Math 381 - Fall 2022

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Week 7

Last Time

1. Matrices
2. Transpose
3. Special matrices (diagonal, tridiagonal, triangular, symmetric positive definite, orthogonal)
4. Vector and matrix norms
5. Eigenvalues and eigenvectors
6. Diagonalizable matrices

Today

1. Linear independence
2. Linear systems
3. Singular matrices
4. Nullspace and range of a matrix

Linear independence

Linear independent set of vectors

A set of n vectors $v_j \in \mathbb{R}^n$, $j = 1, \ldots, n$ is linearly independent if and only if

$$\sum_{j=1}^{n} \alpha_j v_j \neq 0,$$

for every set of constants α_j where at least one of the constants in nonzero.

Linear independence

Linear independent set of vectors (Version 2)

A set of n vectors $v_j \in \mathbb{R}^n$, $j = 1, \dots, n$ is linearly independent if and only if the matrix with columns given by the vectors v_j is non-singular.

Singular Matrix

A matrix $A \in \mathbb{R}^{n \times n}$ is singular if and only if there exists a vector $\rho \in \mathbb{R}^n$ with $\rho \neq 0$ such that $A\rho = 0$.

$A\rho = \lambda\rho, \quad \lambda = 0$

Existence and uniqueness of solutions to linear systems

Theorem

Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. The equation $Ax = b$ has a unique solution if and only if A is nonsingular.

Theorem

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The equation $Ax = b$ has a solution (not necessarily unique) if and only if $\eta^T b = 0$ for every $\eta \in \mathbb{R}^m$ such that $\eta^T A = 0$.

Corollary

If a solution to $Ax = b$ exists, it is either the only solution or there are infinitely many solutions.

$Ax_1 = b, \quad Ap = 0$

$x_2 = x_1 + cp$

$Ax_2 = Ax_1 + cAp = b + 0$

$Ax = b, \quad \eta^T A = 0, \quad A^T \eta = 0$

$\eta^T A x = \eta^T b$

$\Rightarrow 0 = \eta^T b$

Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

$|A| = ad - bc$

Nullspace and range of a matrix

$A : \mathbb{R}^n \to \mathbb{R}^m$

$\text{range}(A) = R(A)$

Range of a matrix

Let $A \in \mathbb{R}^{m \times n}$. The range of A is the space spanned by the column vectors of A. In other words, the vector $Ax \in \text{range}(A)$ for all $x \in \mathbb{R}^n$.

Nullspace of a matrix

The nullspace of a matrix $A \in \mathbb{R}^{m \times n}$ contains all null vectors. In other words, if $A\rho = 0$ then $\rho \in \text{null}(A)$.

$\text{Null}(A) = N(A)$

Singular value decomposition (SVD)

$VV^T = I_n \qquad UU^T = I_m$

For every matrix $A \in \mathbb{C}^{m\times n}$ there exists unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ such that

Reduced SVD

$A = \hat{U}\hat{\Sigma}\hat{V}^T$

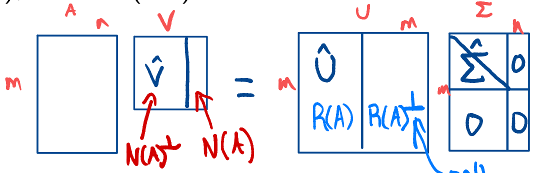
$$A = U\Sigma V^* \qquad A = U\Sigma V^T$$

This is a construction of orthonormal bases for range(A), range(A^T), null(A), and null(A^T).

Rank-Nullity Theorem

$\dim(R(A)) + \dim(N(A)) = n$

$\dim(R(A^T)) + \dim(N(A^T)) = m$



ortho onal complement