Current – Resistance – and Electromotive force

Charges: From rest (electrostatic) to **motion**

Concept of current (charge in **MOTION**)

Concept of magneto**STATIC**

Does **static** magnetic situation exist?: **Static** and **motion** are antonymic

- There must be a current to get a magnetic field
- Current can only come from moving charges

Magnetostatic = approximation

Special kind of dynamic situation where a large number of charges are in **steady flow motion**

NO ACCELERATION

In which material can charges move?

Insulator?
NO

Conductor?
YES

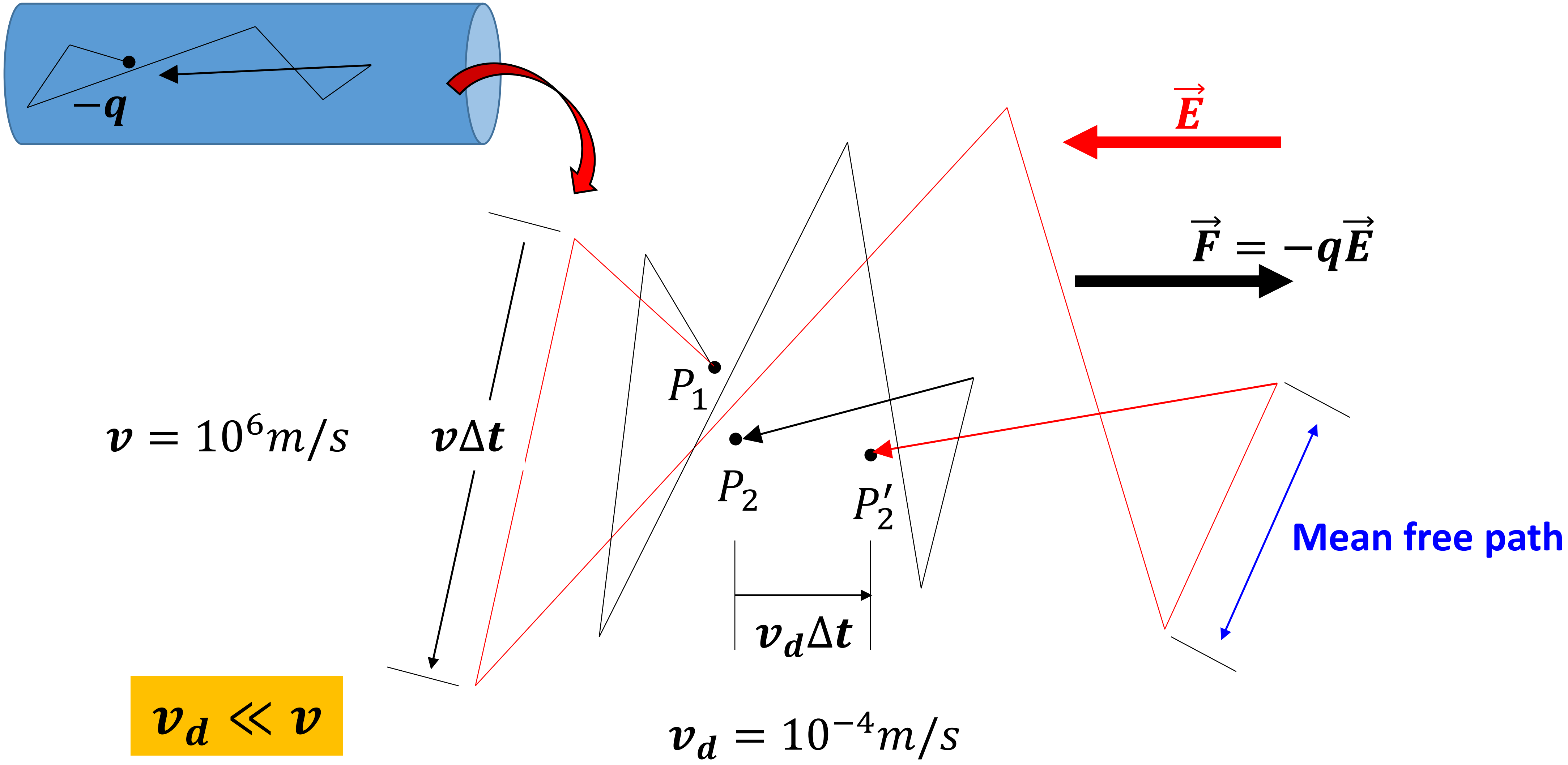
BUT

Moving electrons requires a $\vec{F}$ thus a $\vec{E}$

$\vec{E} = 0$ always in conductors

?

Random versus drift motion: towards Ohm's law



A charge moving in vacuum under the action of $\vec{F} = -q\vec{E}$ accelerates

In a conductor **collisions** with atoms prevents acceleration

Provide **heat** to the conductor

- Useful for a toaster: conversion of energy
- Harmful to solar cell: loss of conversion efficiency

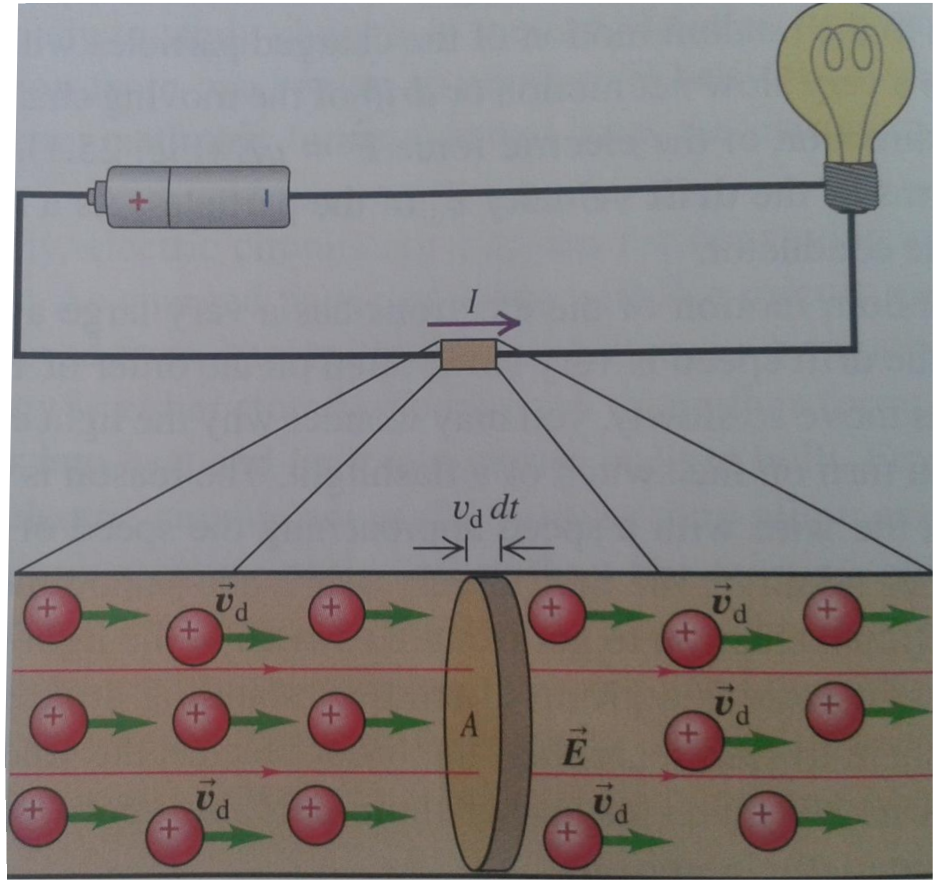
$$I = \frac{dQ}{dt}$$

$$dQ = (Nq\ ).(Av_d dt)$$

Volume of the disk

charges/unit volume

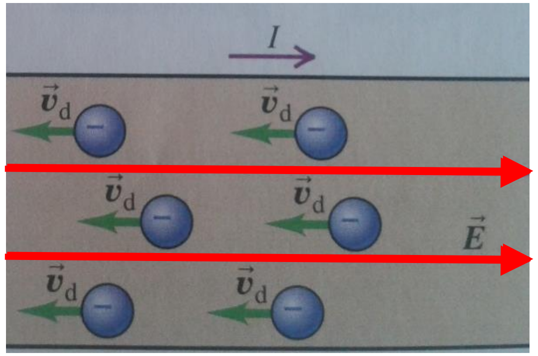
From University Physics, 11th edition



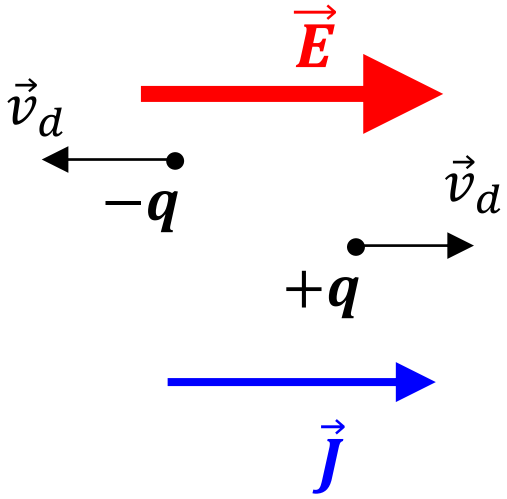
$dQ\ inside\ the\ disk$

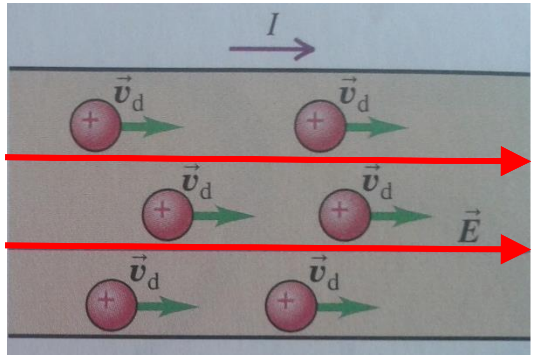
$$I = \frac{dQ}{dt} \qquad dQ = (Nq).(Av_d dt)$$

Vector current density $\boxed{\vec{J} = nq\vec{v}_d}$



$-q$





$+q$

$$\vec{J} = Nq\vec{v}_d = \rho\vec{v}_d$$

Charge density

For most conditions, $\vec{v}_d = \mu_e\vec{E}$

Mobility

$$\boxed{\vec{J} = \rho\mu_e\vec{E} = \sigma_e\vec{E}}$$

Conductivity

From University Physics, 11th edition

Conductivity – Resistivity – and Resistance: Ohm's law

It is important to make a clear distinction between charge density $\boldsymbol{\rho}$ and resistivity $\boldsymbol{\rho_e}$

Ohm's **"law"** discovered in 1826 (Ohm German physicist 1787 – 1854)

- Like Hooke's law
- Ideal gas equation

$$\boxed{\vec{J} = \sigma_e \vec{E} = \frac{1}{\rho_e}\vec{E}}$$

$$[\rho_e] = \Omega m$$

$$[\sigma_e] = (\Omega m)^{-1}$$

Idealized model
Not always valid

$$\frac{I}{A} = \sigma_e \frac{V}{L} = \frac{1}{\rho_e}\frac{V}{L}$$

$$\boxed{\rho_e = \frac{1}{\sigma_e} = R\frac{A}{L}}$$

Example: Calculating resistance

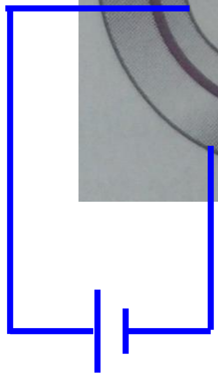
Each of the inner and outer surface is an equipotential so that the current flows radially. **What is the resistance to this radial current flow?**

Identify – execute – evaluate

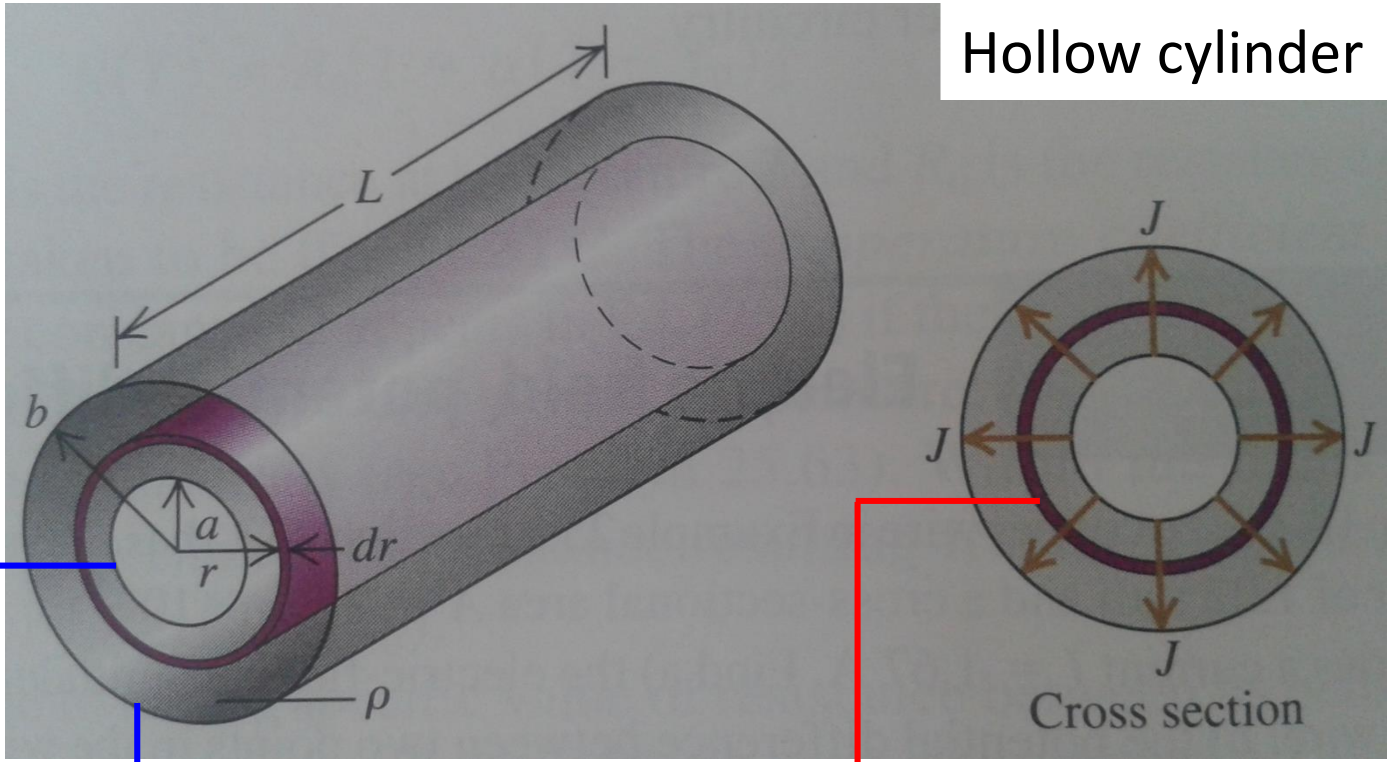
The current is not flowing along the length

Can we use the relation Directly ? $R = \rho_e \dfrac{L}{A}$

$$R = \int_a^b \rho_e \frac{dr}{2\pi r L} = \frac{\rho_e}{2\pi L} Ln \frac{b}{a}$$



Hollow cylinder



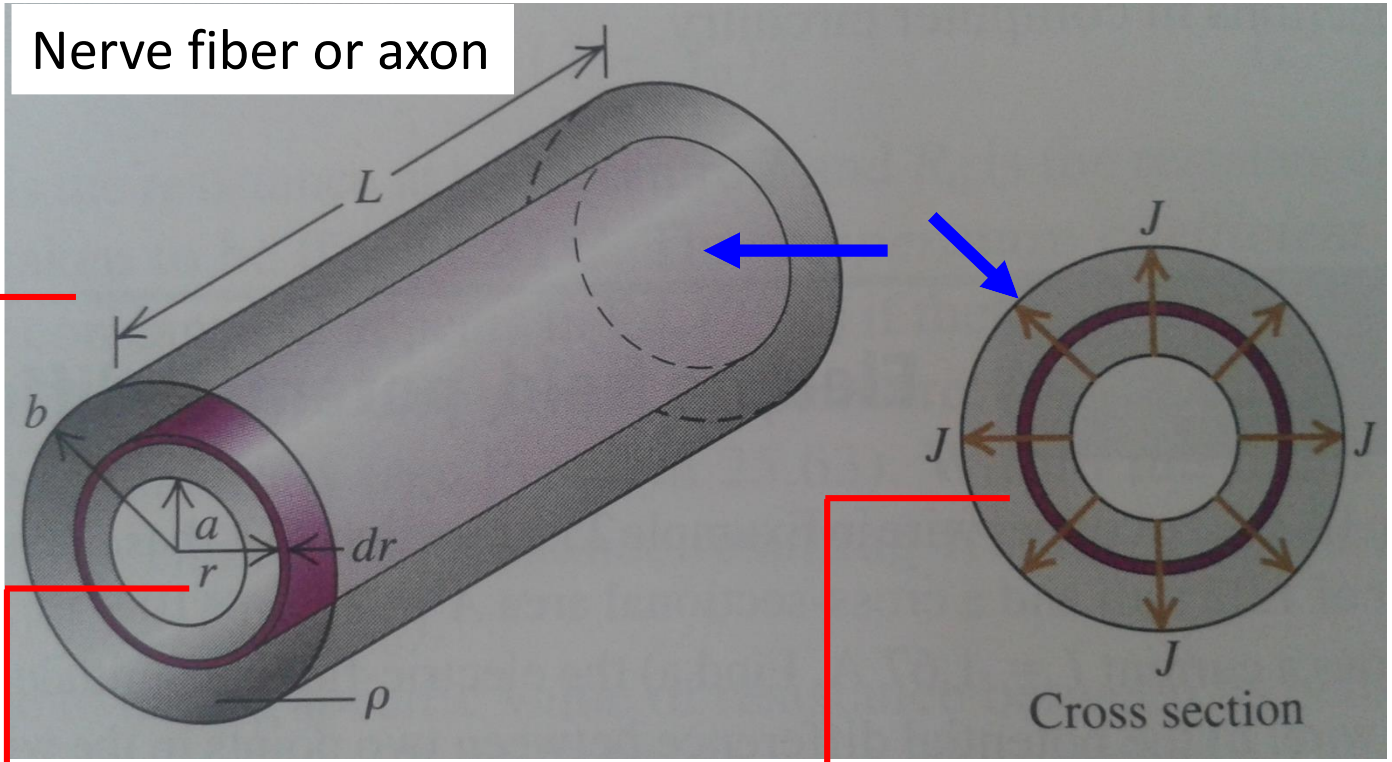
shell

$$dR = \rho_e \frac{dr}{2\pi r L}$$

From University Physics, 11th edition

Identify – execute – evaluate

Stimulation at this point induces radial flow of ions through the membrane
⇨ Potential difference
⇨ Flow of nerve signal



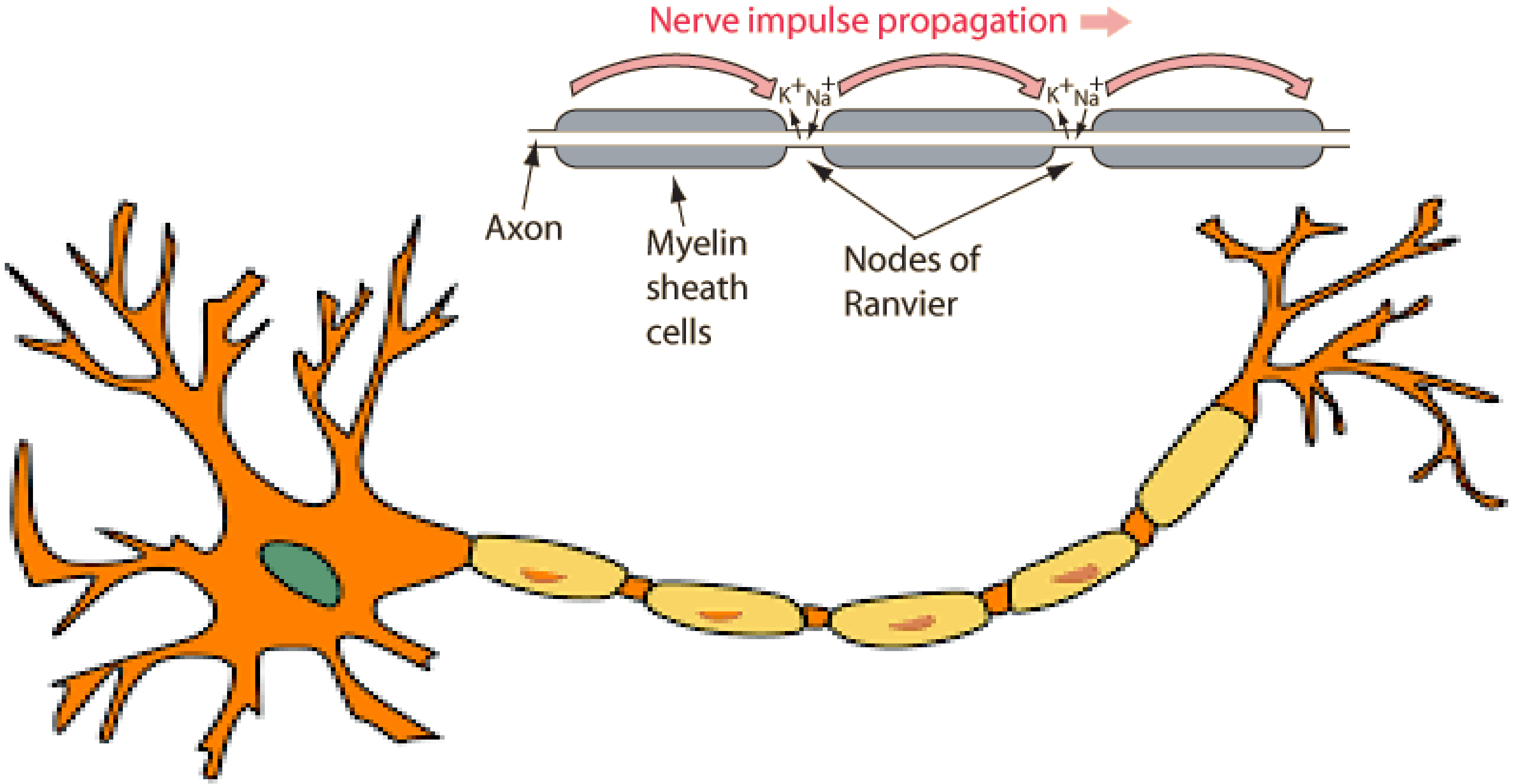
Here flows an outer liquid

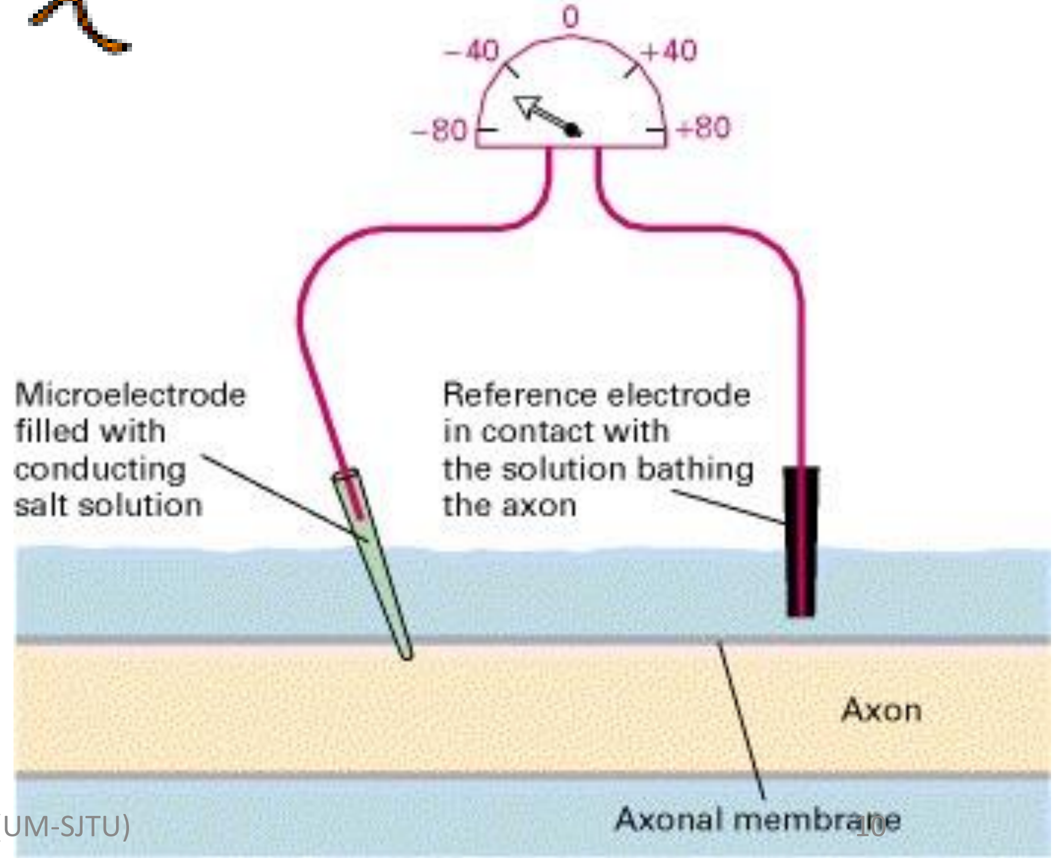
Both at the same potential

Here flows an inner liquid

Cylindrical membrane

No current flows along the cylinder membrane


Nerve impulse propagation
K+ Na+
K+ Na+
Axon
Myelin sheath cells
Nodes of Ranvier


0
−40
+40
−80
+80
Microelectrode filled with conducting salt solution
Reference electrode in contact with the solution bathing the axon
Axon
Axonal membrane

Electromotive force

Closed loop Electrostatic ⇨ Conservative field ⇨ $\vec{\nabla} \times \vec{E} = \vec{0}$ ⇨ $\boxed{\oint_{\Gamma} \vec{E}.d\vec{l} = 0}$

Physical meaning of this relationship?

The work done across a closed loop by the electrostatic field is zero:

$$\boldsymbol{W(\Gamma) = 0}$$

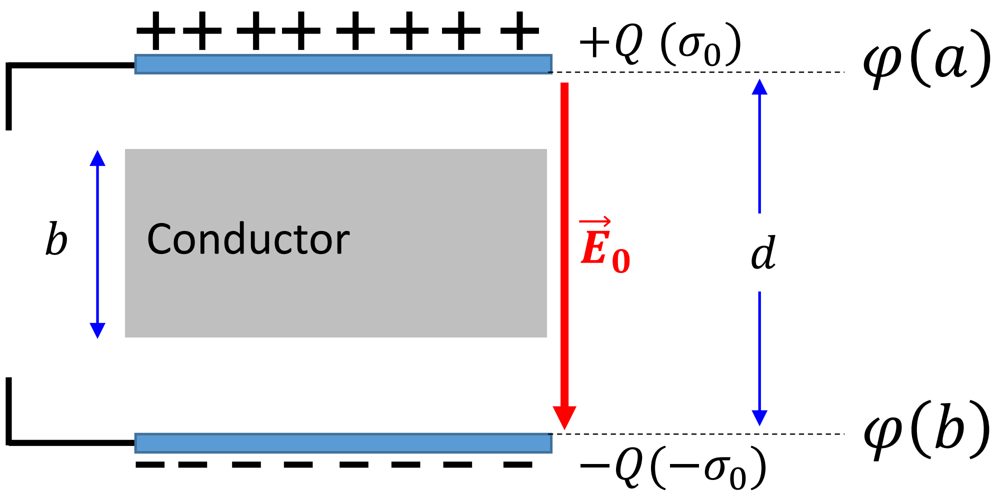
⇨ If the only acting field is **electrostatic**, electron can neither **loose nor gain** energy after completing one loop !

Problem ! Moving electrons collide with atoms ⇨ lose energy ⇨ current stops

$\vec{J} = \sigma_e \vec{E}$ ⇨ $\oint_{\Gamma} \frac{\vec{J}}{\sigma_e}.d\vec{l} = \frac{1}{\sigma_e} \boxed{\oint_{\Gamma} \vec{J}.d\vec{l} = 0}$

Current vanishes very quickly in $\sim 10^{-16}\ s$

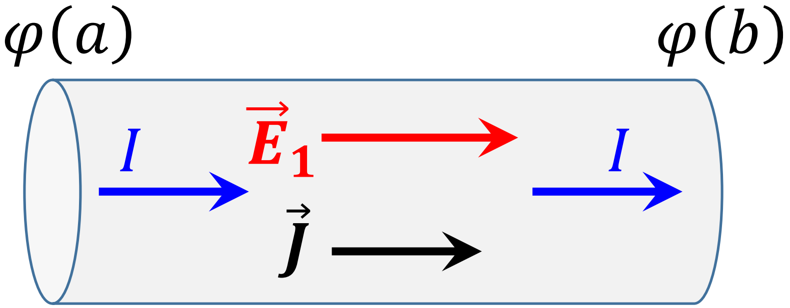
In a piece of conductor



Slide #8 in E_Lecture 9&10_Dielectric

$$\Delta\varphi = \varphi(b) - \varphi(a) = E(d-b) = \frac{\sigma_0}{\varepsilon_0}(d-b)$$

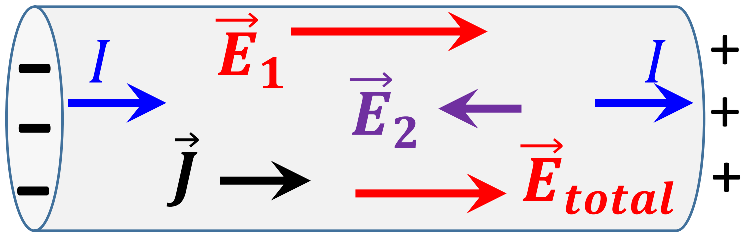
Conservative field $\vec{E}_1$ inside a piece of conductor causes current



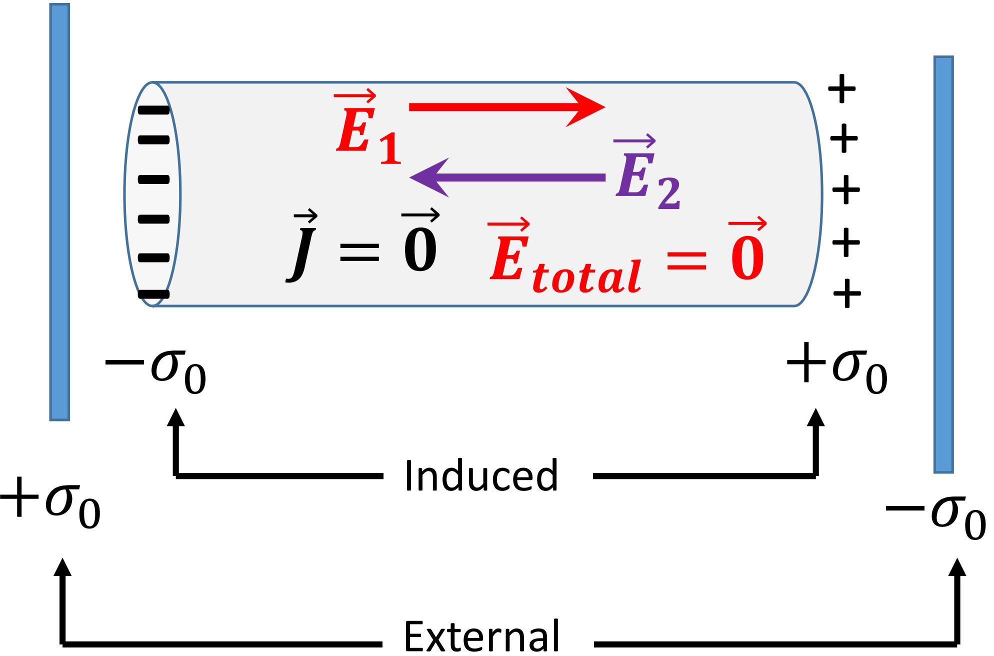
$$\int_a^b \vec{E}.d\vec{l} = \varphi(b) - \varphi(a) \neq 0 \Rightarrow \frac{1}{\sigma_e}\int_a^b \vec{J}.d\vec{l} = \varphi(b) - \varphi(a) \neq 0$$

$$\rho_e \int_a^b \frac{I}{A}\vec{n}.d\vec{l} = \varphi(b) - \varphi(a) = \rho_e \frac{I}{A} L \qquad R = \rho_e \frac{L}{A} \Rightarrow \boxed{\varphi(a) - \varphi(b) = RI}$$

Problem !

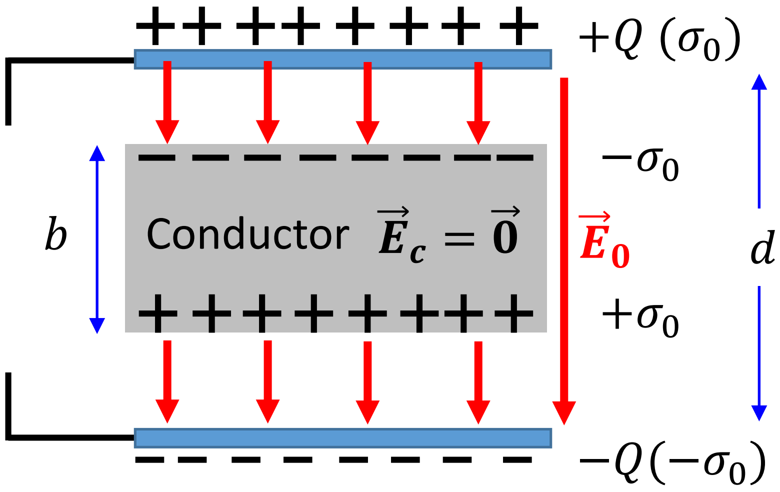


Conservative field $\vec{E}_1$ inside conductor causes charge to build up at ends producing opposing field $\vec{E}_2$ reducing the current



After a very short time $\vec{E}_1$ and $\vec{E}_2$ have the same and the total field inside the conductor is $\vec{E} = \vec{0}$

$$\frac{1}{\sigma_e}\int_a^b \vec{J}.d\vec{l} = \varphi(b) - \varphi(a) = 0 \Rightarrow \boxed{\vec{J} = \vec{0}}$$



Slide #8 in E_Lecture 9&10_Dielectric

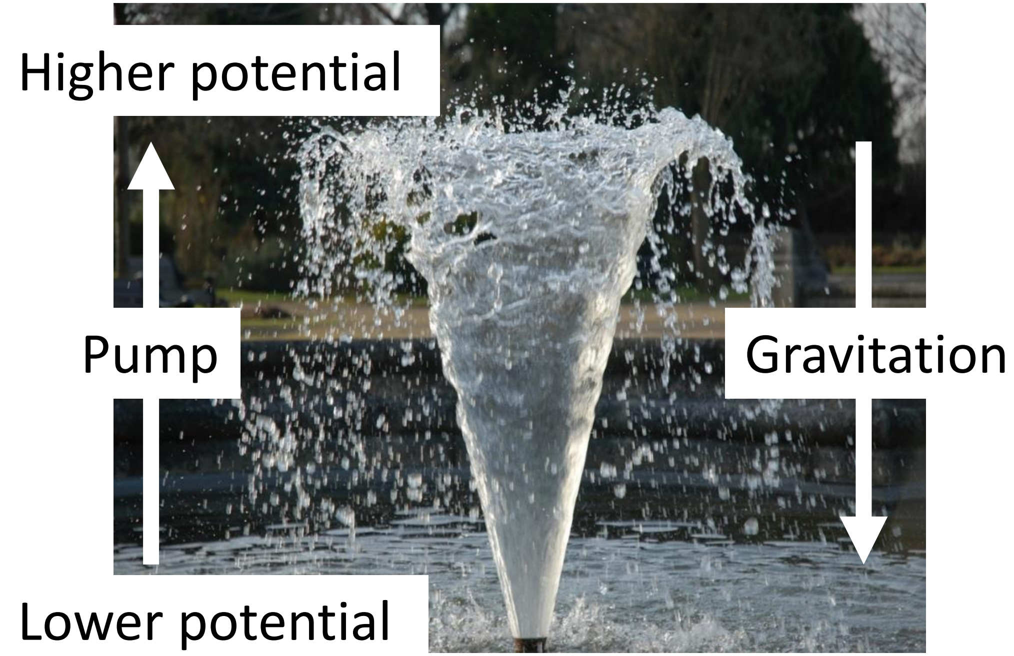
Although collisions with atoms still takes place, the current stops because of the opposing electric field built-up in the conductor

Solution ! Energy must be supplied by an external **non conservative** field

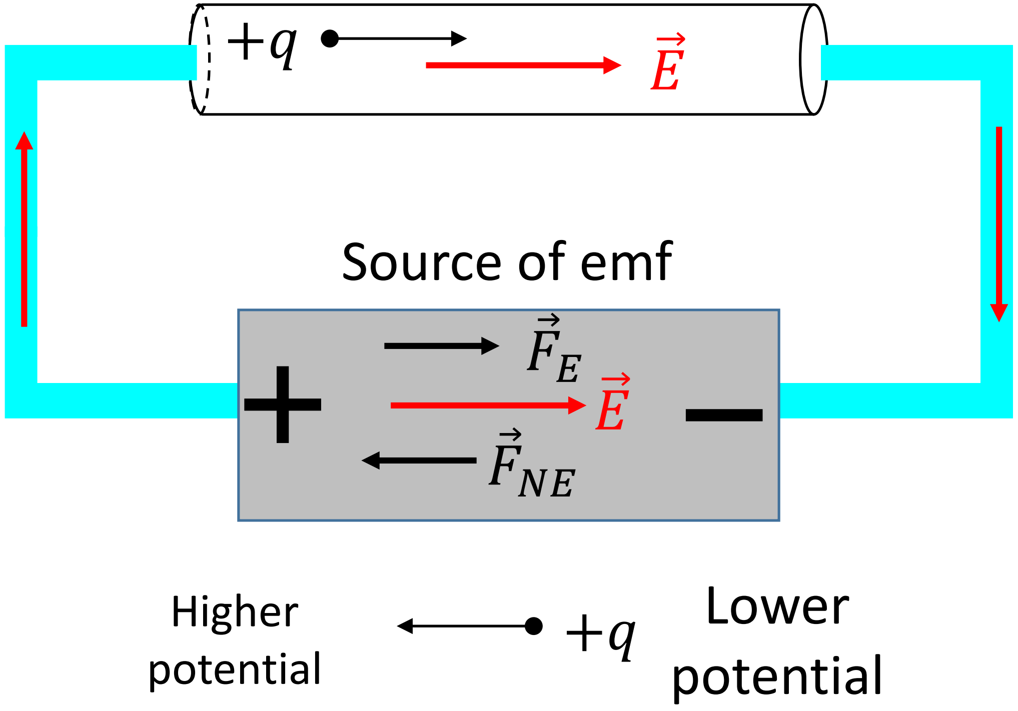
Fountain → Requires a pump to sustain a steady water flow

Electric circuit → Requires an electromotive to sustain a steady current

Electromotive force: **emf** $\mathcal{E}$



Pump does work against gravitation



Non electrostatic force $\vec{F}_{NE}$ does work against the electrostatic force $\vec{F}_E$. For an **ideal** source of emf:

$$\vec{F}_{NE} + \vec{F}_E = \vec{0}$$

$$\boxed{\vec{J} = \sigma_e \vec{E} = \frac{1}{\rho_e} \vec{E}}$$

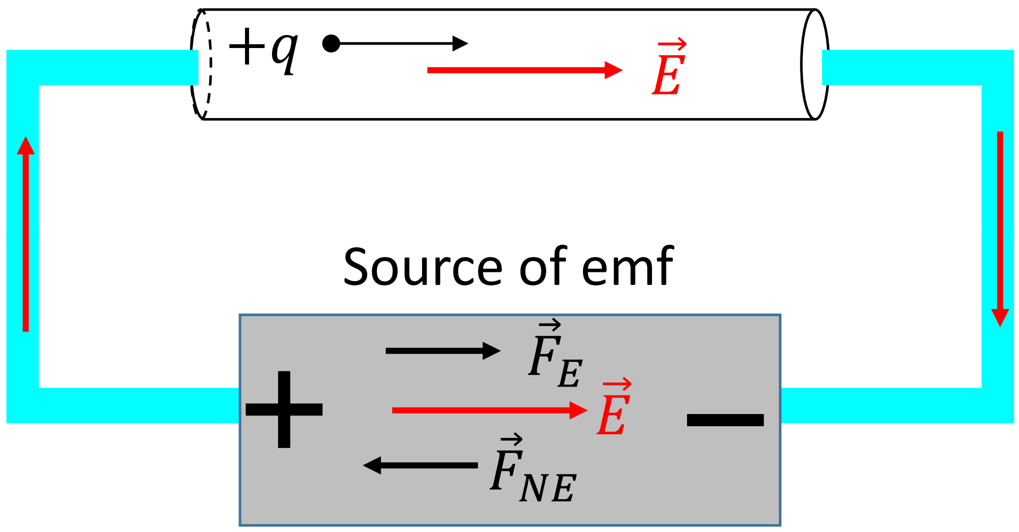
$$[\rho_e] = \Omega m$$

$$[\sigma_e] = (\Omega m)^{-1}$$

$$\boxed{\rho_e = \frac{1}{\sigma_e} = R\frac{A}{L}}$$

$\sigma_e = 0 \Rightarrow \rho_e = \infty \Rightarrow$ perfect insulator

$\sigma_e = \infty \Rightarrow \rho_e = 0 \Rightarrow$ perfect conductor



During the motion of $+q$ in the external circuit its potential energy **decreases**

In the source (pump) the potential energy of $+q$ **increases**

Non electrostatic force $\vec{F}_{NE}$

- **Battery:** from chemical to electrical
- **Induction:** from magnetic to electrical
- **Generator:** from mechanical to electrical
- **Thermocouple:** from thermal to electrical
- **Photovoltaic:** from light to electrical

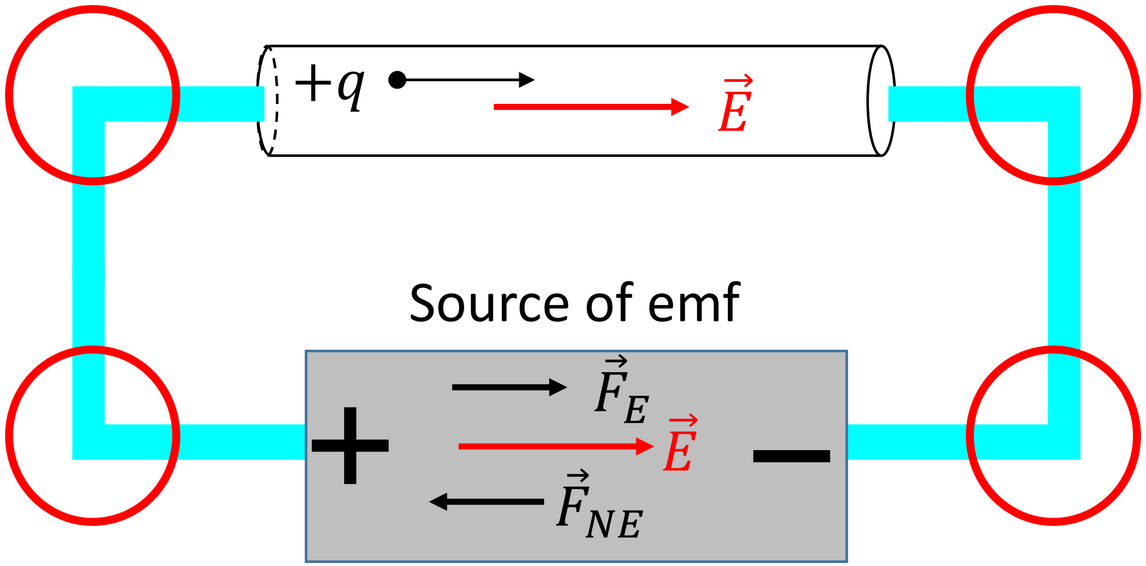
A positive Work is done by $\vec{F}_{NE}$ against the electric force in the source, increasing the potential

$$W(\vec{F}_{NE}) + W(\vec{F}_E) = 0 \quad \Rightarrow \quad \frac{W}{q} = \int \vec{E}_{NE}.d\vec{l} = -\int \vec{E}_E.d\vec{l} = \underset{\substack{\uparrow \\ \boldsymbol{emf}}}{\mathcal{E}} = \varphi(b) - \varphi(a) = \Delta\varphi$$

The potential energy of the charge $+q$ is brought back to its initial value

$$\boxed{\mathcal{E} = \Delta\varphi = RI}$$

Question:



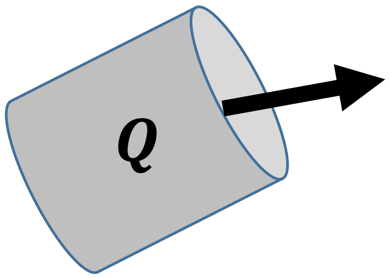
1) Where the wire bends, what causes the current to follow the bends?

2) According to the definition of the current density: $\vec{J} = Nq\vec{v}_d$, the moving charges must accelerate at the bends. Can we still consider the current as steady?

2) What happens to the emf in the real case where there is dissipation inside the source?

Equation of continuity and Kirchhoff's current law

Principle of conservation of charge: **A charge can neither be created not destroyed**



Volume V bounded by a closed surface A

Net current flows out ⇒ Net flux of $\vec{J}$ out of the closed surface

⇒ Net decrease of Q inside the closed surface

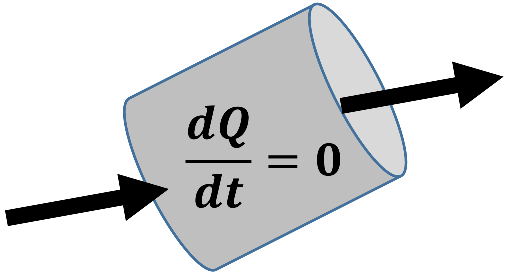
$$Q = \int_V \rho dV$$

$$I = \oint_A \vec{J}.d\vec{A} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho dV$$

Divergence or Gauss's theorem ⇒

$$\boxed{\int_V \vec{\nabla}.\vec{J} dV = -\int_V \frac{\partial \rho}{\partial t} dV}$$

Equation of continuity $\boxed{\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}}$ → For steady current $\boxed{\vec{\nabla}.\vec{J} = 0}$



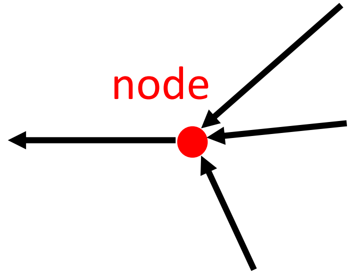
$$I = \oint_A \vec{J}.d\vec{A} = -\frac{d}{dt}\int_V \rho dV = 0$$

Kirchhoff's current law

Kirchhoff's current and potential laws

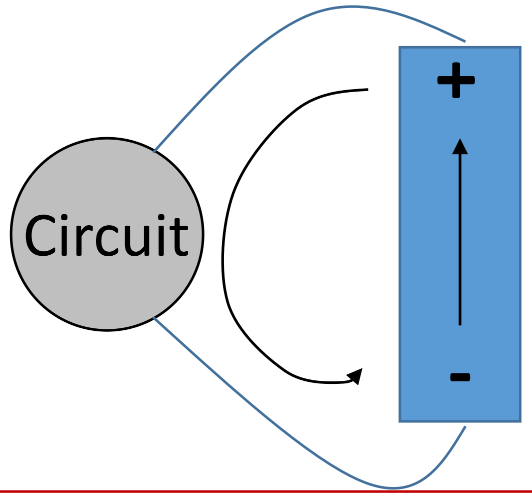
$$\oint \vec{J}\,.\,d\vec{A} = 0$$

Closed A



$$\sum_j I_j(node) = 0$$

Conservation of charge = no current can accumulate at a junction

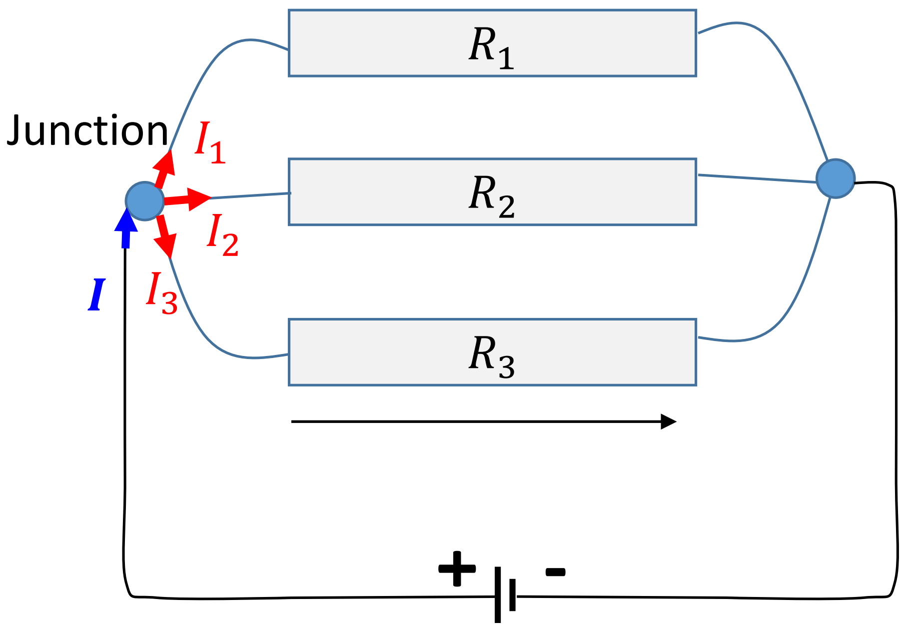


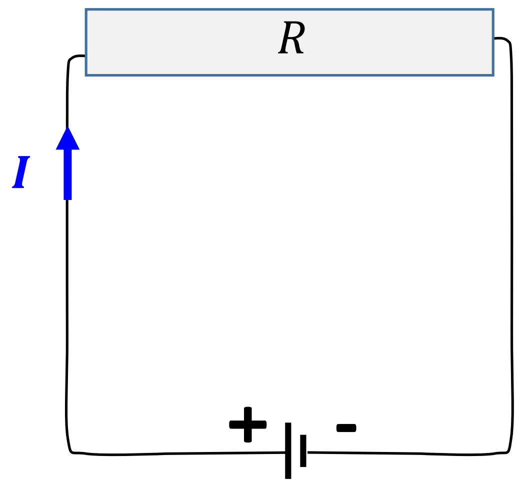
Emf source

In the source we go from – to +
Outside the source we go from + to -

$$\sum_i \Delta\varphi_i = 0$$

Application: resistances in parallel





$\mathcal{E} = RI$

$\mathcal{E} = R_1 I_1$

$\mathcal{E} = R_2 I_2$ **+** $\boxed{\sum_j I_j(node) = 0}$ $\Rightarrow$ $\frac{\mathcal{E}}{R_1} + \frac{\mathcal{E}}{R_2} + \frac{\mathcal{E}}{R_3} = \frac{\mathcal{E}}{R}$ $\Rightarrow$ $\boxed{\sum_j \frac{1}{R_i} = \frac{1}{R}}$

$\mathcal{E} = R_3 I_3$

For resistances in series we use the second Kirchhoff's law

$$\boxed{\sum_i \Delta\varphi_i = 0}$$

Interesting consequence of the continuity equation

$$\boxed{\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}}$$

$$\vec{J} = \sigma_e \vec{E}$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon}$$

Gauss's law in electrostatic (always valid even when charges are moving)

Will be proved later

$$\sigma_e \vec{\nabla}.\vec{E} = -\frac{\partial \rho}{\partial t} \Rightarrow \sigma_e \frac{\rho}{\varepsilon} = -\frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma_e}{\varepsilon}\rho = 0$$

$$\boxed{\rho = \rho_0 e^{-\left(\sigma_e/\varepsilon\right)t}}$$

ρ_0	Initial charge density
$\rho(t)$	Time dependent charge density
$\sigma_e = 1/\rho_e$	σ_e is the conductivity and ρ_e is the resistivity

For a good conductor $\sigma_e \approx 10^7\ (\Omega m)^{-1}, \varepsilon = \varepsilon_0 = 8.85 \times 10^{-12} F m^{-1}$

$$\rho = \rho_0 e^{-\left(\sigma_e/_{\varepsilon}\right)t} \Rightarrow \boxed{\tau = \frac{\varepsilon}{\sigma_e} \approx 10^{-18} s}$$

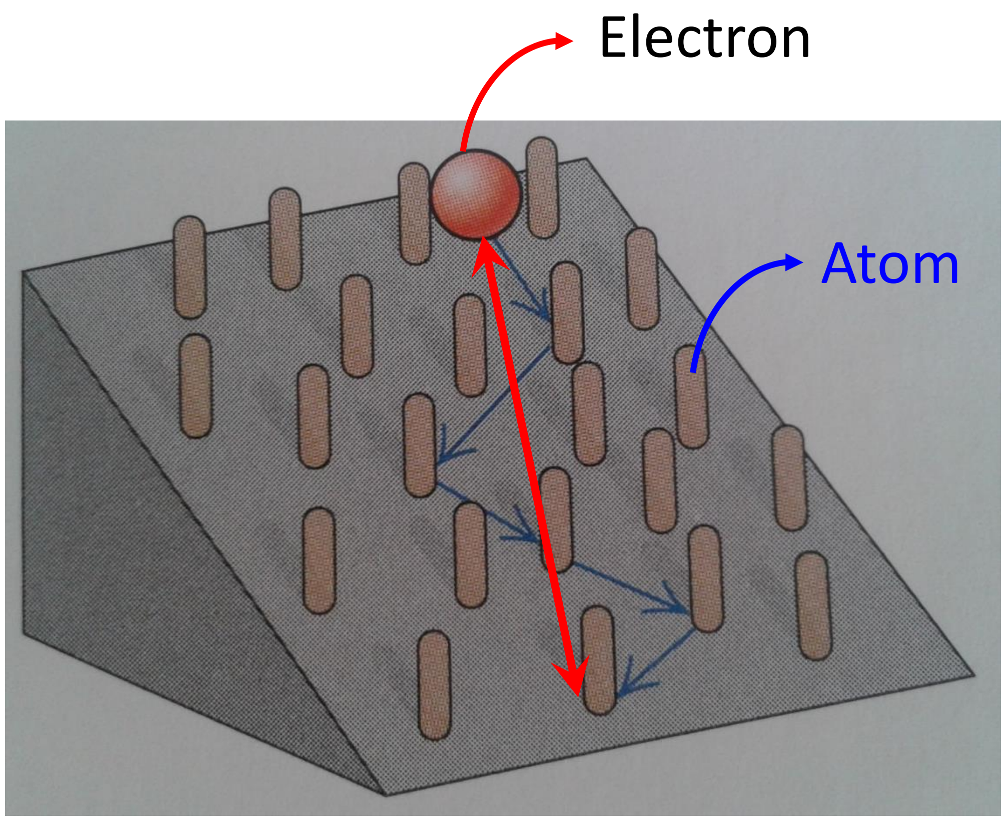
Relaxation time or time required for equilibrium to be established

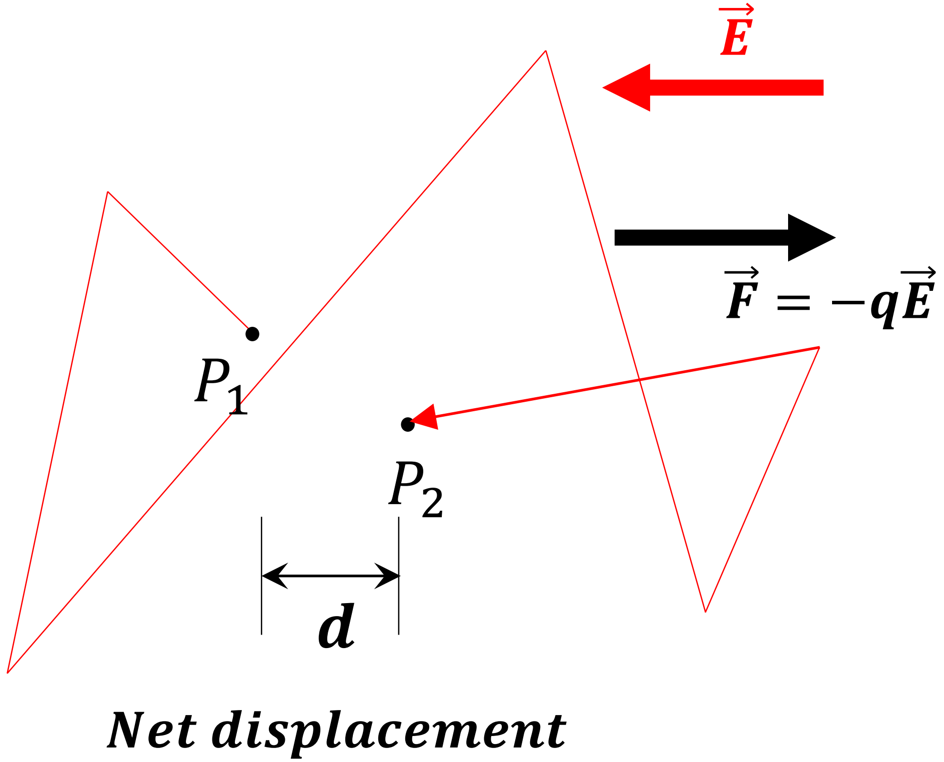
From seconds to hours or days in semiconductors and insulators

Inside conductors the field vanishes in not time and at the surface the same for the tangential component

BUT this is not the whole story !

Theory of metallic conduction





Motion of a ball rolling an inclined plane and bouncing off pegs in its path

From University Physics, 11th edition

Applying an external field tends to accelerate the electrons according to : $\vec{F} = m\vec{a} = q\vec{E}$

But the multiple collisions tend to slow down the electrons

$\vec{v} = \vec{v}_0 + \vec{a}\tau$ τ = average time between collision

$$\langle \vec{v} \rangle = \langle \underbrace{\vec{v}_0 \rangle}_{=0} + \vec{a}\langle \tau \rangle \Rightarrow \langle \vec{v} \rangle = \vec{a}\langle \tau \rangle = \frac{q\langle \tau \rangle}{m}\vec{E}$$

$$\vec{J} = Nq\vec{v}_d = Nq\langle \vec{v} \rangle = \sigma_e \vec{E} \Rightarrow \langle \vec{v} \rangle = \frac{\sigma_e}{Nq}\vec{E}$$

$$\boxed{\langle \tau \rangle = \frac{m\sigma_e}{Nq^2}}$$

For copper: $N = 8.5 \times 10^{28}\ m^{-3}, \sigma_e = 5.8 \times 10^{7}\ (\Omega m)^{-1}, q = 1.6 \times 10^{-19} C, m = 9.11 \times 10^{-31} kg$

$\langle \tau \rangle = 2.4 \times 10^{-14} s$ ⇒ Each electron collides in average $1/\langle \tau \rangle = 4 \times 10^{14}\ times\ /s$

Power dissipation and Joule's law

During their motion in matter electrons collide with atoms $\Rightarrow$ They loose energy

$\vec{E}$ $\Rightarrow$ electrons $\Rightarrow$ atoms

"vector" for energy transmission

Collisions with atoms create heat
= thermal vibration or (phonons)

Work done by the field on moving charges

$$dw = q\vec{E}.d\vec{l} \Rightarrow$$ Power = energy /unit time $$p_i = \frac{dw}{dt} = q\vec{E}.\frac{d\vec{l}}{dt} = q\vec{E}.\vec{v}_i$$

$$\frac{dP}{dV} = \sum_i p_i = \vec{E}.\underbrace{\left(\sum_i N_i.q_i.v_i\right)}_{\vec{J}} \Rightarrow \frac{dP}{dV} = \vec{E}.\vec{J} \Rightarrow \boxed{P = \int_V \vec{E}.\vec{J}.dV}$$

Joule's law

$$P = \int E.JdV = \int_L Edl \int_A JdA$$

I

$\Delta\varphi = RI$

+ Ohm's law

$$\boxed{P = I^2R = \Delta\varphi I}$$

Summary for the equations governing **steady** current density

Differential form

$$\left.\begin{aligned}\vec{\nabla}.\vec{J} &= 0\\ \vec{J} &= \sigma_e \vec{E}\end{aligned}\right\} \quad \vec{\nabla}\times\vec{E} = 0 \quad \Rightarrow \quad \vec{\nabla}\times\left(\frac{\vec{J}}{\sigma_e}\right) = 0$$

Integral form

$$\oint_A \vec{J}.d\vec{A} = 0 \qquad\qquad \oint_\Gamma \frac{\vec{J}}{\sigma_e}.d\vec{l} = 0$$

Boundary conditions are trivial

1) Through any interface, the tangential components of a curl-free vector field is continuous

$$\vec{\nabla} \times \left(\frac{\vec{J}}{\sigma_e} \right) = 0$$

$\frac{\vec{J}}{\sigma_e}$ is a curl free-vector field $\Rightarrow$ $\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}}$

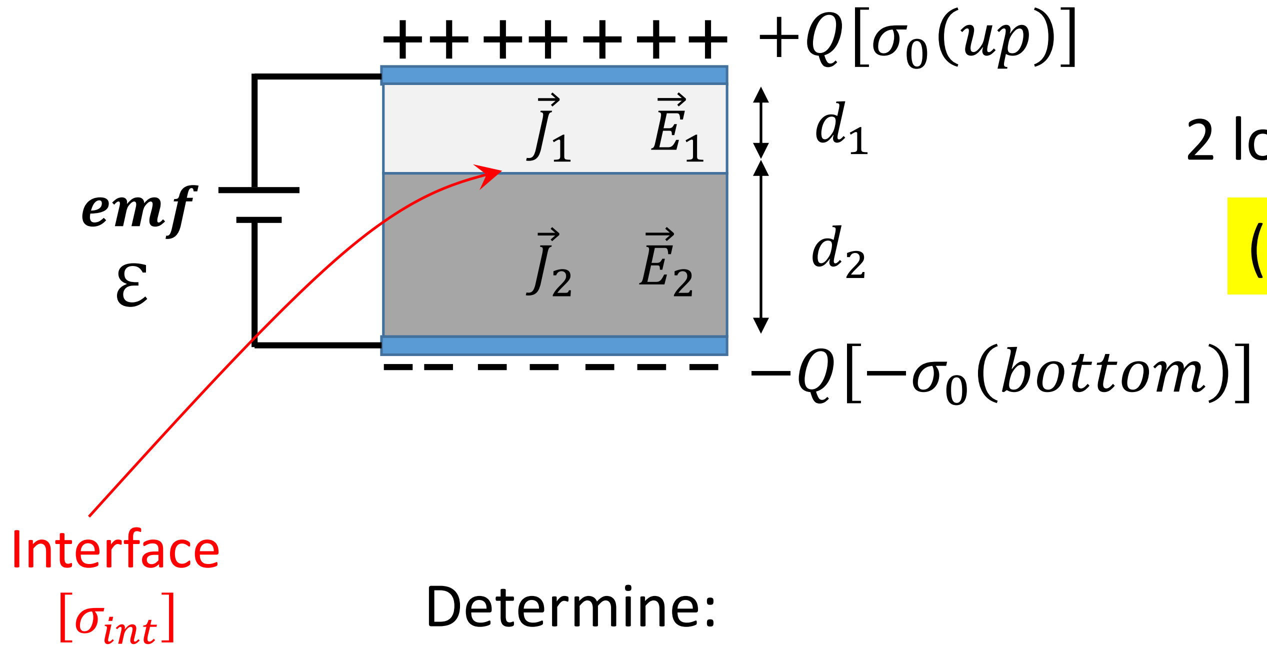
2) Through any interface, the normal components of a divergence-free vector field is continuous

$$\boxed{J_{1n} = J_{2n}}$$

Analogy with vector field $\vec{D}$

When no free charges at the interface $\boxed{D_{1n} = D_{2n}}$

Example of using boundary conditions (see Cheng's book)



2 lossy dielectrics $\varepsilon_1, \sigma_{e1}$ and $\varepsilon_2, \sigma_{e2}$

(Why lossy ?)

Determine:

1) Current density between the conducting plates
2) The electric field in each lossy dielectric
3) Surface charge densities on the plates and at the interface

1) Current density between the conducting plates

$$\mathcal{E} = (R_1 + R_2)I = \left(\frac{d_1}{\sigma_{e1}A} + \frac{d_2}{\sigma_{e2}A}\right)I = \left(\frac{d_1}{\sigma_{e1}} + \frac{d_2}{\sigma_{e2}}\right)\underbrace{\frac{I}{A}}_{J}$$

$$\frac{1}{\sigma_e} = R\frac{A}{d}$$

$$\Rightarrow \quad \boxed{J = \frac{\sigma_{e1}\sigma_{e2}\mathcal{E}}{\sigma_{e1}d_2 + \sigma_{e2}d_1}}$$

What would be J if at least one of the dielectric were not lossy (perfect dielectric)?

2) The electric field in each lossy dielectric

2 electric fields are to be found thus two equations are needed

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = E_1 d_1 + E_2 d_2$$

Through any interface, the normal components of a <u>divergence-free vector field</u> is continuous

$$J_{1n} = J_{2n} \Rightarrow \sigma_{e1} E_{1n} = \sigma_{e2} E_{2n} \qquad E_{1n} = E_1 \qquad E_{2n} = E_2$$

$$\Rightarrow \boxed{E_1 = \frac{\sigma_{e2}\mathcal{E}}{\sigma_{e1} d_2 + \sigma_{e2} d_1}} \qquad \boxed{E_2 = \frac{\sigma_{e1}\mathcal{E}}{\sigma_{e1} d_2 + \sigma_{e2} d_1}}$$

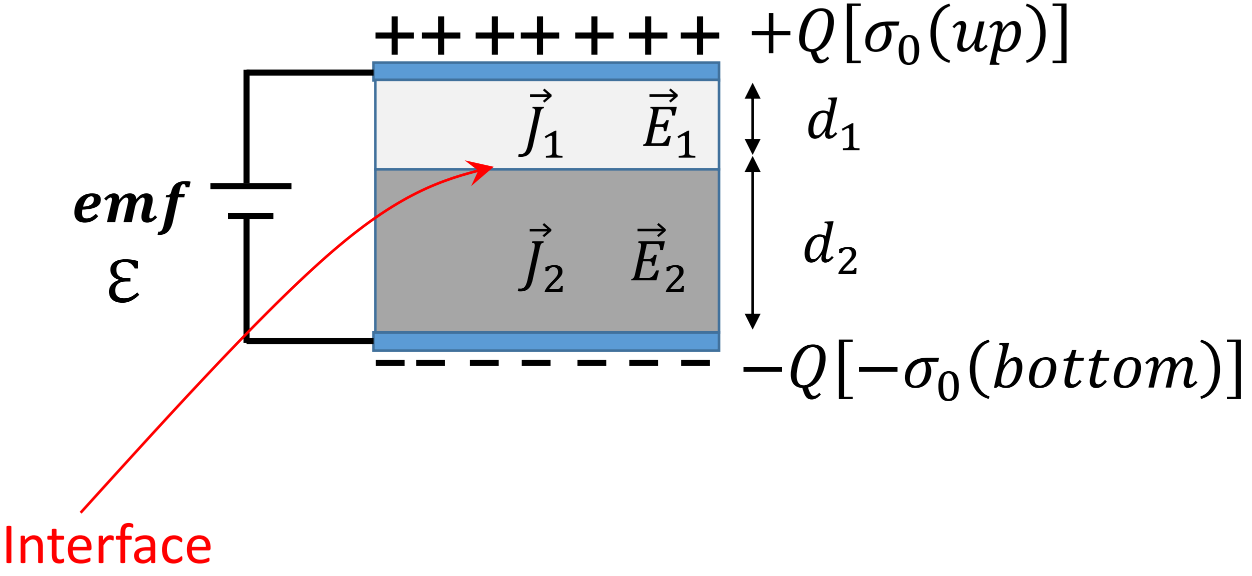
What would be field if both dielectric were identical? $E_1 = E_2 = \mathrm{E} = \dfrac{\mathcal{E}}{d}$ $\quad d = d_1 + d_2$

3) Surface charge densities on the plates and at the interface σ_s's

a) Surface charge density on the plates

Use boundary conditions at the interfaces:

- *Dielectric 1 / upper conducting plate*
- *Dielectric 2 / bottom conducting plate*



$$\sigma_0(up) = \varepsilon_1 E_{1n} = D_{1n} \quad \Rightarrow \quad \sigma_0(up) = \frac{\varepsilon_1 \sigma_{e2} \mathcal{E}}{\sigma_{e1} d_2 + \sigma_{e2} d_1} \quad C/m^2$$

$$\sigma_0(bottom) = \varepsilon_2 E_{2n} = D_{1n} \quad \Rightarrow \quad \sigma_0(bottom) = -\frac{\varepsilon_2 \sigma_{e1} \mathcal{E}}{\sigma_{e1} d_2 + \sigma_{e2} d_1} \quad C/m^2$$

b) Surface charge density at the interface **[σ_{int}'s]**

A charge is induced at the interface because the two dielectrics are not the same ***AND are lossy***

Again we use boundary conditions at the interface dielectric 1 / dielectric 2

$$D_{1n} - D_{2n} = \sigma_{int} \qquad \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \sigma_{int}$$

$$\boxed{\sigma_{int} = \frac{(\varepsilon_2 \sigma_{e1} - \varepsilon_1 \sigma_{e2})\mathcal{E}}{\sigma_{e1} d_2 + \sigma_{e2} d_1}}$$

$$\sigma_0(up) + \sigma_0(bottom) + \sigma_{int} = 0$$