RC for Mid 1
Chapter 3

Outline

- Some tips
- Coulomb’ s Law & Gauss’ s Law
- Electric Potential
- Conductors & Dielectrics in Static Electric Field
- Boundary Condition
- Capacitance
- Energy
- Force & Torque

Fundamental Postulates

Postulates of Electrostatics in Free Space	
Differential Form	**Integral Form**
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$

Gauss’ s Law

Coulomb’ s Law

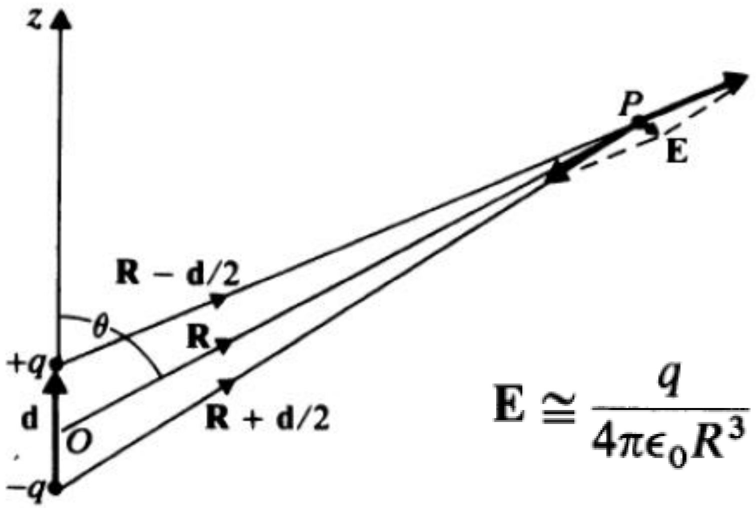
1. Electric Field due to a System of Discrete Charges

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R}-\mathbf{R}')}{4\pi\epsilon_0|\mathbf{R}-\mathbf{R}'|^3} \qquad \text{(V/m)}.}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k(\mathbf{R}-\mathbf{R}'_k)}{|\mathbf{R}-\mathbf{R}'_k|^3} \qquad \text{(V/m)}.}$$

$$\boxed{\vec{\mathbf{F}}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R\frac{q_1q_2}{4\pi\epsilon_0R^2} \qquad \text{(N)}.}$$

Electric Dipole (important!)



$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right].$$

Dipole moment:

Definition - The product of the charge q and the vector d

$$\mathbf{p} = q\mathbf{d}.$$

2. For a surface or line charge

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m)}.$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m)},$$

Summing scalars is easier than summing vector!!!

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

To calculate the field

Potential:

$V(\vec{r}) = \sum_{i=1} \frac{q_i}{4\pi\varepsilon_0\|\vec{r}-\vec{r_i}\|}$	Point Charges
$V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0\|\vec{r}-\vec{r_s}\|} dl_s$	Line Charges
$V(\vec{r}) = \iint_S \frac{\sigma(\vec{r_s})}{4\pi\varepsilon_0\|\vec{r}-\vec{r_s}\|} dS_s$	Surface Charges
$V(\vec{r}) = \iiint_V \frac{\rho(\vec{r_s})}{4\pi\varepsilon_0\|\vec{r}-\vec{r_s}\|} dV_s$	Volume charges

Gauss' s Law

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m)}.$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m)},$$

or

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.$$

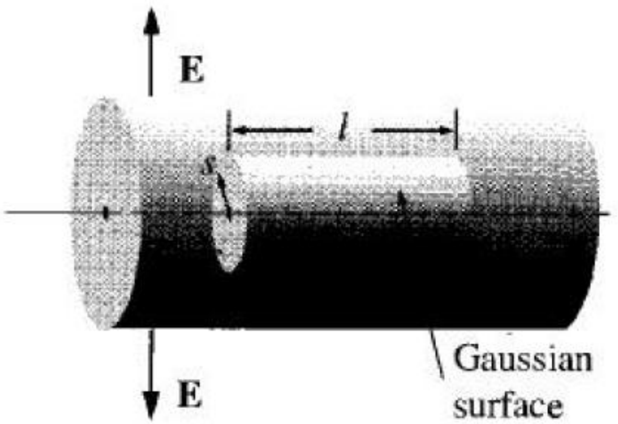
S : can be any hypothetical closed surface

⋆When Gauss' s Law is useful

⋆The examples on the lecture slides

Practice

A long cylinder (Fig. 2.21) carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.



$$\mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$$

Electric Potential

$\nabla \times \mathbf{E} = 0 \Rightarrow \boxed{\mathbf{E} = -\nabla V}$

Calculation !

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad (\mathrm{V}).}$$

In going **against** the E field the electric potential V **increases**

V(*R*) of **a point charge at origin**

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(V)}.}$$

With reference point at infinity

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \right) \cdot (\mathbf{a}_R \, dR),$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \qquad \text{(V)}.}$$

Potential difference between any two points

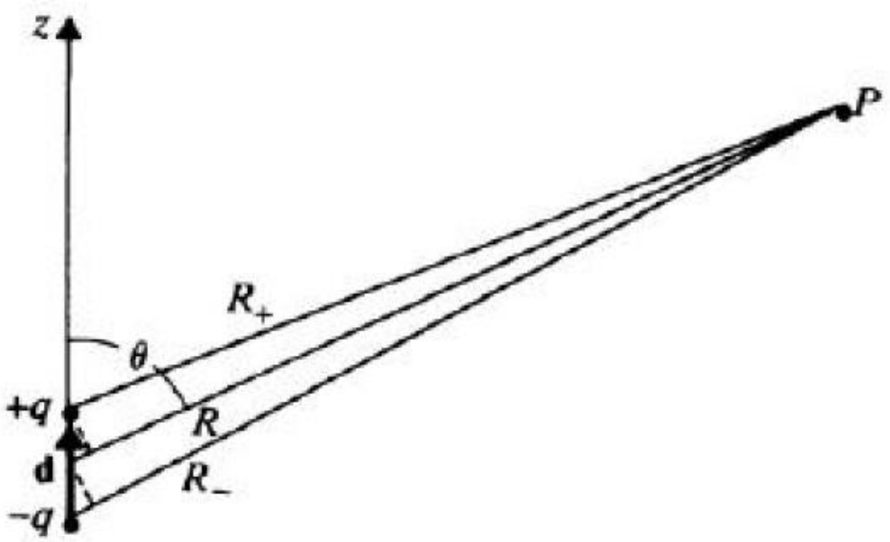
$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_1} \right).$$

A charge is moved from P_1 to P_2 (against the E field if $V_{21}>0$)

V due to *n* Discrete Point Charges

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \qquad \text{(V)}.}$$

V due to dipole



$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},}$$

where $\mathbf{p} = q\mathbf{d}$.

$\boxed{\mathbf{d}\cdot\mathbf{a}_R = d\cos\theta}$

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2} \Rightarrow \mathbf{E} = -\nabla V = -\mathbf{a}_R\frac{\partial V}{\partial R} - \mathbf{a}_\theta\frac{\partial V}{R\,\partial\theta}$$

$$= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta).$$

Conductors in Static Electric field

Inside a Conductor (Under Static Conditions)
$\rho = 0$
$\mathbf{E} = 0$

By Gauss's law

At a state of equilibrium

The tangential component of the E field on a conductor surface is zero.

Boundary Conditions at a Conductor/Free Space Interface
$E_t = 0$
$E_n = \dfrac{\rho_s}{\epsilon_0}$

***Eg 3-11**

EXAMPLE 3–13 Consider two spherical conductors with radii b_1 and b_2 ($b_2 > b_1$) that are connected by a conducting wire. The distance of separation between the conductors is assumed to be very large in comparison to b_2 so that the charges on the spherical conductors may be considered as uniformly distributed. A total charge Q is deposited on the spheres. Find (a) the charges on the two spheres, and (b) the electric field intensities at the sphere surfaces.

Solution

a) Refer to Fig. 3–22. Since the spherical conductors are at the same potential, we have

$$\frac{Q_1}{4\pi\epsilon_0 b_1} = \frac{Q_2}{4\pi\epsilon_0 b_2}$$

or

$$\frac{Q_1}{Q_2} = \frac{b_1}{b_2}.$$

Hence the charges on the spheres are directly proportional to their radii. But, since

$$Q_1 + Q_2 = Q,$$

we find that

$$Q_1 = \frac{b_1}{b_1 + b_2} Q \qquad \text{and} \qquad Q_2 = \frac{b_2}{b_1 + b_2} Q.$$

b) The electric field intensities at the surfaces of the two conducting spheres are

$$E_{1n} = \frac{Q_1}{4\pi\epsilon_0 b_1^2} \qquad \text{and} \qquad E_{2n} = \frac{Q_2}{4\pi\epsilon_0 b_2^2},$$

so

$$\frac{E_{1n}}{E_{2n}} = \left(\frac{b_2}{b_1}\right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1}.$$

The electric field intensities are therefore inversely proportional to the radii, being higher at the surface of the smaller sphere which has a larger curvature.

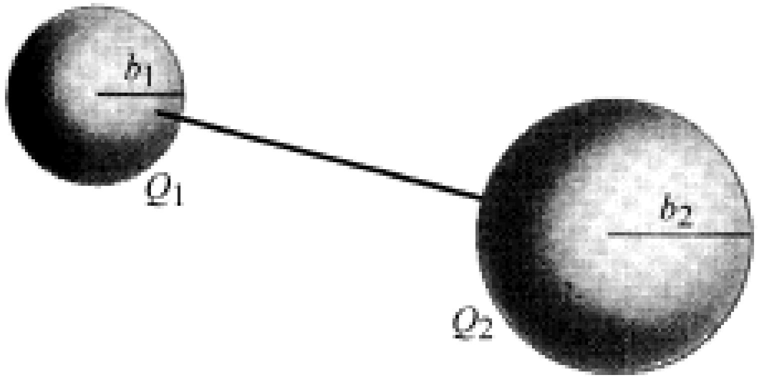


FIGURE 3–22
Two connected conducting spheres (Example 3–13).

Dielectrics in Static Electric field

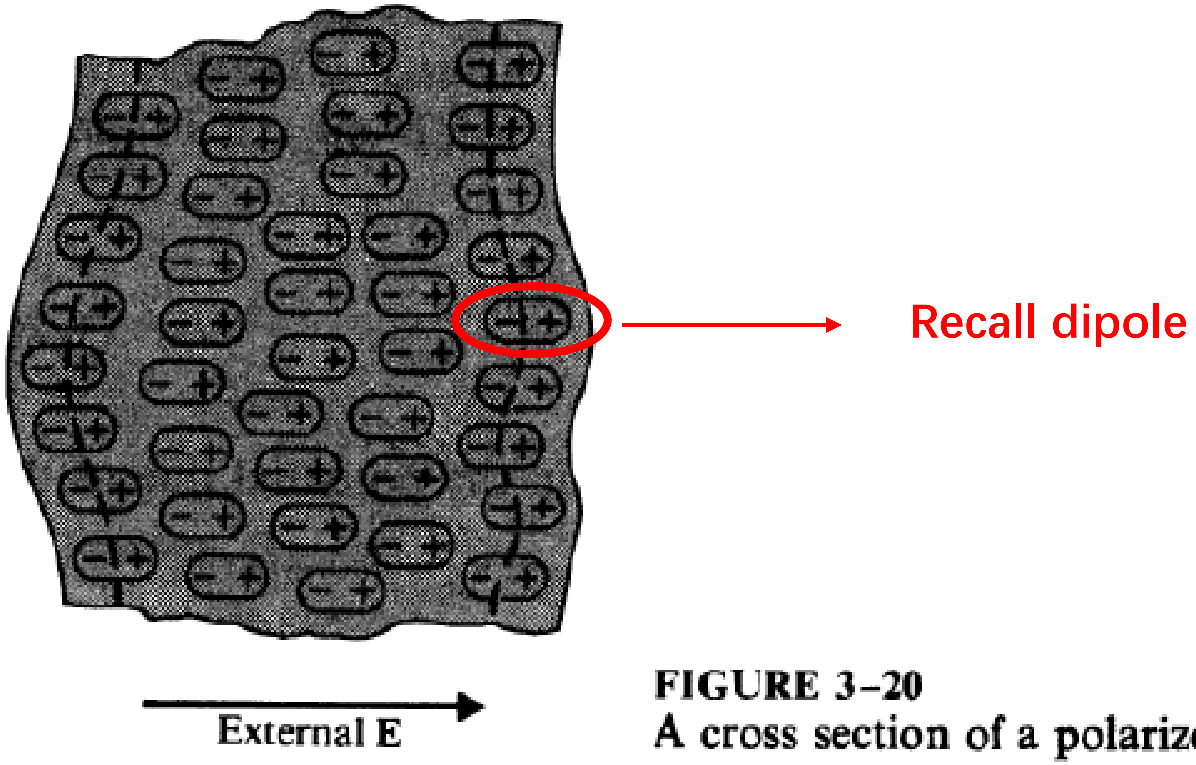


FIGURE 3–20
A cross section of a polarized dielectric medium.

Polarization

Surface Charge Distribution ← $\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$

Volume Charge Distribution ← $\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$

Polarization charge densities, or bound-charge densities

- Polarization vector:

N = nΔv,
where N is the Total # in a volume (Δv);
n is the number density

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \qquad (\mathrm{C/m^2}),$$

p=qd
P=np=nqd

* **Lecture slides p15-18**

Modified Maxwell' s Equations

Equations of Electrostatics in Any Medium

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}),}$$

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

Important!

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).}$$

Where D: electric flux density, electric displacement

Permittivity

– For linear and isotropic medium,

$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E},$ χ_e dimensionless quantity called ***electric susceptibility***

$$\boxed{\begin{aligned}\mathbf{D} &= \epsilon_0(1 + \chi_e)\mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}),\end{aligned}}$$

Relative permittivity (dielectric constant) ← $\epsilon_r = 1 + \chi_e = \dfrac{\epsilon}{\epsilon_0}$

ε: absolute permittivity (or simply permittivity)

***Eg 3-12**

Anisotropic Medium

The ε_r is different for different directions of the electric field

- **D** and **E** vectors generally have different directions
- $\overline{\overline{\varepsilon}}$ is a tensor

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

Biaxial: $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

$$D_x = \epsilon_1 E_x, \quad D_y = \epsilon_2 E_y, \quad D_z = \epsilon_3 E_z.$$

Uniaxial: $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$

EXAMPLE 3–12 A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine $\mathbf{E}$, V, $\mathbf{D}$, and $\mathbf{P}$ as functions of the radial distance R.

Solution The geometry of this problem is the same as that of Example 3–11. The conducting shell has now been replaced by a dielectric shell, but the procedure of solution is similar. Because of the spherical symmetry, we apply Gauss's law to find $\mathbf{E}$ and $\mathbf{D}$ in three regions: (a) $R > R_o$; (b) $R_i < R < R_o$; and (c) $R < R_i$. Potential V is found from the negative line integral of $\mathbf{E}$, and polarization $\mathbf{P}$ is determined by the relation

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \epsilon_0(\epsilon_r - 1)\mathbf{E}. \tag{3–107}$$

The $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{P}$ vectors have only radial components. Refer to Fig. 3–21(a), where the Gaussian surfaces are not shown in order to avoid cluttering up the figure.

a) $R > R_o$
The situation in this region is exactly the same as that in Example 3–11. We have, from Eqs. (3–73) and (3–74),

$$E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$V_1 = \frac{Q}{4\pi\epsilon_0 R}.$$

From Eqs. (3–102) and (3–107) we obtain

$$D_{R1} = \epsilon_0 E_{R1} = \frac{Q}{4\pi R^2} \tag{3–108}$$

and

$$P_{R1} = 0. \tag{3–109}$$

b) $R_i < R < R_o$

The application of Gauss's law in this region gives us directly

$$E_{R2} = \frac{Q}{4\pi\epsilon_0\epsilon_r R^2} = \frac{Q}{4\pi\epsilon R^2}, \tag{3-110}$$

$$D_{R2} = \frac{Q}{4\pi R^2}, \tag{3-111}$$

$$P_{R2} = \left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R^2}. \tag{3-112}$$

Note that D_{R2} has the same expression as D_{R1} and that both E_R and P_R have a discontinuity at $R = R_o$. In this region,

$$\begin{aligned} V_2 &= -\int_{\infty}^{R_o} E_{R1}\, dR - \int_{R_o}^{R} E_{R2}\, dR \\ &= V_1\Big|_{R=R_o} - \frac{Q}{4\pi\epsilon}\int_{R_o}^{R}\frac{1}{R^2}\, dR \\ &= \frac{Q}{4\pi\epsilon_0}\left[\left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_o} + \frac{1}{\epsilon_r R}\right]. \end{aligned} \tag{3-113}$$

c) $R < R_i$

Since the medium in this region is the same as that in the region $R > R_o$, the application of Gauss's law yields the same expressions for E_R, D_R, and P_R in both regions:

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2},$$

$$D_{R3} = \frac{Q}{4\pi R^2},$$

$$P_{R3} = 0.$$

To find V_3, we must add to V_2 at $R = R_i$ the negative line integral of E_{R3}:

$$\begin{aligned} V_3 &= V_2\Big|_{R=R_i} - \int_{R_i}^{R} E_{R3}\, dR \\ &= \frac{Q}{4\pi\epsilon_0}\left[\left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_o} - \left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]. \end{aligned} \tag{3-114}$$

The variations of $\epsilon_0 E_R$ and D_R versus R are plotted in Fig. 3-21(b). The difference $(D_R - \epsilon_0 E_R)$ is P_R and is shown in Fig. 3-21(c). The plot for V in Fig. 3-21(d) is a composite graph for V_1, V_2, and V_3 in the three regions. We note that D_R is a continuous curve exhibiting no sudden changes in going from one medium to another and that P_R exists only in the dielectric region. ∎

Boundary Conditions

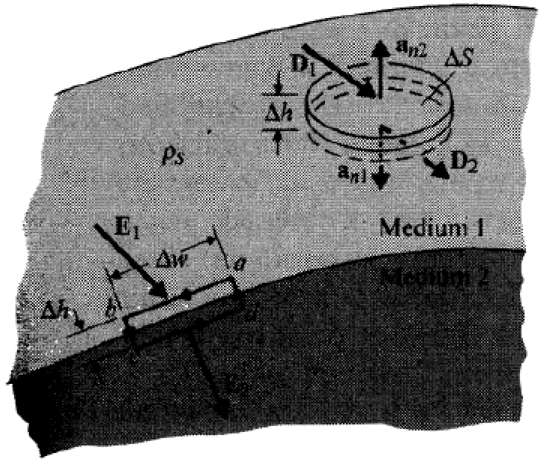


FIGURE 3–23
An interface between two media.

Tangential components,	$E_{1t} = E_{2t}$;
Normal components,	$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$.

1. For a dielectric (Medium 1)/conductor (Medium 2) interface:

$$\mathbf{D}_2 = 0 \Rightarrow D_{1n} = \epsilon_1 E_{1n} = \rho_s,$$

2. For no charge existing at the interface

$$\rho_s = 0, \Rightarrow \begin{aligned} D_{1n} &= D_{2n} \\ \epsilon_1 E_{1n} &= \epsilon_2 E_{2n}. \end{aligned}$$

Capacitance

The ratio Q/V unchanged

$$\boxed{Q = CV,}$$

C: capacitance (C/V, or Farad)

Capaticitor

E ⊥ conductor surfaces (equipotential surfaces)

$$\boxed{C = \frac{Q}{V_{12}} \qquad \text{(F).}}$$

Series

$$\boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}.}$$

Parallel

$$\boxed{C_{||} = C_1 + C_2 + \cdots + C_n.}$$

$$\begin{aligned}
V_1 &= p_{11}Q_1 + p_{12}Q_2 + \cdots + p_{1N}Q_N, \\
V_2 &= p_{21}Q_1 + p_{22}Q_2 + \cdots + p_{2N}Q_N, \\
&\vdots \\
V_N &= p_{N1}Q_1 + p_{N2}Q_2 + \cdots + p_{NN}Q_N.
\end{aligned}$$

p_{ij}: **coefficients of potential**; depends on
1. Shape and position of the conductor
2. Permittivity of surroundings

$$\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\
Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\
&\vdots \\
Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,
\end{aligned}$$

c_{ii}: **coefficients of capacitance**
c_{ij}: **coefficients of induction (i≠j)**

Electrostatic Energy & Forces

A charge Q_1 in free space. Work required to bring a **second** charge Q_2 from infinity to a distance R_{12} (position 2): $W=Q_2V_{2\infty}=Q_2V_2$

$$W_2 = Q_2V_2 = Q_2\frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

Against E field of charge Q_1
(V_2 is due to charge Q_1)

Rewrite $W_2 = Q_2V_2 = Q_2\dfrac{Q_1}{4\pi\epsilon_0 R_{12}}$

$$W_2 = Q_1\frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1V_1.$$

$Q_1V_1=Q_2V_2$

➔ $Q_1V_1+Q_2V_2= 2Q_1V_1=2W_2$

$$W_2 = \tfrac{1}{2}(Q_1V_1 + Q_2V_2).$$

General Expression

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_kV_k \qquad \text{(J)},}$$

Potential V_k is caused by all the other charges

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\(j\neq k)}}^{N}\frac{Q_j}{R_{jk}}.$$

Energy Density

For a continuous charge distribution of density ρ

$$W_e = \frac{1}{2}\int_{\mathcal{V}'} \rho \underline{V}\, d\underline{v} \qquad \text{(J).}$$

Volume

Electrical potential

$$W_e = \frac{1}{2}\int_{\mathcal{V}'} (\nabla \cdot \mathbf{D}) V\, dv.$$

$$W_e = \frac{1}{2}\int_{\mathcal{V}'} \mathbf{D} \cdot \mathbf{E}\, dv \qquad \text{(J).}$$

$$W_e = \frac{1}{2}\int_{\mathcal{V}'} \epsilon E^2\, dv \qquad \text{(J)}$$

$$W_e = \frac{1}{2}\int_{\mathcal{V}'} \frac{D^2}{\epsilon}\, dv \qquad \text{(J).}$$

$$W_e = \int_{\mathcal{V}'} w_e\, dv.$$

$$w_e = \tfrac{1}{2}\mathbf{D} \cdot \mathbf{E} \qquad (\text{J/m}^3)$$

$$w_e = \tfrac{1}{2}\epsilon E^2 \qquad (\text{J/m}^3)$$

$$w_e = \frac{D^2}{2\epsilon} \qquad (\text{J/m}^3).$$

Force & Torque

Mechanical **work** done **by the system**:

$$dW = \mathbf{F}_Q \cdot d\ell,$$

$\mathbf{F_Q}$: total electric force acting on the body

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad (\mathrm{N}).}$$

$$\boxed{(T_Q)_z = -\frac{\partial W_e}{\partial \phi} \qquad (\mathrm{N \cdot m}).}$$

Practice

P.3–36 Determine the capacitance of an isolated conducting sphere of radius b that is coated with a dielectric layer of uniform thickness d. The dielectric has an electric susceptibility χ_e.

P.3–42 Find the electrostatic energy stored in the region of space $R > b$ around an electric dipole of moment $\mathbf{p}$.

$$\mathbf{E} = -\nabla V = -\mathbf{a}_R \frac{\partial V}{\partial R} - \mathbf{a}_\theta \frac{\partial V}{R\,\partial \theta}$$

$$= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta).$$

P3.36

Consider the electric field $\mathbf{E}_1$ in the region $b < R < b+d$.

$$\mathbf{E}_1 = \mathbf{a}_R \frac{Q}{4\pi \epsilon_0 \left(R^2 + X_e R^2\right)} \quad \ldots\ldots (1)$$

Where,

Electric susceptibility is denoted as X_e,

Absolute permittivity is denoted as ϵ_0,

Charge is denoted as Q.

Consider the electric field $\mathbf{E}_1$ in the region $R > b+d$.

$$\mathbf{E}_2 = \mathbf{a}_R \frac{Q}{4\pi \epsilon_0 R^2} \quad \ldots\ldots (2)$$

Consider the formula and find V.

$$V = -\int_{b+d}^{b} \mathbf{E}_1 \cdot d\mathbf{R} - \int_{\infty}^{b+d} \mathbf{E}_2 \cdot d\mathbf{R} \quad \ldots\ldots (3)$$

Substitute equations (1) and (2) in (3).

$$V = -\int_{b+d}^{b} \left(\mathbf{a}_R \frac{Q}{4\pi \epsilon_0 \left(R^2 + X_e R^2\right)}\right) \cdot d\mathbf{R} - \int_{\infty}^{b+d} \left(\mathbf{a}_R \frac{Q}{4\pi \epsilon_0 R^2}\right) \cdot d\mathbf{R}$$

$$= -\int_{b+d}^{b} \left(\frac{Q}{4\pi \epsilon_0 \left(R^2 + X_e R^2\right)}\right) dR - \int_{\infty}^{b+d} \left(\frac{Q}{4\pi \epsilon_0 R^2}\right) dR$$

$$= -\int_{b+d}^{b} \left(\frac{Q}{4\pi \epsilon_0 (1+X_e) R^2}\right) dR - \int_{\infty}^{b+d} \left(\frac{Q}{4\pi \epsilon_0 R^2}\right) dR$$

$$= -\left(\frac{Q}{4\pi \epsilon_0}\left[-\frac{1}{(1+X_e)R}\right]_{b+d}^{b} - \left[\frac{1}{R}\right]_{\infty}^{b+d}\right)$$

$$V = \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)R}\right]_{b+d}^{b} + \left[\frac{1}{R}\right]_{\infty}^{b+d}$$

$$= \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)(b)} - \frac{1}{(1+X_e)(b+d)}\right] + \left[\frac{1}{b+d}\right]$$

$$= \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{1}{b} - \frac{1}{b+d}\right)\right] + \left[\frac{1}{b+d}\right]$$

$$= \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{d}{b(b+d)}\right)\right] + \left[\frac{1}{b+d}\right]$$

$$V = \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{d}{b(b+d)} + \frac{1+X_e}{b+d}\right)\right]$$

$$= \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{d}{b(b+d)} + \frac{b(1+X_e)}{b(b+d)}\right)\right]$$

$$= \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{b(1+X_e)+d}{b(b+d)}\right)\right]$$

$$= \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{b(X_e)+b+d}{b(b+d)}\right)\right]$$

$$V = \frac{Q}{4\pi \epsilon_0}\left[\frac{1}{(1+X_e)}\left(\frac{X_e}{b+d} + \frac{1}{b}\right)\right]$$

$$V = \frac{Q}{4\pi \epsilon_0 (1+X_e)}\left[\frac{X_e}{b+d} + \frac{1}{b}\right] \quad \ldots\ldots (4)$$

Consider the formula and find capacitance (C).

$$C = \frac{Q}{V}$$

Substitute equation (4) for V.

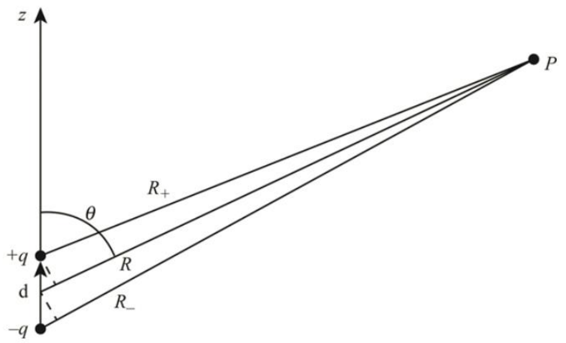
$$C = \frac{Q}{\frac{Q}{4\pi \epsilon_0 (1+X_e)}\left[\frac{X_e}{b+d} + \frac{1}{b}\right]}$$

$$= \frac{1}{\frac{1}{4\pi \epsilon_0 (1+X_e)}\left[\frac{X_e}{b+d} + \frac{1}{b}\right]}$$

$$= \frac{4\pi \epsilon_0 (1+X_e)}{\left(\frac{X_e}{b+d} + \frac{1}{b}\right)}$$

Thus, the capacitance value of an isolated conducting sphere is $\boxed{\frac{4\pi \epsilon_0 (1+X_e)}{\left(\frac{X_e}{b+d} + \frac{1}{b}\right)}}$.

P3.42



Consider the value of electric field intensity in spherical coordinates.

$$\mathbf{E} = \frac{p}{4\pi \epsilon_0 R^3}(2\mathbf{a}_R \cos\theta + \mathbf{a}_\theta \sin\theta) \quad \ldots\ldots (1)$$

Where,

Absolute permittivity is denoted as ϵ_0.

Distance between the field points (*P*) to charge (*q*) is denoted as *R*.

Angle between the axis and field point line is denoted as θ.

Consider the formula to find electrostatic energy.

$$W_e = \frac{1}{2}\int_{V'} \epsilon_0 \, E^2 dv$$

Rewrite the equation.

$$W_e = \frac{1}{2}\int_{V'} \epsilon_0 \, \mathbf{E}^2 d\mathbf{v} \quad \ldots\ldots (2)$$

Substitute equation (1) in (2).

$$\begin{aligned} W_e &= \frac{1}{2}\int_{V'} \epsilon_0 \left(\frac{p}{4\pi \epsilon_0 R^3}(2\mathbf{a}_R \cos\theta + \mathbf{a}_\theta \sin\theta) \right)^2 d\mathbf{v} \\ &= \frac{\epsilon_0}{2}\left(\frac{p}{4\pi \epsilon_0 R^3} \right)^2 \int_{V'} (2\mathbf{a}_R \cos\theta + \mathbf{a}_\theta \sin\theta)^2 \, d\mathbf{v} \\ &= \frac{\epsilon_0}{2}\left(\frac{p}{4\pi \epsilon_0 R^3} \right)^2 \int_{V'} \left(4\cos^2\theta + \sin^2\theta\right) dv \end{aligned}$$

Expand the volume integral.

$$W_e = \frac{\epsilon_0}{2}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_b^{\infty} \frac{1}{R^6}\left(4\cos^2\theta + \sin^2\theta\right) R^2 \sin\theta dR$$

$$= \frac{\epsilon_0}{2}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_b^{\infty} \frac{1}{R^4}\left(4\cos^2\theta + \sin^2\theta\right) \sin\theta dR$$

$$= \frac{\epsilon_0}{2}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[\frac{1}{-3R^3}\left(4\cos^2\theta + \sin^2\theta\right) \sin\theta\right]_b^{\infty}$$

$$= \frac{\epsilon_0}{2}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} \left[\frac{1}{3b^3}\left(4\cos^2\theta + \sin^2\theta\right) \sin\theta\right] d\theta$$

$$W_e = \frac{\epsilon_0}{2(3b^3)}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \left[\frac{-7\cos\theta - \cos 3\theta}{4}\right]_0^{\pi}$$

$$= \frac{\epsilon_0}{2(3b^3)}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \left[\frac{-7(-1)-(-1)}{4} - \frac{-7-1}{4}\right]$$

$$= \frac{\epsilon_0}{2(3b^3)}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi \left[\frac{8}{4} - \frac{-8}{4}\right]$$

$$= \frac{\epsilon_0}{2(3b^3)}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi [4]$$

$$W_e = \frac{2\epsilon_0}{3b^3}\left(\frac{p}{4\pi\epsilon_0}\right)^2 \int_0^{2\pi} d\phi$$

$$= \frac{2\epsilon_0}{3b^3}\left(\frac{p}{4\pi\epsilon_0}\right)^2 [\phi]_0^{2\pi}$$

$$= \frac{2\epsilon_0}{3b^3}\left(\frac{p}{4\pi\epsilon_0}\right)^2 [2\pi]$$

$$= \frac{4\pi\epsilon_0}{3b^3}\left(\frac{p}{4\pi\epsilon_0}\right)^2$$

$$W_e = \frac{4p^2}{16(3b^3)\pi\epsilon_0}$$

$$= \frac{p^2}{12\pi\epsilon_0 b^3}$$

Thus, the value of the electrostatic energy is $\boxed{\frac{p^2}{12\pi\epsilon_0 b^3}}$.

Thank you very much!