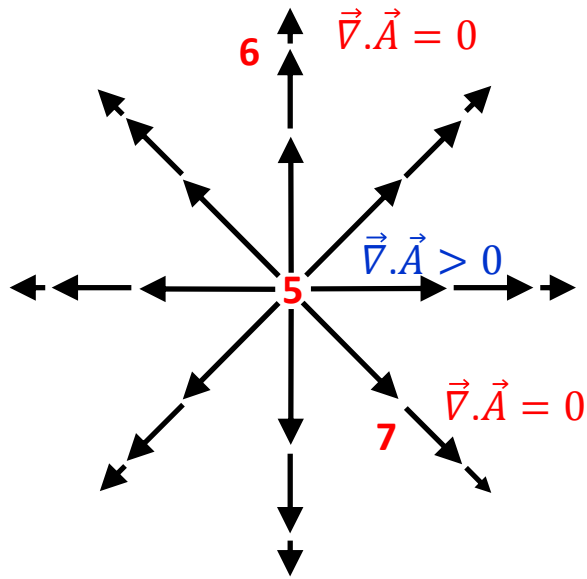


# Gauss law

## Outline

- Demonstrate that Gauss theorem does not depend on the shape of the Gauss surface
- Complete the first Maxwell's and Poisson equations
- Demonstrate that Coulomb's law is exactly inversely proportional to  $r^2$
- Demonstrate that a static charge cannot be in equilibrium in the electric field of other charges
- Concept of work – Electric potential – Electric potential energy
- Apply Gauss law in various circumstances



$\vec{A} \propto (1/r^2)\vec{u}_r$   
Coulomb's law

- The field lines are clearly spreading from points **6** and **7**
- **BUT** they are getting shorter away from the center

**Locally:** Trade-off between spreading and intensity of the field

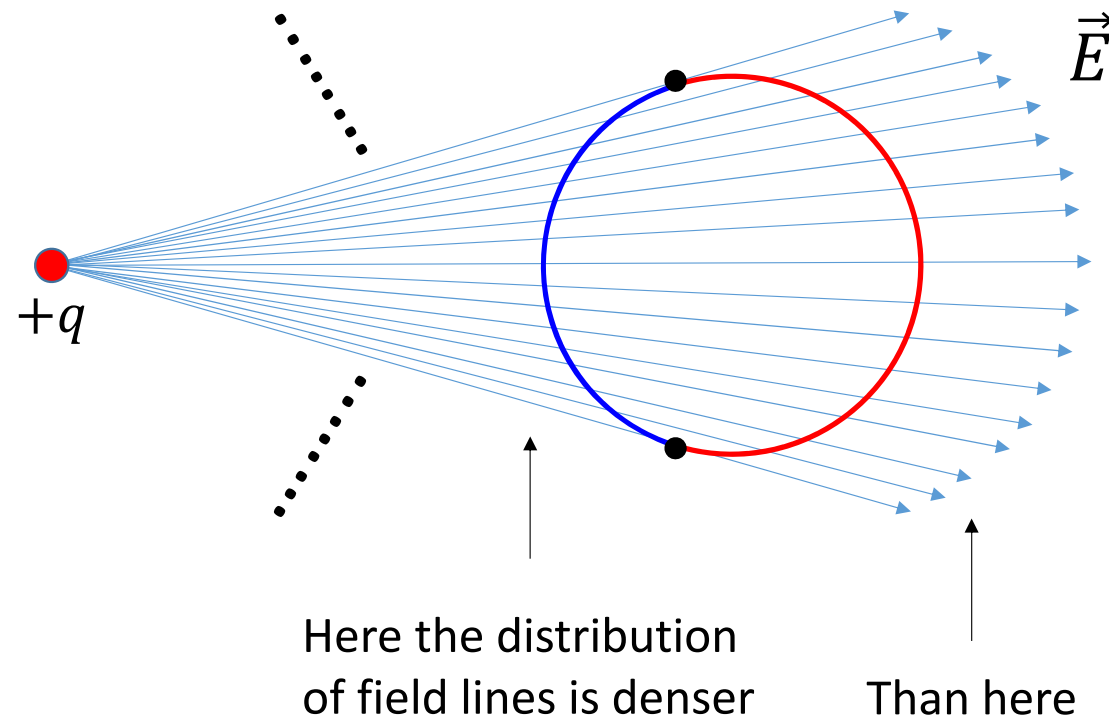
Perfect compensation between **spreading** and **intensity** of the field at any point except around the source **(5)**

$\vec{\nabla} \cdot \vec{A} = 0$  everywhere in space EXCEPT around the origin (source of the field)

$+q$  emits an electric field radially with an intensity given by Coulomb's law

$$E \propto \frac{1}{r^2}$$

Points of tangent

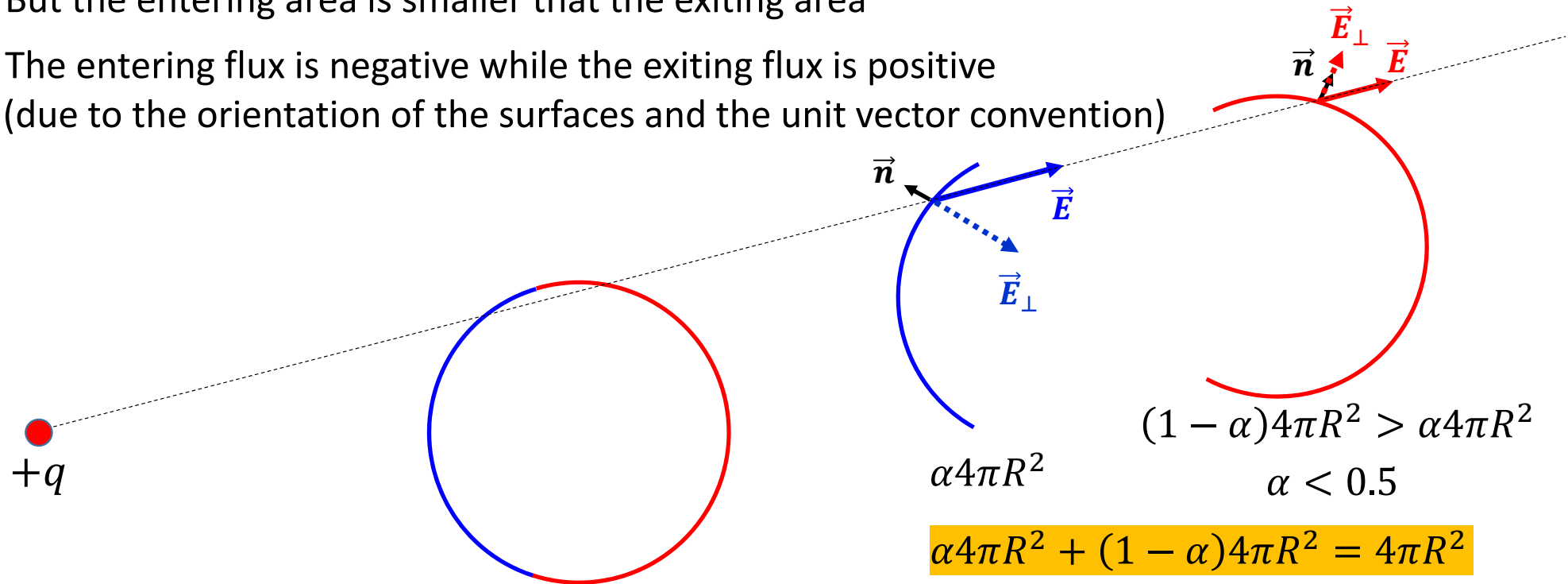


**Coulomb's law  $\Rightarrow$**

$\vec{E}$  entering the sphere from the left (blue portion) is stronger than  $\vec{E}$  exiting from the right (red portion)

**By splitting the entering and exiting parts (spherical caps) we see can several things**

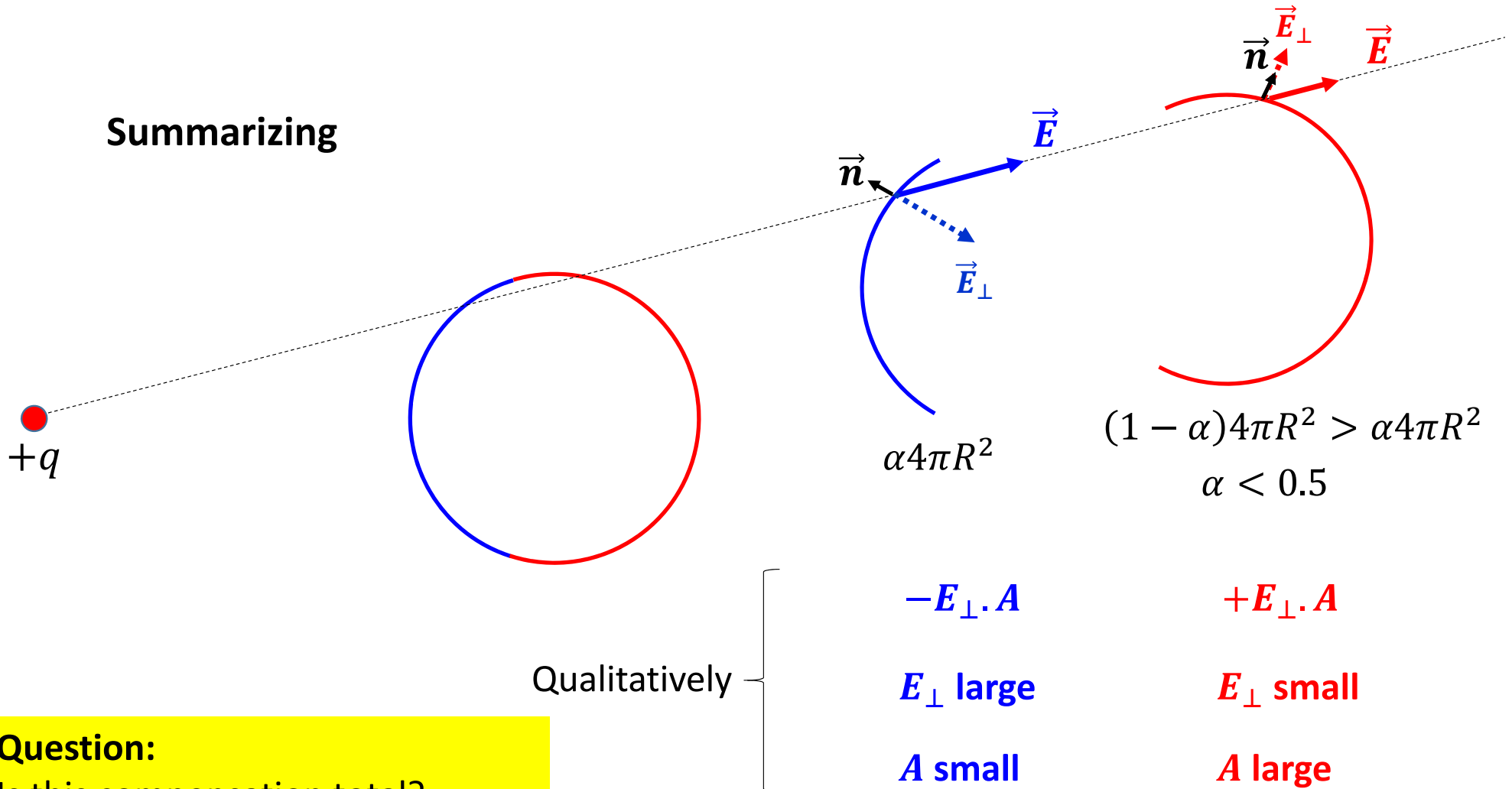
1. The entering field is larger in magnitude than the exiting field according to Coulomb's law
2. But the entering area is smaller than the exiting area
3. The entering flux is negative while the exiting flux is positive (due to the orientation of the surfaces and the unit vector convention)



Therefore we may expect a compensation of the exiting flux by the entering flux

Remember that the flux is given by the **scalar product**  $\vec{E} \cdot \vec{A} = EA \cos(\theta) = E_\perp \cdot A$

## Summarizing



### Question:

Is this compensation total?  
If the answer is yes, then the flux through the closed sphere is zero

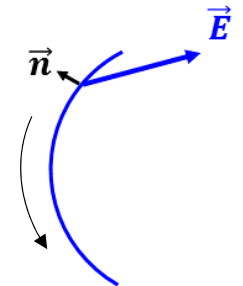
## How to demonstrate that there is a total compensation?

Here the choice of the proper closed surface is crucial. Remember that Gauss's theorem does not specify any requirement regarding the shape of the surface.

We can easily see from previous slides that the choice of a spherical surface is not appropriate at all !

The reason is that to get the **TOTAL** flux from each spherical portion we need to integrate a scalar product

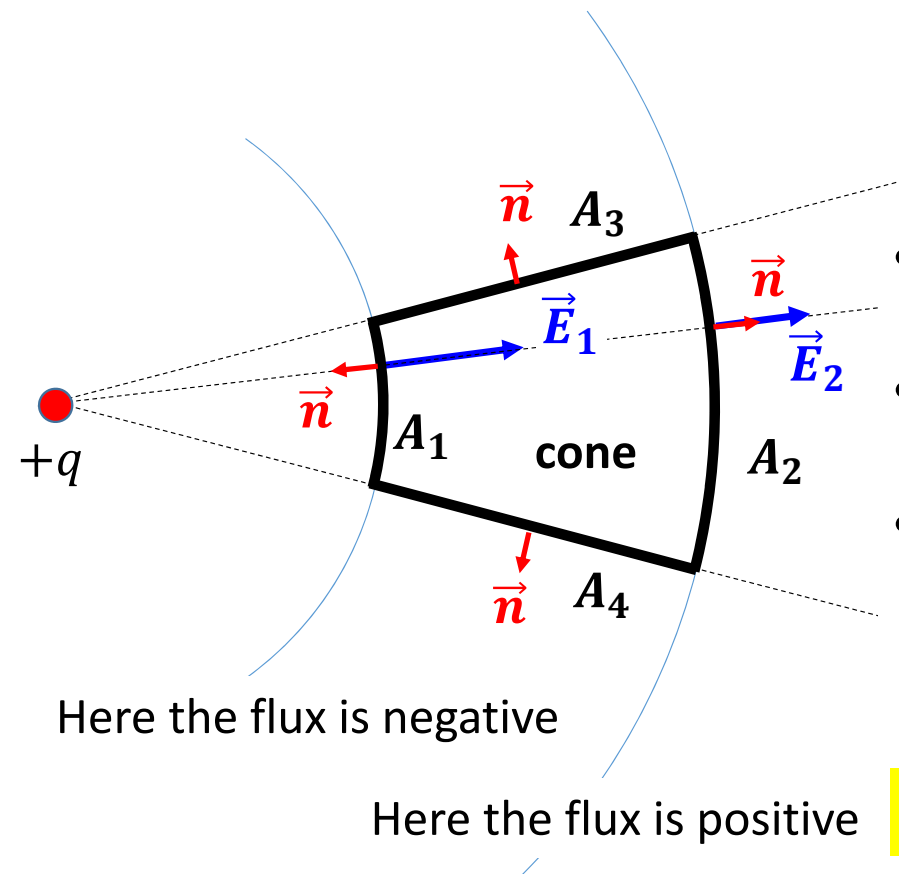
This operation requires to consider a changing angle between  $\vec{E}$  and  $\vec{n}$  while the vector field scans the whole area (blue or red)



Mathematically it is very tedious. Therefore another choice is demanded

As Gauss's theorem allows to take any **closed** surface let us consider two concentric spheres with the center located at the charge itself. These two surfaces cut a cone having its apex at the charge

The closed surface in bold is defined by a cone cut by two concentric spheres  $A_1$  and  $A_2$



### Convention:

For a closed surface the unit vector is directed outwards

- At any point on  $A_1$  and  $A_2$ ,  $\vec{E}$  is parallel to the unit vector
- At each point on  $A_1$  and  $A_2$  the magnitude  $|\vec{E}|$  is constant
- On the two other sides  $A_3$  and  $A_4$  of the cone  $\vec{E}$  is  $\perp$  to the unit vector  $\Rightarrow$  **the flux through these two areas is zero**



This makes life much easier as the scalar product  $\vec{E} d\vec{A} = E dA$

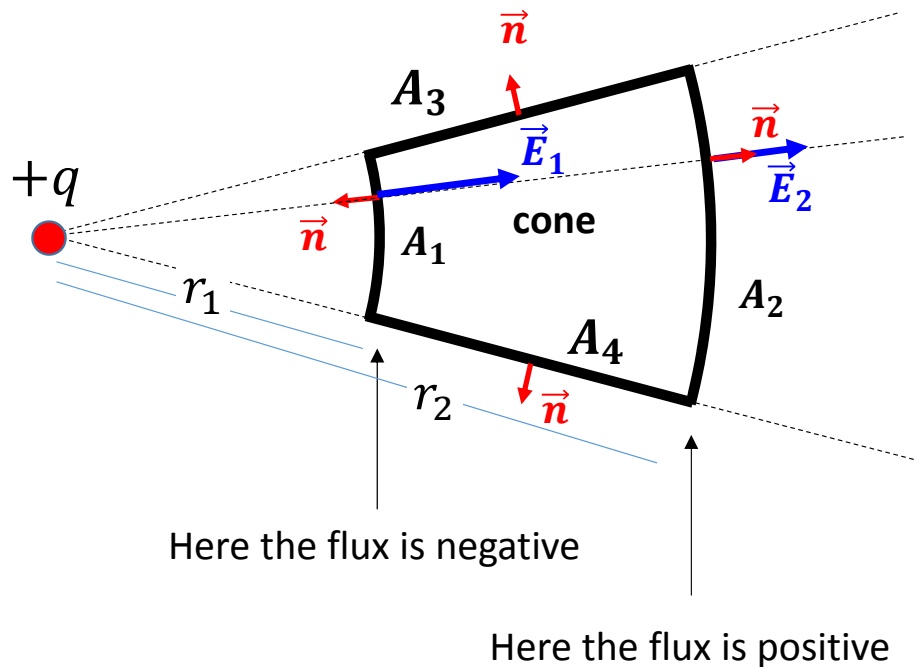


$$Flux = - \int_{A_1} E_1 dA + \int_{A_2} E_2 dA$$

As  $E_1$  and  $E_2$  are constant along their respective areas  $A_1$  and  $A_2$

$$Flux = (-E_1 A_1 + E_2 A_2 + 0 + 0) \propto -\frac{A_1}{r_1^2} + \frac{A_2}{r_2^2}$$

What are  $A_1$  and  $A_2$ ?



From the solid angle of the cone

$$\Omega = \frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$$



$$A_2 = r_2^2 \frac{A_1}{r_1^2}$$

Plugging this in the flux given above, we get

$$Flux \propto -\frac{A_1}{r_1^2} + \frac{A_2}{r_2^2} = -\frac{A_1}{r_1^2} + \frac{A_1}{r_1^2} = 0$$

Gauss's theorem



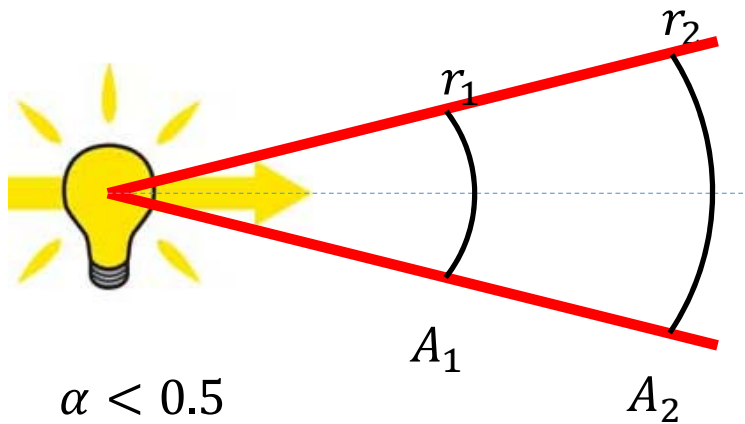
$$\vec{\nabla} \cdot \vec{E} = 0$$

## Conclusion

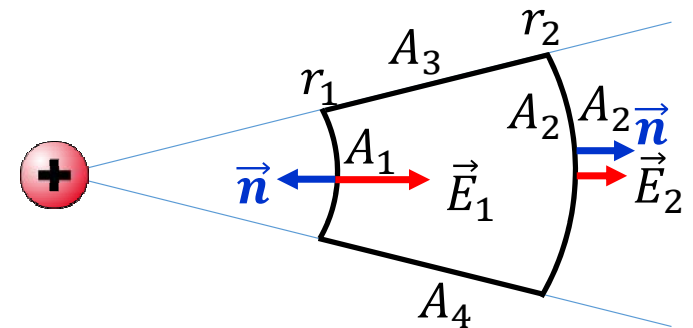
- Gauss's law would not be valid without Coulomb's law ( $E \propto \frac{1}{r^2}$ )
- The flux through a closed surface not containing any charge is zero no matter the shape of that surface
- Coulomb's law would not be valid without Gauss's law

## Consequence

- These two laws are in perfect agreement with the law of energy conservation



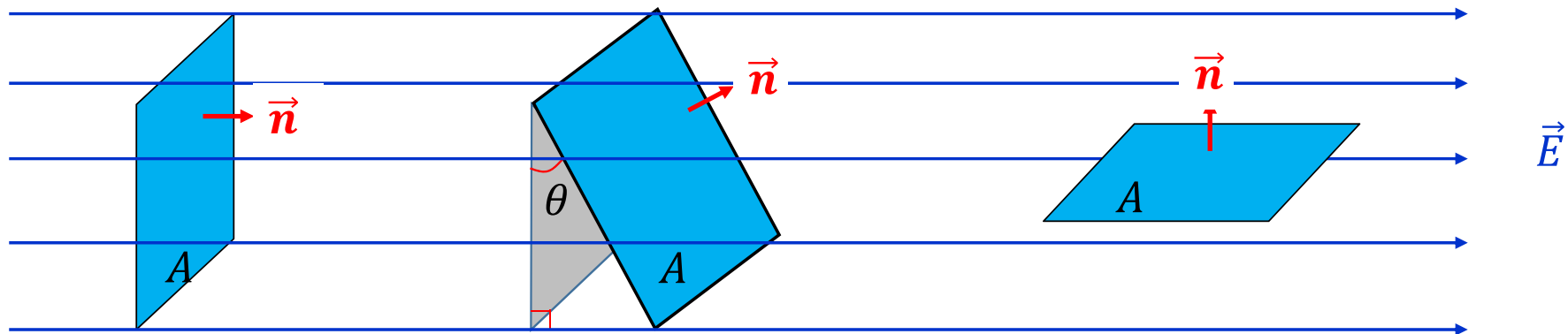
$$\frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$$



Energy passing through  $A_1$  = Energy passing through  $A_2$

Demonstrate that Gauss theorem does not depend on the shape of the Gauss surface

## Tilting the surface with respect to the field changes the flux



$$A = A_{\perp}$$

$$E_{\perp} = E$$

$$A = A_{\perp} / \cos\theta$$

$$E_{\perp} = E \cos\theta$$

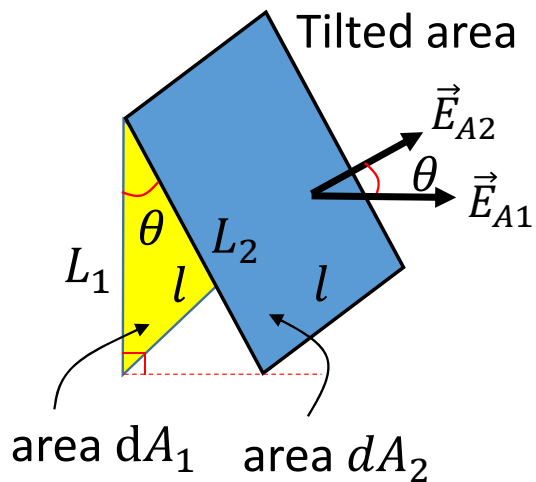
$$A_{\perp} = 0$$

$$E_{\perp} = 0$$

$$\Phi_E = \int_A \vec{E} d\vec{A} \quad \text{Unchanged}$$

Perfect compensation

Tilting the surface does not change anything



$$\left. \begin{aligned} dA_1 &= lL_1 \\ dA_2 &= lL_2 \\ L_1 &= L_2 \cos \theta \end{aligned} \right\} dA_1 = dA_2 \cos \theta$$

$$\vec{E}_{A2} = \vec{E}_{A1} \cos \theta$$

$$\vec{E}_{A1} = \frac{\vec{E}_{A2}}{\cos \theta}$$

$$\vec{E}_{A1} \cdot d\vec{A}_1 = \vec{E}_{A2} \cdot d\vec{A}_2$$



$$\int_{A_1} \vec{E}_{A1} \cdot d\vec{A}_1 = \int_{A_2} \vec{E}_{A2} \cdot d\vec{A}_2$$

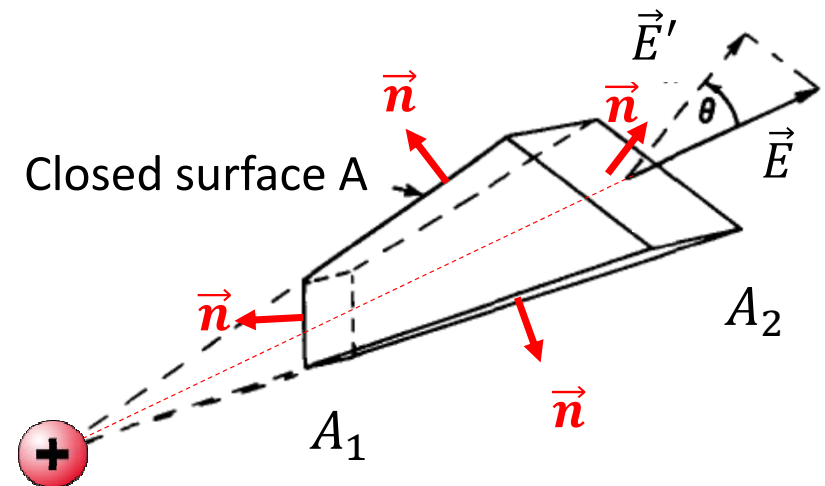
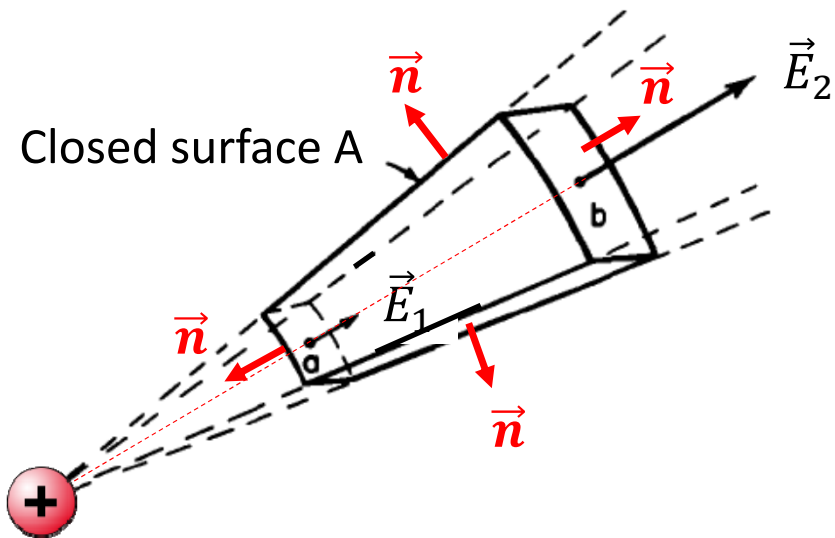
The flux through a closed surface does not depend on the shape of the surface

# The flux of the electric field through closed surface

## The miracle behind the $1/r^2$ dependence

$$\text{Flux } \Phi_E = \oiint \vec{E} d\vec{A}$$

Closed surface



Tilting the surfaces does it bring anything new ?

Demonstrate that the fluxes through these two surfaces are the same

## First part of the first Maxwell's equation

Applying Gradient

$$\vec{E} = -\vec{\nabla}V$$

Applying Divergence

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla}V)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = -\nabla^2 V$$

Towards Poisson and Laplace equations

**Not complete**

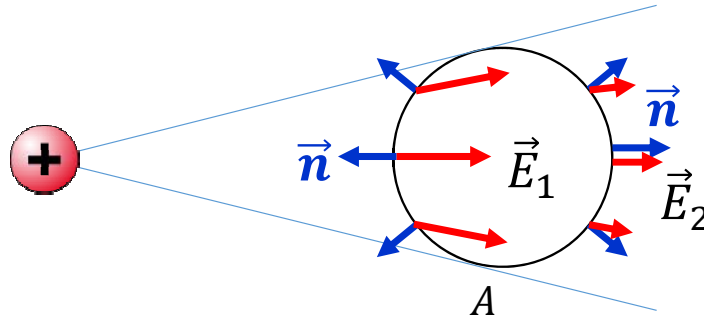
Valid in Electrostatic only !

## Second Maxwell's equation

Applying Curl

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

## Charge outside

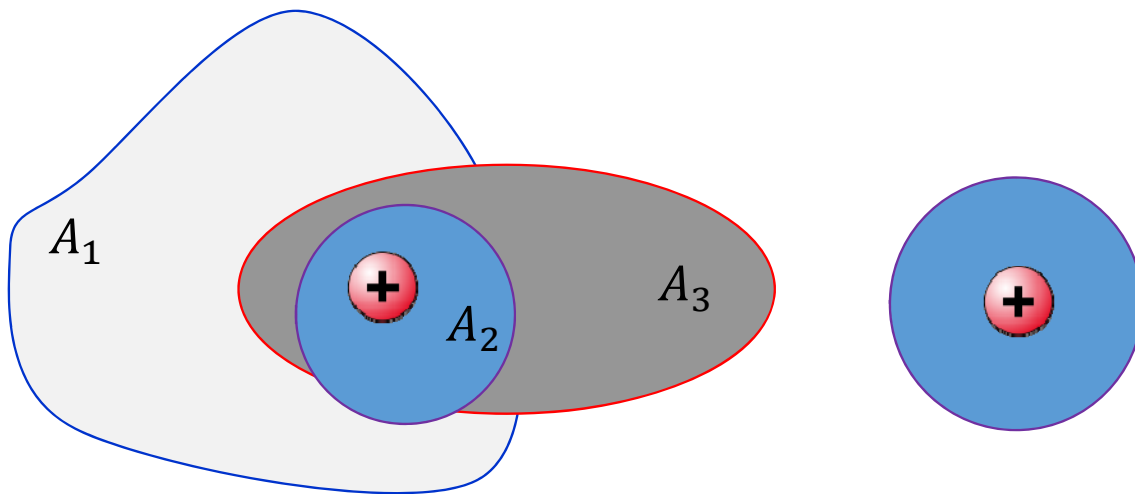


$$\text{Flux } \Phi = \oiint \vec{E} d\vec{A} = 0$$

Closed surface

## Charge inside

The flux of the field produced by the charge  $+q$  through the 3 surfaces is the same



$$\text{Flux } \Phi = \oiint \vec{E} d\vec{A} = \left( \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \right) (4\pi r^2) = \frac{q}{\epsilon_0}$$

Closed surface

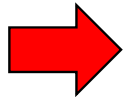


## Gauss theorem

$$\text{Flux } \Phi = \oiint \vec{E} \cdot d\vec{A} = \frac{\sum_i q_i}{\epsilon_0}$$

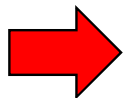
Enclosed in the volume

$$\sum_i q_i = Q = \int_{\text{Volume}} \rho dV$$



$$\oiint_{\text{surface}} \vec{E} d\vec{A} = \int_{\text{Volume}} \frac{\rho}{\epsilon_0} dV$$

## Divergence theorem



$$\oiint_{\text{Surface}} \vec{E} d\vec{A} = \int_{\text{Volume}} \vec{\nabla} \cdot \vec{E} dV$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

**Gauss Law**

## Electrostatic field

If  $\frac{\partial}{\partial t} = 0$  no time dependence

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{E} = -\vec{\nabla}V \Leftrightarrow \text{Electrostatic field is conservative}$$

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot (\vec{\nabla}V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

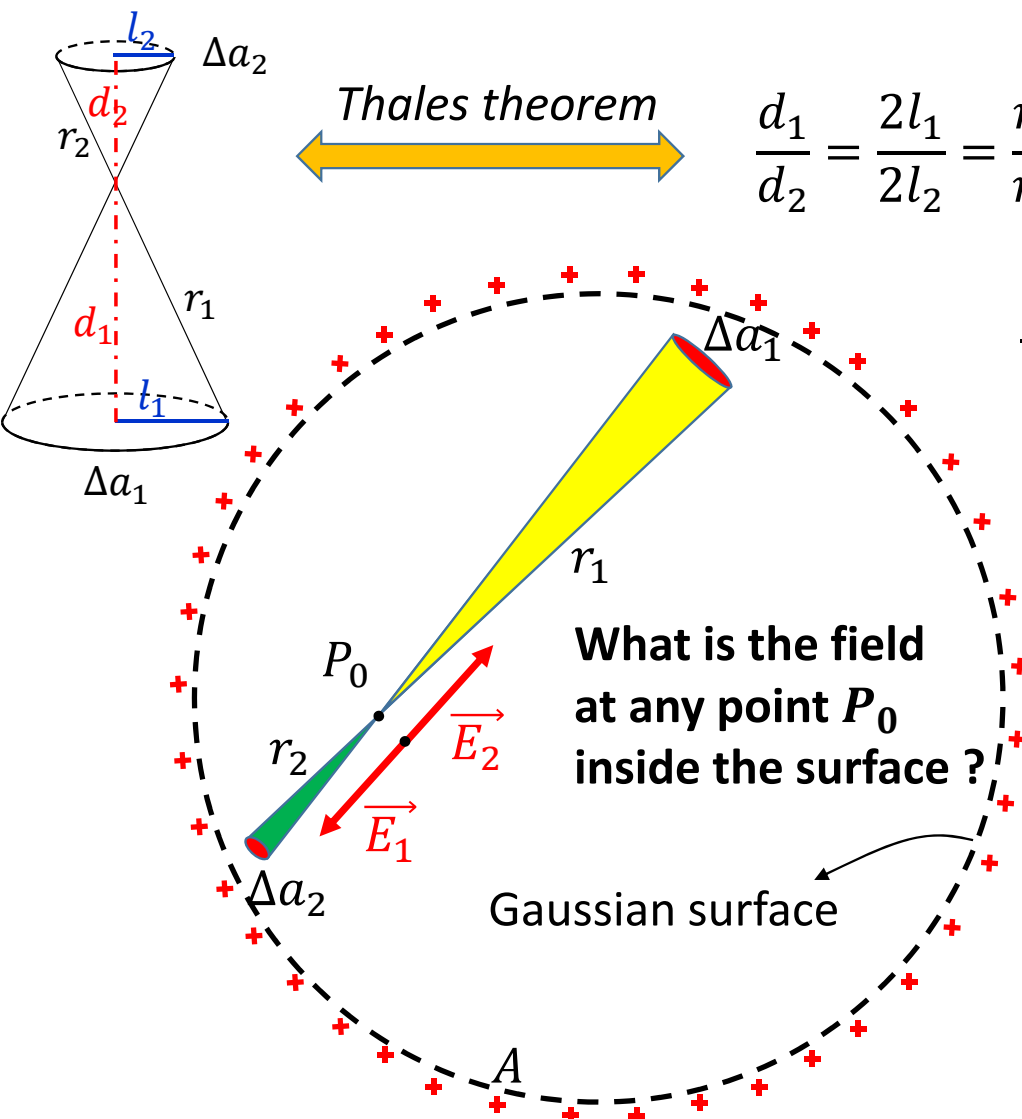
$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Poisson equation

$$\vec{\nabla} \times \vec{E} = 0$$

$$V = -\int_a^b \vec{E} \cdot d\vec{l}$$

Is the field of a point charge exactly  $1/r^2$ ?



Uniformly charged sphere

Thales theorem  $\longleftrightarrow \frac{d_1}{d_2} = \frac{2l_1}{2l_2} = \frac{r_1}{r_2} \rightarrow \frac{\pi l_1^2}{\pi l_2^2} = \frac{r_1^2}{r_2^2} \rightarrow \frac{\Delta a_1}{\Delta a_2} = \frac{r_1^2}{r_2^2}$

If the charge on the sphere is uniform  $\frac{\Delta a_1}{\Delta a_2} = \frac{\Delta q_1}{\Delta q_2}$

$\frac{\Delta q_1}{\Delta q_2} = \frac{r_1^2}{r_2^2} \rightarrow \frac{\Delta q_1}{r_1^2} = \frac{\Delta q_2}{r_2^2} \rightarrow E_1 = E_2$

Electric Field inside the sphere = 0  $E \propto 1/r^2$

Gauss law =  $\oint_A \vec{E} \cdot d\vec{A} = 0 \rightarrow \vec{E} = \vec{0}$

Coulomb law derives from Gauss law

## Problem at midterm II fall 2017

A charged balloon expands as it is blown up, increasing in size from the initial to final diameter as shown. Do the **electric fields** at point 1, 2 and 3 increase, decrease, or stay the same? Explain

- Draw 3 Gaussian surfaces

- Symmetry argument is crucial here otherwise Gauss law would be of no help

**Point 1:** Remains always inside the charged balloon.  
 $\Phi = E_1 \cdot \mathbf{A}_1 = 0$  as no charge inside Gaussian surface 1

$$E_1 = 0$$

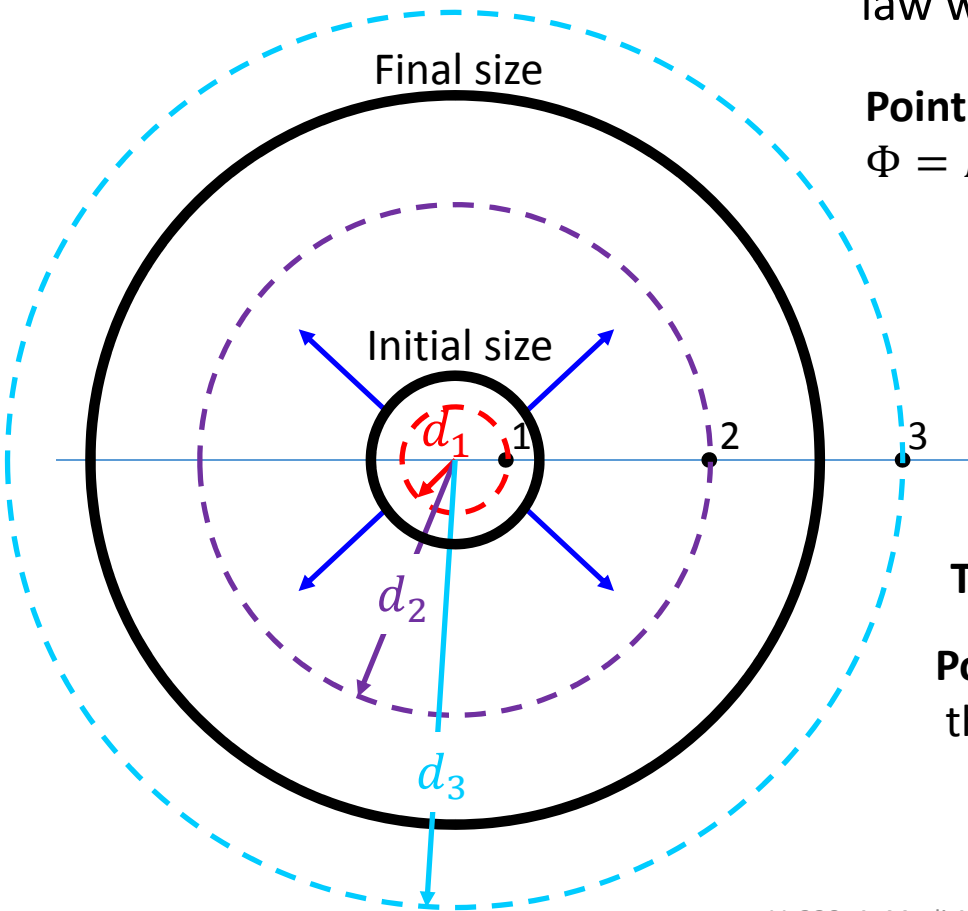
**Point 2:** As long as the charged balloon remains inside Gaussian surface 2 and because of symmetry argument  
 $\Phi = E_2 \cdot \mathbf{A}_2 = Q/\epsilon_0 = Cte$

$$E_2 = \frac{Q}{4\pi d_2^2 \epsilon_0} = Cte$$

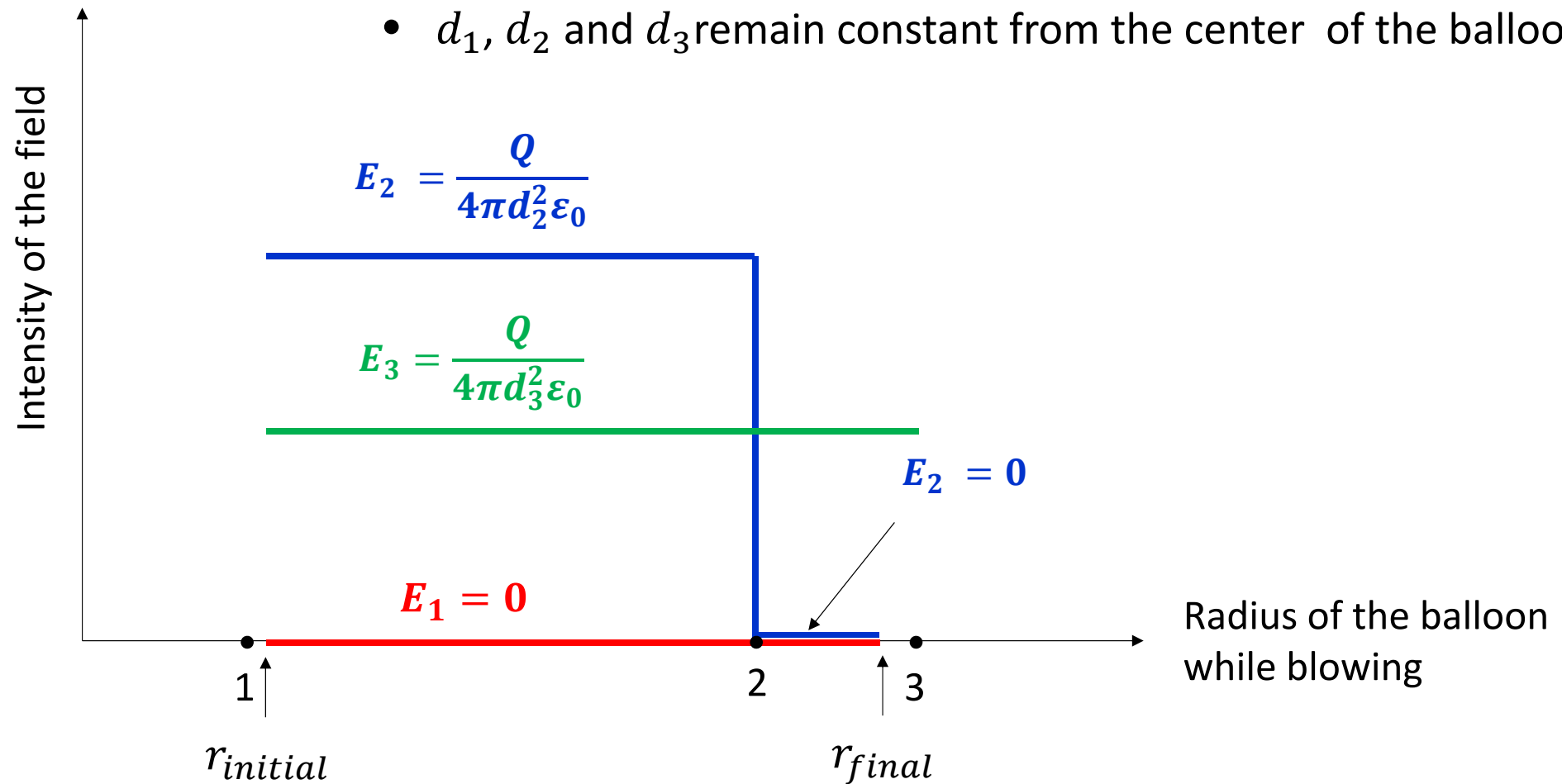
**Then it drops to 0 when the balloon reaches point 2 and beyond**

**Point 3:** The charged balloon remains inside Gaussian surface 3 all the time.  $\Phi = E_3 \cdot \mathbf{A}_3 = Q/\epsilon_0 = Cte$

$$E_3 = \frac{Q}{4\pi d_3^2 \epsilon_0} = Cte$$



- The charge  $Q$  on the balloon does not change during the expansion
- $d_1$ ,  $d_2$  and  $d_3$  remain constant from the center of the balloon



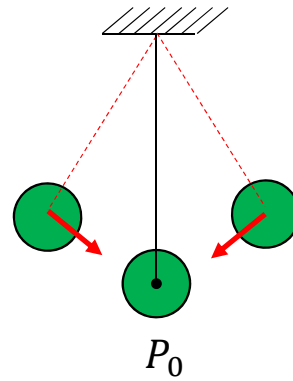
## Consequence of Gauss's law

A static charge **CANNOT** be in equilibrium  
in the electric field of other **STATIC** charges?

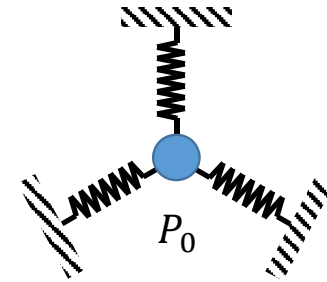


# Mechanical equilibrium

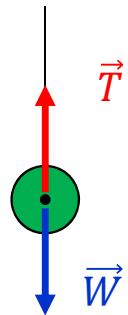
pendulum



Stable equilibrium  
position



Mass – spring system



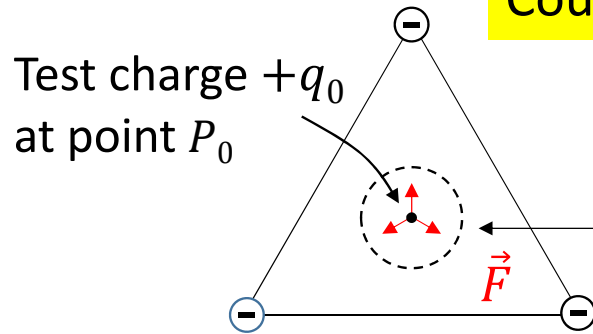
$$\sum \vec{F}_j = \vec{0}$$

Two conditions must be fulfilled



Restoring forces  
directed towards  $P_0$

Could a positive charge test  $+q_0$  placed at  $P_0$  be in equilibrium ?

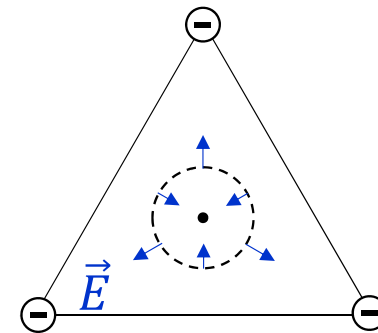
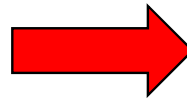
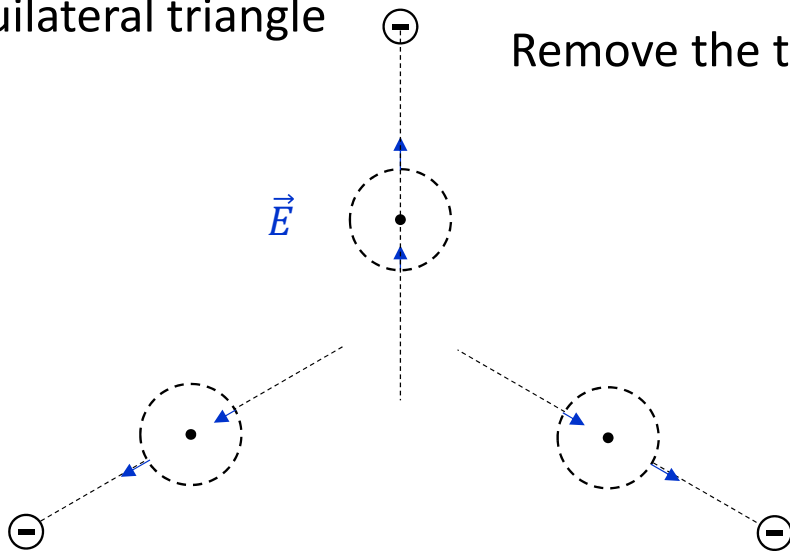


First condition fulfilled

$$\sum_i \vec{F}_i(P_0) = \vec{0}$$

Equilateral triangle

Remove the test charge from point  $P_0$



$\Rightarrow$  Flux = 0 through area



Principle of superposition

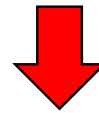
$$\vec{\nabla} \cdot \vec{E} = 0$$

Could a positive charge test  $+q_0$  placed at  $P_0$  be in equilibrium ?

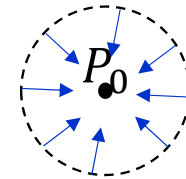
Second condition:

*The restoring forces should bring the charge back to  $P_0$  if slightly displaced*

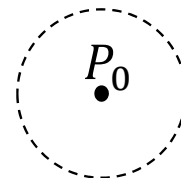
**NOT** fulfilled



The electric flux must be negative around  $P_0$ ?



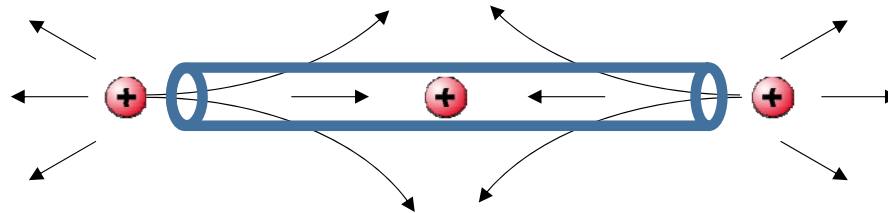
BUT this is **impossible** ! as there are no charges in the area



$$\vec{\nabla} \cdot \vec{E} = 0$$

**Static** atom cannot be in stable equilibrium. It cannot exist

Devising a system in which stable equilibrium is possible although the flux of the field is zero?



This charge is in stable mechanical equilibrium although  $\vec{\nabla} \cdot \vec{E} = 0$

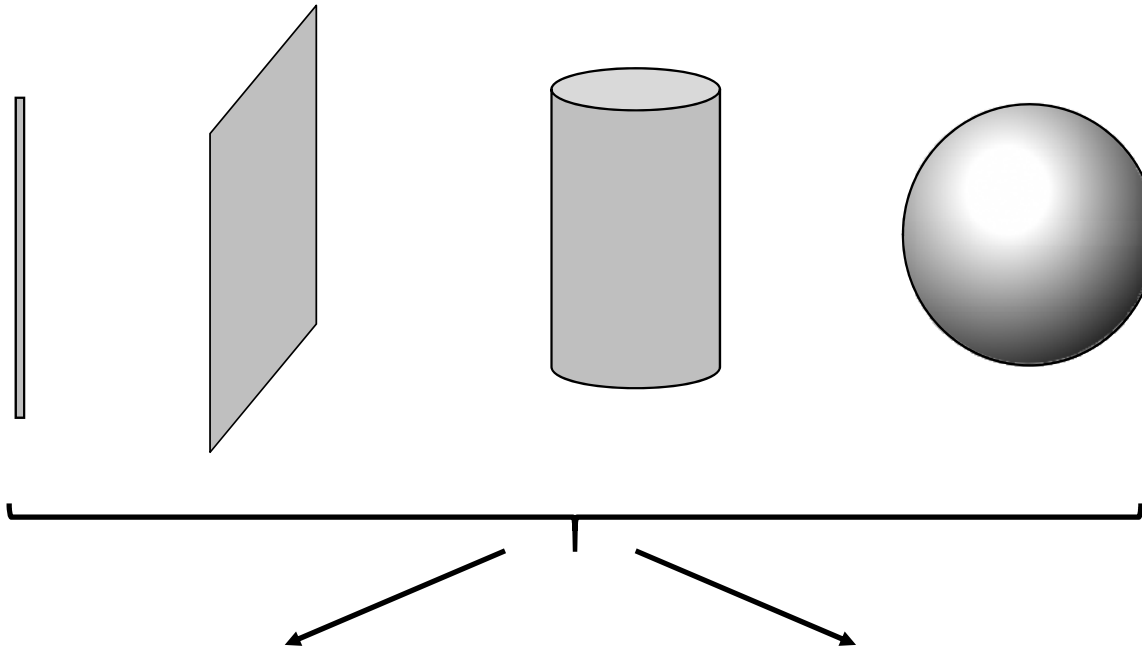
Why?

*Because a mechanical constraint has been added*

## What is the Gauss's law made for?

**Coulomb's law** is the fundamental law for calculating electric field in any configuration...

*In principle...*



Complicated with Coulomb's law

Due to symmetry  $\Rightarrow$  straightforward with Gauss's law

**But is sometimes useless**

# Electrostatics

- Coulomb law: superposition principle
  - Is superposition principle an easy concept when dealing with vectors?
- Concept of work – Potential Energy
- Electrostatic potential and implication:  $\vec{E} = -\vec{\nabla}V$
- Electric field and potentials for various distributions of charges

## Case of point charge

$$\vec{e}_{qP} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

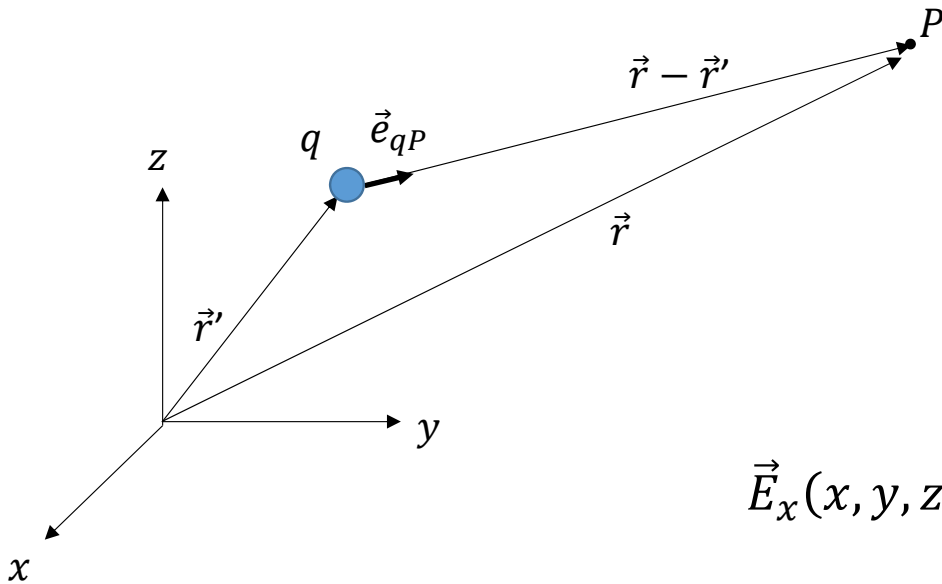
$$\vec{r} - \vec{r}' = (x - x')\vec{i} + (y - y')\vec{j} + (z - z')\vec{k}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}'|^2} \underbrace{\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}}_{\vec{e}_{qP}} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$\vec{E}(\vec{r}) = \vec{E}_x(x, y, z) + \vec{E}_y(x, y, z) + \vec{E}_z(x, y, z)$$

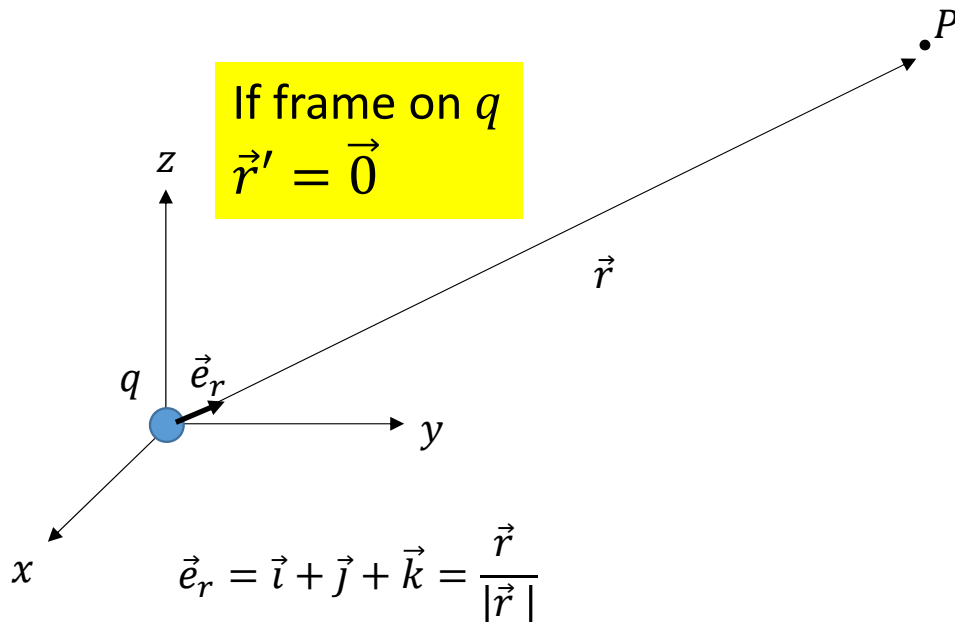
$$\vec{E}_x(x, y, z) = \frac{q}{4\pi\epsilon_0} \frac{x - x'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \vec{i}$$

$$\vec{E}_x(x, y, z) = \text{function of 3 variables}$$





For a single point charge simplification is possible



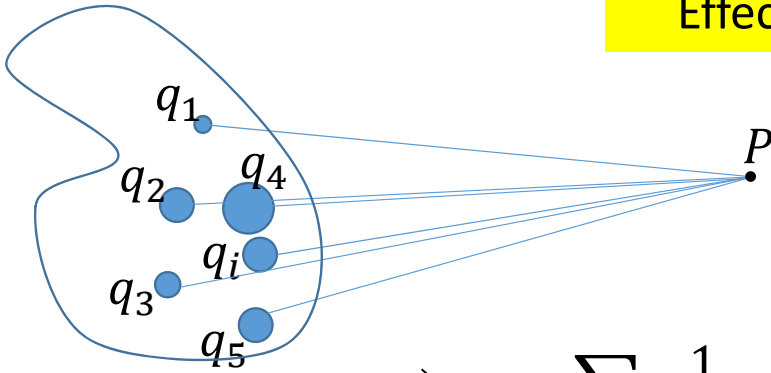
$$\vec{E}_x(x, y, z) = \frac{q}{4\pi\epsilon_0} \frac{x}{[x^2 + y^2 + z^2]^{3/2}} \vec{i}$$

The same for the other components

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \vec{e}_r$$

## Principle of superposition

### Effect of charge distribution



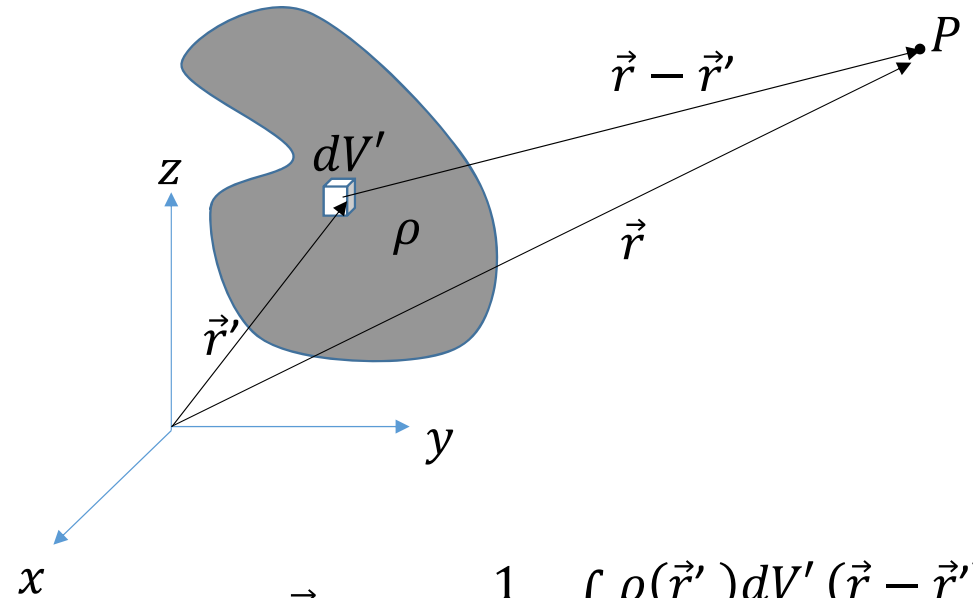
$$\vec{E}(P) = \sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_{iP}^2} \vec{e}_{iP}$$

$\Sigma$  to  $\int$

**Provided we look far away from the distribution**

*Remember the dipole: A test charge close to it may feel one or the other charge, but far away it feels nothing*

$$\sum q_i = 0$$



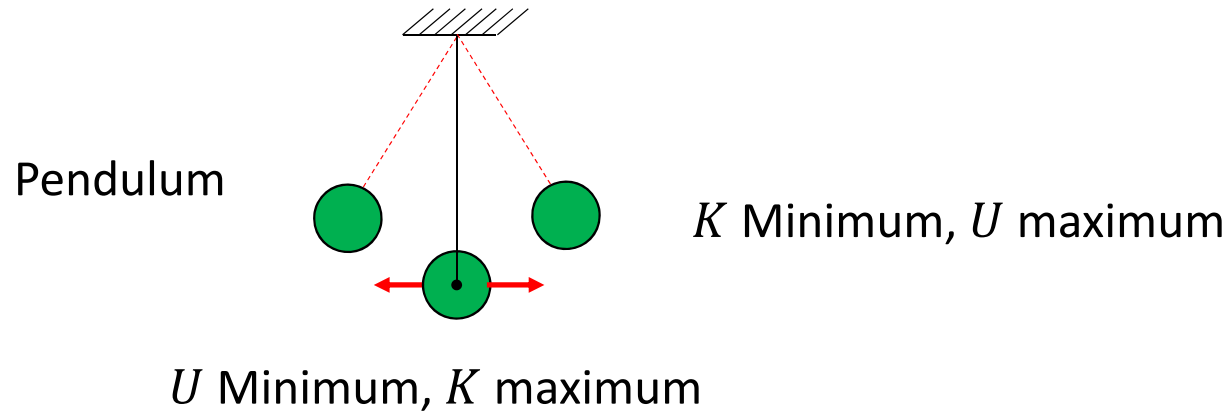
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') dV'}{|\vec{r} - \vec{r}'|^2} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|}$$

**Difficult to carry integration on vectors**

## Concept of work – Electrostatic Potential Energy

What does a conservative force mean and imply?

Conservative force keeps **REVERSIBLE** the exchange between Kinetics and Potential energies



**Conservative force conserves the total energy**

$$K_1 + U_1 = K_2 + U_2 \quad \Rightarrow \quad \Delta K_{1 \rightarrow 2} = -\Delta U_{1 \rightarrow 2} \text{ or } K_2 - K_1 = -(U_2 - U_1)$$

Work – energy theorem For **Conservative Force (CF)**  $\Rightarrow$   **$W(CF)_{1 \rightarrow 2} = \Delta K_{1 \rightarrow 2}$**

$$\Rightarrow \quad \mathbf{W(CF)_{1 \rightarrow 2} = -\Delta U_{1 \rightarrow 2}}$$

In the absence of friction, there are 3 fundamental conservative forces

- Gravitation
- Elastic
- Electric

If other external forces are involved  $\Rightarrow$  work – energy theorem says

$$\Rightarrow W(CF)_{1 \rightarrow 2} + W(ext)_{1 \rightarrow 2} = \Delta K_{1 \rightarrow 2}$$

When the work is done slowly to keep  $\Delta K_{1 \rightarrow 2} = 0$

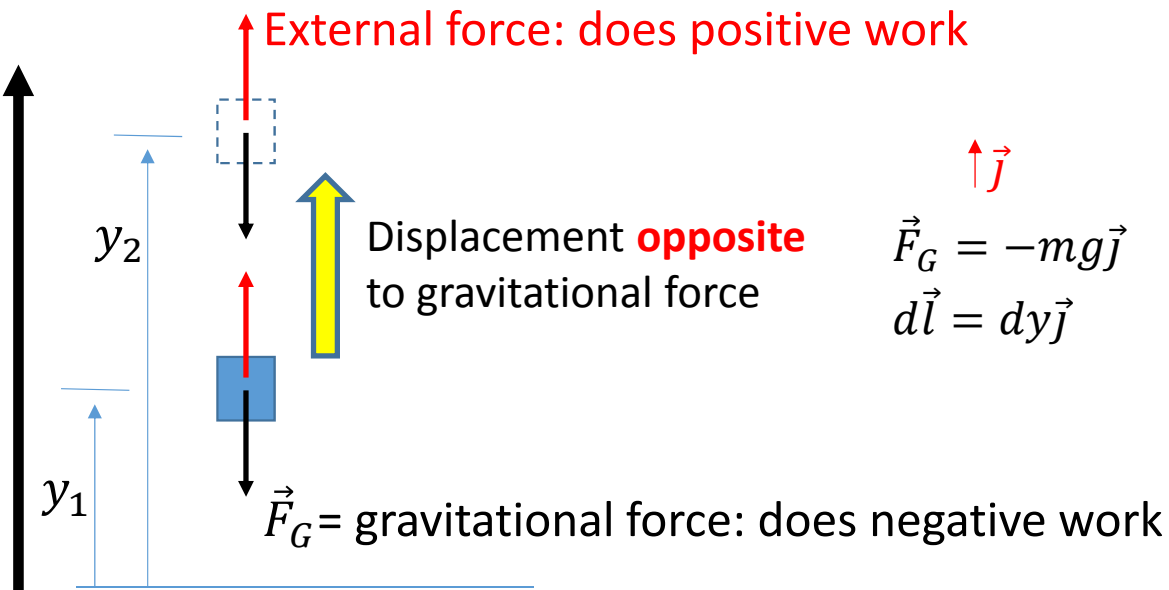
$$\Rightarrow W(ext)_{1 \rightarrow 2} = -W(CF)_{1 \rightarrow 2}$$

External forces do work **AGAINST** conservative forces

$$W(ext)_{1 \rightarrow 2} = -W(CF)_{1 \rightarrow 2} = -(U_2 - U_1)$$

Lifting slowly a body

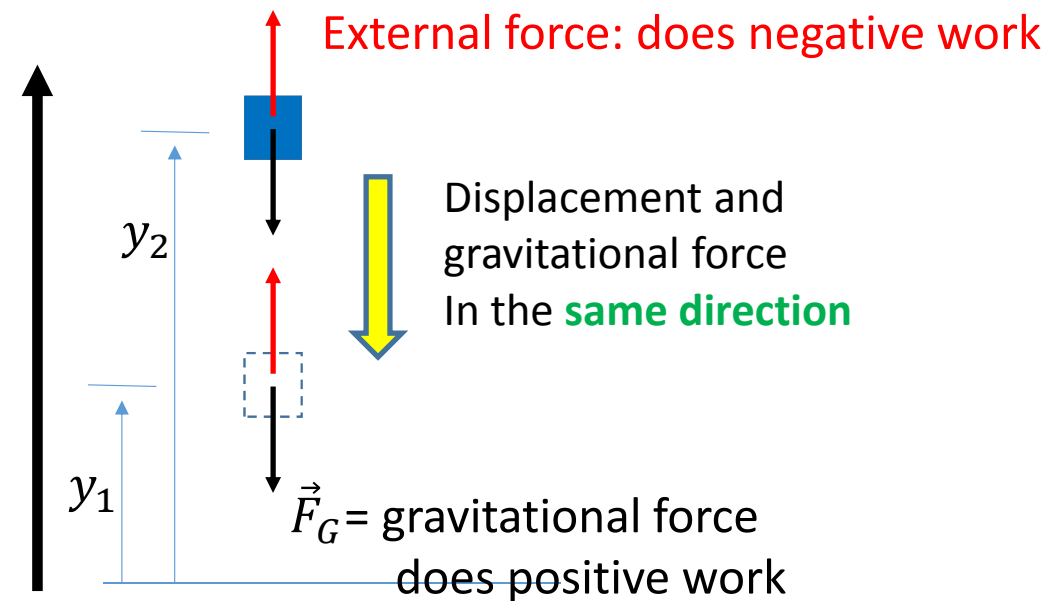
Energy is injected into the system



$$W(CF)_{1 \rightarrow 2} = \int_{y_1}^{y_2} \vec{F}_G d\vec{l} = -mg(y_2 - y_1) < 0$$

Lowering slowly a body

Energy is extracted from the system



$$W(CF)_{2 \rightarrow 1} = \int_{y_2}^{y_1} \vec{F}_G d\vec{l} = -mg(y_1 - y_2) > 0$$

**Conservative mechanical force always acts naturally to push the system towards **lower potential energy****

In case of one dimension  $W_{CF} = -\Delta U \Rightarrow F_{CF} \cdot dx = -dU \Rightarrow \vec{F}_{CF} = -\frac{dU}{dx} \vec{i}$

In case of three dimensions

$$\Rightarrow \vec{F}_{CF} = -\vec{\nabla} U$$

Gradient directed towards higher potential energy

Nature acts to lower the potential energy

In the case of electrostatic  $\Rightarrow \vec{F}_{el} = -\vec{\nabla} U \Rightarrow \textcolor{red}{q} \vec{E}_{el} = -\vec{\nabla} \textcolor{red}{q} V \Rightarrow \vec{E}_{el} = -\vec{\nabla} V$

## Concept of work and Electric potential

In the presence of an electrostatic field  $\vec{E}$ , a test charge  $q_0$  feels a conservative force  $\vec{F}_{el}$

work – energy theorem including external force  $\Rightarrow W(\vec{F}_{el})_{1 \rightarrow 2} + W(\vec{F}_{ext})_{1 \rightarrow 2} = \Delta K_{1 \rightarrow 2}$

$\Rightarrow W(\vec{F}_{el})_{1 \rightarrow 2} = -\Delta U_{1 \rightarrow 2} = -(U_2 - U_1)$

To keep the test charge moving very slowly from 1 to 2

$$\Delta K_{1 \rightarrow 2} = 0 \quad \Rightarrow \quad W(\vec{F}_{ext})_{1 \rightarrow 2} = -W(\vec{F}_{el})$$



Work must be done against the electric force

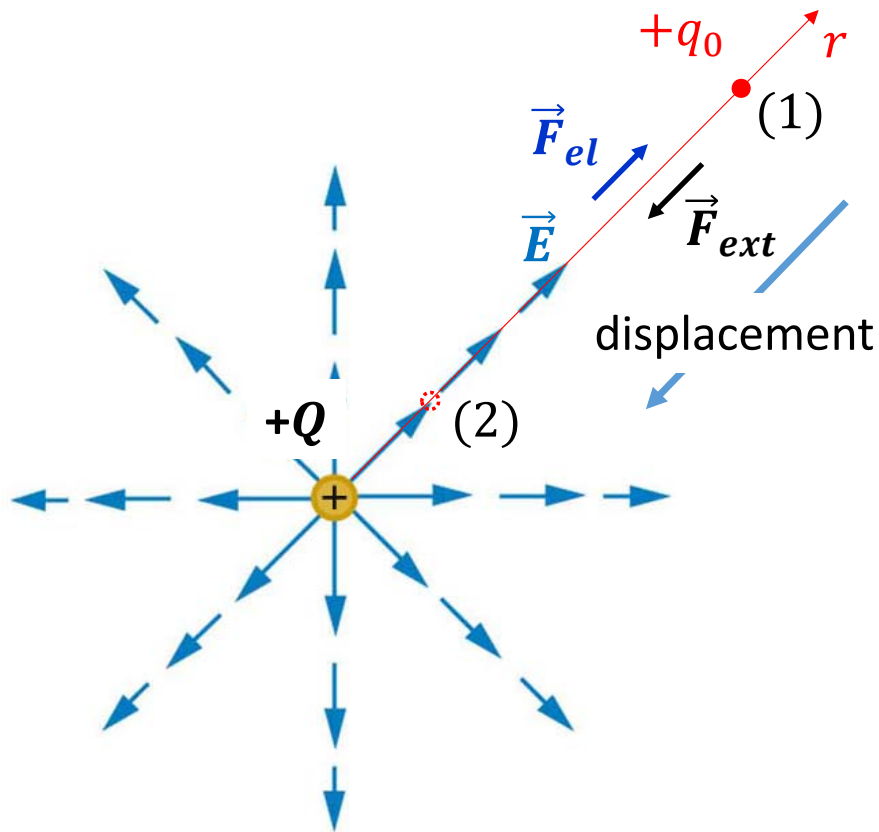
$$W(\vec{F}_{ext})_{1 \rightarrow 2} = - \int_1^2 \vec{F}_{el} \cdot d\vec{r}$$



$$\Rightarrow W(\vec{F}_{el})_{1 \rightarrow 2} = -\Delta U_{1 \rightarrow 2} = -(U_2 - U_1)$$

As external force acts against conservative force

$$\Rightarrow W(\vec{F}_{ext})_{1 \rightarrow 2} = +\Delta U_{1 \rightarrow 2} = +(U_2 - U_1)$$

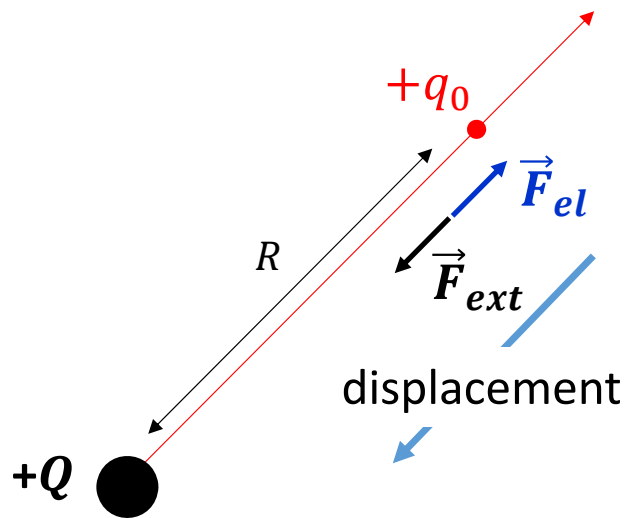


$+q_0$  is the charge test

Work done along a radial path

$\vec{F}_{ext}$  brings the test charge from: (1) to (2)

(1) Could be  $\infty$



$W(\vec{F}_{ext}) > 0$  because energy is **injected** into the system: **Why?**

If charge  $q_0$  is released, it will be repelled  
(Energy is released by the system)

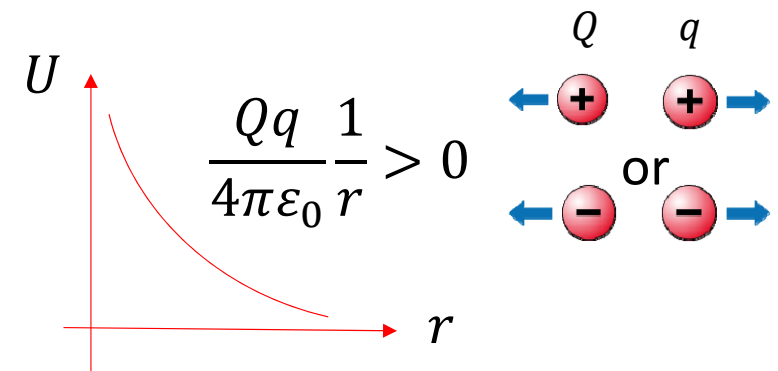
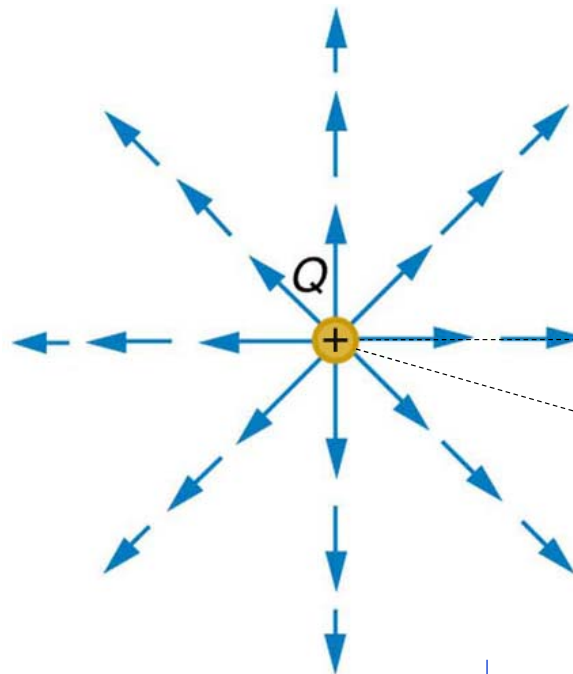
$$W(\vec{F}_{ext})_{\infty \rightarrow R} = \int_{\infty}^R \vec{F}_{ext} \cdot d\vec{r} = - \int_{\infty}^R \vec{F}_{el} \cdot d\vec{r}$$

$$W(\vec{F}_{ext})_{\infty \rightarrow R} = \int_R^{\infty} \vec{F}_{el} \cdot d\vec{r} = \frac{Qq_0}{4\pi\epsilon_0} \int_R^{\infty} \frac{1}{r^2} dr$$

$$W(\vec{F}_{ext})_{\infty \rightarrow R} = \frac{Qq_0}{4\pi\epsilon_0} \frac{1}{R}$$

$$= U \text{ Potential energy} \left\{ \begin{array}{ll} Q \text{ and } q_0 > 0 \text{ Or } Q \text{ and } q_0 < 0 & U > 0 \\ Q > 0 \text{ and } q_0 < 0 \text{ Or vice versa} & U < 0 \end{array} \right.$$

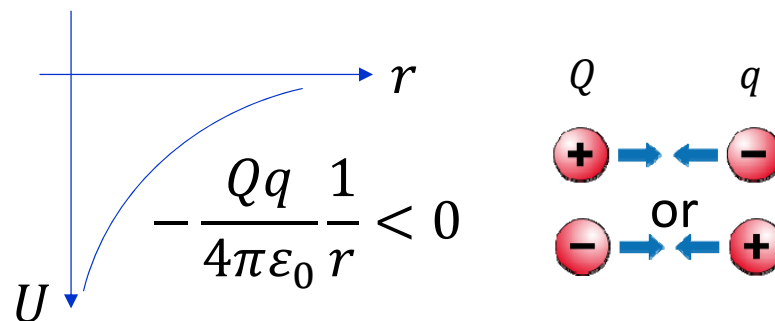
# Potential energy



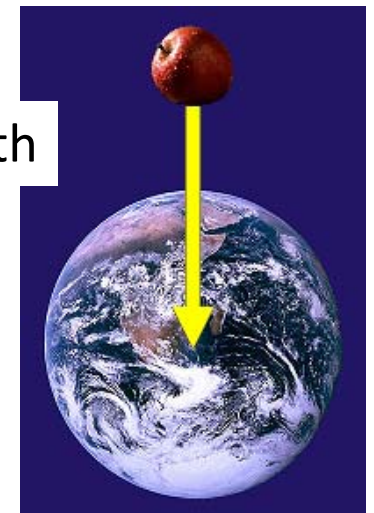
$+q_0$   
Natural path

$-q_0$   
Natural path

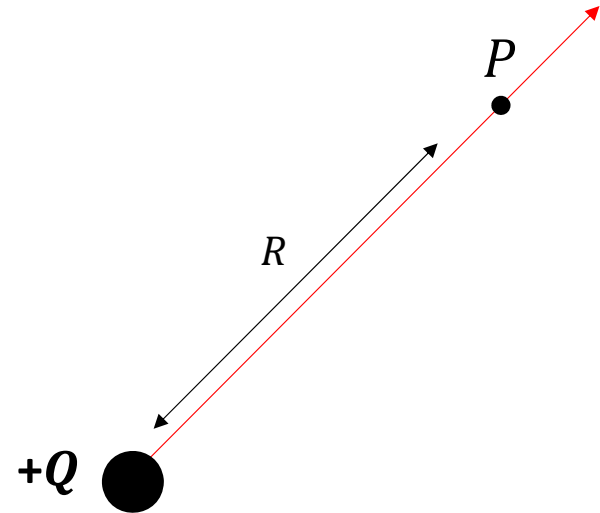
Towards decreasing  
potential energy



Natural path



## Concept of Electric potential



Concept of **Electric field**  $\Rightarrow$  what is the effect of  $Q$  on  $P$  when there is no test charge  $q_0$  ?  $\Rightarrow$  There is  $\vec{E}(P) \Rightarrow \vec{F} = q_0 \vec{E} = m \vec{a} \Rightarrow$  Energy

Likewise we define an **Electric Potential** of doing work on a test charge  $q_0$  once placed at  $P$

Work done on a unit charge  $q_0$ ,  $U = q_0 V$

$$W(\vec{F}_{\text{ext}})_{1 \rightarrow 2} \Big|_{q_0} = \frac{W}{q_0} (\vec{F}_{\text{ext}})_{1 \rightarrow 2} = (V_2 - V_1)$$

$$W(\vec{F}_{\text{el}})_{1 \rightarrow 2} \Big|_{q_0} = \frac{W}{q_0} (\vec{F}_{\text{el}})_{1 \rightarrow 2} = -(V_2 - V_1)$$

$$V(R) = \frac{W}{q_0} = \frac{Q}{4\pi\epsilon_0} \frac{1}{R}$$

$$V(\infty) = 0$$

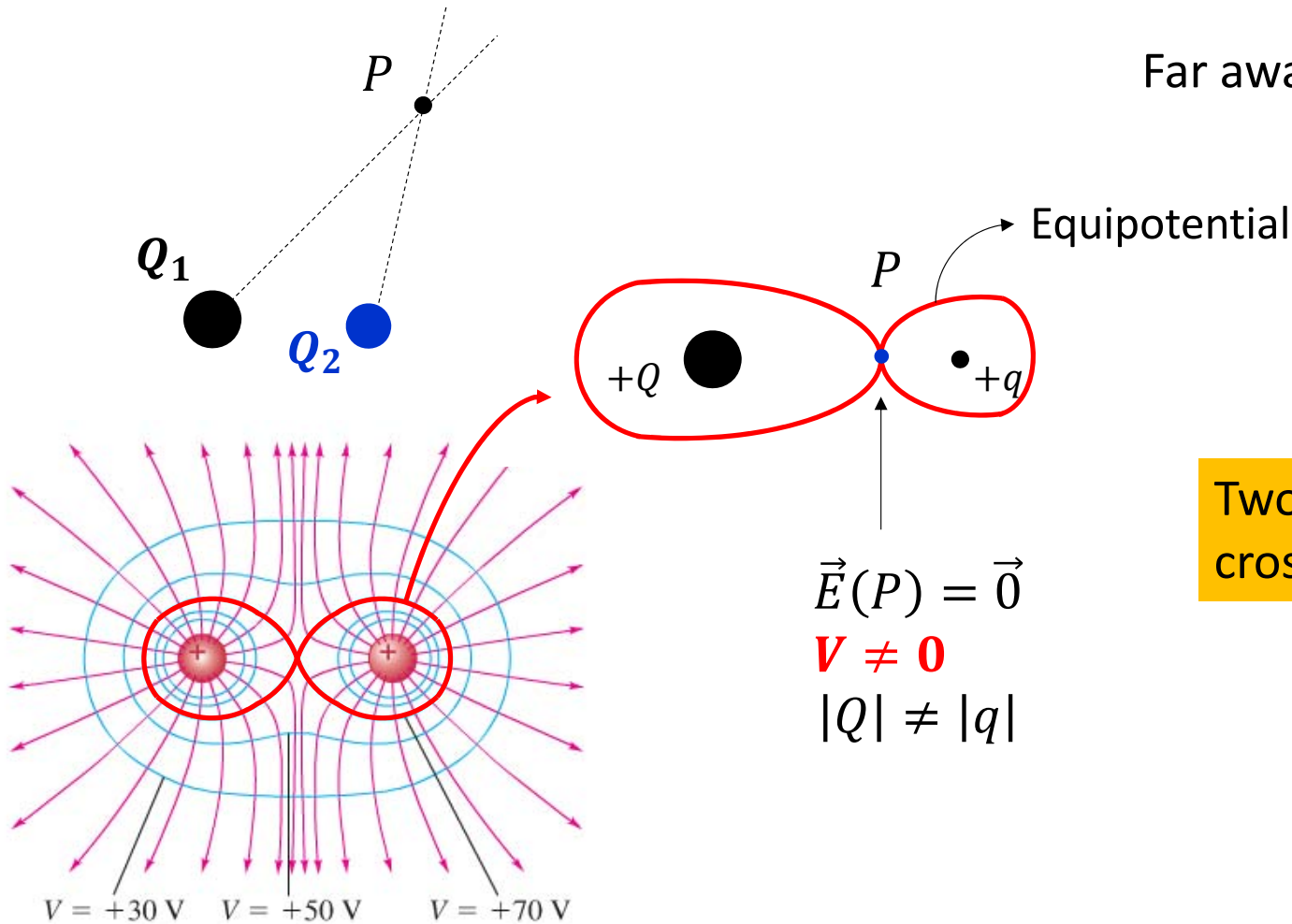
## Superposition principle

$$V(P) = V_{Q_1} + V_{Q_2}$$

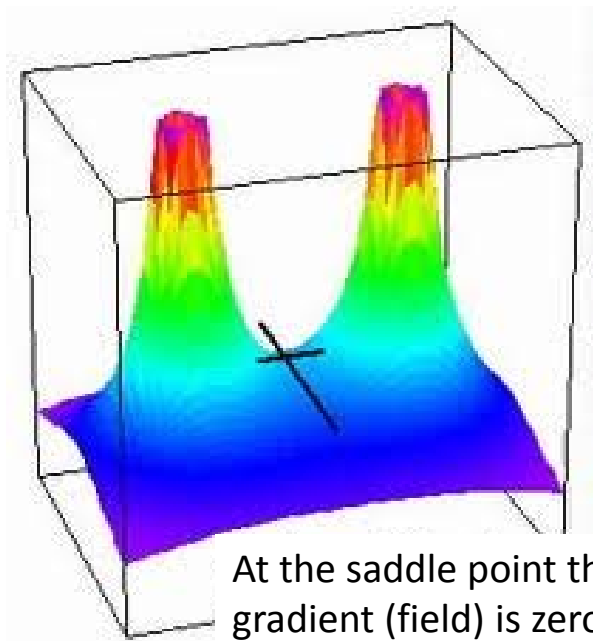
If charge at  $P$  is closer to  $Q_1$  it will feel  $V_{Q_1}$

If charge at  $P$  is closer to  $Q_2$  it will feel  $V_{Q_2}$

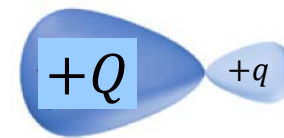
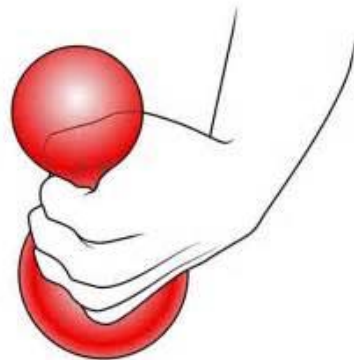
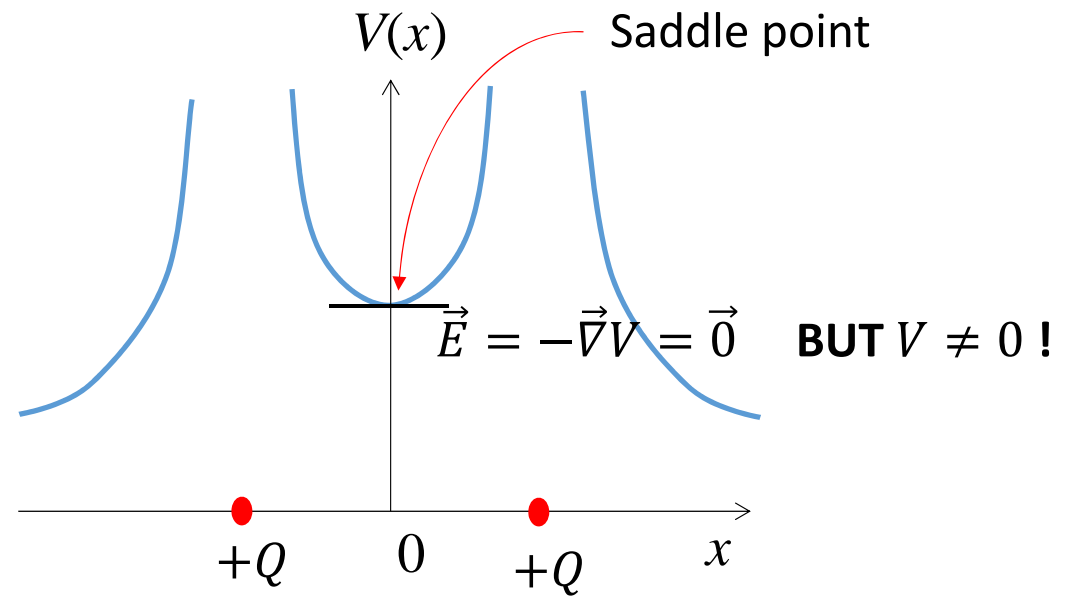
Far away it will feel  $V(Q_1 + Q_2)$



Two **different** equipotentials **NEVER** cross like two electric field lines



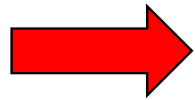
At the saddle point the gradient (field) is zero



## Summarizing

The key word is the work done by the field

The potential energy  $U = qV$  is zero when the charges are infinitely separated



So  $U$  is the work done on the test charge  $q_0$  by the **field** of  $Q$  ( $\sum_{i=1}^n q_i$ ) when moving  $q_0$  from  $r$  to  $\infty$

$W(\vec{F}_{el})_{r \rightarrow \infty} > 0$  if  $Q$  and  $q_0$  are of the same sign and  $U(r) > 0$

$W(\vec{F}_{el})_{r \rightarrow \infty} < 0$  if  $Q$  and  $q_0$  are of opposite sign and  $U(r) < 0$

Electrostatic potential and implication:  $\vec{E} = -\vec{\nabla}V$

Work done along an arbitrary path

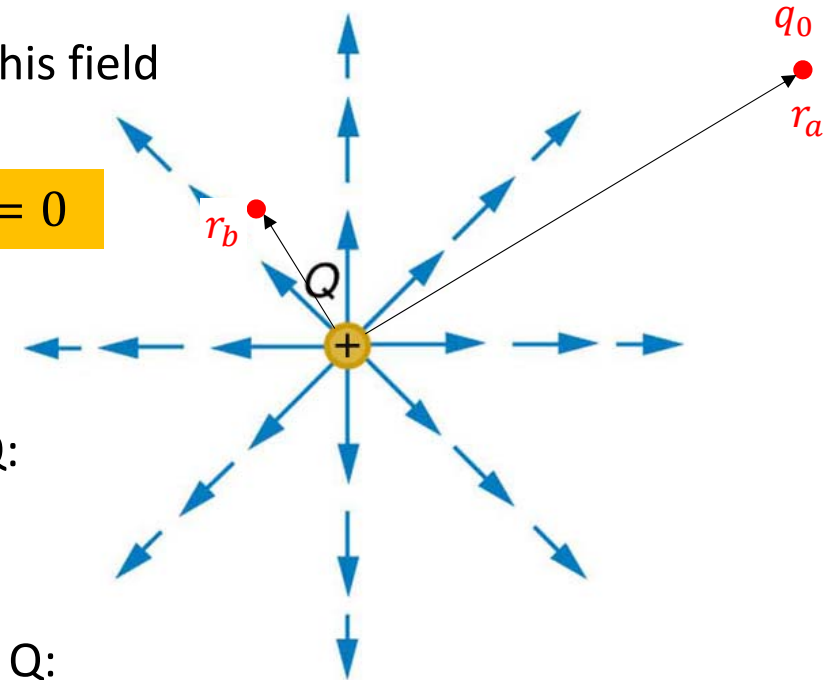


The positive charge  $Q$  creates a field all around

A test charge  $q_0$  will undergo a force (attractive or repulsive) in this field

To keep moving very slowly the charge  $q_0$  from  $r_a$  to  $r_b$

$$\Delta K = 0$$



If  $q_0 > 0$ :

a repulsion takes place with increasing intensity as we approach Q:

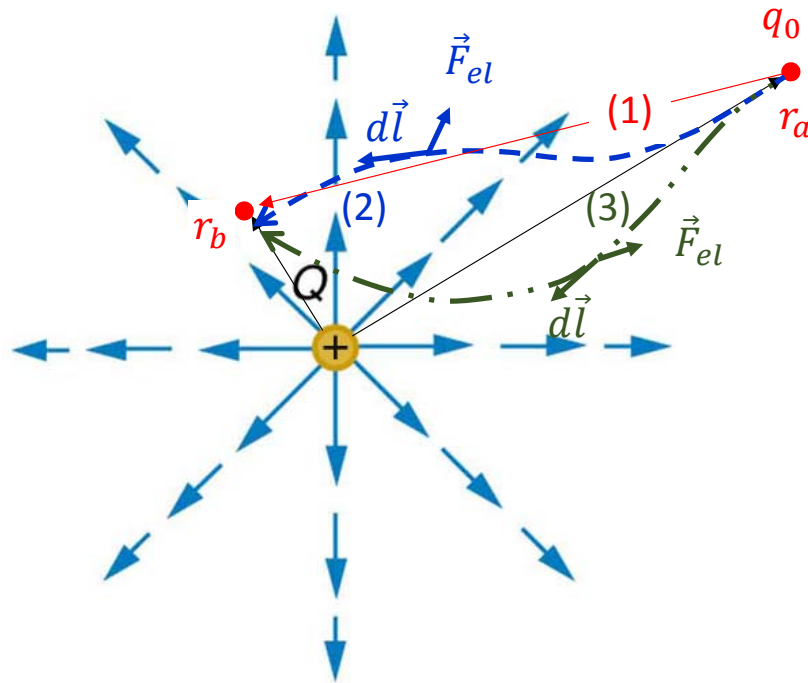
We need to push harder and harder

If  $q_0 < 0$ :

an attraction takes place with increasing intensity as we approach Q:

We need to retain harder and harder

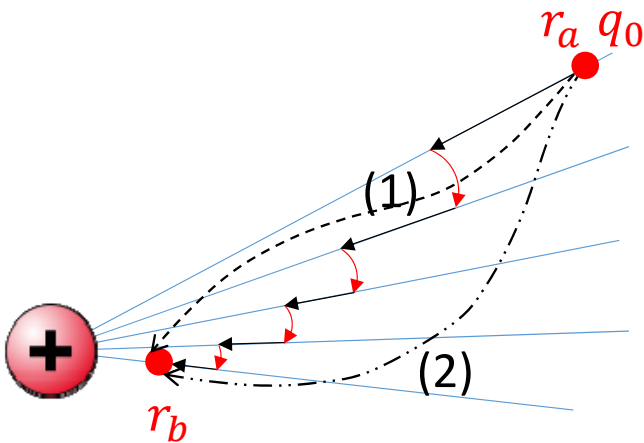
- The opposite happens if we need to go away from  $Q$
- In both cases the work is done against the electric force



$$W = - \int_{r_a}^{r_b} \vec{F}_{el} \cdot d\vec{l}$$

*Does the work depend on the path from  $r_a$  to  $r_b$  ?*

$$W = - \int_{r_a}^{r_b} \vec{F}_{el} \cdot d\vec{l} \quad \longrightarrow \quad \frac{W}{q_0} = W(\text{unit})_{a \rightarrow b} = - \int_{r_a}^{r_b} \vec{E} \cdot d\vec{l}$$



Along the radial lines  $\vec{E} \parallel d\vec{l}$  Work = maximum

Along the curved lines  $\vec{E} \perp d\vec{l}$  Work = 0



**Work does not depend on the path**

$$W(\text{unit})_{a \rightarrow b} = - \int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} = \int_{r_a}^{r_b} dV$$

$$W(\text{unit})_{a \rightarrow b} = V(r_b) - V(r_a)$$

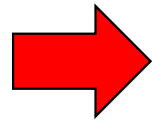
Another way of demonstrating the relation

$$\vec{E} = -\vec{\nabla}V$$

Application of vector calculus !

$$\int_{r_a}^{r_b} dV = \int_{r_a}^{r_b} \left( \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = \int_{r_a}^{r_b} \left( \frac{\partial V}{\partial x} \vec{i} + \frac{\partial V}{\partial y} \vec{j} + \frac{\partial V}{\partial z} \vec{k} \right) (dx\vec{i} + dy\vec{j} + dz\vec{k})$$
$$\int_{r_a}^{r_b} \vec{\nabla}V \cdot d\vec{l}$$

$$\int_{r_a}^{r_b} dV = \int_{r_a}^{r_b} -\vec{E} \cdot d\vec{l}$$



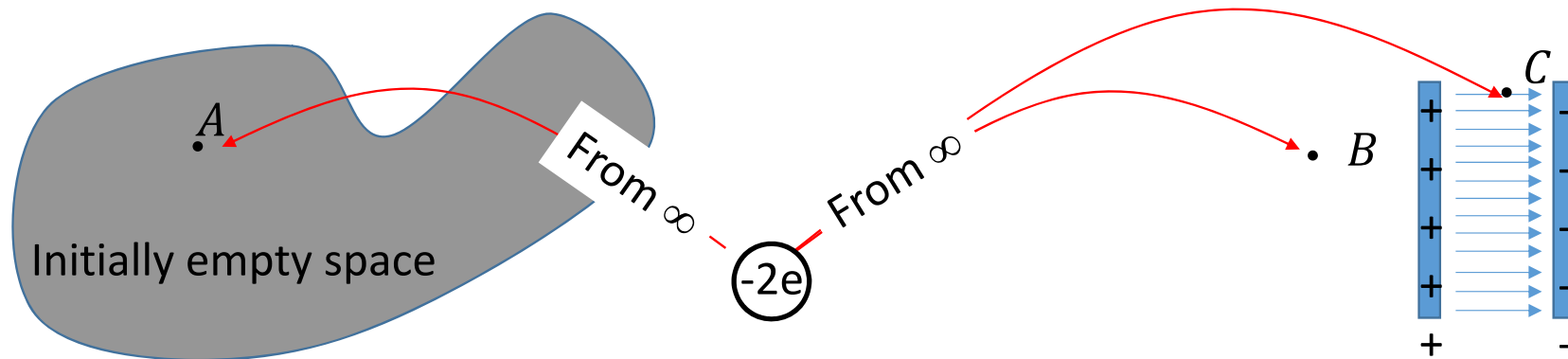
$$\vec{E} = -\vec{\nabla}V$$

Static Electric field is conservative

$$\int_{r_a}^{r_b} dV = \int_{r_a}^{r_b} \vec{\nabla}V \cdot d\vec{l}$$

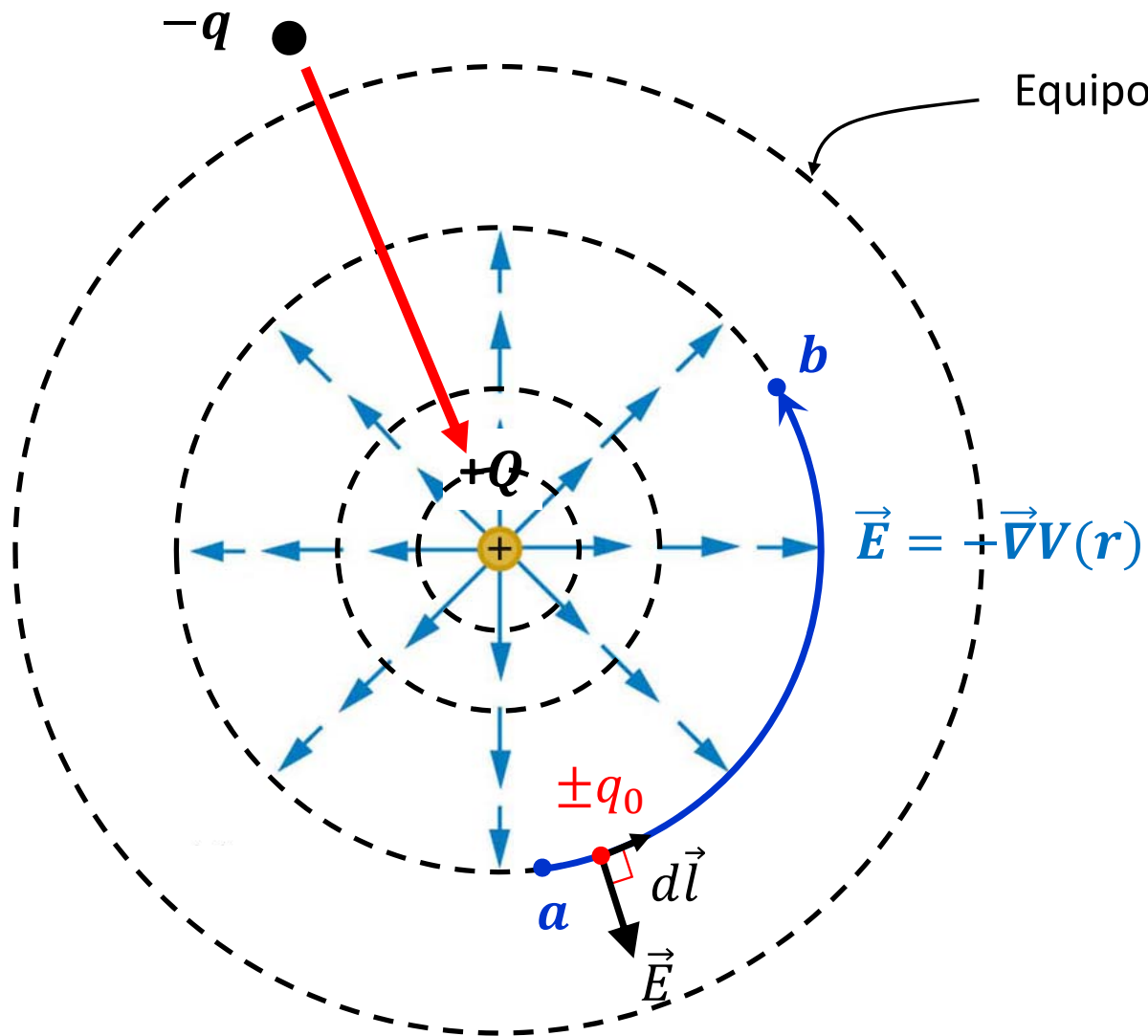
## Questions

Does it require any work to bring the charge  $-q$  from  $\infty$  to point  $A, B$  and  $C$ ?



If  $\vec{E} \propto \frac{1}{r^n} \vec{e}_r$  with  $n \neq 2$  would the work done be path independent?

For any radial and spherically symmetric force the work done does not depend on the path and there exist a potential. There could be any  $r$  dependence of  $\vec{E} \propto \frac{1}{r^n} \vec{e}_r$ . The existence of the potential and the fact that  $\vec{\nabla} \times \vec{E} = \vec{0}$  is due to that only



$$\int_a^b \vec{E} \cdot d\vec{l} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$\Gamma$  = equipotential path

From Stoke's theorem



$$\vec{\nabla} \times \vec{E} = 0$$

**Curl-free**

## Applications of Coulomb and Gauss law

Electric field and potentials for various distributions of charges

# Electric field and potentials for various distributions of charges

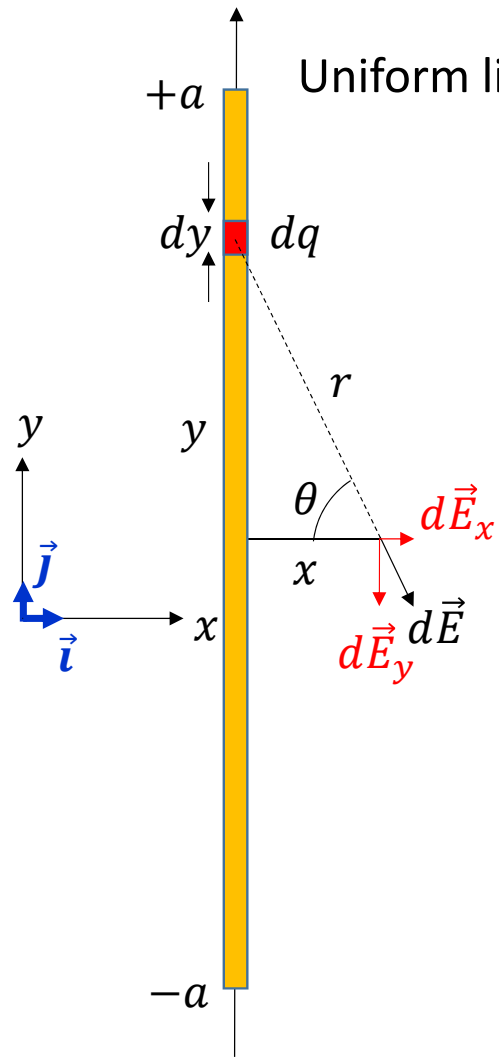
Make use of symmetry whenever possible

- Point charge
- Line of charges
- Ring of charges
- Disk of charges
- Sheet of charges
- Plane (infinite disk) of charges
- Cylindrical distribution of charges
- Spherical distribution of charges
- Rectangular distribution of charges

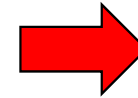


# The field of a line charge

By Coulomb' law



Uniform linear charge density  $\lambda = \frac{dq}{dy} = \frac{Q}{2a}$



$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \frac{dy}{x^2 + y^2}$$

$$dE_x = dE \cos\theta = \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \frac{xdy}{(x^2 + y^2)^{3/2}}$$



$$E_x = \int_{-a}^{+a} dE_r = \frac{1}{4\pi\epsilon_0} \frac{Q}{x} \frac{1}{\sqrt{(x^2 + y^2)}}$$



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{x} \frac{1}{\sqrt{(x^2 + a^2)}} \vec{i}$$

$$dE_y = -dE \sin\theta = \frac{1}{4\pi\epsilon_0} \frac{Q}{2a} \frac{ydy}{(x^2 + y^2)^{3/2}}$$



$$E_y = \int_{-a}^{+a} dE_y = 0$$

Why is this result trivial ?

$\left\{ \begin{array}{l} \text{If } a \ll x \\ \text{If } a = \infty \end{array} \right.$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{x^2} \vec{i} \quad \text{Point charge}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \left( \frac{Q}{2a} \right) \frac{1}{x} \vec{u} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{x} \vec{i}$$

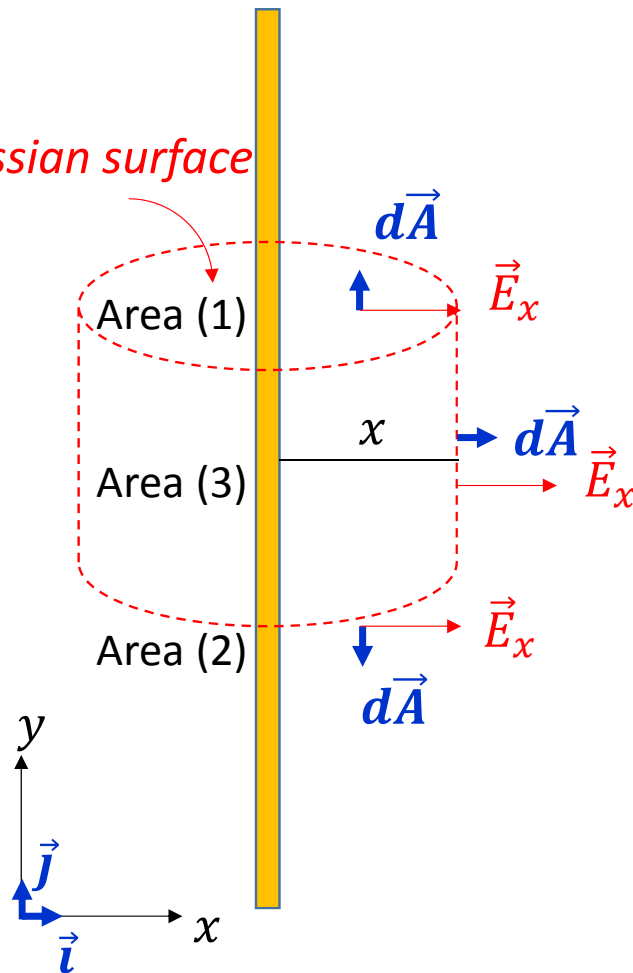
# The field of a line charge

By Gauss law

Line positively charged

Linear charge density  $\lambda = \frac{Q}{l}$  Charge enclosed in Gaussian surface  $Q = \lambda l$

Gaussian surface



Through areas (1) and (2)  
 $\vec{E}_x \perp d\vec{A}$

$$\Phi_E = \int_{\text{Area (1)}} \vec{E}_x d\vec{A} - \int_{\text{Area (2)}} \vec{E}_x d\vec{A} = 0$$

0                      0

Through area (3)  $\vec{E}_x \parallel d\vec{A}$

$$\Phi_E = \int \vec{E}_x d\vec{A} = E_x \cdot 2\pi x l = \frac{Q}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

$$E_x = \frac{\lambda}{2\pi\epsilon_0 x}$$

$$\vec{E}_x = \frac{\lambda}{2\pi\epsilon_0 x} \vec{i}$$

## Potential at distance r from a very long line of charge

$$V_b - V_a = - \int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} = - \int_{r_a}^{r_b} \frac{\lambda}{2\pi\epsilon_0 r} dr = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_a}{r_b}$$

If we set  $r_a = \infty$  and  $V_a = 0$   $V_b = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{\infty}{r_b} = \infty$  !

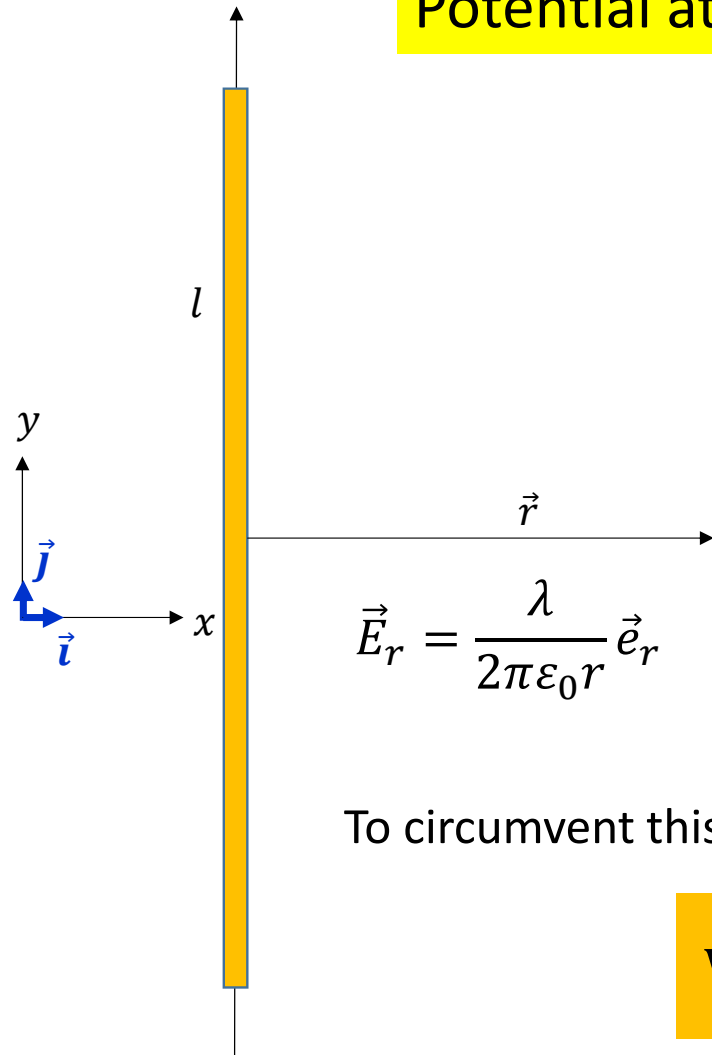
The problem comes for the fact that  $\vec{E}_r = \frac{\lambda}{2\pi\epsilon_0 r} \vec{e}_r$

Assumes that the charge distribution extends to infinity

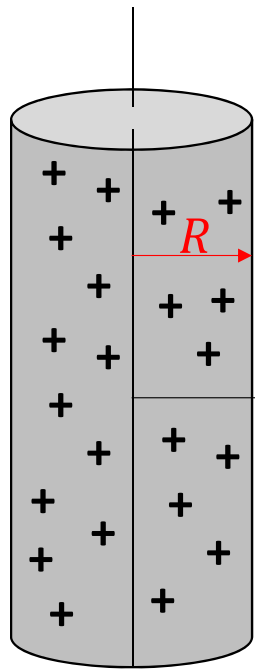
To circumvent this difficulty we consider that  $V_a = 0$  at an arbitrary radial distance  $r_0$

$$V_r = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

Where  $V_r$  decreases when  $r$  increases



# Potential at distance $r$ from a very long charged conducting cylinder



It is preferable to consider that the linear charge density has been transferred to the cylinder  $\Rightarrow$  we can still use  $\lambda$

Gauss law



$$V_r = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{R}{r}$$

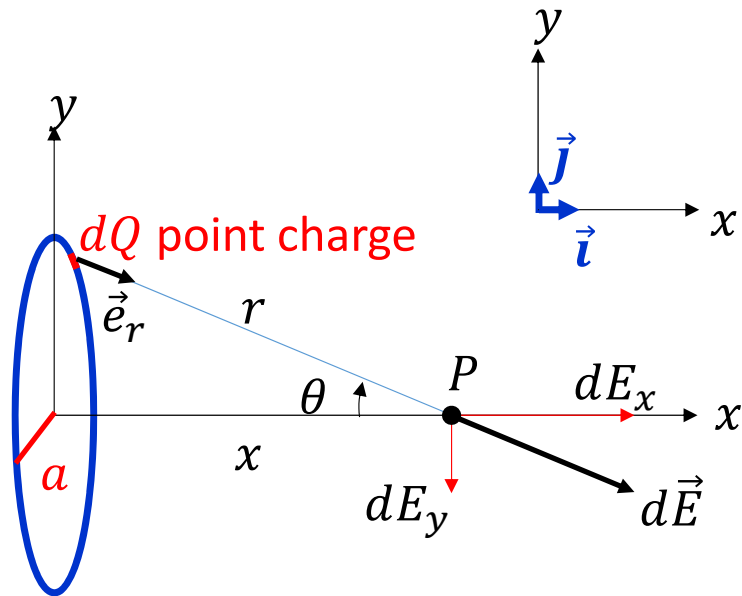
$$\vec{E}_r = -\vec{\nabla} V_r$$

$$\vec{E}_r = \frac{\lambda}{2\pi\epsilon_0 r} \vec{e}_r \quad r > R$$

$$r < R \quad V_r = 0 \quad \Rightarrow \quad \vec{E}_r = 0$$

$E, V = 0$

## The field of a charged ring



$$r = \sqrt{x^2 + a^2}$$

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + a^2}}$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dQ}{x^2 + a^2} \vec{e}_r = dE \cos\theta \vec{i} + dE \sin\theta \vec{j}$$

||  
0

symmetry

$$\vec{E} = E_x \vec{i} = \frac{1}{4\pi\epsilon_0} \int \frac{x dQ}{(x^2 + a^2)^{3/2}} \vec{i}$$

Running around the ring keeps x unchanged

$$\vec{E} = E_x \vec{i} = \frac{1}{4\pi\epsilon_0} \frac{xQ}{(x^2 + a^2)^{3/2}} \vec{i}$$

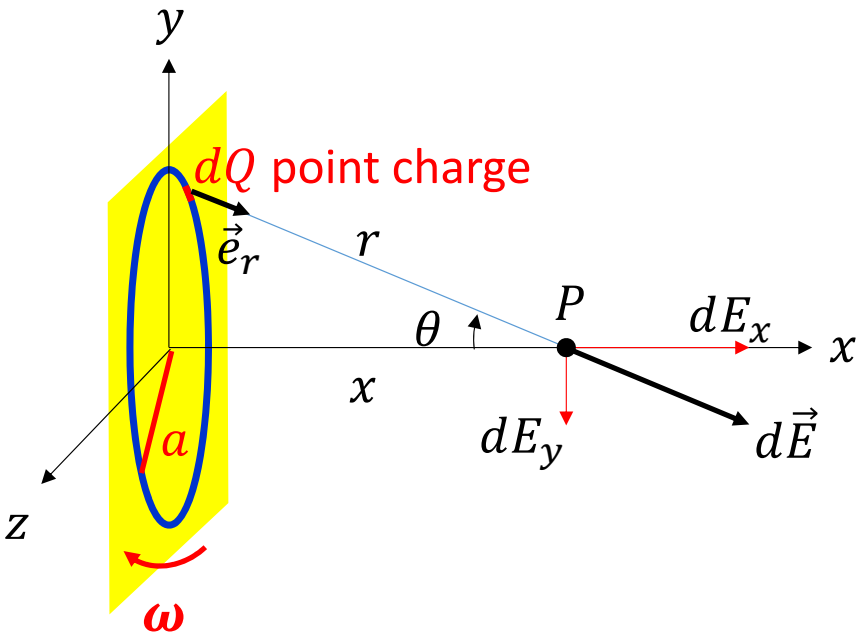
$$\text{If } x \gg a \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{x^2} \vec{i}$$

Far away the ring appears like a point charge

At the center of the ring  $x = 0 \Rightarrow \vec{E} = \vec{0}$

Why this is trivial ?

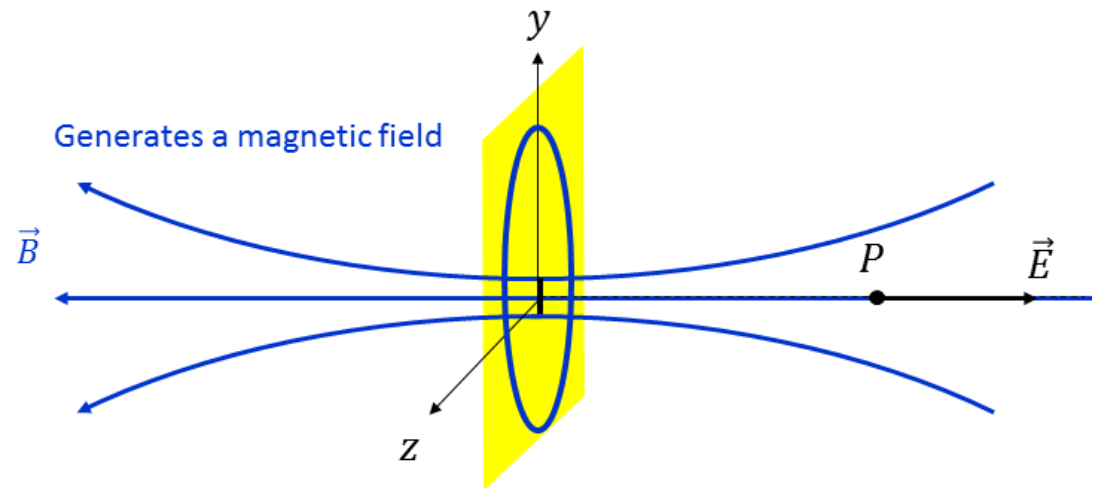
What should we expect if the this ring starts rotating with an angular velocity  $\omega$  ?



Charges are set into motion



Current



Wait for Magnetostatic

## The potential of a charged ring

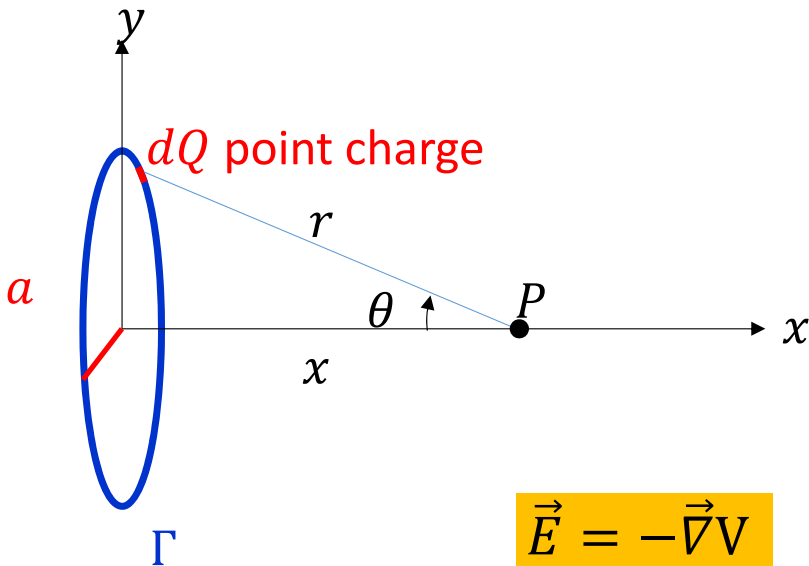
$$dV = \frac{1}{4\pi\epsilon_0} \frac{dQ}{r}$$

$$V = \frac{1}{4\pi\epsilon_0} \oint_{\Gamma} \frac{dQ}{r} = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \oint_{\Gamma} dQ = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{x^2 + a^2}}$$

$$\text{If } x \gg a \quad V = \frac{1}{4\pi\epsilon_0} \frac{Q}{x}$$

Far away the ring appears like a point charge

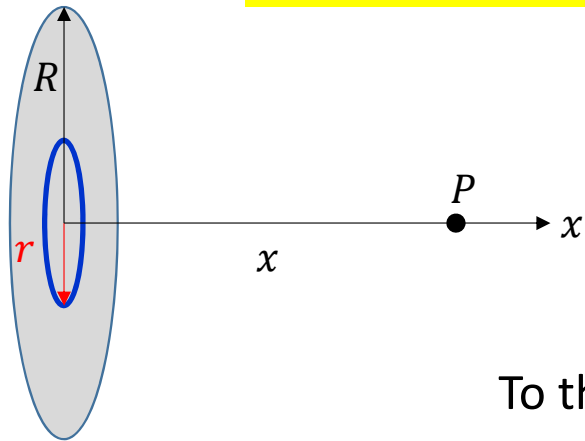
$$\text{At the center of the ring } x = 0 \Rightarrow V = \frac{1}{4\pi\epsilon_0} \frac{Q}{a}$$



$$\vec{E} = -\vec{\nabla} V$$

Works both ways: We can find  $\vec{E}$  from  $V$  and vice versa

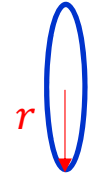
## The field of an infinite sheet of charge: from uniformly charged disk



$$\sigma = \frac{Q}{A}$$

Charge element of the ring  
 $dQ = \sigma 2\pi r dr$

From the ring



$$d\vec{E}_x = \frac{1}{4\pi\epsilon_0} \frac{x dQ}{(x^2 + r^2)^{3/2}} \vec{i}$$

To the disk



$$\vec{E}_x = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{x\sigma 2\pi r dr}{(x^2 + r^2)^{3/2}} \vec{i} = \frac{\sigma x}{2\epsilon_0} \int_0^R \frac{r dr}{(x^2 + r^2)^{3/2}} \vec{i}$$

$$\vec{E}_x = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{x\sigma 2\pi r dr}{(x^2 + r^2)^{3/2}} \vec{i} = \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{1}{\sqrt{R^2/x^2 + 1}} \right] \vec{i}$$

*The field lines are not straight for a finite disk*

If  $R \gg x$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \vec{i}$$



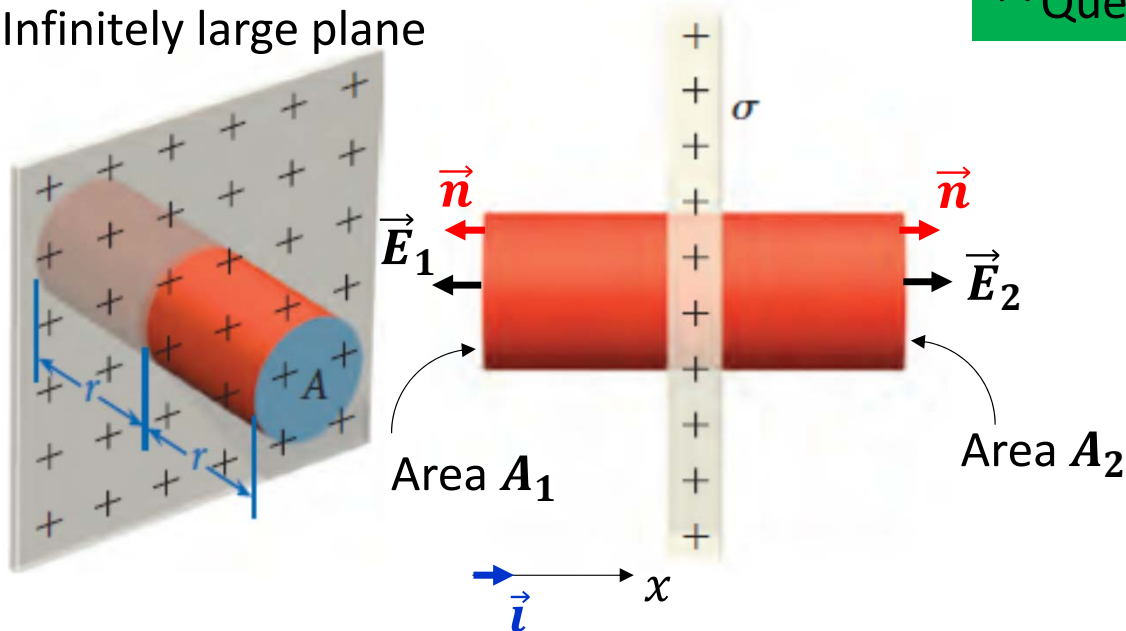
# The field of an infinite sheet of uniformly distributed charges: Using Gauss law

Main tool = symmetry argument  $\Rightarrow$  Gauss's surface = cylinder

$$\Phi_E = \oint \vec{E} d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

Total area

Infinitely large plane



**\*\*Question #1:**

Why is the field horizontal thus  
flux = 0 through the cylinder wall?

**Answer to Question #1:**

Because we consider **infinitely large plane**  
and look at distance close to the plane

$$\vec{E} \text{ uniform} \quad Q_{enc} = \sigma A$$

**\*\*Question #2:**

How does the field look like very far away from the plate,  
Knowing that it has a finite size?

**Answer to Question #2:**

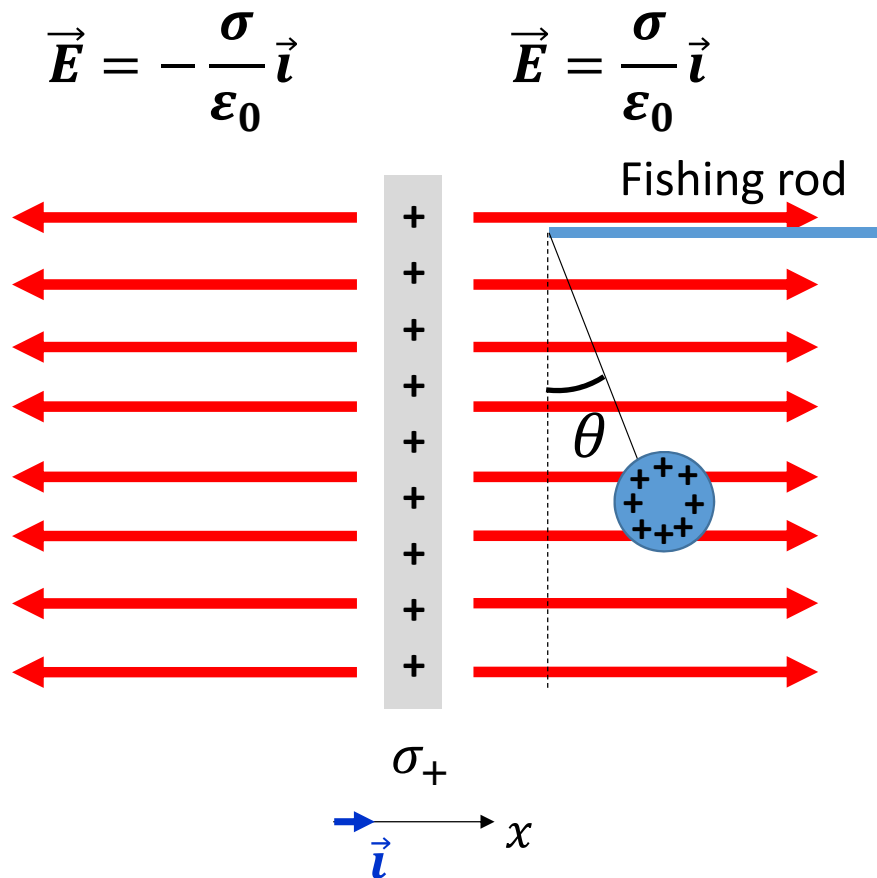
Decaying as  $\sigma A / r^2$  as the plate looks as a point charge

**\*\*\*Question #3:**

What kind of simple experiment can we do to prove that the field is uniform  
at reasonable distance from the charged plate?

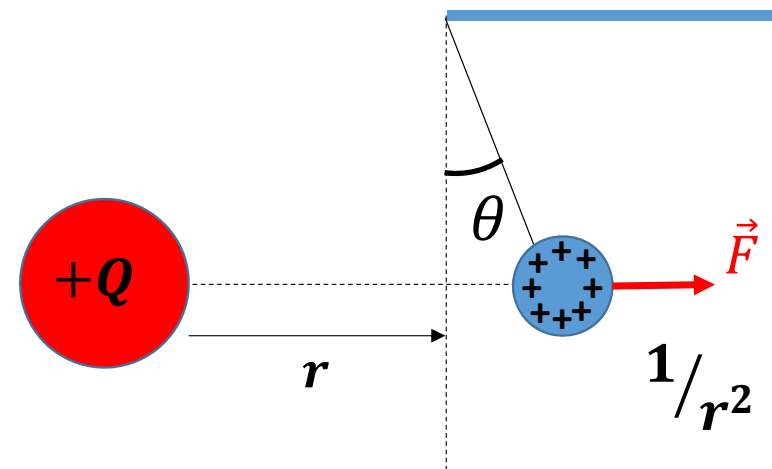
**Answer to Question #3:**

Use charge induction concept



The angle  $\theta$  remains constant for reasonable distance of the pendulum from the plate

A spherical metal initially charged by induction



$\theta$  drops quickly when  $r$  increases

**\*\*Question #4:**

How do we proceed to charge a conducting sphere by induction?

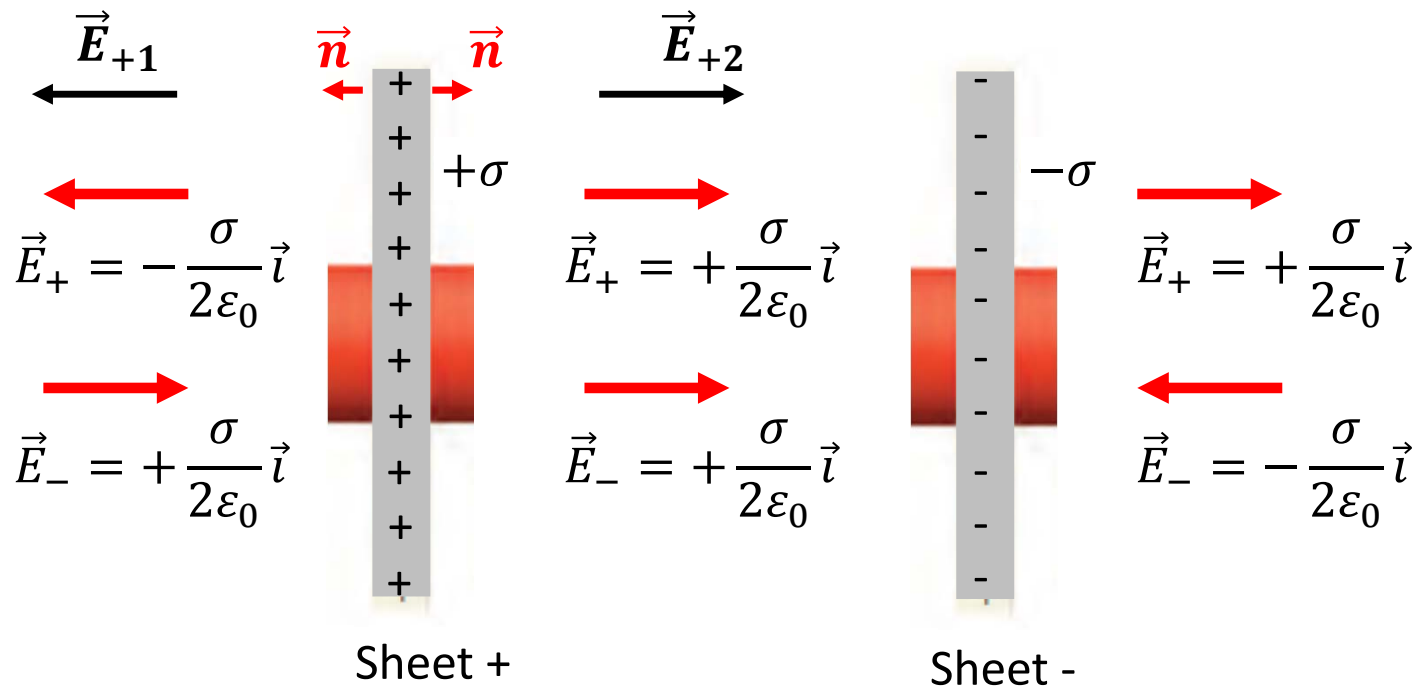
**Answer to Question #4:**

**See slide 73**

# The field due to two parallel sheets of opposite charges: Superposition principle

$$\vec{E}_{+1} \cdot \vec{A}_1 + \vec{E}_{+2} \cdot \vec{A}_2 = E_{+1} \cdot A + E_{+2} \cdot A = \frac{\sigma A}{\epsilon_0}$$

$$\vec{E}_+ = \frac{\sigma}{2\epsilon_0} \vec{l}$$



From positive sheet (plate)

From negative sheet (plate)

$$\vec{E} = \frac{\sigma}{\epsilon_0} \vec{l}$$

## One single type of charge distributed uniformly on a conducting sphere

\*Question #5: How can we obtain such distribution?

Answer to \*Question #5:

There are two possibilities

By friction or contact

All deposited charges will spread uniformly around the sphere as total energy must be minimized (repulsion will do the job).

Stable equilibrium\*

By induction via an external field

**BUT** induction induces charges by pair to keep neutrality (conservation of charge)

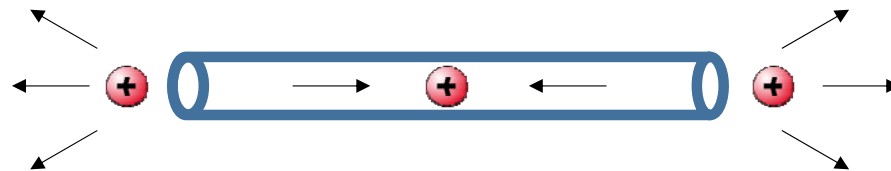
Unstable equilibrium\*

Stable equilibrium\*

??

We have claimed several times that there is **NO** equilibrium for electrostatic charges

**BUT** remember



**Slide #29**

Other forces are acting to maintain equilibrium

- In the hollow tube above mechanical force
- In the case of the conducting sphere, forces at the surface prevent the charges from leaving the conductor

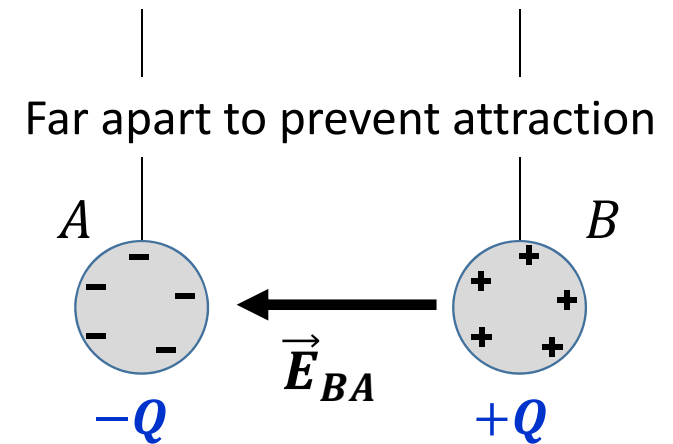
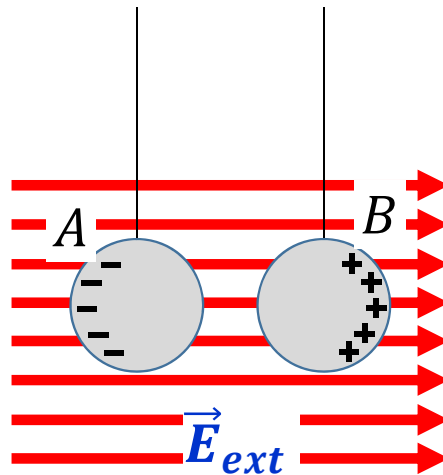
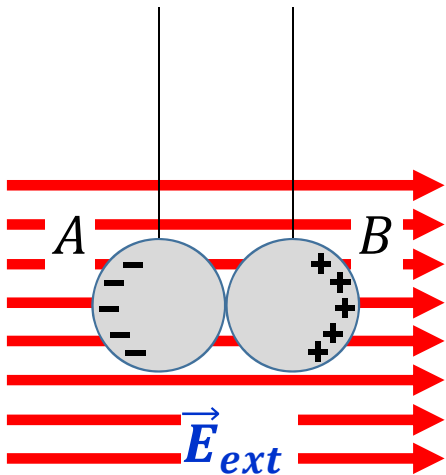
Why induction by an external field leads to

Unstable equilibrium\*

??

Because if we switch off the external field, the charges will redistribute themselves and annihilation takes place bringing the conducting sphere to neutrality

But we want to induce permanent charge on the conducting sphere by an external field !



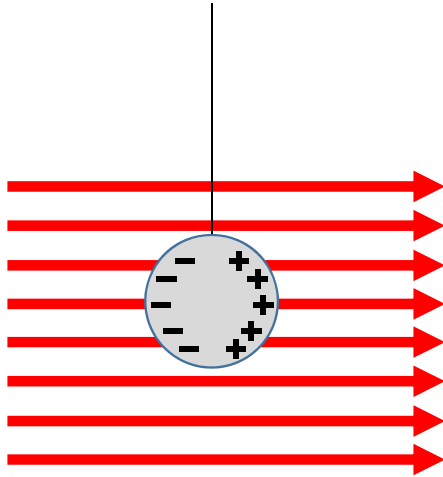
**Step 1:** Contact and induction

**Step 2:** Separation of the spheres

**Step 3:** Switch off the external field

\*\*\*Question #6:

Something is **FUNDAMENTALLY WRONG** with this representation?



Answer to \*\*\*Question #6:

The field lines must bend close to the conducting spheres . Why?

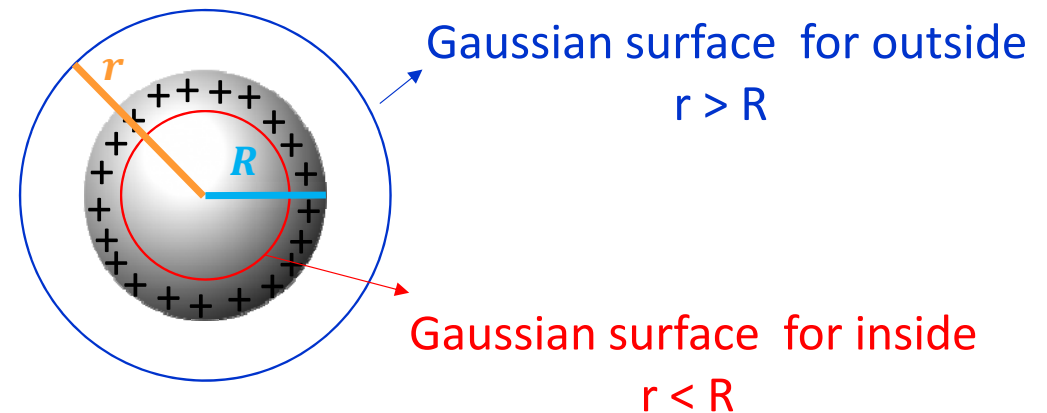
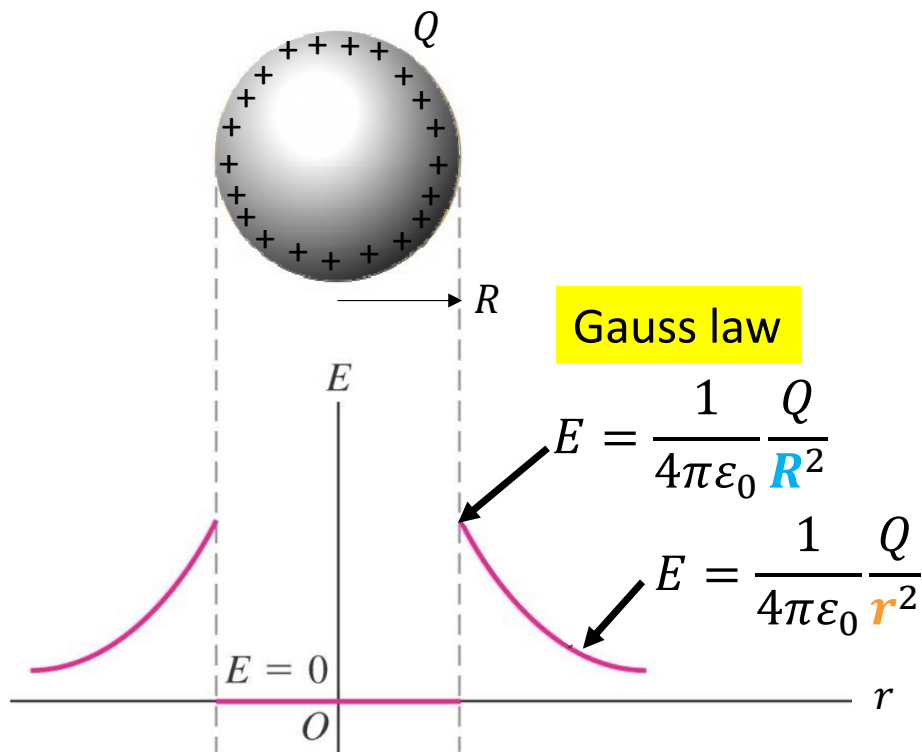
See next



# The field of a surface charged conducting sphere: Using Gauss law

Electric field and potential from the center of the sphere to infinity

Could be a hollow metallic sphere

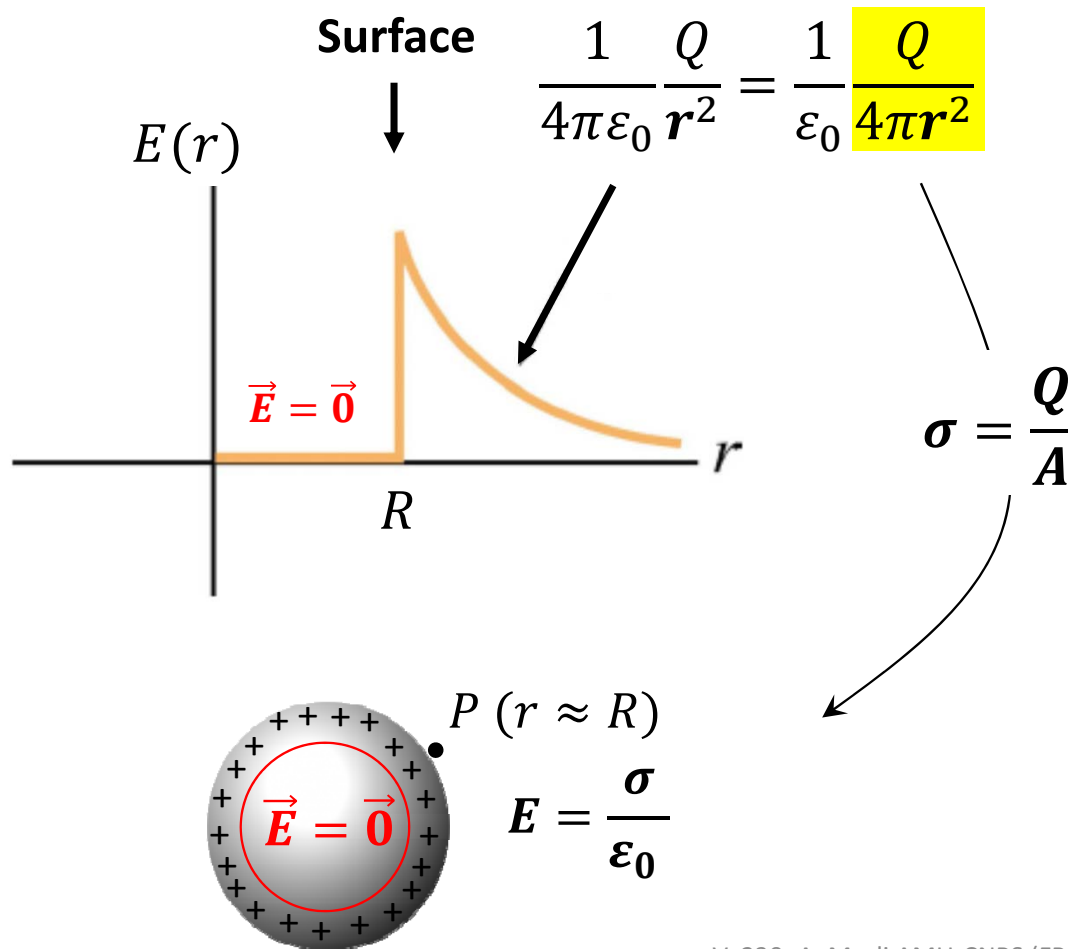


$$\vec{E} = -\vec{\nabla} \cdot V(r)$$

$$V(\text{sphere}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R}$$

The conducting sphere is an equipotential body  
 $V = cte$

The hollow or solid conducting sphere is an example where

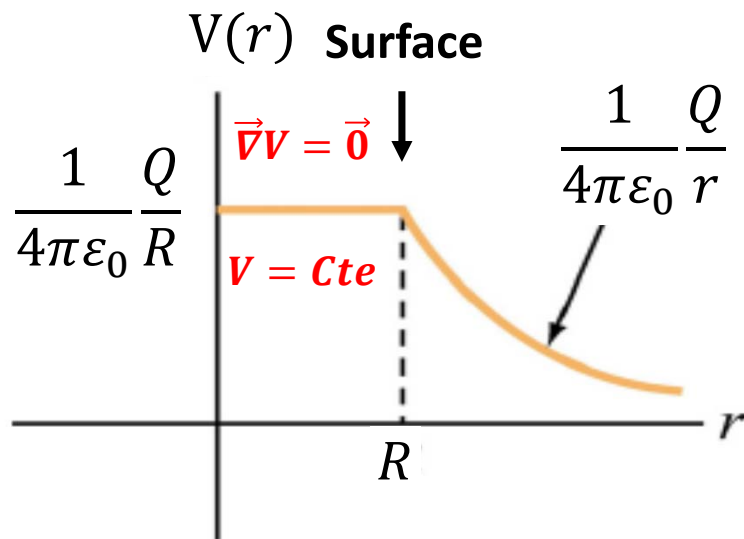


$$\oint \vec{E} \cdot d\vec{A} = \int \vec{\nabla} \cdot \vec{E} dV = 0$$

Closed surface inside the sphere      Volume enclosed by the surface

What does this mean?

The hollow or solid conducting sphere is an example where



$$\Leftrightarrow \text{Volume } \vec{E} = \vec{0} \quad \vec{\nabla} \cdot \vec{E} = 0$$

Diagram of a hollow conducting sphere with positive charges on its surface. The interior is labeled with  $V = Cte$  and  $\vec{\nabla}V = \vec{0}$ . The surface is labeled with  $V = Cte$  and  $\vec{\nabla}V = \vec{0}$ .

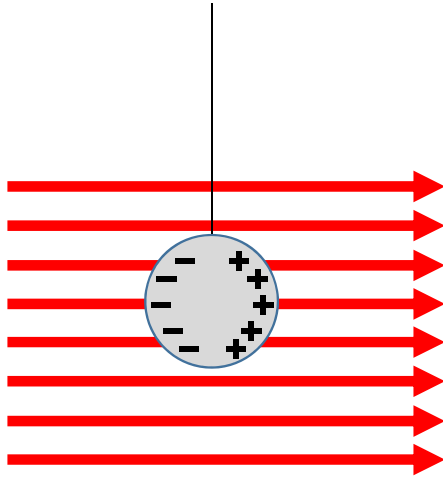
$$dV = \vec{\nabla}V \cdot d\vec{l} = 0$$

$$V = \int_a^b \vec{\nabla}V \cdot d\vec{l} = - \int_a^b \vec{E} \cdot d\vec{l} = 0$$

Along any **open** or **closed** path on the surface

What does this mean?

**\*\*\*Question #7:** Still no clue as to why this representation is **FUNDAMENTALLY WRONG?**

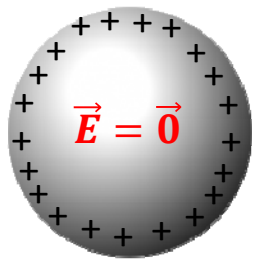


**Answer to \*\*\*Question #7:**

See chapter on conductors

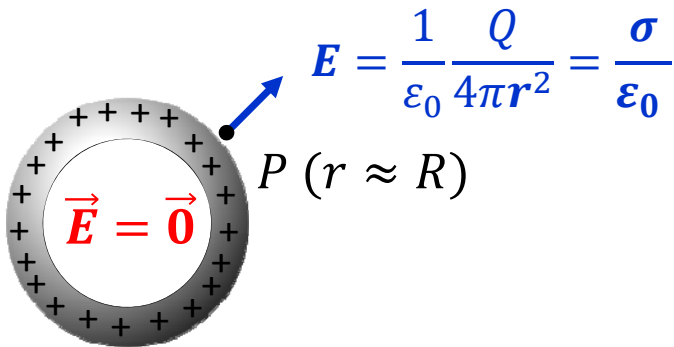
That  $\vec{E} = \vec{0}$  inside the conducting sphere is NOT trivial at all !

**“Conspiracy” principle**



Every charge on the sphere creates a field inside and outside the sphere **BUT** all individual effects when added **cancel out inside**

Consequence  
For a hollow  
sphere



Whereas

$$E = \frac{\sigma}{2\epsilon_0}$$



$\sigma_+$

$$E = \frac{\sigma}{2\epsilon_0}$$

See lectures on conductors

## An interesting point

Outside the sphere nature cannot decide whether the charge is distributed uniformly over the sphere or concentrated on a point at the center because

Electric field  $\propto 1/r^2$

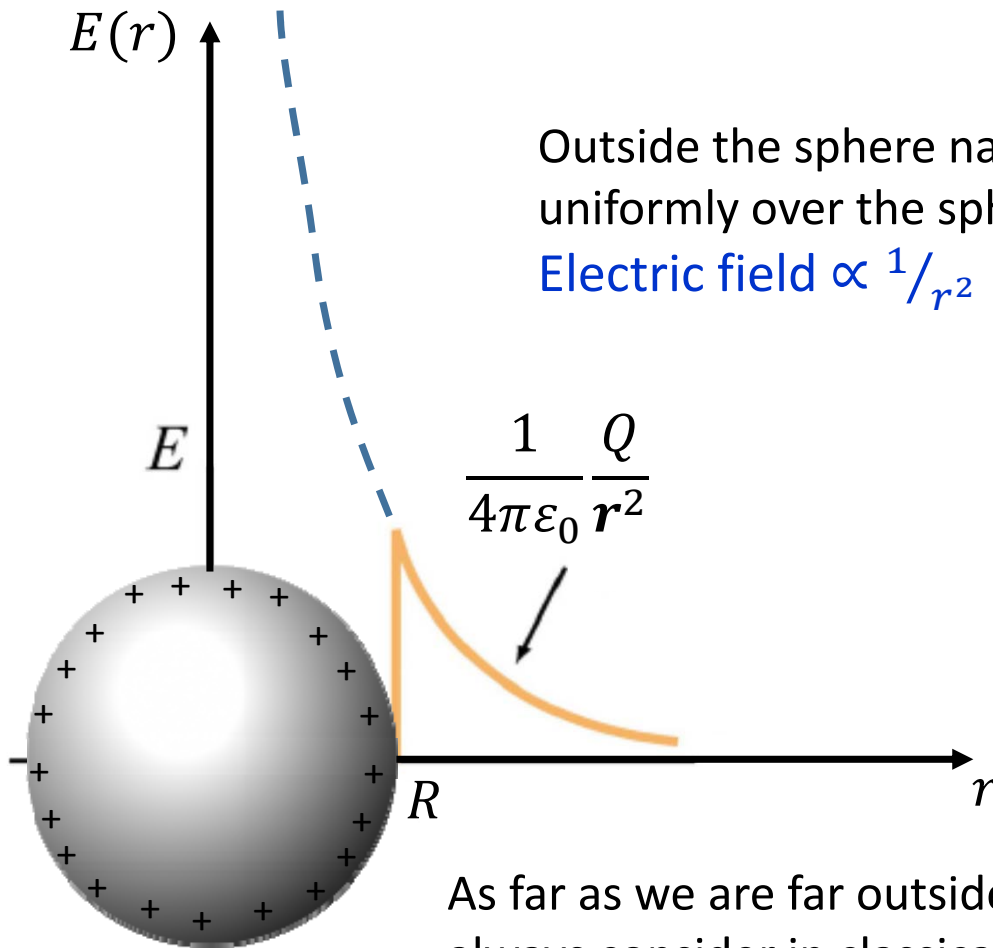
Gravitational field  $\propto 1/r^2$



Planet in a form of a hollow sphere



No gravitation inside



As far as we are far outside the planet with a mass uniformly distributed, we always consider in classical mechanics that the point mass approximation is valid.

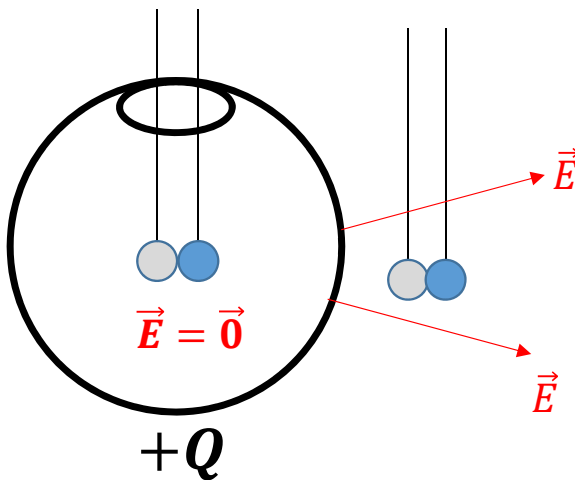
- It took 20 years to Newton to prove this statement
- 100 years later Gauss's law proved it in 3 seconds

**\*\*Question #8:**

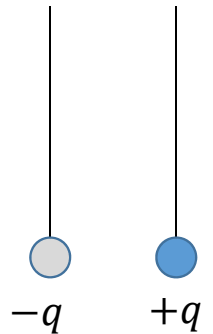
How can we check experimentally that the field inside a hollow sphere is really zero? Theoretically it will be done in the next lecture

**Answer to Question #8:**

Using the double pendulum with two little metallic spheres and a small opening in the hollow sphere



- 1) Bring the double pendulum close to the hollow sphere  
 $\Rightarrow$  charges are induced on the double pendulum



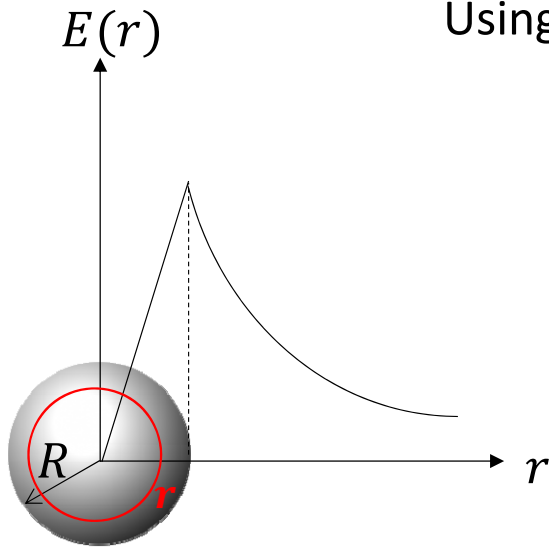
- 2) Inserted into the hollow sphere no charges are induced  
 $\Rightarrow$  field inside is zero

*Rigorously speaking the charge distribution on the hollow sphere is no longer uniform because of the opening. There is field inside but very weak*

# The field of a bulk charged **non**-conducting sphere (dielectric): Using Gauss law

Uniformly charged

Using a Gaussian surface inside and outside the charged sphere



$$r < R \Rightarrow Q = \rho \left( \frac{4\pi}{3} r^3 \right), \quad E(r) 4\pi r^2 = \frac{Q}{\epsilon_0},$$

$$\Rightarrow E(r) = \frac{\rho}{3\epsilon_0} r \quad \Rightarrow \quad E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r$$

$$r > R \Rightarrow Q = \rho \left( \frac{4\pi}{3} R^3 \right), \quad E(r) 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\text{Charge density } \rho = Q / \left( \frac{4\pi}{3} R^3 \right)$$

$$\Rightarrow E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$