Math 381 - Fall 2022

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Week 4

Last Week

1. Solving scalar nonlinear equations
2. Fixed point iteration
3. Newton's method
4. Convergence rate
5. Landau notation

This Week: Polynomial interpolation

1. Motivation
2. Approximation theory
3. Lagrange polynomials
4. Barycentric formula
5. Approximation error
6. Runge Phenomenon

Wouldn't it be nice if we only needed polynomials?

Pros:

- Derivatives are easy
- integration is easy
- root finding is easy
- minimization/maximization is easy

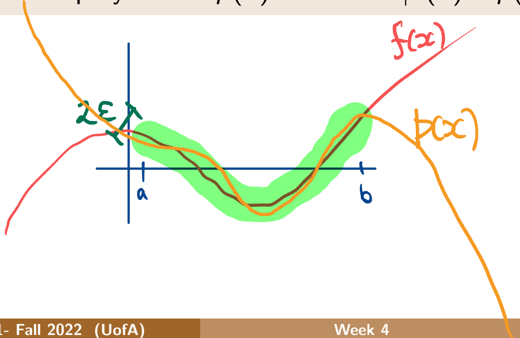
Cons:

- ...

Weierstrass approximation theorem

Weierstrass approximation theorem

Let $f(x)$ be a continuous function on the interval $[a, b]$. For every $\epsilon > 0$ there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$.



The derivative of a polynomial is a polynomial

$p(x) = 5x^2 \qquad p'(x) = 10x$

Given

$$p(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n,$$

we have

$$p'(x) = c_1 + 2c_2 x \cdots + (n-1)c_{n-1}x^{n-2} + nc_n x^{n-1},$$

We have explicit formulas for nth derivative.

The antiderivative of a polynomial is a polynomial

Given

$$p(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n,$$

we have

$$\int p(x) dx = c_0 x + \frac{c_1}{2} x^2 + \cdots + \frac{c_{n-1}}{n} x^n + \frac{c_n}{n+1} x^{n+1} + C,$$

We have explicit formulas for nth antiderivative.

Roots

multiplicity

e.g. $(x-3)^2 = (x-3)(x-3)$

Companion matrix

The roots of the polynomial,

$$p(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + x^n,$$

are given by the eigenvalues of the *Companion matrix*,

$$C(p) = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}$$

identity matrix

$\det(C - \lambda I) = 0$ Characteristic polynomial

Optimization

Solve

$$p'(x) = 0$$

roots can be minima/maxima

If we can compute derivatives and roots then we can solve optimization problems

Polynomial approximations can be used to numerically solve differential equations

There is a connection between finite differences and polynomial approximations (more on this in future lectures and homework)

Centered finite difference formula

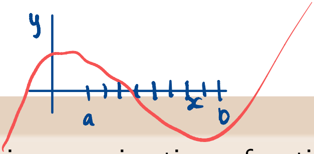
$$\frac{dy}{dx} = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

Exact for *any* second degree polynomial. $f(x) = 5x^2$, $f'(x) = 10x$

$$D_{\text{centered}} f = \frac{5(x+h)^2 - 5(x-h)^2}{2h} = \frac{5}{2h}\left(\cancel{x^2} + 2hx + \cancel{h^2} - \cancel{x^2} + 2hx - \cancel{h^2}\right)$$

$$= \frac{5}{2h} 4hx = 10x$$

How do we approximate a function by a polynomial?



Interpolation nodes:

Suppose we are interested in approximating a function $f(x)$ on the interval $x \in (a, b)$. We pick $n+1$ distinct points x_j, $j = 0, 1, \ldots, n$ such that

$$a \leq x_0 < x_1 < \cdots < x_n \leq b$$

Example: uniform nodes

$$x_j = a + (b - a)\frac{j}{n}$$

Nodes are specific fixed values. We want to find a polynomial that approximates $f(x)$ for any $x \in (a, b)$

How do we approximate a function by a polynomial?

Polynomial Interpolation:

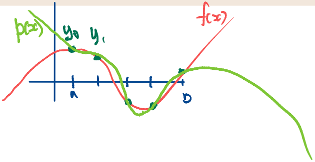
Suppose we are given a function $f(x)$ and we want to compute a polynomial $p(x)$ such that

y_i

$$p(x_j) = f(x_j), \quad j = 0, 1, \ldots, n. \tag{1}$$

$C_0 + C_1 x + C_2 x^2 + \ldots$

Let



$$p(x) = \sum_{j=0}^{n} c_j x^j,$$

The constants c_j are chosen so that (1) is satisfied.

Does equation (1) appear in the Weierstrass approximation theorem?

Lagrange Polynomials

We will use a special polynomial basis that is very well suited for interpolation. Let $y_j = f(x_j)$. The Lagrange interpolation is given by

basis fn : independent of y_j and $f(x)$

$$p(x) = \sum_{j=0}^{n} y_j L_j(x)$$

at $x = x_i$

The basis functions $L_j(x)$ are polynomials of degree n that satisfy

$$L_j(x_i) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$p(x_i) = \sum_{j=0}^{n} y_j L_j(x_i)$$

$$= y_i L_i(x_i)$$

$$= y_i$$

Example $x_0 = 1,\ x_1 = 2,\ x_2 = 4$ $\leftarrow$ 3 nodes

$y_0 = 1,\ y_1 = 3,\ y_2 = 3$

$$p(x) = 1 \cdot L_0(x) + 3 \cdot L_1(x) + 3 \cdot L_2(x)$$

$$L_0(x) = A(x-2)(x-4)$$

$$L_0(x_0) = 1 = A(1-2)(1-4) \Rightarrow A = \frac{1}{3}$$

$$L_1(x) = B(x-x_0)(x-x_2)$$

$$L_1(x_1) = 1$$

Given interpolation nodes x_j, $j = 0, 1, \ldots, n$, with $x_i \neq x_j$ for $i \neq j$, the basis functions are

$$
\begin{aligned}
L_j(x) &= \frac{(x - x_0)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{(x_j - x_0)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n)} \\
&= \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}
\end{aligned}
$$

The Barycentric formula is for practical use

$$p(x) = \sum_{j=0}^{n} y_j L_j(x) \quad , \quad L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

Barycentric interpolation formula

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j y_j}{(x - x_j)}}{\sum_{j=0}^{n} \frac{w_j}{(x - x_j)}},$$

where the “weights” are

$$w_j = \frac{1}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i)}.$$