RC 3

Ch4 &5

Outline

- Poisson’ s and Laplace’ s Equations
- Method of images
- Boundary- value problems

Poisson’ s Equation

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$

In a homogeneous medium

ε is a constant over space

Poisson’s equation $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}.}$

Laplace's Equation

- In a simple medium where there is no free charge, $\rho=0$

Laplace's Equation $\boxed{\nabla^2 V = 0,}$

- Example to use Laplace's equation: a set of conductors at different potentials

Solve V by Laplace's equation ➔ $\mathbf{E}=-\nabla V$ ➔ $\rho_s=\varepsilon E_n$

(see example 4-1)

Also HW 4 p4-1

In Different Coordinates

Cartesian Coordinate $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

Spherical:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

EXAMPLE 4–2 Determine the **E** field both inside and outside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \le R \le b$ and $\rho = 0$ for $R > b$ by solving Poisson's and Laplace's equations for V.

Solution We recall that this problem was solved in Chapter 3 (Example 3–7) by applying Gauss's law. We now use the same problem to illustrate the solution of one-dimensional Poisson's and Laplace's equations. Since there are no variations in θ and ϕ directions, we are dealing only with functions of R in spherical coordinates.

a) Inside the cloud,

$$0 \leq R \leq b, \qquad \rho = -\rho_0.$$

In this region, Poisson's equation ($\nabla^2 V_i = -\rho/\epsilon_0$) holds. Dropping $\partial/\partial\theta$ and $\partial/\partial\phi$ terms from Eq. (4–9), we have

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dV_i}{dR}\right) = \frac{\rho_0}{\epsilon_0},$$

which reduces to

$$\frac{d}{dR}\left(R^2\frac{dV_i}{dR}\right) = \frac{\rho_0}{\epsilon_0}R^2. \tag{4–16}$$

Integration of Eq. (4–16) gives

$$\frac{dV_i}{dR} = \frac{\rho_0}{3\epsilon_0}R + \frac{C_1}{R^2}. \tag{4–17}$$

The electric field intensity inside the electron cloud is

$$\mathbf{E}_i = -\nabla V_i = -\mathbf{a}_R\left(\frac{dV_i}{dR}\right).$$

Since $\mathbf{E}_i$ cannot be infinite at $R = 0$, the integration constant C_1 in Eq. (4–17) must vanish. We obtain

$$\mathbf{E}_i = -\mathbf{a}_R\frac{\rho_0}{3\epsilon_0}R, \qquad 0 \leq R \leq b. \tag{4–18}$$

b) Outside the cloud,

$$R \geq b, \qquad \rho = 0.$$

Laplace's equation holds in this region. We have $\nabla^2 V_o = 0$ or

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{dV_o}{dR}\right) = 0. \tag{4-19}$$

Integrating Eq. (4–19), we obtain

$$\frac{dV_o}{dR} = \frac{C_2}{R^2} \tag{4-20}$$

or

$$\mathbf{E}_o = -\nabla V_o = -\mathbf{a}_R \frac{dV_o}{dR} = -\mathbf{a}_R \frac{C_2}{R^2}. \tag{4-21}$$

The integration constant C_2 can be found by equating $\mathbf{E}_o$ and $\mathbf{E}_i$ at $R = b$, where there is no discontinuity in medium characteristics.

$$\frac{C_2}{b^2} = \frac{\rho_0}{3\epsilon_0} b,$$

from which we find

$$C_2 = \frac{\rho_0 b^3}{3\epsilon_0} \tag{4-22}$$

and

$$\mathbf{E}_o = -\mathbf{a}_R \frac{\rho_0 b^3}{3\epsilon_0 R^2}, \qquad R \geq b. \tag{4-23}$$

Since the total charge contained in the electron cloud is

$$Q = -\rho_0 \frac{4\pi}{3} b^3,$$

Eq. (4–23) can be written as

$$\mathbf{E}_o = \mathbf{a}_R \frac{Q}{4\pi\epsilon_0 R^2}, \tag{4–24}$$

which is the familiar expression for the electric field intensity at a point R from a point charge Q. ∎

1. Proper Coordinates

2. Boundary condition (Value on special points)

Uniqueness of Electrostatic Solutions

- Uniqueness theorem: a solution of Poisson's equation that satisfies the given boundary conditions is a unique solution.

Methods of Images (Important!)

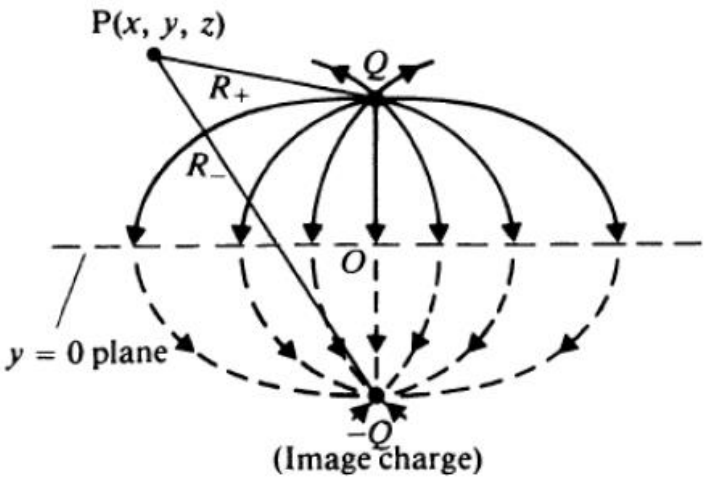
Methods of images: replacing boundaries by appropriate **image charges** in lieu of a formal solution of Poisson's or Laplace's equation

- Condition on boundaries unchanged
- V(**R**) can be determined easily

Four cases

1. Point Charge and Conducting Planes
2. Line Charge and Parallel Conducting Cylinder
3. Point Charge and Conducting Sphere
4. Charged sphere & grounded plane

Point charge and conducting planes



(b) Image charge and field lines.

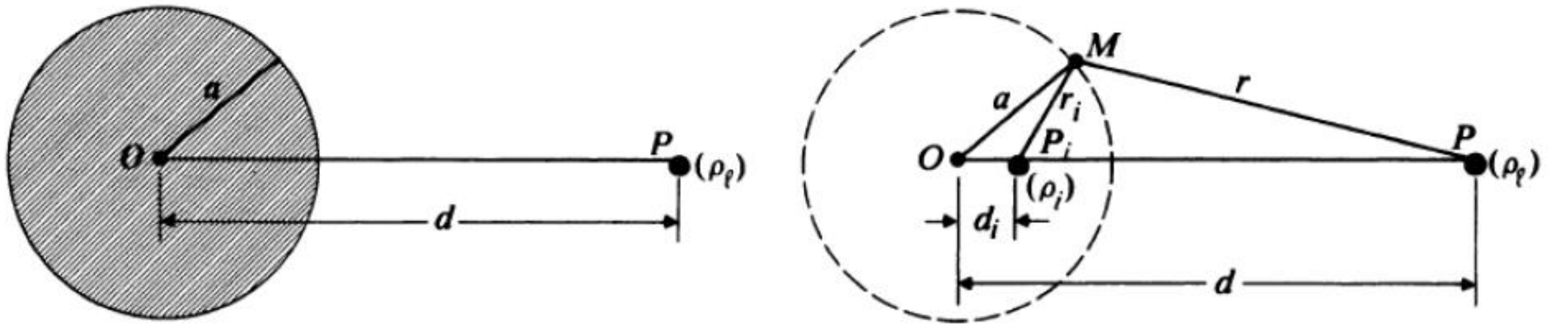
$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right),$$

$$R_+ = [x^2 + (y-d)^2 + z^2]^{1/2},$$
$$R_- = [x^2 + (y+d)^2 + z^2]^{1/2}.$$

****What about y< o region?***

Solution $V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right),$

Line charge and parallel conducting cylinder



(a) Line charge and parallel conducting cylinder.

(b) Line charge and its image.

FIGURE 4–5
Cross section of line charge and its image in a parallel, conducting, circular cylinder.

Assume $\boxed{\rho_i = -\rho_\ell.}$ (intelligent guess, also see figure 3-15)

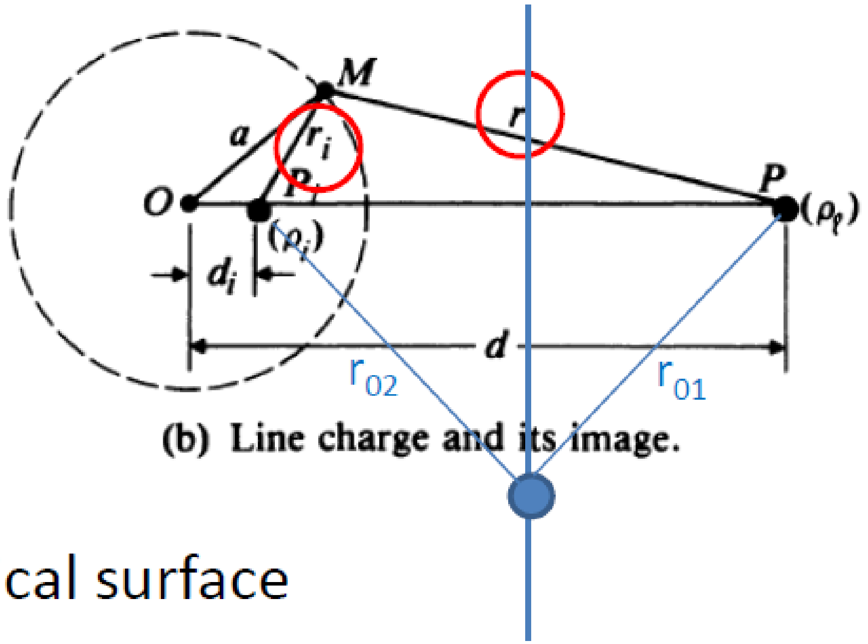
- Voltage due to ρ_l

$$V = -\int_{r_0}^{r} E_r\,dr = -\frac{\rho_\ell}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r}.$$

Reference point, V=0

Point of interest



(b) Line charge and its image.

- Voltage due to ρ_l and ρ_i on cylindrical surface

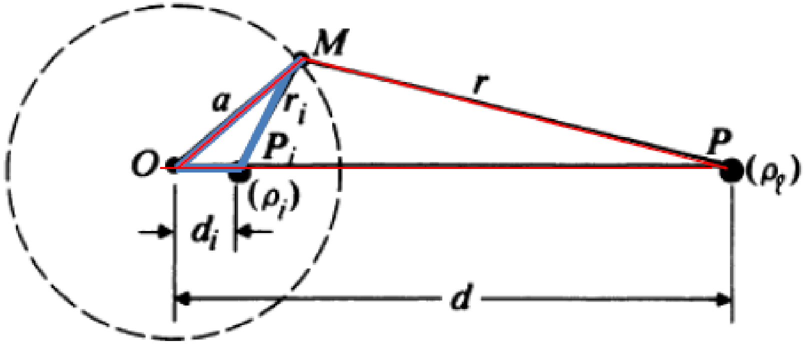
$$V_M = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_{01}}{r} - \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_{02}}{r_i}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_i}{r}.$$

Choosing the same reference point with equidistance from ρ_l and ρ_i so that $\ln r_0$ terms cancel.

- To make V_M=constant

$$\frac{r_i}{r} = \text{Constant.}$$



(b) Line charge and its image.

- To make M coincide with the cylindrical surface (OM=a), P_i should be chosen to make the two triangles OMP_i and OPM similar. (Otherwise, r_i/r=constant over the cylindrical surface cannot be satisfied.)

$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$

$$\boxed{d_i = \frac{a^2}{d}}$$

P_i is called the **inverse point** of P

Point charge and conducting sphere

$$V_M = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0,$$

$$\Downarrow$$

$$\frac{r_i}{r} = -\frac{Q_i}{Q} = \text{Constant.}$$

Similar to the case in 4-4.2

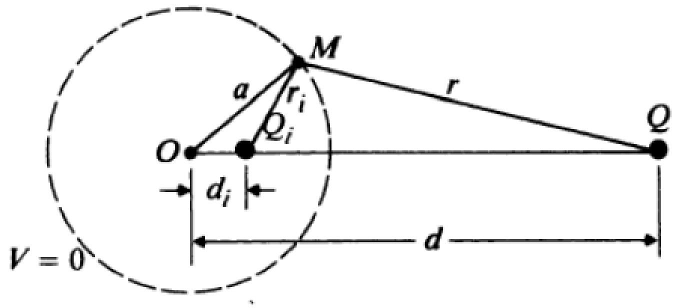
$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$

$$\Downarrow$$

$$-\frac{Q_i}{Q} = \frac{a}{d}$$

$$\Downarrow$$

$$\boxed{Q_i = -\frac{a}{d}Q} \qquad \boxed{d_i = \frac{a^2}{d}.}$$



(b) Point charge and its image.

Q_i is called the **inverse point** of Q

1. Know what exactly how each of the solution comes

2. Know what kind of case it is

Boundary Value problems

1. **Generally speaking, it will not relate with all three dimensions**

2. **Choose proper coordinates**