

## A Model With Clean and Dirty Capital Stocks and and R&D in Green Innovation

Assume there are two capital sectors, each with AK production technology ( $Y_i = A_i K_i$ ,  $i = d, g$ ) and each with its own capital stock that evolves with quadratic adjustments costs and Brownian shocks as follows:

$$\begin{aligned} dK_d/K_d &= [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] dt + \sigma_d dW \\ dK_g/K_g &= [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] dt + \sigma_g dW \end{aligned}$$

We also assume there is R & D investment that leads to an increased arrival rate of a one time jump in Sector 2 productivity. The arrival rate is denoted as  $\lambda$  and evolves as follows

$$d\lambda_t/\lambda_t = (\varphi i_\lambda \frac{K_d + K_g}{\lambda} - \alpha_\lambda) dt + \sigma_\lambda dW$$

The key difference between the sectors is that production from Sector 1 generates emissions. As a result, the evolution of atmospheric temperature is give by the Matthews Approximation, so that temperature  $Y_t$  and cumulative carbon emissions are given by

$$dY_t = E_t(\beta_f dt + \varsigma dW)$$

where  $\beta_f$  is the Matthews parameter and  $\eta$  is the scaling factor converting Sector  $d$  output  $A_d K_d$  into emissions such that

$$E_t = \eta A_d K_d$$

Output can be used in for consumption, investment in either capital stock, or for R&D into improving the productivity of Sector  $g$  :

$$C = A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda (K_d + K_g)$$

We assume exponential-quadratic damages to preferences so that our utility is augmented when accounting for climate damages. Flow utility is a log function over consumption, assuming perfect substitutability over output from the two sectors so that

$$U(C) = \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda (K_d + K_g)) - \delta \log N_t$$

where the log  $N_t$  follows from BBH2 as

$$\begin{aligned} \log N_t &= \Gamma(Y) \\ \Gamma(y) &= \gamma_1 y + \frac{\gamma_2}{2} y^2 + \frac{\gamma_3}{2} \mathbb{I}_{y \geq \bar{y}} (y - \bar{y})^2 \end{aligned}$$

Taking these pieces together we get the HJB equation

$$\begin{aligned} \delta V(K_d, K_g, \lambda, Y, \log N) &= \max_{i_g, i_d, i_\lambda} \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda (K_d + K_g)) - \delta \log N \\ &+ \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} V_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} V_g K_g + \frac{\sigma_d^2 K_d^2}{2} V_{dd} + \frac{\sigma_g^2 K_g^2}{2} V_{gg} \\ &+ \beta_f E_d V_Y + \frac{1}{2} \varsigma^2 E_d^2 V_{YY} + [\{\gamma_1 + \gamma_2 Y + \gamma_3 (Y - \bar{y}) \mathbf{I}\{Y_t > \bar{y}\}\} \beta_f E_d \\ &+ \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y > \bar{y}\}}{2} \varsigma^2 E_d^2] V_{\log N} + \frac{\varsigma^2 E_d^2}{2} V_{\log N \log N} \\ &+ (\varphi i_\lambda - \alpha_\lambda) \lambda V_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} V_{\lambda\lambda} + \lambda (V(K_d, K_g, \lambda, Y, \log N; A'_g) - V(K_d, K_g, \lambda, Y, \log N; A_g)) \end{aligned}$$

The FOC for investment and R&D are given by

$$\begin{aligned} 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + (1 - \phi_d i_d) V_d \\ 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + (1 - \phi_d i_d) V_d \\ 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + \varphi V_\lambda \end{aligned}$$

Using the fact that  $\varphi V_\lambda = \delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1}$ , we can simplify to

$$\begin{aligned} i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{V_\lambda}{V_d} \\ i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{V_\lambda}{V_g} \\ i_\lambda &= \frac{1}{\lambda} \left( (A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{V_\lambda}{V_d})) K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{V_\lambda}{V_g})) K_g - \frac{\delta}{\varphi V_\lambda} \right) \end{aligned}$$

We can analytically simplify out  $\log N_t$  to get a simplified HJB

$$\begin{aligned} \delta v(K_d, K_g, \lambda, Y) &= \max_{i_g, i_d, i_\lambda} \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) \\ &\quad + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\ &\quad + \beta_f E_d v_Y + \frac{1}{2} \varsigma^2 E_d^2 v_{YY} \\ &\quad - [\{\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - 2) \mathbf{I}\{Y_t > 2\}\} \beta_f E_d + \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y_t > 2\}}{2} \varsigma^2 E_d^2] \\ &\quad + (\varphi i_\lambda - \alpha_\lambda) \lambda v_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} v_{\lambda\lambda} + \lambda (v(K_d, K_g, \lambda, Y; A'_g) - v(K_d, K_g, \lambda, Y; A_g)) \end{aligned}$$

FOC

$$\begin{aligned} i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_d} \\ i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_g} \\ i_\lambda &= \frac{1}{\lambda} \left( (A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_d})) K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_g})) K_g - \frac{\delta}{\varphi v_\lambda} \right) \end{aligned}$$

From this we can layer on different forms of uncertainty.

## 1 Post jump

### 1.1 Desired model

Replace  $E_d$  with  $\eta A_d K_d$ . We solve post jump HJB:

$$\begin{aligned} \delta v(K_d, K_g, Y; A'_g) &= \max_{i_g, i_d} \delta \log(A_d K_d - i_d K_d + A'_g K_g - i_g K_g) \\ &\quad + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\ &\quad + \beta_f (\eta A_d K_d) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d K_d)^2 v_{YY} - [\{\gamma_1 + \gamma_2 Y + \gamma_3 (Y - \bar{y}) \mathbf{I}\{Y > \bar{y}\}\} \beta_f (\eta A_d K_d)] \\ &\quad - \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y > \bar{y}\}}{2} [\varsigma^2 (\eta A_d K_d)^2] \end{aligned}$$

denote

$$mc = \delta(A_d K_d - i_d K_d + A'_g K_g - i_g K_g)^{-1}$$

FOC

$$\begin{aligned} i_d &= \frac{1}{\phi_d} - \frac{mc}{\phi_d v_d} \\ i_g &= \frac{1}{\phi_g} - \frac{mc}{\phi_g v_g} \end{aligned}$$

## 1.2 Transformation of state variables

$$X = [\log K, R, Y]', \quad \log K = \log(K_d + K_g), \quad R = \frac{K_g}{K_d + K_g}$$

$$\begin{aligned} d(K_d + K_g) &= \left( [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] K_d + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] K_g \right) dt + (\sigma_d K_d + \sigma_g K_g) dW \\ dK/K &= \left( [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] (1 - R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R \right) dt + (\sigma_d (1 - R) + \sigma_g R) dW \\ d\log K &= \left( [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] (1 - R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R - \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} \right) dt \\ &\quad + (\sigma_d (1 - R) + \sigma_g R) dW \end{aligned}$$

$$dR = \left( (\alpha_d + i_d - \frac{\sigma_d^2}{2})(1 - R) + (\alpha_g + i_g - \frac{\sigma_g^2}{2})R - R\sigma_g^2 + (1 - R)\sigma_d^2 \right) dt + R(1 - R)(\sigma_d + \sigma_g) dW$$

$$dY = \eta A_d (1 - R) K (\beta_f dt + \varsigma dW)$$

### 1.2.1 $\log K, R, Y$

$\log K \in [4., 8.5]$ ,  $R \in [0.14, 0.99]$  and  $Y \in [0., 3.]$

$$\begin{aligned} \delta v(\log K, R, Y; A'_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1 - R) + (A'_g - i_g)R) + \delta \log K \\ &\quad + \left( \left\{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right\} (1 - R) + \left\{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right\} R - \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} \right) v_k \\ &\quad + \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} v_{kk} \\ &\quad + \left( (\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - R\sigma_g^2 - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) + (1 - R)\sigma_d^2 \right) [R(1 - R)] v_r \\ &\quad + \frac{1}{2} R^2 (1 - R)^2 (\sigma_g^2 + \sigma_d^2) v_{rr} \\ &\quad + \beta_f (\eta A_d (1 - R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1 - R) K)^2 v_{YY} \\ &\quad - [\{\gamma_1 + \gamma_2 Y + \gamma_3 (Y - \bar{y}) \mathbb{I}\{Y > \bar{y}\}\} \beta_f (\eta A_d (1 - R) K) + \frac{\gamma_2 + \gamma_3 \mathbb{I}\{Y_t > \bar{y}\}}{2} \varsigma^2 (\eta A_d (1 - R) K)^2] \end{aligned}$$

FOC

$$mc = \delta((A_d - i_d)(1 - R) + (A'_g - i_g)R)^{-1}$$

$$\begin{aligned} mc &= (1 - \phi_d i_d) ((v_k - Rv_r)) & \Rightarrow i_d &= \frac{1}{\phi_d} \left[ 1 - \frac{mc}{v_k - Rv_r} \right] \\ mc &= (1 - \phi_g i_g) (v_k + (1 - R)v_r) & \Rightarrow i_g &= \frac{1}{\phi_g} \left[ 1 - \frac{mc}{v_k + (1 - R)v_r} \right] \end{aligned}$$

Future plans:

- Non-zero  $\gamma_3$

## 2 Pre jump

R& D investment,  $I_\lambda$ , R& D investment ratio  $i_\lambda = \frac{I_\lambda}{K}$   
Technology jump intensity :

$$d\log \lambda = (-\alpha_\lambda + \varphi i_\lambda \frac{K}{\lambda} - \frac{\sigma_\lambda^2}{2}) dt + \sigma_\lambda dW$$

State variables  $\log K, R, Y, \log(\lambda)$ , controls  $i_d, i_g$  and  $i_\lambda$ , HJB:

$$\begin{aligned} \delta v(\log K, R, Y, \log(\lambda); A_g) &= \max_{i_g, i_d, i_\lambda} \delta \log((A_d - i_d)(1 - R) + (A_g - i_g)R - i_\lambda) + \delta \log K \\ &+ \left( \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} (1 - R) + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} R - \frac{|\sigma_d(1 - R) + \sigma_g R|^2}{2} \right) v_k \\ &+ \frac{|\sigma_d(1 - R) + \sigma_g R|^2}{2} v_{kk} \\ &+ \left( (\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - R\sigma_g^2 - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) + (1 - R)\sigma_d^2 \right) [R(1 - R)] v_r \\ &+ \frac{1}{2} R^2 (1 - R)^2 [\sigma_d^2 + \sigma_g^2] v_{rr} \\ &+ \beta_f (\eta A_d (1 - R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1 - R) K)^2 v_{YY} \\ &- [\{\gamma_1 + \gamma_2 Y + \gamma_3 (Y - \bar{y}) \mathbb{I}\{Y > \bar{y}\}\} \beta_f (\eta A_d (1 - R) K) + \frac{\gamma_2 + \gamma_3 \mathbb{I}\{Y > \bar{y}\}}{2} \varsigma^2 (\eta A_d (1 - R) K)^2] \\ &+ (\varphi i_\lambda \frac{K}{\lambda} - \alpha_\lambda - \frac{\sigma_\lambda^2}{2}) v_\lambda + \frac{\sigma_\lambda^2}{2} v_{\lambda\lambda} + \lambda (v^{\text{post}}(\log K, R, Y; A'_g) - v(\log K, R, Y, \log(\lambda); A_g)) \end{aligned}$$

FOC

$$\begin{aligned} mc &= \delta((A_d - i_d)(1 - R) + (A_g - i_g)R - i_\lambda)^{-1} \\ mc &= \frac{K}{\lambda} \varphi v_\lambda \end{aligned}$$

$$\begin{aligned} mc &= (1 - \phi_d i_d) ((v_k - Rv_r)) & \Rightarrow i_d &= \frac{1}{\phi_d} \left[ 1 - \frac{K}{\lambda} \frac{\varphi v_\lambda}{v_k - Rv_r} \right] \\ mc &= (1 - \phi_g i_g) (v_k + (1 - R)v_r) & \Rightarrow i_g &= \frac{1}{\phi_g} \left[ 1 - \frac{K}{\lambda} \frac{\varphi v_\lambda}{v_k + (1 - R)v_r} \right] \end{aligned}$$

$$i_\lambda = (A_d - (\frac{1}{\phi_d} - \frac{K}{\lambda} \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_k - Rv_r}))(1 - R) + (A_g - (\frac{1}{\phi_g} - \frac{K}{\lambda} \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_k + (1 - R)v_r}))R - \frac{\lambda\delta}{K\varphi v_\lambda}$$

## A State variables

$$\begin{aligned}\log(K) &\in [4, 8.5] \\ R &\in [0.01, 0.99] \\ Y &\in [0, 4] \\ \log(\lambda) &\in [-4, 0]\end{aligned}$$

## B Parameters

Economy

Parameters	values
$\delta$	0.01
$(\alpha_d, \phi_d, \sigma_d)$	(-0.02, 8, 0.016)
$(\alpha_g, \phi_g, \sigma_g)$	(-0.02, 8, 0.016)
$(\alpha_\lambda, \varphi, \sigma_\lambda)$	(0, 0.01, 0.016)
$A_d$	0.12
$(A_g, A'_g)$	(0.10, 0.15)

Temperature and damage

Parameters	values
$\beta_f$	1.86 / 1000
$\varsigma$	1.2 * 1.86 / 1000
$\gamma_1$	0.00017675
$\gamma_2$	2 * 0.0022
$\gamma_3$	$\frac{1}{3} \frac{m-1}{19}, m = 1, 2, \dots, 20$
$\bar{y}$	2

$\eta$  are determined as follows:

- Initial capital:  $K_0 = 85/0.115 \approx 739$
- $K_{d,0} = K_0 \times \frac{2}{3} \approx 493$
- Choose  $\eta$  such that  $10 = \eta \times 0.12 \times 493 \Rightarrow \eta \approx 0.17$