Math 381 - Fall 2022

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Week 3

Today

1. Landau notation

Motivation

Operation count

An algorithm for solving $Ax = b$, with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, requires

$$\frac{1}{3}n^3 + 6n^2 + 100n$$

floating point operations (FLOPS)

Motivation

Fixed point iteration

The error at iteration k of a fixed point iteration is $E_k = |x_k - \hat{x}|$

$$E_{k+1} = E_k |g'(\hat{x})| + \frac{E_k^2}{2} |g''(\hat{x})| + \cdots$$

Goal:

Rank the terms of an expression by significance.

Which terms can we ignore? Which terms are insignificant compared to the most significant term?

Motivation

Exponential function near zero

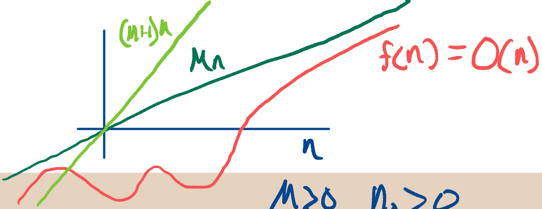
$$e^{\epsilon} = 1 + \epsilon + \frac{\epsilon^2}{2} + \sum_{j=3}^{\infty} \frac{\epsilon^j}{j!}$$

Examples

We want to understand more than the limit $n \to \infty$, we want to understand *how* a function approaches the limit.

- $\frac{n^2}{1+n^2}$
- $1 + \frac{1}{n}\sin(n)$
- $1 - e^{-n}$ $e^{-n} \to 0$ faster than n^{-j}, $j > 0$

"Big O" Notation



Big O

$M>0 \quad n_0>0$

We write $f(n) = O(n^k)$ if there exists constants M and n_0 such that

$$|f(n)| \leq Mn^k, \quad n > n_0$$

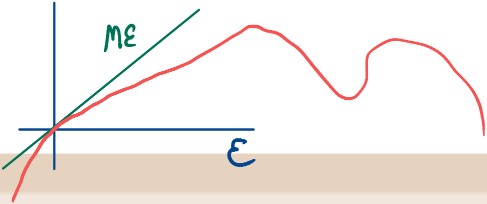
Alternate definition

We write $f(n) = O(n^k)$ if

$$\lim_{n\to\infty} \frac{|f(n)|}{n} = M_*$$

$$\limsup_{n\to\infty} \frac{|f(n)|}{n^k} < \infty$$

“Big O” Notation



Big O

We write $f(\epsilon) = O(\epsilon^k)$ if there exists constants M and $\epsilon_0 > 0$ such that

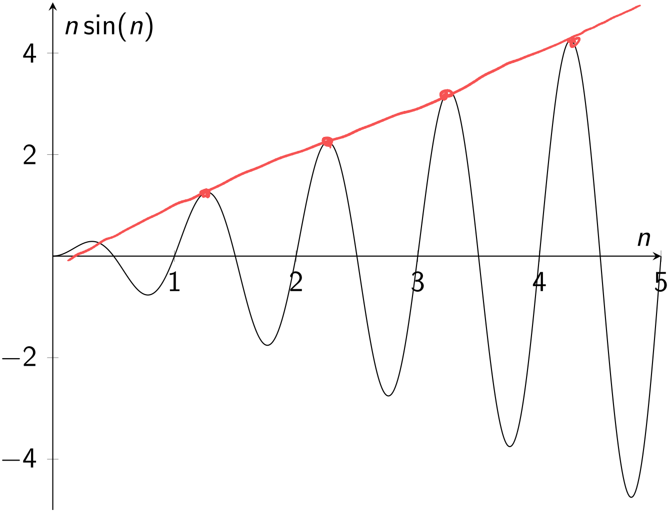
$$|f(\epsilon)| \leq M\epsilon^k, \quad 0 < \epsilon < \epsilon_0$$

Alternate definition

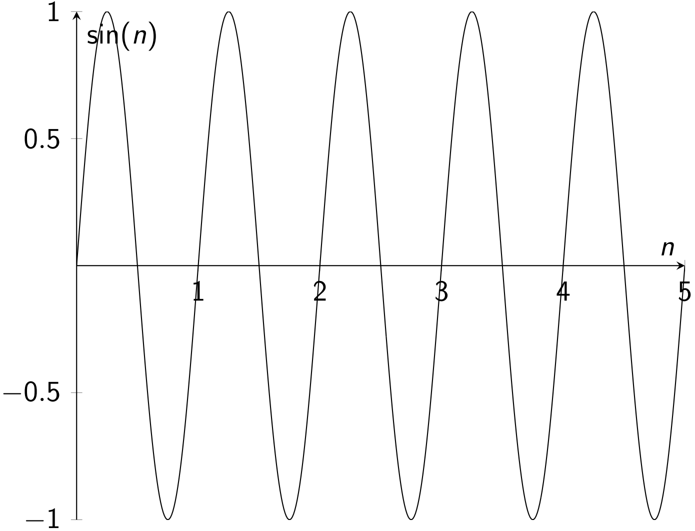
We write $f(\epsilon) = O(\epsilon^k)$ if

$$\limsup_{\epsilon \to 0^+} \frac{|f(\epsilon)|}{\epsilon^k} < \infty$$

Example using the limit superior



Example using the limit superior



Example: $f(n) = O(n^2)$

"f grows like n^2."

Example: $f(\epsilon) = O(\epsilon^2)$

"f goes to zero like ϵ^2."

“Little o” Notation

Little o

We write $f(n) = o(n^k)$ if for every $h > 0$ there exists a $n_0 > 0$ such that

$$|f(n)| \leq hn^k, \quad n > n_0$$

Alternate definition

We write $f(n) = o(n^k)$ if

$$\lim_{n\to\infty} \frac{f(n)}{n^k} = 0$$

"Little o" Notation

Little o

We write $f(\epsilon) = o(\epsilon^k)$ if for every $h > 0$ there exists a $\epsilon_0 > 0$ such that

$$|f(\epsilon)| \leq h\epsilon^k, \quad 0 < \epsilon < \epsilon_0$$

Alternate definition

We write $f(\epsilon) = o(\epsilon^k)$ if

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\epsilon^k} = 0.$$

Example: $f(n) = o(n^2)$

"f grows slower than n^2."

Example: $f(\epsilon) = o(\epsilon^2)$

"f goes to zero faster than ϵ^2."