The concept of vector potential

Pushing the parallel between Electrostatic and Magnetostatic to the ultimate limit

The question that brings to the concept of vector potential

Giving the following equations

$$\boxed{\begin{gathered}\vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}\end{gathered}}$$

Can we solve them without requiring any special symmetry or intuitive guessing ?

In electrostatic we have:

Divergence

$$\vec{\nabla}.\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$\rho(\vec{r})$ source of the field

Curl

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

Gradient

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

Electric potential

If we know the charge distribution $\rho(\vec{r})$

With symmetry ➡ Gauss law $\vec{\nabla}.\boldsymbol{\vec{E}} = \frac{\rho(\vec{r})}{\varepsilon_0}$

Without refereeing to symmetry ➡ $\boldsymbol{V(\vec{r})} = \frac{1}{4\mu\varepsilon_0}\int \frac{\rho(\vec{r}-\vec{r}')}{\vec{r}-\vec{r}'} d\mathcal{V}'$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

In magnetostatic:

By analogy

Divergence	Curl	Gradient	
$\vec{\nabla}.\vec{B} = 0$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r})$	???	"Magnetic potential"

$\vec{J}(\vec{r})$ is for magnetostatic what $\rho(\vec{r})$ is for electrostatic: Source of the field

Is there any equivalent way to solving magnetostatic problems if we know $\vec{J}(\vec{r})$?

The difference with electrostatic comes from these two sides

Concept of Magnetic Potential

Stokes theorem

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Longleftrightarrow \quad \oint \vec{B}.d\vec{l} = \mu_0 I_{encl}$$

Differential form | Integral form

BUT

Can we define a potential for magnetic field

… as we have a potential for an electric field?

Electrostatic

Electric field $\vec{E}$

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

⬇

Scalar **potential** $\boldsymbol{V}, \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\boldsymbol{V}$

B_Lecture 4&7_Coordinate system_Scalar versus Vector fields_Operators

Magnetostatic

Magnetic field $\vec{B}$

$\vec{\nabla} \times \vec{B}$ is not <u>always</u> zero

⬇

Scalar potential $\boldsymbol{\psi}, \vec{\boldsymbol{B}} \cancel{=} -\vec{\boldsymbol{\nabla}}\boldsymbol{\psi}$

NOT ALWAYS POSSIBLE

$\vec{\nabla}.\vec{B} = 0$ **ALWAYS** $\Rightarrow$ $\vec{B} = \vec{\nabla} \times \vec{A}$

Mathematics ⇨ $\vec{\nabla}.\left(\vec{\nabla} \times \vec{A}\right) = 0$

$\vec{A}$ = **<u>Vector</u>** potential

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The parallel with electrostatic

In magnetostatic if we know the current density $\vec{J}(\vec{r}) \Rightarrow \vec{A}(\vec{r}) \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$

$$B_x = \left(\vec{\nabla} \times \vec{A}\right)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \qquad B_y = \left(\vec{\nabla} \times \vec{A}\right)_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \qquad B_z = \left(\vec{\nabla} \times \vec{A}\right)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

By analogy

In electrostatic

Charge distribution $\rho(\vec{r})$ $\Rightarrow$ $V(\vec{r})$ $\Rightarrow$ $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

In magnetostatic

Current density $\vec{J}(\vec{r})$ $\Rightarrow$ $\vec{A}(\vec{r})$ $\Rightarrow$ $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$

For electrostatic Poisson equation $$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

For Magnetostatic … $$\vec{A}(\vec{r}) \overset{?}{\longleftrightarrow} \vec{J}(\vec{r})$$

Neither of the quantities $V(\vec{r})$and $\vec{A}(\vec{r})$ is unique !

Vector potential $\vec{A}$ exists: Physical interpretation

$\vec{B} = \vec{\nabla} \times \vec{A}$ ⇒ Dimension Equation ⇒ $[A] = [B].[L]$

Lorentz force $F = qvB$ ⇒ $[B] = \dfrac{[F]}{[qv]}$

$[qA] = [\text{Momentum}]$

Although the vector potential result from mathematical considerations it does have a physical meaning

Non uniqueness of Vector potential $\vec{\boldsymbol{A}}$ in magnetostatic

The vector potential $\vec{A}$ is **NOT** unique

A lot of different $\vec{A}$'s give the same $\vec{B}$

We can measure locally $\vec{B}$ **BUT NOT** $\vec{\boldsymbol{A}}$

In electrostatic $V' = V + C$

C does not give rise to $\vec{E}$

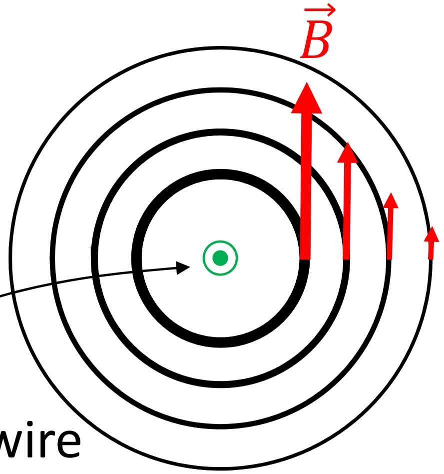
$V(\infty) = 0$

V is always defined within a constant C

Non uniqueness of Vector potential $\vec{\boldsymbol{A}}$ in magnetostatic

$\vec{B} = \vec{\nabla} \times \vec{A}$

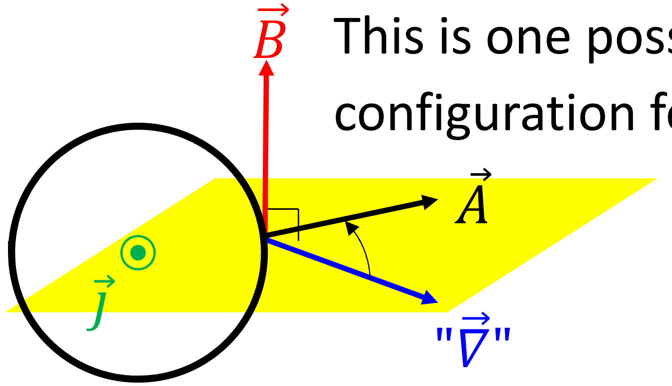
$\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{A}}$



Straight current carrying wire

Infinite number of configurations in which $\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{A}}$

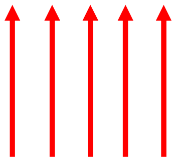
$\vec{\nabla}.\vec{B} = 0$



This is one possible configuration for $\vec{A}$...

... Among infinite possibilities

Proof that the vector potential $\vec{A}$ is **NOT** unique

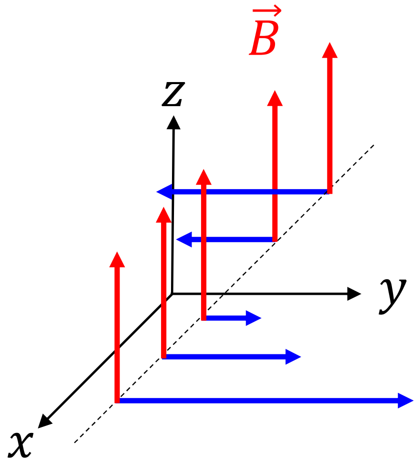


$$\vec{B} = \vec{B}_z = B_z\vec{k} = \boldsymbol{b}\vec{k}$$
$$\vec{B}_y = 0$$
$$\vec{B}_x = 0$$

What is $\vec{A}$?

$$B_z = \left(\vec{\nabla} \times \vec{A}\right)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \boldsymbol{b}$$

Possibility 1



$\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{A}}$

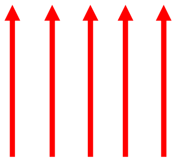
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{A} = \vec{A}_y = A_y\vec{j} = \mathbf{b}\mathrm{x}\vec{j}$$

Any of the blue vectors $\vec{A}$ is okay: an infinity

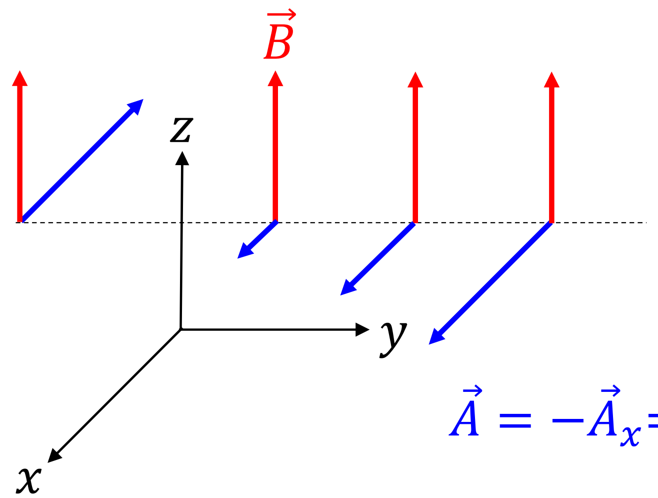
Proof that the vector potential $\vec{A}$ is **NOT** unique

Possibility 2



$$\vec{B} = \vec{B}_z = B_z\vec{k} = \boldsymbol{b}\vec{k}$$
$$\vec{B}_y = 0$$
$$\vec{B}_x = 0$$

$\boldsymbol{\vec{B} \perp \vec{A}}$



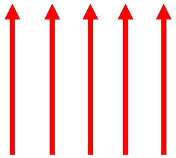
What is $\vec{A}$?

$$\vec{A} = -\vec{A}_x = -A_x\vec{\imath} = -\mathbf{b}\mathrm{y}\vec{\imath}$$

$$B_z = \left(\vec{\nabla} \times \vec{A}\,\right)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \boldsymbol{b}$$

Any of the black vectors $\vec{A}$ is okay: an infinity

Proof that the vector potential $\vec{A}$ is **NOT** unique



$$\vec{B} = \vec{B}_z = B_z\vec{k} = \boldsymbol{b}\vec{k}$$
$$\vec{B}_y = 0$$
$$\vec{B}_x = 0$$

What is $\vec{A}$?

Any combination of the previous ones

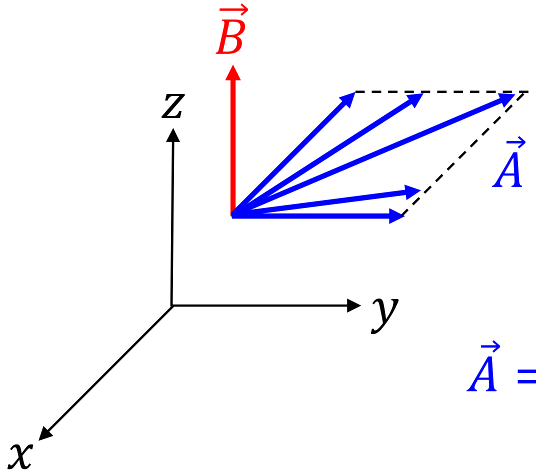
$$\vec{B} = \vec{B}_z = (\boldsymbol{\alpha} + \boldsymbol{\beta})\boldsymbol{b}\vec{k}$$

$$(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \mathbf{1}$$

Infinity of combinations

Possibility 3

$\vec{\boldsymbol{B}} \perp \vec{\boldsymbol{A}}$



$$\vec{A} = -A_x\vec{\imath} + A_y\vec{\jmath}$$

$$\vec{A} = -\boldsymbol{\alpha b}y\vec{\imath} + \boldsymbol{\beta b}x\vec{\jmath}$$

$$B_z = \left(\vec{\nabla} \times \vec{A}\,\right)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\boldsymbol{\alpha} + \boldsymbol{\beta})\boldsymbol{b}$$

Any BLUE vectors $\vec{A}$ is okay: an infinite number

In electrostatic

Solution is not unique for V **BUT** must be unique for $\vec{E}$

$\vec{\nabla} \times \vec{E} = \vec{0}$ ⇒ $\vec{E} = -\vec{\nabla} V$ $\quad$ $\mathrm{V}' = V + C$

Unique $\vec{E}$ for any arbitrary C

Useful condition for $V' = V + C$ that does not affect $\vec{E}$ $\quad$ $\vec{\nabla}.C = \vec{0}$

$V(\infty) = 0$

Likewise Solution is not unique for $\vec{A}$ **BUT** must be unique for $\vec{B}$

Using Curl

$$\vec{B} = \vec{\nabla} \times \overrightarrow{A'} = \vec{\nabla} \times \vec{A}$$

Two possible solutions

$\overrightarrow{A'}$ and $\vec{A}$ have the same curl

$$\vec{\nabla} \times \overrightarrow{A'} - \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\overrightarrow{A'} - \vec{A}\right) = \vec{0}$$

Looks like

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

For which

$$\vec{E} = -\vec{\nabla} V$$

$$\left(\overrightarrow{A'} - \vec{A}\right) = \vec{\nabla}\psi$$

$$\vec{\nabla} \times \left(\vec{\nabla}\psi\right) = \vec{0}$$

$\vec{\nabla}\psi$ does not give rise to $\vec{B}$

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$$\overrightarrow{A'} = \vec{A} + \vec{\nabla}\psi$$

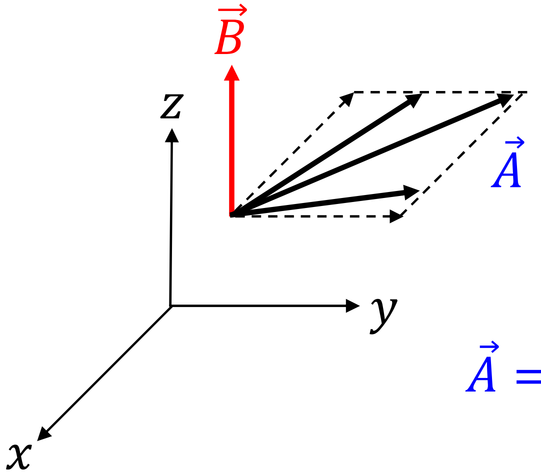
Unique $\vec{B}$ for any arbitrary ψ

Electrostatic $V' = V + C$

Example for $\overrightarrow{A'} - \vec{A} = \vec{\nabla}\psi$

Possibility 3

$\vec{B} \perp \vec{A}$

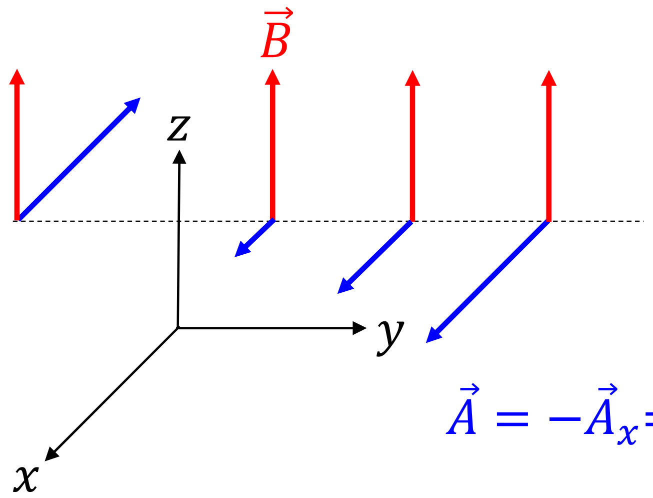


$$\vec{A} = -A_x\vec{\imath} + A_y\vec{\jmath}$$

$$\vec{A} = \underbrace{-\boldsymbol{\alpha b}y\vec{\imath} + \boldsymbol{\beta b}x\vec{\jmath}}_{\vec{A}_1}$$

Possibility 2

$\vec{B} \perp \vec{A}$



$$\vec{A} = -\vec{A}_x = -A_x\vec{\imath} = -\mathbf{b}\mathrm{y}\vec{\imath}$$

$$\vec{A}_2$$

$$\vec{A}_1 - \vec{A}_2 = by(1-\alpha)\vec{\imath} + \beta bx\vec{\jmath} = \vec{\nabla}\left(\frac{bxy}{2}[1-\alpha+\beta]\right) = \vec{\nabla}(\underbrace{\boldsymbol{bxy}[\mathbf{1}-\boldsymbol{\alpha}]}_{\boldsymbol{\psi}}) = \vec{\nabla}\boldsymbol{b}\beta\boldsymbol{xy}$$

$\overrightarrow{A'}$ and $\vec{A}$ <u>have the same **Curl**</u> ⇨ $\vec{B}$ is unique

BUT

Using Div $\overrightarrow{A'} = \vec{A} + \vec{\nabla}\psi$ ⇒ $\vec{\nabla}.\overrightarrow{A'} = \vec{\nabla}.\vec{A} + \nabla^2\psi$

$$\vec{\nabla}.\overrightarrow{A'} = \nabla^2\psi$$

$\overrightarrow{A'}$ and $\overrightarrow{A}$ do not need to <u>have the same **Divergence**</u>

In magnetostatic, the greatest mathematical convenience requires

$$\vec{\nabla}.\vec{A} = 0$$
$$\vec{\nabla} \times \vec{A} = \vec{B}$$

Final step: From Mathematics to Physics

$$\vec{U} \times (\vec{V} \times \vec{W}) = \vec{V}(\vec{U}.\vec{W}) - \vec{W}(\vec{U}.\vec{V})$$

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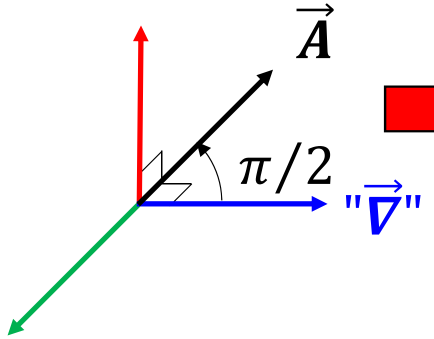
$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$$



$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \boldsymbol{\mu_0}\vec{\boldsymbol{J}}$$

$$\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{A}} = \boldsymbol{0}$$

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})} = \vec{\nabla}.(\vec{\nabla}.\vec{A}) \boxed{- \nabla^2 \vec{A}}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \vec{0}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Miracle of analogy

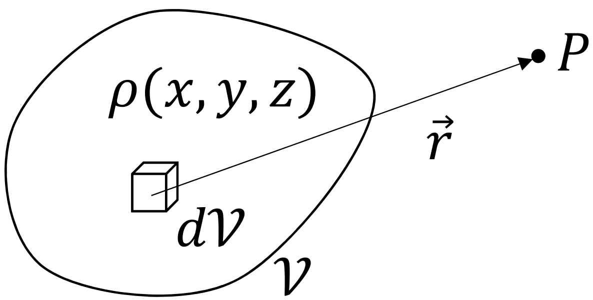
What relationship between $\vec{A}$ and $\mu_0\vec{j}$? $\nabla^2\vec{A} = -\mu_0\vec{j}$

$$\begin{cases} \nabla^2 A_x = -\mu_0 j_x \\ \nabla^2 A_y = -\mu_0 j_y \\ \nabla^2 A_z = -\mu_0 j_z \end{cases}$$

For electrostatic
Poisson equation $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$

$$(A_x, A_y, A_z) \quad \Rightarrow \quad (B_x, B_y, B_z)$$

Electrostatic $\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$



P can be inside $\mathcal{V}$

V *and* $\mathcal{V}$ should not be confused

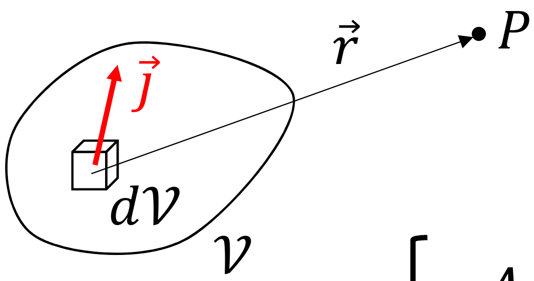
Potential

Volume

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(x,y,z)\, d\mathcal{V}}{r}$$

Magnetostatic $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

Steady current ⇨ $\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{j}} = \mathbf{0}$



Imaginary "electrostatic" problem

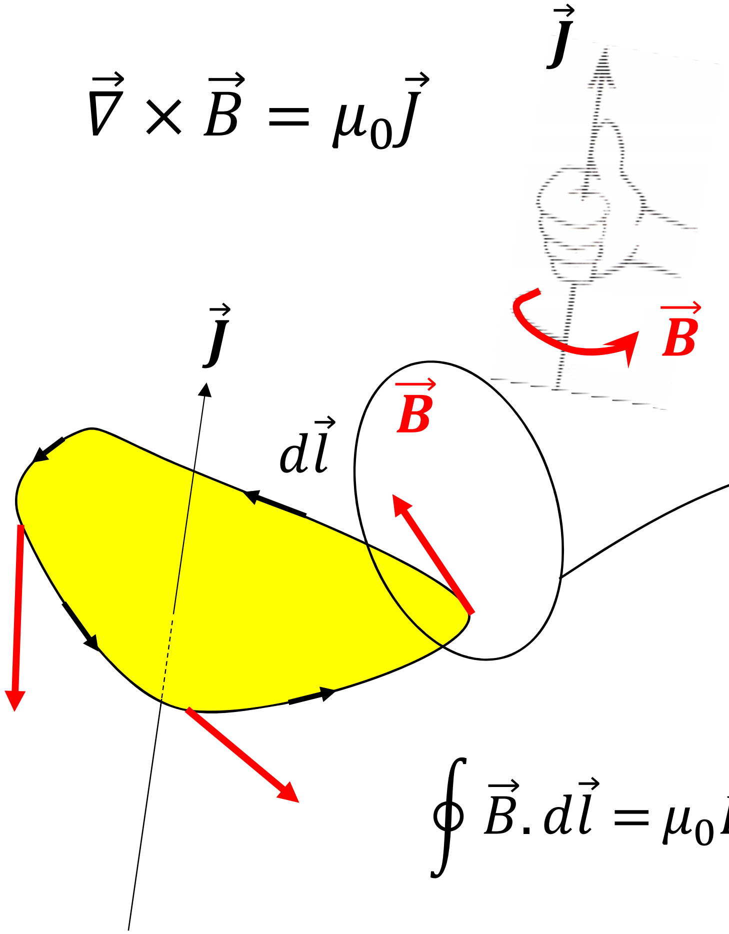
$$\begin{cases} A_x(P) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{j_x(x,y,z)\, d\mathcal{V}}{r} \\ A_y(P) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{j_y(x,y,z)\, d\mathcal{V}}{r} \\ A_z(P) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{j_z(x,y,z)\, d\mathcal{V}}{r} \end{cases}$$

Knowing $A_x(P)$ $A_y(P)$ $A_z(P)$

$$\vec{\boldsymbol{B}}(\boldsymbol{P}) = \vec{\nabla} \times \vec{A}(P) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \underbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}_{\boldsymbol{B_x(P)}} \vec{i} + \underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{\boldsymbol{B_y(P)}} \vec{j} + \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{\boldsymbol{B_z(P)}} \vec{k}$$

Stoke's theorem: Power of analogy

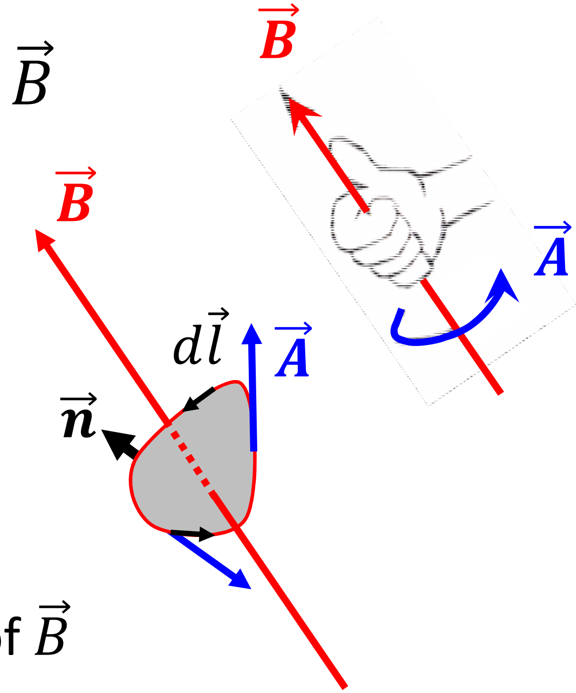
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$



$$\oint \vec{B}.d\vec{l} = \mu_0 I_{encl}$$

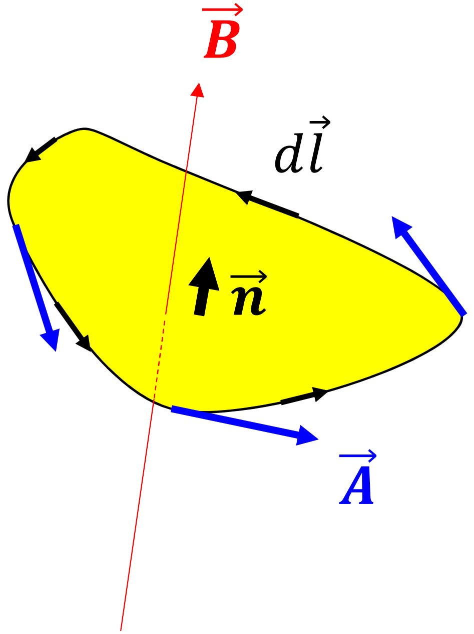
$\vec{A} \perp \vec{B}$

$$\vec{\nabla} \times \vec{A} = \vec{B}$$



$$\oint \vec{A}.d\vec{l} = \text{Flux of } \vec{B}$$

Surface = S instead of A (to avoid giving the same label to the area and vector potential)



$$\oint \vec{A}.d\vec{l} = \text{Flux of } \vec{B}$$

$$= \int (\vec{\nabla} \times \vec{A})\, d\vec{S}$$

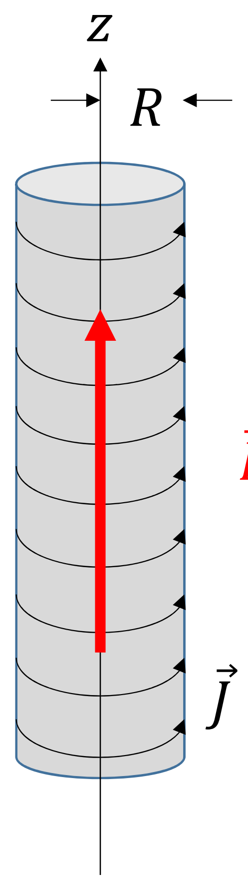
$$= \int \vec{B}\, d\vec{S}$$

Thomb = direction of the flux of $\vec{B}$

2 examples

Vector potential for a solenoid

Vector potential for a current carrying wire



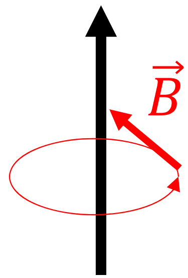
$|\vec{B}| = \mu_0 n I$

$\vec{B} = \mu_0 n I \vec{k}$

$n = N/l$ # turns/length

Analogy

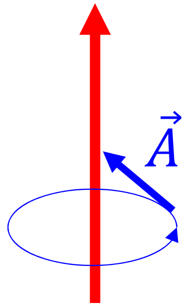
$\vec{J}$ In $\vec{k}$ direction



$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ In $\vec{e}_\theta$ direction

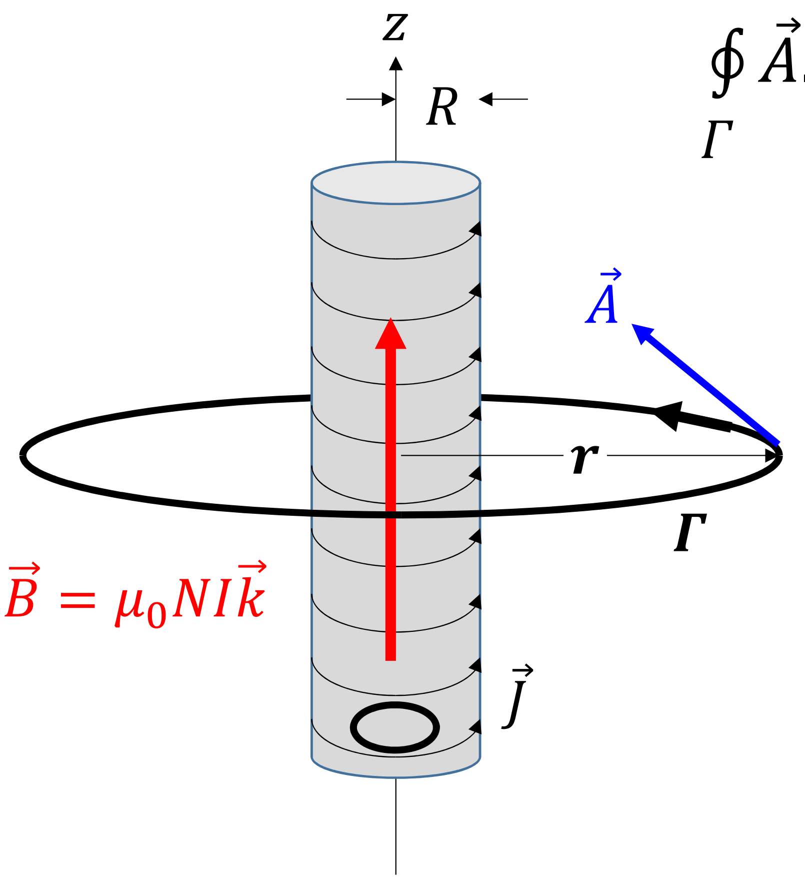
Wire carrying current

$\vec{B}$ In $\vec{k}$ direction



$\vec{\nabla} \times \vec{A} = \vec{B}$ $\vec{A}$ in $\vec{e}_\theta$ direction

Solenoid



$$\oint_{\Gamma} \vec{A}.d\vec{l} = \text{flux passing through the loop } \Gamma$$

$r > \mathrm{R}$ $\quad A_\theta 2\pi r = B\pi R^2 = \mu_0 n I \pi R^2$

$$\boxed{\vec{A} = \frac{\mu_0 n I R^2}{2r}\vec{e}_\theta}$$

$r < \mathrm{R}$ $\quad A_\theta 2\pi r = B\pi r^2 = \mu_0 n I \pi r^2$

$$\boxed{\vec{A} = \frac{\mu_0 n I r}{2}\vec{e}_\theta}$$

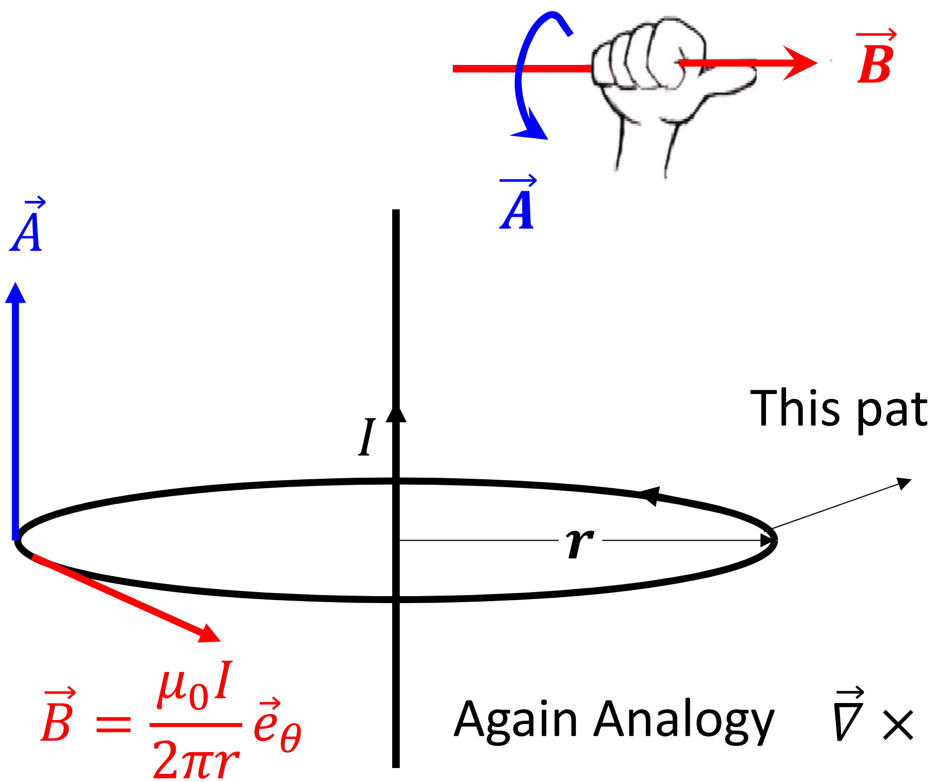
$r = \mathrm{R} \Leftrightarrow$ continuity

Check: $\vec{\nabla} \times \vec{A}\Big|_z = \frac{1}{r}\left(\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)$ $\begin{cases} r < \mathrm{R} & \vec{B} = \mu_0 n I \vec{k} \\ r > \mathrm{R} & \vec{\boldsymbol{B}} = \vec{\mathbf{0}} \end{cases}$

The vector potential $\vec{\boldsymbol{A}}$ exists in a region where there is no magnetic field **!**

$$\boxed{\vec{\nabla} \times \vec{A} = \vec{B}}$$

The existence of $\vec{\boldsymbol{A}}$ has been proven by quantum mechanics and a beautiful experiment:
Bohm and Aharanov 1956



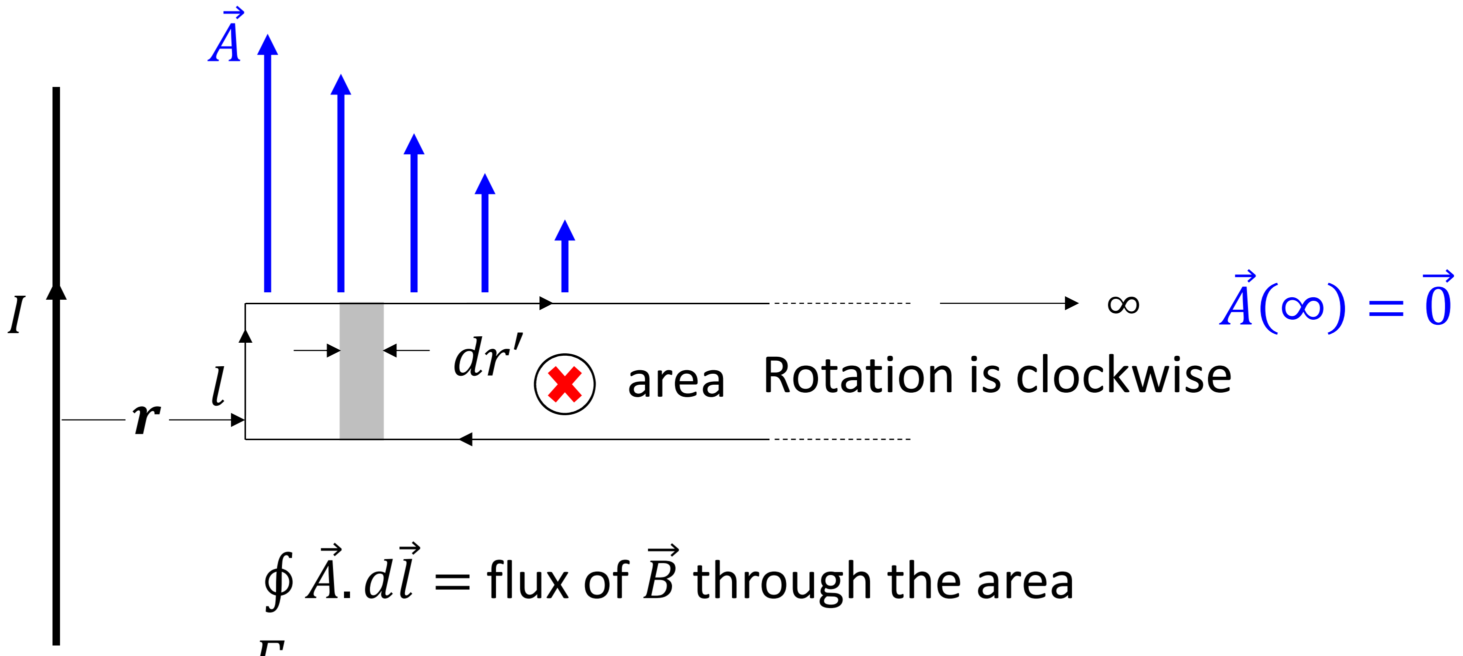
$$\vec{B} = \frac{\mu_0 I}{2\pi r}\vec{e}_\theta$$

This path is the one corresponding to the current in a solenoid

For which $\vec{B}$ is **vertical**

Again Analogy $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ $\quad$ $\vec{\nabla} \times \vec{A} = \vec{B}$ ⇒ $\vec{A}$ is **vertical**

Direction of $\vec{A}$ is known before any calculation



$$\oint_{\Gamma} \vec{A}.d\vec{l} = \text{flux of } \vec{B} \text{ through the area}$$

$$A(r)l = l\int_r^{\infty} \frac{\mu_0 I}{2\pi r'} dr' = \frac{l\mu_0 I}{2\pi} Lnr' \Big|_r^{\infty} = -\frac{l\mu_0 I}{2\pi} Lnr$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\vec{e}_\theta$$

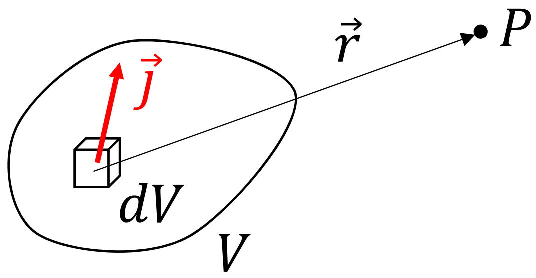
Disregarding infinity

$$\boxed{A(r) = -\frac{\mu_0 I}{2\pi} Lnr}$$

Applications of the Vector potential

- Wire carrying current
- Magnetic dipole

Magnetostatic $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{j}}$



Ampere's law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

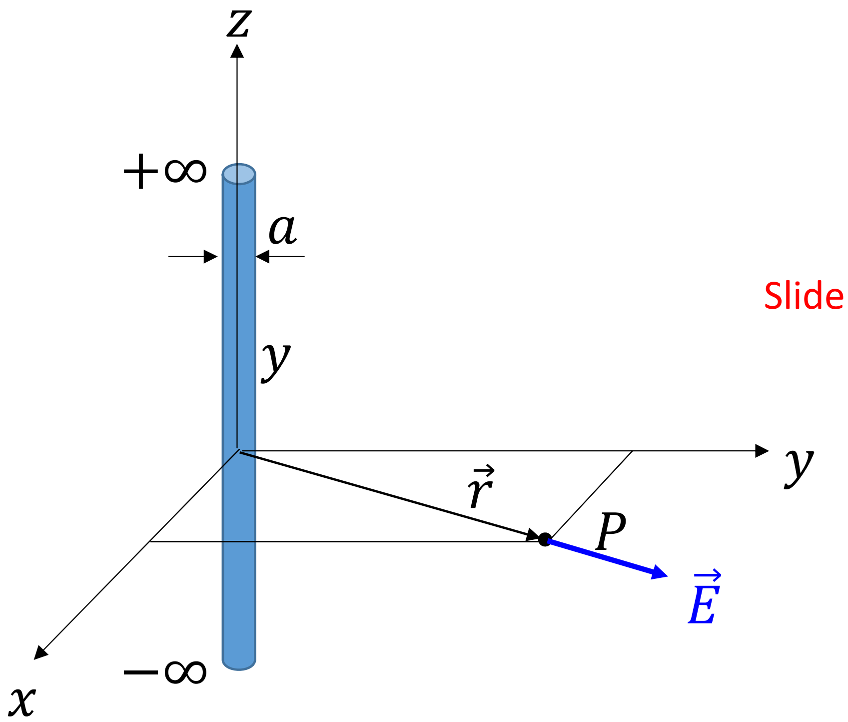
<u>Steady current</u> ⇨ $\boldsymbol{\vec{\nabla}.\vec{j} = 0}$

Imaginary electrostatic problem

$$\left\{ \begin{aligned} A_x(P) &= \frac{\mu_0}{4\pi} \int \frac{j_x(x,y,z)\, dV}{r} \\ A_y(P) &= \frac{\mu_0}{4\pi} \int \frac{j_y(x,y,z)\, dV}{r} \\ A_z(P) &= \frac{\mu_0}{4\pi} \int \frac{j_z(x,y,z)\, dV}{r} \end{aligned} \right.$$

Electrostatic versus magnetostatics: Analogy

A straight wire



Electrostatic

Volume charge density

$$\rho.\pi a^2 = \frac{dQ}{dl} = \lambda \text{ (linear charge density)}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \Rightarrow V(r) = \frac{\lambda}{2\pi\varepsilon_0} Ln\frac{R}{r}$$

Slide #59 E_Lectures 8&9 Gauss law in Electrostatics

$$\Downarrow$$

$$\vec{E} = -\vec{\nabla} V$$

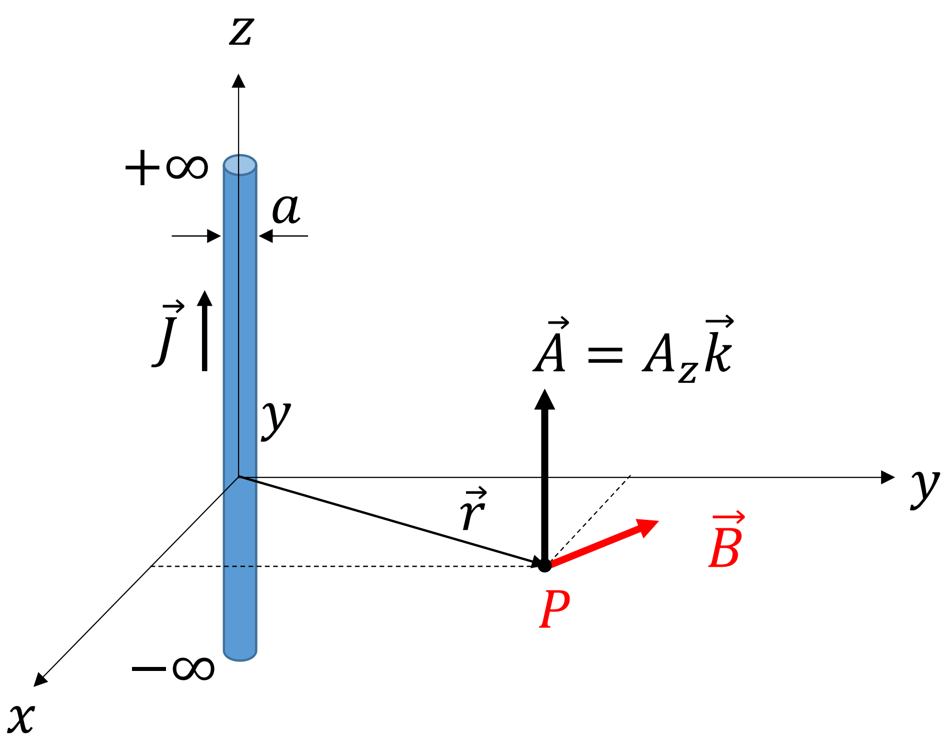
$$\Downarrow$$

$$\vec{E} = E_x\vec{i} + E_y\vec{j}$$

$$E_z = 0$$

Electrostatic versus magnetostatics: Analogy

A straight wire



Magnetostatic

$$j_x = 0 \qquad \vec{A}_x(P) = 0$$

$$j_y = 0 \qquad \vec{A}_y(P) = 0$$

Electrostatic $\quad V = -\dfrac{\lambda}{2\pi\varepsilon_0} Lnr + LnR$

Constant to be dropped

By analogy

$$j_z = \frac{I}{\pi a^2} \qquad \vec{A}_z(P) = -\frac{\mu_0 j_z \pi a^2}{2\pi} Lnr$$

Knowing $A_x(P)$ $A_y(P)$ $A_z(P)$

$$\vec{\boldsymbol{B}}(\boldsymbol{P}) = \vec{\nabla} \times \vec{A}(P) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \underbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}_{\boldsymbol{B_x}(\boldsymbol{P})}\vec{\imath} + \underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{\boldsymbol{B_y}(\boldsymbol{P})}\vec{\jmath} + \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{\boldsymbol{B_z}(\boldsymbol{P})}\vec{k}$$

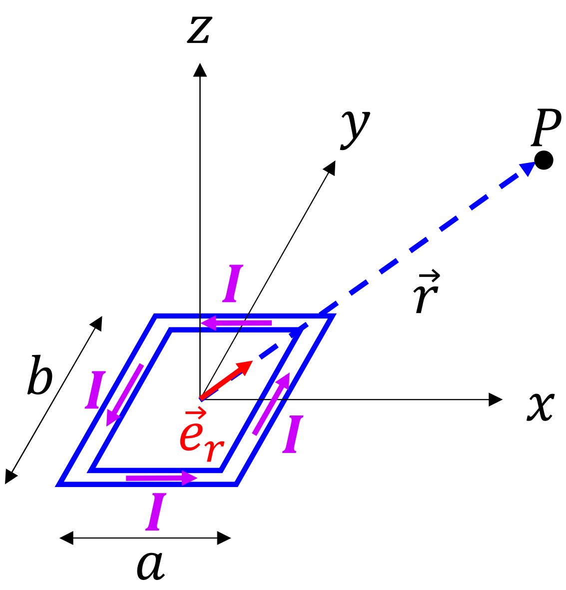
$$B_x = -\frac{\mu_0 I}{2\pi}\frac{y}{r^2} \qquad B_y = +\frac{\mu_0 I}{2\pi}\frac{x}{r^2} \qquad B_z = 0$$

$$B = \sqrt{\left(\vec{B}_x\right)^2 + \left(\vec{B}_y\right)^2 + \left(\vec{B}_z\right)^2} = \frac{\mu_0 I}{2\pi r}$$

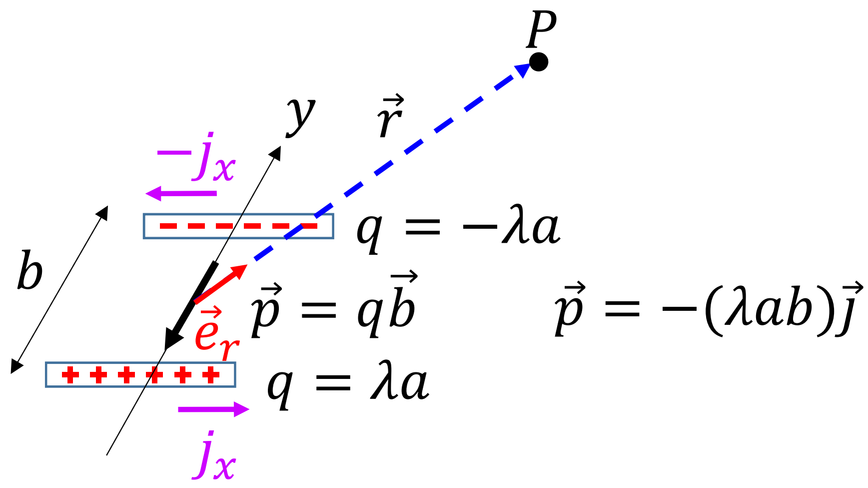
Of course more complicated than via simple Ampere's law !

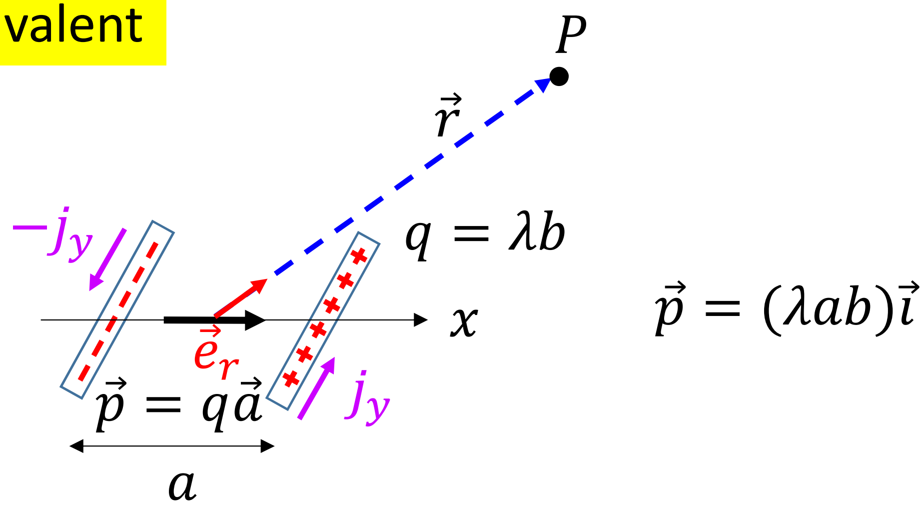
Current loop: Magnetic dipole

Current loop: Magnetic dipole

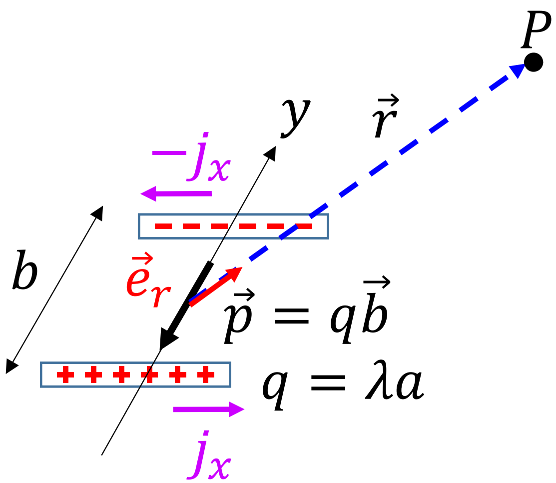


Electrostatic equivalent

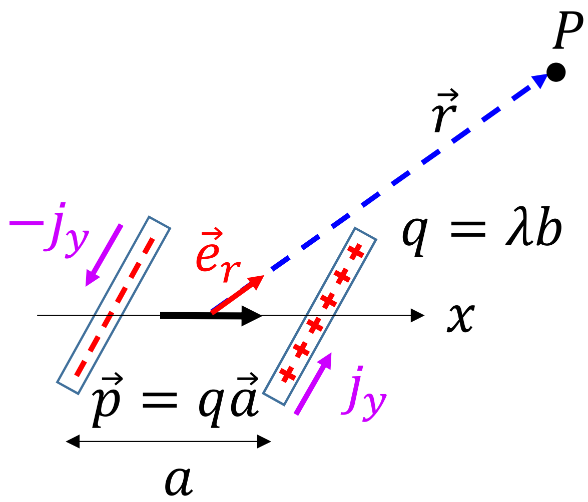




Electrostatic equivalent



$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{e}_r}{r^2}$$



$$\vec{p} = -(\lambda ab)\vec{j}$$

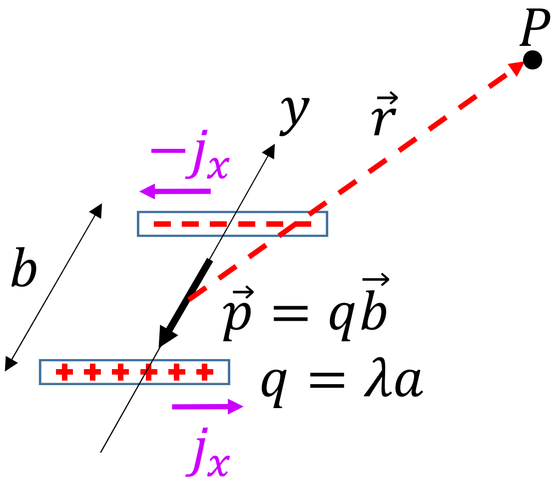
$$\vec{p} = (\lambda ab)\vec{\imath}$$

$$V(P) = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda ab}{r^2}\frac{y}{r}$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ab}{r^2}\frac{x}{r}$$

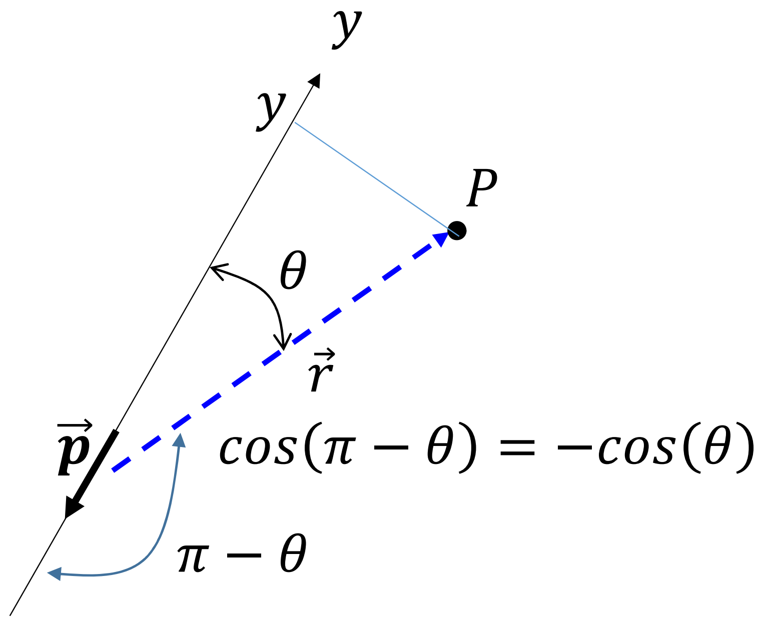
$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{e}_r}{r^2}$$

There is a $cos(\theta)$ here



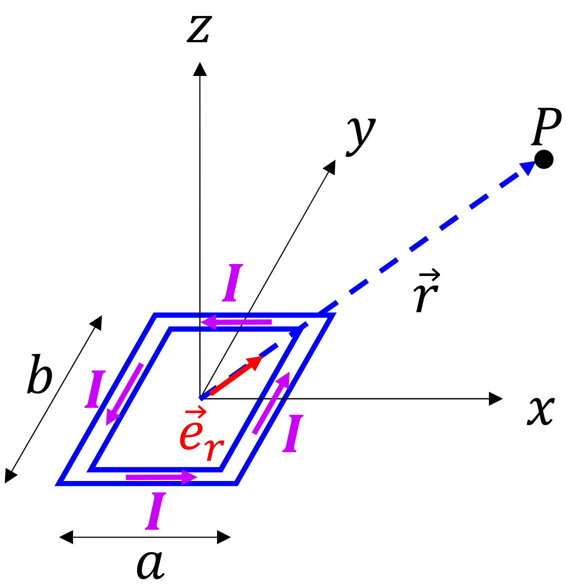
$$\vec{p}.\vec{e}_r = \lambda ab\vec{j}.\vec{e}_r$$

$$\vec{j}.\vec{e}_r = cos(\theta)$$



$$\boxed{cos(\theta) = \frac{y}{r}}$$

Current loop: From electrostatic to equivalent magnetostatic



$$V(P) = -\frac{\lambda ab}{4\pi\varepsilon_0}\frac{y}{r^3}$$

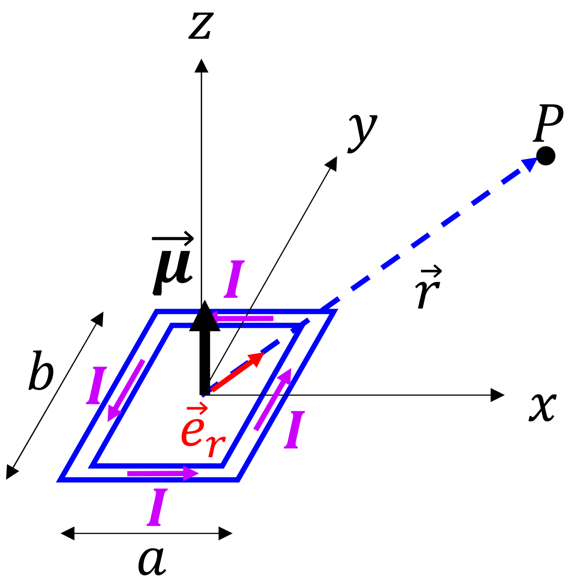
$$V(P) = \frac{\lambda ab}{4\pi\varepsilon_0}\frac{x}{r^3}$$

$$\boxed{A_x(P) = -\frac{\mu_0 Iab}{4\pi}\frac{y}{r^3}}$$

$$\boxed{A_y(P) = \frac{\mu_0 Iab}{4\pi}\frac{x}{r^3}}$$

$A_z(P) = 0$ as $j_z(x, y, z) = 0$

Current loop: Magnetic dipole



$A_z(P) = 0$ as $j_y(x,y,z)=0$

$$\boxed{A_x(P) = -\frac{\mu_0 Iab}{4\pi}\frac{y}{r^3}}$$

$$\boxed{A_y(P) = \frac{\mu_0 Iab}{4\pi}\frac{x}{r^3}}$$

$\boldsymbol{\mu = Iab}$ $\longrightarrow$ **Vector $\vec{\mu}$ along the z-axis**

$$\vec{\mu} = 0\vec{\imath} + 0\vec{\jmath} + \mu\vec{k}$$

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

The vector potential has a real physical meaning

$$\boxed{\vec{A}(P) = \frac{\mu_0}{4\pi}\frac{\vec{\mu}\times\vec{r}}{r^3}}$$

Magnetic dipole

$$B_x = \left(\vec{\nabla} \times \vec{A}\right)_x$$

$$B_y = \left(\vec{\nabla} \times \vec{A}\right)_y$$

$$B_z = \left(\vec{\nabla} \times \vec{A}\right)_z$$

$$B_x = \frac{\mu_0 \mu}{4\pi} \frac{3xz}{r^5}$$

$$B_y = \frac{\mu_0 \mu}{4\pi} \frac{3yz}{r^5}$$

$$B_z = -\frac{\mu_0 \mu}{4\pi} \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right)$$

$\vec{\mu} = I\vec{A}$ magnetic dipole moment

Electric dipole

$$E_x = \frac{p}{4\pi\varepsilon_0} \frac{3xz}{r^5}$$

$$E_y = \frac{p}{4\pi\varepsilon_0} \frac{3yz}{r^5}$$

$$E_z = -\frac{p}{4\pi\varepsilon_0} \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right)$$

$\vec{p} = q\vec{d}$ electric dipole moment

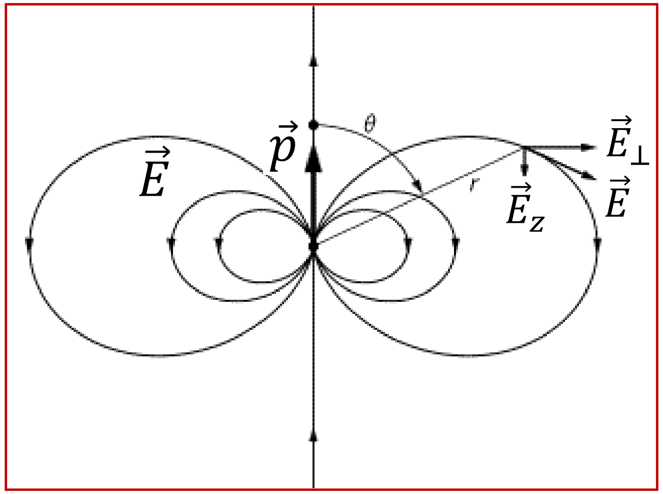
Electrostatic

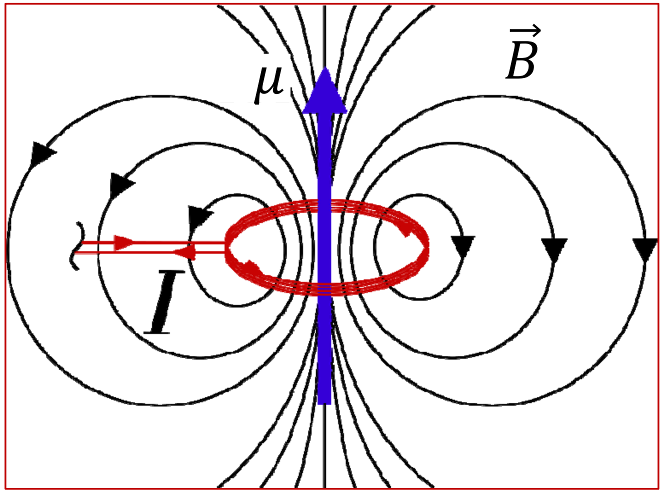
$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

Magnetostatic

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Starting with <u>completely different laws</u> and ending up with the <u>same kind of field</u>





Some important remarks

In the case of dipoles

Electric $\vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0}$

Magnetic $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Completely different laws

Lead to the same kind of fields

Why ?

Because the dipole fields appear only when we are far from the charges and currents

Through most of the relevant space there are no charges nor currents

$\vec{\nabla}.\vec{E} = 0$

$\vec{\nabla} \times \vec{E} = 0$

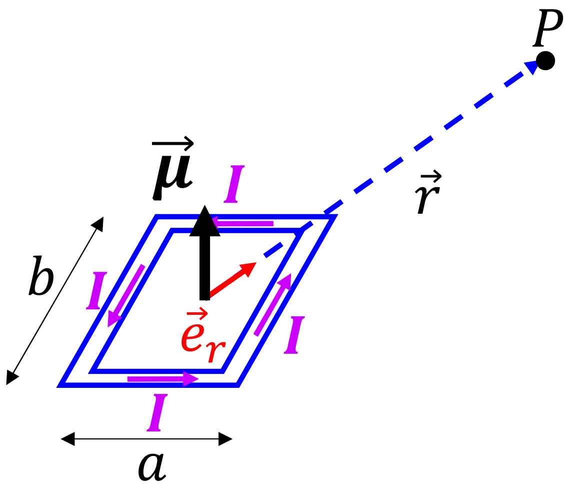
Both give the same solutions

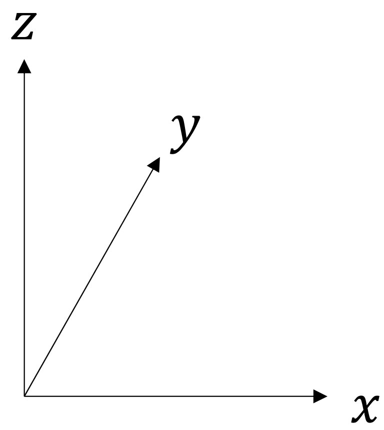
$\vec{\nabla}.\vec{B} = 0$

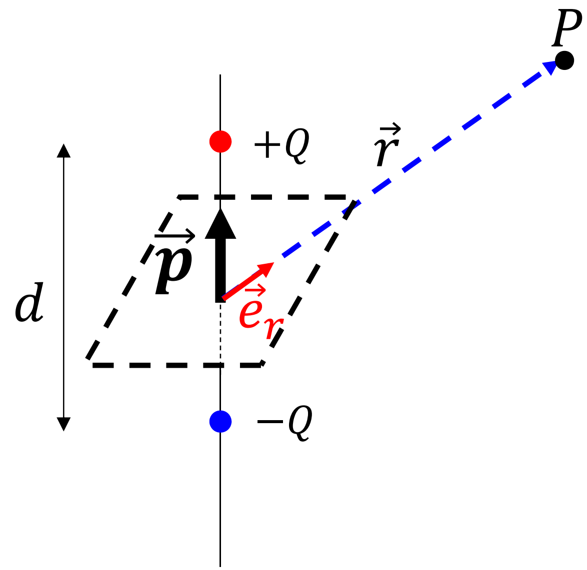
$\vec{\nabla} \times \vec{B} = 0$

$\vec{E}$ and $\vec{B}$ identical

What do we observe far away?







$\boldsymbol{\mu = Iab}$

$\boldsymbol{p = qd}$

Far away we see only dipole moments $\vec{\mu}$ or $\vec{p}$

What is the advantage of the vector potential?

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ ⇒ Integrals of $\vec{B}$ are very complicated

$\vec{B} = \vec{\nabla} \times \vec{A}$ ⇒ Derivatives of $\vec{A}$ much simpler

Integrals of $\vec{A}$ are those of electrostatics **(known)**

Static Maxwell's equations in vacuum

electrostatic

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

Magnetostatic

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

What is true only in a static world and what is true always?

<u>Electric</u> and **<u>Magnetic</u>** phenomena are no longer two distinct subjects in a **DYNAMIC** world: Maxwell

What is true and what is wrong in what we have learned so far?

Wrong in general (true only for static)

Coulomb' s law $$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{e}_r$$

$$\vec{\nabla} \times \vec{E} = \vec{0}$$

$$\vec{E} = -\vec{\nabla}V$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r})\vec{e}_r}{r^2}dV$$

For conductor, $E = 0, V = constant. Q = CV$

True always

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$ Lorentz force

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$ Gauss' law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

In conductor, E makes currents

What is true and what is wrong in what we have learned so far?

Wrong in general (true only for static)

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampere's law

$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}) \times \vec{e}_r}{r^2} dV$ Biot & Savart's law

True always

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ Maxwell's law

$\vec{\nabla} . \vec{B} = 0$ No magnetic charges

$\vec{B} = \vec{\nabla} \times \vec{A}$

What is true and what is wrong in what we have learned so far?

Wrong in general (true only for static)

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson's law

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$ With

$$\vec{\nabla}.\vec{A} = \vec{0}$$

True always

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$ And

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$ With

$$\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0$$

End of static world