Math 381 - Fall 2022

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Week 10

Last Week

1. Least Squares problems (overdetermined and full rank)
2. QR decomposition
3. Gram-Schmidt orthogonalization
4. Householder reflections $F = I - 2uu^T$

This Week

1. Stability and accuracy of the Householder Triangularization Method for the QR decomposition
2. Conditioning of least squares problems
3. Stability and accuracy of least squares algorithms

Review from last week: Householder Triangularization

QR: $Q_n Q_{n-1} \cdots Q_2 Q_1 A = R$

LU: $L_{n-1} L_{n-2} \cdots L_2 L_1 A = U$

The following modifies the elements of A in place and stores the vectors $v_1, \ldots, v_n$.

Algorithm: Householder Triangularization method QR decomposition

for k in $1, 2, \ldots, n$:

$x = A_{k:m,k}$

$v_k = \mathrm{sign}(x_1)\|x\|_2 e_1 + x$

$v_k = v_k / \|v_k\|_2$

$A_{k:m,k:n} = A_{k:m,k:n} - 2v_k(v_k^T A_{k:m,k:n})$

The vectors v_k are stored instead of forming Q explicitly (expensive). They can be used in an algorithm to compute $Q^T b$ for some vector b or Qx for some vector x. If forming Q is needed, then this algorithm can be applied to the canonical vectors e_j that form the columns of the identity matrix.

Example in Jupyter

A copy will be posted in the Week 10 notebook after lecture

Backward stability of Householder Triangularization

We want to understand the phenomena we observed in our experiment, the result

$$\frac{\|A - A_2\|}{\|A\|} = O(\epsilon),$$

where $A_2 = Q_2 R_2$. We will need to study *backward stability*.

Theorem:

Let the QR factorization $A = QR$ of a matrix $A \in \mathbb{R}^{m\times n}$ be computed by Householder triangularization (on a computer with suitable finite precision arithmetic) to obtain $\tilde{R}$ and the sequence of Householder reflection vectors $\tilde{v}_k$, $k = 1, 2, \ldots, n$. Let $\tilde{Q}$ be the exactly unitary matrix corresponding to the vectors $\tilde{v}_k$ (i.e., no approximation error in generating $\tilde{Q}$ from the vectors $\tilde{v}_k$). Then we have that

$$\tilde{Q}\tilde{R} = A + \delta A, \qquad \frac{\|\delta A\|}{\|A\|} = O(\epsilon),$$

for some $\delta A \in \mathbb{R}^{m\times n}$.

Questions

The stability result is interesting, but our end goal is rarely to compute the QR decomposition.

- Does backward stability of Householder triangularization extend to other tasks that use the inaccurate Q and R matrices, like, for example, a least squares problem or even a nonsingular linear system?
- When do we need accuracy in Q and R individually?

How about for solving nonsingular linear systems?

Algorithm: Solving $Ax = b$ using QR Factorization

Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and $b \in \mathbb{R}^n$.

1. $QR = A$ with Householder Triangularization
2. $y = Q^T b$ (using the algorithm in the next slide)
3. $x = R^{-1}A$ solve the upper triangular linear system using Backward Substitution

Algorithm: transformations by Q^T using Householder reflectors

$y = b.copy()$
for k in $1, 2, \ldots, n$:
    $p = v_k^T y_{k:m}$
    $y_{k:m} = y_{k:m} - 2v_k p$

Backward Stability

Let $y = Q^T b$. The output of the above algorithm generates a $\tilde{y}$ that exactly solves

$$(Q + \delta Q)\tilde{y} = b,$$

for some $\delta Q \in \mathbb{R}^{n\times n}$ such that $\|\delta Q\| = O(\epsilon)$.

Backward Stability of Backward Substitution

Let $R \in \mathbb{R}^{n \times n}$ be upper triangular and nonsingular, and let $x = R^{-1}y$ for some $y \in \mathbb{R}^n$. The output of Backward Substitution generates a $\tilde{x}$ that exactly solves

$$(R + \delta R)\tilde{x} = y,$$

for some $\delta R \in \mathbb{R}^{n \times n}$ such that $\frac{\|\delta R\|}{\|R\|} = O(\epsilon)$.

If each step is backward stable, is the entire algorithm also backward stable?