Math 381 - Fall 2022

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Week 6

Last Week

1. Differentiation of Lagrange polynomials
2. Started integration of Lagrange polynomials

This Week

1. Finish integrals of Lagrange polynomials
2. Solving differential equations with Lagrange polynomials
3. Analysis of integration approximation accuracy
4. Roots of Lagrange polynomials
5. Review for the Midterm on Friday

Question:

If an nth degree polynomial $q(x)$ has antiderivative

$$p(x) = \int q(x)dx,$$

is $p(x)$ a polynomial, and if so, what is its degree?

yes, n+1 degree

EX

$q(x) = Ax^2 + Bx + C$

$p(x) = \frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx + D$

Questions:

Why do we not need to compute new nodes for the derivative of a polynomial?

$n+1$ nodes $\Rightarrow$ Polynomial of degree at most n

Do we need to compute new nodes for the antiderivative (indefinite integral) of a polynomial?

yes, the antideravative is a polynomial of degree $n+1$, which requires $n+2$ nodes.

From last time: calculating the indefinite integral

Want to compute $p(x) = \int_a^x q(u)\,du$

$$\mathcal{D}[p](x) = q(x) \quad \Rightarrow \quad p(x) = \mathcal{D}^{-1}[q](x) \text{ ???}$$

$p(a) = 0$

$$\underbrace{\begin{array}{c} n+2 \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \hline & & & \\ & \multicolumn{2}{c}{n+2 \times n+2 \text{ matrix}} & \\ & \multicolumn{2}{c}{D \text{ (except top row)}} & \\ & & & \end{bmatrix} \end{array}}_{D} \; \underbrace{\begin{bmatrix} p(a) \\ p(\tilde{x}_1) \\ p(\tilde{x}_2) \\ \vdots \\ p(b) \end{bmatrix}}_{p} = \underbrace{\begin{bmatrix} 0 \\ q(x_1) \\ q(x_2) \\ \vdots \\ q(b) \end{bmatrix}}_{RHS}$$

new nodes

old nodes

n+2   n+2   n+2

$\underbrace{q(x_1), q(x_2), \ldots, q(b)}_{n+1}$ q

Solve for vector p

Derivative matrix

$$d_{ij} = \frac{w_j}{w_i(x_i - x_j)}, \quad i \neq j$$

$$d_{ii} = -\sum_{\substack{k=0 \\ k\neq i}}^{n+1} d_{ik}.$$

Indefinite integral

The antiderivative nodes $p(x_j) = p_j$, $j = 0, 1, \dots, n+1$ are the solution to the following linear system of equations,

$$\hat{D}p = b,$$

where $\hat{d}_{0j} = \delta_{0j}$ and $\hat{d}_{ij} = d_{ij}$ for $i = 1, \dots, n+1$ is given by the formula above. The right hand side is given by $b_0 = 0$ and $b_j = q(x_j)$ for $j = 1, \dots, n+1$.

Procedure for antiderivatives (indefinite integral)

new nodes $\tilde{x}$
require new weights

Given n nodes $\tilde{x}_i$ on $[a, b]$, with weights $\tilde{w}_i$.

1. Compute new nodes x_j and weights w_j for $n + 1$ nodes on $[a, b]$
2. Evaluate $q(x_j)$ at the new nodes x_j (using the old nodes and weights)
3. Construct the matrix D defined on previous slide (using new nodes and weights)
4. Construct right hand side vector b with $b_0 = 0$ and $b_j = q(x_j)$ for $j = 1, \ldots, n + 1$
5. Solve $Dp = b$ for new y-nodes $p_j = p(x_j)$ using Python 'solve' function

Example: $u' = ru$, $t \in (0, \infty)$ (Exponential Growth)

$r > 0$ $\quad u(0) = u_0 > 0$ initial condition

Solve for $u(t)$, $t \in [0, 10]$.

$Du = ru \Rightarrow Du - ru = 0$

$\Rightarrow [D - rI]u = 0$

$u(t) = u_0 e^{rt}$

$u(0) = u_0$

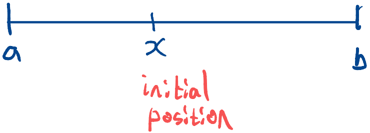
$$\begin{bmatrix} 1 \quad 0 \quad \cdots \quad 0 \\ \hline \\ D - rI \\ \\ \end{bmatrix} \begin{bmatrix} u(0) \\ \hline u(t_1) \\ \vdots \\ u(10) \end{bmatrix} = \begin{bmatrix} u_0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

Example: $u''(x) = -1, \quad x \in (a,b)$ (mean first passage time)

$u(a) = 0, \quad u(b) = 0$

$u(x) = Ax + B - \frac{1}{2}x^2$

$= -\frac{1}{2}(x-a)^2 - \frac{1}{2}(b-a)^2$



$u(x)$: mean first passage time

$\inf\{t>0 \,;\, X(t)=a, X(t)=b\}$

$DDu = -1$

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & D^2 & \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{} \underbrace{\begin{bmatrix} u(a) \\ \\ u(b) \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ \vdots \\ -1 \\ 0 \end{bmatrix}}_{RHS}$$

Approximation accuracy for integrals

Theorem

If $f(x) = f(a) + \int_a^x g(u)du$ and $p(x) = f(a) + \int_a^x q(u)du$ on $[a, b]$, then

$$\|f - p\|_\infty \leq (b - a)\|g - q\|_\infty$$

assume pos

Review: eigenvalues and eigenvectors

Given a square matrix $A \in \mathbb{R}^{n\times n}$, an eigenvector $v \in \mathbb{C}^n$ and eigenvalue $\lambda \in \mathbb{C}$ satisfy

$$Av = \lambda v.$$

$z = \mu + i\omega$ $\quad i = \sqrt{-1}$ $\quad z = re^{i\theta}$

real $\quad$ imaginary

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Roots of Lagrange Polynomials

Let $p_j = p(x_j)$ for nodes x_j and let w_j be the barycentric weights. Define the matrices

$$A = \begin{bmatrix} 0 & -p^T \\ w & X \end{bmatrix}, \quad X_{ij} = \begin{cases} 0, & i \neq j \\ x_i & i = j \end{cases}$$

$$B = \begin{bmatrix} 0 & \\ & I \end{bmatrix}$$

The roots $\hat{x}$ of the polynomial $p(x)$ are the generalized ~~eigenvectors~~ eigenvalues of the characteristic equation $\det(\hat{x}B - A) = 0$.

Normally ...

$\det(A - \lambda I) = 0$