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Notation and Definitions

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Abstract
In the following, a set of suggestions for notations and definitions is presented. We might want to adopt some of this for future work. Let's discuss!

Keywords
Notation — Definitions — Transformations — Rotations — Quaternions

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Contents

1. Symbols

1.1 Basic Symbols

a	A lower-case symbol denotes a scalar (with common capital exceptions).
$\mathbf{a}$	A bold lower-case symbol denotes a vector.
$\boldsymbol{a}$	A bold lower-case italic symbol denotes a homogeneous vector.
$\mathbf{A}$	A bold capital symbol denotes a matrix.
$\mathbf{1}$	The identity matrix, optionally with dimension as subscript.
$\mathbf{0}$	A zero matrix, optionally with dimensions as subscripts.
$\mathscr{A}$	A set.

1.2 Spaces and Manifolds

$\mathbb{R}$	The Real numbers.
$\mathbb{C}$	The Complex numbers.
$\mathbb{R}^3$	The 3D Euclidean space.
S^3	The 3-Sphere group.
$SO(3)$	The 3D rotation group.
$SE(3)$	The Special Euclidean group.

In general the states live in a manifold, therefore we use a perturbation in tangent space $\mathfrak{g}$ and employ the group operator $\boxplus$, that is not commutative in general, the exponential exp and logarithm log. Now, we can define the perturbation $\delta\mathbf{x} := \mathbf{x} \boxplus \bar{\mathbf{x}}^{-1}$ around the estimate $\bar{\mathbf{x}}$. We use a minimal coordinate representation $\delta\boldsymbol{\chi} \in \mathbb{R}^{\dim \mathfrak{g}}$. A bijective mapping $\Phi : \mathbb{R}^{\dim \mathfrak{g}} \to \mathfrak{g}$ transforms from minimal coordinates to tangent space. Thus we obtain the transformations from and to minimal coordinates:

$$\delta\mathbf{x} = \exp(\Phi(\delta\boldsymbol{\chi})), \tag{1}$$
$$\delta\boldsymbol{\chi} = \Phi^{-1}(\log(\delta\mathbf{x})). \tag{2}$$

1.3 Stochastics

$\mathscr{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	The Normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$.
$E[.]$	The expectation operator.

1.4 Physical Parameters

g	Earth's acceleration due to gravity.

2. Coordinates and Transformations

2.1 Frames

- $\underrightarrow{\mathcal{F}}_A$ A frame of reference in $\mathbb{R}^3$.
- ${}_A\mathbf{a}$ The vector $\mathbf{a}$ expressed in $\underrightarrow{\mathcal{F}}_A$.
- ${}_A\mathbf{r}_P$ The position vector from the origin of $\underrightarrow{\mathcal{F}}_A$ to point P expressed in $\underrightarrow{\mathcal{F}}_A$.
- ${}_A\mathbf{r}_{BC}$ The position vector from the point of ${}_A\mathbf{r}_B$ to point ${}_A\mathbf{r}_C$ expressed in $\underrightarrow{\mathcal{F}}_A$, as shown in Fig.1.

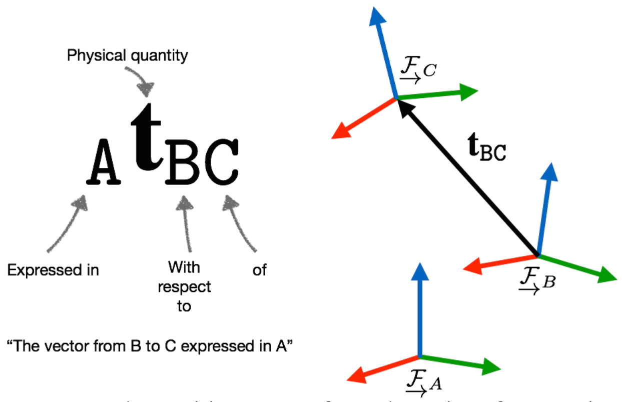

Figure 1. The position vector from the point of B to point C expressed in the frame A.

Generally speaking, we use $\underrightarrow{\mathcal{F}}_{C_i}$ for a camera frame (the i^{th}), $\underrightarrow{\mathcal{F}}_S$ for an IMU frame, and $\underrightarrow{\mathcal{F}}_B$ for a body frame. As for fixed frames, we may use $\underrightarrow{\mathcal{F}}_W$ as a World frame (any orientation) or $\underrightarrow{\mathcal{F}}_N$ for a navigation frame (fixed directions, e.g. East-North-Up (ENU) as complyant with the ROS standard).

2.2 Ellipsoid Coordinates

- ϕ The WGS 84 latitude.
- λ The WGS 84 longitude.
- h The WGS 84 altitude.

2.3 Orientation

- $\mathbf{C}_{AB}$ The rotation matrix $\mathbf{C}_{AB}$, represents the orientation of the frame, $\underrightarrow{\mathcal{F}}_B$, with respect to another frame, $\underrightarrow{\mathcal{F}}_A$.
  It also means that a vector expressed in the body frame, ${}_B\mathbf{a}$, to be rotated into the world frame as ${}_A\mathbf{a}$ by ${}_A\mathbf{a} = \mathbf{C}_{AB}\,{}_B\mathbf{a}$, as shown in Fig.2.
- $\mathbf{q}_{AB}$ A quaternion of rotation.
- φ roll angle (Tait-Bryan).
- θ pitch angle (Tait-Bryan).
- ψ yaw angle (Tait-Bryan).

2.3.1 Tait-Bryan Angles

Sometimes, it is useful to use a minimal global orientation parameterisation, e.g. for orientation plots. We adopt the Euler Angles ZYX $[\psi, \theta, \phi]$, aka Tait-Bryan angles (Flight convention: yaw, pitch, roll). Singularities are at $\theta = \pm\frac{\pi}{2}$.

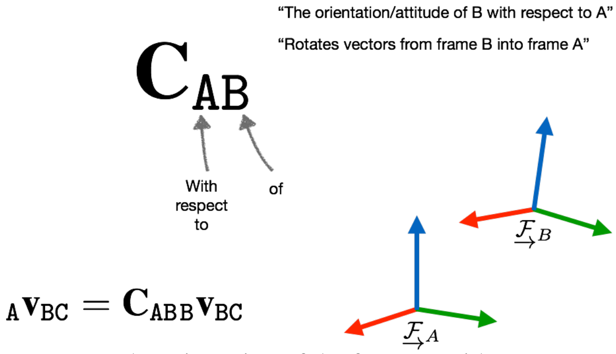

Figure 2. The orientation of the frame B with respect to frame A.

2.3.2 Quaternions

Quaternions provide a singularity-free alternative to Euler Angles. We use the Hamilton quaternion convention, i.e. :

$$Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, \tag{3}$$

$$i^2 = j^2 = k^2 = ijk = -1, \tag{4}$$

$$\|Q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1. \tag{5}$$

Note that $\mathbf{q}_{AB}$ and $-\mathbf{q}_{AB}$ represent the same rotation, but not the same unit quaternion. Rotation quaternion as vector: $\mathbf{q} = [q_0, q_1, q_2, q_3]^T = [a(\mathbf{q}), \mathbf{v}(\mathbf{q})]^T$ with $a(\mathbf{q}) = q_0$ and $\mathbf{v}(\mathbf{q}) := [q_1, q_2, q_3]^T$.

2.3.3 Angle-Axis and Rotation Vector

We use the Angle α and Axis $\mathbf{n}$ representation $[\alpha, \mathbf{n}^T]^T_{AB}$ as an alternative to a Quaternion $\mathbf{q}_{AB}$. The identity Angle-Axis: $[0, 1, 0, 0]^T$.

We may also write the same quantity as a Rotation Vector $\boldsymbol{\alpha}_{AB} = \alpha\mathbf{n}$.

2.3.4 Operators

- $\otimes$ The quaternion multiplication, such that $\mathbf{q}_{AC} = \mathbf{q}_{AB} \otimes \mathbf{q}_{BC}$.
- $[.]^+$ The left-hand quaternion matrix, such that $\mathbf{q}_{AC} = [\mathbf{q}_{AB}]^+ \mathbf{q}_{BC}$.
- $[.]^\oplus$ The right-hand quaternion matrix, such that $\mathbf{q}_{AC} = [\mathbf{q}_{BC}]^\oplus \mathbf{q}_{AB}$.
- $[.]^\times$ The cross-product matrix of a vector, such that $\mathbf{a} \times \mathbf{b} = [\mathbf{a}]^\times \mathbf{b}$.

Quaternion multiplication:

$$\mathbf{q} \otimes \mathbf{p} = \begin{bmatrix} (a(\mathbf{q})a(\mathbf{p}) - \mathbf{v}(\mathbf{q})^T\mathbf{v}(\mathbf{p}) \\ a(\mathbf{q})\mathbf{v}(\mathbf{p}) + a(\mathbf{p})\mathbf{v}(\mathbf{q}) + \mathbf{v}(\mathbf{q}) \times \mathbf{v}(\mathbf{p})) \end{bmatrix}. \tag{6}$$

Written in matrix multiplication form, we get

$$[\mathbf{q}_{AB}]^{+} = \begin{bmatrix} a & -\mathbf{v}^T \\ \mathbf{v} & a\mathbf{1} - [\mathbf{v}]^{\times} \end{bmatrix}, \tag{7}$$

$$[\mathbf{q}_{AB}]^{\oplus} = \begin{bmatrix} a & -\mathbf{v}^T \\ \mathbf{v} & a\mathbf{1} + [\mathbf{v}]^{\times} \end{bmatrix}, \tag{8}$$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]^{\times}\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{10}$$

2.3.5 Local Parameterisation

We use the minimal (3D) axis-angle perturbation of orientation $\delta\boldsymbol{\alpha} \in \mathbb{R}^3$ which can be converted into its quaternion equivalent $\delta\mathbf{q}$ via the exponential map:

$$\delta\mathbf{q} := \exp\left(\begin{bmatrix} 0 \\ \frac{1}{2}\delta\boldsymbol{\alpha} \end{bmatrix}\right) = \begin{bmatrix} \cos\left\|\frac{\delta\boldsymbol{\alpha}}{2}\right\| \\ \operatorname{sinc}\left\|\frac{\delta\boldsymbol{\alpha}}{2}\right\| \frac{\delta\boldsymbol{\alpha}}{2} \end{bmatrix}. \tag{11}$$

Therefore, using the group operator $\otimes$, we write

$$\mathbf{q}_{WS} = \delta\mathbf{q} \otimes \bar{\mathbf{q}}_{WS}. \tag{12}$$

Note that linearization of the exponential map around $\delta\boldsymbol{\alpha} = \mathbf{0}$ yields:

$$\delta\mathbf{q} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\alpha} \end{bmatrix} = \boldsymbol{\iota} + \frac{1}{2}\begin{bmatrix} \mathbf{0}_{1\times 3} \\ \mathbf{1}_3 \end{bmatrix}\delta\boldsymbol{\alpha}, \tag{13}$$

where $\boldsymbol{\iota}$ denotes the identity quaternion.

2.3.6 Transforming Representations

Quaternion to rotation matrix:

$$\mathbf{C}_{AB} = \mathbf{1}_{3\times 3} + 2a(\mathbf{q}_{AB})[\mathbf{v}(\mathbf{q}_{AB})]^{\times} + 2\left([\mathbf{v}(\mathbf{q}_{AB})]^{\times}\right)^2 =$$

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_0q_2+2q_1q_3 \\ 2q_0q_3+2q_1q_2 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_0q_1+2q_2q_3 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}. \tag{14}$$

Using the above perturbation, we obtain the following perturbation of the rotation matrix:

$$\mathbf{C}_{AB} = (\mathbf{1} + [\delta\boldsymbol{\alpha}]^{\times})\overline{\mathbf{C}}_{AB}. \tag{15}$$

For the inverse transformation, we get

$$\mathbf{C}_{BA} = \mathbf{C}_{BA}^T = \overline{\mathbf{C}}_{BA}(\mathbf{1} - [\delta\boldsymbol{\alpha}]^{\times}). \tag{16}$$

These two expressions are extremely useful when linearizing terms involving quantities undergoing coordinate transformations.

Tait-Bryan Angles to rotation matrix:

$$\mathbf{C}_{WB} = \tag{17}$$

$$\begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{18}$$

Quaternions to Angle-Axis: We get the Angle-Axis representation $[\alpha, \mathbf{n}^T]^T$ for $\|\mathbf{v}\| \geq \varepsilon$:

$$\alpha = 2\arccos a, \tag{19}$$

$$\mathbf{n} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{20}$$

for $\|\mathbf{v}\| < \varepsilon$, we use the identity Angle-Axis $[0, 1, 0, 0]^T$

2.4 Homogeneous Transformation

$\boldsymbol{T}_{AB}$ A homogeneous transformation matrix.

As a composition of rotation matrix and position vector, we write a homogeneous transformation matrix as

$$\boldsymbol{T}_{AB} = \begin{bmatrix} \mathbf{C}_{AB} & {}_A\mathbf{r}_B \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{AB} & {}_A\mathbf{r}_{AB} \\ \mathbf{0}^T & 1 \end{bmatrix}. \tag{21}$$

The resulting transformation matrix, $\boldsymbol{T}_{AB}$, represents the pose of the frame $\underrightarrow{\mathcal{F}}_B$ with respect to the frame $\underrightarrow{\mathcal{F}}_A$, such that a point expressed in the frame $\underrightarrow{\mathcal{F}}_B$, ${}_B\mathbf{p}$, can be transformed into the frame $\underrightarrow{\mathcal{F}}_A$, ${}_A\mathbf{p}$ by ${}_A\mathbf{p} = \boldsymbol{T}_{AB}\,{}_B\mathbf{p}$.

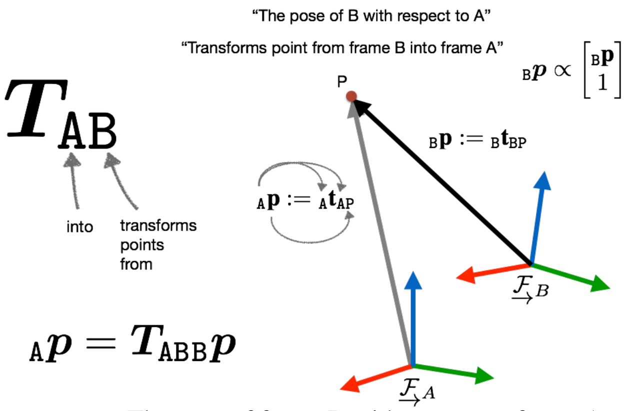

Figure 3. The pose of frame B with respect to frame A.

2.5 Image warping

${}_A\mathbf{I}_B$ Render the image from $\underrightarrow{\mathcal{F}}_B$ to $\underrightarrow{\mathcal{F}}_A$.
It can also be seen as viewing $\underrightarrow{\mathcal{F}}_A$ from $\underrightarrow{\mathcal{F}}_B$.

2.6 Lie algebra

$$\xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in \mathbb{R}^6 \tag{22}$$

$$\xi^{\wedge} = \begin{bmatrix} \phi^{\wedge} & \rho \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{23}$$

$$T = \exp(\xi^{\wedge}) = \begin{bmatrix} \exp(\phi^{\wedge}) & J\rho \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{24}$$

$$\approx 1 + \xi^{\wedge} \tag{25}$$

$$T \rightarrow \exp(\delta\boldsymbol{\xi})T \tag{26}$$

$$T^{-1} \rightarrow (\exp(\delta\boldsymbol{\xi})T)^{-1} = T^{-1}(\exp(\delta\boldsymbol{\xi}))^{-1} \tag{27}$$

$$\approx T^{-1}(\exp(-\delta\boldsymbol{\xi})) \approx T^{-1}(1 - \delta\boldsymbol{\xi}) \tag{28}$$

The above three approximations work for small perturbation $\delta\boldsymbol{\xi} \rightarrow 0$ only.

3. Time Derivatives of Position & Orientation

${}_A\mathbf{v}_{BC}$	The velocity of $\mathcal{F}_C$ as seen from $\mathcal{F}_B$ represented in $\mathcal{F}_A$.
${}_A\boldsymbol{\omega}_{BC}$	The rotational velocity of $\mathcal{F}_C$ w.r.t. $\mathcal{F}_B$ represented in $\mathcal{F}_A$.

In the following, we use $\mathcal{F}_I$ as an inertial frame and $\mathcal{F}_B$ as a (moving) body frame.

3.1 Linear Velocity

Velocity of point Q on rigid body $\mathcal{F}_B$ that rotates with ${}_B\boldsymbol{\omega}_{IB}$, w.r.t. to the inertial frame $\mathcal{F}_I$: ${}_B\mathbf{v}_{IQ} = {}_B\mathbf{v}_{IB} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_Q$.

3.2 Angular Velocity

${}_B\boldsymbol{\omega}_{IB}$	Absolute angular velocity of rigid body B expressed in frame B.
${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	Inverse of angular velocity.
${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$	Coordinate transformation of angular velocity from frame B to frame W.
${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$	Composition of angular velocity: .

3.2.1 Rotation Matrix Time Derivative vs. Angular Velocity

$$[{}_I\boldsymbol{\omega}_{IB}]^\times = \dot{\mathbf{C}}_{IB}\,\mathbf{C}_{IB}^T = \dot{\mathbf{C}}_{BI}^T\,\mathbf{C}_{BI}\,, \tag{29}$$

$$[{}_B\boldsymbol{\omega}_{IB}]^\times = \mathbf{C}_{IB}^T\,\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\,\dot{\mathbf{C}}_{BI}^T\,, \tag{30}$$

$$\dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\,[{}_B\boldsymbol{\omega}_{IB}]^\times. \tag{31}$$

3.2.2 Rotation Quaternion Time Derivative vs. Angular Velocity

$$\dot{\mathbf{q}}_{IB} = \boldsymbol{\Xi}(\mathbf{q}_{IB})\,{}_B\boldsymbol{\omega}_{IB} = \boldsymbol{\Omega}({}_B\boldsymbol{\omega}_{IB})\mathbf{q}_{IB}\,, \tag{32}$$

$$\boldsymbol{\Xi}(\mathbf{q}_{IB}) = \begin{bmatrix} \mathbf{v}(\mathbf{q}_{IB}) & a(\mathbf{q}_{IB})\mathbf{1} + [\mathbf{v}(\mathbf{q}_{IB})]^\times \end{bmatrix}, \tag{33}$$

$$\boldsymbol{\Omega}({}_B\boldsymbol{\omega}_{IB}) = \begin{bmatrix} 0 \\ {}_B\boldsymbol{\omega}_{IB} \end{bmatrix}^\oplus. \tag{34}$$

3.2.3 Angle-Axis Time Derivative vs. Angular Velocity

To rotation velocity:

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\alpha} + \dot{\mathbf{n}}\sin\alpha + [\mathbf{n}]^\times\dot{\mathbf{n}}(1-\cos\alpha), \tag{35}$$

$${}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\alpha} + \dot{\mathbf{n}}\sin\alpha - [\mathbf{n}]^\times\dot{\mathbf{n}}(1-\cos\alpha), \tag{36}$$

and back:

$$\dot{\alpha} = \mathbf{n}^T\,{}_I\boldsymbol{\omega}_{IB}\,, \tag{37}$$

$$\dot{\mathbf{n}} = \left(-\frac{1}{2}\frac{\sin\alpha}{1-\cos\alpha}([\mathbf{n}]^\times)^2 - \frac{1}{2}[\mathbf{n}]^\times\right){}_I\boldsymbol{\omega}_{IB}\,, \tag{38}$$

$$\dot{\alpha} = \mathbf{n}^T\,{}_B\boldsymbol{\omega}_{IB}\,, \tag{39}$$

$$\dot{\mathbf{n}} = \left(-\frac{1}{2}\frac{\sin\alpha}{1-\cos\alpha}([\mathbf{n}]^\times)^2 + \frac{1}{2}[\mathbf{n}]^\times\right){}_B\boldsymbol{\omega}_{IB}\,, \tag{40}$$

$$\forall\alpha \in \mathbb{R}\backslash\{0\}. \tag{41}$$

3.2.4 Rotation Vector Time Derivative vs. Angular Velocity

To rotation velocity:

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbf{1} + [\boldsymbol{\alpha}]^\times\left(\frac{1-\cos\|\boldsymbol{\alpha}\|}{\|\alpha\|^2}\right) + ([\boldsymbol{\alpha}]^\times)^2\left(\frac{\|\alpha\|-\sin\|\boldsymbol{\alpha}\|}{\|\boldsymbol{\alpha}\|^3}\right)\right)\dot{\boldsymbol{\alpha}}, \tag{42}$$

$${}_B\boldsymbol{\omega}_{IB} = \left(\mathbf{1} - [\boldsymbol{\alpha}]^\times\left(\frac{1-\cos\|\boldsymbol{\alpha}\|}{\|\alpha\|^2}\right) + ([\boldsymbol{\alpha}]^\times)^2\left(\frac{\|\alpha\|-\sin\|\boldsymbol{\alpha}\|}{\|\boldsymbol{\alpha}\|^3}\right)\right)\dot{\boldsymbol{\alpha}}, \tag{43}$$

$$\forall\|\boldsymbol{\alpha}\| \in \mathbb{R}\backslash\{0\}, \tag{44}$$

and back:

$$\dot{\boldsymbol{\alpha}} = \left(\mathbf{1} - \tfrac{1}{2}[\boldsymbol{\alpha}]^\times + ([\boldsymbol{\alpha}]^\times)^2\frac{1}{\|\boldsymbol{\alpha}\|^2}\left(1 - \frac{\|\boldsymbol{\alpha}\|}{2}\frac{\sin\|\boldsymbol{\alpha}\|}{1-\cos\|\boldsymbol{\alpha}\|}\right)\right){}_I\boldsymbol{\omega}_{IB}\,, \tag{45}$$

$$\dot{\boldsymbol{\alpha}} = \left(\mathbf{1} + \tfrac{1}{2}[\boldsymbol{\alpha}]^\times + ([\boldsymbol{\alpha}]^\times)^2\frac{1}{\|\boldsymbol{\alpha}\|^2}\left(1 - \frac{\|\boldsymbol{\alpha}\|}{2}\frac{\sin\|\boldsymbol{\alpha}\|}{1-\cos\|\boldsymbol{\alpha}\|}\right)\right){}_B\boldsymbol{\omega}_{IB}\,, \tag{46}$$

$$\forall\|\boldsymbol{\alpha}\| \in \mathbb{R}\backslash\{0\}. \tag{47}$$

3.2.5 Tait-Bryan Angles Time Derivatives vs. Angular Velocity

$$\begin{bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \\ 0 & \cos\phi & -\sin\phi \\ 1 & \frac{\sin\phi\sin\theta}{\cos\theta} & \frac{\cos\phi\sin\theta}{\cos\theta} \end{bmatrix}{}_B\boldsymbol{\omega}_{IB}\,, \tag{48}$$

$$\forall\theta \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z}.$$

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