Math 381 - Fall 2022

Jay Newby

University of Alberta

Week 3

Root finding

$f(\hat{x}) = 0$

Fixed point

$g(\hat{x}) = \hat{x}$

Fixed point iteration

$x_{n+1} = g(x_n)$ Difference equation

Rate of convergence

given a f^n $f(x)$ & x_0

Goal: $f(\hat{x}) = 0$

Convergence rate: linear

If the sequence x_n converges to $\hat{x}$ and if n is sufficiently large then the convergence rate is linear if there exists a constant $C > 0$ and integer $N > 0$ such that

$$\frac{|x_n - \hat{x}|}{|x_{n-1} - \hat{x}|} \leq C, \quad \forall n > N.$$

Absolute Error

$$E_n = |x_n - \hat{x}|$$

Linear convergence (for E_0 sufficiently small)

$$E_n \approx C E_{n-1} \Rightarrow E_n = C^n E_0$$

Can we derive a method with faster convergence

Given $f(x)$ & x_0, Want $x_{n+1} = g(x_n)$

Taylor Expand around $\hat{x}$

$$g(x) = g(\hat{x}) + (x-\hat{x})g'(\hat{x}) + \frac{1}{2}(x-\hat{x})^2 g''(\hat{x}) + R_3(x)$$

$$x_{n+1} = g(x_n) = g(\hat{x}) + (x_n-\hat{x})g'(\hat{x}) + \frac{1}{2}(x_n-\hat{x})^2 g''(\hat{x}) + R_3(x_n)$$

$$x_{n+1} = \hat{x} + (x_n-\hat{x})g'(\hat{x}) + \cdots$$

$$x_{n+1} - \hat{x} = (x_n-\hat{x})g'(\hat{x}) + \cdots$$

$$|x_{n+1}-\hat{x}| \leq |x_n-\hat{x}|\,|g'(\hat{x})| + |x_n-\hat{x}|^2 \left|\frac{g''(\hat{x})}{2}\right| + \cdots$$

What if $g'(\hat{x}) = 0$?

Quadratic convergence

$E_{n+1} \approx C E_n^2$

$g(x) = x - \frac{f(x)}{f'(x)}$ | $g(\hat{x}) = \hat{x}$, $g'(x) = 1 - \frac{f'f' - ff''}{(f')^2}$

$g'(\hat{x}) = 1 - \frac{(f')^2}{(f')^2} = 0$

Assume (for now) $f'(\hat{x}) \neq 0$

Newton's Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Rate of convergence

Convergence rate: quadratic

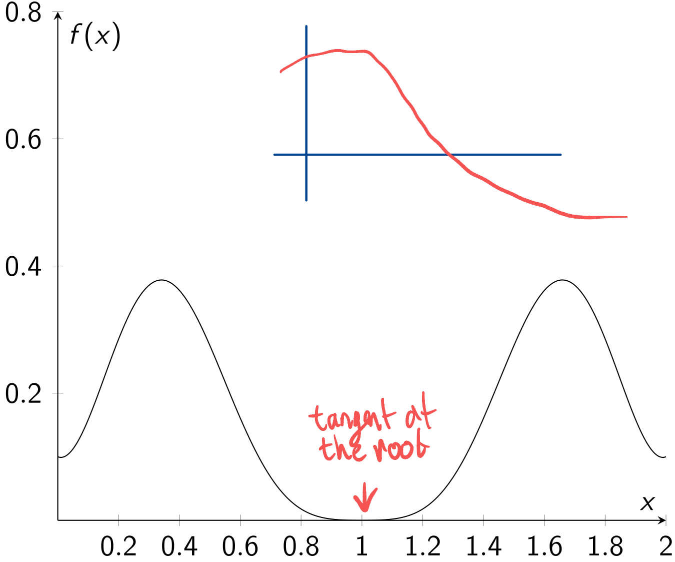
If the sequence x_n converges to $\hat{x}$ and if n is sufficiently large then the convergence rate is ~~linear~~ quadratic if there exists a constant $C > 0$ such that

$$\cancel{C_{\text{rate}} =} \frac{|x_n - \hat{x}|}{|x_{n-1} - \hat{x}|} \leq C|x_{n-1} - \hat{x}|. \qquad n \to \infty$$

$$\frac{E_n}{E_{n-1}} = O(E_{n-1})$$

Newton's method for the case of repeated roots

EX
$f(x) = x^2$



Newton's method for the case of repeated roots

$$g'(x) = 1 - \frac{(f')^2 - ff''}{(f')^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

Consider a function that has a repeated root such as

$$f(x) = (x - \hat{x})^2 h(x), \quad h(x) > 0.$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

In this case, the ratio

$$\frac{|f(x)f''(x)|}{[f'(x)]^2} \to C,$$

as $x \to \hat{x}$. If $f \to 0$; $f' \to 0$

L'Hopital $\frac{f'f'' + ff'''}{2f'f''} \approx \frac{\cancel{f'}f''}{2\cancel{f'}f''} = \frac{1}{2}$

Hence, $g'(\hat{x}) = \frac{1}{2} \Rightarrow$ Linear Convergence

Newton's method for the case of repeated roots

goes to one as $m \to \infty$

$$E_{n+1} = C E_n \qquad E_n = C^n E_0$$

$$e^{\log(c^n)} = e^{n \log c}$$

$$e^{\varepsilon n}$$

Example

Let $f(x) = x^m$, for $m \geq 2$. What is the convergence rate for Newton's Method applied to $f(x)$ to find the multiple root at $x = 0$?

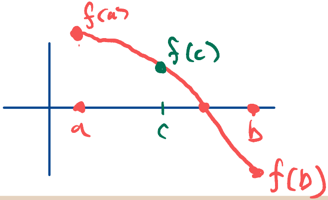
$$f'(x) = m x^{m-1}$$

$$f''(x) = m(m-1)x^{m-2}$$

$$g'(x) = \frac{f f''}{(f')^2} = \frac{x^m \, m(m-1)x^{m-2}}{m^2 (x^{m-1})^2} = \frac{m(m-1)}{m^2} \frac{\cancel{x^{2m-2}}}{\cancel{x^{2m-2}}} = \frac{m(m-1)}{m^2}$$

$$\lim_{m \to \infty} g'(0) =$$

Bisection Method



Bisection Algorithm

Given a continuous function $f(x)$ and an open interval (a, b) such that $a < b$ and $f(a)f(b) < 0$,

1. Set $c = \frac{a+b}{2}$
2. If $|c - a| < \mathrm{tol}$ end and return c
3. If $f(c)f(b) < 0$ then set $a = c$, otherwise set $b = c$
4. repeat step 1

Secant Method

Suppose we use a finite difference approximation for $f'(x)$? Assuming $|x_n - x_{n-1}|$ is small, we have

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} + O(x_n - x_{n-1}).$$

superlinear (not quite quadratic)

Secant Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Steffensen's Method

$$h(x) \approx \frac{f(x+f(x)) - f(x)}{f(x)}.$$

Quadratic convergence

Secant Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{h(x_n)}$$

Optimization

Optimization of a continuous scalar function

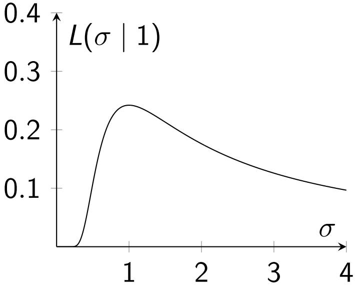
Let (a, b) be an open interval containing $x_{\min}$, and let $f(x)$ be continuous on (a, b). Find the minimum at the value $x_{\min}$ such that $f(x_{\min}) < f(x)$ (or $f(x_{\max}) > f(x)$ for maximization) for all $x \in (a, b)$ such that $x \neq x_{\min}$. In other words, find

$$f(x_{\min}) = \min_{x \in (a,b)} f(x).$$

Example: Maximum Likelihood

Derive the Maximum Likelihood estimator for $\sigma > 0$

$$L(\sigma \mid u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}$$



$$f(\sigma) = \log(L(\sigma)) = -\frac{u^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)$$

Solve

$$f'(\sigma) = 0$$