Conductors and electrostatics

Main questions related to conductors

How do static charges distribute in a conductor?

How does the electric field look like <u>inside</u> and <u>outside</u> a conductor? Does the shape matter?

What is the potential inside a conductor?

What impact on an external field pattern when a conductor is placed in that field?

Conductor

- In a conductor the charges are free to move if subjected to a force: <u>very high conductivity</u>

- **<u>When isolated</u>** ⇨ charge must be conserved: <u>local</u> as well as <u>overall</u> neutrality prevail

  ➢ If charges are <u>added</u>, they redistribute minimizing the total energy but they remain in the conductor which is charged

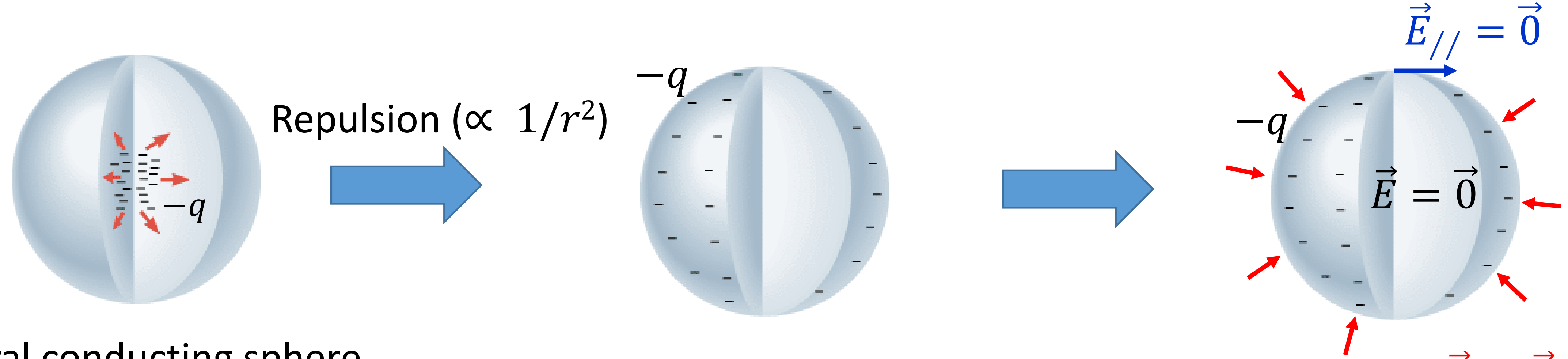
  ➢ If charges are <u>induced</u>, they redistribute: **overall** neutrality prevails **BUT NOT locally** and once the field is removed the conductor retrieves its initial condition

Consequence of coulomb ($\propto\ 1/r^2$ law)

Charging a conductor

Isolated conductor: Charge conservation

Uniformly distributed **BECAUSE** of the spherical shape !

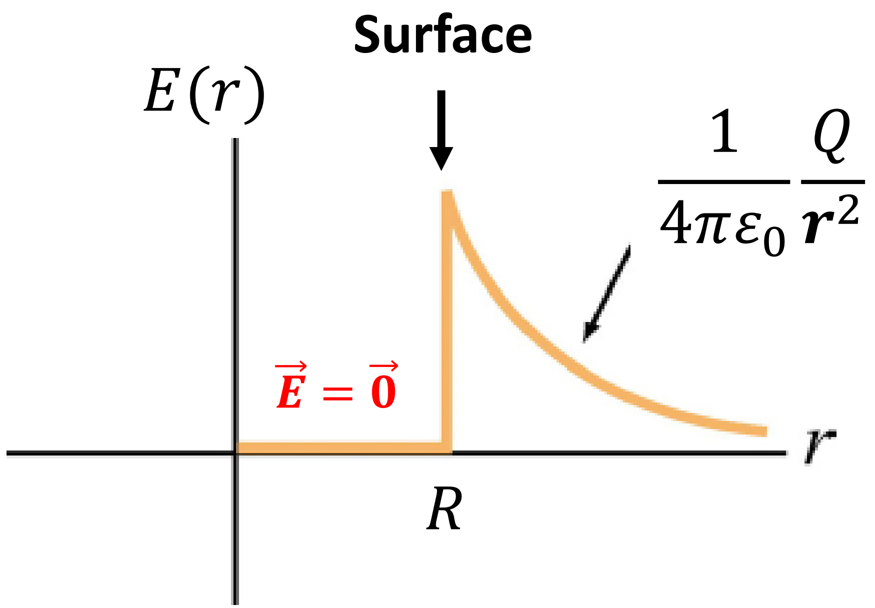


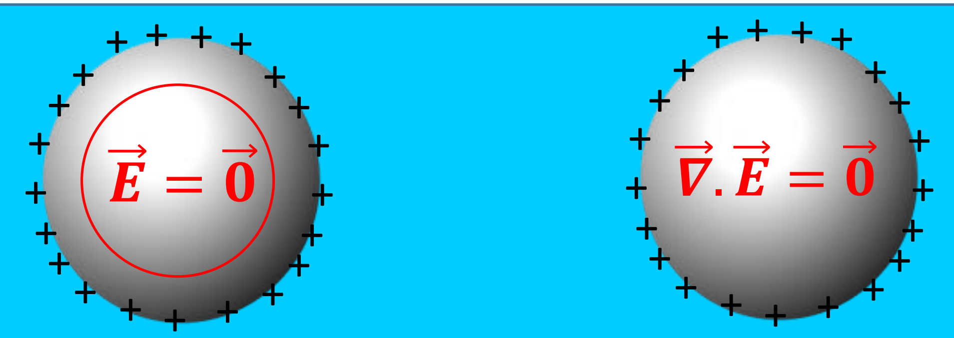
Neutral conducting sphere
+
Excess electrons added inside

Excess charges pushed towards surface but cannot leave the conductor

At equilibrium and electrostatic conditions, excess charges **MUST** reside on the surface of the conductor

The hollow or solid conducting sphere is an example where:



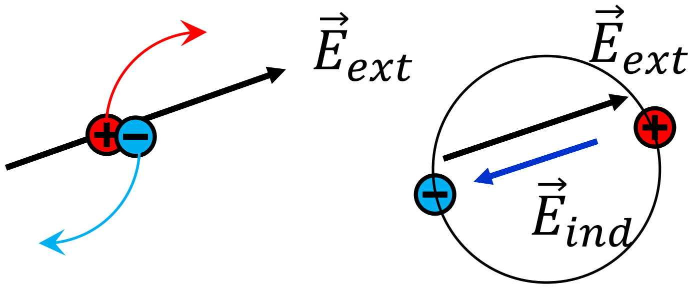


$$\oint \vec{E}.d\vec{A} = \int \vec{\nabla}.\vec{E}\, dV = 0$$

Closed surface inside the sphere — Volume enclosed by the surface

What does this mean?

Why is the electric field zero inside the conductor ?



$\vec{E}_{ext} + \vec{E}_{ind} = \vec{0}$ Charges accumulate at the surface

From Gradient

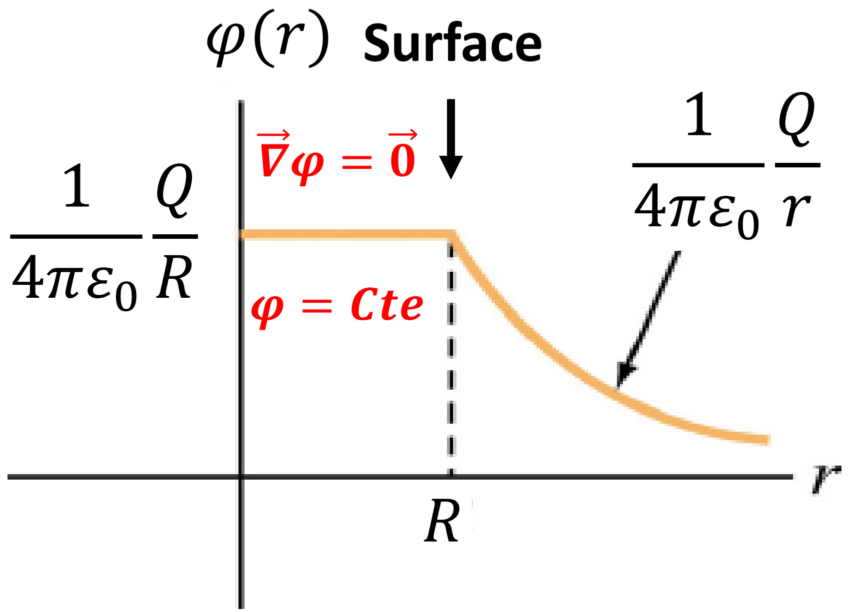
$\vec{E} = \vec{0} \Rightarrow -\vec{\nabla}\varphi = 0 \Rightarrow \varphi = Cte \Rightarrow$ The whole conductor is an equipotential **Including the surface**

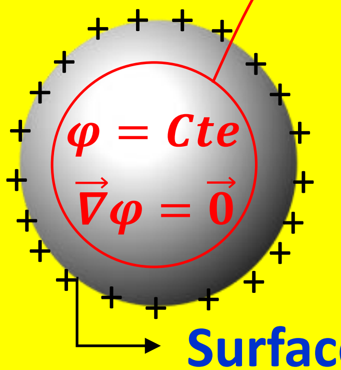
From Divergence

$\vec{E} = \vec{0} \Rightarrow \vec{\nabla}.\vec{E} = 0 \Rightarrow$ **No charge inside**

The hollow or solid conducting sphere is an example where:

$\Leftrightarrow$ **Volume** $\vec{E} = \vec{0}$, $\vec{\nabla}.\vec{E} = \vec{0}$





$\varphi = Cte$, $\vec{\nabla}\varphi = \vec{0}$

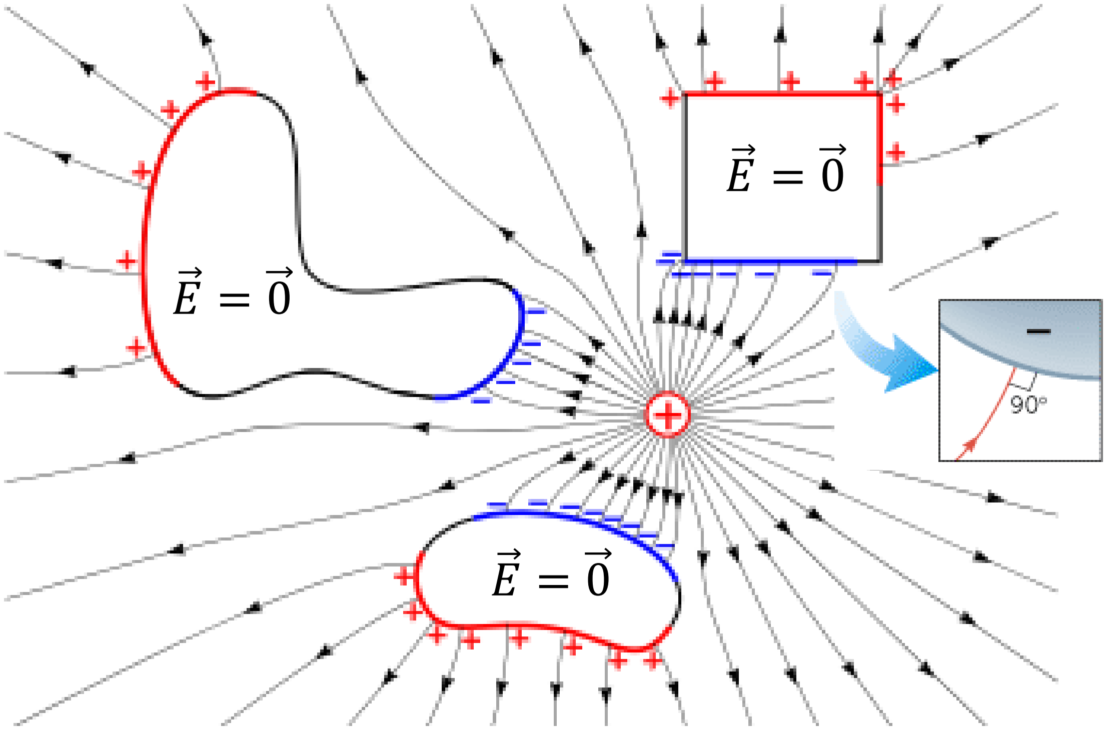
$$d\varphi = \vec{\nabla}\varphi . d\vec{l} = 0$$

$$\varphi = \int_a^b \vec{\nabla}\varphi . d\vec{l} = -\int_a^b \vec{E} . d\vec{l} = 0$$

*Along any **open** or **close**d path on the surface*

What does this mean?

Why the electric field at the surface must be perpendicular?



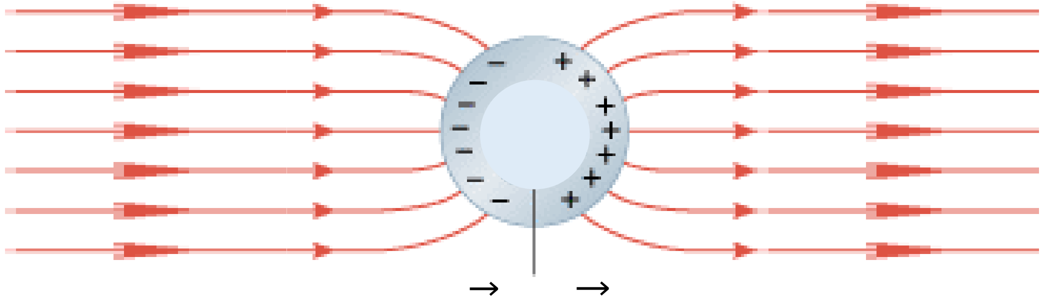
If not, there would be a current flowing on the surface,
and that surface would cease to be an equipotential

$$\int \vec{E}.d\vec{l} = \int -\vec{\nabla}\varphi.d\vec{l} = 0$$

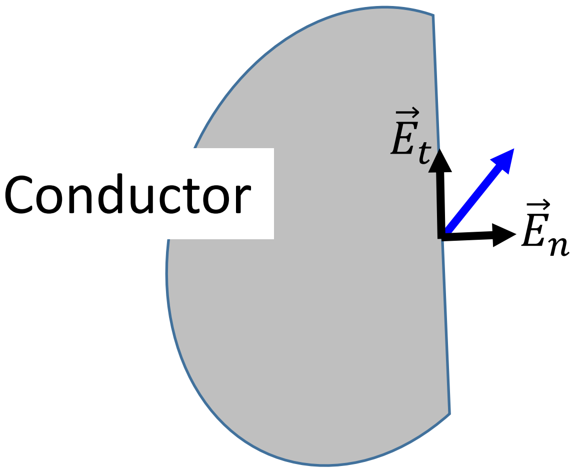
Path on the surface

$$\boxed{\vec{E} \perp d\vec{l}}$$

The external field lines MUST curve when approaching the conductor to hit it perpendicularly



$\vec{E}_t$ cannot exist on the surface of a conductor for another physical reason



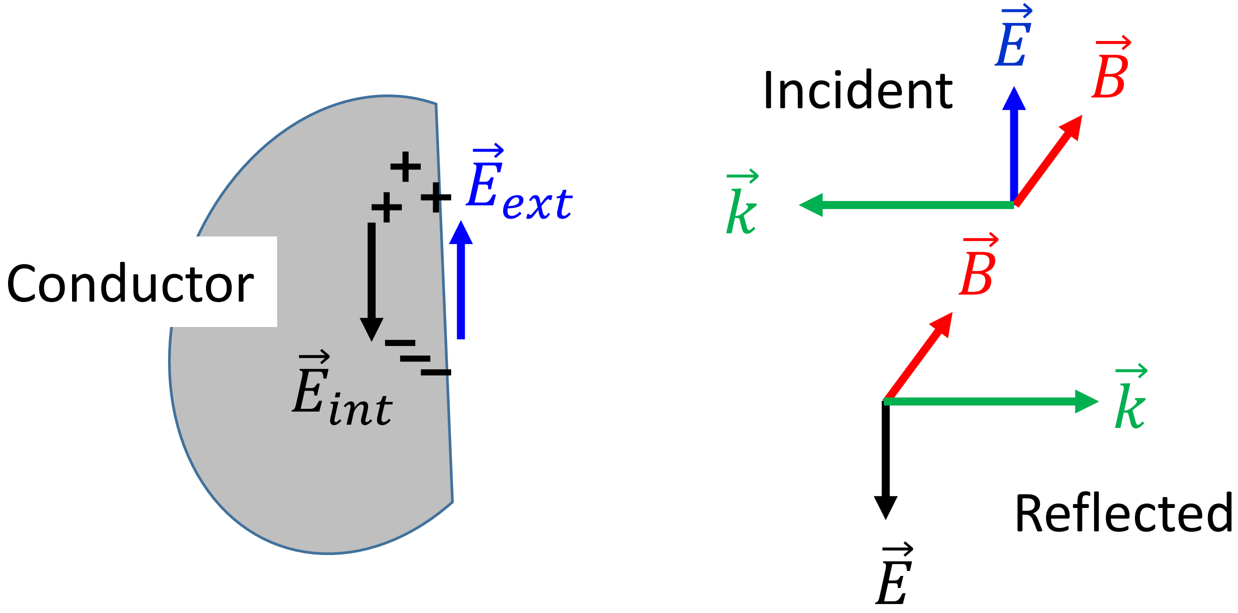
If $E_t \neq 0$ ⇨ charges will move along the surface

⇨ perpetual motion can be created with a charge outside !

***Question #1:

What does this property imply regarding the interaction of a conductor with Electromagnetic waves

Answer to Question #1: Conductors are very good reflectors

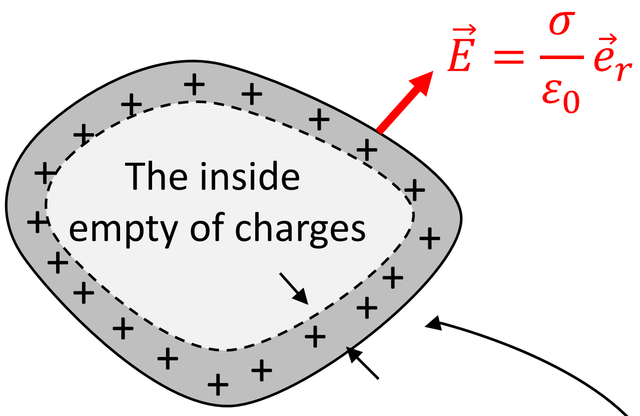


$$\vec{E}_{int} + \vec{E}_{ext} = \vec{0}$$

As the external field E_{ext} oscillates so does the internal field E_{int} and the conductor transforms into a source of Electromagnetic wave

Perfect reflector if the conductivity is infinite

Conductor and charge distribution



Excess charge in a conductor distribute on the surface with $\vec{E} = \vec{0}$ inside

How thick is the surface layer where all charges distribute?

- $\sigma = 8.85 \times 10^{-12} \times 10^{6} = 10\mu C/m^2$
- Charge of electron q $= \ 1.6 \times 10^{-19}C$
- Concentration of free electron in a metal N = $10^{28}m^3$
- $\sigma = \frac{nq}{S} = \frac{nqd}{Sd} = Nqd$
- $E = \frac{\sigma}{\varepsilon_0} = 10^5 V/m$

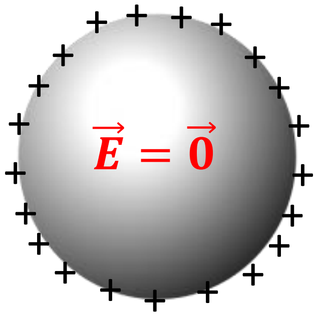
Thickness $\mathrm{d} = \frac{\sigma}{qN} = 10^{-14}m$ *size of a nucleus!*

Here σ does not stem for surface charge density but for conductivity. The same symbols can be used for different physical quantities

Charges distribute really at the surface

That $\vec{E} = \vec{0}$ inside the conducting sphere is NOT trivial at all !

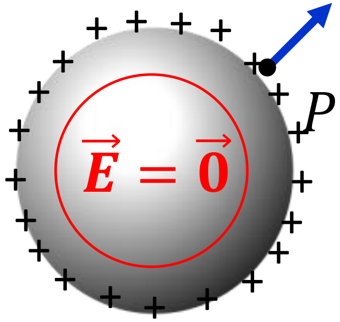
<u>Conspiracy principle</u>



Every charge on the sphere creates a field inside and outside the sphere **BUT** all individual effects when added cancel out inside.

$$\frac{1}{\varepsilon_0}\frac{Q}{4\pi \boldsymbol{r}^2} \qquad \boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\varepsilon_0}}$$

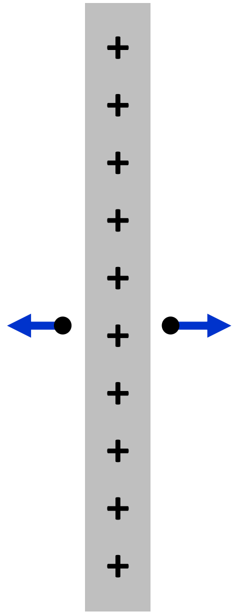
Consequence



$P\ (r \approx R)$

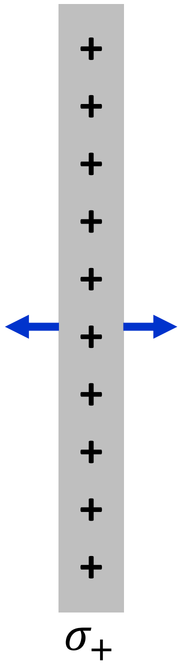
Whereas

$$\boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{2\varepsilon_0}} \qquad \boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{2\varepsilon_0}}$$



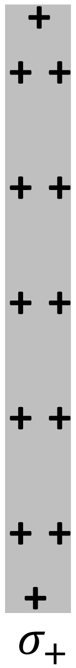
σ_+

$$E = \frac{\boldsymbol{\sigma}}{2\varepsilon_0}$$

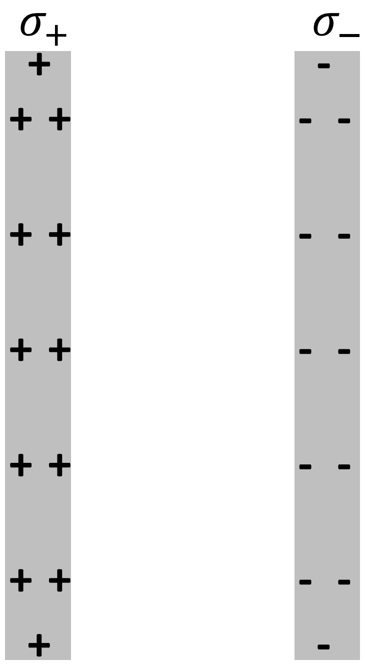


$$E = \frac{\boldsymbol{\sigma}}{2\varepsilon_0}$$

Real representation is this

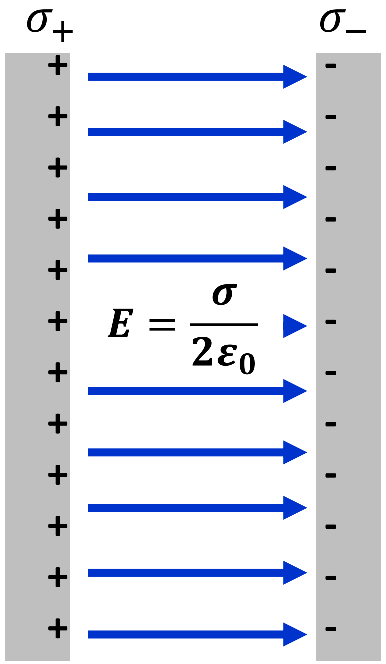


Remember that a charge added to a conductor spreads all around the surface !



Turns out to be this

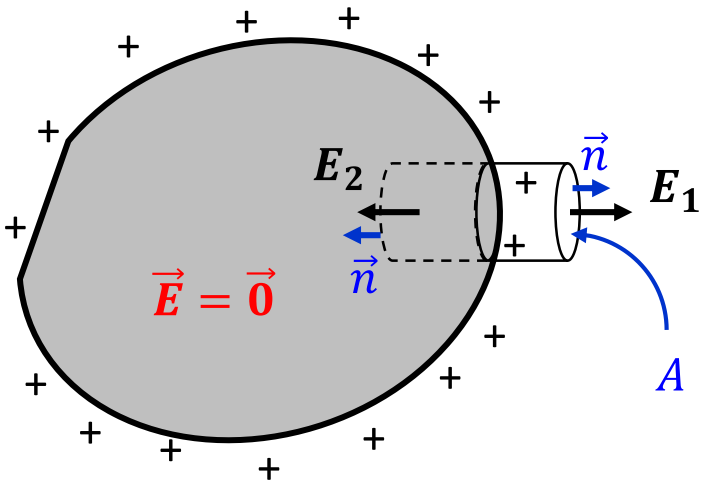
$$E = 0$$



$$E = 0$$

It is the mutual influence that brings all charges on one side!

In both cases $E_1 A + E_2 A = \dfrac{\sigma A}{\varepsilon_0}$

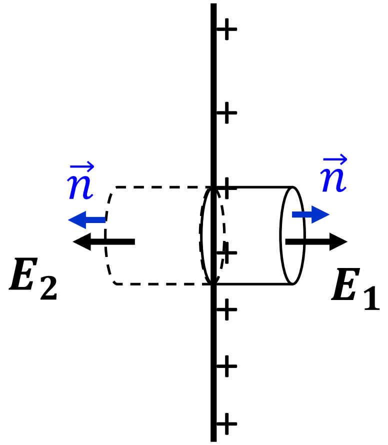


We evaluate the field in the immediate neighborhood of the conductor

All charges outside the pill box conspire to "kill" the field E_2

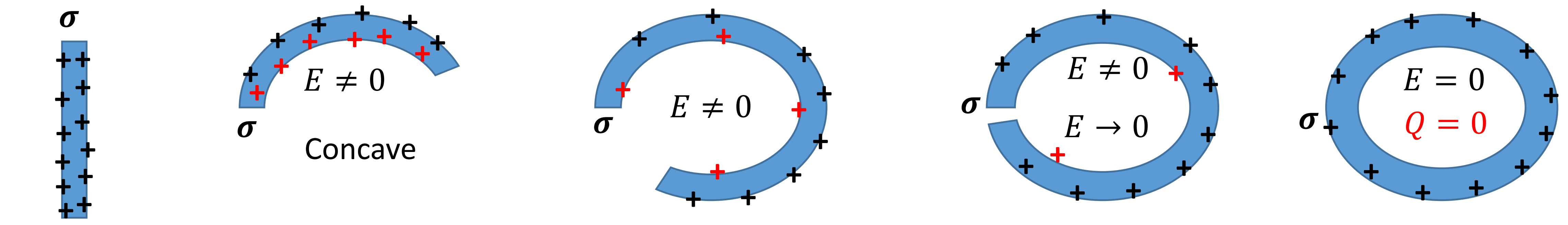
$$E_1 + E_2 = E_1 + 0 = \boxed{E = \frac{\sigma}{\varepsilon_0}}$$

"Conspiracy principle"



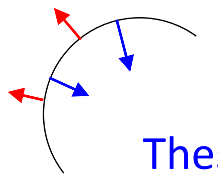
$$E_1 = E_2 = \boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

"Conspiracy" principle

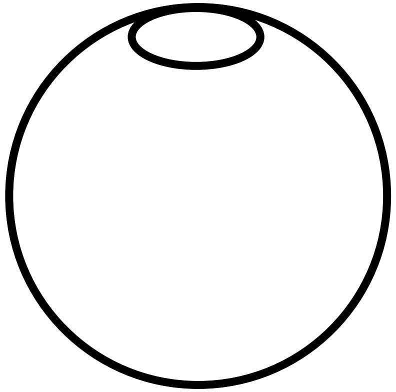


$\boldsymbol{E_1 = \dfrac{\sigma}{2\varepsilon_0}}$ $\quad$ $\boldsymbol{E_2 > E_1}$ $\quad$ $\boldsymbol{E_3 > E_2}$ $\quad$ $\boldsymbol{E_4 > E_3}$ $\quad$ $\boldsymbol{E_5 = \dfrac{\sigma}{\varepsilon_0}}$

These unit vectors diverge



These unit vectors converge

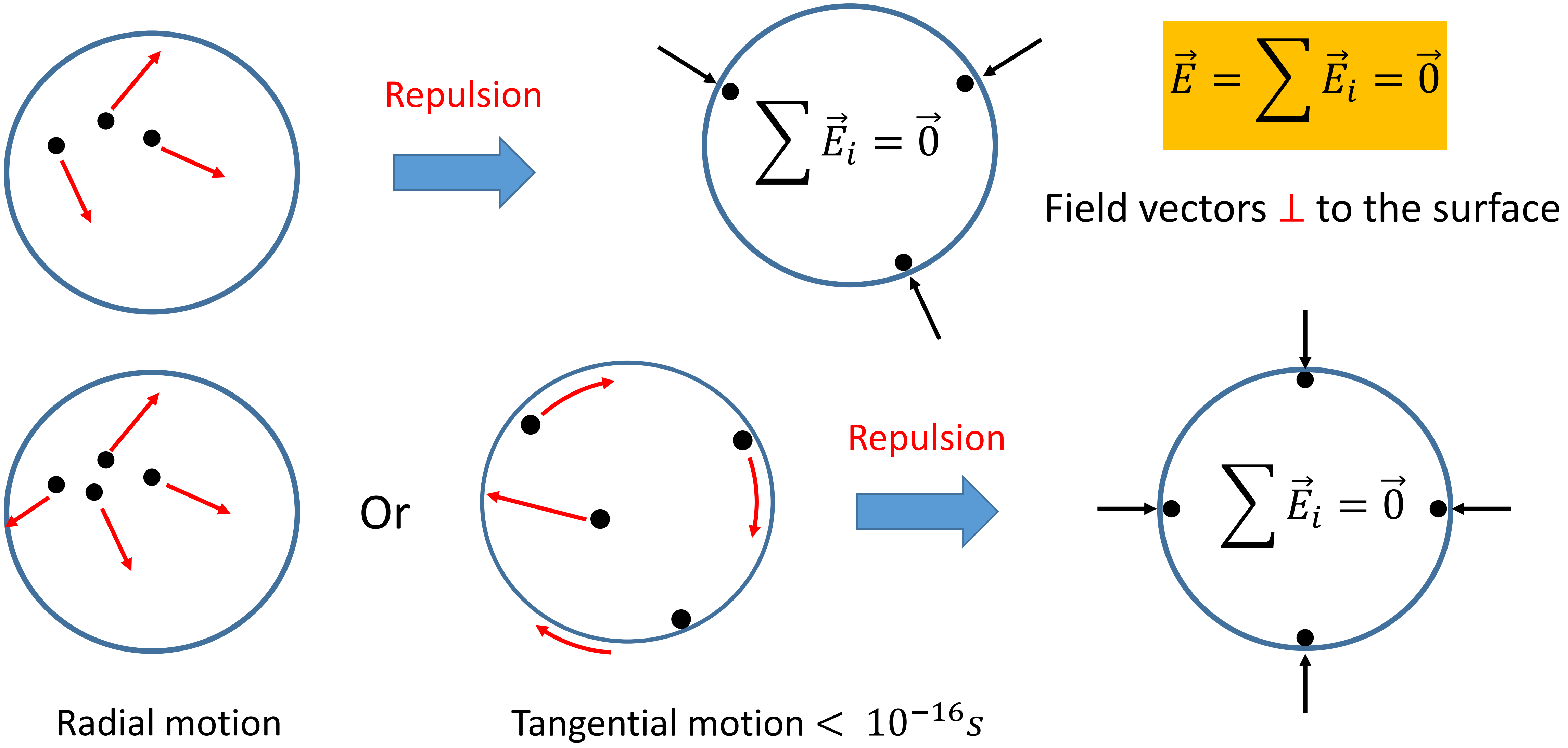


Remember the hollow sphere with an opening:
Field inside is not completely canceled

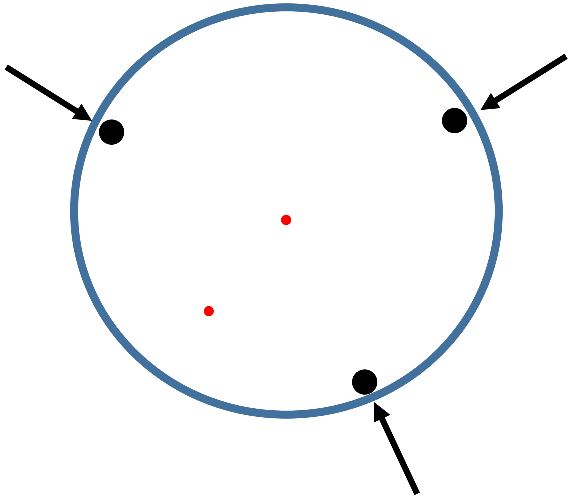
See slide #90, E_Lectures 8&9_Electrostatics_Gauss law

Consider a 2D conducting disc

● Negative charge

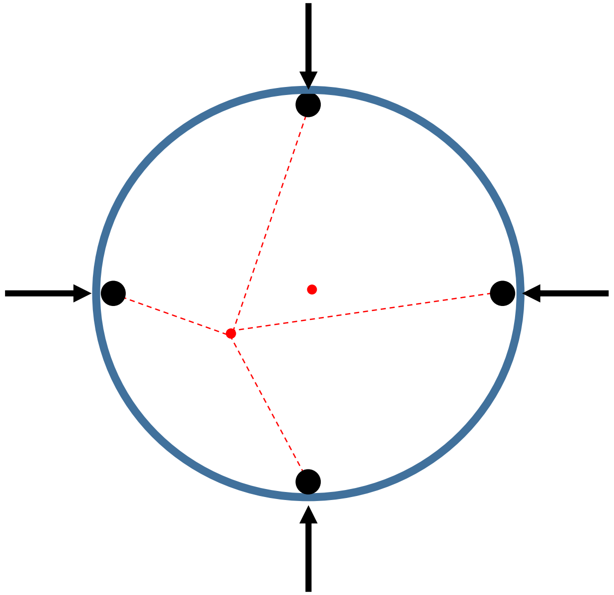


What about the potential in the conducting disc?



Inside the conducting disc

$$\varphi = \sum \varphi_i = \sum \frac{1}{4\pi\varepsilon_0}\frac{q}{r_i} =$$



r_i is the distance from each charge to the point of consideration: In the case of the center

$$\varphi = \frac{3q}{4\pi\varepsilon_0 R}$$

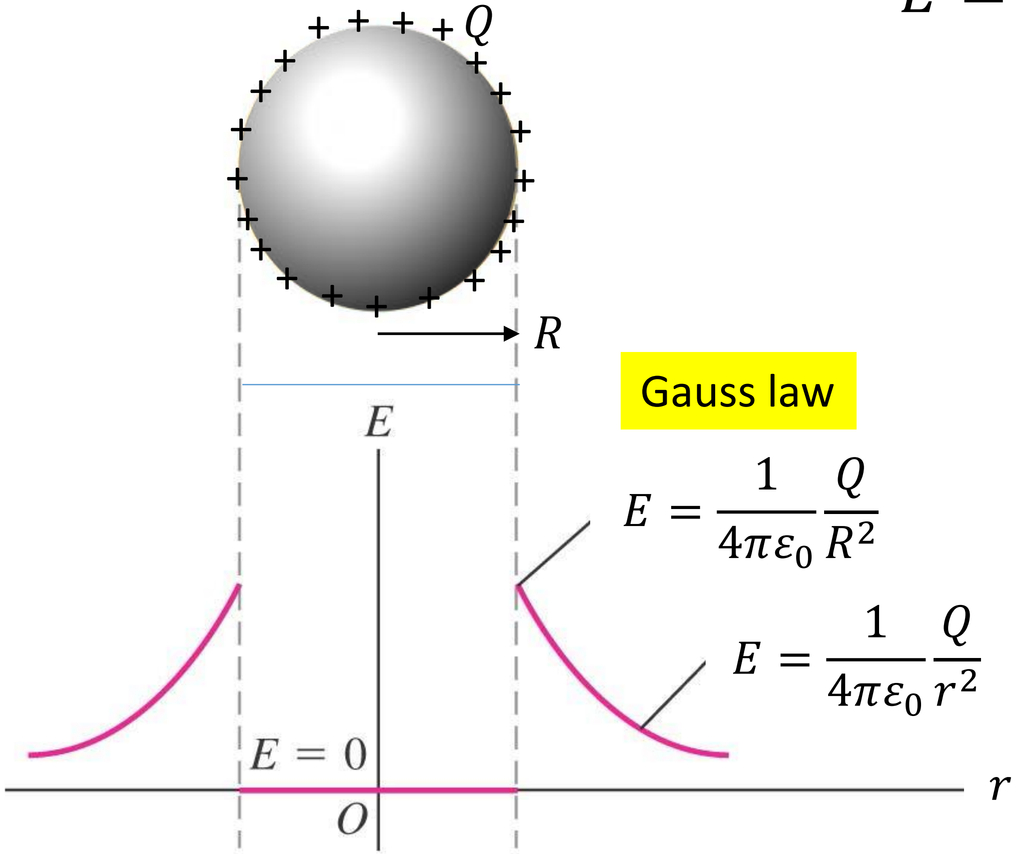
Everywhere inside the disc as $\sum \vec{E}_i = \vec{0}$ inside

$$\varphi = \frac{4q}{4\pi\varepsilon_0 R}$$

The electric field and potential from the center of the sphere to infinity

$\vec{E} = -\vec{\nabla}.\varphi(r)$

$$\text{here}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

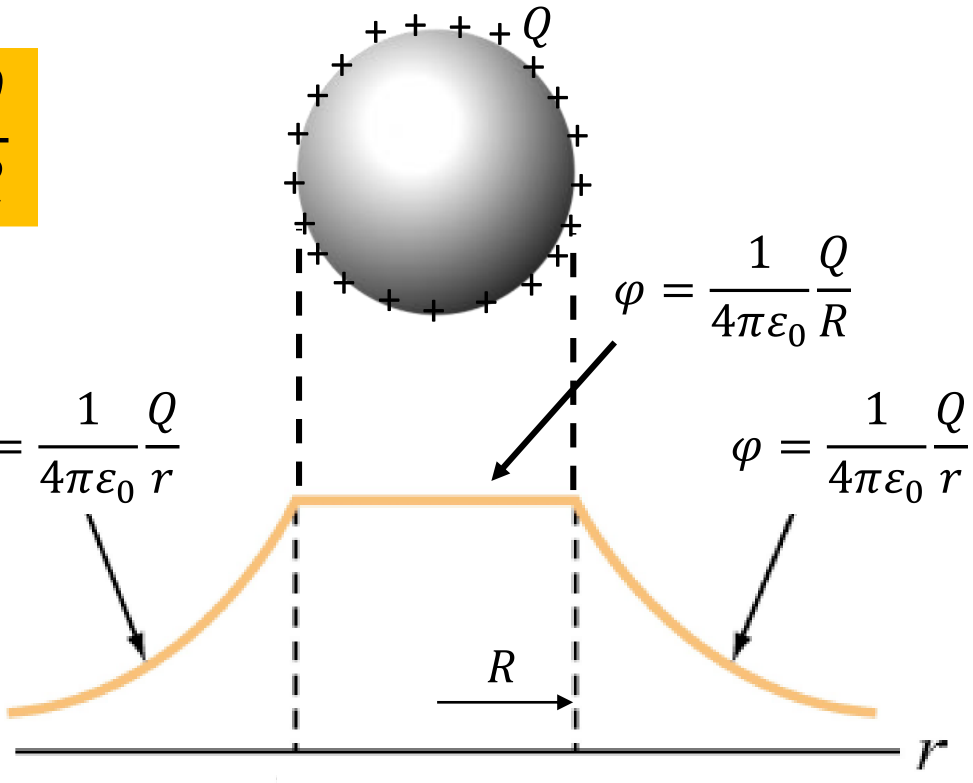


Gauss law

$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}$

$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$

Discontinuity for E

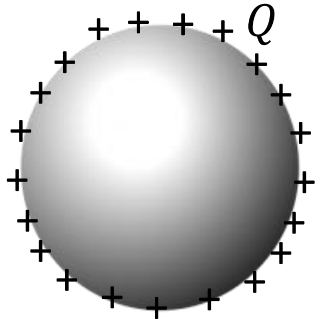


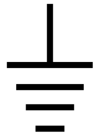
$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$

$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

The earth as an infinite sink for charges



$=$ 

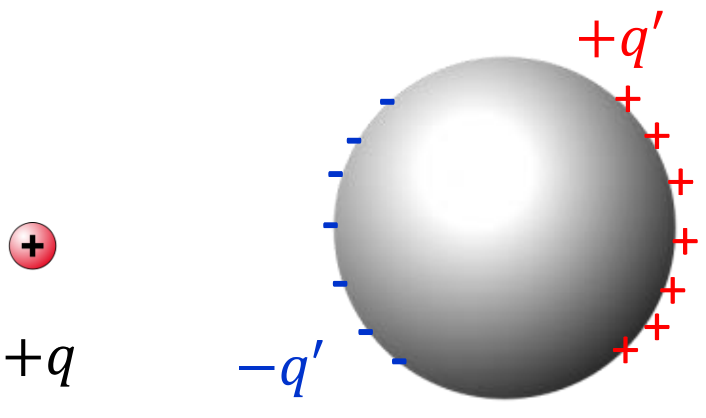
$R = 6.4 \times 10^{3} m$

$$\varphi_G(earth) = 1.4 \times 10^{-12} Q \ (V) \approx 0$$

$$Q = 10^{9} C \quad \rightarrow \quad \boldsymbol{\varphi_G(earth) = 1.4 \times 10^{-3}\ V}$$

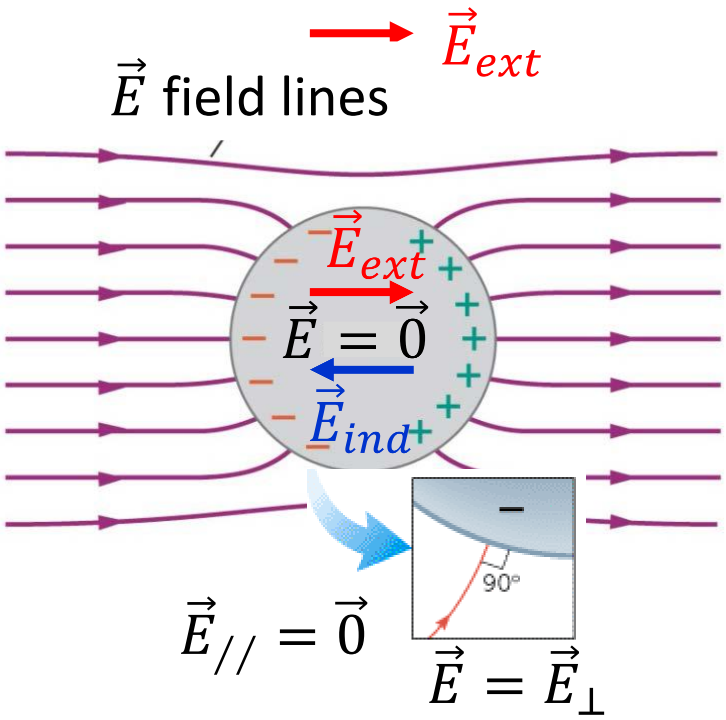
Inducing charges in a conductor

Isolated conductor: Charge conservation ⇨ overall conductor remains neutral **BUT** not locally



Neutral conducting sphere
+
A charge outside (Induction)

In both case, q_{ind} results from the forces involved. Equilibrium must reached



Non uniform field outside

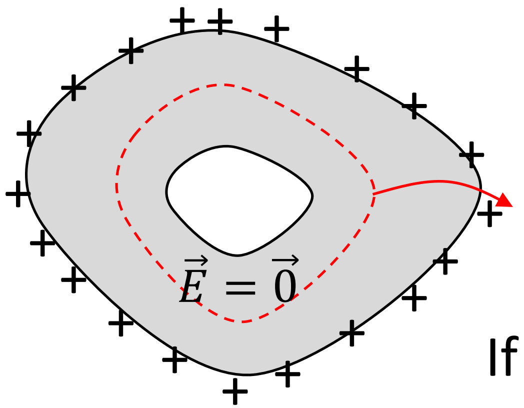
Uniform field outside

**Question 1: Does the sphere as a whole feel a force from these two external fields?

Answer to **Question 1:

In the case of non uniform field. Attraction because $-q_{ind}$ is closer to $+q$ than $+q_{ind}$

Conductor with a hollow

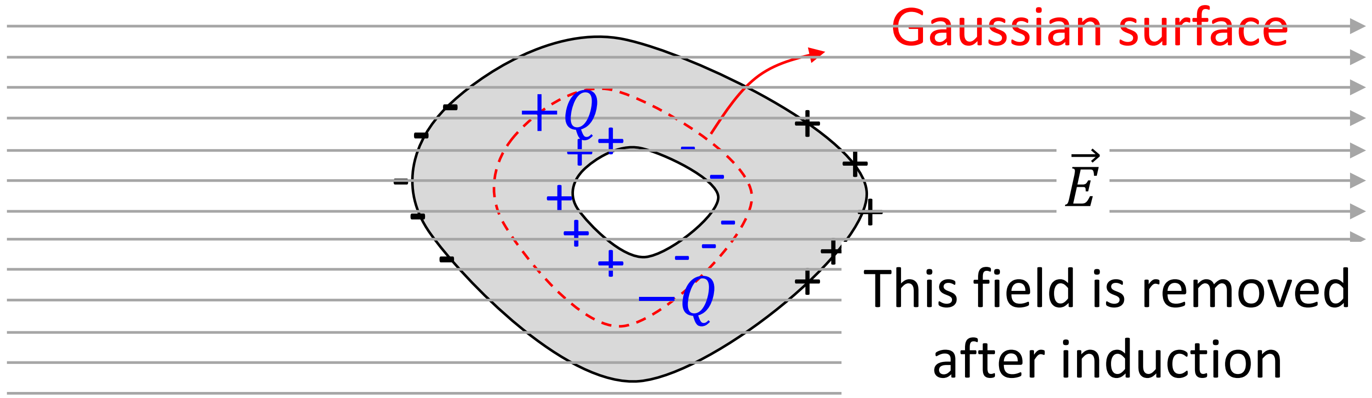


If **no charge** in the cavity AND because $\vec{E} = \vec{0}$ in the conductor

⇨ **no charge in the inner surface of the conductor**

How to prove it ?

Do charges put on the outer surface of a hollow conductor or an external field induce charges inside?



Charge conservation is respected

$\Phi = 0$ because $E = 0$ in conductor

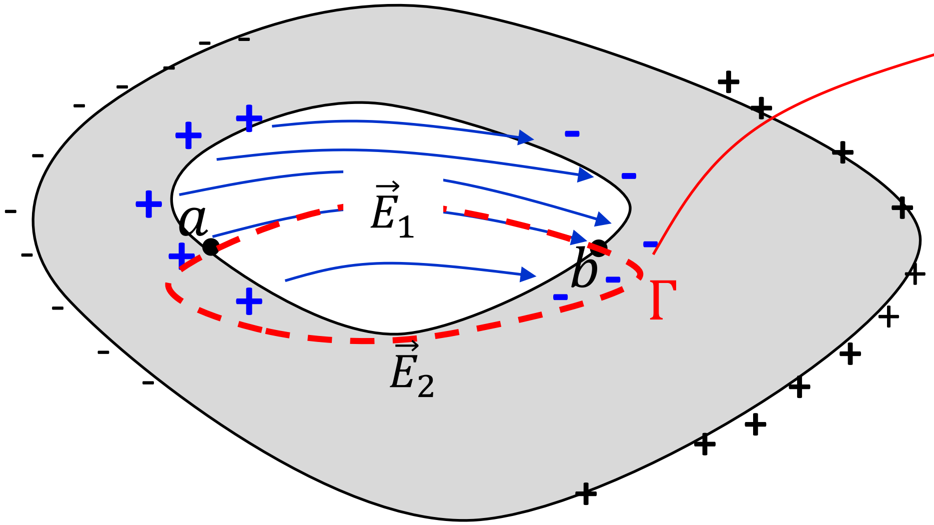
- No **NET** charge in the cavity and in the inner surface which is okay with the charge conservation law.
- Shape not spherical, Gauss' law cannot exclude the case $-Q$ and $+Q$

If we had a sphere, we could argue from symmetry ($\vec{E} \parallel \vec{A}$, and E constant along the surface) that there will be **NO** charge at **ALL**

$E.A = 0 \Rightarrow E = 0 \Rightarrow Q = 0$

But the most beautiful and elegant demonstration comes from Stoke's theorem

If charges are induced in the inner surface there must be field lines



Closed loop along a field line going from + to -

$$\oint_\Gamma \vec{E}.d\vec{l} = ?$$

$$\oint \vec{E}.d\vec{l} = \int_a^b \vec{E}_1.d\vec{l} + \int_b^a \vec{E}_2.d\vec{l}$$

? $\quad$ = 0 Conductor

Cannot be 0 if there are field lines because there are charges!

Stoke's theorem

$$\oint_\Gamma \vec{E}.d\vec{l} = \iint_A (\vec{\nabla} \times \vec{E}).\vec{n}\, dA = \mathbf{0}$$

Because the field derives from a potential ! $\vec{\nabla} \times \vec{\nabla}\varphi = 0$

To make $\int_a^b \vec{E}_1 . d\vec{l} = 0$, we must clear up the field lines thus the charges

In an **EMPTY** cavity of a conductor there can be

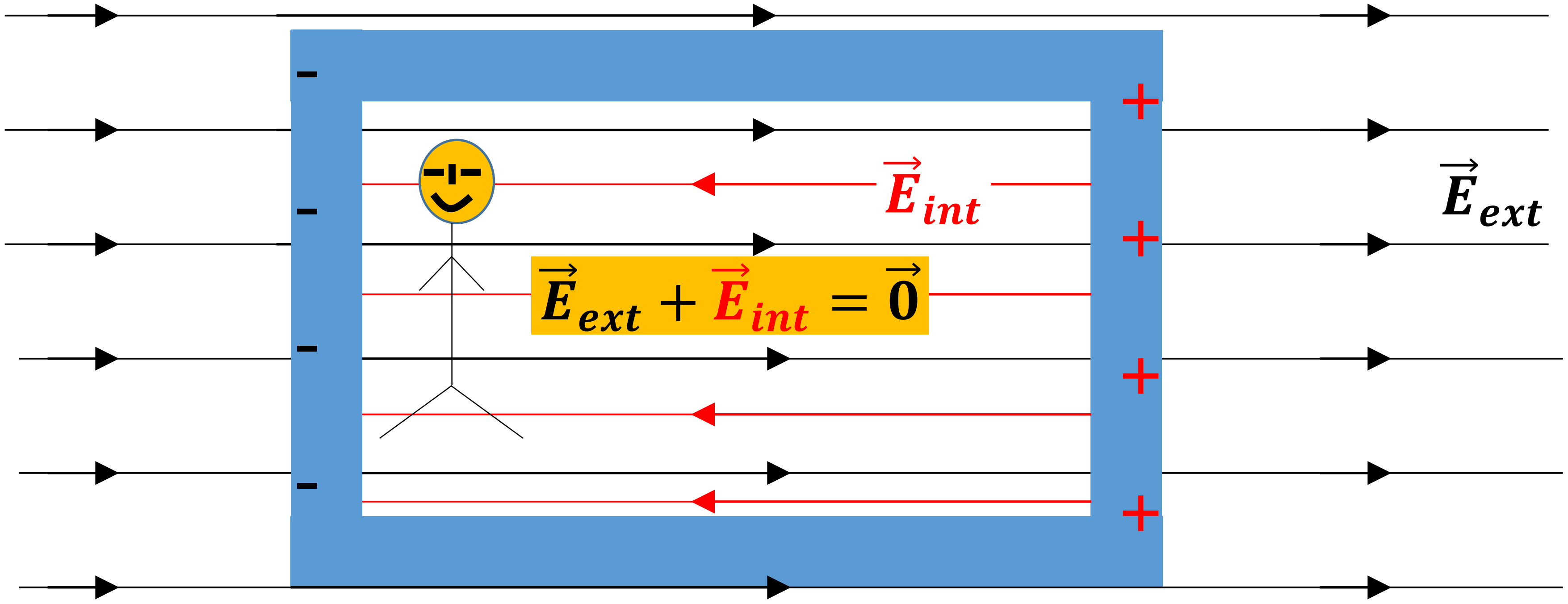
- **NO** charges in the inner surface thus **NO** field
- **NO** matter what is outside or on the outer surface

***Question 2: What can technology make out with this physical property?

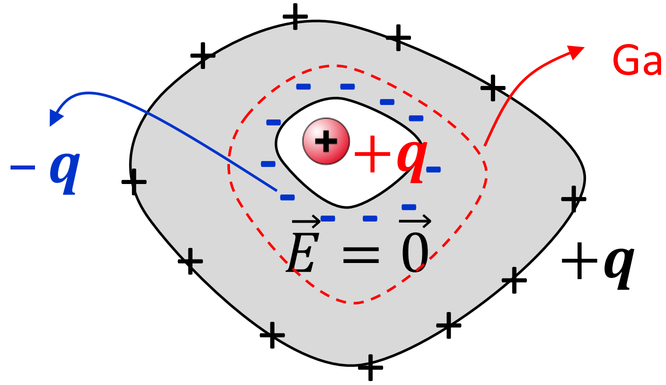
Answer to ***Question 2: Faraday cage (shielding) which prevents you from electric shocks

Shielding Electric Fields

A box or room made of metal liner can shield its interior from external electric fields. Mobile electrons in the metal will respond to the field and reorient themselves until the field inside the box no longer exists. The external field (black) points right. This causes a charge separation in the box (e^{-}'s migrating left), which produces its own field (red), negating the external field. Thus, the net field inside is zero. **Outside, the field persists.**



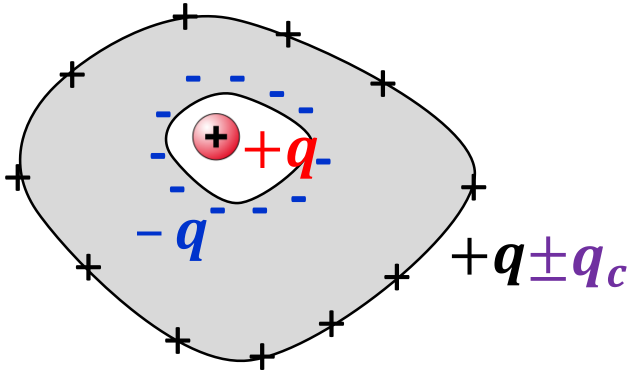
What about the opposite: Hollow conductor containing a charge in the cavity and **NO** charge or field outside



1) Gauss theorem AND because $\vec{E} = \vec{0}$ in the conductor

⇨ A negative charge $-q$ builds up in the inner surface of the conductor

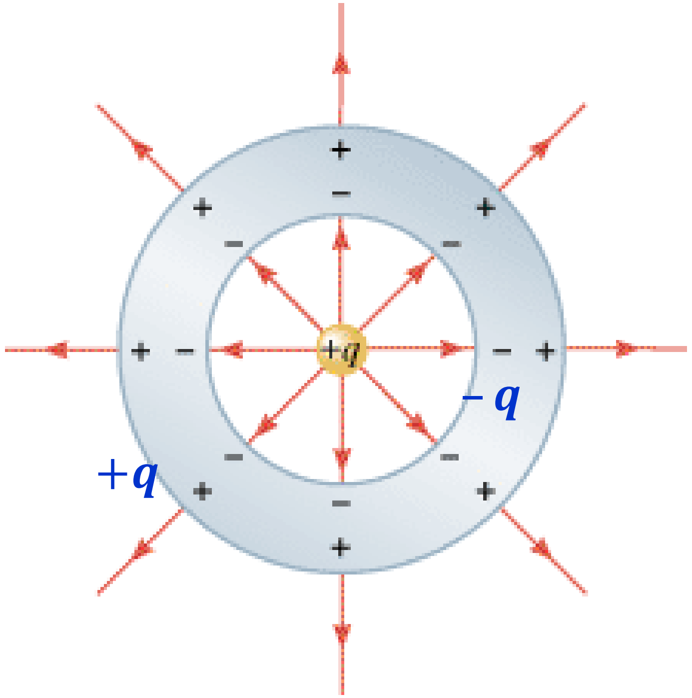
2) And because of the neutrality of the conductor positive charge $+q$ must build up at the outer surface



3) If a charge $\pm q_c$ is added on the outer surface, **NOTHING will change inside the hollow conductor containing charge** $+q$

Hollow conductor containing a charge in the cavity and **NO** charge or field outside

Case 1: charge at the center of the cavity

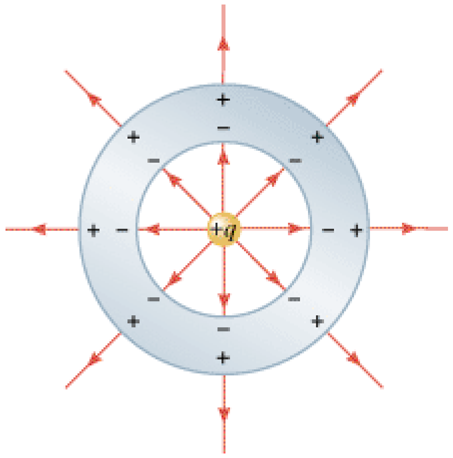


- The field lines emanate from positive charge $+\boldsymbol{q}$ and stop at the inner surface of the conductor (In the conductor $E = 0$).
- As they must stop at negative charges, a charge $-\boldsymbol{q}$ is induced in the inner surface of the cavity.
- Because the conductor is initially neutral, it must remain so. A positive charge $+q$ is then induced in the outer surface

It looks like the field outside is just and extension of the field inside, in spite of the discontinuity brought by the conductor

But nature is much subtle than that !

Spherical shell with a charge in the center of the cavity



- The field lines hit the inner surface of the conductor <u>perpendicularly</u>
- Outside they emerge <u>perpendicularly</u> from the outer to the surface

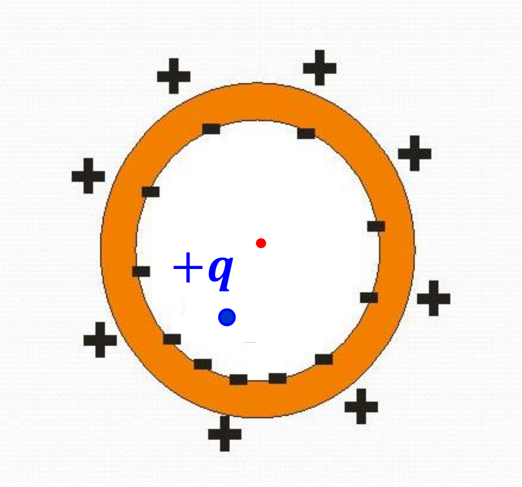
- The conducting shell introduces **discontinuity** in the field lines **BUT** does **NOT** <u>seem</u> to perturb the field pattern created by the charge inside the cavity

$+q$ in the center = $+q$ on the outer surface ➡ # fields lines in the cavity = # lines outside the sphere

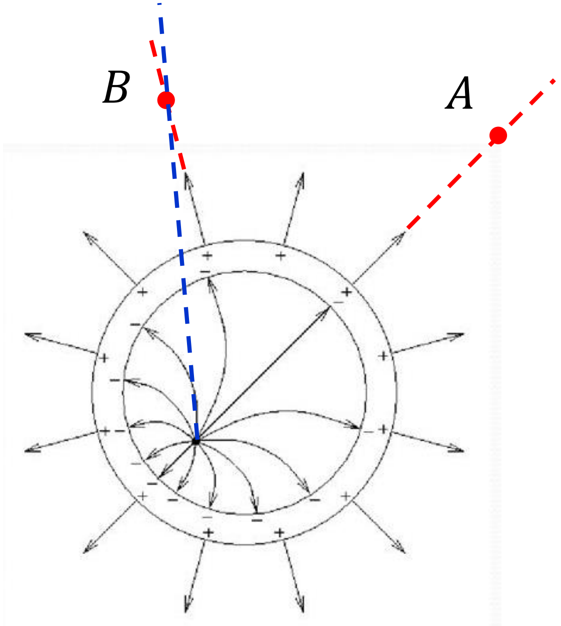
Do the fields inside the cavity and outside the sphere have something to do with each other?

Are they dependent or independent from each other?

What happens if the charge is moved off the center of the cavity?



- The field lines **MUST** hit the inner surface of the conductor perpendicularly
- The induced charges on the outer surface **MUST** distribute uniformly to minimize the energy
- And **MUST** emerge outside perpendicularly from the outer the surface

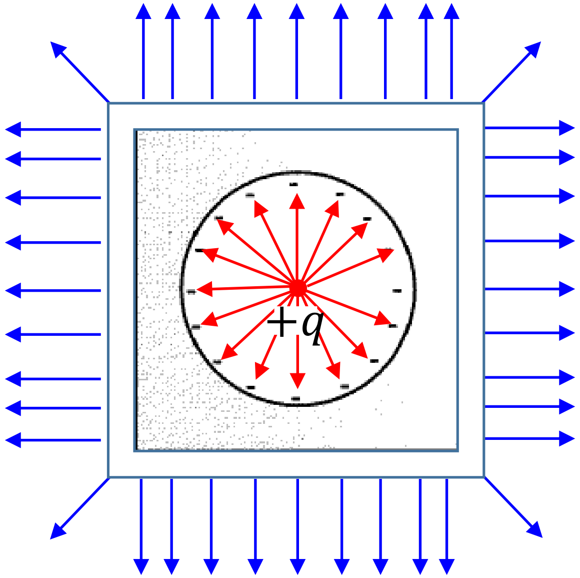


The outer field acting at points A and B is totally different from the inner field due to the charge inside the cavity

These two fields are completely independent from each other

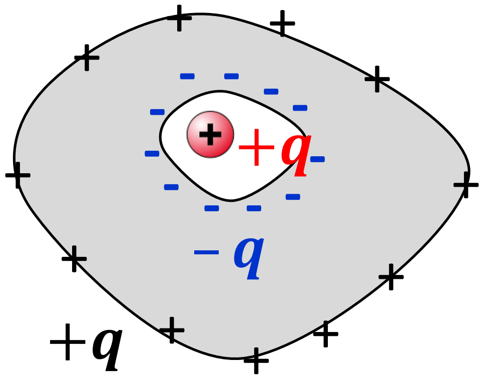
Inside the cavity the charge can be moved in any direction, outside the field stays unperturbed

What happens if we change the shape of the conductor: cubic shape?

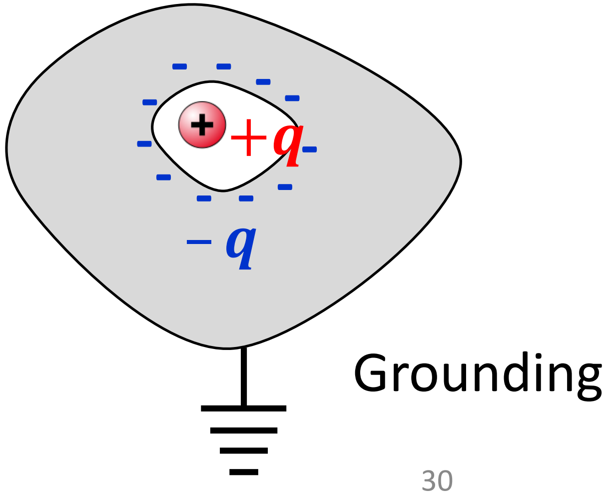


Here the field lines show a spherical symmetry when emanating from the charge at the **center** of the cavity

Outside the cube shaped conductor the field lines are parallel on each side, except at the corners where they show a different orientation and have greater density near the corners due to stronger bending of the surface



$+q$ is **responsible** for the field outside **BUT does not shape it***. How can we avoid that field while the charge is still in the cavity?



** The fields inside and outside are independent !*

Induction: How it works?

- How do the induced charges distribute at the surface of the conductor so that the potential of the surface is constant ?

- What is the net force between the inductor and the conductor?

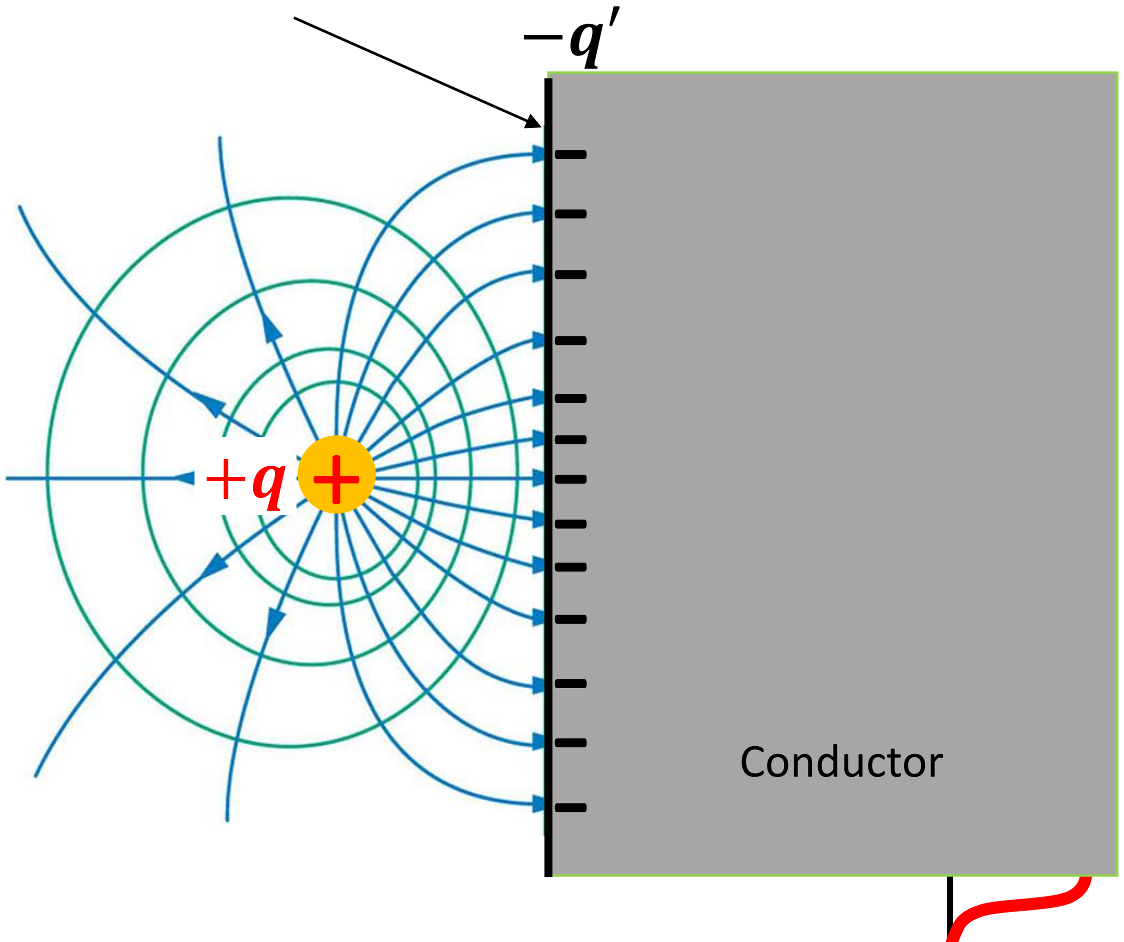
Source of the field inducing the charge

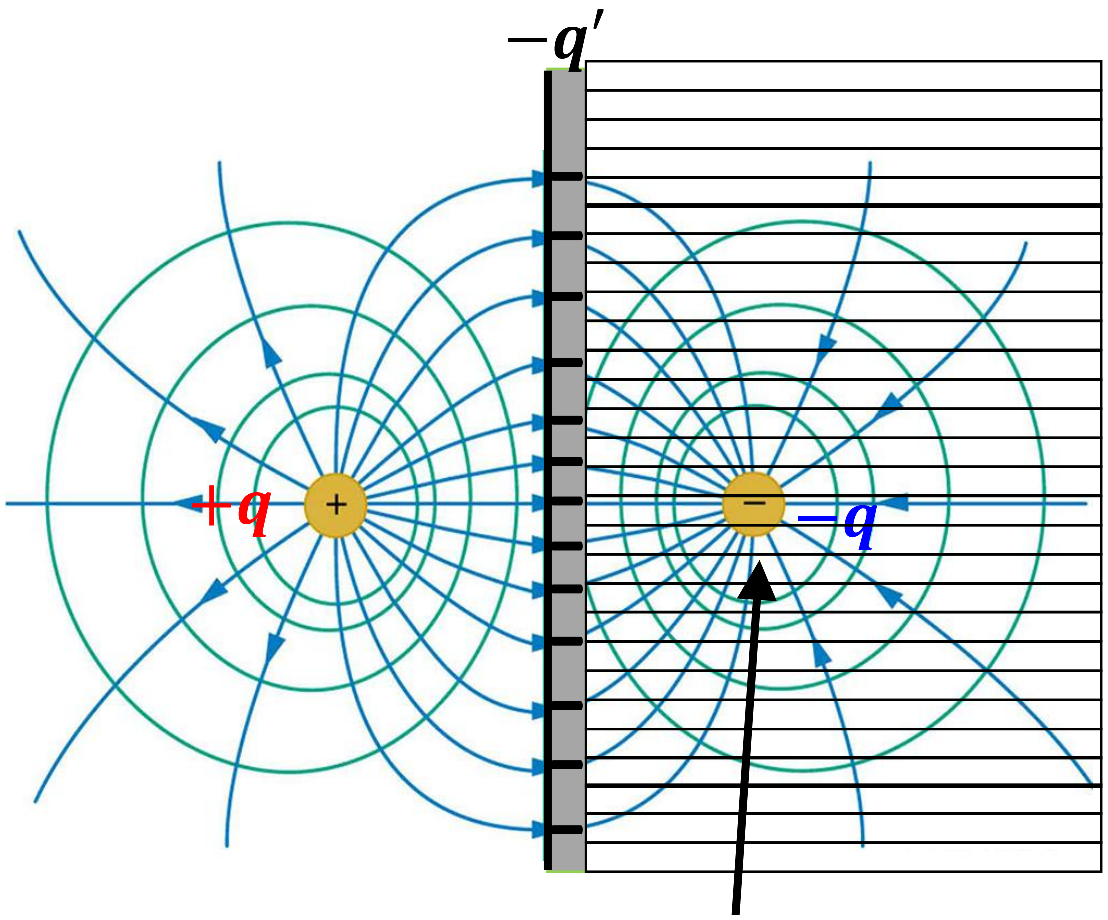
This is not a trivial question at all !!

Halfway between the two Charges the potential is zero

The charges induced on the **left** do not distribute uniformly along the surface of the conductor

The distribution follows the field lines



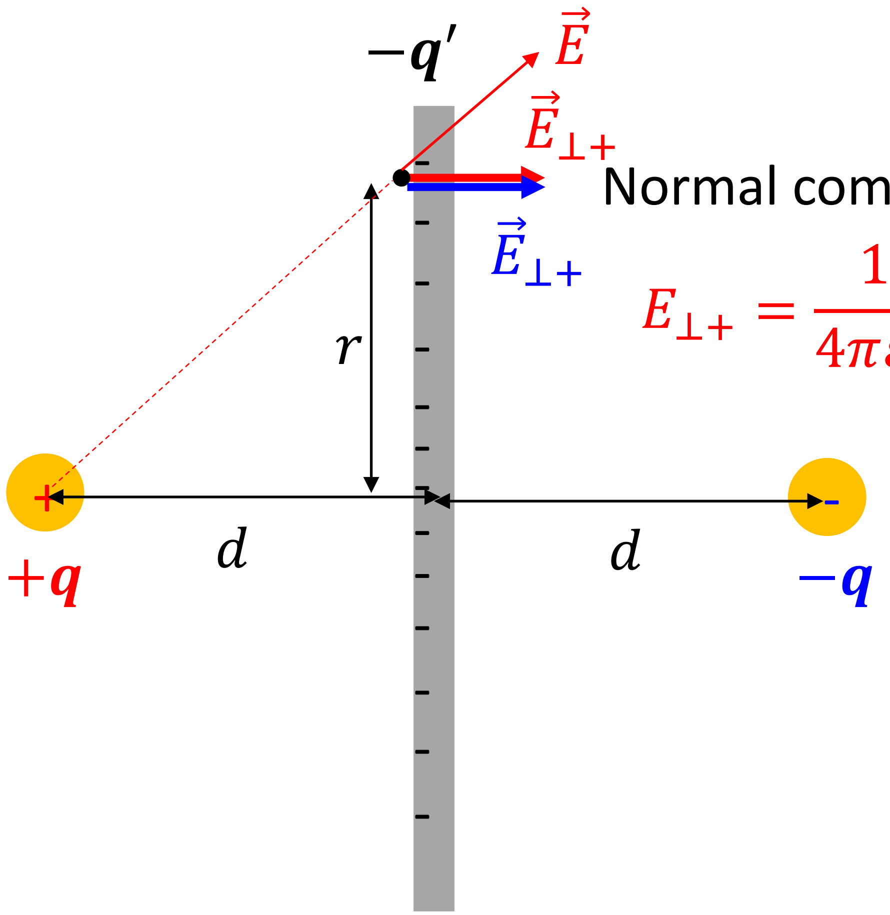


Remove charge $-q$

All induced positive charges flow to the ground

This becomes the image charge for $-q'$

The conducting plane has an infinite size



Normal component of the field du to charge $+\boldsymbol{q}$ (perpendicular to the plane)

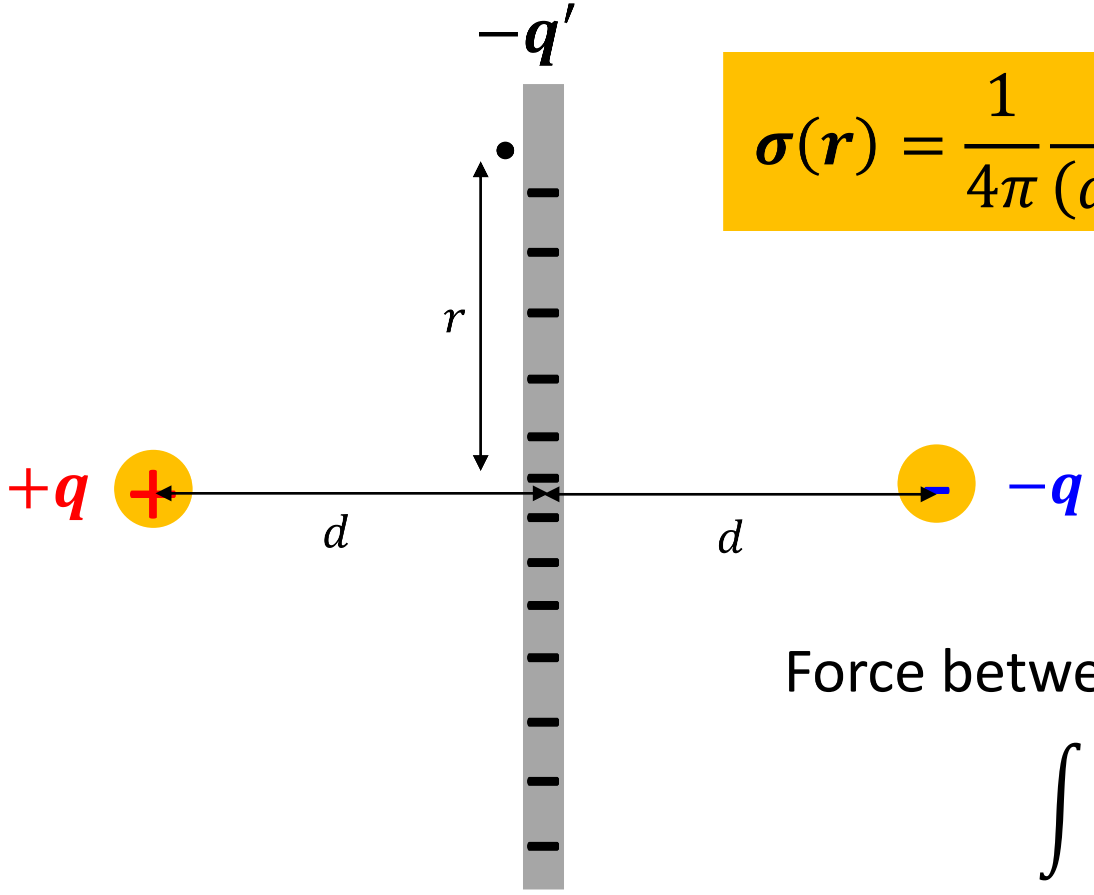
$$E_{\perp +} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{(d^2+r^2)^{3/2}}$$

To this field we must add the contribution from charge – $\boldsymbol{q}$' due to the charged conducting plane, whose image charge is $-q$ on the other side of the plane

$$E_{\perp -} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(d^2+r^2)^{3/2}}$$

$$E(r) = E_{\perp +} + E_{\perp -} = \frac{1}{4\pi\varepsilon_0}\frac{2qd}{(d^2+r^2)^{3/2}} = \frac{\boldsymbol{\sigma}(\boldsymbol{r})}{\boldsymbol{\varepsilon_0}}$$

From which we get the **NONUNIFORM** charge distribution



$$\boxed{\boldsymbol{\sigma}(\boldsymbol{r}) = \frac{1}{4\pi}\frac{2qd}{(d^2+r^2)^{3/2}}} \Rightarrow \int \boldsymbol{\sigma}(\boldsymbol{r}) = q = q'$$

Surface of the plane

Force between charge $(+\boldsymbol{q})$ and the conductor $(-\boldsymbol{q}')$

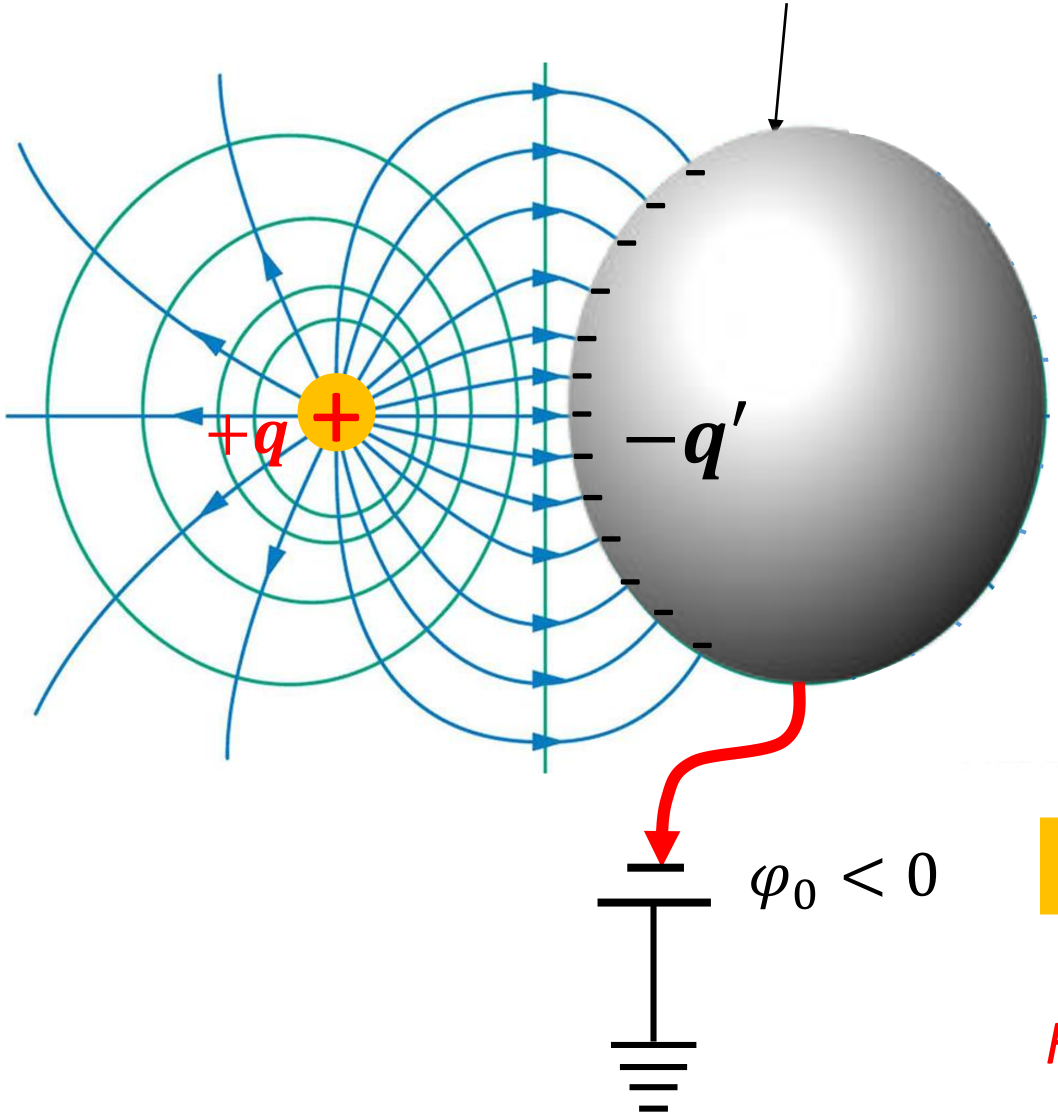
$$\int qE(r)dr = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}$$

Coulomb's law between two opposite charge $+q$ an $-q$
as if the conducting plane was not there !

What a big deal !!!!!

All these considerations to end up with Coulomb's law ?

This is a negative equipotential surface φ_0



- We replace the charge $-q$ by a conductor fitting with this surface potential
- Charges $-\boldsymbol{q}'$ and $+\boldsymbol{q}'$ are induced on the surface of the conductor due to the field of charge $+\boldsymbol{q}$
- The conductor is placed at the same potential φ_0

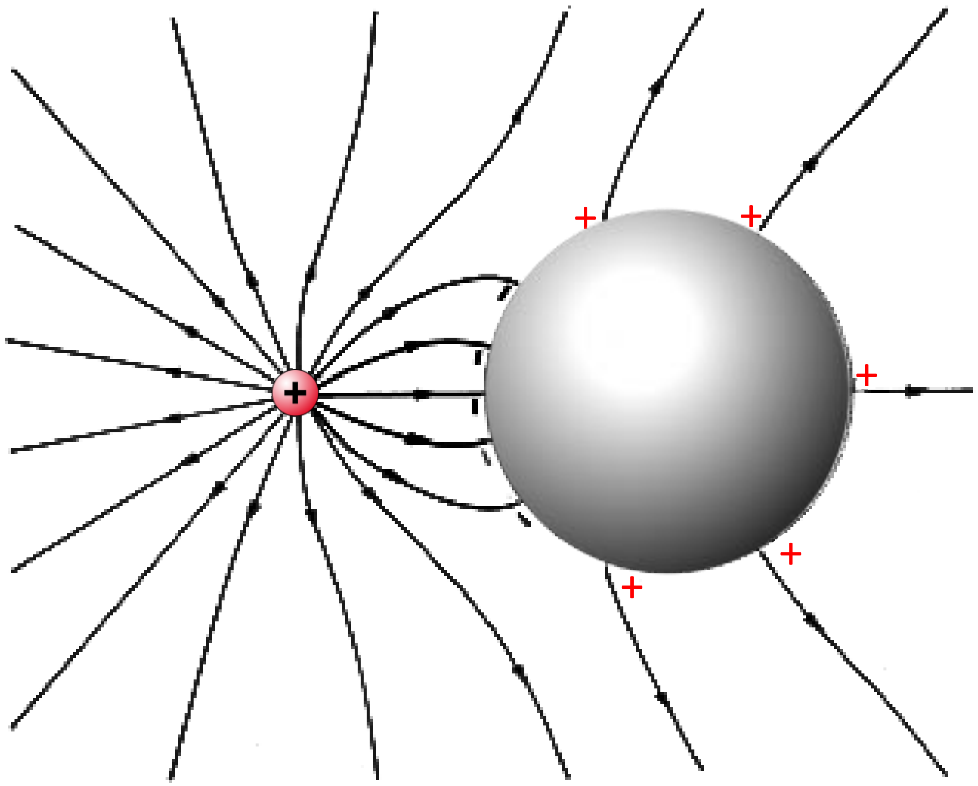
This configuration reproduces exactly the dipole

However, the equipotential is not exactly a sphere

However, the field of two **unequal** point charges has a spherical potential

The external charge (field) induces both types of charges on the sphere

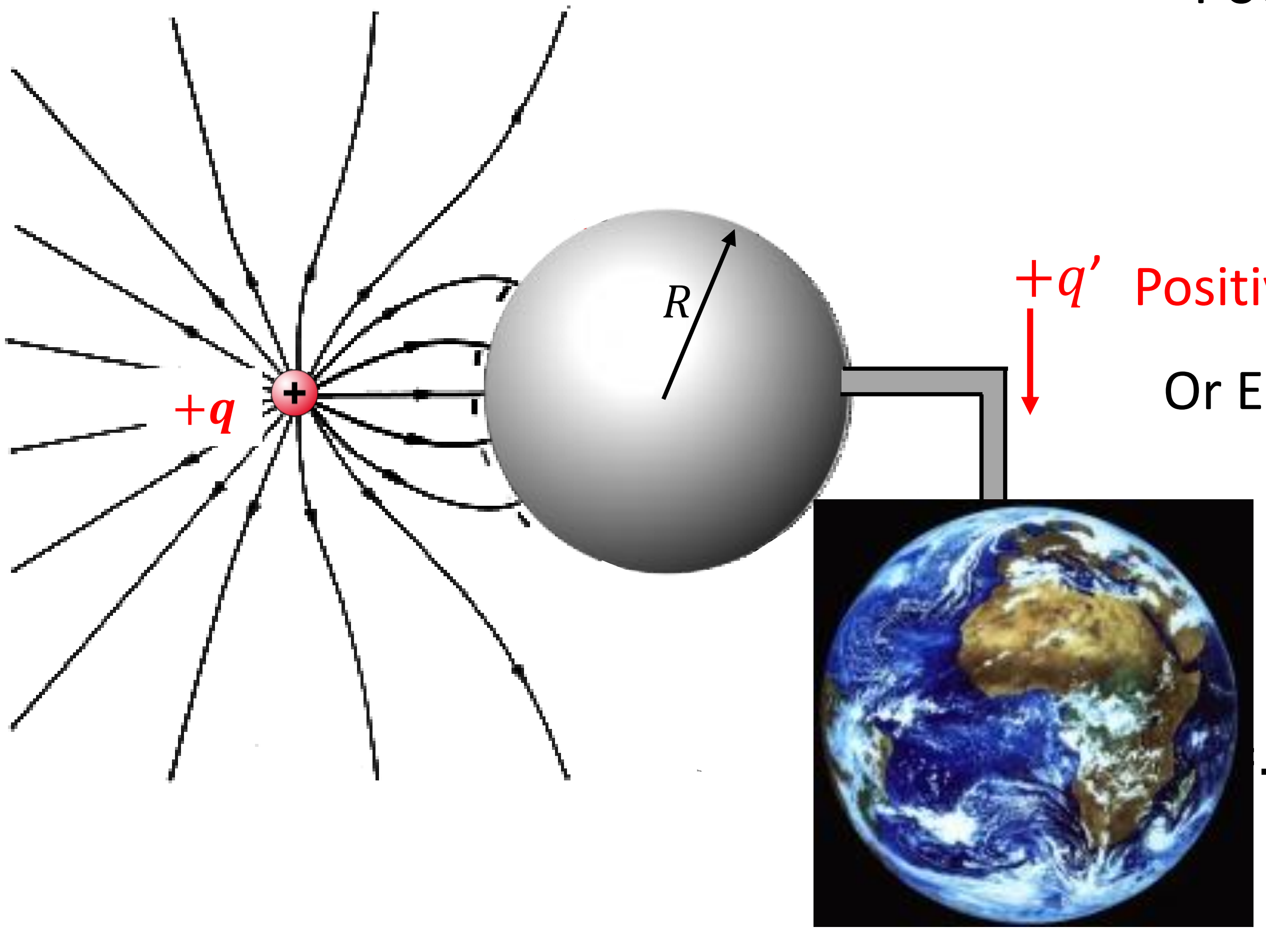
⬄ Charge conservation



What happens if we ground the sphere?

Inducing charges on a **grounded** conductor

Potential of any sphere $\varphi = \frac{Q}{4\pi\varepsilon_0 R}$



$+q'$ Positive charges flow to the earth

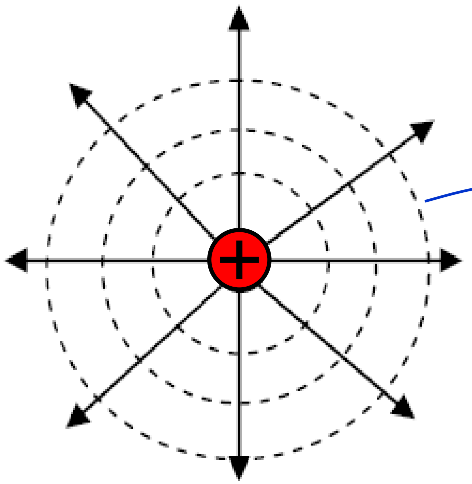
Or Electrons are provided to the sphere

$-q'$

The process is inverted if the external polarizing charge is negative

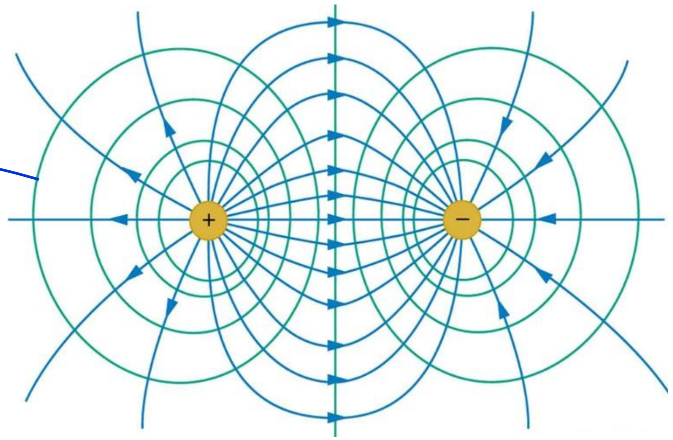
For the earth $R \to \infty \Rightarrow \varphi = 0$ for any charge q

The concept of image force

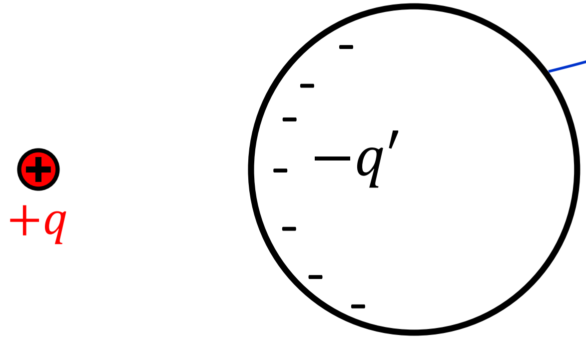


A single charge generates spherical equipotential shapes

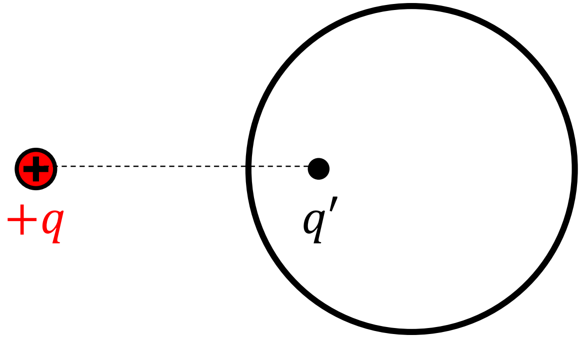
Equipotential



Two equal and opposite charges do not generate spherical equipotential shapes



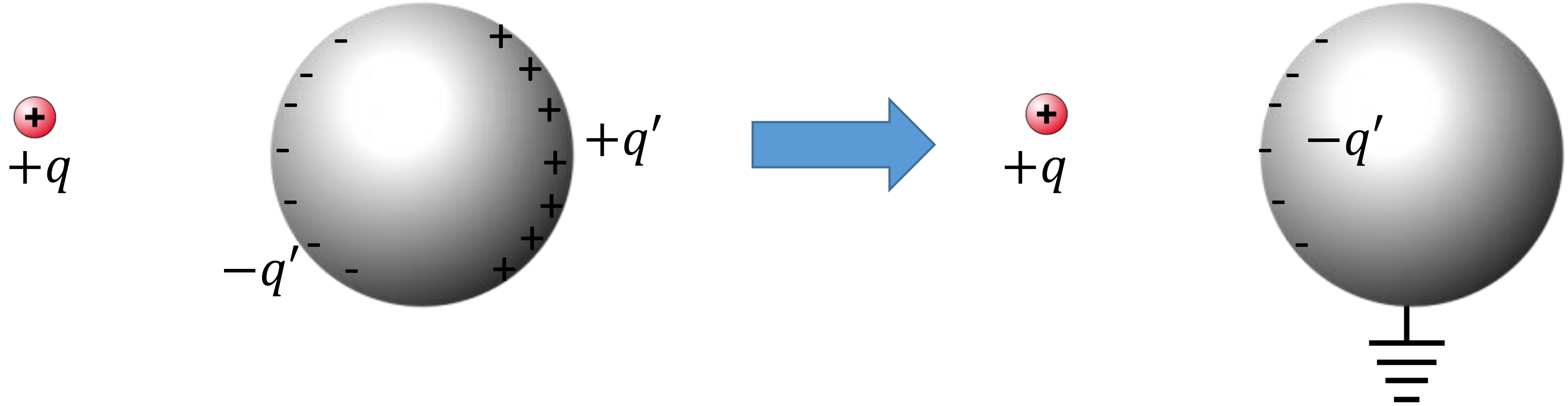
Two ***unequal*** and opposite charges may generate spherical equipotential



q' = Image charge representing all negative charges induced at the surface of the conductor

How do the induced charges distribute at the surface of the conductor so that the spherical potential of the surface is constant ?

Image method

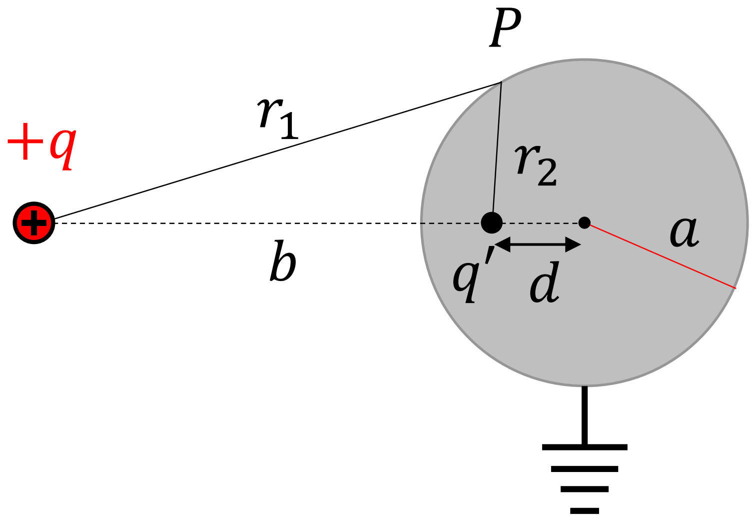


$\varphi = 0$

Everywhere in the sphere

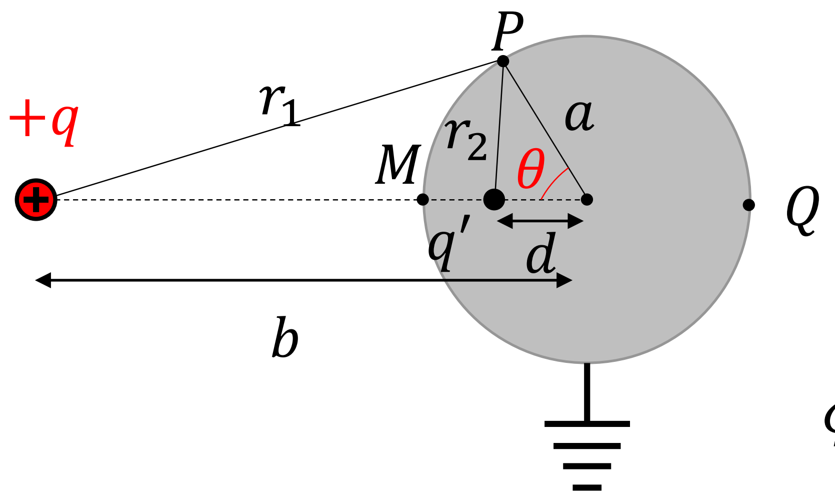
The grounding removes the <u>positive charges</u>

- The charge $-\boldsymbol{q}$ induced on the sphere is equivalent to an image charge q' placed at d from the center



$$\varphi(P) = 0 \Rightarrow \varphi(P) = \frac{q}{r_1} + \frac{q'}{r_2} = 0 \qquad \left(\text{we drop } \frac{1}{4\pi\varepsilon_0}\right)$$

The negative charge induced on the surface is equivalent to q' placed at position $b - d$



$\varphi = 0$ everywhere in the sphere

⇨ $\varphi(P) = \varphi(M) = \varphi(Q) = 0$

Image charge $q' < 0$

$\varphi(M)$ ⇨ $\dfrac{+q}{b-a} + \dfrac{q'}{a-d} = 0$ ⇨ $q' = -\dfrac{a-d}{b-a}q$

We know a and b but **NOT** d

On any point on the surface ⇨ use **Al-Kashi's theorem**

$$r_2 = \sqrt{a^2 + d^2 - 2adcos\theta}$$

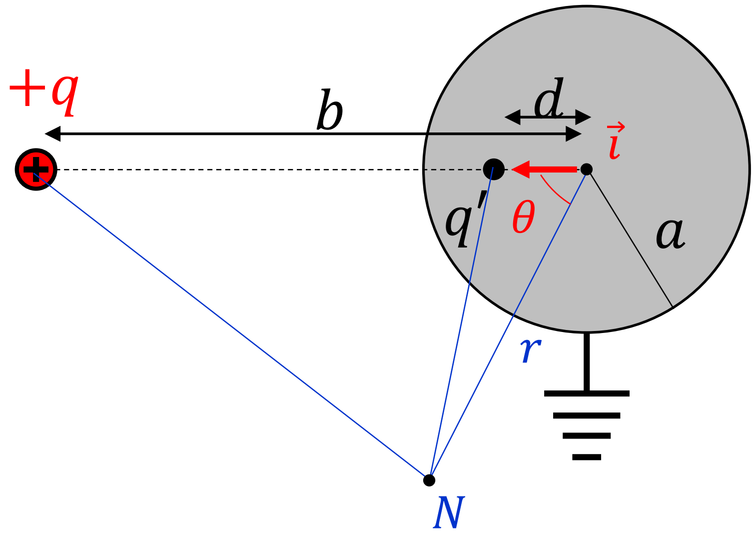
$$r_1 = \sqrt{a^2 + b^2 - 2abcos\theta}$$

$\varphi(M) = \varphi(Q) = 0$ ⇨ $d = \dfrac{a^2}{b}$

$$\boxed{q' = -\frac{a}{b}q}$$

q' increases with radius of the sphere (more lines hit the sphere)and decreases with distance b

⇨ If $a = b$ ⇨ $q' = -q$
Classical dipole



On a point N Outside the sphere at any point (r, θ, ϕ) from the center of the sphere

$$\varphi(N) = q\left[\frac{1}{\sqrt{r^2 + b^2 - 2brcos\theta}} - \frac{1}{\sqrt{\left(\frac{rb}{a}\right)^2 + a^2 - 2brcos\theta}}\right]$$

If N is on the surface

$$\boxed{\varphi(N) = 0 \quad \Rightarrow \quad r = a}$$

What is the force between $+q$ *and* the image charge q' ?

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(b-d)^2}\vec{\imath} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2ab}{(b^2-a^2)^2}\vec{\imath}$$

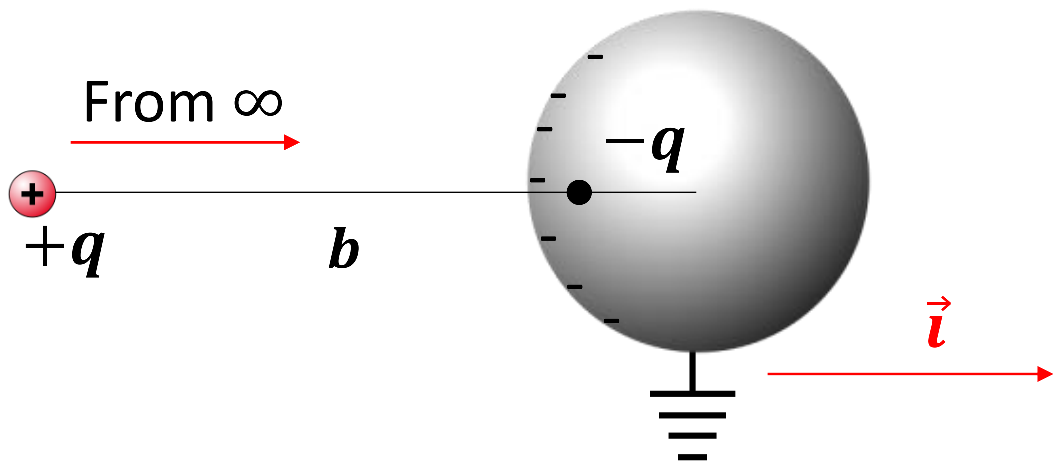
The work done to bring the charge $+q$ from ∞ to a distance $\boldsymbol{b}$ to the center of the grounded conductor: ***b* is the variable**

The origin is taken at the center of the sphere

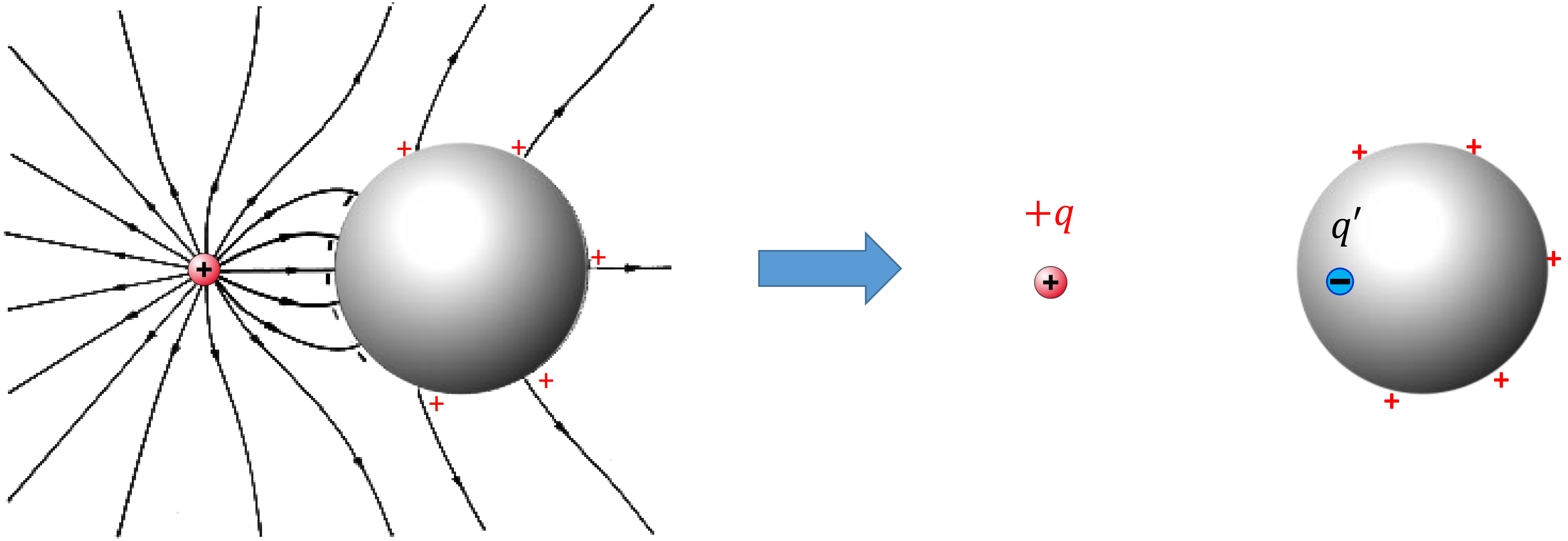
$$W = -\int_{\infty}^{b}\vec{F}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{b}\frac{q^2ax}{(x^2-a^2)^2}dx$$

$$\boxed{W = -\frac{1}{8\pi\varepsilon_0}\frac{aq^2}{b^2-a^2} < 0}$$

Energy is extracted from the system



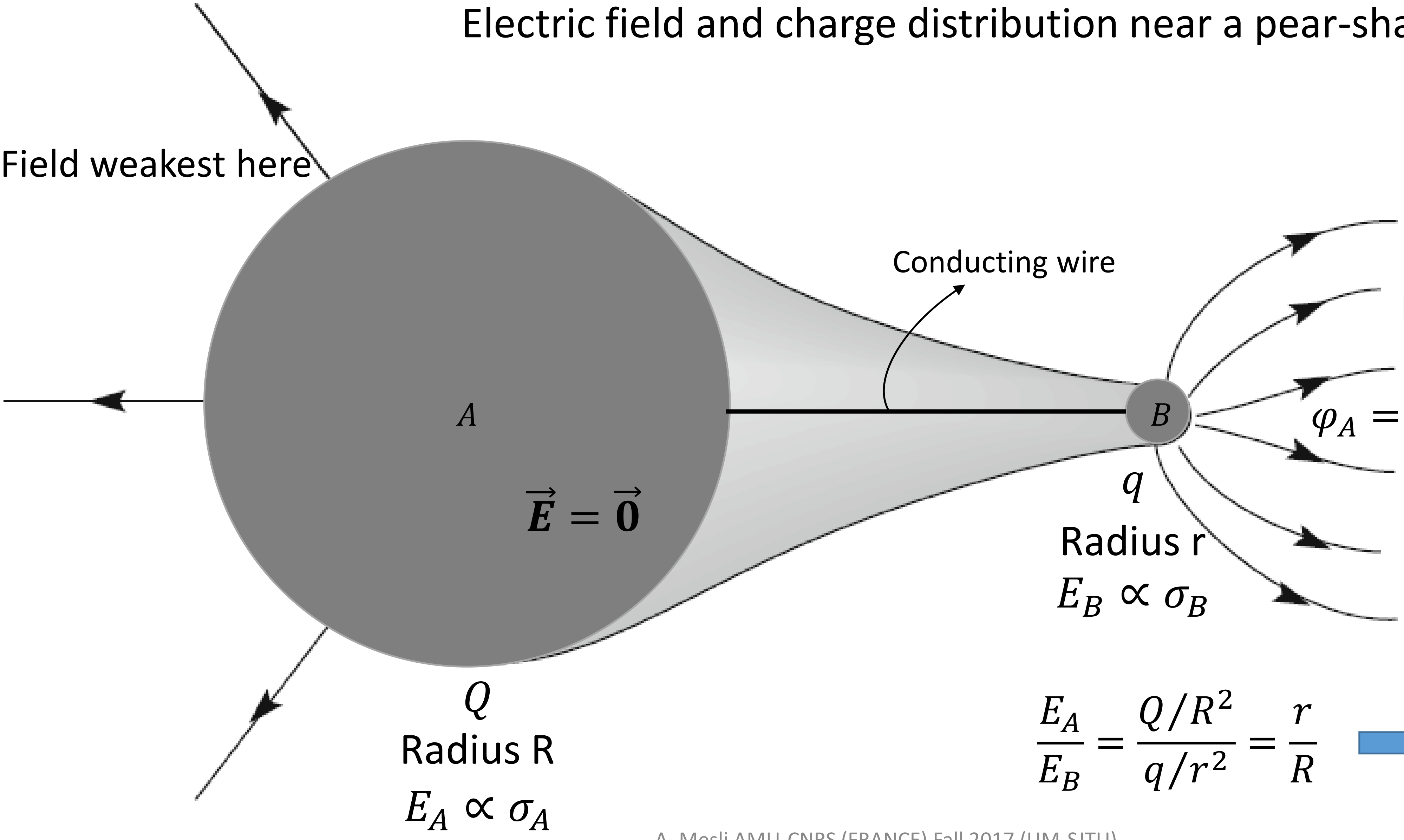
Question: What would happen if we do not ground the conducting sphere



- How should we account for the induced positive charges?
- Should we use another image charge q''?
- And if so where should it be placed?

Conductor shape dependent electric field

Electric field and charge distribution near a pear-shaped conductor



$$\varphi_A = \varphi_B \Rightarrow \frac{Q}{R} = \frac{q}{r}$$

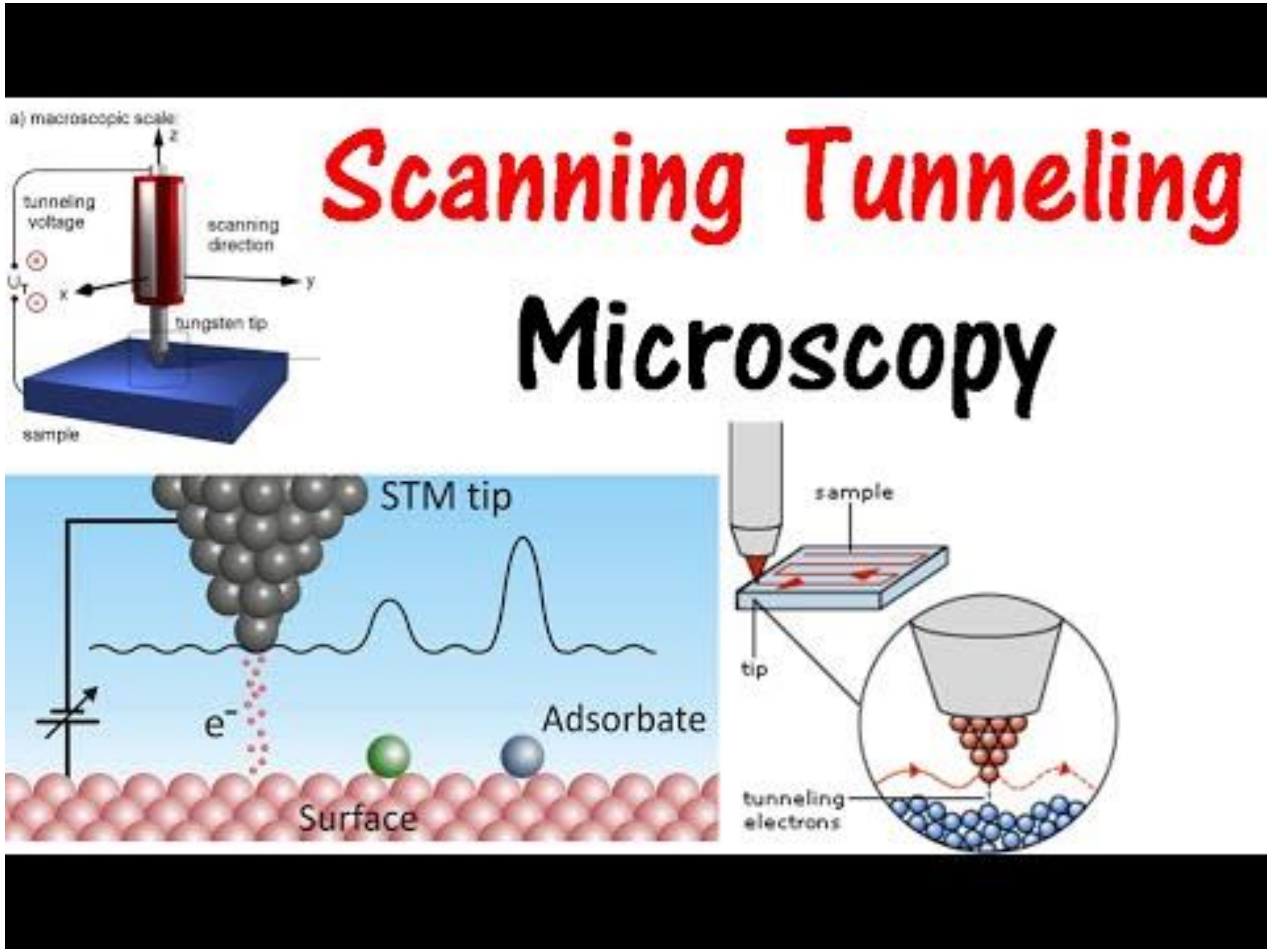
$$\frac{E_A}{E_B} = \frac{Q/R^2}{q/r^2} = \frac{r}{R} \Rightarrow \boxed{E_B = E_A \frac{R}{r}}$$

***Question 3: Are the charges distributed uniformly on each sphere?

Answer to ***Question 3:

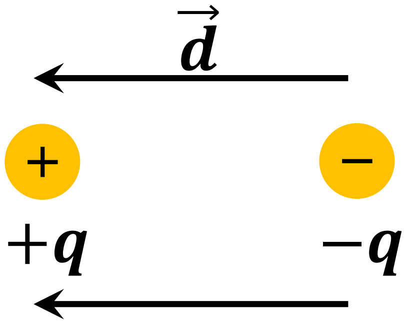
We have just seen that two charge distributions disturb each other

One major application of a tip-shaped conductor: observing atoms



<u>Nobel price in 1986:</u> Gerd Binning and Heinrich Roher (IBM Zürich, Switzerland)

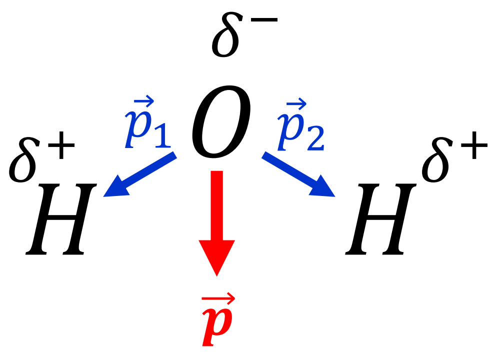
Electric dipole



$\vec{p} = q\vec{d}$ Dipole moment

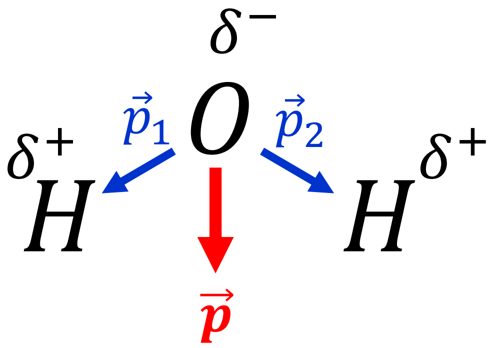
Why is the dipole so important?

Water molecules

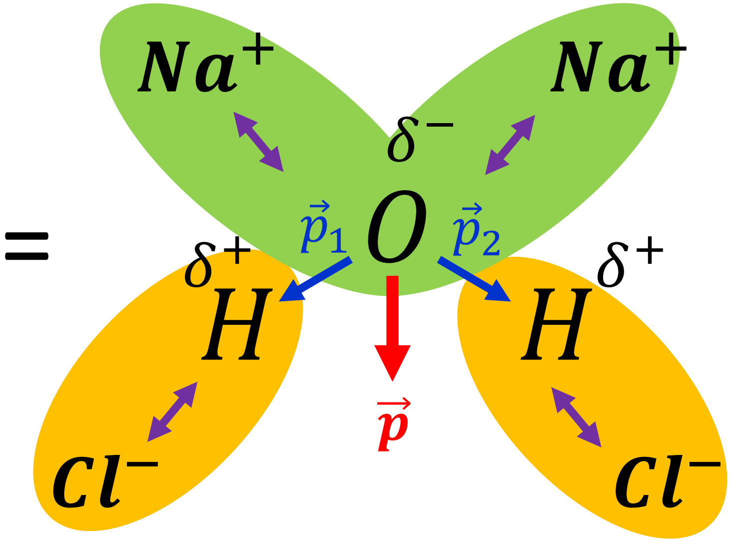


$\vec{p} = \vec{p}_1 + \vec{p}_2$ → Net dipole moment

Why is water is a good solvent for Na^+Cl^- ?



$+ \quad Na^+Cl^- \quad =$



If water were not polar it would have been a poor solvent

…. **BUT not only !** The microwave would have been useless

Electrostatic

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

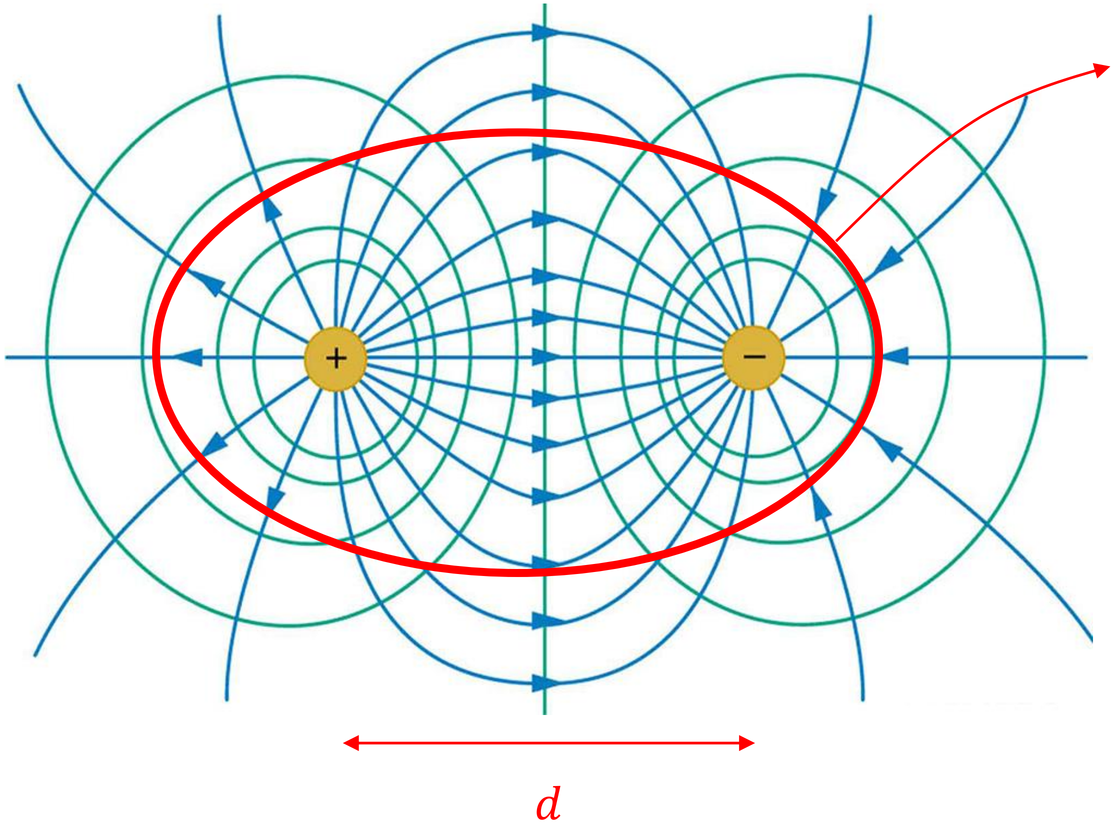
$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\varphi \quad \Rightarrow \quad \vec{\nabla}.\vec{E} = -\vec{\nabla}.\left(\vec{\nabla}\varphi\right) = -\nabla^2\varphi = \frac{\rho}{\varepsilon_0}$$

Poisson equation $\nabla^2\varphi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$

If we know $\rho(\vec{r})$ ⇨ we extract $\varphi(\vec{r})$ and ⇨ we get $\vec{E}(\vec{r})$

<u>Superposition principle</u> $\varphi(\vec{r}) = \int \frac{\rho\left(\vec{r'}\right)dV\prime}{4\pi\varepsilon_0\left|\vec{r}-\vec{r'}\right|} \frac{\vec{r}-\vec{r'}}{\left|\vec{r}-\vec{r'}\right|}$

Gauss' law is of no help for a dipole



Gauss surface $\Phi_E = \int \vec{E}.d\vec{A} = \frac{\rho}{\varepsilon_0} = 0$

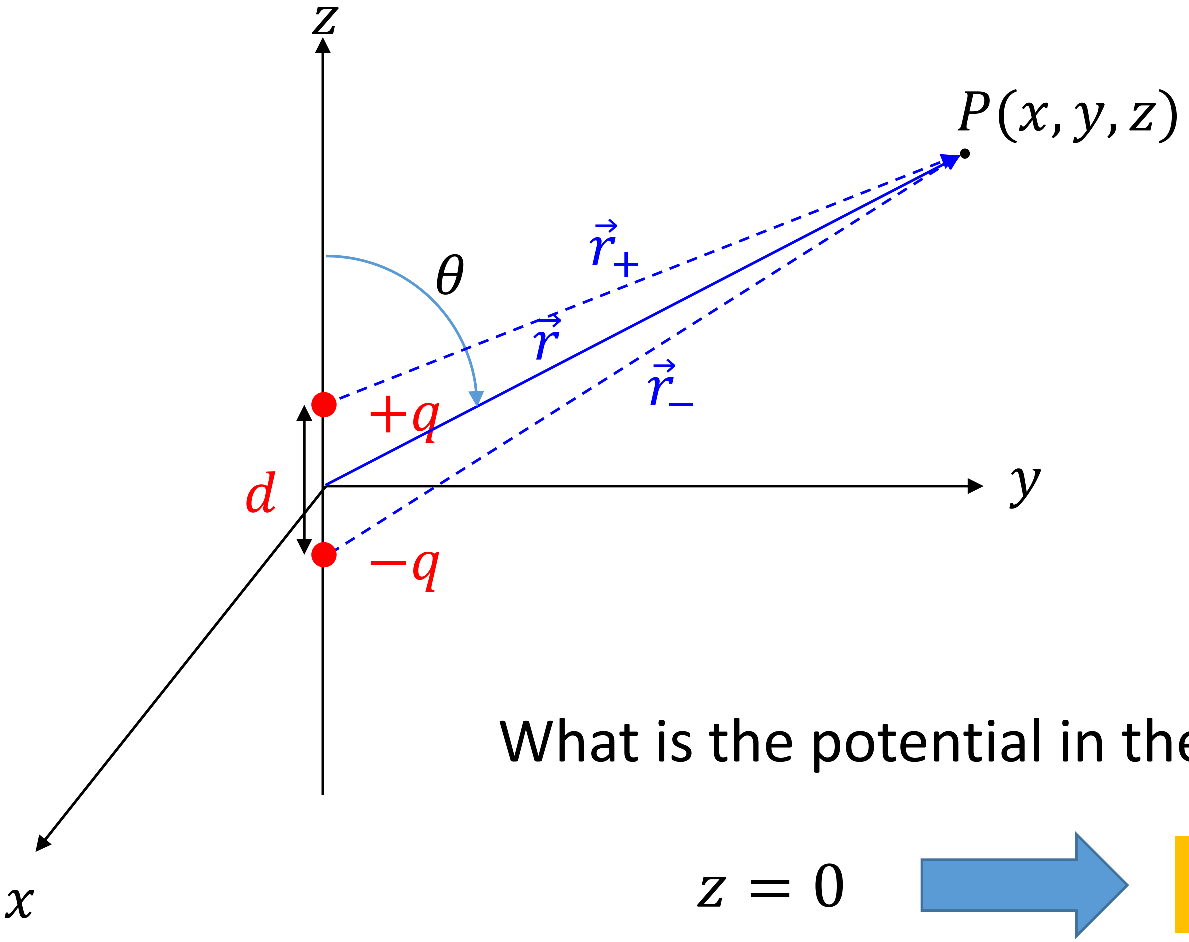
It does not mean that $\vec{E} = \vec{0}$

To have $\vec{E} = \vec{0}$ everywhere

$d = 0$ charges $+q$ and $-q$ on top of each other

But this is no longer a dipole !

Superposition principle



$$\varphi(P) = \sum_i \varphi_i(P)$$

$$\varphi(P) = \varphi_+ + \varphi_-$$

$$\varphi(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{q}}{\underbrace{\sqrt{\left(z - \frac{d}{2}\right)^2 + x^2 + y^2}}_{\boldsymbol{r}_+}} - \frac{\boldsymbol{q}}{\underbrace{\sqrt{\left(z + \frac{d}{2}\right)^2 + x^2 + y^2}}_{\boldsymbol{r}_-}}\right]$$

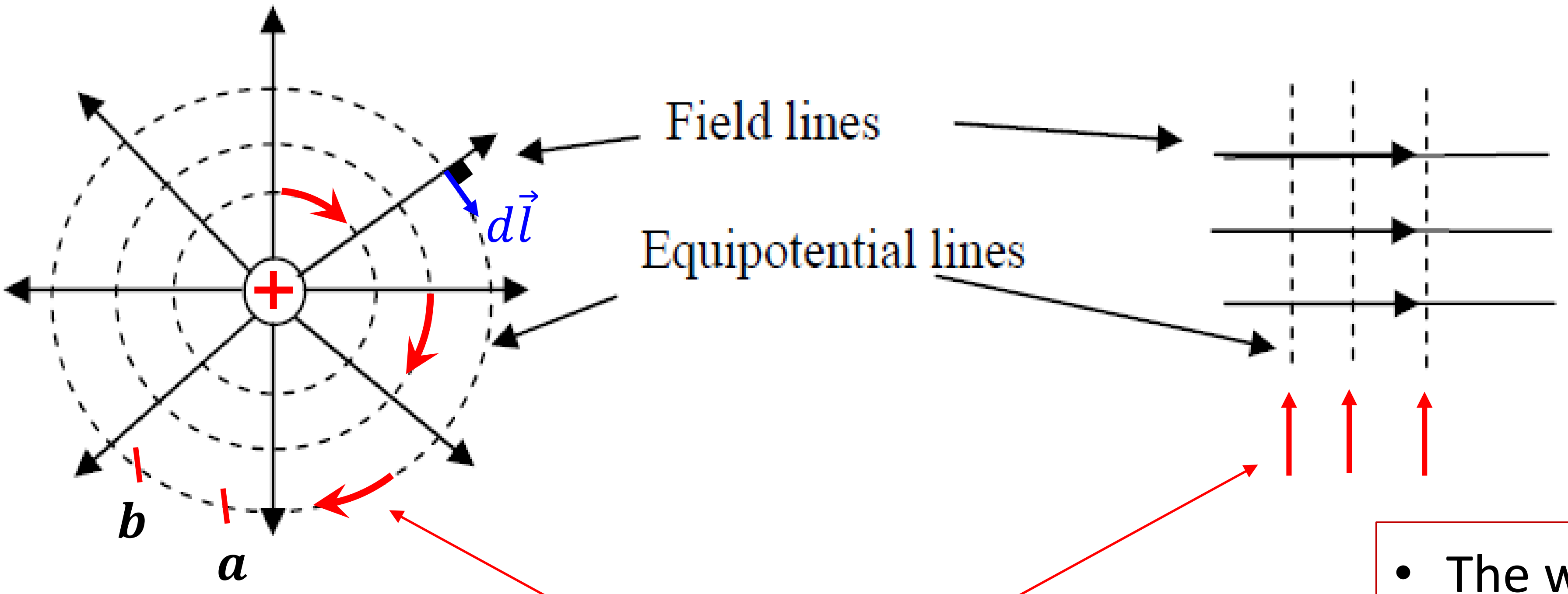
What is the potential in the $x - y$ plane ?

$z = 0$ ⟶ **Along the $x - y$ plane $\varphi = 0$**

BUT only because the two charges are equal in magnitude

<u>Definition of a dipole</u>

Field and equipotential lines



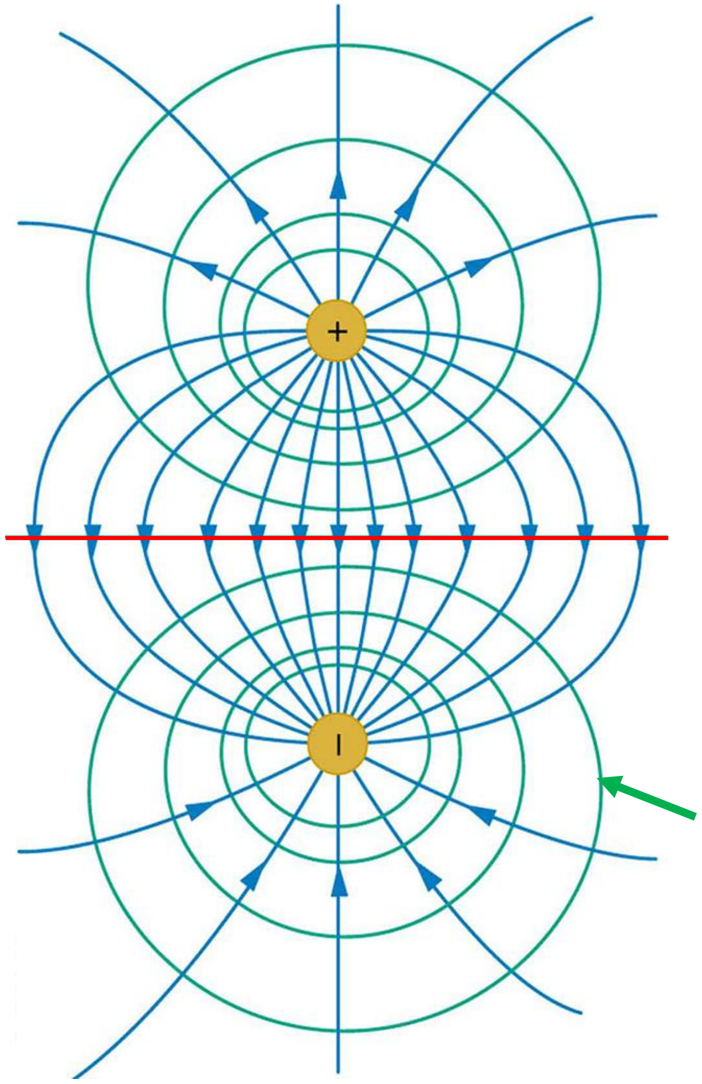
$\boldsymbol{\varphi(b) = \varphi(a)}$

Moving charge along these
Lines requires no work

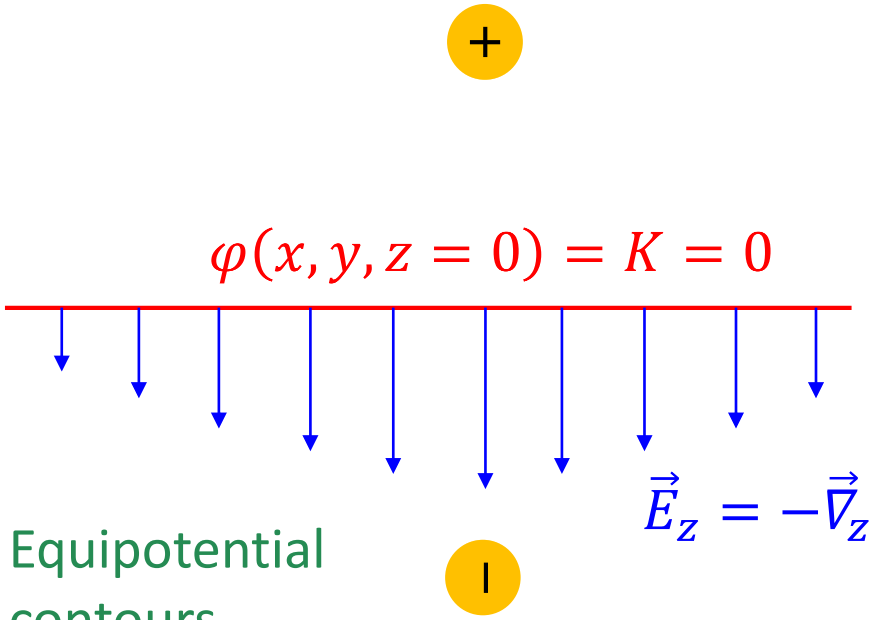
- The work is done against $\vec{E}$,
  $W = -\int \vec{E}.d\vec{l}$
- $\vec{E} \perp d\vec{l}$
- $W_{a\to b} = \varphi(b) - \varphi(a) = 0$
  $\varphi(b) = \varphi(a)$

$$\vec{E} = -[\vec{\nabla}_x\varphi(x,y,z)\vec{\imath} + \vec{\nabla}_y\varphi(x,y,z)\vec{\jmath} + \vec{\nabla}_z\varphi(x,y,z)\vec{k}]$$

$$\vec{\nabla}_x\varphi(x,y,z)\Big|_{z=0} = \vec{\nabla}_y\varphi(x,y,z)\Big|_{z=0} = 0 \text{ and } \boldsymbol{\vec{E}} = -\boldsymbol{E_z\vec{k}}$$

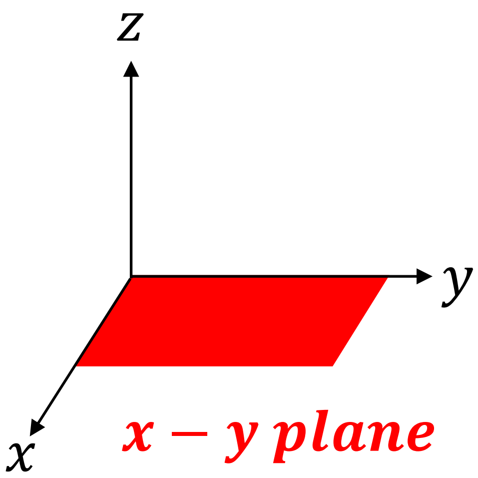


Equipotential contours



$$\varphi(x,y,z=0) = K = 0$$

$$\vec{E}_z = -\vec{\nabla}_z\varphi$$



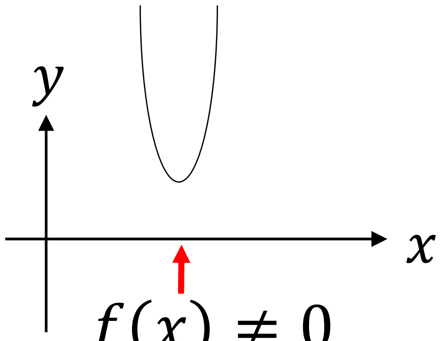
$$\varphi(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\left(z-{}^{d}/_{2}\right)^2 + x^2 + y^2}} - \frac{1}{\sqrt{\left(z+{}^{d}/_{2}\right)^2 + x^2 + y^2}}\right] = K$$

If at $\vec{r} = \vec{r}_0$, $\boldsymbol{\varphi(\vec{r}_0) = 0}$, what happens to $\boldsymbol{\vec{\nabla}\varphi(\vec{r}_0)}$?

φ is a local property

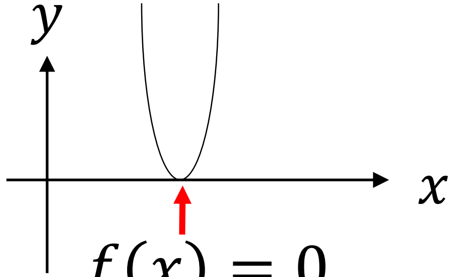
In 1D

If $f(x)$ is a function that cancels at $x = x_0$, it does not mean that $\left.\frac{df}{dx}\right|_{x=x_0} = 0$



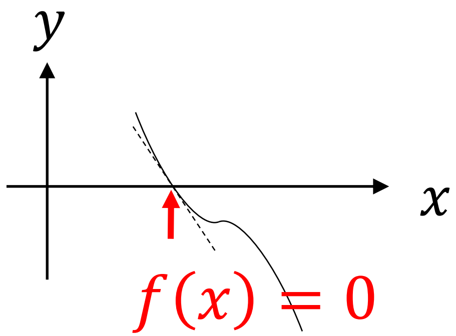
$f(x) \neq 0$

$$\frac{df}{dx} = 0$$



$f(x) = 0$

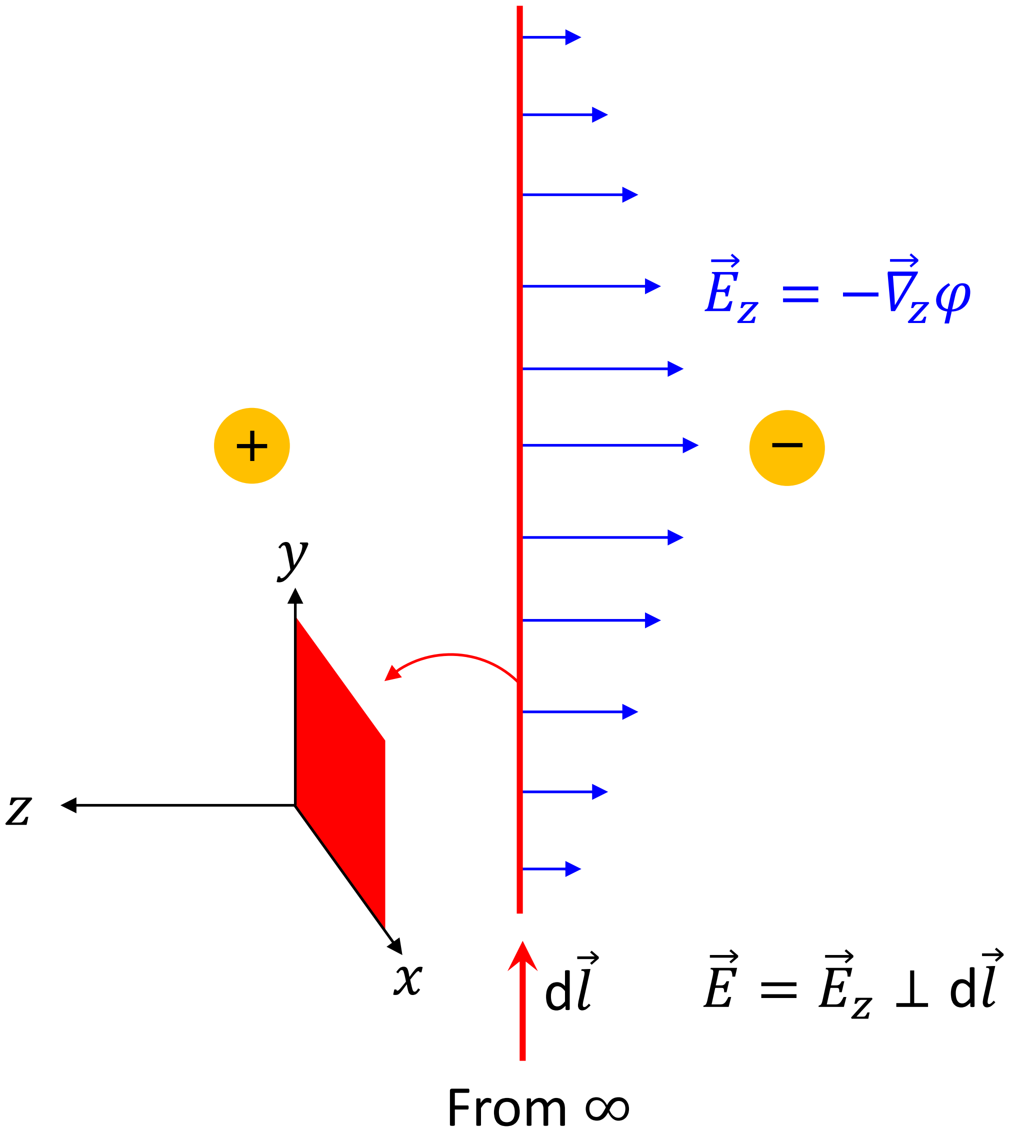
$$\frac{df}{dx} = 0$$



$f(x) = 0$

$$\frac{df}{dx} \neq 0$$

Work done from infinity to the middle along the line or the plane of equidistance is **zero**



In the $\boldsymbol{x - y\ plane}$, φ is equal to zero

$$\begin{array}{ccccccc} \vec{E} = E_x\vec{\imath} & + & E_y\vec{\jmath} & + & \boxed{E_z\vec{k}} & = & -\vec{\nabla}\varphi \\ \parallel & & \parallel & & \parallel & & \\ -\vec{\nabla}_x\varphi & & -\vec{\nabla}_y\varphi & & -\vec{\nabla}_z\varphi & & \\ \parallel & & \parallel & & \nparallel & & \\ 0 & & 0 & & 0 & & \end{array}$$

Although $\boldsymbol{\varphi = 0}$

The work done between two points in space depends on the potentials at that points only **NOT** on the path

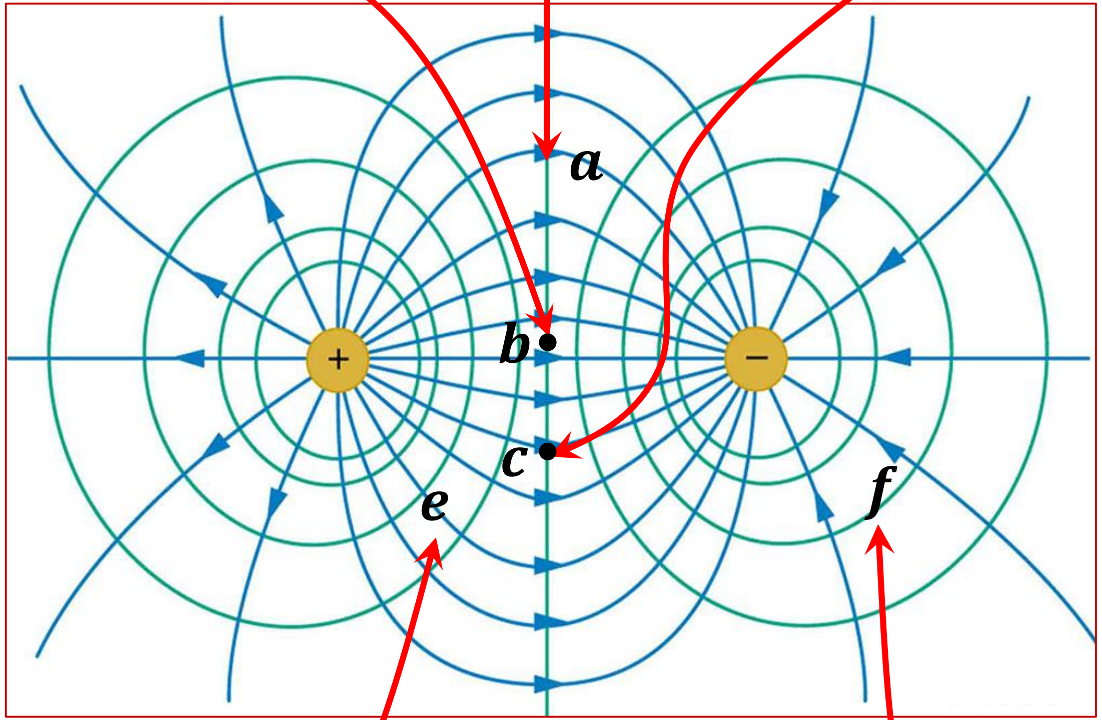
$$\varphi(b) - \varphi(a) = -\int_a^b \vec{E}\,d\vec{l}$$

Any path

Because electrostatic field is conservative

Work done to bring a positive charge

$W_{\infty \to b} = 0$ $W_{\infty \to a} = 0$ $W_{\infty \to c} = 0$



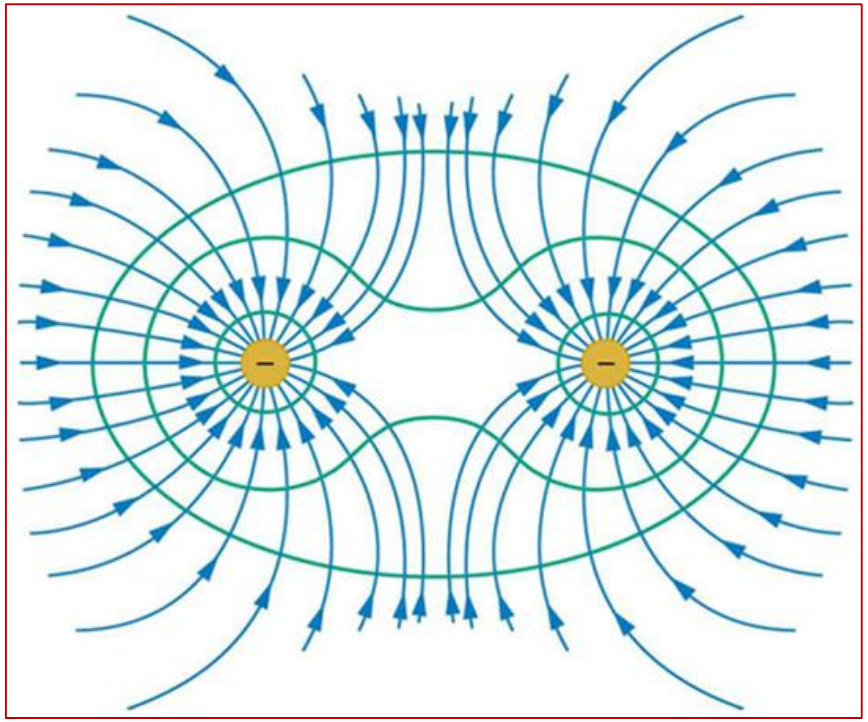
$\varphi > 0$

$W_{\infty \to y} > 0$

$\varphi = 0$

$\varphi < 0$

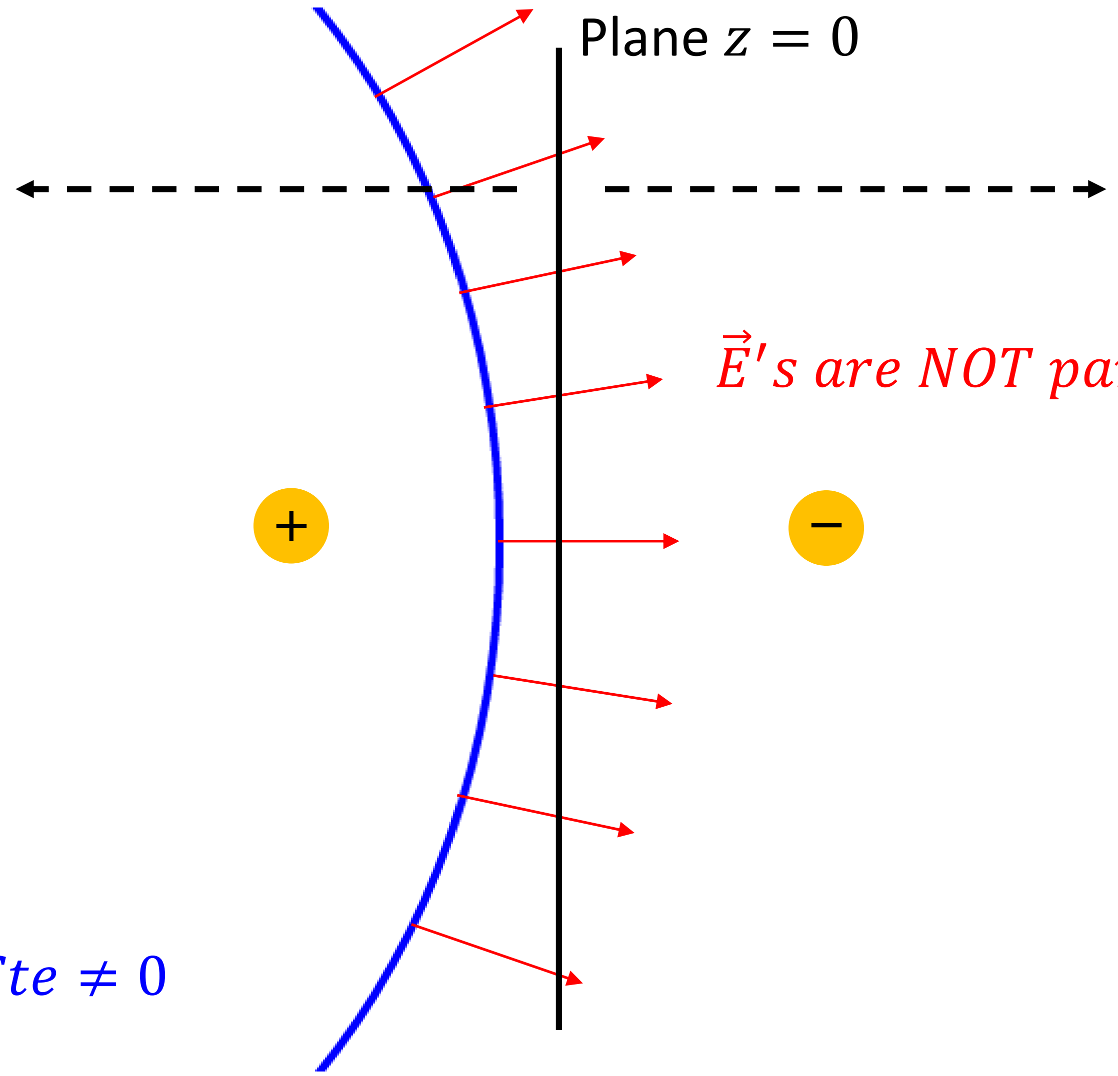
$W_{\infty \to x} < 0$



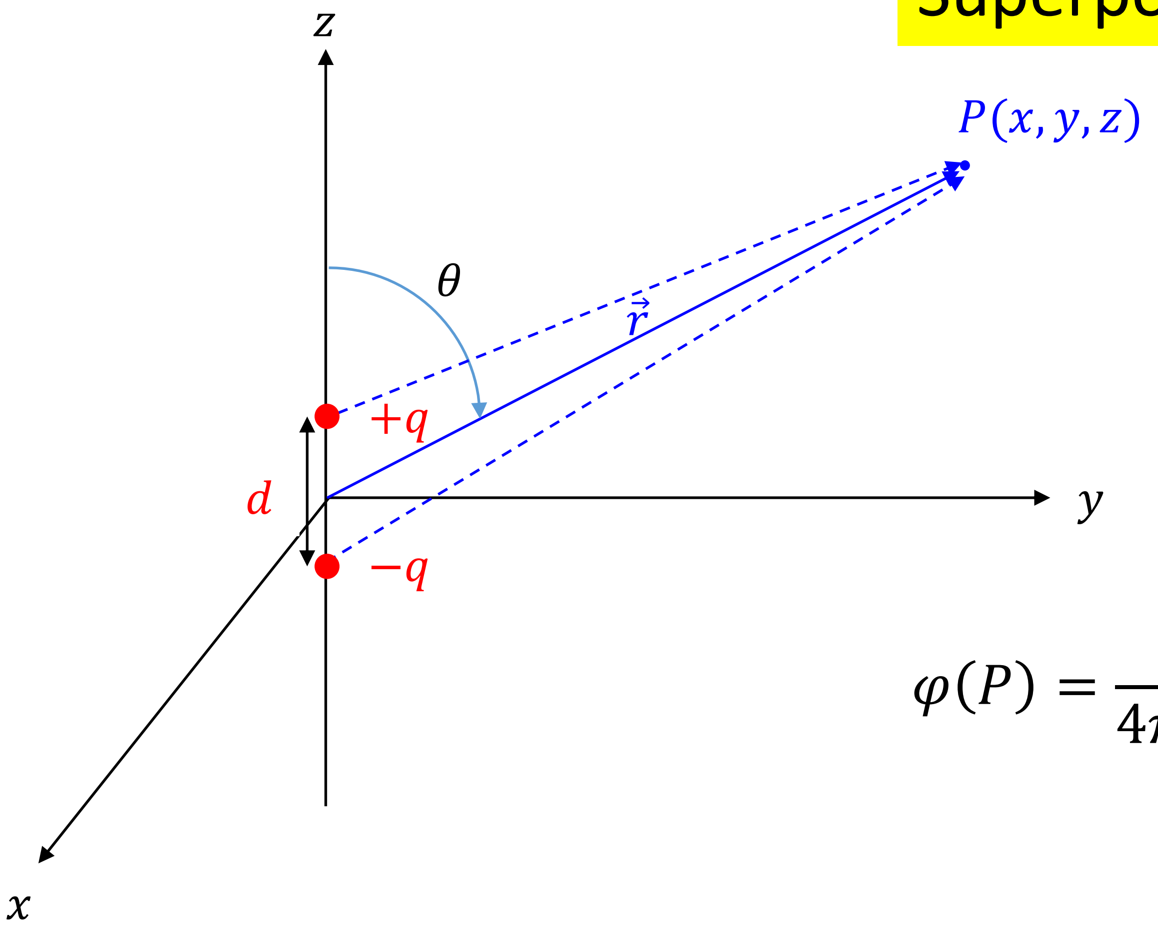
Work done from ∞ to any point is never 0

$\varphi < 0$

Equipotential surfaces



Superposition principle



$$\varphi(P) = \sum_i \varphi_i(P)$$

$$\varphi(P) = \varphi_+ + \varphi_-$$

$$\varphi(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{\left(z-\frac{d}{2}\right)^2 + x^2 + y^2}} - \frac{q}{\sqrt{\left(z+\frac{d}{2}\right)^2 + x^2 + y^2}}\right]$$

- Close to the dipole, $P(x, y, z)$ “sees” $+q$ and $-q$
- What does $P(x, y, z)$ see at distances $r >> d$

Looking far from the dipole

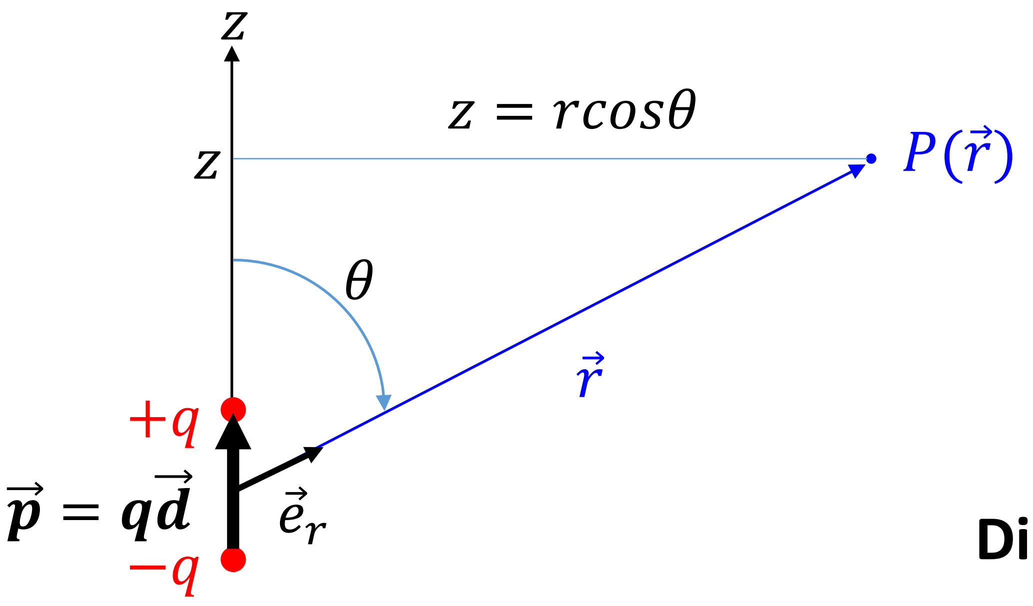
d small but not 0

$$\left(z-\frac{d}{2}\right)^2 \approx z^2 - zd \quad \Rightarrow \quad \sqrt{\left(z-\frac{d}{2}\right)^2 + x^2 + y^2} = \sqrt{r^2 - zd} = r\left(1-\frac{zd}{r^2}\right)^{1/2} = r\left(1 - \frac{zd}{2r^2}\right)$$

$$\left(z+\frac{d}{2}\right)^2 \approx z^2 + zd \quad \Rightarrow \quad \sqrt{\left(z+\frac{d}{2}\right)^2 + x^2 + y^2} = \sqrt{r^2 + zd} = r\left(1+\frac{zd}{r^2}\right)^{1/2} = r\left(1 + \frac{zd}{2r^2}\right)$$

$$\left.\begin{aligned} \varphi_+(P) &= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \frac{zd}{2r^2}\right) \\ \varphi_-(P) &= -\frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 - \frac{zd}{2r^2}\right) \end{aligned}\right\} \qquad \boxed{\varphi(P) = \frac{q}{4\pi\varepsilon_0}\frac{zd}{r^3}} \qquad \boxed{r >> d}$$

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{zd}{r^3} \quad + \quad \begin{matrix} z = r\cos\theta \\ p = qd \end{matrix} \quad \Rightarrow \quad \varphi(r) = \frac{p\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2}$$



In the $x - y$ plane, $\theta = \frac{\pi}{2}, \varphi(r)|_{z=0} = 0$

$$p\cos\theta\ 1 = \vec{p}.\vec{e}_r$$

Dipole $\varphi(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}.\vec{e}_r}{r^2}$ ← Spherical coordinate

Dipole $\varphi(r) \propto \frac{1}{r^2}$

Point charge $\varphi(r) \propto \frac{1}{r}$

Point charge $E(r) \propto \frac{1}{r^2}$

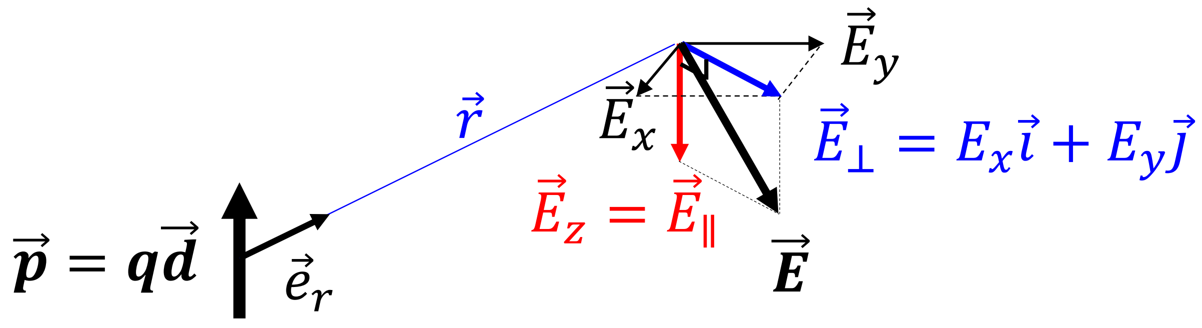
What are the three Cartesian components of the field of a dipole?

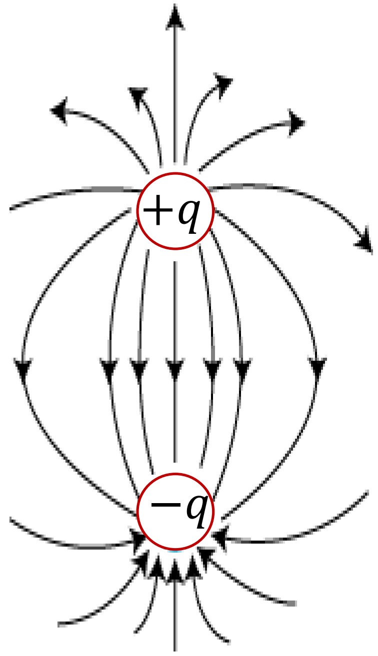
$$\vec{E}(x,y,z) = -\vec{\nabla}\varphi(x,y,z) = -\left(+\underbrace{\frac{\partial\varphi}{\partial x}}_{E_x}\vec{i} + \underbrace{\frac{\partial\varphi}{\partial y}}_{E_y}\vec{j} + \underbrace{\frac{\partial\varphi}{\partial z}}_{E_z}\vec{k}\right)$$

$$\boxed{\varphi(r) = \frac{\boldsymbol{q}}{4\pi\varepsilon_0}\frac{\boldsymbol{d}z}{r^3} = \frac{\boldsymbol{p}}{4\pi\varepsilon_0}\frac{z}{r^3} \qquad r^2 = x^2 + y^2 + z^2}$$

$$\boxed{E_z = \frac{p}{4\pi\varepsilon_0}\frac{3\cos^2\theta - 1}{r^3}}$$

$$\left.\begin{array}{l} \boxed{E_x = \dfrac{p}{4\pi\varepsilon_0}\dfrac{3zx}{r^5}} \\ \\ \boxed{E_y = \dfrac{p}{4\pi\varepsilon_0}\dfrac{3zy}{r^5}} \end{array}\right\} = \boldsymbol{0 \; if \; z = 0 \; (x - y \; plane)}$$





Dipole in spherical coordinates

$\vec{\nabla}$ in spherical coordinates

$$\vec{\nabla} = \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r sin\theta} \frac{\partial}{\partial \varphi} \right)$$

$$\varphi(r) = \frac{p cos\theta}{4\pi\varepsilon_0} \frac{1}{r^2}$$

$$E_r(r) = -\frac{\partial \varphi(r)}{\partial r} = \frac{2p cos\theta}{4\pi\varepsilon_0} \frac{1}{r^3}$$

$$E_\theta(r) = -\frac{1}{r} \frac{\partial \varphi(r)}{\partial \theta} = \frac{p sin\theta}{4\pi\varepsilon_0} \frac{1}{r^3}$$

$$E_\varphi(r) = 0$$

Gauss Law = Flux through a closed surface

$$\Phi = \int \vec{E}.\vec{n} dA = 0$$

$$\vec{E} = E_r \vec{e}_r + E_\theta \vec{e}_\theta \qquad dA = r^2 sin\theta d\theta d\varphi$$

$$0 < \theta < \pi$$
$$0 < \emptyset < 2\pi$$

Gauss law can also be checked through divergence

$$\vec{\nabla}.\vec{E} = \underbrace{\frac{1}{r^2}\frac{\partial r^2 E_r}{\partial r}} + \underbrace{\frac{1}{rsin\theta}\frac{\partial sin\theta E_\theta}{\partial \theta}} + \underbrace{\frac{1}{rsin\theta}\frac{\partial E_\varphi}{\partial \varphi}}_{\substack{\parallel \\ 0}} = 0$$

$$\underbrace{-\frac{2pcos\theta}{4\pi\varepsilon_0}\frac{1}{r^3} \quad + \frac{2pcos\theta}{4\pi\varepsilon_0}\frac{1}{r^3}}_{\substack{\parallel \\ 0}}$$

A beautiful illustration of the principle of superposition

Dipole potential as a gradient: An interesting aspect

$$\varphi(\vec{r}) = \frac{\vec{p}.}{4\pi\varepsilon_0}\boxed{\frac{\vec{e}_r}{r^2}} = \frac{1}{4\pi\varepsilon_0}\vec{p}.\boxed{\vec{\nabla}\left(-\frac{1}{r}\right)}$$

The gradient is taken along $\vec{e}_r$

$$\vec{\nabla}\left(-\frac{1}{r}\right) = \frac{\vec{e}_r}{r^2}$$

$$\varphi(\vec{r}) = -\vec{p}.\vec{\nabla}\boxed{\left(\frac{1}{4\pi\varepsilon_0 r}\right)}$$

Potential of a **unit** charge

$$\varphi(\vec{r}) = \vec{p}.\boxed{\vec{E}_{unit\ charge}}$$

$$\vec{\nabla}\left(\varphi_{unit\ charge}\right)$$

Potential of a dipole = scalar product of
the dipole moment and the gradient of the potential of a **unit** charge

For a dipole: $\varphi(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}.\vec{e}_r}{r^2}$ $\Rightarrow$ $\varphi(\vec{r}) = -\vec{p}.\vec{\nabla}\left(\dfrac{1}{4\pi\varepsilon_0 r}\right)$ Scalar

Gradient of the potential of a **unit** charge

Formal Similarity

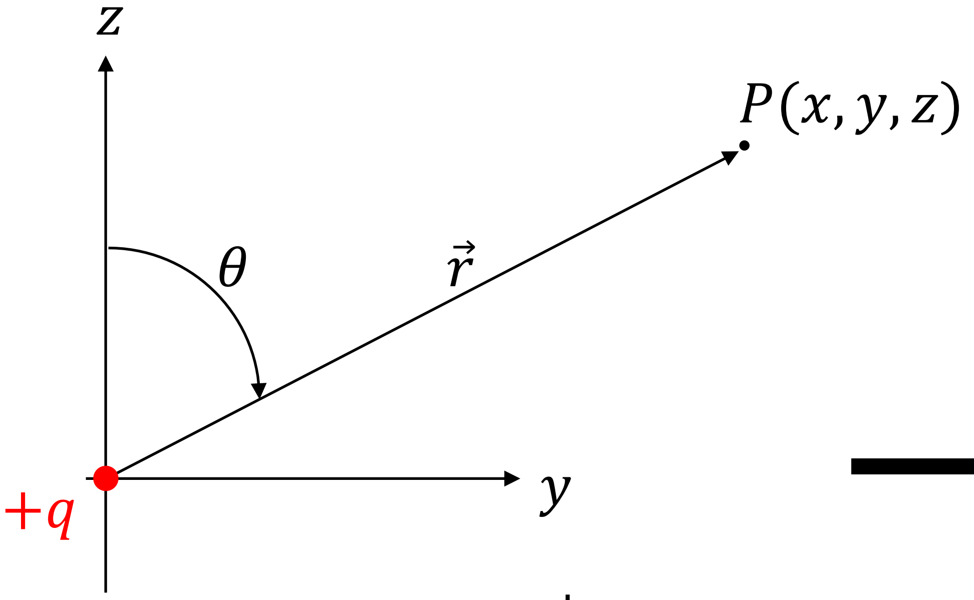
For a point charge: $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q\vec{e}_r}{r^2}$ $\Rightarrow$ $\vec{E}(\vec{r}) = -q\vec{\nabla}\left(\dfrac{1}{4\pi\varepsilon_0 r}\right)$ Vector

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}) =$$

Is there any physical reason behind such a formal similarity?

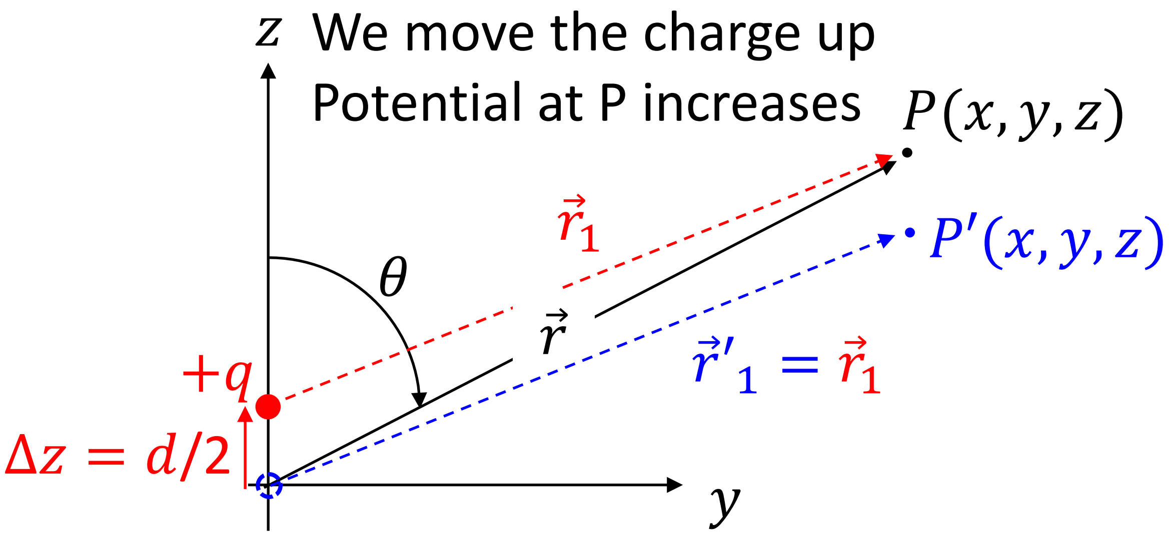
Let's us consider a 2D case

In the calculation below we drop $\frac{1}{4\pi\varepsilon_0}$

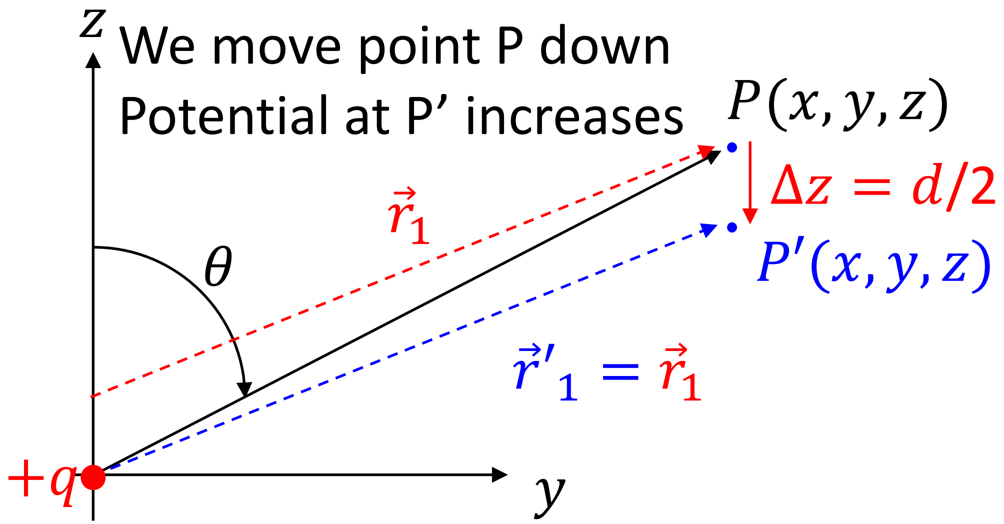


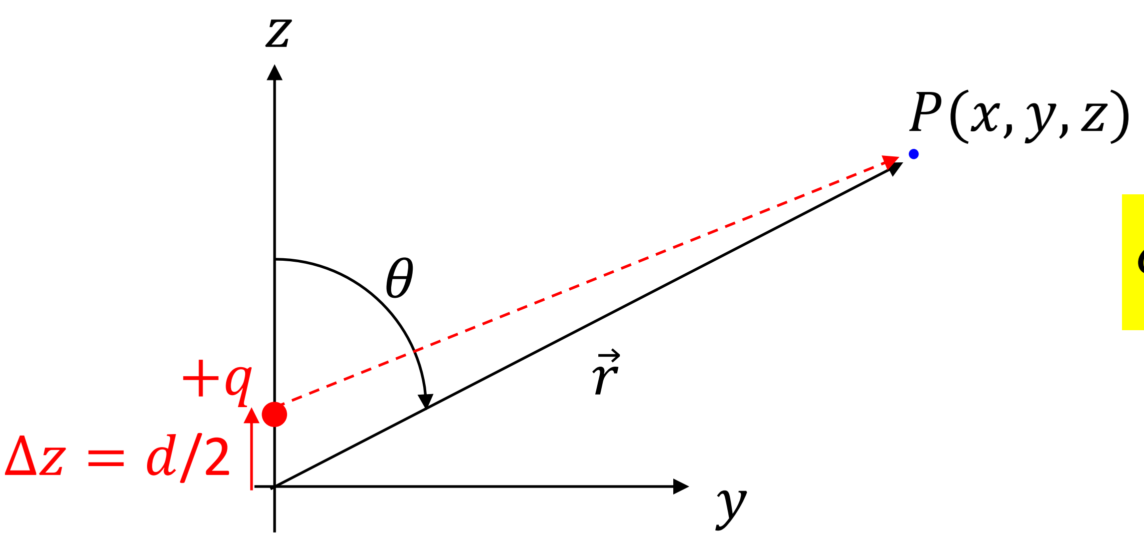
$$\varphi(P) \propto \frac{+q}{r}$$

charge at origine



$$\boldsymbol{\varphi(P)}\Big|_{\boldsymbol{charge\ at+d/2}} = \boldsymbol{\varphi(P')}\Big|_{\boldsymbol{charge\ at\ 0}}$$





$$\varphi(P)\Big|_{charge\ at\ d/2} = \varphi(P)\Big|_{charge\ at\ 0} + \Delta\varphi$$

$$\frac{+q}{r} \qquad\qquad \frac{\partial\varphi}{\partial r}dr$$

From r to z: other coordinate remain unchanged

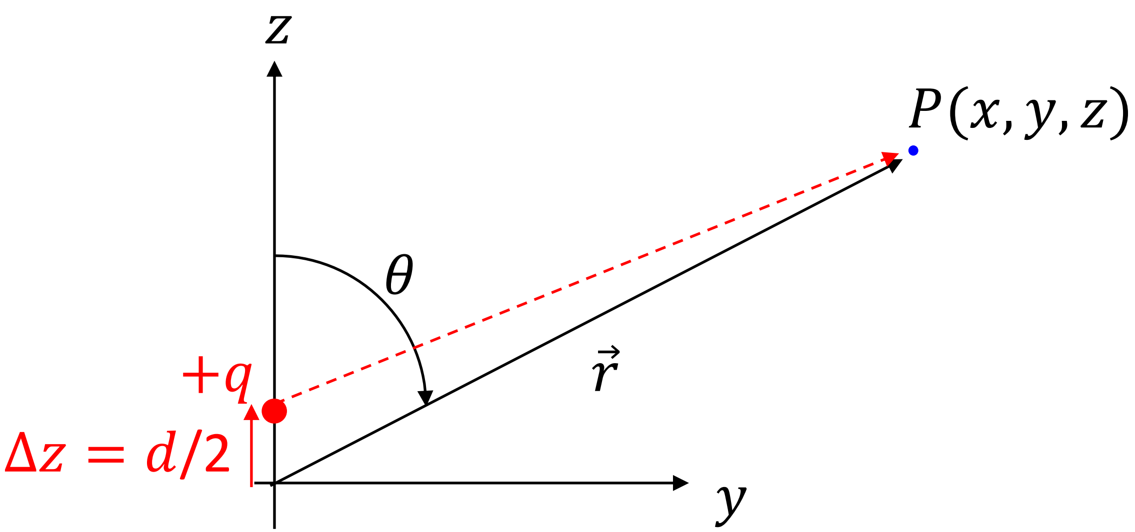
$= {}^{d}/_{2}$

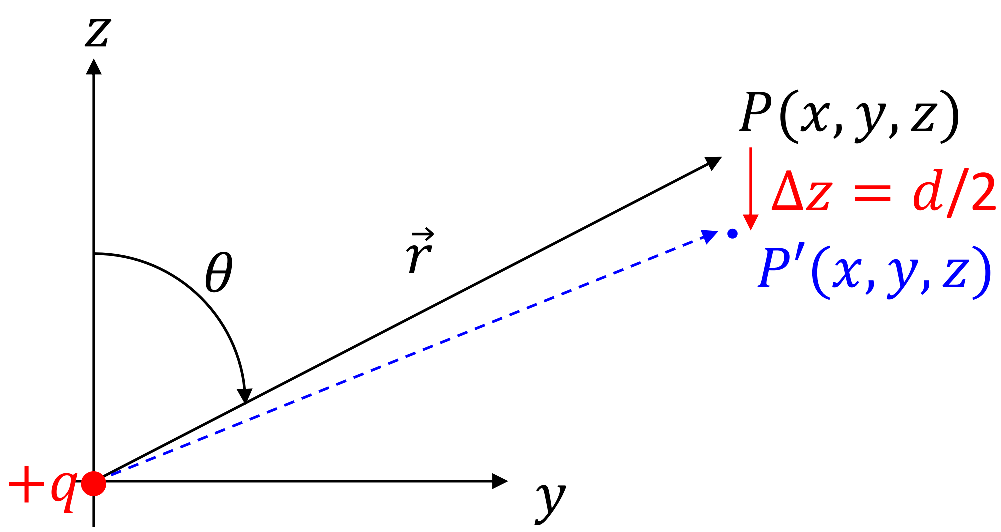
$$-\frac{\partial}{\partial z}\left(\frac{+q}{r}\right)\Delta z$$

We want the potential to increase

$$\varphi(P) \propto +q\left[\frac{1}{r} - \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]$$

charge at + d/2





$$\varphi(P) \propto +q\left[\frac{1}{r}-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]$$

charge moves to $\boldsymbol{+d/2}$

$\boldsymbol{P(x,y,z)}$ ***remains fixed***

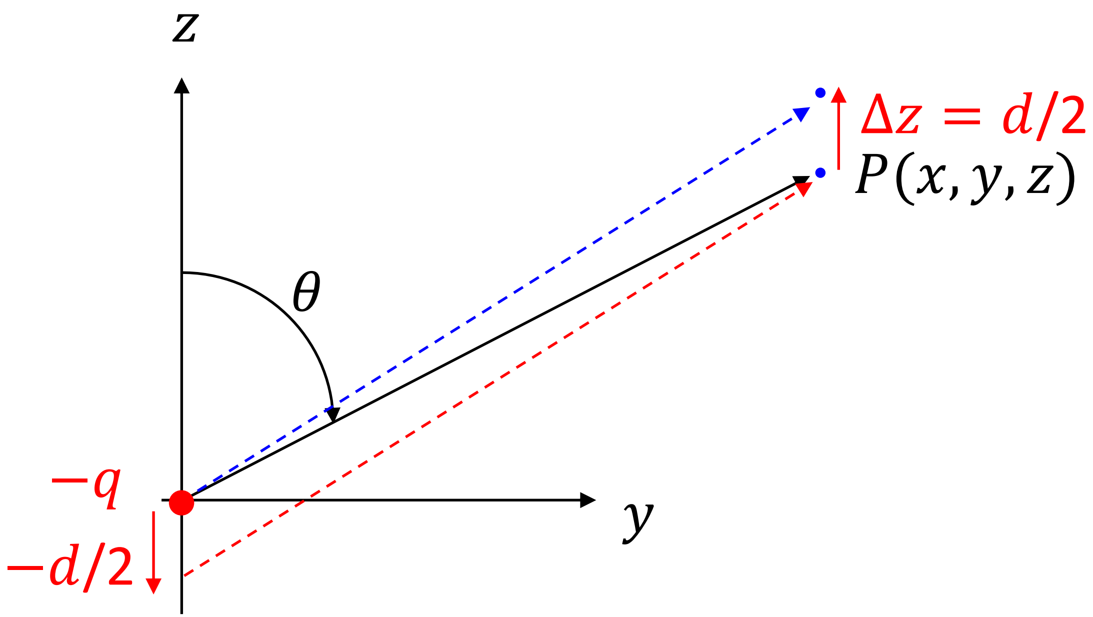
$$\varphi(P') \propto +q\left[\frac{1}{r}-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]$$

charge remains at origine

$\boldsymbol{P(x,y,z)}$ ***moves to*** $\boldsymbol{P'(x,y,z)}$

$$\boldsymbol{\varphi(P)}\Big|_{\boldsymbol{charge\ at+d/2}} = \boldsymbol{\varphi(P')}\Big|_{\boldsymbol{charge\ at\ 0}}$$

$$\boxed{\varphi_+(r) \propto +q\left[\frac{1}{r}-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]\frac{d}{2}\right]}$$



Decrease of the potential

$$\boxed{\varphi_-(r) \propto \frac{-q}{r} + \frac{\partial}{\partial z}\left(\frac{-q}{r}\right)\frac{d}{2}}$$

principle of superposition $\varphi(r) = \varphi_+(r) + \varphi_-(r)$ $\Rightarrow$ $\varphi(r) = -\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\boldsymbol{qd}$

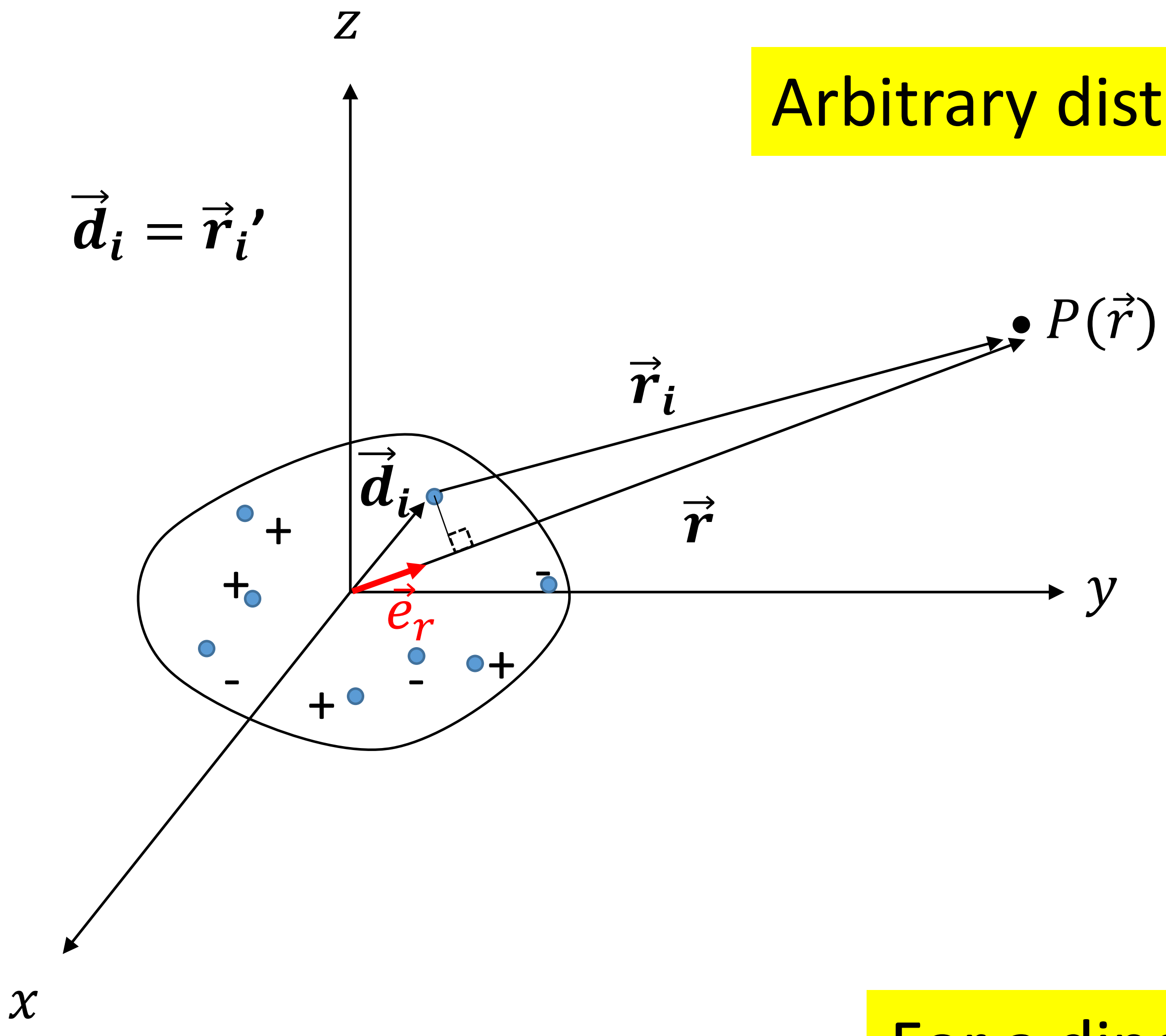
$\boldsymbol{p}$

Potential of a dipole

Extension to 3D

$$\boldsymbol{\varphi(r) = -\vec{p}.\vec{\nabla}\left(\frac{1}{r}\right)}$$

Arbitrary distribution of dipoles



$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\left|\vec{r} - \vec{d}_i\right|} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

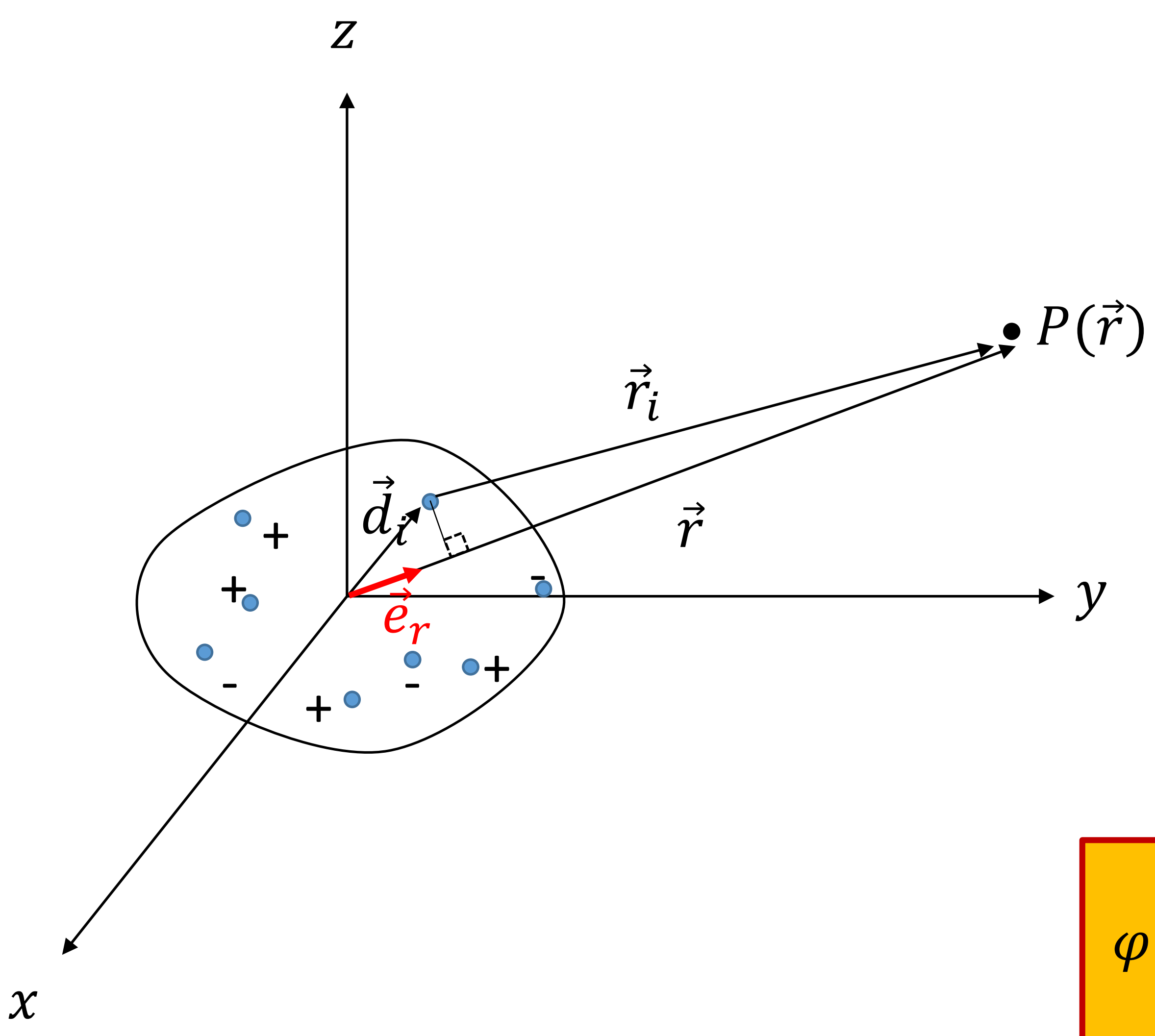
For $r \;>>\; d_i$

$$\varphi(r) = \frac{Q}{4\pi\varepsilon_0 r} \qquad Q = \int \rho(d')dV'$$

If $Q = 0$ ⇨ $\varphi = 0$!

For a dipole, $r \;>>\; d/2$

$Q = 0$ but $\varphi \neq 0$ **!**



To a good approximation

$$r - d_i = r - \vec{d}_i . \vec{e}_r$$

$$\frac{1}{|\vec{r} - \vec{r}_i|} \approx \frac{1}{|r - \vec{d}_i.\vec{e}_r|} \approx \frac{1}{r}\left(1 + \frac{\vec{d}_i.\vec{e}_r}{r}\right)$$

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \sum_i q_i \frac{\vec{d}_i.\vec{e}_r}{r^2} \dots\right)$$

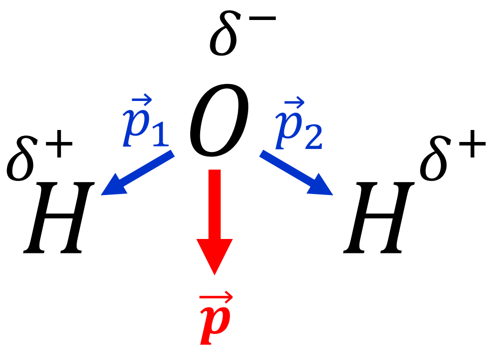
For a distribution of dipoles $Q = 0$

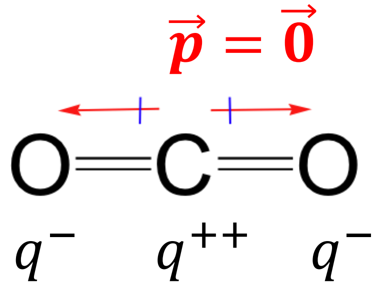
$$\boxed{\varphi(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i\vec{d}_i.\vec{e}_r}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{(\sum_i \vec{p}_i).\vec{e}_r}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}.\vec{e}_r}{r^2}}$$

Dipole moment of the distribution $\boxed{\vec{P} = \left(\sum_i \vec{p}_i\right)}$

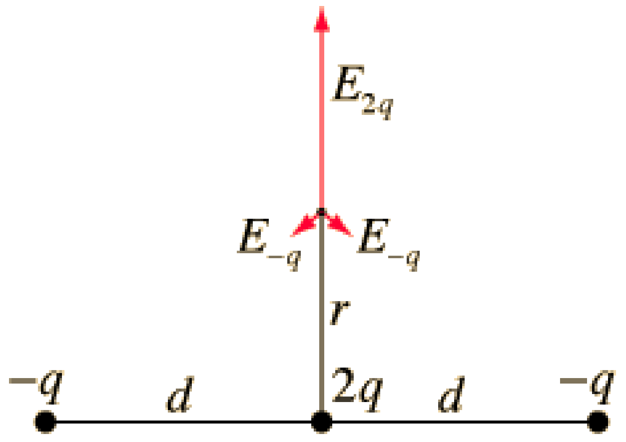
$$\boxed{\vec{P} = \left(\sum_i \vec{p}_i\right)}$$

Example water: Net dipole moment $\vec{\boldsymbol{p}}$





NO net dipole moment



Linear quadrupole

$$E_r = \frac{2q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{r}{(r^2+d^2)^{3/2}}\right]$$

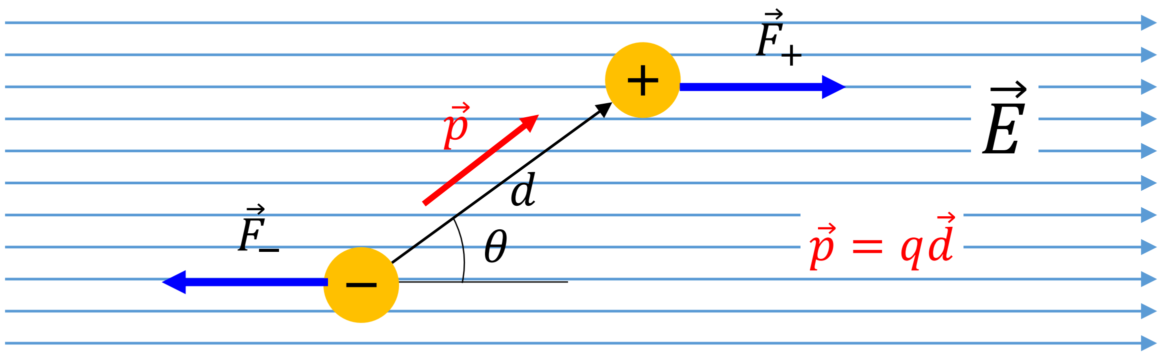
$$E_{2q}\ at\ r = \frac{2q}{4\pi\varepsilon_0 r^2}$$

For $\frac{d^2}{r^2} \ll 1$ $\quad \left(1+\frac{d^2}{r^2}\right)^{-3/2} = 1 - \frac{3}{2}\frac{d^2}{r^2}$

$$E_{-q}\ at\ r = \frac{-q}{4\pi\varepsilon_0 (r^2+d^2)^{3/2}}$$

$$E_r = \frac{3qd^2}{4\pi\varepsilon_0 r^4} \qquad \boxed{\varphi(r) = \frac{qd^2}{4\pi\varepsilon_0 r^3}}$$

Force, Torque and work done on dipole: **Uniform field**



Net force $\sum_i \vec{F}_i = \vec{0}$ ⟹ No translation

BUT the forces do not act on the same line ⟹ **Torque is exerted** ⟹ Rotation

$$\vec{\tau}_+ = \frac{\vec{d}}{2} \times \vec{F}_+$$

$$\vec{\tau}_- = \frac{\vec{d}}{2} \times \vec{F}_-$$

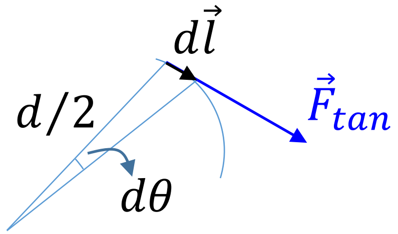
$$\vec{\tau} = \sum_i \vec{\tau}_i = \vec{d} \times \vec{F}_+ = qdE_+ sin\theta$$

⟹ $\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$

No torque if $\vec{p} \parallel \vec{E}$

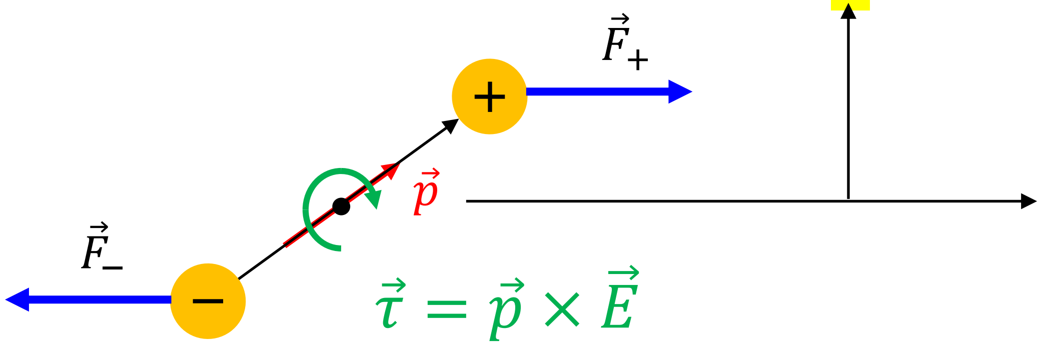
Force, Torque and work done on dipole: **Uniform field**

Work done by torque $dW = F_{tan}dl = F_{tan}\left(\frac{d}{2}\right)d\theta = \tau d\theta$



$$dW = \tau d\theta = -pEsin\theta d\theta$$

+ Convention for angle orientation



$\vec{\tau} = \vec{p} \times \vec{E}$

Because $\vec{\tau}$ tends to decrease θ

⇨ lowering the energy of the system

$$W = \tau d\theta = \int_{\theta_1}^{\theta_2} -pEsin\theta d\theta = pEcos\,\theta_2 - pEcos\theta_1$$

Work – energy theorem $\quad W = -\Delta U \;\rightarrow\; \Delta U = -pEcos\theta_2 - pEcos\theta_1$ $\quad \boxed{U = -pEcos\theta}$

Dipole in a uniform field

Microwave oven

- Water molecules in an external electric field ⇨ Experience a torque ⇨ Begin to rotate
- Direction of the external electric field changes very rapidly ⇨ water molecules perform rotational oscillations

- Water molecules gain energy which they dissipate in the form of heat

- Frequency of 2.45 GHz for the oscillating electric field

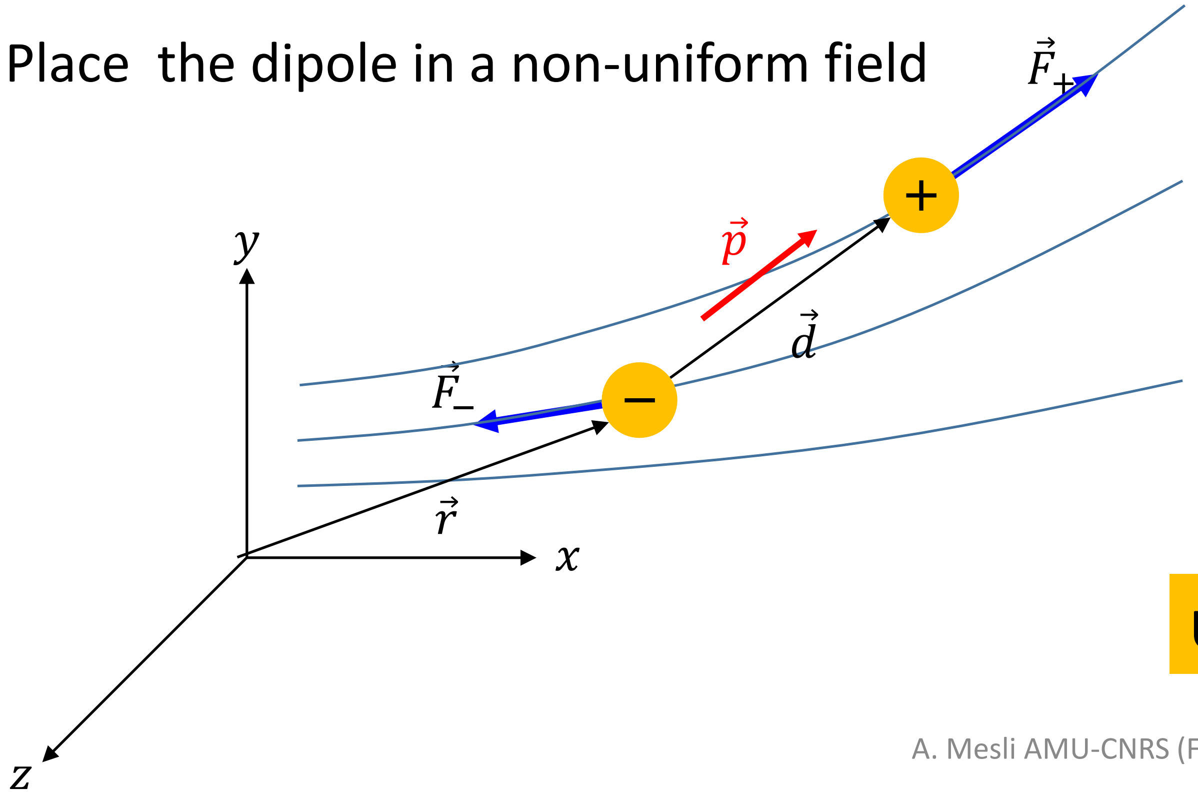
Torque done on dipole: **Nonuniform field**

Question

How can we do to attract a neutral (polarizable) object?

⇨ how do we do to make the dipole translate?

Place the dipole in a non-uniform field



$$\vec{F} = \sum_i \vec{F}_i = q\vec{E}(\vec{r}+\vec{d}) - q\vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}+\vec{d}) = \vec{E}(\vec{r}) + \vec{d}.\nabla\vec{E}(\vec{r})$$

Scalar matrix

$$\vec{F} = q\vec{d}.\vec{E}(\vec{r}) = \vec{p}.\nabla\vec{E}(\vec{r})$$

Uniform field ⇔ $\nabla\vec{E}(\vec{r}) = 0$ ⇔ $\vec{F} = \vec{0}$

To observe an effect of nonuniform field on a dipole, the scale of $\vec{E}$ nonuniformity MUST be large compared to $\boldsymbol{d}$

Torque and work done on dipole: **nonuniform field**

The principle of the boomerang