Math 381 - Fall 2022

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Week 8

Last Time

1. Forward/backward substitution
2. LU decomposition
3. Gaussian Elimination (tridiagonal matrices)

This Time

1. Gaussian Elimination (general case)
2. Gaussian Elimination with partial pivoting

Outer Product

Definition: inner product

$$\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i$$

Definition: outer product

rank 1 matrix

$$xy^T = Z, \qquad Z_{ij} = x_i y_j$$

Inner product

$$x^T y = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\cdot 0 + 2\cdot 1 = \boxed{2}$$

outer product

$$y x^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 & 2\cdot 1 \\ 0\cdot 1 & 2\cdot 1 \end{bmatrix}$$

$$xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} y_1 x & y_2 x & \cdots & y_n x \end{bmatrix}$$

$$Z$$

Gaussian Elimination (general case)

Recall $A = L U$

L: lower Δ U: upper Δ

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \frac{a_{21}}{a_{11}} r_1^T$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} \end{bmatrix}}_{U}$$

$L_1 A = U \Rightarrow A = L_1^{-1} U$

IS $L = L_1^{-1}$?

$$L_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{a_{21}}{a_{11}} & 1 \end{bmatrix}$$

Lower triangular matrix encodes the Gaussian elimination operations with

$$l_{ij} = \frac{y_{ij}}{y_{jj}}$$

where Y is the matrix after j steps applied to the matrix A.

Gaussian Elimination (general case)

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -l_{n,k} & & & 1 \end{bmatrix} = I - l_k e_k^T$$

$$l_k = \begin{array}{c} \\ \\ \\ k \\ k+1 \\ \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{n,k} \end{bmatrix} \qquad e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$L_{n-1} L_n \cdots L_2 L_1 A = U$$

Guassian Elimination

$$L = [L_{n-1} \cdots L_1]^{-1} = L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}$$

Gaussian Elimination (general case)

Want to show that

$$L_k^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & l_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & l_{n,k} & & & 1 \end{bmatrix}$$

Want to show

$$L_k^{-1} = I + l_k e_k^T$$

$$L_k^{-1} L_k = I$$

$$[I + l_k e_k^T][I - l_k e_k^T]$$

$$= I + \cancel{l_k e_k^T} - \cancel{l_k e_k^T} - (l_k e_k^T)(l_k e_k^T)$$

$$= I - l_k \underbrace{\cancel{(e_k^T l_k)}}_{0} e_k^T = I$$

Gaussian Elimination (general case)

Want to show that $L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1} = L$ where

$$L = \begin{bmatrix} 1 & & & & & \\ l_{2,1} & \ddots & & & & \\ l_{3,1} & \ddots & 1 & & & \\ l_{4,1} & & l_{k+1,k} & \ddots & & \\ \vdots & & \vdots & \ddots & 1 & \\ l_{n,1} & \dots & l_{n,k} & \dots & l_{n,n-1} & 1 \end{bmatrix}$$

Want to show

$$L = I + \sum_{j=1}^{n} l_j e_j^T$$

$$L_k^{-1} L_{k+1}^{-1} = [I + l_k e_k^T][I + l_{k+1} e_{k+1}^T] = I + l_k e_k^T + l_{k+1} e_{k+1}^T + l_k (\underbrace{e_k^T l_{k+1}}_{0}) e_{k+1}^T$$

$$A = \overbrace{L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}}^{L} U$$

Warning:

$$e_k^T l_{k-1} \neq 0$$

Some matrices work well with Gaussian Elimination

Definition: Diagonally dominant matrix

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Example

$$\begin{bmatrix} -2 & 1 & 0.5 \\ 1 & -3 & 0 \\ 4 & 0 & -7 \end{bmatrix}$$

$2 > 1.5$

$3 > 1+0$

$7 > 4+0$

Example: breakdown of Gaussian elimination and the need for pivoting

Generalized LU

$$A = PLU$$

↑ permutation matrix $\quad PP^T = I$

Example:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad P^T = \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \Rightarrow P^T A = \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} = \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}$

$Ax = b \Rightarrow PLUx = b$. Let ② $Ux = y$

Then $PLy = b \Rightarrow$ ① $Ly = P^T b$

Example: breakdown of Gaussian elimination and the need for pivoting

Example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution:

$$Ax = b \Rightarrow x = \boxed{\begin{matrix} 1 \\ -1 \\ 1 \end{matrix}}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{array}\right]} \; - \qquad \boxed{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{array}\right]}$$

Next Time

Example: breakdown of Gaussian elimination and the need for pivoting

GE is unstable for matrices near to a matrix that requires pivoting

Example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\epsilon & 2 \\ 1 & 2 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \epsilon & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]} \qquad \boxed{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \epsilon & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]}$$

Gaussian elimination with partial pivoting