Math 381 - Fall 2022

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Week 8

Last Time

1. Gaussian Elimination (general case)
2. Gaussian Elimination with partial pivoting

This Time

1. Condition number of a matrix
2. Error bounds
3. Stability
4. Backward error analysis

Determinants are a bad measure of distance to a singular matrix

Even though $\det(A) = 0$ if and only if A is singular, it does not follow that A is closer to singular than B if

$$|\det(A)| < |\det(B)|$$

.

Example $A = 0.1I$:

$$A = 0.1I = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$\det(A) = \prod_{i=1}^{n} \lambda_i$ does not depend on the eigenvectors

Condition number of a matrix

$\|A\|_2 = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \|Ax\|$

Definition: condition number of a matrix

$$\kappa(A) = \|A\| \|A^{-1}\|.$$

- Depends on the norm used (when specified we use $\kappa_p(A)$)
- $1 \le \kappa(A) < \infty$ $\quad I = AA^{-1} \Rightarrow 1 = \|I\| = \|AA^{-1}\| \le \|A\| \|A^{-1}\|$
- $\kappa(A) = \infty$ if A is singular

Claim: If $A \in \mathbb{R}^{n \times n}$ is orthogonal then $\kappa_2(A) = 1$

proof: Let $x \in \mathbb{R}^n$ s.t. $\|x\| = 1$.

$\|A\| \|A^{-1}\| = \|Ax\|_2^2 = x^T A^T A x = x^T I x = x^T x = \|x\|_2^2 = 1$

Condition number of a matrix

$$\sigma_i = \sqrt{\lambda_i} \quad \Big| \ \lambda_i \text{ eigenvalue of } A^TA$$

- $\kappa_2(A) = \frac{\max_{1\leq i\leq n}\sigma_i}{\min_{1\leq i\leq n}\sigma_i}$
- For diagonal matrices $\kappa(A) = \frac{\max_{1\leq i\leq n}|a_{ii}|}{\min_{1\leq i\leq n}|a_{ii}|}$

The 2-norm condition number is inversely proportional to the 2-norm distance to the nearest singular matrix

Claim:

Let

$$\epsilon = \min_{E \in \mathbb{R}^{n \times n}} \{\|E\|_2; \det(A+E) = 0\}.$$

We have that

$$\frac{\epsilon}{\|A\|_2} = \frac{1}{\kappa_2(A)}.$$

Backward error analysis

Root finding

$|x-\hat{x}| < C\varepsilon$ if $f'(\hat{x}) \neq 0$

$|x-\hat{x}| < M\sqrt{\varepsilon}$ if $f'(\hat{x}) = 0$

Lagrange interp

$|\hat{p}(x) - p(x)| \leq \kappa_n \varepsilon$

Goal: bound the relative error of the approximate solution

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa\epsilon.$$

Two steps:

1. Investigate the sensitivity of the problem to perturbations
2. Establish the backward stability of the algorithm

Step 1: Investigate the sensitivity of the problem to perturbations

Conditioning of the problem:

Suppose that the problem $Ax = b$ is perturbed so that

$$[A + \delta A]\hat{x} = b + \delta b,$$

where

$$\frac{\|\delta A\|}{\|A\|} = O(\epsilon), \quad \frac{\|\delta b\|}{\|b\|} = O(\epsilon).$$

Then the resulting perturbation to the solution is bounded by

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa\epsilon.$$

Note that this does not depend on any specific algorithm.

Step 2: Establish the backward stability of the algorithm

Definition: backward stable algorithm

An algorithm to solve $Ax = b$ is said to be backward stable if it produces an approximation $\hat{x}$ satisfying

$$[A + \delta A]\hat{x} = b + \delta b,$$

for some $\delta A \in \mathbb{R}^{n \times n}$ and $\delta b \in \mathbb{R}^n$ such that

$$\frac{\|\delta A\|}{\|A\|} = O(\epsilon), \quad \frac{\|\delta b\|}{\|b\|} = O(\epsilon).$$

In other words, the algorithm that is backwards stable yields the exact answer to a perturbed problem.

Backward error analysis

Combining the two steps:

If the problem $Ax = b$ is well posed and the algorithm is backward stable then it will produce approximate solutions $\hat{x}$ such that,

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa\epsilon.$$

We cannot expect accurate answers to badly conditioned problems

Backward error analysis

Step 1:

Investigate the sensitivity of the problem to perturbations

Consider the perturbations $\delta A \in \mathbb{R}^{n\times n}$ and $\delta b \in \mathbb{R}^{n}$ such that

$$(A + \delta A)\hat{x} = b + \delta b.$$

Backward error analysis

$Ax = b \Rightarrow \|A\|\|x\| \geq \|b\|$

$\Rightarrow \boxed{\frac{1}{\|b\|} \geq \frac{1}{\|A\|\|x\|}}$

$\kappa(A) = \|A\|\|A^{-1}\|$

Definition: residual

The amount by which the approximate solution fails to satisfy the equation:

$$\hat{r} = A\hat{x} - b.$$

Claim:

Let $\hat{x}$ be the numerical solution to $Ax = b$. The relative error is bounded by

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa(A)\frac{\|\hat{r}\|}{\|b\|}$$

Proof: $\hat{r} = A\hat{x} - b = A\hat{x} - Ax = A(\hat{x} - x)$.

$\Rightarrow \hat{x} - x = A^{-1}\hat{r} \Rightarrow \frac{\|\hat{x} - x\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\hat{r}\|}{\|x\|}\frac{\|A\|}{\|A\|} = \frac{\kappa(A)\|\hat{r}\|}{\|A\|\|x\|} \leq \kappa(A)\frac{\|\hat{r}\|}{\|b\|}$

Backward error analysis

Claim:

Consider the perturbations $\delta A \in \mathbb{R}^{n\times n}$ and $\delta b \in \mathbb{R}^n$ such that

$$(A+\delta A)\hat{x} = b + \delta b.$$

$$\frac{\|x-\hat{x}\|}{\|x\|} \le \frac{\kappa \|\hat{r}\|}{\|b\|}$$

The relative error is bounded by

$$\frac{\|x-\hat{x}\|}{\|x\|} \le \kappa(A)\frac{\|\delta b\| + \|\delta A\|\|\hat{x}\|}{\|b\|}$$

Implicit Bound

Proof: $(A+\delta A)\hat{x} = b + \delta b$

$\Rightarrow A\hat{x} + \delta A\hat{x} = b + \delta b$

$\hat{r} = -\delta A\hat{x} + \delta b$

$\Rightarrow \|\hat{r}\| \le \|\delta A\|\|\hat{x}\| + \|\delta b\|$

$$\frac{\|x-\hat{x}\|}{\|x\|} \le \kappa(A)\frac{\|\delta b\| + \|\delta A\|\|\hat{x}\|}{\|b\|}$$

Explicit

Backward error analysis

Claim:

Assuming that $\|\delta A\| < 1/\|A^{-1}\|$ and that $A + \delta A$ is invertible (see homework), the previous relative error bound can be written as

$$\frac{\|x - \hat{x}\|}{\|x\|} \le \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right).$$

Proof: Let $\hat{x} = x + \delta x$

$(A+\delta A)(x+\delta x) = b + \delta b$

$\Rightarrow \cancel{Ax} + \delta A x + (A+\delta A)\delta x = \cancel{b} + \delta b$

$\Rightarrow [A+\delta A]\delta x = -\delta A x + \delta b$

$\Rightarrow \delta x = [A+\delta A]^{-1}[-\delta A x + \delta b]$

$\Rightarrow \|\delta x\| \le \|[A+\delta A]^{-1}\|(\|\delta A\|\|x\| + \|\delta b\|)$

$\frac{\|\delta x\|}{\|x\|} \le \|[A+\delta A]^{-1}\|\|A\|\left(\frac{\|\delta b\|}{\|A\|\|x\|} + \frac{\|\delta A\|}{\|A\|}\right)$

$\frac{1}{\|b\|} \ge \frac{1}{\|A\|\|x\|}$

$\frac{\|\delta x\|}{\|x\|} \le \|[A+\delta A]^{-1}\|\|A\|\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)$

Homework problem

Backward error analysis