A : set $g(a) = \dfrac{f(a+h) - f(a)}{h}$

the total computational error include approximation error and absolute round off error in machine epsilon

$$\hat{g}(a) = \frac{\hat{f}(a+h) - \hat{f}(a)}{h}$$

the total computational error is $|f'(a) - \hat{g}(a)|$

$$= |f'(a) - g(a)| + \underbrace{|g(a) - \hat{g}(a)|}_{\text{round off}}$$

approximation error

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + O(h^3)$$

$$\frac{f(a+h) - f(a)}{h} - f'(a) = \frac{h}{2} f''(a)$$

$$\therefore \quad |f'(a) - g(a)| \le \frac{h}{2}|f''(a)| = \frac{Mh}{2}$$

round off:

$$|g(a) - \hat{g}(a)| \le \left|\left(\frac{f(a+h) - f(a)}{h} - \frac{\hat{f}(a+h) - \hat{f}(a)}{h}\right)\right|$$

$$\le \left|\frac{f(a+h) - \hat{f}(a+h)}{h}\right| + \left|\frac{\hat{f}(a) - f(a)}{h}\right|$$

$$\le \frac{\epsilon}{h} + \frac{\epsilon}{h} = \frac{2\epsilon}{h}$$

$\therefore \quad |f'(a) - \hat{g}(a)| \leq \frac{2\epsilon}{h} + \frac{Mh}{2}$

proved

B. want minimized

$T(h) = \frac{Mh}{2} + \frac{2\epsilon}{h}$

$T'(h) = 0 = \frac{M}{2} - \frac{2\epsilon}{h^2}$

$\frac{2\epsilon}{h^2} = \frac{M}{2}$

$h^2 = \frac{4\epsilon}{M}$

$h = \sqrt{\frac{4\epsilon}{m}} \quad (M > 0 \ \epsilon > 0)$.

C. Yes, It is better

Because the estimated curve obtained by part a is more similar than that of discretization error.

However, the minimum of the empirical error is smaller than computational error.