Recall: Binomial(n, p)
⇒ counts the number of successes in n trials.
- # of trials is fixed
- # of successes is random

Stat 265 – Unit 3 – Video Set 4 – Class Examples

Summary:

1) Bernoulli Trial: A trial resulting in a "success" with probability p and a "failure" with probability $1 - p$.

2) Geometric(p) Random Variable:

- X = the number of **trials** until the first success.
- $p_X(x) = P(X = x) = (1-p)^{x-1}p, \quad x = 1, 2, 3 \ldots$
- $\mu_X = \dfrac{1}{p}, \quad \sigma_X^2 = \dfrac{1-p}{p^2}$
- Cumulative Distribution Function: $F_X(x) = P(X \le x) = 1-(1-p)^x, \quad x = 1, 2, 3, \ldots$
- Memoryless Property: $P(X > a+b \mid X > a) = P(X > b),$ for any positive integers, a and b.

3) NegativeBinomial(r, p) Random Variable:

- Y = the number of **trials** until the r-th success.
- $p_Y(y) = P(Y = y) = \dbinom{y-1}{r-1} p^r (1-p)^{y-r}, \quad y = r, r+1, r+2 \ldots$
- $\mu_Y = \dfrac{r}{p}, \quad \sigma_Y^2 = \dfrac{r(1-p)}{p^2}$
- If $X_i \overset{ind}{\sim} Geometric(p),\ i = 1, \ldots, r,$ then $Y = \sum_{i=1}^{r} X_i \sim NegativeBinomial(r, p).$

Example A: A particular hockey team will award a financial bonus to their players depending on how many games it takes them to score their first goal of the season. Suppose the probability any player scores at least one goal in any given game is 0.30 and that all players and games are independent and identical. Let *Y* denote the number of games needed until a player scores a goal. $\Rightarrow Y \sim \text{Geometric}(p = 0.30)$, $p_Y(y) = (0.7)^{y-1}(0.3)$, $y = 1, 2, \ldots$

$\mu_Y = 1/0.3 = 10/3$, $\sigma_Y^2 = \frac{1-p}{p^2} = 70/9$

a) A players financial bonus (in thousands of dollars) is given by $B = 120 - 3Y^2$. What is the expected value of any randomly selected players' bonus? Note: Suppose their bonus can be negative (then they owe the team money for their poor play).

$$\mu_B = E[B] = E[120 - 3Y^2] = 120 - 3E[Y^2].$$

Note: $\sigma_Y^2 = E[Y^2] - \mu_Y^2 \Rightarrow E[Y^2] = \sigma_Y^2 + \mu_Y^2$

$$= \frac{1-p}{p^2} + \left(\frac{1}{p}\right)^2 = \frac{1-0.3}{0.3^2} + \left(\frac{1}{0.3}\right)^2 = \frac{170}{9}$$

$$E[B] = 120 - 3\left(170/9\right) = 190/3 \approx 63.3333.$$

$$\Rightarrow \$63{,}333.33,$$

b) What is the probability a randomly selected player is awarded a negative bonus (which we can now consider to be "no bonus"?

$$P(B < 0) = P(120 - 3Y^2 < 0) = P(Y^2 > 40) = P(Y > 6.32)$$

$$= P(Y \geq 7) = p_Y(7) + p_Y(8) + p_Y(9) + \cdots$$

$$= 1 - P(Y \leq 6) = 1 - F_Y(6) = 1 - (1 - (1-p)^6)$$

$$= (1-p)^6 = 0.7^6 \approx 0.117649.$$

c) Suppose they cannot be awarded a negative bonus. Thus, if B < 0, then their bonus is simply 0. What is the expected value of any randomly selected players' bonus?

y	$b = 120 - 3y^2$	Prob $(0.7)^{y-1}(0.3)$
1	117	0.3
2	108	0.21
3	93	0.147
4	72	0.1029
5	45	0.07203
6	12	0.050421
7	~~-27~~ 0	0.117649
8	~~-72~~ 0	
⋮	0 ⋮	

$$E[B] = \sum_{\text{all } b} b\,P(B=b)$$

$$= 117(0.30) + 108(0.21) + 93(0.147) + \cdots + 12(0.050421) + 0(0.117649)$$

$$= 82.70620$$

OR $82,706.20.

d) Given a randomly selected player has $\underbrace{\text{not scored in their first 3 games}}_{Y>3 \text{ or } Y \geq 4}$, what is the probability they are awarded a $\underbrace{\text{positive bonus}}_{Y \leq 6}$?

$$P(Y \leq 6 \mid Y \geq 4) = \frac{P(4 \leq Y \leq 6)}{P(Y \geq 4)} = \frac{P(Y \leq 6) - P(Y \leq 3)}{1 - P(Y \leq 3)}$$

$$= \frac{F_Y(6) - F_Y(3)}{1 - F_Y(3)} = \frac{(1-(0.7)^6) - (1-(0.7)^3)}{1-(1-(0.7)^3)}$$

$$= \frac{0.7^3 - 0.7^6}{0.7^3} = 1-(0.7)^3 = P(Y \leq 3) = 0.657.$$

e) Consider a random sample of $\underset{\to n}{10}$ players. What is the probability at least 2 of them are $\underbrace{\text{awarded no bonus}}_{\text{"success"}}$?

$p = 0.117649$

Let X = # who are awarded no bonus

$\sim Bin(n=10, p=0.117649)$

$$P(X \geq 2) = 1 - p_X(0) - p_X(1)$$

$$= 1 - (1-p)^{10} - \binom{10}{1} p (1-p)^9 = 0.33259.$$

$<8 \Rightarrow \leq 7$

f) What is the probability it takes a randomly selected player under 8 games to score at least one goal in 3 of those games?

Let T = # of games until the player scores at least one goal 3 times.

$\sim \text{Neg Bin}(r = 3,\ p = 0.30)$

$$P(T=t) = \binom{t-1}{2}(0.30)^3(0.70)^{t-3}, \quad t = 3, 4, 5, \ldots$$

$$P(T \leq 7) = p_T(3) + p_T(4) + p_T(5) + p_T(6) + p_T(7)$$

$$= 0.3^3 + \binom{3}{2}(0.3)^3(0.7) + \binom{4}{2}(0.3)^3(0.7)^2 + \binom{5}{2}(0.3)^3(0.7)^3 + \binom{6}{2}(0.3)^3(0.7)^4$$

$$= \cdots = 0.352930S.$$

g) What is the expected value and variance for the total number of games needed for a randomly selected player to score at least one goal in 3 different games?

$$\mu_T = \frac{r}{p} = \frac{3}{0.3} = 10$$

$$\sigma_T^2 = \frac{r(1-p)}{p^2} = \frac{3(0.7)}{0.3^2} = \frac{70}{3} \approx 23.333.$$

Example B: A gambler will win a game with a probability of 0.45 (and lose otherwise) and all games are independent and identically distributed. What is the probability they $\underbrace{\text{will win their 3rd game before having their 3rd loss}}_{E}$?

Recall:

Let Y = # of wins in their first five games.

$\sim Bin(n=5, p=0.45)$

$$p_Y(y) = \binom{5}{y}(0.45)^y(0.55)^{5-y}, \quad y = 0, 1, 2, \ldots, 5.$$

$$P(E) = P(Y \geq 3) = p_Y(3) + p_Y(4) + p_Y(5)$$

$$= 0.275653 + 0.112767 + 0.018453$$

$$= 0.406873.$$

OR

Let X = # of games until their 3rd win $\sim NegBin(r=3, p=0.45)$

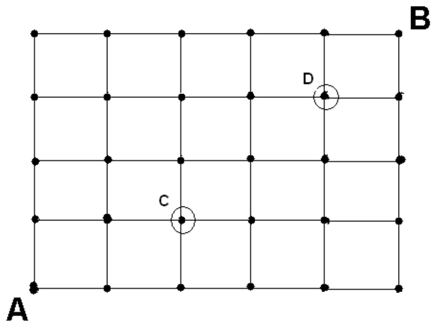
$$p_X(x) = \binom{x-1}{2}(0.45)^3(0.55)^{x-3}, \quad x = 3, 4, 5, \ldots$$

$$P(E) = P(X \leq 5) = p_X(3) + p_X(4) + p_X(5)$$

$$= 0.091125 + 0.150356 + 0.165392$$

$$= 0.406873.$$

Example C: Consider the grid below. Suppose that starting at point A you can go one step up or one step to the right at each move. This is continued until the point labeled B is reached. Assume each path is equally likely.



$N = \binom{9}{4} = 126$ equally likely paths.

$n_c = \binom{3}{1}\binom{6}{3} = 60$

$P(C) = \underbrace{\frac{60}{126}}_{1-p}$, $P(\bar{C}) = \frac{66}{126} = p$

Suppose that if you land on point C you have to start all over. Thus, if (after 3 steps) you are on point C, you will start back at point A, and your next step is your 4th step. If you hit C again, then you start all over at point A and your next step is now your 7th step, and so on. What is the expected value and variance for the number of steps needed to reach B?

Let Y = total # of steps needed to get to point B.
$(y = 9, 12, 15, 18, \ldots)$

Let X = # of attempts to get to point B until the first attempt that misses point C.
$\sim Geo(p = P(\bar{C}) = 66/126)$.

Then $Y = 9 + 3(X-1) = 6 + 3X$.

$\mu_Y = E[Y] = 6 + 3\mu_X = 6 + 3\left(\frac{1}{p}\right) = 6 + 3\left(\frac{126}{66}\right) = \frac{129}{11} \approx 11.727.$

$\sigma_Y^2 = V[Y] = V[6+3X] = V[3X] = 9\sigma_X^2$

$= 9\left(\frac{1-p}{p^2}\right) = \frac{1890}{121} \approx 15.6198.$

$p = 66/126$

Example D: A plane is missing and is presumed to have equal probability of going down in any of three regions. If a plane is actually down in region i, let 0.5^i denote the probability that the plane will be found upon a search of the $i-th$ region, $i = 1, 2, 3$. Suppose they will alternate searching the regions until the plane is found (region 1 then 2 then 3 then repeat). What is the expected value and variance for the number of searches needed until the plane is found? $\Rightarrow X$

Let $p_i = 0.5^i = P(\text{Found in region } i \text{ if in region } i)$, $i = 1, 2, 3$.

$p_1 = 1/2$, $p_2 = 1/4$, $p_3 = 1/8$.

Let Y_i = # of searches of region i until the plane is found if it is down in region i.

$\sim \text{Geometric}(p_i)$, $i = 1, 2, 3$.

Let Z_i = total # of searches (of all regions) until the plane is found if in region i.

$\Rightarrow$ $i=1$: $Z_1 = 3Y_1 - 2$, $E[Z_1] = 3E[Y_1] - 2 = 3(2) - 2 = 4$.

$i=2$: $Z_2 = 3Y_2 - 1$, $E[Z_2] = 3E[Y_2] - 1 = 3(4) - 1 = 11$.

$i=3$: $Z_3 = 3Y_3$, $E[Z_3] = 3E[Y_3] = 3(8) = 24$.

$\Rightarrow E[X]$?

Consider $P(X=x)$ by partitioning on the 3 regions.

Let $I_i = \begin{cases} 1, & \text{if plane is in region } i \\ 0, & \text{else} \end{cases}$, $i = 1, 2, 3$.

$$P(X=x) = P(X=x \cap I_1 = 1) + P(X=x \cap I_2 = 1) + P(X=x \cap I_3 = 1)$$

$$= P(I_1=1)P(X=x \mid I_1=1) + P(I_2=1)P(X=x \mid I_2=1)$$

$$+ P(I_3=1)P(X=x \mid I_3=1)$$

$$= \tfrac{1}{3}P(Z_1=x) + \tfrac{1}{3}P(Z_2=x) + \tfrac{1}{3}P(Z_3=x).$$

$$\therefore E[X] = \sum_{x=1}^{\infty} xP(X=x) = \sum_{x=1}^{\infty} x\left(\tfrac{1}{3}P(Z_1=x) + \tfrac{1}{3}P(Z_2=x) + \tfrac{1}{3}P(Z_3=x)\right)$$

$$= \tfrac{1}{3}\left(\sum_{x=1}^{\infty} xP(Z_1=x) + \sum_{x=1}^{\infty} xP(Z_2=x) + \sum_{x=1}^{\infty} xP(Z_3=x)\right)$$

$$= \tfrac{1}{3}\left(E[Z_1] + E[Z_2] + E[Z_3]\right)$$

$$= \tfrac{1}{3}(4 + 11 + 24) = 13.$$

$$E[X^2] = \sum_{x=1}^{\infty} x^2 P(X=x) = \cdots = \frac{1}{3}\left(E[Z_1^2] + E[Z_2^2] + E[Z_3^2]\right).$$

$\sigma_{Y_1}^2 + \mu_{Y_1}^2$

- $Z_1 = 3Y_1 - 2$, $E[Z_1^2] = E[(3Y_1-2)^2] = 9E[Y_1^2] - 12E[Y_1] + 4$

$$= 9(6) - 12(2) + 4 = 34.$$

$Y_1 \sim Geo(p=0.5)$

- $Z_2 = 3Y_2 - 1$, $E[Z_2^2] = 9E[Y_2^2] - 6E[Y_2] + 1 = 9(28) - 6(4) + 1 = 229.$

- $Z_3 = 3Y_3$, $E[Z_3^2] = 9E[Y_3^2] = 9(120) = 1080.$

$$\therefore E[X^2] = \frac{1}{3}(34 + 229 + 1080) = \frac{1343}{3} \approx 447.6667.$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \frac{1343}{3} - 13^2 = \frac{836}{3} \approx 278.6667.$$