Math 381 - Fall 2022

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Week 4

Last Time

1. Existence and uniqueness theorem for interpolating polynomial
2. Derived the Barycentric formula
3. Error bound for $|f(x) - p(x)|$, $x \in [a, b]$
4. Examples in Jupyter Week 4 notebook

Today

1. Error bound (continued)
2. Runge Phenomena
3. Chebyshev nodes

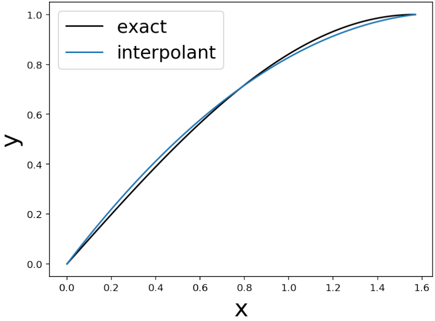
Error Bound

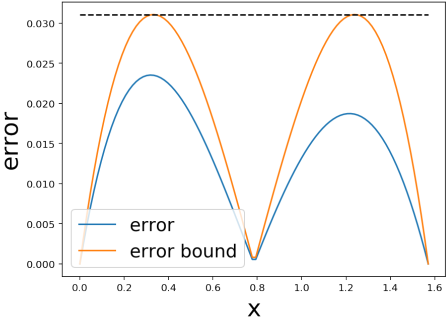
Let $f \in C^{n+1}[a, b]$, and let p be the interpolating polynomial for f on distinct nodes $x_0, \ldots, x_n \in [a, b]$. Then, for every $x \in [a, b]$, we have

$$|f(x) - p(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \prod_{i=0}^{n} |x - x_i|.$$

Example

Let $f(x) = \sin(x)$, $x \in [0, \pi/2]$, and consider $n = 2$ points at $x_0 = 0$, $x_1 = \pi/4$, and $x_2 = \pi/2$.





Uniform nodes

We will write formula for $a=-1, b=1$ $[-1,1]$

For the interval $[-1, 1]$,

$$x_j = -1 + \frac{2j}{n}.$$

For the interval $[a, b]$,

$$x_j = a + \frac{(b-a)j}{n}.$$

Barycentric weights for uniform nodes

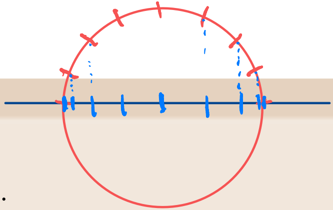
Homework problem

$$w_j = \frac{\left(\frac{n}{2}\right)^n (-1)^{n-j}}{n!} \binom{n}{j}$$

$x_j \in [-1, 1]$

$$w_j = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{1}{(x_j - x_i)}$$

Chebyshev nodes



For the interval $[-1,1]$,

$$x_j = -\cos\left(\frac{j\pi}{n}\right).$$

For the interval $[a,b]$,

$$x_j = \frac{a+b}{2} - \frac{b-a}{2}\cos\left(\frac{j\pi}{n}\right).$$

j = arange(n+1)

Barycentric weights for Chebyshev nodes

$\frac{1}{2}, -1, 1, -1, \cdots, \pm\frac{1}{2}$

$$w_j = \begin{cases} (-1)^j, & 1 \le j \le n-1 \\ \frac{(-1)^j}{2}, & j = 0, n \end{cases}$$

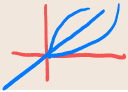
endpoints

$\frac{(-1)**j}{2}$

Stability to roundoff error

Condition number

~~Lebesgue constant~~



$$\kappa_n = \max_{a \le x \le b} \sum_{j=0}^{n} |L_j(x)|.$$

Theorem: Stability

Let $x_0, x_1, \ldots x_n$ be distinct nodes, and suppose $p(x)$ and $\hat{p}(x)$ are polynomials of degree at most n satisfying $p(x_j) = y_j$ and $\hat{p}(x_j) = \hat{y}_j$, $j = 0, 1, \ldots, n$. If

$$|y_j - \hat{y}_j| \le \epsilon, \quad j = 0, 1, \ldots, n,$$

then

$$\|p - \hat{p}\|_\infty \le \kappa_n \epsilon. \qquad O(\varepsilon)$$

$y_j = f(x_j)$

$\hat{y}_j = y_j + \varepsilon$

$$\|f(x)\|_\infty = \max_{x \in [a,b]} |f(x)|$$

Proof:

$$p(x) - \hat{p}(x) = \overbrace{\sum_{j=0}^{n} y_j L_j(x)}^{p(x)} - \overbrace{\sum_{j=0}^{n} \hat{y}_j L_j(x)}^{\hat{p}(x)} = \sum_{j=0}^{n} (y_j - \hat{y}_j) L_j(x).$$

It follows that

$$\begin{aligned} |p(x) - \hat{p}(x)| &= \left| \sum_{j=0}^{n} (y_j - \hat{y}_j) L_j(x) \right| \\ \text{Triangle inequality} \quad &\leq \sum_{j=0}^{n} |y_j - \hat{y}_j| |L_j(x)| \\ &\leq \epsilon \sum_{j=0}^{n} |L_j(x)| \\ &\leq \epsilon \kappa_n. \end{aligned}$$

Homework prob

$$\sum_{j=0}^{n} L_j(x) = 1$$

not the same

Theorem

The Lebesgue constant κ_n for uniform nodes is bounded from below by

$$\kappa_n \geq \frac{2^n}{4n^2}.$$

Theorem

The Lebesgue constant κ_n for Chebyshev nodes is

$$\kappa_n = O(\log(n)).$$

$$\lim_{n \to \infty} \log(n) = \infty$$

A stable algorithm is one that gives nearly the right answer to nearly the right question

Condition of a problem

Consider an abstract problem as a function $f : X \to Y$ mapping problem parameters to problem solutions. A *well-conditioned* problem is one with the property that all small perturbations to the problem parameters $x \in X$ lead to small well behaved changes in the solution. That is, for $\|\delta x\|$ sufficiently small with $x + \delta x \in X$,

$$\|\delta f\| = \|f(x + \delta x) - f(x)\| \leq \hat{\kappa}\|\delta x\|,$$

where $\hat{\kappa}$ is called the *absolute condition number*.

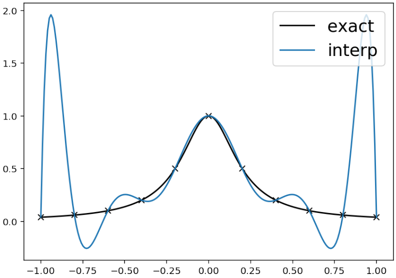
In practice, problems might be ill-conditioned or unstable if $\hat{\kappa}$ is very large or if the above bound does not exist (i.e., $\hat{\kappa} = \infty$).

Runge Phenomenon

$f^{(n)} \to \infty$ as $n \to \infty$

Runge function

$$f(x) = \frac{1}{1+25x^2}$$



Error Bound:

$$|f(x) - p(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \times \prod_{i=0}^{n} |x - x_i|.$$