Math 381 - Fall 2022

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Week 5

Last Time

$y_0 = p(x_0)$

$D[f](x) = f'(x)$

$Dp = p'$

$$\boxed{\begin{matrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{matrix}}$$

1. Differentiation of Lagrange polynomials

Today

1. Finish differentiation of Lagrange polynomials
2. Integration of Lagrange polynomials

Differentiation with Lagrange polynomials

We will represent differentiation with a matrix

$$q(x_i) = \sum_{j=0}^{n} d_{ij} \overbrace{p(x_j)}^{y_j},$$

where

Step 1 $$d_{ij} = \frac{w_j}{w_i(x_i - x_j)}, \quad i \neq j$$

Step 2 $$d_{ii} = -\sum_{\substack{k=0 \\ k \neq i}}^{n} d_{ik}.$$

Proof: preliminaries

Definition we will use this week

$$L_j(x) = w_j l_j(x), \qquad l_j(x) = \prod_{\substack{i=0 \\ i\neq j}}^{n} (x - x_i)$$

Recall

$L_j(x_i) = \delta_{ij}$

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$

$$w_j = \frac{1}{\prod_{\substack{i=0 \\ i\neq j}}^{n} (x_j - x_i)}$$

$l_j(x_i) = \frac{1}{w_j}\delta_{ij}$

$$p(x) = \sum_{j=0}^{n} p(x_j) w_j l_j(x)$$

$$p'(x) = \sum_{j=0}^{n} p(x_j) w_j l_j'(x)$$

Proof: $(i \neq j)$

We wil use $l_j(x_j) = 1/w_j$, $l_j(x_i) = 0$.

$$d_{ij} = \frac{w_j}{w_i(x_i - x_j)}$$

$$q(x_i) = p'(x_i) = \sum_{j=0}^{n} \overbrace{p(x_j)}^{y_j} \overbrace{\boxed{w_j l_j'(x_i)}}^{d_{ij}}.$$

$$\boxed{\text{Want to show } l_j'(x_i) = \frac{1}{w_i(x_i - x_j)}}$$

$$\log(l_j(x)) = \log\Big(\prod_{\substack{k=0\\k\neq j}}^{n}(x - x_k)\Big) = \sum_{\substack{k=0\\k\neq j}}^{n}\log(x - x_k) \Rightarrow l_j'(x) = l_j(x)\sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x - x_k}$$

can't evaluate at x_i

$$l_j'(x) = \sum_{\substack{k=0\\k\neq j}}^{n}\frac{(x-x_j)\,l_j(x)}{(x-x_j)(x-x_k)} = \sum_{\substack{k=0\\k\neq i}}^{n}\frac{(x-x_i)\prod_{\substack{r=0\\r\neq j}}^{n}(x-x_r)}{(x-x_j)(x-x_k)}$$

$$= \sum_{\substack{k=0\\k\neq i}}^{n}\frac{\cancel{(x-x_k)}\prod_{\substack{r=0\\r\neq k}}^{n}(x-x_r)}{(x-x_j)\cancel{(x-x_k)}} = \sum_{\substack{k=0\\k\neq i}}^{n}\frac{l_k(x)}{x-x_j}$$

$$\Rightarrow l_j'(x_i) = \sum_{\substack{k=0\\k\neq i}}^{n}\frac{l_k(x_i)}{x_i - x_j} = \frac{1}{w_i(x_i - x_j)}$$

∎

Proof: $(i = j)$

Recall Homework $\sum_{k=0}^{n} L_k(x) = \sum_{k=0}^{n} w_k \ell_k(x) = 1$

$$\Rightarrow \sum_{k=0}^{n} w_k \ell_k'(x) = 0 = \sum_{k=0} w_k \ell_k'(x_i)$$

$$\Rightarrow w_i \ell_i'(x_i) + \sum_{\substack{k=0 \\ k \neq i}}^{n} w_k \ell_k'(x_i) = 0$$

$$w_i \ell_i'(x_i) = -\sum_{\substack{k=0 \\ k \neq i}}^{n} w_k \ell_k'(x_i)$$

$$\boxed{d_{ij} = w_j \ell_j'(x_i)}$$

$$d_{ii} = -\sum_{\substack{k=0 \\ k \neq i}}^{n} d_{ik}$$

We will study solving linear systems of equations in Week 7-9

$x = \text{solve}(A, b)$

When solving a linear system of equations $Ax = b$, we do not compute the matrix inverse A^{-1} explicitly. It is more efficient to employ a solver algorithm that computes x given A and b.

$x = A^{-1}b$

Goal for integration of Lagrange polynomials

Goal: compute indefinite integral

For Lagrange polynomial $q(x)$, compute

$$p(x) = \int_a^x q(u)du.$$

From calculus

$$\frac{d}{dx}p(x) = q(x)$$

$$\int q(x)dx = \int \frac{d}{dx}p(x)dx = p(x) + \underline{C}$$

$$\int_a^x q(u)du = \int_a^x \frac{d}{dx}p(u)du = p(x) - \underline{p(a)}$$

Can we follow the same approach as polynomial differentiation?

$$\int_a^x q(u)du = p(x) - p(a)$$

$$Rq = p - p(a)$$

$$\boxed{\quad R \quad} \begin{bmatrix} q(x_0) \\ q(x_1) \\ \vdots \\ q(x_n) \end{bmatrix} = \begin{bmatrix} p(x_0) \\ p(x_1) \\ \vdots \\ p(x_n) \end{bmatrix} - p(a)$$

$$r_{ij} = \frac{1}{l_i'(x_i)} \int_a^{x_i} l(u)du$$

But there a better way...

$$\mathcal{D}[p](x) = q(x) \quad \Rightarrow \quad p(x) = \mathcal{D}^{-1}[q](x) \text{ ???}$$

Idea:

$Dp = q$

Can we compute?

$p = D^{-1}q$

Recall: for some $p \neq 0$

$Dp = 0$

$D1 = \sum_{i=0}^{n} d_i \cdot 1 = 0$

Next time...

Next Time:

- Derive linear system for integration
- Analysis of integration approximation accuracy
- Roots of Lagrange polynomials (maybe)
- Solving differential equations
- The PyCheb package (from the makers of Chebfun for Matlab)