Math 381 - Fall 2021

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Week 11

Last Time

1. Low rank approximations
2. Gaussian elimination with complete pivoting

Today

1. Eigenvalue problems

Eigenvalue problems

$Ar = \lambda r$

$\ell^* A = \lambda \ell^*$

complex conjugate and transpose

Eigenvalues and eigenvectors

For square matrices A, an eigenvalue λ and eigenvector v satisfy

$$Av = \lambda v.$$

Characteristic equation

Eigenvalues of A are the roots of the characteristic polynomial; that is, they satisfy

$$\det(A - \lambda I) = 0.$$

$z = x + iy$	$\bar{z} = x - iy$
$z = re^{i\theta}$	$\bar{z} = re^{-i\theta}$

$z = r\cos(\theta) + ir\sin(\theta)$

Conditioning of the eigenvalue problem (simple eigenvalues)

$(A + \delta A)r = \lambda r$. $A_0 = A$, $\epsilon A_1 = \delta A$

Rewrite as $(A_0 + \epsilon A_1)r = \lambda r$ for $0 < \epsilon \ll 1$.

Will show that $\lambda \sim \lambda_0 + \epsilon \frac{l_0^* A_1 r_0}{l_0^* r_0}$.

Let $\lambda \sim \lambda_0 + \epsilon \lambda_1$ and $r \sim r_0 + \epsilon r_1$. want to know

know know know

$$(A_0 + \epsilon A_1)(r_0 + \epsilon r_1) = (\lambda_0 + \epsilon \lambda_1)(r_0 + \epsilon r_1)$$

unperturbed

$O(1)$: $A_0 r_0 = \lambda_0 r_0$

$O(\epsilon)$: $\cancel{A_0 r_0} + \epsilon A_0 r_1 + \epsilon A_1 r_0 = \cancel{\lambda_0 r_0} + \epsilon \lambda_0 r_1 + \epsilon \lambda_1 r_0$ $l^* A_0 = \lambda_0 l^*$

ignore $O(\epsilon^2)$ terms

$\Rightarrow A_0 r_1 + A_1 r_0 = \lambda_0 r_1 + \lambda_1 r_0$ Fredholm Alternative

$\Rightarrow [A_0 - \lambda_0 I] r_1 = -[A_1 - \lambda_1 I] r_0$ $l_0^* [A_0 - \lambda_0 I] = 0$

Need "solvability condition" for λ_1 $\Rightarrow l_0^* [A_1 - \lambda_1 I] r_0 = 0$

Conditioning of the eigenvalue problem (continued)

$$\ell_0^*[A_1 - \lambda_1 I]r_0 = 0 \Rightarrow \ell_0^* A_1 r_0 - \lambda_1 \ell_0^* r_0 = 0$$

$$\Rightarrow \lambda_1 = \boxed{\frac{\ell_0^* A_1 r_0}{\ell_0^* r_0}}$$

Assume $\|\ell_0\| = \|r_0\| = 1$

Condition number of eigenvalue

$\mathcal{K}(\lambda_0)$

Conditioning of the eigenvalue problem (simple eigenvalues)

Let λ_0 be a simple eigenvalue of the matrix A_0. Consider

$$(A_0 + \epsilon A_1)r = \lambda, \quad 0 < \epsilon \ll 1.$$

Assuming that $\|r_0\| = \|l_0\| = 1$ we have

$$|\lambda - \lambda_0| \leq \frac{\epsilon \|A_1\|}{|l_0^* r_0|}.$$

Condition number of an eigenvalue

Let r and l be a right and left eigenvector of A (respectively) corresponding to the simple eigenvalue λ. The condition number of λ is

$$\kappa(\lambda) = \frac{\|r\| \|l\|}{|l^* r|}.$$

Why don't we simply solve for the roots of the characteristic polynomial?

Example

$\lambda^2 - 2\lambda + 1 - \epsilon = 0$

Example

$\lambda^2 - 2\lambda + 1 + \epsilon = 0$

unperturbed

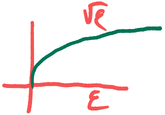
$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$\lambda = 1$

$\lambda_\pm \sim 1 \pm \sqrt{\varepsilon}$

$\lambda_\pm \sim 1 \pm i\sqrt{\varepsilon}$

Previous examples we have seen (big number)ε of ill-conditioned problems



$\lim_{\varepsilon \to 0^+} \frac{d\lambda}{d\varepsilon} = \infty$

Ill conditioned eigenvalue problems: eigenvalues with degenerate eigenspaces are ill conditioned

Example: $A = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$

Unperturbed problem

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \lambda = 1$

$r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Finish Friday

General eigenvalue solvers must be iterative

Theorem: Abel 1824

For any $n \geq 5$, there is a polynomial $p(z)$ of degree n with rational coefficients that has a real root $p(r) = 0$ with the property that r cannot be written using any expression involving rational numbers, addition, subtraction, multiplication, division, and kth roots.

Daily Linear Algebra

Definition: similar matrix

A matrix $A \in \mathbb{C}^{n\times n}$ is *similar* to a matrix $B \in \mathbb{C}^{n\times n}$ if there exists a nonsinglular $S \in \mathbb{C}^{n\times n}$ such that

$$A = SBS^{-1}$$

Daily Linear Algebra

Claim:

If $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ are similar, then they have the same eigenvalues.

Proof:

Eigenvalue revealing decompositions

1. Diagonalization $A = X\Lambda X^{-1}$ (only if the matrix is diagonalizable)
2. Unitary diagonalization $A = Q\Lambda Q^*$ (only normal matrices $A^T A = AA^T$)
3. Schur factorization $A = QTQ^*$ (all square matrices)

Theorem: Schur factorization

Every Square matrix A has a Schur factorization such that

$$A = QRQ^*$$

where Q is unitary and R is upper triangular.

Proof: (time permitting)