FINANCIAL MATHEMATICS MONEY MARKET

1. Methods of Interest Calculation, Yield Curve and Quotation

1.1 Methods to Calculate Interest

While calculating interests, the general question is how the interest for one period is determined. The interest calculation methods employed can vary, depending on national and product markets.

As a rule, interest can be calculated in the following manner:

$$\boxed{I = C \times r \times \frac{D}{B}}$$

I	= amount of interest
C	= capital amount
r	= interest rate in decimals (i.e. 5 % = 0.05; 10.3 % = 0.103; etc.)
D	= number of days of the term of interest
B	= day basis for calculation (fixed number of days per year) (= day base)

There are three ways to determine the **number of days** (D).

a) **Actual:** Counting the actual numbers of days that elapse.

Term of interest: 1 March – 31 March ➔ 30 days
Term of interest: 1 March – 1 April ➔ 31 days

b) **30**: Each month counts as 30 days (remaining days in a month are subtracted).

Term of interest: 1 March – 31 March ➔ 30 days
Term of interest: 1 March – 30 March ➔ 29 days
Term of interest: 1 March – 1 April ➔ 30 days

c) **30E**: Each month counts as 30 days (the 31st is treated as if it was the 30th; remaining days are subtracted).

Term of interest: 1 March – 31 March ➔ 29 days
Term of interest: 1 March – 30 March ➔ 29 days
Term of interest: 1 March – 1 April ➔ 30 days

This method is used in the Euromarket as well as in some continental European markets.

The correct number of days can be determined by using the ISDA formula for the 30-method and 30E-method:

$$D = (y_2 - y_1) \times 360 + (m_2 - m_1) \times 30 + (d_2 - d_1)$$

D = number of days
y_1 = year in which the period starts
y_2 = year in which the period end
m_1 = start month
m_2 = maturity month
d_1 = start day
d_2 = maturity day

where for d_1 and d_2:

	30-method	30E-method
d_1	$d_1 = 31 \rightarrow 30$	$d_1 = 31 \rightarrow 30$
d_2	$d_2 = 31 \rightarrow 30$, if d_1 is 30 or 31	$d_2 = 31 \rightarrow 30$

Example

interest period 1st–31st March 2004

30-method:

D = (2004 – 2004) x 360 + (3-3)x30 + (31-1) = 30

30E-method:

D = (2004 – 2004) x 360 + (3-3)x30 + (30-1) = 29

There are three ways to determine the day basis (B).

a) **360**: Assuming that each year has 360 days.

Annual term: 1 March XY – 1 March XZ are 365 days ➔ day base is 360 days

Annual term: 1 March XY – 3 March XZ*) are 367 days ➔ day base is 360 days

*) e.g. with a weekend

b) **365**: Assumption that each year has 365 days.

Just as with **360**, but generally ➔ day base is 365 days

c) **Actual**:

Example

- in the money market (isda-method):

The actual days per year are counted (leap year 366 days, "normal" year 365 days). If a deal runs over two years (one of them being a leap year), the interest calculation is divided into two parts.

2 January 2003 – 2 January 2004 interest calculation Actual/Actual

$$x \cdot \frac{364}{365} + x \cdot \frac{1}{366}$$

Example

- in the capital market (isma-method):

A year is counted with actual days of the term of interest (multiplied by the number of terms of interest).

Bond with semi-annual interest payments:
1 March XY – 1 September XY are 184 days ➔ day basis is 368 days (184 x 2)

$$x \cdot \frac{184}{368}$$

Therefore, theoretically nine combinations of days (D) and basis (B) are possible but only 5 of them are practically used: Actual / 365; Actual / 360; 30 / 360; 30E / 360; and Actual / Actual.

Daily conventions vary from market to market. In the table below the conventions for money markets and capital markets are listed. In the capital markets, however, these conventions may differ in their specifications regarding the international and domestic market and regarding different financial instruments.

Therefore, please clarify these conditions before you trade!

	Money market		**Money market**		**Capital Market**
Australia	Actual / 360	Norway	Actual / 360	Euro	Actual / Actual
Euro	Actual / 360	Poland	Actual / 365	Great-Britain	Gilts: s.a.*) Actual / Actual
New Zealand	Actual / 360	Sweden	Actual / 360	Japan	30 / 360 or Actual / Actual
Great Britain	Actual / 365	Switzerland	Actual / 360	Sweden	30 / 360 or 30E / 360
Hong Kong / Singapore	Actual / 365	Czech Republic	Actual / 360	Switzerland	30 / 360 or 30E / 360
Japan	Actual / 360	USA	Actual / 360	USA	30 / 360 or Actual / Actual

*) semi-annual

Example

Semi-annual bond, principal 10,000 at an interest rate of 7,5 % p.a., last coupon on 1 May, next coupon on 1 November (number of days: 184). On 31 May, the following interest is due:

Calculation method	Days of term/Days per year	Calculation of interest
Actual / 365	30 / 365	$10{,}000 \cdot 0.075 \cdot \frac{30}{365} = 6164$
Actual / 360	30 / 360	$10{,}000 \cdot 0.075 \cdot \frac{30}{360} = 62.50$
30 / 360	30 / 360	$10{,}000 \cdot 0.075 \cdot \frac{30}{360} = 62.50$
30E / 360	29 / 360	$10{,}000 \cdot 0.075 \cdot \frac{29}{360} = 60.42$
Actual / Actual	30 / 368	$10{,}000 \cdot 0.075 \cdot \frac{30}{368} = 61.14$

1.2 The Yield Curve

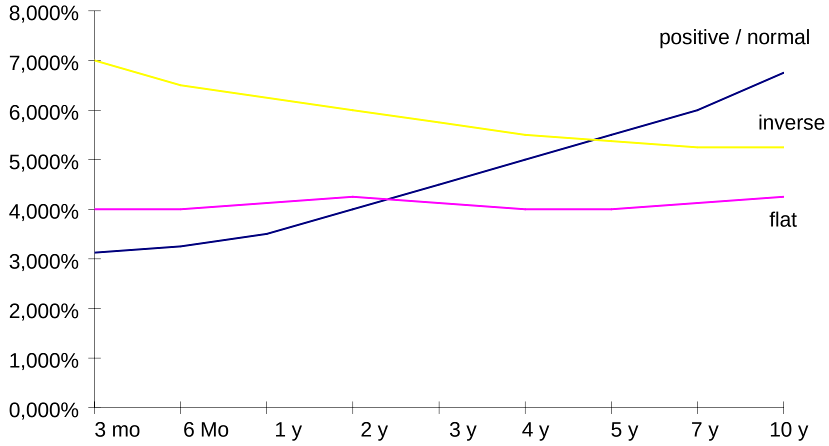
The yield curve (also known as interest rate structure) displays interest rates of a specific financial instrument for different terms to maturity. For example the T-bond curve shows the yields of US Treasury bonds (“T-bond”) with various terms to maturity. Due to the number of different instruments there are a lot different yield curves such as interest rate swap (IRS) curve, LIBOR-curve, Bund-curve, mortgage bond-curve, etc..

There are three different types of yield curves:

a) **Steep yield curve**: (normal or positive) short-term interest rates are lower than long-term interest rates.

b) **Flat yield curve**: interest rates for different terms are the same.

c) **Inverse yield curve**: short-term interest rates are higher than long-term interest rates.



Hypothesis on the Shape of the Yield Curve

There are several theories on the shape of a yield curve. One of them is the so-called **Interest Expectation Theory**. Therefore, the yield curve represents the expectations of market participants regarding future yields. If the market expects rising interest rates, the yield curve will slope upwards, because market participants will anticipate higher future rates for long-term investments.

According the **Liquidity Preference Theory** the market participants demand a premium for long-term investments compared to short term deals. Therefore, even in a situation where no interest rate change is expected, the curve will slightly slope upwards. This constellation is called "normal interest curve".

The **Market Segmentation Theory** says that different groups of market participants are active in market segments with different duration. This supply and demand mismatch in the various segments leads to different yields for different terms.

Certainly, there are several other factors that have an influence on the yield curve, e.g. interventions of the central bank, preferences of liquidity among market participants, etc.

1.3 Interpolation

Since there is not always a benchmark at hand, interest rates must sometimes be estimated.

In the following, we want to show the most simple method to calculate interest rates for unusual terms: straight-line interpolation.

$$r = r_s + \left[\frac{r_l - r_s}{D_l - D_s}\right] \times (D - D_s)$$

r	= interest rate in decimals
D	= number of days of the term of interest
r_s	= interest rate in decimals, short-term
D_s	= number of days, short-term
r_l	= interest rate in decimals, long-term
D_l	= number of days, long-term

Example

We calculate the interest rate for a deposit for 1½ months (46 days) on the basis of the following interest rates:

1 month	3½ %	(31 days)
3 months	3¾ %	(92 days)

$$r = 0.035 + \left[\frac{0.0375 - 0.035}{92 - 31}\right] \times (46 - 31) = 3{,}56148\%$$

The method of straight-line interpolation has the advantage that it is very easy to apply.
To give a complete overview, we would like to point out, that there exist also methods which take the non-linearity of interest rates into account. These are:

- logarithmic interpolation
- cubic splines

1.4 Quotation

Interest rates are often quoted in basis points that lie either below or above a certain benchmark rate. A bank could lend money at LIBOR – 3 basis points; or a Dollar bond is issued at T + 50: this would mean 50 basis points above the yield of a comparable US Treasury bill. One basis point is equal to $^{1}/_{100}$ th of 1 %, i.e. 0.01 %.

The quotation of interest rates in the money market is done on a p.a. basis, where the interest is paid at the end of the term. The interest that is thereby paid is called simple interest (e.g. CHF $2^{1}/_{4}$ – $2^{3}/_{8}$ % for 6 months).

In the money market, most transactions have terms of not more than one year. Where the term exceeds 12 months, interest is paid first after one year and then at maturity. For example, interest payments on an 18-month deposit are due after 12 months for the first time, and finally after 18 months, i.e. at the end of the 6-month period of the second year. Simple interest for a definite period is calculated on the basis of the principal.

Note: There are different methods to calculate interest for individual instruments.

Example

What is the value of 1 BP of a USD 1,000,000 deposit yielding 5 %?

Since USD is calculated on ACT /360 basis but the interest is quoted on the basis of a 360-day year, the value of 1 BP is

$$1{,}000{,}000 \times 0.0001 \times \frac{365}{360} = 101.38889$$

2. Financial Arithmetics in the Money Market

2.1 Calculating Simple Interest

The formula for calculating simple interest (single payment of interest and a term of less than one year) is:

$$\boxed{I = C \times r \times \frac{D}{B}}$$

I = amount of interest

C = capital amount

r = interest rate in decimals (i.e. 5 % = 0.05; 10.3 % = 0.103; etc.)

D = number of days of interest period

B = day basis of calculation (fixed number of days per year)

Example

Bank A gives a 1-month deposit of Euro 5 Mio at 3 %. Start date of this credit is October 1st and end date is November 1st. The actual number of days for this period is 31. Basis of term calculation is 360 days per year. The absolute interest of this credit is:

$$I = 5{,}000{,}000 \times 0.03 \times \frac{31}{360} = 12{,}916.67$$

On November 1st, the borrower will either

a) pay interest of Euro 12,916.67 and roll over the credit, or

b) settle the credit by paying back the principal plus interest:
Euro 5,012,916.67

2.2 Average Interest

If different interest rates apply over several interest periods while giving or taking money, the average interest rate may be calculated like this:

$$rAV=\left[\left(r_1\times\frac{D_1}{B}\right)+\left(r_2\times\frac{D_2}{B}\right)+\left(r_3\times\frac{D_3}{B}\right)+........+\left(r_n\times\frac{D_n}{B}\right)\right]\times\frac{B}{D_N}$$

r_{AV} = average interest rate

$r_1, r_2, \ldots r_n$ = interest rate of the respective terms of interest, in decimals

B = day basis of calculation

D_N = number of days of the total term

$D_1, D_2, \ldots D_n$ = number of days of the respective term

Example

You lent EUR at the following rates during the last year:

2 January - 2 April	(90 days) at 2.5 %
2 April - 2 July	(91 days) at 2.75 %
2 July - 2 October	(92 days) at 2.875 %
2 October - 2 January	(92 days) at 3 %

Calculate the average interest rate.

$$\left[\left(0.025\times\frac{90}{360}\right)+\left(0.0275\times\frac{91}{360}\right)+\left(0.02875\times\frac{92}{360}\right)+\left(0.03\times\frac{92}{360}\right)\right]\times\frac{360}{365}=2.78288\%$$

2.3 Calculating Compound Interest (Effective Interest)

If an amount is lent or borrowed over several terms and the interest payments are not paid out at the end of each term, the amount that is the basis for the interest calculation is raised by the amount of accrued interest. This is commonly known as capitalisation or compound interest.

The general formula for the calculation of compound interest is:

$$ER=\left\{\left[\left(1+r_1\times\frac{D_1}{B}\right)\times\left(1+r_2\times\frac{D_2}{B}\right)\times\left(1+r_3\times\frac{D_3}{B}\right)\times\ldots\ldots\times\left(1+r_n\times\frac{D_n}{B}\right)\right]-1\right\}\times\frac{B}{D_N}$$

ER	=	effective interest rate
$D_1,\ldots.D_n$	=	number of days of the respective term
B	=	day basis of calculation
D_N	=	number of days of the total term
$r_{1..}..r_n$	=	interest rate of the respective terms of interest, in decimals

Example

Last year you invested CHF at the rates given below. The subsequent investment is made up of the original amount and the accrued interest.

2 January - 2 April	(90 days) at 2.50 %
2 April - 2 July	(91 days) at 2.75 %
2 July - 2 October	(92 days) at 2.875 %
2 October - 2 January	(92 days) at 3.0%

What is the effective interest rate?

$$ER=\left\{\left[\left(1+0.025\times\frac{90}{360}\right)\times\left(1+0.0275\times\frac{91}{360}\right)\times\left(1+0.02875\times\frac{92}{360}\right)\times\left(1+0.03\times\frac{92}{360}\right)\right]-1\right\}\times\frac{360}{365}=2.8124\%$$

2.4 Calculating Forward Rates (for terms < 1 year)

A forward-forward rate (or simply forward rate) is an interest rate for a future term of interest, e.g. an interest rate for a 6-month investment that will begin in 3 months. These forward rates can be derived from the interest rates prevailing in the market. By investing for 9 months and refinancing for 3 months the same effects can be achieved today.

The formula to calculate forward rates is as follows:

$$FR = \left\{\left[\frac{1 + \left(r_l \times \frac{D_l}{B}\right)}{1 + \left(r_S \times \frac{D_S}{B}\right)}\right] - 1\right\} \times \frac{B}{D_{l-S}}$$

FR = forward interest rate

r_l = interest rate in decimals, long-term

r_s = interest rate in decimals, short-term

D_l = number of days, long-term

D_s = number of days, short-term

B = day basis of calculation

$D_{l\text{-}s}$ = difference between short term and long term (in days)

Forward rate calculation for GBP, starting in 3 months for a term of 3 months.

Interest rates GBP: 3 months = 7½ % (91 days)
6 months = 7¾ % (183 days)

2.5 Calculating Forward Rates (for terms > 1 year)

Given a short-term and a long-term interest rate it is possible to calculate a so-called forward rate (also called forward-forward rate).

Formula:

$$FR = \left[\frac{(1+r_l)^N}{(1+r_s)^n} \right]^{\frac{1}{(N-n)}} - 1$$

FR	= forward rate of interest
r_l	= rate of interest in decimals, long-term
N	= term in years, long term
r_s	= rate of interest in decimals, short-term
n	= term in years, short term

Note: The exact calculation is done on the basis of zero rates. For long terms, differences may become too big without using zeros.

2.6 Calculating the Future Value (for terms < 1 year)

Starting with the present value today (principal), the future value can be determined. The amount of money that is due at the end of the loan's or deposit's term is made up of the original amount of capital plus the interest. This amount is called the future value.

For example, if, today, you invest GBP 100 for 1 year at 4 % p.a., you receive an amount of GBP 104 (Actual / 365) at the end of the year.

The present value of this investment is GBP 100.
The future value of this investment is GBP 104.

Formula for the simple calculation of the future value:

$$\boxed{FV = PV \times \left(1 + \left(r \times \frac{D}{B}\right)\right)} \quad \text{or} \quad \boxed{FV = PV + \left(PV \times r \times \frac{D}{B}\right)}$$

FV = future value of the investment
PV = present value
r = interest rate in decimals
D = number of days of the term of interest
B = day basis of calculation

In other words, the future value of an investment is the amount of capital (PV) plus accrued interest.

Example

You take a deposit of EUR 1 Mio at a rate of 6 % p.a. for 92 days (Actual / 360). At the end of its term, the value of the deposit will be:

$$FV = 1{,}000{,}000 \times \left(1 + \left(0.06 \times \frac{92}{360}\right)\right) = 1{,}015{,}333.33$$

Note: When calculating future values that include compound interest, one has to first compute the effective interest rate from the nominal interest rate. Only after this has been done, can one use the above formula.

2.7 Calculating the Future Value (for terms > 1 year)

The formula for calculating the future value for terms > 1 year is:

$$\boxed{FV = C \cdot (1 + r)^N}$$

FV = future value
C = capital amount
r = interest rate p.a. in decimals
N = total term in years

Note: It is also assumed that the re-investment is done at the same rate.

2.8 Calculating the Present Value (for terms < 1 year)

The current value of a future cash flow is called present value. The present value is calculated by discounting the future value. Many markets, e.g. that for US T-bills, conventionally use discount rates.

$$\boxed{PV = \frac{FV}{1 + \left(r \times \frac{D}{B}\right)}}$$

PV = present value (capital)
FV = future value
r = interest rate in decimals
D = number of days
B = day basis of calculation

Example

We know that the current yield of a US treasury bill is 5.50 %. The future value is 1,000,000 in 2 months (61 days). Calculate the present value of the T bill.

$$PV = \frac{1{,}000{,}000}{1 + \left(0.055 \times \frac{61}{360}\right)} = 990{,}766.61$$

2.9 Calculating the Present Value (for terms > 1 year)

With a term of more than a year, the present value can be calculated in the following way:

$$PV = \frac{FV}{(1+r)^N}$$

Example

The present value of EUR 1 in five years at an interest rate of 6 % is:

$$PV = \frac{1}{(106)^5} = \mathbf{0.74726}$$

2.10 Interest Calculation with PV and FV (for terms < 1 year)

If we know the future value, the present value as well as the term of an investment and if there are no cash flows during the term , we can calculate the current yield (current market interest rate).

$$r = \frac{(FV - PV)}{PV} \times \frac{B}{D}$$

FV = future value
PV = present value
B = day basis of calculation
D = number of days
r = current yield (current market interest rate)

Example

A EUR Commercial Paper with a time to maturity of 82 days has the stated cash flows. What is the yield?

present value: = 987,627

future value: = 1,000,000

$$r = \frac{(1{,}000{,}000 - 987{,}627)}{987{,}627} \times \frac{360}{82} = 5.5001\%$$

Note: You cannot use this formula if the investment has several cash flows.

2.11 Interest Calculation with PV and FV (for terms > 1 year)

If present value, future value and the term of a deal are known, the interest rate can be calculated. Thereby, a re-investment of the coupon payments at the same interest rate is assumed.

$$\boxed{r = \sqrt[N]{\frac{FV}{PV}} - 1} \quad \text{or} \quad \boxed{\left(\frac{FV}{PV}\right)^{\left(\frac{1}{N}\right)} - 1}$$

FV = future value

PV = present value

N = term

Example

At what interest rate do you have to invest EUR 50 mio in order to receive EUR 100 mio (incl. accrued interest) after 10 years?

$$r = \sqrt[10]{\frac{100}{50}} - 1$$

r = 7.17735 %

Note: The re-investment of the coupon payments at the same interest rate is assumed.

2.12 Converting Discount Rates into Yield

Some financial instruments (e.g. US Treasury bills) are quoted on a discount rate basis, i.e. the interest is calculated on the basis of the future value and not on the invested capital.
In order to make such instruments comparable to instruments which are quoted on a yield basis we have to convert the discount rate into an interest rate.

$$\boxed{r = \frac{r_d}{1 - \left(r_d \times \frac{D}{B}\right)}}$$

r = interest rate p.a.
r_d = discount rate, in decimals
D = number of days
B = day basis of calculation

Example

The discount rate of an US T-bill is 5 %, its term is 92 days.
What is the comparable p.a. rate of interest is:

$$r = \frac{0{,}05}{1 - \left(0{,}05 \times \frac{92}{360}\right)} = 5{,}06472\%$$

2.13 Converting from Money Market Basis to Bond Basis and vice versa

Since the basis for interest payments usually is different in the capital market and the money market, we must be able to convert these payments.

$$\boxed{r_{CM} = r_{MM} \times \frac{D_{MM}}{B_{MM}} \times \frac{B_{CM}}{D_{CM}}} \qquad \boxed{r_{MM} = r_{CM} \times \frac{D_{CM}}{B_{CM}} \times \frac{B_{MM}}{D_{MM}}}$$

r_{CM} = interest rate in the capital market
r_{MM} = interest rate in the money market
D_{MM} = number of days per year, money market
B_{MM} = basis of term calculation, money market

D_{CM} = number of days per year, capital market
B_{CM} = basis of term calculation, capital market

Example

An interest rate in the capital market of 3.50% (30/360) is to be compared to an interest rate in the money market.

$$3.50\% \times \frac{360}{360} \times \frac{360}{365} = 3.45205\% \quad \text{money market}$$

2.14 Conversion of Non-Annual Payments into Effective Interest Rate

Interest payments not always are due annually but sometimes daily, weekly, monthly, quarterly, and semi-annual interest payments are also possible. With bonds, it is quite common that interest payments are made semi-annually. To be able to compare these non-annual interest payments to yearly payments (single payment of interest), one converts the nominal interest rate into the effective interest rate. With a single p.a. payment, the nominal rate equals to the effective rate of interest.

$$\boxed{ER = \left(1 + \frac{NR}{FIP}\right)^{FIP} - 1}$$

ER = effective rate of interest p.a. in decimals
NR = nominal rate of interest p.a. in decimals
FIP = frequency of interest payments p.a. (e.g. 2 for semi-annual, 4 for quarterly,....etc.)

Example

You invest EUR at 6.00% for a period of 1 year and agree quarterly interest payments. What is the effective interest rate of this investment?

$$ER = \left(1 + \frac{0.06}{4}\right)^{4} - 1 = 6.13636\%$$

A quarterly interest rate of 6.00 % corresponds to an annual rate of 6.14%.

2.15 Conversion of Annual into Non-Annual Interest Payments

We can also convert non-annual payments into annual payments:

$$r_{NA} = \left(\sqrt[FIP]{1 + rA} - 1\right) \times FIP$$

r_{NA} = non-annual rate of interest p.a., for the term of interest
FIP = frequency of interest payments p.a.
r_A = annual rate of interest p.a., in decimals

Example

You hold a bond with annual interest payments of 6.00%. Calculate the nominal interest rate, based on semi-annual interest payments.

$$r_{SA} = \left(\sqrt[2]{1 + 0.06} - 1\right) \times 2 = 5.9126\ \%$$

The interest rate of 6.00% annual corresponds to a rate of 5.9126% semi annual.