A Model With Clean and Dirty Capital Stocks and and R&D in Green Innovation -

Assume there are two sectors, each with AK production technology ($Y_i = A_i K_i,\ i = d, g$) and each with its own capital stock that evolves with quadratic adjustments costs and Brownian shocks as follows:

$$dK_d/K_d = [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2]dt + \sigma_d dW$$
$$dK_g/K_g = [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2]dt + \sigma_g dW$$

We also assume there is $R\&D$ investment that leads to an increased arrival rate of a one time jump in Sector 2 productivity. The arrival rate is denoted as λ_t and evolves as follows

$$d\lambda_t = (\varphi i_\lambda - \alpha_\lambda)\lambda_t dt + \sigma_\lambda dW_t$$

The key difference between the sectors is that production from Sector 1 generates emissions. As a result, the evolution of atmospheric temperature is give by the Matthews Approximation, so that temperature T_t and cumulative carbon emissions are given by

$$dY_t = \beta_f E_t + \varsigma E_t dW_t$$

where β_f is the Matthews parameter and η is the scaling factor converting Sector d ouput $Y_d = A_d K_d$ into emissions such that $E_t = \eta A_d K_d$.

Output can be used in for consumption, investment in either capital stock, or for R&D into improving the productivity of Sector g :

$$C = A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda$$

We assume exponential-quadratic damages to preferences so that our utility is augmented when accounting for climate damages. Flow utility is a log function over consumption, assuming perfect substitutability over output from the two sectors so that

$$U(C) = \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N_t$$

where the $\log N_t$ follows from BBH2 as

$$\begin{aligned} \log N_t &= \Gamma(Y) \\ \Gamma(y) &= \gamma_1 y + \frac{\gamma_2}{2} y^2 + \frac{\gamma_3}{2} \mathbf{1}_{y \geq \bar{y}} (y - \bar{y})^2 \end{aligned}$$

Taking these pieces together we get the HJB equation

$$\begin{aligned} \delta V(K_d, K_g, \lambda_t, Y_t, \log N_t) = \max_{i_g, i_d, i_\lambda} & \ \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N_t \\ & + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} V_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} V_g K_g + \frac{\sigma_d^2 K_d^2}{2} V_{dd} + \frac{\sigma_g^2 K_g^2}{2} V_{gg} \\ & + \beta_f \eta A_d K_d V_Y + \frac{(\varsigma E_d)^2}{2} V_{YY} + [\{\gamma_1 + \gamma_2 Y_t\} \beta_f \eta A_d K_d + \frac{1}{2} \gamma_2^2 \varsigma^2 E_d^2] V_{\log N} + \frac{\varsigma^2 E_d^2}{2} V_{\log N \log N} \\ & + (\varphi i_\lambda - \alpha_\lambda) \lambda_t V_\lambda + \frac{\sigma_\lambda^2}{2} V_{\lambda\lambda} + \lambda_t [V(K_d, K_g, Y_t, N_t, \lambda_t; A_g') - V(K_d, K_g, Y_t, N_t, \lambda_t; A_g)] \end{aligned}$$

The FOC for investment and R&D are given by

$$\begin{aligned} 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + \{1 - \phi_d i_d\} V_d \\ 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + \{1 - \phi_g i_g\} V_g \\ 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + \varphi V_\lambda \end{aligned}$$

Using the fact that $\varphi V_\lambda = \delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1}$ we can simplify to

$$\begin{aligned}
i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d}\frac{V_\lambda}{V_d} \\
i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g}\frac{V_\lambda}{V_g} \\
i_\lambda &= \frac{1}{\lambda}[(A_d - \{\frac{1}{\phi_d} - \frac{\varphi}{\phi_d}\frac{V_\lambda}{V_d}\})K_d + (A_g - \{\frac{1}{\phi_g} - \frac{\varphi}{\phi_g}\frac{V_\lambda}{V_g}\})K_g - \frac{\delta}{\varphi V_\lambda}]
\end{aligned}$$

We can analytically simplify out $\log N_t$ to get a simplified HJB

$$\begin{aligned}
\delta v(K_d, K_g, \lambda_t, Y_t) = {} & \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) \\
& + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\
& + \beta_f \eta A_d K_d v_Y + \frac{(\varsigma E_d)^2}{2} v_{YY} - [\{\gamma_1 + \gamma_2 Y_t\}\beta_f \eta A_d K_d + \frac{1}{2}\gamma_2^2 \varsigma^2 E_d^2] \\
& + (\varphi i_\lambda - \alpha_\lambda)\lambda_t v_\lambda + \frac{\sigma_\lambda^2}{2} v_{\lambda\lambda} + \lambda_t [v(K_d, K_g, Y_t, \lambda_t; A_g') - v(K_d, K_g, Y_t, \lambda_t; A_g)]
\end{aligned}$$

From this we can layer on different forms of uncertainty.