Computation details
in
Climate projects

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State variables

The first, simplest model has three state variables:

$$
\begin{aligned}
Y_t &: \quad \text{atmospheric temperature anomaly from preindustrial value} \\
\log N_t &: \quad \text{logarithm of economic damages from climate change} \\
K_t &: \quad \text{capital stock used for production of consumption good}
\end{aligned}
$$

The states evolve in the following manner

$$
\begin{aligned}
dY_t =& \mathcal{E}_t(\iota_y \cdot Z_t)\theta dt + \mathcal{E}_t(\iota_y \cdot Z_t)\varsigma' dW_t \\
d\log N_t =& (\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - \bar{y})\mathbb{I}_{Y_t > \bar{y}})\mathcal{E}_t(\iota_y \cdot Z_t)\theta dt \\
&+ \frac{1}{2}(\gamma_2 + \gamma_3 \mathbb{I}_{Y_t > \bar{y}})\mathcal{E}_t^2(\iota_y \cdot Z_t)^2 |\varsigma|^2 dt \\
&+ (\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - \bar{y})\mathbb{I}_{Y_t > \bar{y}})\mathcal{E}_t(\iota_y \cdot Z_t)\varsigma' dW_t \\
dK_t &: \text{omitted because of separability of the problem}
\end{aligned}
$$

Instantaneous utility

Utility or well-being of the agent in the model has a log form:

$$
\begin{aligned}
U(C, \mathcal{E}) =& (1-\eta) \log \tilde{C}_t + \eta \log \mathcal{E}_t \\
=& (1-\eta)(\log C_t - \log K_t) + (1-\eta)(\log K_t - \log N_t) + \eta \log \mathcal{E}_t
\end{aligned}
$$

Active part in the following computation:

$$\eta \log \mathcal{E}_t$$

Optimal Controls

$$
\begin{aligned}
\mathcal{E}_t : & \quad \text{emissions quantity} \\
C_t : & \quad \text{consumption quantity}
\end{aligned}
$$

Notation

$$
\begin{aligned}
y &: \text{realization of } Y_t \\
\tilde{e} &: \text{realization of } (\iota_y \cdot Z_t)\mathcal{E}_t \\
\phi(y), \phi_m(y) &: \text{pre- and post-jump value functions} \\
\delta, \eta &: \text{discounting and utility input share parameters} \\
\theta_\ell &: \text{climate sensitivity parameter, } \ell = 1, 2, \ldots, L \\
\pi_\ell^a &: \text{prior probabilities of } \theta_\ell, \ell = 1, 2, \ldots, L \\
\omega_\ell^a &: \text{distorted probabilities of } \theta_\ell, \ell = 1, 2, \ldots, L \\
\pi_m^p &: \text{prior probabilities of } \gamma_m^3, m = 1, 2, \ldots, M \\
h &: \text{drift distortion for climate model misspecification} \\
g &: \text{jump distortion for damage model misspecification} \\
\xi_a, \xi_b, \xi_p &: \text{uncertainty parameters}
\end{aligned}
$$

Discounted lifetime expected utility

We solve the discounted, lifetime expected utility of our "planner".

Post jump, this is given by

$$
\begin{aligned}
V_m(Y_t, \log N_t, K_t) = & \max_{C, \tilde{\mathcal{E}}} \min_{\omega_\ell^a, h} \\
& E[\int_T^\infty e^{-\delta t} \{ U(C, \tilde{\mathcal{E}}) \\
& + \frac{\xi_b}{2} h' h + \xi_a \sum_{\ell=1}^{L} \omega_\ell^a (\log \omega_\ell^a - \log \pi_\ell^a) \} dt]
\end{aligned}
$$

Discounted lifetime expected utility

We solve the discounted, lifetime expected utility of our "planner".

Pre jump, the problem is given by

$$
\begin{aligned}
V_m(Y_t, \log N_t, K_t) = \max_{C,\tilde{\mathcal{E}}} \min_{\omega_\ell^a, h} \\
& E\Big[\int_0^T e^{-\delta t}\{U(C, \tilde{\mathcal{E}}) \\
& + \frac{\xi_b}{2} h'h + \xi_a \sum_{\ell=1}^{L} \omega_\ell^a (\log \omega_\ell^a - \log \pi_\ell^a)\} dt \\
& + e^{-\delta(T-t)} \sum_{m=1}^{M} \pi_m^p \tilde{V}_m(Y_T, \log N_t, K_t)\Big]
\end{aligned}
$$

Value function simplification

We simplify the HJB equation by a guess and verify approach. Conjecture that

$$V_m(y, \log n, k) = -\frac{1-\eta}{\delta}\log n + \frac{1-\eta}{\delta}k + \phi_m(y)$$

We then derived a two-state HJB equation over (z, y)

Value matching, step I

With baseline probabilities of climate models π^a_ℓ for $\ell = 1, 2, \ldots, L$, and $\bar{y} = 2$ solve the following HJB for $\gamma_3^m \in \{\gamma_3^1, \gamma_3^2, \gamma_3^3\}$ on $y \in [0, 4]$ to get the corresponding $\phi_m(y)$ with $m = 1, 2, 3$:

$$
\begin{aligned}
-\delta\phi_m(y) = \max_{\tilde{e}} \min_{\omega^a_\ell : \sum_{\ell=1}^L \omega^a_\ell = 1} \min_h \quad & \eta \log \tilde{e} + \frac{\xi_b}{2} h'h \\
& + \frac{d\phi_m}{dy} \left(\sum_{\ell=1}^{L} \omega^a_\ell \theta_\ell \tilde{e} + \tilde{e}\varsigma h\right) + \frac{1}{2} \frac{d^2\phi_m(y)}{dy^2} (\tilde{e})^2 |\varsigma|^2 \\
& + \frac{(\eta - 1)}{\delta} (\gamma_1 + \gamma_2 y + \gamma_3^m (y - \bar{y}) \mathbb{I}_{y > \bar{y}}) \left(\sum_{\ell=1}^{L} \omega^a_\ell \theta_\ell \tilde{e} + \tilde{e}\varsigma h\right) \\
& + \frac{1}{2} \frac{(\eta - 1)}{\delta} (\gamma_2 + \gamma_3^m \mathbb{I}_{y > \bar{y}}) (\tilde{e})^2 \mid \varsigma \mid^2 \\
& + \xi_a \sum_{\ell=1}^{L} \omega^a_\ell (\log \omega^a_\ell - \log \pi^a_\ell)
\end{aligned} \tag{1}
$$

Value matching, step II

Assume the baseline probabilities for damage functions are $(\pi_1^p, \pi_2^p, \pi_3^p) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Given the above ϕ_1, ϕ_2 and ϕ_3, solve the following HJB for $\phi(y)$ on $y \in [0, \bar{y}]$:

$$
\begin{aligned}
-\delta\phi_m(y) = \max_{\tilde{e}} \min_{\omega_\ell^a : \sum_{\ell=1}^L \omega_\ell^a = 1} \min_h \min_{g^m \geqslant 0} \quad & \eta \log \tilde{e} + \frac{\xi_b}{2} h' h \\
& + \frac{d\phi_m}{dy} \left(\sum_{\ell=1}^L \omega_\ell^a \theta_\ell \tilde{e} + \tilde{e}\varsigma h\right) + \frac{1}{2} \frac{d^2\phi_m(y)}{dy^2} (\tilde{e})^2 |\varsigma|^2 \\
& + \frac{(\eta - 1)}{\delta} (\gamma_1 + \gamma_2 y) \left(\sum_{\ell=1}^L \omega_\ell^a \theta_\ell \tilde{e} + \tilde{e}\varsigma h\right) \\
& + \frac{1}{2} \frac{(\eta - 1)}{\delta} \gamma_2 (\tilde{e})^2 |\varsigma|^2 \\
& + \xi_a \sum_{\ell=1}^L \omega_\ell^a (\log \omega_\ell^a - \log \pi_\ell^a) \\
& + \mathcal{I}(y) \sum_{m=1}^M g^m \pi_m^p (\phi_m(y) - \phi(y)) + \xi_p \mathcal{I}(y) \sum_{m=1}^M \pi_m^p (1 - g^m + g^m \log g^m)
\end{aligned}
\tag{2}
$$

Value matching, step II

Minimizing over g^m:

$$g^{m,*} = \exp\left[\frac{1}{\xi_p}(\phi(y) - \phi_m(y))\right]$$

and then replace

$$\mathcal{I}(y)\sum_{m=1}^{M} g^m \pi_m^p(\phi_m(y) - \phi(y)) + \xi_p \mathcal{I}(y)\sum_{m=1}^{M} \pi_m^p(1 - g^m + g^m \log g^m)$$

in (2) with

$$-\xi_p \mathcal{I}(y)\frac{\sum_{m=1}^{M} \pi_m^p \exp(-\frac{1}{\xi_p}\phi_m(y)) - \exp(-\frac{1}{\xi_p}\phi(y))}{\exp(-\frac{1}{\xi_p}\phi(y))}$$

Value matching, step II

In computation, the HJB after minimizing over g^m is approximated by the following optimization:

$$
\begin{aligned}
-\delta\phi_m(y) = \max_{\tilde{e}} \min_{\omega_\ell^a : \sum_{\ell=1}^L \omega_\ell^a = 1} \min_h \quad & \eta \log \tilde{e} + \frac{\xi_b}{2} h' h \\
& + \frac{d\phi_m}{dy} \left(\sum_{\ell=1}^L \omega_\ell^a \theta_\ell \tilde{e} + \tilde{e}\varsigma h\right) + \frac{1}{2}\frac{d^2\phi_m(y)}{dy^2} (\tilde{e})^2 |\varsigma|^2 \\
& + \frac{(\eta - 1)}{\delta} (\gamma_1 + \gamma_2 y) \left(\sum_{\ell=1}^L \omega_\ell^a \theta_\ell \tilde{e} + \tilde{e}\varsigma h\right) \\
& + \frac{1}{2} \frac{(\eta - 1)}{\delta} \gamma_2 (\tilde{e})^2 |\varsigma|^2 \\
& + \xi_a \sum_{\ell=1}^L \omega_\ell^a (\log \omega_\ell^a - \log \pi_\ell^a)
\end{aligned}
\tag{3}
$$

$$
\text{with b.c. } \phi(y = \bar{y}) = -\xi_p \log \left(\sum_{m=1}^3 \exp\left(-\frac{1}{\xi_p}\phi_m(y = \bar{y})\right) \right)
$$

For both step I and II

We start with an initial guess of value function $\phi_0(y)$ and initial values of $\{\omega_\ell^a\}_{\ell=1}^L$, and update the value function according to the following way:
1. For a given $\phi_i(y)$, compute the optimizing h according to its first order condition:

$$h = -\frac{1}{\xi_b}\left[\frac{d\phi_i(y)}{dy} + \frac{\eta-1}{\delta}\left(\gamma_1 + \gamma_2 y + \gamma_3 (y-\bar{y})\mathbb{I}_{y>\bar{y}}\right)\right]\varsigma\tilde{e}$$

Replace

$$\left(\frac{d\phi_i(y)}{dy} + \frac{(\eta-1)}{\delta}(\gamma_1 + \gamma_2 y + \gamma_3^m (y-\bar{y})\mathbb{I}_{y>\bar{y}})\right)\tilde{e}\varsigma h + \frac{\xi_b}{2}h'h$$

in the HJB with

$$-\frac{1}{2\xi_b}\left[\frac{d\phi_i(y)}{dy} + \frac{\eta-1}{\delta}(\gamma_1 + \gamma_2 y + \gamma_3 (y-\bar{y})\mathbb{I}_{y>\bar{y}})\right]^2 \mid \varsigma \mid^2 \tilde{e}^2$$

2 . Compute the optimizing $\tilde{e}$ according to its first order condition.
The quadratic equation of $\tilde{e}$:

$$
\begin{aligned}
0 =& \eta + \sum_{\ell=1}^{L} \omega_\ell^a \left(\frac{d\phi_i}{dy} + \frac{(\eta-1)}{\delta}(\gamma_1 + \gamma_2 y + \gamma_3 (y-\bar{y})\mathbb{I}_{y>\bar{y}}) \right) \theta_\ell \tilde{e} \\
&+ \left(\frac{d^2\phi_i}{dy^2} + \frac{(\eta-1)}{\delta}\left(\gamma_2 + \gamma_3 \mathbb{I}_{y>\bar{y}}\right) \right) |\varsigma|^2 \tilde{e}^2 \\
&- \frac{1}{\xi_b}\left[\frac{d\phi_i}{dy} + \frac{(\eta-1)}{\delta}(\gamma_1 + \gamma_2 y + \gamma_3 (y-\bar{y})\mathbb{I}_{y>\bar{y}}) \right]^2 |\varsigma|^2 \tilde{e}^2
\end{aligned}
$$

3. After computing the optimizing $\tilde{e}$ from above, we compute the optimizing ω_ℓ^a according to its first order condition:

$$\omega_\ell = \frac{\pi_\ell^a \exp\left(-\frac{1}{\xi_a}\left[\frac{d\phi_i}{dy} + \frac{(\eta-1)}{\delta}(\gamma_1 + \gamma_2 y + \gamma_3 (y - \bar{y})\mathbb{I}\{y > \bar{y}\})\right]\tilde{e}\cdot\theta_\ell\right)}{\sum_{\ell=1}^{L} \pi_\ell^a \exp\left(-\frac{1}{\xi_a}\left[\frac{d\phi_i}{dy} + \frac{(\eta-1)}{\delta}(\gamma_1 + \gamma_2 y + \gamma_3 (y - \bar{y})\mathbb{I}\{y > \bar{y}\})\right]\tilde{e}\cdot\theta_\ell\right)},$$

$$\ell = 1, 2, \ldots, L$$

4. Plug the above computed $\tilde{e}$ and $\{\omega_\ell^a\}_{\ell=1}^L$ back into the above HJB. Update $\phi_i(y)$ to $\phi_{i+1}(y)$ by solving the following ODE:

$$
\begin{aligned}
\frac{\phi_{i+1}(y)-\phi_i(y)}{\epsilon} = & -\delta\phi_{i+1}(y)+\eta\log\tilde{e} \\
& +\frac{1}{2}\left(\frac{d^2\phi_{i+1}}{dy^2}+\frac{(\eta-1)}{\delta}\left(\gamma_2+\gamma_3\mathbb{I}\{y>\bar{y}\}\right)\right)(\tilde{e})^2|\varsigma|^2 \\
& -\frac{1}{2\xi_b}\left[\frac{d\phi_{i+1}}{dy}+\frac{(\eta-1)}{\delta}(\gamma_1+\gamma_2 y+\gamma_3(y-\bar{y})\mathbb{I}\{y>\bar{y}\})\right]^2\cdot|\varsigma|^2(\tilde{e})^2 \\
& +\sum_{\ell=1}^{L}\omega_\ell^a\left(\frac{d\phi_{i+1}}{dy}+\frac{(\eta-1)}{\delta}(\gamma_1+\gamma_2 y+\gamma_3(y-\bar{y})\mathbb{I}\{y>\bar{y}\})\right)\theta_\ell\tilde{e} \\
& +\xi_a\sum_i\omega_\ell^a(\log\omega_\ell^a-\log\pi_\ell^a)
\end{aligned}
$$

Blued $\tilde{e}$ and ω_ℓ^a indicate they are computed using $\phi_i(y)$.

5. Check whether the convergence condition is satisfied. Set a tolerance level, *tolerance*. We say that the algorithm converges, if:

$$\frac{\mid \phi_{i+1}(y) - \phi_i(y) \mid}{\epsilon} < tolerance$$

and we get the solution $\phi(y) = \phi_{i+1}(y)$. Otherwise, assign $\phi_{i+1}(y)$ to $\phi_i(y)$, and repeat step 1-4.

A richer setting

Clean v.s. Dirty capital model

Apart from the two state variables: Y_t, $\log N_t$
Two new state variables:

$$K_t^D : d\log K_t^D = \left[\alpha + \left(\frac{I_t^D}{K_t^D}\right) - \frac{\kappa}{2}\left(\frac{I_t^D}{K_t^D}\right)^2\right] dt - \frac{|\sigma_d|^2}{2} dt + \sigma_d dW_t$$

$$K_t^C : d\log K_t^C = \left[\alpha + \left(\frac{I_t^C}{K_t^C}\right) - \frac{\kappa}{2}\left(\frac{I_t^C}{K_t^C}\right)^2\right] dt - \frac{|\sigma_c|^2}{2} dt + \sigma_c dW_t$$

Instantaneous utility:

$$\log\left([A_c K_t^C - I_t^C]^{1-\nu} L^\nu + [A_d K_t^D - I_t^D]^{1-\nu} E^\nu\right)$$

New variables and parameters

Two more control variables: i_c, i_d, realization of $\frac{I_t^C}{K_t^C}, \frac{I_t^D}{K_t^D}$

Parameters for capital stocks: $\alpha, A_c, A_d, \sigma_c, \sigma_d$

Value function simplification

We define capital ratio as the fraction of clean capital

$$z = \frac{k_c}{k_c + k_d}$$

We simplify the HJB equation by a guess and verify approach. Conjecture that

$$\phi_m^0(k_c, k_d, y, \log n) = -\frac{1-\eta}{\delta}\log n + \phi_m(z, y) + \frac{1}{\delta}\log(k_c + k_d)$$

We then derived a two-state HJB equation over (z, y)

FOC for controls i_c, i_d, e

$$\frac{(1-\eta)(A_c z - \boldsymbol{i_c} z)^{-\eta} L^\eta}{\{1-\kappa \boldsymbol{i_c}\}[\frac{d\phi_m}{dz}(1-z) + \frac{(1-\eta)}{\delta}]} = (A_c z - \boldsymbol{i_c} z)^{1-\eta} L^\eta + (A_d(1-z) - \boldsymbol{i_d}(1-z))^{1-\eta} \boldsymbol{e}^\eta$$

$$\frac{(1-\eta)(A_d(1-z) - \boldsymbol{i_d}(1-z))^{-\eta} \boldsymbol{e}^\eta}{\{1-\kappa \boldsymbol{i_d}\}[-\frac{d\phi_m}{dz} z + \frac{(1-\eta)}{\delta}]} = (A_c z - \boldsymbol{i_c} z)^{1-\eta} L^\eta + (A_d(1-z) - \boldsymbol{i_d}(1-z))^{1-\eta} \boldsymbol{e}^\eta$$

$$\frac{-\eta(A_d(1-z) - \boldsymbol{i_d}(1-z))^{1-\eta} \boldsymbol{e}^{\eta-1}}{a\boldsymbol{e} + b} = (A_c z - \boldsymbol{i_c} z)^{1-\eta} L^\eta + (A_d(1-z) - \boldsymbol{i_d}(1-z))^{1-\eta} \boldsymbol{e}^\eta$$

Where

$$a = \frac{d^2\phi_m}{dy^2}\zeta^2 - \frac{1}{\delta}\zeta^2(\gamma_2 + \gamma_3^m \mathbb{I}_{y>\bar{y}}) - \frac{\left[\frac{d\phi_m}{dy} - \frac{1}{\delta}\{\gamma_1 + \gamma_2 y + \gamma_3^m (y-\bar{y})\mathbb{I}_{y>\bar{y}}\}\right]^2}{\xi_b}\zeta^2$$

$$b = \left[\frac{d\phi_m}{dy} - \frac{1}{\delta}\{\gamma_1 + \gamma_2 y + \gamma_3^m (y-\bar{y})\mathbb{I}_{y>\bar{y}}\}\right]\sum_{\ell=1}^{L}\omega_\ell \theta_\ell$$

Cobweb style iterations for updating controls

Expand the three equation system by adding an equation defining q

$$q = (A_c z - \boldsymbol{i_c} z)^{1-\eta} L^{\eta} + (A_d(1-z) - \boldsymbol{i_d}(1-z))^{1-\eta} \boldsymbol{e}^{\eta} = g(i_c, i_d, e)$$

Then rewrite the first order conditions as

$$q = \frac{(1-\eta)(A_c \cdot z - \boldsymbol{i_c} z)^{-\eta} L^{\eta}}{\{1 - \kappa \boldsymbol{i_c}\}[\frac{d\phi_m}{dz}(1-z) + \frac{(1-\eta)}{\delta}]} = f_1(i_c)$$

$$q = \frac{(1-\eta)(A_d(1-z) - \boldsymbol{i_d}(1-z))^{-\eta} \boldsymbol{e}^{\eta}}{\{1 - \kappa \boldsymbol{i_d}\}[-\frac{d\phi_m}{dz} z + \frac{(1-\eta)}{\delta}]} = f_2(i_d, e)$$

$$q = \frac{-\eta(A_d(1-z) - \boldsymbol{i_d}(1-z))^{1-\eta} \boldsymbol{e}^{\eta-1}}{a \cdot \boldsymbol{e} + b} = f_3(i_d, e)$$

Iterative schemes

Given (q, i_c, i_d, e) and a learning rate λ, $(0 \leqslant \lambda \leqslant 1)$, compute

1. $i_c^* = f_1^{-1}(q)$
2. $e^* = f_2^{-1}(q, i_d)$
3. $i_d^* = f_3^{-1}(q, e^*)$
4. $q^* = \lambda \cdot g(i_c^*, i_d^*, e^*) + (1 - \lambda) \cdot q$
5. set $(q, i_c, i_d, e) = (q^*, i_c^*, i_d^*, e^*)$
6. iterate until convergence