Sannikov's Problem

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1 Original Problem

We define function $F(W)$ together with the optimal choices of $a(W)$ and $c(W)$ satisfy the HJB equation

$$rF(W) = \max_{a>0,c} r(a-c) + F'(W)r(W - u(c) + h(a)) + \frac{F''(W)}{2} r^2 \gamma(a)^2 \sigma^2. \tag{1}$$

2 Equivalent Problem with Form Suitable for Computation

$$F''(W) = \min_{a>0,c} \frac{F(W) - a + c - F'(W)(W - u(c) + h(a))}{r\gamma(a)^2\sigma^2/2}. \tag{2}$$

To compute the optimal contract, we must solve (2) by setting

$$F(0) = 0$$

and choosing the largest slope $F'(0) \geq F_0'(0)$ such that the solution F satisfies

$$F(1) = F_0(1) \text{ and } F'(1) = F_0'(1)$$

at some point $1 \geq 0$, where $F'(1) = F_0'(1)$ is called the smooth-pasting condition.

Let functions $c : (0,1) \to [0, \infty)$ and $a : (0,1) \to \mathcal{A}$ be the minimizers in equation (2). A typical form of the value function $F(W)$ together with $a(W), c(W)$ and the drift of the agent's continuation value are shown in Figure 1.

3 Bin's experiment and motivation

3.1 Motivation

By looking at equation (1), I found that it is of extremely similar form as the problem that we had but much simpler due to only 1-dimensional value function but two controls.

This, in my understanding, is a good chance to utilize the petsclinearsystem to test my understanding of FOC condition of controls that the team usually used in computation. As you will see, equation (1) contain some interesting questions that I have been thinking in the past.

3.2 Experiments

Since equation (1) is of form acceptable by petsclinearsystem, I utilized this form directly for computation.

3.2.1 FOC of a

After analysis, this is trivial FOC.

3.2.2 FOC of c: Caviar

This is the Caviar and address the question that I have had in mind for a long time!!!

By taking 1st-derivative and 2nd-derivative w.r.t. c in the RHS of equation (1),we obtain FOD

$$-r(1 + F'/2\sqrt{c}) \tag{3}$$

and SOD

$$rF'/4\sqrt{c} \tag{4}$$

Then there are two cases:

- $F' > 0$: This case lead equation (3) to be negative for all c, which directly give the corner solution as $c = 0$
- $F' < 0$: This case lead SOD (4) to be negative and therefore FOD (3) decreasing in c. More inspection is that FOD (3) will intersect with x-axis, which leads to a interior maximizer c^*, which is exactly the intersection between (3) and x-axis.

In other words, in the first case, $c(w) = 0$, which is corresponding to the increasing part of $F(w)$, $F(w)$ before W^*.

And in the second case, $c(w) > 0$, which is corresponding to the decreasing part of $F(w)$,$F(w)$ after W^*.

3.3 Success

The output is Figure 2.

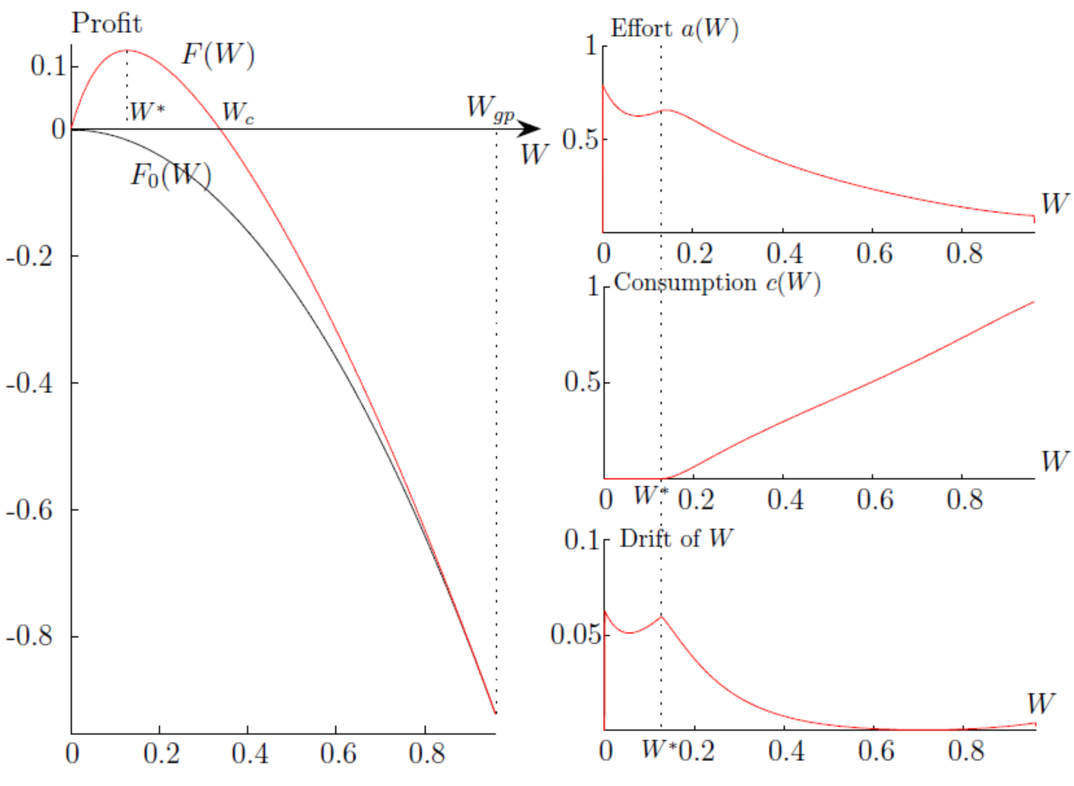

Figure 1: Function F for $u(c) = \sqrt{c}$; $h(a) = 0.5a^2 + 0.4a$; $r = 0.1$ and $\sigma = 1$, $F_0(W) = -W^2$: Point $W^*$is the maximum of F.

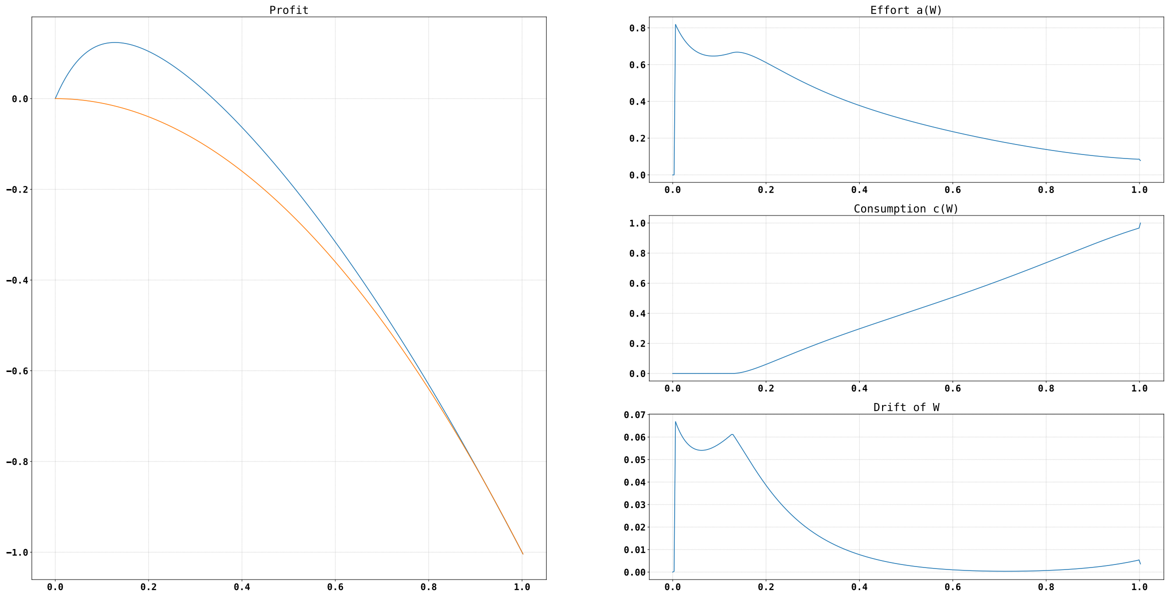

Figure 2: Bin's result of $F(w), a(w), c(w)$ and drift term in HJB