Math 381 - Fall 2022

Jay Newby

University of Alberta

Week 3

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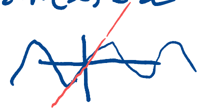
Last Week

1. Programming
2. Hardware
3. Introduced notions of accuracy and error
4. Floating point number system

This Week

$\sin(x) = 0$

$\sin(x) = x$



❶ Approximating solutions to nonlinear scalar equations

Two equivalent ways of formulating

1. Root finding problem

$$f(x) = 0$$

$$f(x) = g(x) - x$$

2. Fixed point problem

$$g(x) = x$$

Iterative methods and difference equations

Sequence: eg $\{x_n\} = x_0, x_1, x_2, \cdots$

Difference equation (first order, autonomous, nonlinear)

$$x_n = g(x_{n-1}), \quad x_0 = a$$

EX: Exponential Growth/Decay

$x_n = C x_{n-1}, \; x_0 = 1.1$

$x_1 = C x_0 = C\,1.1$

$x_2 = C x_1 = C(C\,1.1) = C^2 1.1$

$x_3 = C^3 1.1$

$\vdots$

$x_n = C^n 1.1 = C^n x_0$

Limit $n \to \infty$

$\lim_{n\to\infty} x_n = \hat{x}$ if it exists

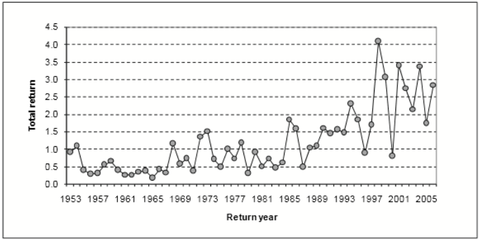
$\hat{x} = g(\hat{x})$

The Ricker model

The Ricker model for the growth of fish populations is

$$x_{t+1} = \underbrace{c\, x_t e^{-x_t}}^{g(x_t)}, \qquad c > 0.$$

FIGURE 2.16. Total return of Fraser River chum, 1953-2006.
Returns consist of catch + escapements of adults.



$$x_{t+1} = c\, x_t e^{-x_t} = g(x_t), \qquad c > 0.$$

Equilibrium points: $g(\hat{x}) = \hat{x}$

$c\hat{x}e^{-\hat{x}} = \hat{x} \Rightarrow \quad \hat{x}\left(ce^{-\hat{x}} - 1\right) = 0$

$\hat{x} = 0$ & $\hat{x} = \log(c)$

$ce^{-\hat{x}} = 1$

$\Rightarrow \log(c) - \hat{x} = 0$

Taylor expand around $x=0$

$g(x) = g(0) + x g'(0) + \frac{x^2}{2} g''(0) + R(x)$

$= xc + \tilde{R}(x)$

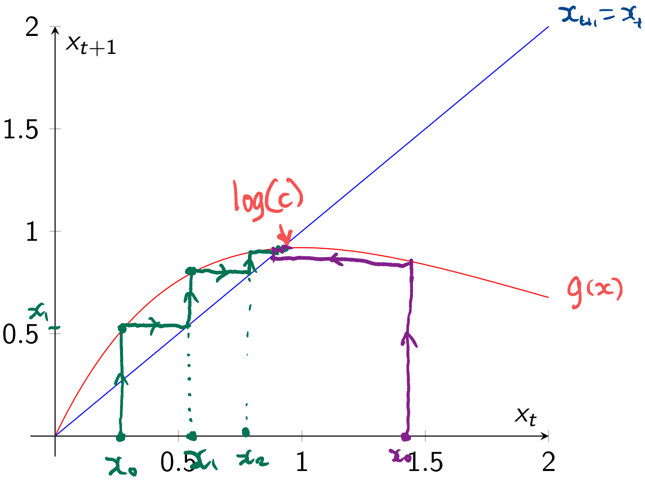
$g(x) \sim cx$ as $x \to 0^+$

$\Rightarrow x_{t+1} \sim c x_t, \quad x_t \to 0^+$

$g'(x) = c(1-x)e^{-x} \Rightarrow g'(0) = c$

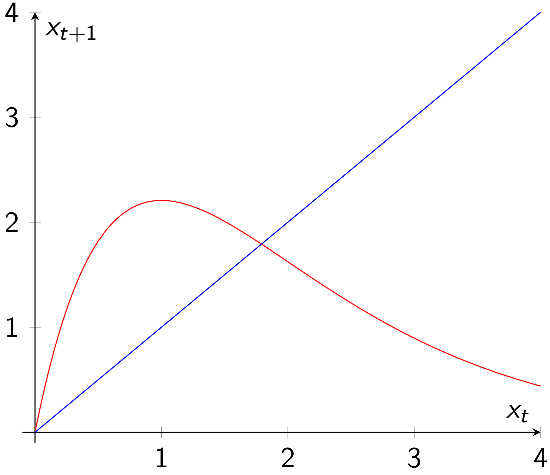
Cobwebbing

$x_{t+1} = g(x_t)$



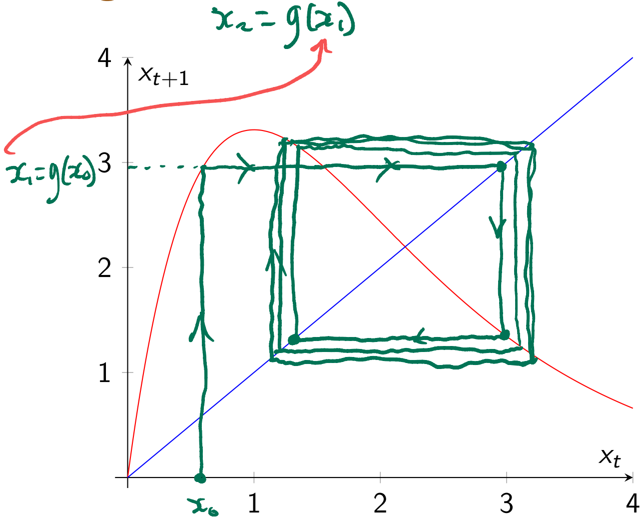
$c = 2.5$

Cobwebbing



$$c = 6$$

Cobwebbing



$c = 9$

Stability criterion

Exponential growth/Decay
$x_{t+1} = C x_t$

Stability criterion

Suppose $\hat{x}$ is an equilibrium of a difference equation $x_{t+1} = g(x_t)$ where g is continuous and differentiable.

- $\hat{x}$ is **asymptotically stable** if $|g'(\hat{x})| < 1$.
- it is **unstable** if $|g'(\hat{x})| > 1$.

The Ricker model

The Ricker model is:

$$x_{t+1} = c\, x_t e^{-x_t}, \qquad c > 0.$$

- Use the stability criterion to determine values of c that lead to asymptotically stable or unstable equilibria.
- Explain your findings in biological terms

$g'(0) = C$

Stability of the Ricker model

Solving a nonlinear equation AKA finding roots of a function

Root finding

Suppose we have a continuous function $f(x)$, $x \in \mathbb{R}$. The root of the function is a solution $\hat{x}$ to the equation $f(\hat{x}) = 0$. In general, there can be a finite, countably infinite, or uncountably infinite number of solutions.

Difference equations and fixed point iteration

$|g'(\hat{x})| < 1$

Fixed point iteration

Suppose we have a continuous function f that is smooth in a neighbourhood of $\hat{x}$ where $f(\hat{x}) = 0$. A numerical scheme for iterative convergence to the root $\hat{x}$ is given by

$$x_n = g(x_{n-1}), \quad g(x) = f(x) + x.$$

$g'(x) = f'(x) + 1$

$|g'(0)| = |f'(0) + 1| < 1$

Are there other methods?

Suppose we use

- $g(x) = x - f(x)$
- $g(x) = x + 2f(x)$
- Many more...

Rate of convergence

$g(x_n) = x_n$

Convergence rate: linear

If the sequence x_n converges to $\hat{x}$ and if n is sufficiently large then the convergence rate is linear if there exists a constant $C > 0$ such that

$$\mathcal{C}_{\mathrm{rate}} = \lim_{n\to\infty} \frac{|x_n - \hat{x}|}{|x_{n-1} - \hat{x}|},$$

$$\frac{|x_n - \hat{x}|}{|x_{n-1} - \hat{x}|} \leq C, \quad n \to \infty.$$

Absolute Error

$E_n = |x_n - \hat{x}|$

$$\frac{E_n}{E_{n-1}} \leq C \quad n \to \infty$$

$$E_n \approx C E_{n-1}$$

$$E_n \approx C^n E_0 \; , \; C < 1$$

Can we derive a method with faster convergence

$$E_n = C E_{n-1}^2$$

Quadratic convergence