Basic Operations

Addition: $\vec{v} + \vec{w}$

$$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$

Subtraction: $\vec{w} + (\vec{v} - \vec{w}) = \vec{v}$

$$\begin{bmatrix}3\\1\end{bmatrix} + (\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

Scalar Multiplication

$(\vec{v} \cdot \vec{w})$

$$2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$-1/2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}-3/2\\-1\\-1/2\end{bmatrix}$$

Magnitude & Direction

2 Dimensions $\|\vec{v}\|^2 = v_x{}^2 + v_y{}^2$ $\qquad \|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2}$

3 Dimensions $\qquad \|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$

n Dimensions $\qquad \|\vec{v}\| = \sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$

Normalization - *Process of finding a unit vector in the same direction as a given vector*

Unit vector in direction of $\vec{v}$ $\qquad \frac{1}{\|\vec{v}\|} = \vec{v}$

Normalize $\qquad \vec{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Magnitude $\qquad \|\vec{v}\| = \sqrt{-1^2 + 1^2 + 1^2} = \sqrt{3}$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

Verify unit vector of u is 1 $\qquad \|\vec{v}\| = \sqrt{\frac{-1}{\sqrt{3}}^2 + \frac{1}{\sqrt{3}}^2 + \frac{1}{\sqrt{3}}^2} = 1$

Zero Vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Magnitude of 0 Vector is 0 $\|\vec{0}\| = 0$

Cannot be normalized, has no direction $\frac{1}{\|\vec{0}\|} = undefined$

Dot Products

Inner product: angle between 2 vectors $\vec{v} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

Solve for the angle using inverse function $\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}\right)$

$= \arccos\left(\frac{1}{\|\vec{v}\|}\vec{v} \cdot \frac{1}{\|\vec{w}\|}\vec{w}\right)$

Shorter formula: $(\vec{v} \cdot \vec{w}) = v_1 w_1 + v_2 w_2 + \cdots + v_{n1} w_n$

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 1 \cdot -1 \cdot 0 = 5$

Angle between 2 vectors: $\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}\right)$

$\theta = \arccos\left(\frac{5}{\sqrt{6}\sqrt{10}}\right)$

$\approx .87\ radians$

$\approx 50°$

Same direction: $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\|$ $\cos\theta = 1$

$\theta = 0$ & radians, $0°$

Opposite direction: $\vec{v} \cdot \vec{w} = -\|\vec{v}\| \cdot \|\vec{w}\|$ $\cos\theta = -1$

$\theta = \pi$ radians, $180°$

Right angle: $\vec{v} \cdot \vec{w} = 0$ $\cos\theta = 0$

$\theta = \frac{\pi}{2}$ radians, $90°$

Of a vector: $\vec{v} \cdot \vec{v} = \|\vec{v}\| \cdot \|\vec{v}\| \cos 0$ $\cos 0 = 1$

$= \|\vec{v}\|^2$

$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

Parallel & Orthogonal Vectors

Parallel: $\vec{v}$ *and* $\vec{w}$ *are parallel if one is a scalar multiple of the other*

$\vec{v}$ *is parallel to*: $2\vec{v}, \frac{1}{2\vec{v}}, -\vec{v}$

Orthogonal: $\vec{v}$ *and* $\vec{w}$ *are orthogonal if* $\vec{v} \cdot \vec{w} = 0$

$Case\ 1$: $\vec{v} = \vec{0}$, or

$Case\ 2$: $\vec{w} = \vec{0}$, or

$Case\ 3$: $\vec{v} \cdot \vec{w} = 0$, *the lines are perpendicular*

Special case: $\vec{0}$, is the only vector orthogonal to itself.

Projecting Vectors

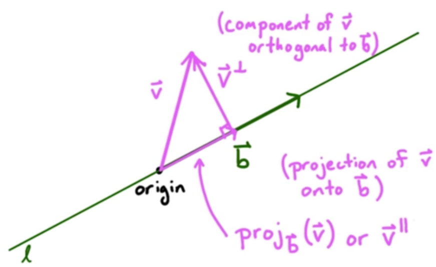
Orthogonality: Allows for structured decomposition of objects into combinations of simpler objects.

Basis vector: $\vec{b}$, the projection of $\vec{v}$ *onto* $\vec{b}$, $proj_{\vec{b}}$ *or* $\vec{v}^{\parallel}$

$$\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$$

$\vec{v}^{\parallel}$, parallel to $\vec{b}$

$\vec{v}^{\perp}$, orthogonal to $\vec{b}$



$$\|\vec{v}^{\parallel}\| = \|\vec{v}\| \cos\theta \ \rightarrow\ \|\vec{v}^{\parallel}\| = \|\vec{v}\| \frac{\vec{v}\cdot\vec{b}}{\|\vec{v}\|\cdot\|\vec{b}\|} = \frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|} = \vec{v} \cdot \vec{u}_{\vec{b}}$$

if $\theta \leq 90°$,

$\|\vec{v}^{\parallel}\| = \vec{v} \cdot \vec{u}_{\vec{b}}$

$\vec{v}^{\parallel}$ points in same direction as $\vec{b}$

$\vec{u}_{\vec{b}}$ has a magnitude of 1

$\vec{v}^{\parallel} = \left(\vec{v} \cdot \vec{u}_{\vec{b}}\right) \cdot \vec{u}_{\vec{b}}$

if $\theta > 90°$,

$\|\vec{v}^{\parallel}\| = -\ \vec{v} \cdot \vec{u}_{\vec{b}}$

$\vec{v}^{\parallel}$ points in opposite direction as $\vec{b}$

$\vec{u}_{\vec{b}}$ has a magnitude of 1

$\vec{v}^{\parallel} = -\|\vec{v}^{\parallel}\|\, \vec{u}_{\vec{b}} = \vec{v}^{\parallel}$

$\vec{v}^{\parallel} = \left(\vec{v} \cdot \vec{u}_{\vec{b}}\right) \cdot \vec{u}_{\vec{b}}$

Cross Product $\vec{v} \times \vec{w}$ orthogonal to both $\vec{v}$ and $\vec{w}$

$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta$

Note: The output of a cross product is a vector, not a number

if $\theta \leq 0$ or π, $(0^\circ$ or $180^\circ)$,

$$\Longrightarrow \|\vec{v} \times \vec{w}\| = 0$$
$$\Longrightarrow \vec{v} \times \vec{w} = \vec{0}$$

$\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$,

$$\Longrightarrow \vec{v} \times \vec{w} = \vec{0}$$

$\vec{v}$ is parallel to $\vec{w}$,

$$\Longrightarrow \vec{v} \times \vec{w} = \vec{0}$$

Right-hand rule: $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

anticommutative: Switching the order, negates the product.

Parallelogram

$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta$

Reminder: SOH-CAH-TOA

height	$\sin\theta = \frac{height}{\|\vec{w}\|} \Longrightarrow \|\vec{w}\| \sin\theta = height$
area of parallelogram (base ⋅ height)	$\|\vec{v}\|\|\vec{w}\| \sin\theta = \|\vec{v} \times \vec{w}\|$
area of triangle	$\frac{\|\vec{v} \times \vec{w}\|}{2}$

$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \ \vec{w} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, then \ \vec{v} \times \vec{w} = \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ -(x_1 z_2 - x_2 z_1) \\ x_1 y_2 - x_2 y_1 \end{bmatrix},$$

$$\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 3 - 0 \cdot -2 \\ -(5 \cdot 3 - (-1) \cdot (-2)) \\ 5 \cdot 0 - (-1) \cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -13 \\ 3 \end{bmatrix}$$

area of a parallelogram Spanned by $\vec{v} \times \vec{w}$	$\sqrt{9^2 + (-13)^2 + 3^2} \approx 16.093$