

More Fun With Automata  
Homework 2, CS500, Fall 2014

Aaron Gonzales (group 16), Ahmad Darki (group 11), Manasa Navada (group 30)

February 3, 2015

## Ex 7

**Show that if  $L$  is regular then  $L^*$  is regular. Why does it not suffice to use the fact that the regular languages are closed under concatenation and union?**

**Answer:**

We know that  $L^* = \epsilon \cup L \cup LL \cup \dots$ , the enumerated set of all possible combinations of strings made up of the language  $L$ . We also know that for a language to be regular, there must be some DFA or NFA that can accept it.

If we let  $M_L$  be the machine that recognizes language  $L$ , then let  $M_{L^*}$  be the machine that recognizes  $L^*$ . To construct  $M_{L^*}$ , add a new start state  $S'$  that is also an accepting state to  $M_L$ . Connect  $S'$  to  $M_L$  with an  $\epsilon$  transition and add another  $\epsilon$  edge out of  $M_L$  such that it connects to  $S'$ . This machine can accept any possible combinations of valid instances of  $L$  and includes  $\epsilon$ , showing that  $L^*$  is indeed regular.

Since we are adding an  $\epsilon$  transition, concatenation and union do not cover  $\epsilon$  closure and we need to account for that.

## Ex 8

**Given a string  $w$ , let  $w^R$  denote  $w$  written in reverse. Given a language  $L$ , let  $L^R = \{w^R | w \in L\}$ . Prove that  $L$  is regular if and only if  $L^R$  is regular. Why is this harder to prove with DFAs?**

**Answer:**

We know that a language is regular if there is a DFA that can accept it. This tells us that in order to prove  $L^R$ 's regularity, there must be some DFA that can accept it. Generally put, we can map our current DFA to a DFA that can accept the new reversed language by modifying the start state, accepting states, and transition functions as follows, which will be formalized below.

*Proof.* Let  $M = \{S, A, s^0, S^{yes}, \delta\}$  be the machine that recognizes  $L$ . Let  $M' = \{S', A', S'^0, S'^{yes'}, \delta_*'\}$  be the NFA that recognizes  $L^R$ , constructed as below:

Let  $S_{new} \in S$  be a new start state and connect it to all  $s \in S^{yes}$  with an  $\epsilon$  label. Make  $S^0$  into an  $S'^{yes'}$ . Make  $s \in S^{yes} \rightarrow S'$  (make accepting states into normal states). Make  $S^0 \rightarrow S'^{yes'}$  (make the original initial state into the sole accepting state).

This new NFA will only accept languages that begin in the first char of the  $w^R$  and end in the last char of  $w^R$ . As we have constructed an NFA that recognizes this  $L^R$ , we can deduce that  $L^R$  is regular, meaning that  $L \Leftrightarrow L^R$ .  $\square$

The NFA epsilon ability allows us to more easily create new NFAs from existing automata.

## Ex 9

**A for-all NFA is one such that  $L(M)$  is the set of strings where every computation path ends in an accepting state. Show how to simulate an for-all NFA with a DFA, and thus prove that a language is recognized by some for-all NFA if and only if it is regular.**

**Answer:**

This is similar to converting any NFA to a DFA. let

$$A_n = \{Q, q_0 \in Q, F \subseteq Q, \delta : Q \times \Sigma \rightarrow P(Q)\}$$

be the  $\forall$  NFA we wish to convert.

Let

$$A_d = \{Q' = P(Q), \{q_0\}, \delta'(S, a) = \cup_{q \in S} \delta(q, a), F' = \{S \subseteq Q : S \cap F \neq \emptyset\}\}$$

be the DFA to which we are mapping  $A_n$ .

note that

$$Q' = \mathcal{P}(Q)$$

$Q'$  is power set of  $Q$

$$F' = \{S \subseteq Q : S \cap F \neq \emptyset\}$$

DFA accepting states as subsets of  $Q$  with all elements accepting

So all of the DFA's accepting states are on a computation path and that the All-paths NFA recognizes a regular language as we have build its DFA.

## Ex 10

**A parity finite-state automaton, or PFA for short, is like an NFA except that it accepts a string  $w$  if and only if the number of accepting paths induced by reading  $w$  is odd. Show how to simulate a PFA with a DFA, and thus prove that a language is recognized by a PFA if and only if it is regular. Hint: this is a little trickier than our previous simulations, but the number of states of the DFA is the same.**

**Answer:**

Assume an NFA being regular:

$$M_2 = (S_2, A, S_2^0, S_2^{yes}, \delta_2)$$

From this NFA we will have a DFA as:

$$M = (S, A, S^0, S^{yes}, \delta)$$

Where  $S = 2^{S_2}$ ,  $S^{yes} = S_2^{yes}$ ,  $\delta(T, a) = \cup_{t \in T} \delta_2(t, a)$ , and  $S^{yes} = \{T \cap S_2^{yes}\}$  in which this DFA is regular as well. For this problem we need to prove that for an odd number of  $\delta_2(S_2^0, w) = S_2^{yes} \in S_2^{yes}$  we will be able to make another DFA. From this we know that there will be an odd number of transitions of  $\delta_2(S_2^0, w)$  which means  $|w|$  is an odd number as in  $w = w_1 w_2 \dots w_{(2n+1)}$ .

## Ex 11

**Given finite words  $u$  and  $v$ , say that a word  $w$  is an interweave of  $u$  and  $v$  if I can get  $w$  by peeling off symbols of  $u$  and  $v$ , taking the next symbol of  $u$  or the next symbol of  $v$  at each step, until both are empty. (Note that  $w$  must have length  $|w| = |u| + |v|$ .) For instance, if  $u = cat$  and  $v = tapir$ , then one interleave of  $u$  and  $v$  is  $w = ctaapitr$ . Note that, in this case, we don't know which  $a$  in  $w$  came from  $u$  and which came from  $v$ . Now given two languages  $L_1$  and  $L_2$ , let  $L_1 \wr L_2$  be the set of all interweaves  $w$  of  $u$  and  $v$ , for all  $u \in L_1$  and  $v \in L_2$ . Prove that if  $L_1$  and  $L_2$  are regular, then so is  $L_1 \wr L_2$ .**

**Answer:**

The machine for  $L_1$  can be defined as:

$$M_1 = \{S_1, A, S_1^0, S_1^{yes}, \delta_1\}$$

and the machine for  $L_2$  as:

$$M_2 = \{S_2, A, S_2^0, S_2^{yes}, \delta_2\}$$

The product construction of the two languages allows us to run them in parallel. Let  $L' = L_1 \times L_2$  to be the Cartesian product of the two languages  $L_1$  to  $L_2$ , with its machine defined as:

$$M' = \{S' = \{S_1 \times S_2\}, A, S'^0 = \{S_1^0 \cup S_2^0\}, S'^{yes} = \{S_1^{yes} \cup S_2^{yes}\}, \delta' = \{\delta_1 \times \delta_2\}\}$$

Now, in order to handle the potential for various combinations of valid interweaves, we need to use the power set of  $L'$ , which would result in this final machine,  $M$ , that recognizes  $L_1 \wr L_2$ .

$$\begin{aligned}
M &= (S, A, S_0, S^{yes}, \delta) \\
S &= \mathcal{P}(S') \\
S_0 &= S_{L_1}^0 \cup S_{L_2}^0 \\
S^{yes} &= s \in S'^{yes} \\
\delta &= \{ \{ \delta_1, S_n, a \in A \}, \{ \delta_2, S_n, a \in A \} \}
\end{aligned}$$

This allows us to accept words that are valid in and order of  $M_1$  and  $M_2$ , such as the word “cat”  $\in L_1$  and “cat”  $\in L_2$  allows us to accept the first word in any valid accepting state of  $M_1$  and continue weaving it with the rest of the possible states of  $M_2$ , e.g., “cacatt”.

## Ex 12

Given a language  $L$ , let  $L_{1/2}$  denote the set of words that can appear as first halves of words in  $L$ :

$$L_{1/2} = \{x \mid \exists y : |x| = |y| \text{ and } xy \in L\}$$

where  $|w|$  denotes the length of a word  $w$ . Prove that if  $L$  is regular, then  $L_{1/2}$  is regular. Generalize this to  $L_{1/3}$ , the set of words that can appear as middle thirds of words in  $L$ :

$$L_{1/3} = \{y \mid \exists x, z : |x| = |y| = |z| \text{ and } xyz \in L\}$$

**Answer A:**

We know that  $|x| = |y|$  and must show that the set of words in  $L_{1/2}$  can be represented by a DFA to be proved regular. FA's limit us in that we cannot go back in time or record with external memory, and we may not know how large a word  $w$  is before it is read. As such, we must (A) build a FA that allows us to give  $|x| = |y|$  and allows us to verify that (B)  $xy \in L$ . This is complicated but not impossible.

Chris Moore's hint about River Song and the Doctor is a reference to Dr. Who, in which the characters are moving toward each other in time, one going forward and the other going backward. Given that hint, let us construct a FA that allows us to accomplish A and B, mostly given by the product construction of two FAs.

*Proof.* Let us define two FAs:

$$M = \{Q, \Sigma, \delta, S_0, S^{yes}\} \quad (1)$$

$$M^R = \{Q^R, \Sigma, \delta^R, S^{yes^R}\} \quad (2)$$

$$Q^R = Q \cup \{S_0^R\} \quad (3)$$

$M$  is the DFA which recognizes  $L$  and  $M^R$  is the NFA that recognizes  $L^R$ . We have seen that the reverse operator is closed under regularity.  $Q^R$  is the union of states of  $M$  and accepting states of  $L^R$ . Now introduce  $M_{1/2}$ , the machine that recognizes  $L_{1/2}$ , which is the product of  $M \times M^R$  and we define all of  $M_{1/2}$  member's below.

$$M_{1/2} = M \times M^R \quad (4)$$

$$= \{Q_{1/2}, \Sigma, \delta_{1/2}, S_{0_{1/2}}, S_{1/2}^{yes}\} \quad (5)$$

$$Q_{1/2} = Q \times Q^R \quad \text{States} \quad (6)$$

$$\delta_{1/2} = \{(q_1, q_2), a\} = \cup_{c \in \Sigma} \{\delta(q_1, a), \delta^R(q_2, c)\} \quad \text{transition function} \quad (7)$$

$$S_{0_{1/2}} = \{S_0, S_0^R\} \quad \text{initial states} \quad (8)$$

$$S_{1/2}^{yes} = \{(q_1, q) \mid q \in Q\} \quad \text{accepting states} \quad (9)$$

Our new machine,  $M_{1/2}$  will read the string  $w \in L$  and only accept it if the strings  $x, y$  are equal in length and are a part of this new  $L_{1/2}$  language.  $\square$

**(a) Answer B**

Proving this for  $L_{1/3}$  would involve repeating the process from above and getting a triple product construction for the three FA for which we have interest, though the construction of this is laborious. A possible proof by induction for an  $L_{1/n}$  for which any word in the language can be chopped into  $n$  discreet and equal chunks could follow.

*Proof.* Via induction over the size of a word

$$|x_1|, |x_2|, \dots, |x_n| \mid \sum_{i=1}^n |x_i| = w \in L$$

, where  $L$  is regular and  $|w| \% n = 0$ .

□

**Ex 13**

Show that if  $u \sim_L v$ , then  $ua \sim_L va$  for any  $a \in A$ .

**Answer:**

*Proof.* L-equivalence is defined by Definition 3<sup>1</sup> as given some language  $L \subseteq A^*$ ,  $u, v \in A^*$  are  $u \sim_L v$  if  $\forall w \in A^*$   $uw \in L$  iff  $vw \in L$ . Since this is a forall notion, we can say that this would include words  $w = ax$  or a word comprised with a prefix  $a$  on  $x$ .  $\forall x, uax \in L$  iff  $vax \in L \implies ua \sim_L va$ . □

**Ex 14**

Describe the equivalence classes of the three languages from Exercise 2. Use them to give the minimal DFA for each language, or prove that the DFA you designed before is minimal.

t

**Part 1**

**Answer:**

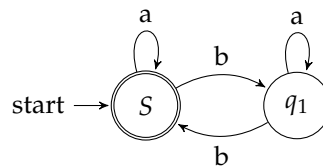


Figure 1: DFA for Exercise 2(a), the set of words in  $\{a, b\}^*$  with an even number of b's.

We know that there are only two equivalence classes, thanks to this DFA being in its minimal state.

$$\{z \mid \forall w \in L \Leftrightarrow zw \in L\}$$

Note that the above describes the class with  $z$  having even b's and  $w$  having even b's.

we have a class  $\{bb\}$ , the class where we have an even number of b's and a class  $\{b\}$  in which we have an odd number of b's. All other combinations in the language do not matter.

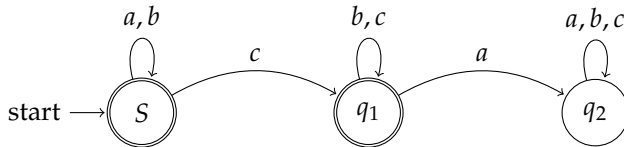
<sup>1</sup>Automata notes

## Part 2

Describe the equivalence classes of the three languages from Exercise 2. Use them to give the minimal DFA for each language, or prove that the DFA you designed before is minimal.

**Exercise 2.2:** The set of strings in  $\{a, b, c\}^*$  where there is no  $c$  anywhere to the left of an  $a$ .

As it has been proven that the following DFA is a minimal, with respect to the number of states the *equivalence class* are 3 as followed:



1.  $[ε] = [a] = [ab] = [aba]$ , The set of strings in  $w$  without any  $c$ .
2.  $[c] = [cb] = [cbc]$ , The set of strings in  $w$  starting with  $c$ .
3.  $[c] = [cb] = [cbc] = [cbcb] = [cbcb a]$ , The set of strings in  $w$  where there is  $c$  to the left of  $a$

## Part 3

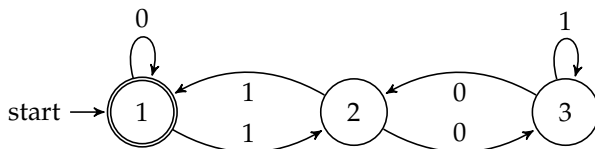
The set of strings in  $\{0, 1\}^*$  that encode, in binary, an integer  $w$  that is a multiple of 3. Interpret the empty string  $ε$  as the number zero.

**Answer:**

A minimal DFA for this language has three states. Each state represents the remainder of the number when it is divided by 3. The equivalence classes for this language are:

1.  $w$  containing the remainder 0. (Also the start state and accepting state)
2.  $w$  containing the remainder 1.
3.  $w$  containing the remainder 2.

Considering the following transition between these gives the following DFA.



Hence it is proved that the DFA we designed in HW1 is minimal.

## Ex 20

Consider the language

$$L_{a=b,c=d} = \{w \in \{a, b, c, d\}^* \mid \#_a(w) = \#_b(w) \text{ and } \#_c(w) = \#_d(w)\}$$

What are its equivalence classes? What does its minimal infinite-state machine look like?

**Answer:**

Clearly, we can see that we would require an infinite number of states to count the number of each letter in  $A$ . Formally, we can see the equivalence classes of  $L_{a=b, c=d}$  as such:

Consider the set of words

$$\{a^i | i \geq 0\} = \{\epsilon, a, aa, aaa, \dots\}$$

. If  $i \neq j \implies a^i \not\sim a^j$ , as

$$a^i b^i \in L_{a=b}, \text{ but } a^j b^i \notin L_{a=b}$$

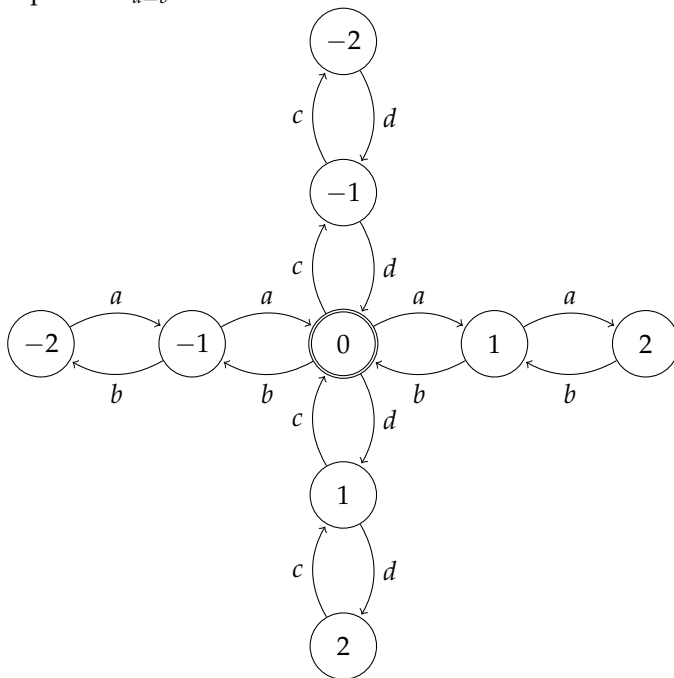
We need a similar set of words for  $c$ :

$$\{c^n | n \geq 0\} = \{\epsilon, c, cc, ccc, \dots\}$$

. If  $i \neq j \implies c^i \not\sim c^j$ , as

$$c^n d^n \in L_{c=d}, \text{ but } c^m d^n \notin L_{c=d}$$

Each possible  $i, n$  gives us an equivalence class. Note that this logic followed directly from the automata notes example for  $L_{a=b}$ .

**Group critiques****(b) Aaron Gonzales**

- Ex 7: Forgot to explain 2nd part of the question; otherwise correct.
- Ex: 8: Mentioned lack of both sides of "iff", but this was shown succinctly. Didn't explicitly mention difficulty of using DFAs.
- 9: complete and no critiques
- 12: Good approach; lacking slightly in clarity
- 13: No comments
- 14: Correct
- 20: Correct

**(c) Ahmad Darki**

- blah

**(d) Manasa Navada**

- Ex7: Regular expressions were not appropriate to use at that point in the assignment, though it was a correct answer.
- Ex8: difficult to follow explanation for part two and lack of proving both directions for the “iff”.
- Ex 9: No issues.
- Ex 12: Incorrect approach as the automata cannot keep track of a previous state. Difficult to follow.
- Ex 13: Correct; no issues.
- Ex 14: Correct; no issues.
- Ex 20: Poor explanation and lack of evidence as to why there are infinite equivalence classes.

**Group contributions****(e) Aaron Gonzales**

- Used explanations from his problems, typeset and compiled final report, extra work done on odd problems that were not initially completed or attempted.

**(f) Ahmad Darki**

- blah

**(g) Manasa Navada**

- blah