$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$e^{ix} = \cos(x) + i\sin(x)$$

Math 381 - Fall 2022

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Jay Newby

University of Alberta

$$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Week 7

$$\frac{\|f^{(n)}\|_\infty}{n!}\prod_{i=0}^{n}|x-x_n| \sim M\rho^n$$

Last Week

1. We had our midterm exam

This Week

1. Review of linear algebra

Matrices

Matrix

A real matrix $A \in \mathbb{R}^{n\times m}$ is

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}$$

Transpose

Matrix transpose

Given a real matrix $A \in \mathbb{R}^{n \times m}$, its transpose $A^T \in \mathbb{R}^{m \times n}$ has elements

$$A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & & \vdots \\ a_{1,m} & a_{2,m} & \cdots & a_{n,m} \end{bmatrix}$$

Rule

Given two matrices $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times q}$,

$AB \neq BA$

$$(AB)^T = B^T A^T.$$

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

Given three matrices

$$(ABC)^T = C^T B^T A^T.$$

Symmetric matrix

Symmetric matrix

$A = A^T$ or $a_{ij} = a_{ji}$.

Diagonal matrices

Eg $I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

Diagonal Matrix

A real matrix $A \in \mathbb{R}^{n \times n}$ is

$$A = \begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{bmatrix}$$

Diagonal matrices

We will omit the zeros when writing matrices in the future.

Diagonal Matrix

A real matrix $A \in \mathbb{R}^{n\times n}$ is

$$A = \begin{bmatrix} a_{1,1} & & & \\ & a_{2,2} & & \\ & & \ddots & \\ & & & a_{n,n} \end{bmatrix}$$

Tridiagonal matrices

Tridiagonal Matrix

A real matrix $A \in \mathbb{R}^{n\times n}$ is

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots & & 0 \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & 0 & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & & & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Tridiagonal matrices

We will omit the zeros when writing matrices in the future.

Tridiagonal Matrix

A real matrix $A \in \mathbb{R}^{n \times n}$ is

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & & \\ & a_{3,2} & a_{3,3} & a_{3,4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & a_{n-1,n} \\ & & & & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Triangular matrices

Upper Triangular Matrix

A real matrix $A \in \mathbb{R}^{n \times n}$ is

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{bmatrix}$$

Triangular matrices

We will omit the zeros when writing matrices in the future.

Upper Triangular Matrix

A real matrix $A \in \mathbb{R}^{n\times n}$ is

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ & a_{2,2} & \cdots & a_{2,n} \\ & & \ddots & \vdots \\ & & & a_{n,n} \end{bmatrix} \underbrace{\boxed{\begin{matrix} \\ \\ \\ x_n \end{matrix}}}_{x} = \underbrace{\boxed{\begin{matrix} \\ \\ \\ b_n \end{matrix}}}_{b}$$

Symmetric positive definite matrices

Symmetric positive definite matrix

A matrix $A \in \mathbb{R}^{n \times n}$ is Symmetric positive definite if and only if $A = A^T$ and

$$x^T A x > 0, \quad \forall x \neq 0.$$

Orthogonal matrices

related concept

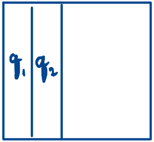
$A^{-1}A = I$

(Unitary Matrix)

Orthogonal matrix

A matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal if and only if

$$Q^T Q = I$$



$$q_i^T q_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Vector norms

2-norm

$$\|x\|_2 = \sqrt{\sum_{j=1}^{n} x_j^2}$$

Vector norms

1-norm

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|.$$

∞-norm

$$\|x\|_\infty = \max_{1\le j\le n} |x_j|.$$

Vector norms

p-norm

$$\|x\|_p = \left(\sum_{j=1}^{n} x_j^p \right)^{1/p}$$

Vector norms

Definition of a norm

A norm satisfies three requirements

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in \mathbb{R}$.
3. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$ Triangle inequality

Matrix norms

Induced matrix norm

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Submultiplicative property

$$\|AB\| \leq \|A\| \|B\|$$

$$\|Ax\| \leq \|A\| \|x\|$$

Spectral Radius

Spectral radius

Let $A \in \mathbb{R}^{n\times n}$ and let Ω be the set of eigenvalues of A. The spectral radius is defined as

$$\rho(A) = \max\{|\lambda|; \lambda \in \Omega\}$$

$\rho(A) \leq \|A\|$

Eigenvalues and eigenvectors

Given a square matrix $A \in \mathbb{R}^{n\times n}$, an eigenvector $v \in \mathbb{C}^n$ and eigenvalue $\lambda \in \mathbb{C}$ satisfy

$$Av = \lambda v.$$

Diagonalizable matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if it can be decomposed with

$$A = V \Lambda V^{-1},$$

where $V \in \mathbb{C}^{n \times n}$ is the matrix formed with columns given by the eigenvectors and $\Lambda \in \mathbb{C}^{n}$ is a diagonal matrix with diagonal elements given by the eigenvalues.