Math 381 - Fall 2022

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Week 4

Last Time

1. Polynomial Interpolation: introduction and motivation
2. Lagrange polynomials
3. Barycentric formula

Today

1. Barycentric formula (continued)
2. Interpolation error
3. Runge phenomenon

Lagrange Polynomials

We will use a special polynomial basis that is very well suited for interpolation. Let $y_j = f(x_j)$. The Lagrange interpolation is given by

$$p(x) = \sum_{j=0}^{n} y_j L_j(x)$$

The basis functions $L_j(x)$ are polynomials of degree n that satisfy

$$L_j(x_i) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

These basis polynomials are

$$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

Existence and uniqueness theorem

For a given set of points (x_j, y_j), $j = 0, 1, \ldots n$, with distinct $x_0, \ldots, x_n$, there exists a unique Lagrange interpolating polynomial,

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Proof: By construction $p(x)$ is a polynomial of degree at most n and interpolates the points (x_j, y_j) (see previous lecture slides).

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Proof: By construction $p(x)$ is a polynomial of degree at most n and interpolates the points (x_j, y_j) (see previous lecture slides). To prove uniqueness, suppose that $q(x)$ is a ~~Lagrange~~ polynomial interpolating the points. We need to show that $r(x) = p(x) - q(x) = 0$.

$q(x_j) = y_j$ and $p(x_j) = y_j$

$\Rightarrow r(x_j) = p(x_j) - q(x_j) = 0$

$\Rightarrow r(\cdot)$ has $n+1$ roots at $x_0, \dots, x_n$

$\boxed{\text{The difference of two polynomials of degree (at most) } n \text{ is a polynomial of degree at most } n}$

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degree zero polynomial

$p(x) = C$

includes $C=0$

Proof: By construction $p(x)$ is a polynomial of degree at most n and interpolates the points (x_j, y_j) (see previous lecture slides). To prove uniqueness, suppose that $q(x)$ is a ~~Lagrange~~ polynomial interpolating the points. We need to show that $r(x) = p(x) - q(x) = 0$.The polynomial $r(x)$ has roots at the $n + 1$ points x_j. Hence, we must have

polynomial

polynomial of degree n+1

$$r(x) = C\tilde{r}(x)\boxed{\prod_{j=0}^{n}(x - x_j)},$$

unknown constant

However, $r(x)$ is a polynomial of degree at most n. It follows that $C = 0$.

The Barycentric formula is for practical use

$$p(x) = \sum_{j=0}^{n} y_j L_j(x), \quad L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x-x_i)}{(x_j-x_i)}$$

Barycentric interpolation formula

only place y appears

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j y_j}{(x-x_j)}}{\sum_{j=0}^{n} \frac{w_j}{(x-x_j)}},$$

where the "weights" are

$$w_j = \frac{1}{\prod_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i)}.$$

How do we derive the Barycentric formula?

Independent of y_j

$$\boxed{\psi(x) = \prod_{i=0}^{n} (x - x_i)}$$

We want to show that

Standard $\qquad$ Barycentric

$$p(x) = \sum_{j=0}^{n} y_j \underbrace{\prod_{\substack{i=0\\ i\neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}}_{L_j(x)} = \frac{\sum_{j=0}^{n} \frac{w_j y_j}{(x-x_j)}}{\sum_{j=0}^{n} \frac{w_j}{(x-x_j)}}, \quad w_j = \frac{1}{\prod_{\substack{i=0\\ i\neq j}}^{n}(x_j - x_i)}$$

$$= \sum_{j=0}^{n} w_j y_j \prod_{\substack{i=0\\ i\neq j}}^{n} (x-x_i) \frac{(x-x_j)}{(x-x_j)} = \psi(x) \sum_{j=0}^{n} \frac{w_j y_j}{x-x_j}$$

want to show $\psi(x) = \dfrac{1}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}}$.

Homework Problem: show

$$\psi(x) \sum_{j=0}^{n} \frac{w_j}{x-x_j} \equiv 1$$

Hint:
suppose $f(x) \equiv 1$

Theorem

Let $f \in C^{n+1}[a, b]$, and let p be the interpolating polynomial for f on distinct nodes $x_0, \ldots, x_n \in [a, b]$.Then, for every $x \in [a, b]$, we have

$$f(x) - p(x) = \frac{f^{(n+1)}(\eta_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

for some $\eta_x \in [a, b]$.
We will sketch proof next time.

Corollary

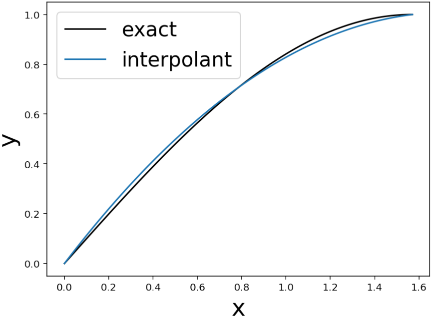
$$|f(x) - p(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \prod_{i=0}^{n} |x - x_i|$$

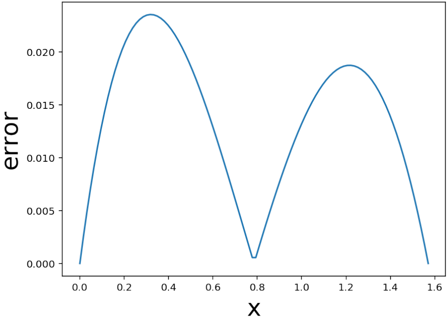
Approximation error of polynomial interpolant

We want to understand (bound) the approximation error $|f(x) - p(x)|$, $x \in [a, b]$.

Example

Let $f(x) = \sin(x)$, $x \in [0, \pi/2]$, and consider $n = 2$ (3 points) at $x_0 = 0$, $x_1 = \pi/4$, and $x_2 = \pi/2$.





Look at Week 4 Jupyter notebook