Stat 265 – Unit 3 – Video Set 6 – Class Examples

Summary:

Hypergeometric(N,r,n) Random Variable:

Consider a population with a finite number of elements, N, where r of those elements are considered "successes". Consider selecting a random sample of n elements without replacement.

↳ selections are dependent.

- Y = the number of successes in the sample.
- $p_Y(y) = P(Y = y) = \dfrac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}, \; y = 0,1,2,\ldots, \; y \le r, \; n-y \le N-r.$
- $\mu_Y = E[Y] = \dfrac{nr}{N}.$
- $\sigma_Y^2 = V[Y] = n\left(\dfrac{r}{N}\right)\left(\dfrac{N-r}{N}\right)\left(\dfrac{N-n}{N-1}\right).$

Aside: Suppose we sampled with replacement.

- selections are independent.
- $p = r/N$
- $Y \sim \text{Binomial}(n, p = r/N)$.
- $\mu_Y = np = \dfrac{nr}{N}$
- $\sigma_Y^2 = np(1-p) = n\left(\dfrac{r}{N}\right)\left(\dfrac{N-r}{N}\right)$

$\rightarrow N$

Example A: Small electrical motors are shipped in lots of 50. Before such a shipment is accepted an inspector chooses 5 $\rightarrow n$ motors at random and tests them. If none of the motors are found to be defective, the shipment is accepted. If one or more are found to be defective, then the entire shipment is tested (all 50 motors). Suppose that it is known that there are 3 defective motors in the lot. r

a) Suppose it costs \$100 to test a motor. What is the expected cost for any given shipment?

b) What is the expected cost until a shipment is accepted? Assume all shipments are iid.

Let Y = # of defectives in the sample of $n=5$

$\sim HG(N=50, r=3, n=5)$.

$$p_Y(y) = \frac{\binom{3}{y}\binom{47}{5-y}}{\binom{50}{5}}, \quad y = 0, 1, 2, 3.$$

$$E[Y] = \frac{nr}{N} = 0.30, \quad V[Y] = \cdots = 0.25898.$$

a) Let X = total cost to test motors in a single shipment $= \begin{cases} 500, & \text{if } Y=0 \\ 5000, & \text{if } Y \geq 1 \end{cases}$

$$p_X(500) = P(Y=0) = \frac{\binom{3}{0}\binom{47}{5}}{\binom{50}{5}} = \frac{1419}{1960} \simeq 0.72398.$$

$$p_X(5000) = P(Y \geq 1) = 1 - P(Y=0) = \frac{541}{1960} \simeq 0.27602$$

$$E[X] = 500\left(\frac{1419}{1960}\right) + 5000\left(\frac{541}{1960}\right) \simeq 1742.09.$$

b) Let T = total cost until a shipment is accepted.

Let K = # of shipments until one is accepted.

$\sim \text{Geometric}(p = P(Y=0) = 1419/1960)$.

Then $T = 5000(K-1) + 500 = 5000K - 4500$.

$$E[T] = 5000E[K] - 4500 = 5000\left(\frac{1}{p}\right) - 4500$$

$$= 5000\left(\frac{1960}{1419}\right) - 4500 \approx 2406.27.$$

$$V[T] = 5000^2 V[K] = 5000^2\left(\frac{1-p}{p^2}\right) = 1316525 0.$$

$p = \frac{1419}{1960}$

$\to n$ $\quad$ $\to r$

Example B: A game is played by randomly selecting 10 balls (without replacement) from a box containing 32 white and 18 red balls. Let X be the number of red balls among those selected. Suppose you win \$5 for each red ball selected and lose \$3 for each white ball selected. Let Y = # of whites.

$N = 32 + 18 = 50$

a) What is the probability of selecting at least 8 white balls?

b) What is the expected value and variance of your win amount?

c) If the balls are selected in succession, what is the probability that the second and third balls are both red?

a) $Y \sim HG(N = 50,\ r = 32,\ n = 10)$

$$p_Y(y) = \frac{\binom{32}{y}\binom{18}{10-y}}{\binom{50}{10}}, \quad y = 0, 1, \ldots, 10$$

$$P(Y \geq 8) = p_Y(8) + p_Y(9) + p_Y(10)$$

$$= \frac{\binom{32}{8}\binom{18}{2} + \binom{32}{9}\binom{18}{1} + \binom{32}{10}\binom{18}{0}}{\binom{50}{10}} = \cdots \approx 0.21209.$$

c) Let R_i = ith selection is red, $i = 1, 2, 3, \ldots$

$$P(R_2 \cap R_3) = P(R_1 \cap R_2 \cap R_3) + P(\bar{R}_1 \cap R_2 \cap R_3)$$

partition over $\{R_1, \bar{R}_1\}$

$$= P(R_1)P(R_2 \mid R_1)P(R_3 \mid R_1 \cap R_2) + P(\bar{R}_1)P(R_2 \mid \bar{R}_1)P(R_3 \mid \bar{R}_1 \cap R_2)$$

$$= \left(\frac{18}{50}\right)\left(\frac{17}{49}\right)\left(\frac{16}{48}\right) + \left(\frac{32}{50}\right)\left(\frac{18}{49}\right)\left(\frac{17}{48}\right) = \left(\frac{18}{50}\right)\left(\frac{17}{49}\right) = P(R_1 \cap R_2)$$

b) $X = \#$ of reds , $Y = \#$ of whites.

- Let $W =$ win amount $= 5X - 3Y = 5(10-Y) - 3Y$

$X + Y = n = 10$ $\qquad = 50 - 8Y.$

$$E[W] = 50 - 8E[Y] = 50 - 8(6.4) = -1.2.$$

$$\left\{ \begin{array}{l} Y \sim HG(N=50, r=32, n=10) \\ E[Y] = \frac{nr}{N} = \frac{10(32)}{50} = 6.4 \\ \sigma_Y^2 = \cdots = \frac{2304}{1225} \approx 1.88082 \end{array} \right\}$$

$$V[W] = V[50-8Y]$$
$$= V[-8Y] = (-8)^2 V[Y]$$
$$= 64\left(\frac{2304}{1225}\right) \approx 120.37224.$$

OR

- $W = 5X - 3Y = 5X - 3(10-X) = 8X - 30.$

$$\left\{ \begin{array}{l} X \sim HG(N=50, r=18, n=10) \\ E[X] = \frac{nr}{N} = 3.6, \quad \sigma_X^2 = \frac{2304}{1225} \end{array} \right\}$$

$$E[W] = 8(3.6) - 30 = -1.2$$

$$V[W] = 8^2\left(\frac{2304}{1225}\right) \approx 120.37224.$$