Math 381 - Fall 2022

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Week 11

Last Week

1. Backward error analysis
2. Conditioning of least squares problem
3. Backward stability of least squares algorithms
4. Low rank approximations
5. Gaussian elimination with complete pivoting

This Week

1. Eigenvalue solvers

Eigenvalue problems

Eigenvalues and eigenvectors

For square matrices A, an eigenvalue λ and eigenvector v satisfy

$$Av = \lambda v.$$

Characteristic equation

Eigenvalues of A are the roots of the characteristic polynomial; that is, they satisfy

$$\det(A - \lambda I) = 0.$$

$z = x + iy = re^{i\theta}$
$\bar{z} = x - iy = re^{-i\theta}$

$A^* = \bar{A}^T$ Transpose and complex conjugate of each element

Conditioning of the eigenvalue problem (simple eigenvalues)

$(A+\delta A)r = \lambda r.$

Rewrite as $(A_0+\epsilon A_1)r = \lambda r$ for $0<\epsilon \ll 1$.

Will show that $\lambda \approx \lambda_0 + \epsilon \frac{l_0^* A_1 r_0}{l_0^* r_0}$. $\Rightarrow |\lambda - \lambda_0| \leq \left|\frac{l_0^* A_1 r_0}{l_0^* r_0}\right| \epsilon$

$r \approx r_0 + \epsilon r_1$, $\lambda \approx \lambda_0 + \lambda_1 \epsilon$

unknowns

$$(A_0+\epsilon A_1)(r_0+\epsilon r_1) = (\lambda_0 + \lambda_1 \epsilon)(r_0 + \epsilon r_1)$$

$O(1)$: $(\epsilon = 0)$ $\quad A_0 r_0 = \lambda_0 r_0$ $\quad$ Assume we know r_0, λ_0

$O(\epsilon)$: i) expand ii) keep terms like ϵ iii) ignore terms like ϵ^2

$$\cancel{A_0 r_0} + \epsilon A_1 r_0 + \epsilon A_0 r_1 + \cancel{\epsilon^2 A_1 r_1} = \cancel{\lambda_0 r_0} + \epsilon \lambda_1 r_0 + \epsilon \lambda_0 r_1 + \cancel{\epsilon^2 \lambda_1 r_1}$$

$$\Rightarrow A_1 r_0 + A_0 r_1 = \lambda_1 r_0 + \lambda_0 r_1$$

$$\Rightarrow A_0 r_1 - \lambda_0 r_1 = -A_1 r_0 + \lambda_1 r_0$$

$$\Rightarrow [A_0 - \lambda_0 I] r_1 = -A_1 r_0 + \lambda_1 r_0$$

Conditioning of the eigenvalue problem (continued)

$$[A_0 - \lambda_0 I] r_1 = -A_1 r_0 + \lambda_1 r_0$$

solution r_0 exists?

Fredholm Alternative Theorem

$$l_0^* [A_0 - \lambda_0 I] = 0$$

require

$$l_0^* (-A_1 r_0 + \lambda_1 r_0) = 0$$

$$\Rightarrow -l_0^* A_1 r_0 + \lambda_1 l_0^* r_0 = 0$$

$$\lambda_1 = \frac{l_0^* A_1 r_0}{l_0^* r_0}$$

$$\lambda \sim \lambda_0 + \frac{l_0^* A_1 r_0}{l_0^* r_0} \varepsilon$$

Assume

$$\| l_0 \| = \| r_0 \| = 1$$

Conditioning of the eigenvalue problem (simple eigenvalues)

Let λ_0 be a simple eigenvalue of the matrix A_0. Consider

$$(A_0 + \epsilon A_1)r = \lambda, \quad 0 < \epsilon \ll 1.$$

Assuming that $\|r_0\| = \|l_0\| = 1$ we have

$$|\lambda - \lambda_0| \leq \frac{\epsilon \|A_1\|}{|l_0^* r_0|}.$$

Condition number of an eigenvalue

Let r and l be a right and left eigenvector of A (respectively) corresponding to the simple eigenvalue λ. The condition number of λ is

$$\kappa(\lambda) = \frac{\|r\| \|l\|}{|l^* r|}.$$

Why don't we simply solve for the roots of the characteristic polynomial?

Example

$\lambda^2 - 2\lambda + 1 - \epsilon = 0$

Example

$\lambda^2 - 2\lambda + 1 + \epsilon = 0$

$\lambda \sim 1 + \lambda_1 \varepsilon^\alpha$

$(1+\lambda_1\varepsilon^\alpha)^2 - 2(1+\lambda_1\varepsilon^\alpha) + 1 - \varepsilon = 0$

$\cancel{1} + 2\lambda_1\varepsilon^\alpha + \lambda_1^2\varepsilon^{2\alpha} - \cancel{2} - 2\lambda_1\varepsilon^\alpha + \cancel{1} - \varepsilon = 0$

$\alpha = 1$

$\cancel{2\lambda_1\varepsilon} + \cancel{\lambda_1^2\varepsilon^2} - \cancel{2\lambda_1\varepsilon} - \varepsilon = 0$

$-\varepsilon = 0$

can't solve :(

$\alpha = 1/2$

$\cancel{2\lambda_1\varepsilon^{1/2}} + \lambda_1^2\varepsilon - \cancel{2\lambda_1\varepsilon^{1/2}} - \varepsilon \simeq 0$

$\lambda_1^2 - 1 = 0 \Rightarrow \lambda_1 = \pm 1$

$\boxed{\begin{array}{c}\text{unperturbed problem} \\ (\lambda-1)^2 = 0 \\ \lambda = 1 \\ \text{repeated root}\end{array}}$

$\lambda \sim 1 \pm \sqrt{\varepsilon}$

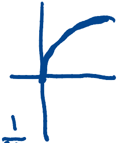
Bad :(

$(\sqrt{\varepsilon})' = \frac{1}{2\sqrt{\varepsilon}}$

$\lambda \sim 1 \pm i\sqrt{\varepsilon}$

$\varepsilon\lambda^2 + b\lambda + c = 0$

singular perturbation



$\left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon \to 0^+} = \infty$

Ill conditioned eigenvalue problems: eigenvalues with degenerate eigenspaces are ill conditioned

Example: $A = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$

$$[A - \lambda I]^2 \eta = r$$

General eigenvalue solvers must be iterative

Theorem: Abel 1824

For any $n \geq 5$, there is a polynomial $p(z)$ of degree n with rational coefficients that has a real root $p(r) = 0$ with the property that r cannot be written using any expression involving rational numbers, addition, subtraction, multiplication, division, and kth roots.

Daily Linear Algebra

Definition: similar matrix

A matrix $A \in \mathbb{C}^{n\times n}$ is *similar* to a matrix $B \in \mathbb{C}^{n\times n}$ if there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that

$$A = SBS^{-1}$$

Daily Linear Algebra

Claim:

If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are similar, then they have the same eigenvalues.

Proof:

Eigenvalue revealing decompositions

1. Diagonalization $A = X\Lambda X^{-1}$ (only if the matrix is diagonalizable)
2. Unitary diagonalization $A = Q\Lambda Q^*$ (only normal matrices $A^T A = AA^T$)
3. Schur factorization $A = \cancel{QTQ^*}$ QRQ^* (all square matrices)

Theorem: Schur factorization

Every Square matrix A has a Schur factorization such that

$$A = QRQ^*$$

where Q is unitary and R is upper triangular.

Proof: (time permitting)