Ve230 RC1

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Definition

Commutative Law of Vector Addition:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Definition

Associative Law of Vector Addition:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

Definition

Distributive Law of Vector Addition:

$$n(\vec{A} + \vec{B}) = n\vec{A} + n\vec{B}$$

Definition

Vector Multiplication - Scalar or Dot Product:

$$\vec{A} \cdot \vec{B} = |A||B| \cos \theta_{AB}$$

Commutative Law of Dot Product: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
Distributive Law of Dot Product: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Definition

Vector Multiplication - Vector or Cross Product:

$$\vec{A} \times \vec{B} = |A||B| \sin \theta_{AB} \hat{C}, \hat{C} = \hat{A} \times \hat{B}$$

Non-Commutative Law: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
Non-Associative Law: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

Vector Product in Matrix Format

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z p \end{vmatrix}$$
$$= \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x)$$

Vector Triple Product

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

General Expression

$$d\ell = \mathbf{a}_{u_1}(h_1 du_1) + \mathbf{a}_{u_2}(h_2 du_2) + \mathbf{a}_{u_3}(h_3 du_3)$$

$$ds_1 = h_2 h_3 du_2 du_3$$

$$ds_2 = h_1 h_3 du_1 du_3$$

$$ds_3 = h_1 h_2 du_1 du_2$$

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin\theta\,dR\,d\theta\,d\phi$

Coordinate Transformation - Cartesian to Cylindrical

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$$

$$x = r\cos\phi, y = r\sin\phi, z = z$$

$$r = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x}, z = z$$

Coordinate Transformation - Cartesian to Spherical

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$

$$\begin{bmatrix} \hat{R} \\ \hat{\theta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 \sin\theta_0 & \sin\varphi_0 \sin\theta_0 & \cos\theta_0 \\ \cos\varphi_0 \cos\theta_0 & \sin\varphi_0 \cos\theta_0 & -\sin\theta_0 \\ -\sin\varphi_0 & \cos\varphi_0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Gradient

General expression:

$$\nabla \equiv \left(\mathbf{a}_{u_1} \frac{\partial}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3 \partial u_3} \right)$$

Cartesian:

$$\nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

Cylindrical:

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$$

Spherical:

$$\nabla = \hat{R} \frac{\partial}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi}$$

Divergence

General Expression:

$$\nabla \cdot \mathbf{A} \equiv \mathrm{div}\mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

Cartesian:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical:

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Sphericpal:

$$\nabla \cdot \vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Definition

Divergence Theorem:

$$\int_v \nabla \cdot \overline{A} dV = \oint_s \overline{A} \cdot d\overline{S}$$

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$\boxed{\text{Flux density} \times \text{volume} = \text{Flux}}$

The net circulation of a vector field around a closed path is:

$$\oint_c \overline{A} \cdot d\bar{l}$$

Curl is the circulation per unit are in short.

$$\nabla \times \overline{A} = \lim \frac{1}{\Delta s} \left(\hat{n} \oint_c \overline{A} \cdot d\bar{l} \right)$$

The magnitude is the maximum net circulation of A per unit area as the area tends to zero.
The direction is the normal direction of the area. (right-hand rule)

Curl

General Expression:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Cartesian:

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial yp} \right)$$

Cylindrical:

$$\nabla \times \mathbf{A} = \mathbf{a}_r \left(\frac{\partial A_z}{r \partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \mathbf{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \mathbf{a}_z \left[\frac{\partial}{\partial r}(r A_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

Spherical:

$$\nabla \times \mathbf{A} = \mathbf{a}_R \frac{1}{R \sin\theta} \left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \mathbf{a}_\theta \frac{1}{R} \left[\frac{1}{\sin\theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(R A_\phi) \right] + \mathbf{a}_\phi \frac{1}{R} \left[\frac{\partial}{\partial R}(R A_\theta) - \frac{\partial A_R}{\partial \theta} \right]$$

Definition

Stoke's Theorem:

$$\int_s (\nabla \times \overline{A}) \cdot d\overline{S} = \oint_c \overline{A} \cdot \overline{l}$$

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

$\boxed{\text{Circulation density} \times \text{area} = \text{Circulation}}$

Two null identities

$$\nabla \times (\nabla V) = 0$$

If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.

$$\nabla \cdot (\nabla \times A) = 0$$

If a vector field is divergenceless, then it can be expressed as the curl of another vector field.

Combination of Vector Operators: Laplacian

$$\nabla \cdot \nabla V = \nabla^2 V$$

Cartesian:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

- Solenoidal and irrotational:
  $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$
  Example: a static electric field in a charge-free region
- Solenoidal but not irrotational:
  $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} \neq 0$
  Example: a steady magnetic field in a current-carrying conductor
- Irrotational but not solenoidal:
  $\nabla \times \mathbf{F} = 0$ and $\nabla \cdot \mathbf{F} \neq 0$
  Example: a static electric field in a charged region
- Neither solenoidal nor irrotational:
  $\nabla \cdot \mathbf{F} \neq 0$ and $\nabla \times \mathbf{F} \neq 0$
  Example: an electric field in a charged medium with a time-varying magnetic field

Helmholtz Theorem: A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

Practice

A vector field $\mathbf{D} = \mathbf{a}_R(\cos^2\phi)/R^3$ exists in the region between two spherical shells defined by $R = 1$ and $R = 2$. Evaluate

a). $\oint \mathbf{D} \cdot ds$

b). $\int \nabla \cdot \mathbf{D} dv$

Postulates of Electrostatics in Free Space

Differential Form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \nabla \times \mathbf{E} = 0$$

Integral Form:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}, \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

Coulomb's Law

$$\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0|\mathbf{R} - \mathbf{R}'|^3}$$

For a system of Discrete charges,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k(\mathbf{R} - \mathbf{R}'_\mathbf{k})}{|\mathbf{R} - \mathbf{R}'_\mathbf{k}|^3}$$

An Electric Dipole

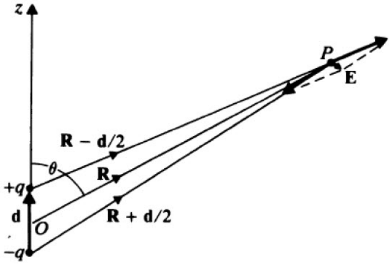

FIGURE 3–5
Electric field of a dipole.

Electric dipole moment: the product of the charge q and the vector **d**, $\mathbf{p} = q\mathbf{d}$.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right]$$

Field:	Potential:	
$\vec{E}(\vec{r}) = \sum_{i=1} \frac{q_i(\vec{r}-\vec{r_i})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_i}\rvert^3}$	$V(\vec{r}) = \sum_{i=1} \frac{q_i}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_i}\rvert}$	Point Charges
$\vec{E}(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})(\vec{r}-\vec{r_s})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_s}\rvert^3} dl_s$	$V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_s}\rvert} dl_s$	Line Charges
$\vec{E}(\vec{r}) = \iint_S \frac{\sigma(\vec{r_s})(\vec{r}-\vec{r_s})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_s}\rvert^3} dS_s$	$V(\vec{r}) = \iint_S \frac{\sigma(\vec{r_s})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_s}\rvert} dS_s$	Surface Charges
$\vec{E}(\vec{r}) = \iiint_V \frac{\rho(\vec{r_s})(\vec{r}-\vec{r_s})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_s}\rvert^3} dV_s$	$V(\vec{r}) = \iiint_V \frac{\rho(\vec{r_s})}{4\pi\varepsilon_0\lvert\vec{r}-\vec{r_s}\rvert} dV_s$	Volume charges

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

Summing scalars is easier than summing vector!!! 34