Math 381 - Fall 2022

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Week 5

Last Week

1. Polynomial interpolation with Lagrange polynomials
2. Existence and uniqueness
3. Barycentric formula
4. Interpolation stability and approximation accuracy
5. Chebyshev nodes

This Week

- Using polynomial interpolation for:
  1. Differentiation
  2. Integration
  3. Root finding
  4. Solving differential equations
- This week we begin to use linear algebra concepts

Review: Matrix vector multiplication

Block Matrix example

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \underbrace{\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}}_{x} = \begin{bmatrix} A\underline{x}_1 + B\underline{x}_2 \\ C\underline{x}_1 + D\underline{x}_2 \end{bmatrix}$$

Given an $(n+1) \times (n+1)$ matrix A with elements a_{ij} and a vector $x \in \mathbb{R}^{n+1}$ with elements x_j, the matrix vector product $y = Ax$ is given by

$$y_i = \sum_{j=0}^{n} a_{ij} x_j, \quad i = 0, 1, \ldots, n.$$

Review: Matrix-matrix multiplication

$C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{n \times p} \Rightarrow (CD) \in \mathbb{R}^{m \times p}$

what about

$AB \neq BA$

DC not unless $p=m$

Given the $(n+1) \times (n+1)$ matrices A and B with elements a_{ij} and b_{ij}, the matrix-matrix product $C = AB$ is

$$c_{ij} = \sum_{k=0}^{n} a_{ik} b_{kj}, \quad i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, n$$

Python

$\cancel{A * B}$

$A @ B$

$A + B$

Review: invertible square matrices

Square matrix

A square matrix has the same number of rows and columns

Inverse matrix

Given a square matrix A, the matrix inverse A^{-1} exists if and only if the determinant $\det(A) \neq 0$. If the matrix is invertible then

$$Ax = y \Leftrightarrow x = A^{-1}y.$$

ie singular

If the matrix is not invertible then there exists a vector $\rho \neq 0$ such that $A\rho = 0$.

Differential operators

The derivative and the integral can be defined as *linear operators* acting on functions; e.g.,

$$\frac{d}{dx}f(x) = \mathcal{D}[f](x), \quad \int_a^x f(u)du = \mathcal{K}[f](x).$$

Differential operators are 'infinite dimensional' versions of matrices. The algebra rules for differential operators are similar to matrices, and we can sometimes approximate infinite dimensional operators with matrices.

Lagrange interpolating polynomial

$$p(x) = \sum_{j=0}^{n} y_j L_j(x)$$

Definition we will use this week

$$L_j(x) = w_j l_j(x), \qquad l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} (x - x_i)$$

$$w_j = \frac{1}{\prod_{\substack{i=0 \\ i \neq j}}^{n} (x_j - x_i)}$$

Evaluation of the Barycentric formula as a matrix vector multiplication

evalutionpoint

Weights

x-nodes

$y_j = p(x_j)$

$$p(\tilde{x}_i) = \boxed{\frac{\sum_{j=0}^{n} \frac{w_j}{\tilde{x}_i - x_j} p(x_j)}{\sum_{k=0}^{n} \frac{w_k}{\tilde{x}_i - x_k}}} = \sum_{j=0}^{n} e_{ij}\, p(x_j),$$

independent of y_j

where

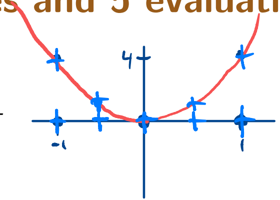
$$e_{ij} = \begin{cases} \boxed{\frac{w_j}{\tilde{x}_i - x_j} \Big/ \sum_{k=0}^{n} \frac{w_k}{\tilde{x}_i - x_k},} & \tilde{x}_i \notin \{x_0, x_1, \ldots, x_n\} \\ 1, & \tilde{x}_i = x_j \\ 0, & \tilde{x}_i = x_k,\ k \neq j \end{cases}$$

$$\tilde{y} = E_{\tilde{x}}\, y$$

Example: 3 nodes and 5 evaluation points

n=2

- $x = [-1, 0, 1]^T$
- $w = [0.5, -1, 0.5]^T$
- $y = [4, 0, 4]^T$



Use 5 evaluation points

$$\tilde{x} = \left[-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right]^T$$

$$\begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \tilde{y}_4 \end{bmatrix} = \begin{bmatrix} e_{00} & e_{01} & e_{02} \\ e_{10} & e_{11} & e_{12} \\ e_{20} & e_{21} & e_{22} \\ e_{30} & e_{31} & e_{32} \\ e_{40} & e_{41} & e_{42} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

Example continued

$$e_{ij} = \begin{cases} \frac{w_j}{\tilde{x}_i - x_j} / \sum_{k=0}^{n} \frac{w_k}{\tilde{x}_i - x_k}, & \tilde{x}_i \notin \{x_0, x_1, \dots, x_n\} \\ 1, & \tilde{x}_i = x_j \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{x}_0 = x_0 = -1$

$$e_{0,0} = 1, \quad e_{0,1} = 0, \quad e_{0,2} = 0,$$

$\tilde{x}_1 = -\frac{1}{2}$

$x_0 = -1$

$$e_{1,0} = \frac{\frac{w_0}{\tilde{x}_1 - x_0}}{\frac{w_0}{\tilde{x}_1 - x_0} + \frac{w_1}{\tilde{x}_1 - x_1} + \frac{w_2}{\tilde{x}_1 - x_2}} = \frac{\frac{0.5}{-0.5-(-1)}}{\frac{0.5}{-0.5-(-1)} + \frac{-1}{-0.5-0} + \frac{0.5}{0.5-1}}$$

$$= \frac{1}{1+2-1} = \frac{1}{2}$$

Example: 3 nodes and 5 evaluation points

- $x = [-1, 0, 1]^T$
- $w = [0.5, -1, 0.5]^T$
- $y = [4, 0, 4]^T$

Use 5 evaluation points

$$\tilde{x} = \left[-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right]^T$$

$$\begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \tilde{y}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & e_{11} & e_{12} \\ 0 & 1 & 0 \\ e_{30} & e_{31} & e_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

Recipe to compute derivative of Lagrange polynomial

Ingredients:

1. x-nodes $x = [x_0, x_1, \cdots, x_n]^T$
2. y-nodes $y = [y_0, y_1, \cdots, y_n]^T$
3. weights $w = [w_0, w_1, \cdots, w_n]^T$

Procedure:

1. Compute matrix D with elements d_{ij} using formula on the next slide
2. Compute the y-nodes for the derivative ($y_i' = p'(x_i)$) with a matrix vector multiplication

$$y' = Dy$$

What can we do with the result?

We can now use x, w, and y' to evaluate the derivative $p'(\tilde{x})$ at any evaluation points $\tilde{x}$ using the Barycentric formula (e.g., to make a nice plot).

Differentiation with Lagrange polynomials

We will represent differentiation with a matrix

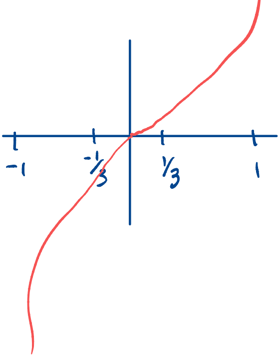
$$p'(x_i) = \sum_{j=0}^{n} d_{ij} p(x_j),$$

where

off diagonal $$d_{ij} = \frac{w_j}{w_i(x_i - x_j)}, \quad i \neq j$$

diagonal $$d_{ii} = -\sum_{\substack{k=0 \\ k \neq i}}^{n} d_{ik}.$$

Example:



1. x-nodes $x = \left[-1, -\frac{1}{3}, \frac{1}{3}, 1\right]^T$
2. y-nodes $y = [-2, -1, 1, 2]^T$
3. weights $w = [1, -3, 3, -1]^T$

$$\begin{bmatrix} \tilde{y}'_0 \\ \tilde{y}'_1 \\ \tilde{y}'_2 \\ \tilde{y}'_3 \end{bmatrix} = \begin{bmatrix} d_{00} & d_{01} & d_{02} & d_{03} \\ d_{10} & d_{11} & d_{12} & d_{13} \\ d_{20} & d_{21} & d_{22} & d_{23} \\ d_{30} & d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$d_{0,1} = \frac{w_1}{w_0(x_0 - x_1)} = \frac{-3}{1 \cdot (-1 - (-1/3))} = \frac{3}{3/3 + (-1/3)} = 9/2$$

$$d_{0,2} = \frac{w_2}{w_0(x_0 - x_2)} = \frac{3}{-1 - 1/3} = -9/4$$

$$d_{0,3} = \frac{w_3}{w_0(x_0 - x_3)} = \frac{-1}{-1 - 1} = 1/2$$

$$d_{0,0} = -d_{0,1} - d_{0,2} - d_{0,3} = -9/2 + 9/4 - \frac{1}{2}$$

Example:

1. x-nodes $x = \left[-1, -\frac{1}{3}, \frac{1}{3}, 1\right]^T$
2. y-nodes $y = [-2, -1, 1, 2]^T$
3. weights $w = [1, -3, 3, -1]^T$

$$\begin{bmatrix} \tilde{y}'_0 \\ \tilde{y}'_1 \\ \tilde{y}'_2 \\ \tilde{y}'_3 \end{bmatrix} = \begin{bmatrix} (-9/2 + 9/4 - 1/2) & 9/2 & -9/4 & 1/2 \\ d_{10} & d_{11} & d_{12} & d_{13} \\ d_{20} & d_{21} & d_{22} & d_{23} \\ d_{30} & d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 3 \\ -1 \end{bmatrix}$$

D@y ~~D*y~~

Approximating accuracy for the derivative of a general function $f(x)$

General problems with differentiation:

- Differentiation amplifies small errors
- Polynomial interpolation can smooth away features in the function $f'(x)$ (see Week 5 notebook for an example)

We do not need to approximate derivatives as often as we need to approximate integrals. It is often easier to simply apply the rules of calculus and calculate $f'(x)$ analytically.

Wednesday: Integration...