C_2 \lambda^2 + C_3 \lambda + C_4 = 0 \quad (18)$$

where each of the constants  $C_i$  are linear functions of  $C_{N\alpha}$

$$\begin{aligned} C_1 &= E_1 C_{N\alpha} + E_2 & C_2 &= E_3 C_{N\alpha} + E_4 \\ C_3 &= E_5 C_{N\alpha} + E_6 & C_4 &= E_7 C_{N\alpha} \end{aligned} \quad (19)$$

and, finally, the constants  $E_i$  are given as follows:

$$\begin{aligned} E_1 &= \cos \alpha_0 [k + i(1+r_c)^2]/2ki(1+r_c)^2 \\ E_2 &= (2C_{T_0} \cos \alpha_0 - C_{T\alpha} \sin \alpha_0)/2k \\ E_3 &= C_{T_0} [k + i(1+r_c)^2 + k^2(1+r_c)]/2k^2i(1+r_c)^2 \\ E_4 &= C_{T_0} r_{ch}/2i(1+r_c) \\ E_5 &= C_{T_0} \cos \alpha_0 (3 + r_{ch})/4ki(1+r_c) \\ E_6 &= C_{T_0} r_{ch} (2C_{T_0} \cos \alpha_0 - C_{T\alpha} \sin \alpha_0)/4ki(1+r_c) \\ E_7 &= C_{T_0}^2 (1 + r_{ch})/4k^2i(1+r_c) \end{aligned} \quad (20)$$

For any given system parameters, the constants  $C_i$  may be calculated and Eq. (18) solved for the four roots. For stability, it is necessary that  $C_4$  be positive (static stability), which is true if  $C_{N\alpha}$  is positive, since  $E_7$  is always positive. This confirms the earlier discussion of stable gliding.

For dynamic stability, the real parts of the four roots must all be negative, which is ensured<sup>14</sup> if

$$C_1, C_2, C_3, \text{ and } C_4 > 0 \quad (21)$$

and if

$$C_3(C_1 C_2 - C_3) - C_1^2 C_4 > 0 \quad (22)$$

Equation (21) is also satisfied if  $C_{N\alpha}$  is positive, a condition already required.

Equation (22) leads to a quadratic equation in  $C_{N\alpha}$

$$K_1 C_{N\alpha}^2 + K_2 C_{N\alpha} + K_3 > 0 \quad (23)$$

where the constants  $K_i$  are

$$K_1 = E_1(E_3E_5 - E_1E_7)$$

$$K_2 = E_5(E_2E_3 - E_1E_4 - E_5) + E_1(E_3E_6 - 2E_2E_7)$$

$$K_3 = E_2(E_4E_5 + E_3E_6 - E_2E_7) + E_6(E_1E_4 - 2E_5)$$

Equation (23) yields the minimum value  $C_{N\alpha}(\min)$  required for dynamic stability,

$$C_{N\alpha}(\min) = -K_2/2K_1 + (K_2^2/4K_1^2 - K_3/K_1)^{1/2} \quad (24)$$

It will be seen that Eq. (24) is fairly restrictive and difficult for most parachute shapes to meet.

### Numerical Stability Results

Equation (24) may be used to make parametric studies of longitudinal stability. However, the parameters  $k$ ,  $i$ ,  $r_c$ , and  $r_{ch}$  are not very significant physically, because the hydrodynamic inertia is mixed into all of them. Perhaps a more logical group would be the following four:

$V_0^2/gL$  = the descent Froude number

$D_0/L$  = the chute slenderness ratio

$m_c/m_p$  = the canopy-to-load mass ratio

$C_{T\alpha}$  = the gliding force coefficient

where  $D_0$  is the nominal diameter of the canopy.

The dimensionless hydrodynamic mass  $B_m$ , defined as

$$B_m = 6m_{ch}/\pi\rho D_0^3 \quad (25)$$

relating  $m_{ch}$  to a sphere of air of diameter  $D_0$ , has been measured<sup>16</sup> and shown to vary considerably with porosity. The hydrodynamic moment of inertia factor

$$B_i = 60I_{ch}/\pi\rho D_0^5 \quad (26)$$

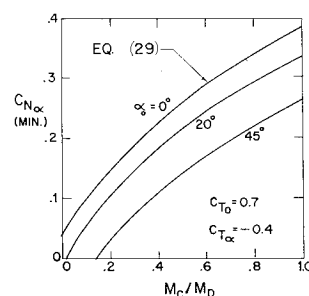
has also been measured<sup>17</sup> as a function of porosity. To be exact, both  $B_m$  and  $B_i$  are tensors with significant directional variation. However, from assumption 4, we shall approximate them here as average scalar values.

Experimental values of  $B_m$  (Ref. 16) are approximately 0.7 at zero porosity, dropping sharply to approximately 0.03 at 40% geometric porosity. Similarly,  $B_i$  is approximately 0.23 at zero porosity,<sup>17</sup> dropping to 0.04 at 40% porosity.

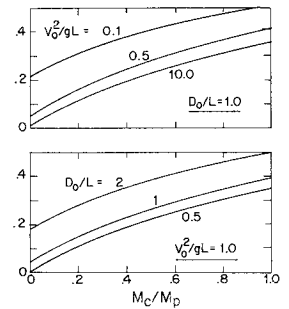
The four new practical parameters are related to those of Eqs. (6) as follows:

$$\left. \begin{aligned} k &= \frac{1}{2}C_{T0}V_0^2/(gL) + \frac{2}{3}B_mD_0/L \\ r_{ch} &= \frac{2B_mD_0/L}{3k - 2B_mD_0/L} \\ r_c &= (m_c/m_p)(1 + r_{ch}) + r_{ch} \\ i &= kr_c/(1 + r_c)^2 + B_i(D_0/L)^3/15 \end{aligned} \right\} \quad (27)$$

**Fig. 3 Effect of glide angle on parachute stability ( $V_0^2/gL = D_0/L = 1.0$ , zero porosity)**



**Fig. 4 Effect of Froude number and slenderness ratio on stability ( $C_{T0} = 0.7$ , zero porosity).**



For comparison with the stability criterion, Eq. (24), experimental values of  $C_{N\alpha}$  and  $C_{T\alpha}$  are needed. Reference 2 gives wind-tunnel results for ten different canopy shapes with varying effective porosity. For all ten shapes, the measured values of  $C_{T\alpha}$  lie between 0.0 and  $-0.4$ , while the values of  $C_{N\alpha}$  lie between 0.0 and  $+0.4$ . The maximum effect of  $C_{T\alpha}$  on glide stability can thus be shown by evaluating Eq. (24) for  $C_{T\alpha} = -0.4$ . A typical result is shown in Fig. 3 for various glide angles of a nonporous canopy. It is seen that a negative  $C_{T\alpha}$  has a stabilizing effect at finite glide angles. If  $C_{T\alpha}$  is set equal to 0.0, the curves collapse to the single curve marked  $\alpha_0 = 0^\circ$ , which is the conservative or maximum value of  $C_{N\alpha}(\min)$ . Figure 3 also shows that a heavily loaded parachute (small  $m_c/m_p$ ) is more stable than a lightly loaded system.

### Conservative Glide-Stability Criterion

A conservative and greatly simplified expression results if one sets  $C_{T\alpha} = 0$  in Eqs. (13-15). The determinant of coefficients reduces to a cubic,

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \quad (28)$$

where

$$C_1 = \frac{1}{2}C_{N\alpha}(1/k + 1/i(1 + r_c)^2)$$

$$C_2 = (C_{N\alpha} + C_{T0}r_{ch})/2i(1 + r_c)$$

$$C_3 = C_{N\alpha}C_{T0}(1 + r_{ch})/4ki(1 + r_c)$$

which is independent of the angle  $\alpha_0$ . The Routh stability criterion for a cubic is  $C_1C_2 = C_3$ , which results in a very simple expression for  $C_{N\alpha}(\min)$

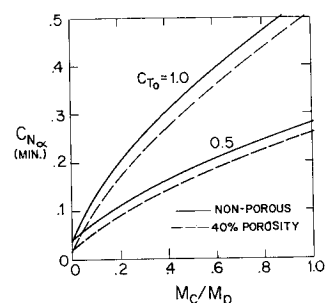
$$C_{N\alpha}(\min) = C_{T0}\{[i(1 + r_c)^2 - kr_{ch}]/[i(1 + r_c)^2 + k]\} \quad (29)$$

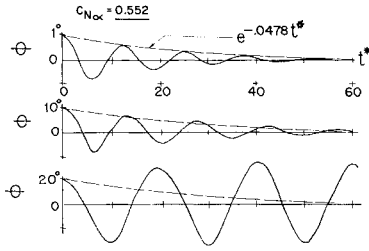
Equation (29) is very similar to a criterion for stable vertical descent given by Heinrich and Rust.<sup>8</sup>

Using Eq. (29) for a conservative estimate, the general effects of Froude number and slenderness ratio are shown in Fig. 4. It is seen that increased stability is achieved by a more rapid descent (large  $V_0^2/gL$ ) and by long lines (small  $D_0/L$ ). Note that the two effects are opposing if one merely changes  $L$ .

Figure 5 illustrates the effect of  $C_{T0}$  and porosity, showing that the high-drag body is less stable. The effect of porosity is a slight increase in stability through a decrease in the hydrodynamic inertia terms. A more important effect of porosity is the fact that the force coefficients themselves

**Fig. 5 Effect of tangent force and porosity on stability ( $V_0^2/gL = D_0/L = 1.0$ ).**





**Fig. 6 Computer solutions for 1°, 10°, and 20° pitch displacements of the stable system.**

change greatly with porosity,<sup>2</sup> which is not accounted for in Fig. 5.

From the data of Ref. 2, we find that most parachutes simply cannot develop sufficient  $C_{N\alpha}$  to be glide stable except at high load masses, high descent speeds, and low effective porosity. For example, the personnel guide surface canopy, which has low  $C_{N\alpha}$  and high  $(m_c/m_p)$ , would be expected to oscillate rather than glide. This is evident from the computer solutions of Ludwig and Heins<sup>12</sup> for personnel parachutes.

### Lateral Dynamic Stability

The lateral motion, Eqs. (16) and (17), is clearly much simpler and easily yields the following stability polynomial:

$$\lambda[\lambda^2 + C_{T_0}r_{ch}/2i(1 + r_c)] = 0 \quad (30)$$

The real part of all three roots is obviously zero. Hence, lateral small disturbances will lead to neutrally stable oscillations, although Eq. (24) must also be satisfied to keep all disturbances small. Thus a coning motion of an otherwise stable system cannot ensue from a small lateral disturbance, but instead is strictly a large-disturbance phenomenon.

### Large-Disturbance Studies

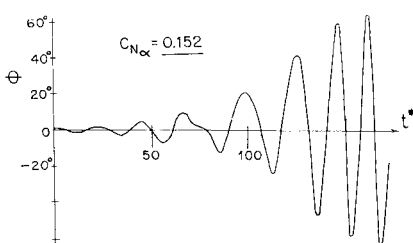
To indicate the limitations of the previous linearized analysis, some exact computer solutions of Eqs. (5) will be shown. To this end, we select a simple but fairly realistic parachute as our basic system.

$$\begin{aligned} C_{T_0} &= 0.7 & C_{T\alpha} &= -0.1 & \alpha_0 &= 25^\circ \\ k &= i = 1.0 & r_c &= r_{ch} = 0.1 \end{aligned} \quad (31)$$

We will vary  $C_{N\alpha}$  to show different effects. Both Eqs. (24) and (29) predict that  $C_{N\alpha}(\min) = 0.352$ . We select for study two systems on either side of this value: 1) stable,  $C_{N\alpha} = 0.552$ ; and 2) unstable,  $C_{N\alpha} = 0.152$ . Simple polynomial expressions will be used for the variation of the force coefficients with angle of attack,

$$\begin{aligned} C_N(\alpha) &= \alpha_0 C_{N\alpha} (\alpha/\alpha_0) (\alpha/\alpha_0 - 1) \\ C_T(\alpha) &= C_{T_0} + \frac{1}{2} C_{T\alpha} \alpha_0 (\alpha^2/\alpha_0^2 - 1) \end{aligned} \quad (32)$$

which closely resemble the curves in Fig. 2.



**Fig. 7 Response of the unstable system to a 1° pitch displacement.**

### Longitudinal Disturbances

We first test the apparently stable system with various-sized longitudinal disturbances:  $\theta' = 1^\circ, 10^\circ$ , and  $20^\circ$ . The complete linear theory, Eq. (18), predicts an exponential damping rate of  $e^{-0.478t^*}$ , with a dominant short period of  $T^* = 13.03$  (the long period mode damps out rapidly). For comparison, the simpler theory, Eq. (28), gives a damping rate  $e^{-0.0518t^*}$  and a period  $T^* = 12.87$ , reasonably accurate considering the reduction in computation.

Figure 6 shows the response  $\theta(t^*)$  from the computer solution for each of the three disturbances. The  $1^\circ$  response is in nearly perfect agreement with the linear theory. The  $10^\circ$  disturbance still damps out, but at a lesser rate than the linear prediction. In contrast, the  $20^\circ$  case does not damp but instead jumps to a large angle ( $\pm 32^\circ$ ) oscillation which persists. The angle of attack also oscillates between  $\pm 32^\circ$ , which is outside the limits  $\pm \alpha_0$ . The explanation is that the  $20^\circ$  disturbance is sufficient to throw the parachute over the lower hump of the  $C_N$  curve and across to the negative  $\alpha$  side. In rare cases, the chute will again damp to a stable glide at  $(-\alpha_0)$ , but usually the final condition is a pitching oscillation at some angle greater than  $\alpha_0$ . A rule of thumb is that a parachute which is stable in the linearized sense will jump to a large angle pitching oscillation if struck with a disturbance greater than one-half its stable angle  $\alpha_0$ .

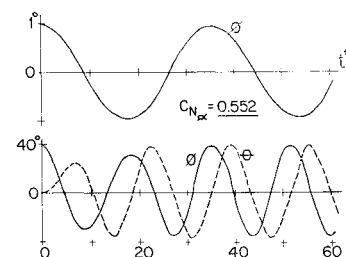
To test the unstable system, we need only try the single disturbance  $\theta' = 1^\circ$  and let the system do the rest. The response is given in Fig. 7, showing that the motion diverges to a very large amplitude oscillation of  $\pm 65^\circ$ . Other size disturbances would approach the same final pitching motion, which is characteristic of the unstable system in longitudinal motion.

### Lateral Disturbances

The lateral behavior of the stable system can be observed by comparing two disturbances,  $\phi' = 1^\circ$  and  $40^\circ$ . The resulting responses are shown in Fig. 8. The motion for  $\phi' = 1^\circ$  is a neutral sinusoid, in agreement with the linear theory. The period  $T^* = 35.2$  can be predicted from Eq. (30). The induced longitudinal motion  $\theta$  is a negligibly small fraction of a degree, too small to show in the figure.

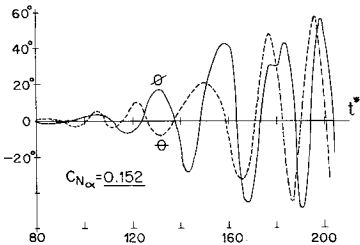
In contrast, the large disturbance of  $40^\circ$  immediately induces a longitudinal motion  $\theta(t^*)$  of comparable magnitude. Both  $\theta$  and  $\phi$  approach a stable oscillation (not a pure sinusoid) with equal amplitudes of  $38.7^\circ$ , with  $\theta$  lagging  $\phi$ . To visualize this motion, we note that the canopy is falling directly downward at a resultant velocity  $V_c^* = 1.038$ , while the load is moving downward along a helical path of constant radius. The parachute centerline maintains a constant angle with the vertical of  $38.7^\circ$  (identical to the amplitude of  $\theta$  and  $\phi$ ) and the centerline rotates about the vertical with a constant angular rate of 0.493. In short, the  $40^\circ$  lateral disturbance has thrown the stable parachute into a uniform vertical coning motion.

The unstable system, when struck with a  $1^\circ$  lateral disturbance, also diverges into a coning motion, as shown in Fig. 9. The lateral motion induces very small glide-plane motions which build up exponentially because of the longitudinal instability. Since the induced motion is only  $\theta' =$



**Fig. 8 Stable system response to 1° and 40° lateral displacements.**

Fig. 9 Unstable system response to a 1° lateral displacement.



0.02°, the build-up time is long; hence, Fig. 9 is begun at  $t^* = 80$ . The final motion is again a pure vertical coning, this time at a larger angle  $\beta = 58.6^\circ$ , where  $\beta$  is the coning angle illustrated in Fig. 10. Since the only difference in the two systems is the value of  $C_{N\alpha}$ , it is clear that  $C_{N\alpha}$  has a large effect on the coning angle. Also, this time the angular coning rate  $\dot{\gamma}$  is 0.606, where  $\gamma$  is the angle in the horizontal plane as illustrated by Fig. 10. The two angles,  $\beta$  and  $\gamma$ , are sufficient to describe a pure vertical coning attitude. In terms of the Euler angles  $\theta$ ,  $\phi$ , and  $\psi$ , these two angles may be written as follows:

$$\cos \beta = \cos \theta \cos \phi$$
$$\cos \gamma = \csc \beta (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)$$

(33)

One may conclude that a linearized stable parachute is neutrally stable to a small lateral disturbance but may jump into a vertical coning motion if hit with a large lateral displacement. A linearized unstable parachute will be driven into a coning motion by any lateral displacement.

Coning Analysis

The computer solutions shown in Figs. 8 and 9 give the key to the mathematical form of the functions involved in vertical coning. From this key, a complete algebraic theory of coning can be constructed. The velocity  $w$  is constant, while  $u$  and  $v$  are sinusoidal with equal amplitude and 90° out of phase. The angular velocities  $p$  and  $q$  are also sinusoidal, of equal amplitude, 90° out of phase, and together lag  $u$  and  $v$  by constant phase angle  $\sigma$ . Hence the proper functions are

$$u = u_0 \cos(\omega t^*) \qquad v = u_0 \sin(\omega t^*)$$
$$w = V_c^* \cos \beta = \text{const}$$
$$p = p_0 \cos(\omega t^* + \sigma) \qquad q = p_0 \sin(\omega t^* + \sigma)$$

(34)

If we substitute Eqs. (34) into the equations of motion, Eqs. (5), using Eqs. (33) to eliminate  $\theta$  and  $\phi$ , we find that each equation of motion is satisfied identically, i.e., Eqs. (34) are the exact solution for coning. What remains are seven algebraic relations involving  $u_0$ ,  $p_0$ ,  $\sigma$ ,  $\beta$ ,  $\omega$ ,  $\dot{\gamma}$ , and  $V_c$ .

After considering manipulation, we are left with a single transcendental equation for finding the coning angle  $\beta$

$C_T/C_N = a \tan \beta + b \cot \beta$

where

$b = [(1 + r_c)^2 + 1]/[(1 + r_c)^2 - r_{ch}]$

and

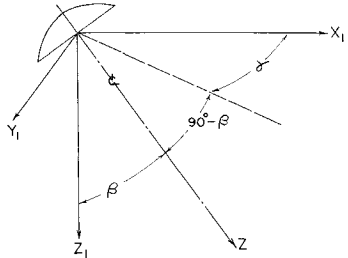
$a = [1 + r_{ch} b k / i]/(1 + r_c)^2$

and where the coefficients  $C_T$  and  $C_N$  are evaluated at the

Table 1 SAND system parameters

Altitude, ft	<i>k</i>	<i>i</i>	<i>r<sub>c</sub></i>	<i>r<sub>ch</sub></i>
180,600	198.0	11.0	0.051	0.0006
143,300	49.8	2.76	0.052	0.0023
121,800	20.0	1.11	0.057	0.0066

Fig. 10 Illustration of coning angles.



angle of attack  $\alpha = \beta$ . The quantities  $a$  and  $b$  are system constants and do not involve  $\beta$ . For our sample parachute, Eq. (31),  $a = 0.99$  and  $b = 1.99$ . The ratio  $C_T/C_N$  depends upon the choice of  $C_{N\alpha}$ , whether 0.552 or 0.152, and the polynomial curve fits, Eqs. (32). Since the right-hand side of Eq. (35) is always positive, the coning angle must be in a region of positive  $C_N$ , which means that  $\beta$  must be greater than  $\alpha_0$ .

Figure 11 shows a graphical solution for the coning angle for each of the two cases of  $C_{N\alpha}$ , 0.552 or 0.152. The solutions are seen to be 38.7° and 58.6°, respectively, in perfect agreement with the computer solutions for coning.

Once  $\beta$  is known, the rate of fall of the canopy can be calculated from the relation

$V_c^{*2} = C_{T_0} \sin \beta / (b C_N)$

(36)

and the angular rate of coning  $\dot{\gamma}$  can be calculated from the expression

$\dot{\gamma}^2 = [(1 + r_c)/2k](C_T V_c^{*2} - C_{T_0} \cos \beta) \csc^2 \beta$

(37)

Equations (35-37) give the most important coning variables. Other supplementary relations are

$$\omega = \dot{\gamma} \cos \beta$$
$$\dot{\gamma} = -(1 + r_c) V_c^* \tan(\sigma)$$

(38)

As in the stability analysis, these coning relations could be studied for parametric effects, using say, the practical parameters from Eqs. (27). It is necessary also to assume some force coefficient curves, say, Eqs. (32), since the coning does not occur at the glide point. One finds that a single coning solution exists for any given system, and that the only two parameters which have any great effect on coning are  $\alpha_0$  and  $C_{N\alpha}$ . There is little point in making charts of coning solutions because, unlike the glide point study, no exact criterion for coning motion to ensue can be established. One might think that the larger coning angles might be more difficult to achieve, but we have seen in Fig. 11 that large  $\beta$  probably implies glide-point instability, and hence, is very susceptible to coning. The glide-stable systems also cone easily if struck with the proper sized lateral displacement. The authors can only conclude that the coning motion as outlined here is always a possibility if the disturbance conditions warrant. However, it is possible that, at these large angles of attack, some unaccounted-for pitching moment variations may develop such as an unusual variation in the canopy center of pressure, thus precluding a stable coning motion. To the authors' knowledge, such center-of-pressure

Fig. 11 Graphical coning angle solution for the two illustrative cases. (A:  $C_{N\alpha} = 0.552$ ; B:  $C_{N\alpha} = 0.152$ ).

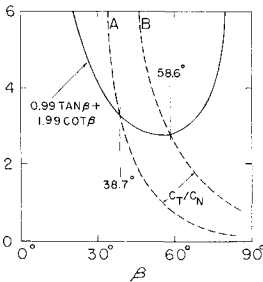


Table 2 Coning calculations for SAND

Altitude, ft	$\beta$	$\dot{\gamma}$	Theoretical coning rate, rps	Observed rate, rps
180,600	39.3°	0.0352	0.105	0.10
143,300	39.0°	0.0702	0.110	0.12
121,800	38.5°	0.114	0.115	0.22

variations have not been measured, hence, the use of our somewhat dubious assumption 6. It is also interesting to note that this coning analysis is not limited to parachutes but is mathematically valid for any free-falling body of revolution. Such a body may cone if it can develop aerodynamic forces which satisfy Eq. (35).

Comparison with Coning Data

A coning motion was observed during a test flight of the Sandia Corporation's SAND sampler vehicle.<sup>18</sup> This system employs a 14.3% full extended skirt parachute of 50.5-ft nominal diam to lower a 700 lb payload from altitudes of several hundred kilofeet. For the test flight, the dimensionless parameters at three different altitudes are presented in Table 1. Note that the hydrodynamic parameter  $r_{ch}$  is very small because of the high altitude. The force coefficients are not known at these altitudes, but another analysis<sup>19</sup> assumed the canopy to have an effective porosity of 0.042, for which Ref. 2 gives  $C_{T_0} \doteq 0.8$ ,  $\alpha_0 \doteq 19^\circ$ , and  $C_{N\alpha} \doteq 0.2$ . Using Eq. (29) and Table 1, we calculate  $C_{N\alpha}(\text{min}) = 0.045$  for all three altitudes; hence, the system is glide-stable. In spite of this, the system coned continuously at an angle  $\beta \doteq 40^\circ$  throughout the altitude range.

Numerical calculation of Eqs. (35) and (37) for the parameters in Table 1 is given in Table 2. The agreement is good except for the coning rate at the lowest altitude. However, the lack of knowledge of the canopy effective porosity could easily account for this discrepancy.

Conclusions

It has been shown that a simplified five-degree-of-freedom analysis can yield considerable information about parachute dynamic stability. Subject to the simplifying assumptions listed in the beginning, the conclusions of this study are:

- 1) A system that is glide-stable from Eq. (24) will damp out small longitudinal glide-point disturbances and be neutrally stable for small lateral fluctuations.
- 2) A system that is glide-unstable will diverge to a large angle pitching oscillation if disturbed longitudinally and to a large angle vertical coning motion if disturbed laterally.
- 3) A stable parachute with finite  $\alpha_0$  will jump into a pitching oscillation if hit with a large longitudinal disturbance, and will jump into vertical coning if given a large lateral displacement. Exact algebraic expressions are given for the coning motion, which is apparently a stable mode of descent.

The validity of these conclusions should be established or disproven by experiment. There is a need not only to test the given stability criteria but also to examine the basic assumptions critically.

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