Math 381 - Fall 2021

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Week 13

Last Time

1. Eigenvalue problem: introduction

Today

1. Conditioning of eigenvalue problems
2. Example of an iill conditioned eigenvalue problem
3. Schur factorization
4. Two phase strategy for computing eigenvalues

Eigenvalue problems

Diagonalizable matrix

$A = V \Lambda V^{-1}$ $\quad AV = V\Lambda$

diagonal

Eigenvalues and eigenvectors

For square matrices A, an eigenvalue λ and eigenvector v satisfy

$$Av = \lambda v.$$

Characteristic equation

Eigenvalues of A are the roots of the characteristic polynomial; that is, they satisfy

$$\det(A - \lambda I) = 0.$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda = 1$ algebraic mult 2

$r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ geometric mult 2

$r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Conditioning of the eigenvalue problem (simple eigenvalues)

Let λ_0 be a simple eigenvalue of the matrix A_0. Consider

$$(A_0 + \overset{\delta A}{\epsilon A_1})r = \lambda, \quad 0 < \epsilon \ll 1.$$

We showed that $\lambda \sim \lambda_0 + \epsilon \frac{l_0^* A_1 r_0}{l_0^* r_0}$.

Assuming that $\|r_0\| = \|l_0\| = 1$ we have

$$|\lambda - \lambda_0| \leq \frac{\epsilon \|A_1\|}{|l_0^* r_0|}.$$

Condition number of an eigenvalue

Let r and l be a right and left eigenvector of A (respectively) corresponding to the simple eigenvalue λ. The condition number of λ is

$$\kappa(\lambda) = \frac{\|r\|\|l\|}{|l^* r|}.$$

Ill conditioned eigenvalue problems: eigenvalues with degenerate eigenspaces are ill conditioned

Example: $A = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$

unperturbed problem

$A_0 = \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}}$ $\quad \lambda_0 = 1$

$r_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[A_0 - I] = \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}$

geo mult $= \dim(N(A_0 - \lambda I))$

$[A_0 - \lambda I] r = 0$

singular

perturbed problem

$(1-\lambda)^2 - \varepsilon = 0$

$\mu = 1 - \lambda$

$\mu^2 = \varepsilon$

$\mu = \pm\sqrt{\varepsilon}$

$\lambda = 1 \mp \sqrt{\varepsilon}$

$$\kappa(\lambda_0) = \frac{\|r_0\| \, \|\ell_0\|}{|\ell_0^* r_0|} = \infty$$

$[A_0 - \lambda_0 I] z = r_0$

By FA Thm

$\ell_0^*[A_0 - \lambda_0 I] = 0 \Rightarrow \ell_0^* r_0 = 0$

General eigenvalue solvers must be iterative

Theorem: Abel 1824

For any $n \geq 5$, there is a polynomial $p(z)$ of degree n with rational coefficients that has a real root $p(r) = 0$ with the property that r cannot be written using any expression involving rational numbers, addition, subtraction, multiplication, division, and kth roots.

Daily Linear Algebra

Definition: similar matrix

A matrix $A \in \mathbb{C}^{n\times n}$ is *similar* to a matrix $B \in \mathbb{C}^{n\times n}$ if there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that

$$A = SBS^{-1}$$

$A = V\Lambda V^{-1}$ $\quad Ax = b$ $\quad \tilde{b} = V^{-1}b$

$V^{-1}x = \tilde{x}$ $\quad V^{-1}AV\tilde{x} = V^{-1}b$

$\Rightarrow V^{-1}V\Lambda V^{-1}V\tilde{x} = V^{-1}b \Rightarrow \boxed{\Lambda\tilde{x} = \tilde{b}}$

Daily Linear Algebra

Claim:

If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are similar, then they have the same eigenvalues.

$A = SBS^{-1}$

Proof:

$Ar = \lambda r$

$Sy = r \Rightarrow y = S^{-1}r$

$\lambda r = Ar = SBS^{-1}r$

$\Rightarrow \lambda S^{-1}r = BS^{-1}r$

$\Rightarrow \lambda y = By$

Eigenvalue revealing decompositions

1. Diagonalization $A = X\Lambda X^{-1}$ (only if the matrix is diagonalizable)
2. Unitary diagonalization $A = Q\Lambda Q^*$ (only normal matrices $A^T A = AA^T$)
3. Schur factorization $A = QTQ^*$ (all square matrices)

Theorem: Schur factorization

Every Square matrix A has a Schur factorization such that

unitary Q

$Q^*Q = I$

$$A = QR\boxed{Q^*} = \tilde{Q}\tilde{R}$$

qr decomp

where Q is unitary and R is upper triangular.

Proof: By induction ($n=1$ $a = qr^{1/q} = r$)

Assume $n-1$ and show n

Let $A \in \mathbb{R}^{n\times n}$ and let $Ar = \lambda r$ with $\|r\|_2 = 1 = r^* r = \|r\|_2^2$

Let

$U = [r | u_2 | \cdots | u_n]$ be unitary

$$U^*AU = U^*[\lambda r | Au_2 | Au_3 | \cdots | Au_n] = \begin{bmatrix} \lambda & b^T \\ 0 & C \end{bmatrix}$$

$(n-1)\times(n-1)$ matrix

we have $C = VTV^*$. Let $Q = U\begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$

$$Q^*AQ = \begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix} U^*AU \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} \lambda & b^TV \\ 0 & T \end{bmatrix}$$

upper triangular

The two phases of computing eigenvalues

Can we use Householder reflections to compute Schur decomposition directly? no!

Phase 1

The matrix A is converted to a similar upper Hessenberg matrix

$$\begin{bmatrix} \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ & \ddots & \ddots & \vdots \\ & & \times & \times \end{bmatrix}$$

Phase 2

The similar upper Hessenberg matrix is iteratively converted into a similar triangular matrix

Idealization

$$Q_k, R_k = qr(A_k)$$
$$A_{k+1} = R_k Q_k$$