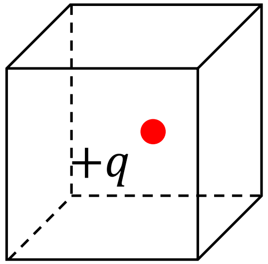
Electrostatic Energy

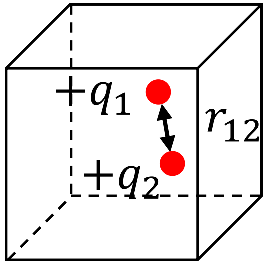
Potential energy



$U = 0$

Open the box → nothing happens

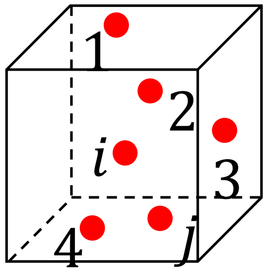
It costs no work to bring this charge in the empty box



$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}$$

Open the box → charges fly apart

Energy stored = work done to bring the second charge is given back



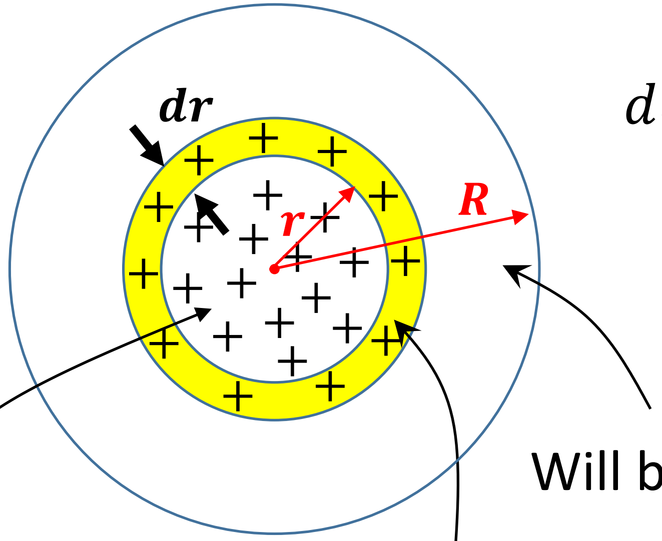
$$U = \sum_{\substack{i \neq j \\ i<j}} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

Sum taken over all pairs

How many pairs do we have in this example?

$$\frac{n(n-1)}{2} = 15 \text{ for } n = 6 \text{ charges}$$

The electrostatic energy of a uniformly charged non conducting sphere



Already charged

$$\sum_i q_i \to Q(r)$$

being charged

$$\sum_j q_j \to dQ$$

$$Q(r) = \rho \frac{4}{3}\pi r^3$$

$$dQ = \rho 4\pi r^2 dr$$

$$\rho = \frac{Q}{V}$$ ⟶ Total volume of the sphere

$$dU = \frac{Q(r)\, dQ}{4\pi\varepsilon_0 r}$$ ⟹ $$U = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$$ ⟹ $$U = \frac{Q^2}{4\pi\varepsilon_0 \left(R/\frac{3}{5}\right)}$$

$$U = \sum_{\substack{i\neq j\\ i<j}} \frac{q_i q_j}{4\pi\varepsilon_0}\frac{1}{r_{ij}} \to U = \frac{Q^2}{4\pi\varepsilon_0}\langle\frac{1}{r_{ij}}\rangle$$

$$\boxed{\langle\frac{1}{r_{ij}}\rangle = \frac{3}{5R}}$$ ⟹ $< r_{ij} >$?

$$= (1/n)^{1/3}$$

n is # charge /volume

The energy of a capacitor

$$dW = \vec{F}.d\vec{l} = dq\left(\vec{E}.d\vec{l}\right)$$

$$\vec{E} = -\vec{\nabla}V$$

$$dW = -dq.d(\Delta V)$$

$$C = \frac{q}{\Delta V}$$

$$dW = -\frac{qdq}{C} = -dU$$

Work energy theorem $dW = -dU$

Work done against the field

$$dU = \frac{qdq}{C}$$

$$\boxed{U = \int_0^Q dU = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}Q\Delta V}$$

Energy of a uniformly charged sphere

Conducting sphere (cs)

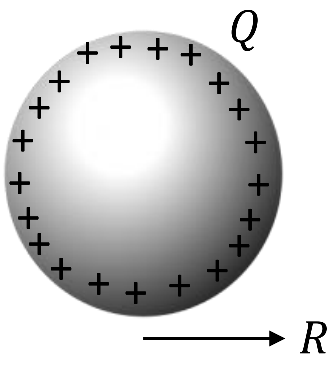
$$V(sphere) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

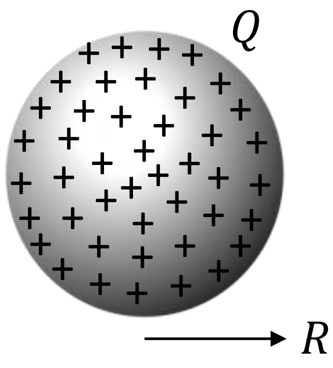
$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

$$U_{cs} = \frac{1}{2}QV$$

$$U_{cs} = \frac{1}{2}\left(\frac{Q^2}{4\pi\varepsilon_0 R}\right)$$

Relative to infinity $\rightarrow \Delta V = V$





Non conducting sphere (ncs)

$$U_{ncs} = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$$

$$\frac{U_{cs}}{U_{ncs}} = \frac{5}{6}$$

Force on charged conductors

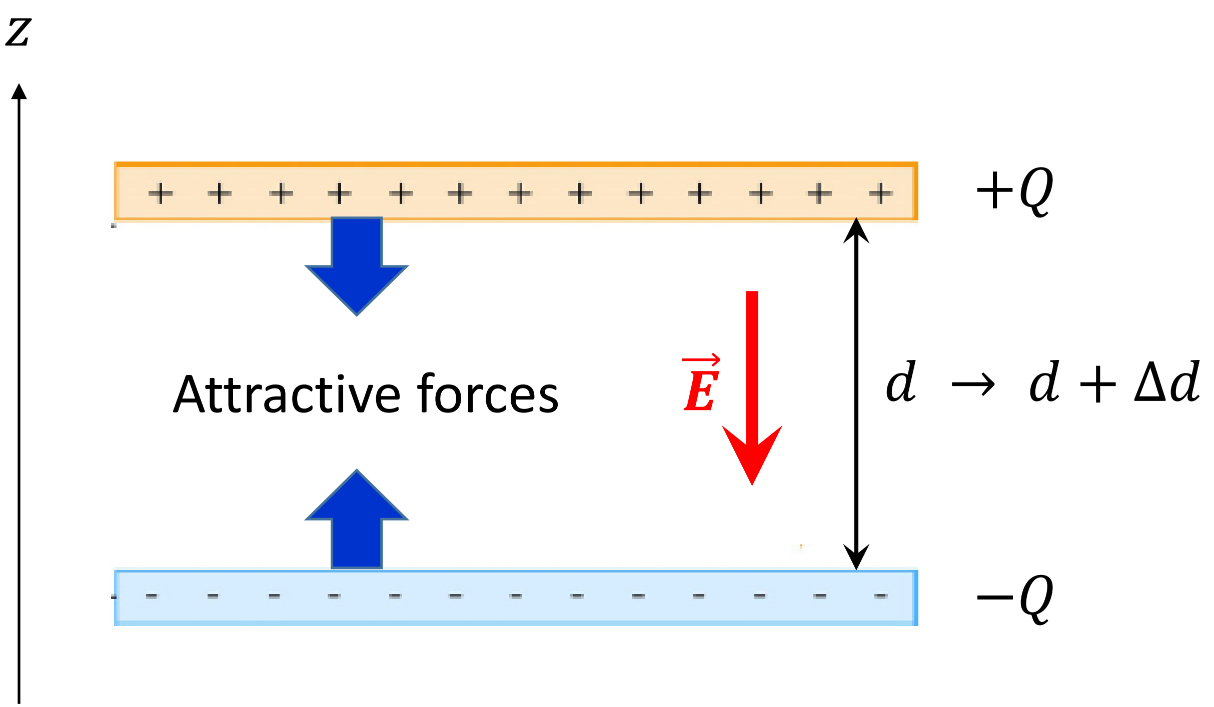
Intuitively the force between the plates

$$\boxed{F = QE}$$

From the definition of the work

$$\Delta W = F\Delta d = \Delta U$$

$$U = \frac{1}{2}\frac{Q^2}{C} \Rightarrow \Delta U = \frac{Q^2}{2}\Delta\left(\frac{1}{C}\right)$$



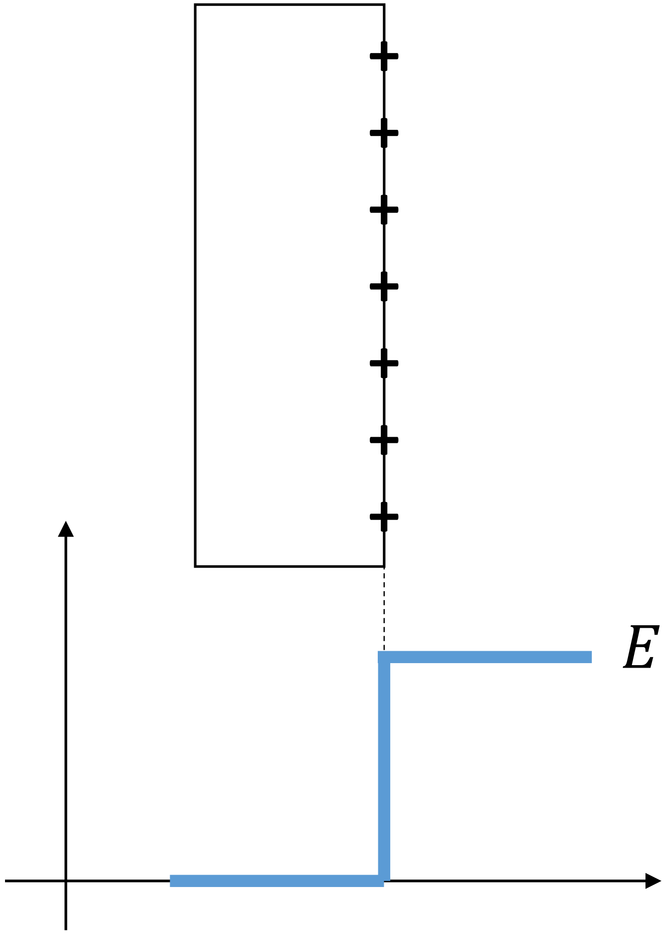
$$C = \frac{\varepsilon_0 A}{d} \qquad \frac{1}{C} = \frac{d}{\varepsilon_0 A} \qquad \Delta\left(\frac{1}{C}\right) = \frac{\Delta d}{\varepsilon_0 A} \Rightarrow F = \frac{Q^2}{2\varepsilon_0 A} = \frac{Q}{2}\underbrace{\left(\frac{Q}{\varepsilon_0 A}\right)}_{E = \frac{\sigma}{\varepsilon_0}}$$

$$\boxed{F = \frac{1}{2}QE} \quad ?$$

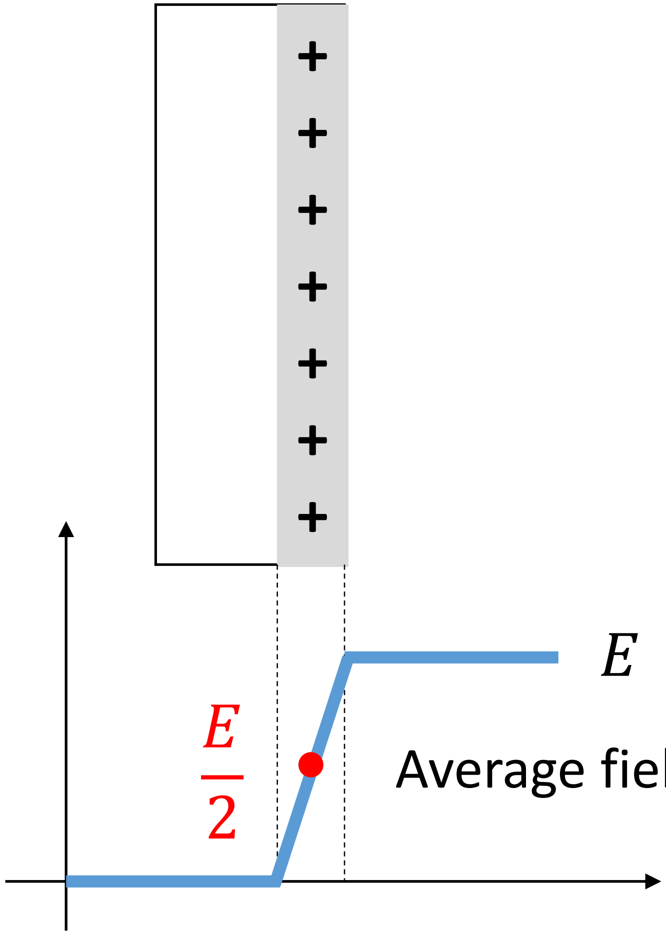
(slide #68 from E_Lectures 8&9 Gauss law in Electrostatics) ⇨ $E = \frac{\sigma}{\varepsilon_0}$

What is exactly the field acting on the charge at the surface?

Ideal case
Infinitely thin sheet of charges



Real case
Finite sheet of charges



Average field acting on the charges

Energy in the electrostatic field

Basic equation $$U = \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

all pairs

For every pair the energy is counted twice

Unless we restrict to $i < j$

$$q_i \rightarrow \rho_1 d\mathcal{V}_1$$

$$q_j \rightarrow \rho_2 d\mathcal{V}_2$$

$$U = \frac{1}{2} \int \frac{\rho_1 \rho_2 d\mathcal{V}_1 d\mathcal{V}_2}{4\pi\varepsilon_0 r_{12}}$$

Over all space

Caution ! $\mathcal{V}$ = volume

V = Potential

$$U = \frac{1}{2} \int \rho_1 d\mathcal{V}_1 \underbrace{\int \frac{\rho_2 d\mathcal{V}_2}{4\pi\varepsilon_0 r_{12}}}$$

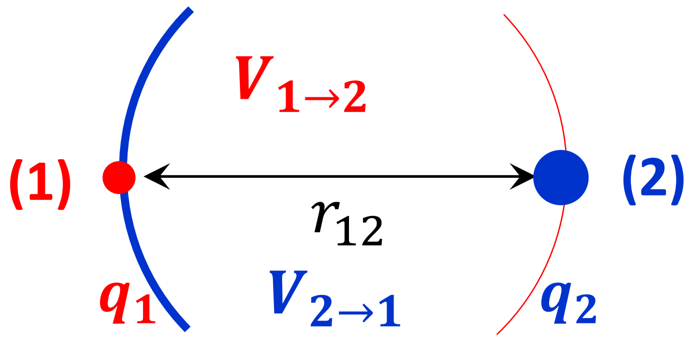
$$\boxed{U = \frac{1}{2} \int \rho V d\mathcal{V}}$$

Why do we need the factor $\frac{1}{2}$?

V_1= Potential at charge distribution (1) created by charge distribution (2)

Slide #4 D_Lectures 4-7 Coordinate system Scalar versus Vector fields Operators

The mutual energy of two charges q_1 and q_2



$$U_1 = q_2 V_{1\to 2} = q_2 \frac{q_1}{4\pi\varepsilon_0 r_{12}}$$

$$U_2 = q_1 V_{2\to 1} = q_1 \frac{q_2}{4\pi\varepsilon_0 r_{12}}$$

$$\boxed{U = \frac{1}{2}[U_1 + U_2]}$$

$$\boxed{U = \frac{1}{2}[q_1 V_{2\to 1} + q_2 V_{1\to 2}]}$$

Question: Where is the electrostatic energy located?

If we consider two charges interacting, is the energy located

- On one or the other charge
- At both of them
- In between

Does it make sense to ask this kind of question?

Corollary: local versus spread energy

It makes sense to talk about local energy when dealing for instance with heat energy.

Then conservation law tells us that if locally the energy is changing there must be an outflow or inflow of heat energy to account for the conservation law in the given volume.

Remember our treatment of heat flow to demonstrate Gauss law

BUT in electromagnetism the energy is every where the fields are.
Electromagnetic fields carry energy with them

$$U = \frac{1}{2}\int \rho V d\mathcal{V} \quad \Rightarrow \quad \vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \vec{E} = -\vec{\nabla}V \qquad \boxed{U = -\frac{\varepsilon_0}{2}\int \mathrm{V}\,\nabla^2 \mathrm{V} d\mathcal{V}}$$

See slide #8

$$V\,\nabla^2 V = \mathrm{V}\underbrace{\left(\frac{\partial^2 \mathrm{V}}{\partial x^2}+\frac{\partial^2 \mathrm{V}}{\partial y^2}+\frac{\partial^2 \mathrm{V}}{\partial z^2}\right)}_{\text{Laplace}} = \frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial \mathrm{V}}{\partial x}\right)-\left(\frac{\partial \mathrm{V}}{\partial x}\right)^2+\frac{\partial}{\partial y}\left(\mathrm{V}\frac{\partial \mathrm{V}}{\partial y}\right)-\left(\frac{\partial \mathrm{V}}{\partial y}\right)^2+\frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial \mathrm{V}}{\partial x}\right)-\left(\frac{\partial \mathrm{V}}{\partial x}\right)^2$$

$$V\,\nabla^2 V = \mathrm{V}\left(\frac{\partial^2\mathrm{V}}{\partial x^2}+\frac{\partial^2\mathrm{V}}{\partial y^2}+\frac{\partial^2\mathrm{V}}{\partial z^2}\right) = \frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial\mathrm{V}}{\partial x}\right)-\left(\frac{\partial\mathrm{V}}{\partial x}\right)^2+\frac{\partial}{\partial y}\left(\mathrm{V}\frac{\partial\mathrm{V}}{\partial y}\right)-\left(\frac{\partial\mathrm{V}}{\partial y}\right)^2+\frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial\mathrm{V}}{\partial x}\right)-\left(\frac{\partial\mathrm{V}}{\partial x}\right)^2$$

$$\underbrace{\left[\frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial\mathrm{V}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mathrm{V}\frac{\partial\mathrm{V}}{\partial y}\right)+\frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial\mathrm{V}}{\partial x}\right)\right]}_{aa'+bb'+cc'} - \underbrace{\left[\left(\frac{\partial\mathrm{V}}{\partial x}\right)^2+\left(\frac{\partial\mathrm{V}}{\partial y}\right)^2+\left(\frac{\partial\mathrm{V}}{\partial x}\right)^2\right]}_{aa+bb+cc}$$

Scalar product

$$\boxed{\vec{\nabla}.(\mathrm{V}\vec{\nabla}\mathrm{V})}$$

Scalar product

$$\boxed{(\vec{\nabla}\mathrm{V}).(\vec{\nabla}\mathrm{V})}$$

$$\mathrm{V}\,\nabla^2\mathrm{V} = \vec{\nabla}.\left(\mathrm{V}\vec{\nabla}\mathrm{V}\right) - \left(\vec{\nabla}\mathrm{V}\right).\left(\vec{\nabla}\mathrm{V}\right)$$

$\Rightarrow$ Substitution into $U = -\frac{\varepsilon_0}{2}\int \mathrm{V}\,\nabla^2\mathrm{V}dV$

$$U = \frac{\varepsilon_0}{2}\int \left(\vec{\nabla}\mathrm{V}\right).\left(\vec{\nabla}\mathrm{V}\right)dV - \frac{\varepsilon_0}{2}\int \vec{\nabla}.\left(\mathrm{V}\vec{\nabla}\mathrm{V}\right)dV$$

$\Downarrow$

Gauss's theorem $\Rightarrow$ $U = \int \vec{\nabla}.\left(\mathrm{V}\vec{\nabla}\mathrm{V}\right)dV = \int \mathrm{V}\vec{\nabla}\mathrm{V}.\vec{n}dA$

Volume　Surface

$$\int \ldots dV \quad \rightarrow \quad \int \ldots dA$$

Volume　Surface

We want to consider the whole infinite space

$$\int_{\text{Surface}} \mathrm{V}\vec{\nabla}\mathrm{V}.\vec{n}dA \rightarrow \mathbf{0}$$

$1/r$ $1/r^2$ r^2

$$U = \frac{\varepsilon_0}{2}\int_{\text{All space}} (\vec{\nabla}\mathrm{V}).(\vec{\nabla}\mathrm{V})dV = \frac{\varepsilon_0}{2}\int_{\text{All space}} \vec{E}.\vec{E}dV$$

$$\boxed{U = \frac{\varepsilon_0}{2}\int_{\text{All space}} E^2 dV}$$

$\frac{\varepsilon_0}{2}E^2 =$ Energy density

The energy contained in the space between the plates of a capacitor

$$\left.\begin{aligned} U &= \frac{1}{2}\frac{Q^2}{C} \\ C &= \frac{Q}{\mathrm{V}} \end{aligned}\right\} \quad U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\mathrm{V}^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2 (Ad)$$

$$\boxed{\frac{U}{Ad} = \frac{1}{2}\varepsilon_0 E^2 = \text{Energy density}}$$

Using the force and work done

$$F = \frac{1}{2}QE = \frac{1}{2}\boldsymbol{\sigma} AE \qquad E = \frac{\boldsymbol{\sigma}}{\varepsilon_0}$$

$$F = \frac{1}{2}QE = \frac{1}{2}\boldsymbol{\sigma} AE = \frac{A}{2}\varepsilon_0 E^2 \qquad W = Fd = \frac{Ad}{2}\varepsilon_0 E^2 \qquad \boxed{\frac{U}{Ad} = \frac{W}{Ad} = \frac{1}{2}\varepsilon_0 E^2}$$

$$\boxed{\frac{1}{2}\varepsilon_0 E^2 = \frac{F}{A} = \text{Electrostatic pressure}}$$

Application to the energy of a point charge

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

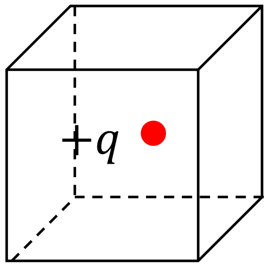
Energy density $\rightarrow \dfrac{1}{2}\varepsilon_0 E^2 = \dfrac{q^2}{32\pi^2\varepsilon_0 r^4}$

Total energy contained in the whole universe containing a single charge q

$$d\mathcal{V} = 4\pi r^2 dr \qquad U = \int_{r=0}^{\infty} \frac{q^2}{32\pi^2\varepsilon_0 r^4}\, dV$$

$$U = -\frac{q^2}{8\pi\varepsilon_0 r}\Big|_{r=0}^{r=\infty} = \infty\ !$$

Remember this



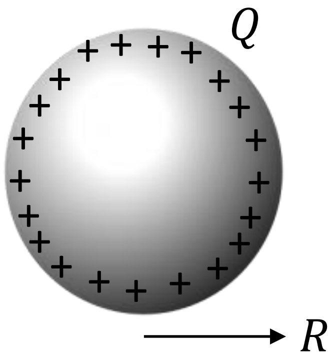
$+q$ does not interact with itself

$U = 0$

Open the box → nothing happens

Big problem !!

Fundamental difficulties in our understanding of nature



$$U_{cs} = \frac{1}{2}\left(\frac{Q^2}{4\pi\varepsilon_0 R}\right)$$

$U_{cs} \rightarrow \infty$ when $R \rightarrow 0$

1) Electron as small as it is, is not a point charge but a distribution of charge

2) Something is wrong with the theory of electricity at very small distances

3) Something is wrong with our idea of local conservation of energy