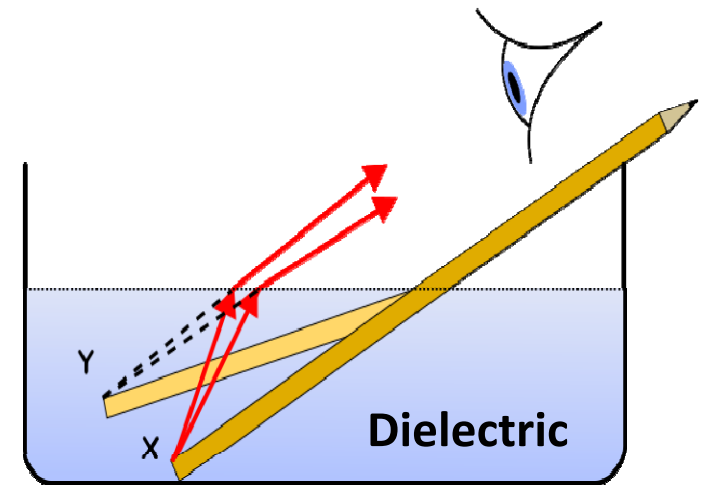


# Dielectrics



Consequence of the behavior of the “electric field” at the interface between two different media

Illustration of the boundary conditions



- How the applied field affects the dielectric ?
- How the dielectric reacts to the field inside, outside and at the **interface** ?

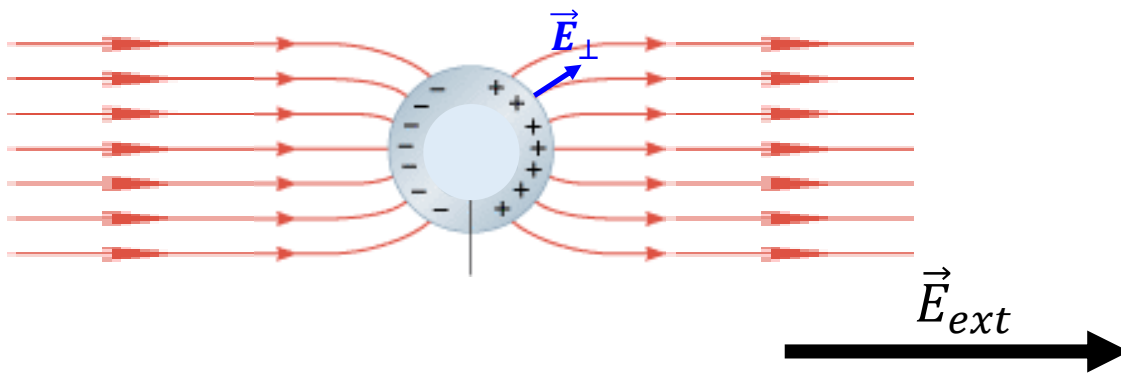
- Free Charge and free charge density (linear, surface, volume)
- Electric field
- Electric Force
- Electric Potential
- Work – energy theorem
- Potential Energy
- Vacuum
- Conductor
- Around

## Dipole and dipole moment

- Polarization
- Bound charges (linear, surface, volume)
- Permittivity of materials  $\varepsilon = \varepsilon_0 \varepsilon_r$  ( $\varepsilon_r = 1$  in vacuum)
- Electric susceptibility  $\chi \Rightarrow \varepsilon = \varepsilon_0(1 + \chi)$  ( $\chi = 0$ )
- Vector displacement
- Dielectric

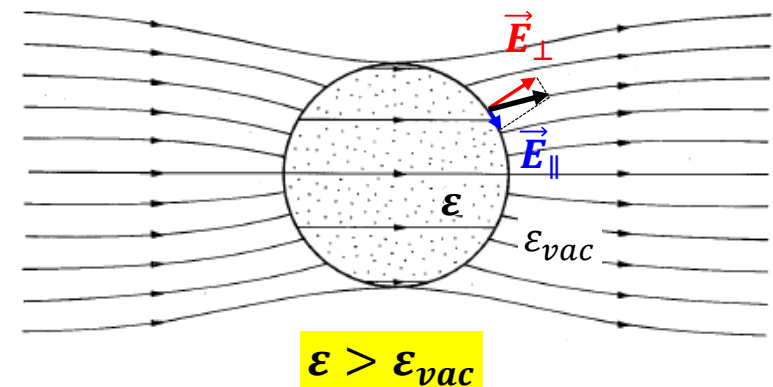
# Differences and similarities between conductor and dielectric in the presence of electric field

## Conductor: Induction



- Charge free to move
- At the surface  $\vec{E}_{\perp} \Rightarrow \vec{E}_{\parallel} = \vec{0}$
- Inside  $\vec{E}_{ind}$  opposes  $\vec{E}_{ext}$
- Total compensation  $\vec{E}_{net} (inside) = \vec{0}$

## Dielectric: Polarization



- Charge **NOT** free to move
- At the surface  $(\vec{E}_{\perp} \text{ \& \; } \vec{E}_{\parallel}) \neq \vec{0}$
- Inside  $\vec{E}_{ind}$  opposes  $\vec{E}_{ext}$
- Partial compensation  $\vec{E}_{net} (inside) \neq \vec{0}$

Both materials are affected by the electric field and both affect its shape

## What is a dielectric ?

- A dielectric has the ability to get **polarized** by an external applied field. This field induces electric dipoles inside the dielectric
- Polarization occurs in both **polar** and **nonpolar** materials
- Although any kind of substance is **polarizable** to some extent, the effect of polarization is important only in insulating materials
- The dielectric is an insulator BUT an insulator is not necessarily a dielectric

## Consequence

The induction of electric dipoles within the dielectric modifies the electric field pattern both **inside** and **outside** the material

Perfect conductor: **infinite conductivity**

Polarization = induction of macroscopic “dipoles” made of free charges

Perfect dielectric: **zero conductivity**

Polarization = induction of microscopic dipoles made of bound charges

## Concept of Permittivity of a medium

From  $\varepsilon_0$ (vacuum) to  $\varepsilon$ (dielectric) =  $\varepsilon_r \varepsilon_0$

From Slides 50 - 52 in C\_Lecture 3 Introduction 2

- Electric permittivity  $\varepsilon_{diel}$  describes how an electric field affects **AND** is affected by a dielectric. It is determined by the ability of a material to polarize in response to an applied field, and thereby to cancel, partially, the field inside the material

In vacuum, Coulomb's law contains the constant

$$\frac{1}{4\pi\varepsilon_0}$$

in which  $\varepsilon_0$  is defined as the permittivity of vacuum

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$$

Coulomb's law

$$F = \frac{1}{4\pi\varepsilon_0} \frac{qq'}{r^2}$$

## Conductor versus dielectric: Induction versus polarization

There is a fundamental difference between induction and polarization by an external field

In both cases an external field generates dipoles

### **Induction applies to conductors:**

It results in the generation of opposite charges residing at the surface

The field inside the conductor = 0  $\Rightarrow \epsilon$  goes from  $\epsilon_0$  (vacuum) to  $\epsilon_r \epsilon_0$  where  $\epsilon_r = \infty$

No dipoles inside

It is  $\epsilon_r = \infty$  that makes a perfect conductor = perfect reflector

Forbids the electrostatic field to penetrate inside

## Polarization applies to dielectrics (insulators):

It results in the generation of dipoles due to **BOUND** charges induced locally

The field inside the dielectric  $\neq 0 \Rightarrow \epsilon$  goes from  $\epsilon_0$  (vacuum) to  $\epsilon_r \epsilon_0$  where  $\epsilon_r = \text{finite}$

The dipoles inside are microscopic

$$\epsilon(\text{permittivity of the dielectric}) = \epsilon_r \epsilon_0$$

$\epsilon_r$  is the relative permittivity or dielectric constant (No dimension)

$$\epsilon_r = 1 \text{ in vacuum}$$

$\epsilon_r$  indicates the strength of the polarizability of a material



Concept of screening



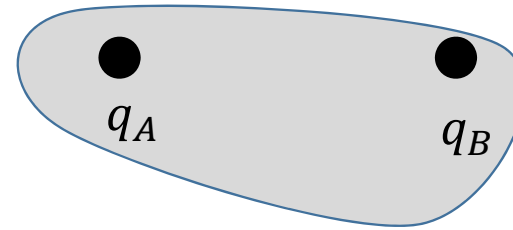
## What is the meaning of the dielectric constant $\epsilon_r$

$\epsilon_r$  is a material property that affects the coulomb force between two point charges



$$F = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{r^2}$$

In vacuum



$$F = \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{q_A q_B}{r^2}$$

Force is lowered  
in a dielectric

In dielectric

Another point of view: As if  $q_A$  is reduced to  $q'_A = q_A/\epsilon_r$

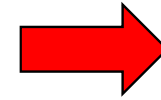
$$F = \frac{1}{4\pi\epsilon_r\epsilon_0} \frac{q_A q_B}{r^2}$$



$$F = \frac{1}{4\pi\epsilon_0} \frac{q'_A q_B}{r^2}$$

Screening effect

What is the reason for the screening effect?



Polarization

## Materials

## Dielectric constants

Vacuum	1
Air	1.005364
Glass, pyrex 7740	5
Mica	5.4
Muscle	58
Skin	33-44
Tongue	38
Water, liquid, 0 °C	87.9 (made of permanent dipoles)
Polyethylene	2.26

## Why do we use the generalized permittivity concept: $\epsilon_0 \epsilon_r$

- The concepts of polarizability and dipole moment distribution are introduced to relate microscopic phenomena to the macroscopic fields
- The introduction of *permittivity* eliminates the need to explicitly consider microscopic effects... *in simple case of linear, homogeneous and isotropic medium*



From the macroscopic point of view

knowing the *permittivity* of a dielectric is all what we need to describe its interaction with the external field

# Three important properties to characterize a Dielectric in Ve230

**Linearity**

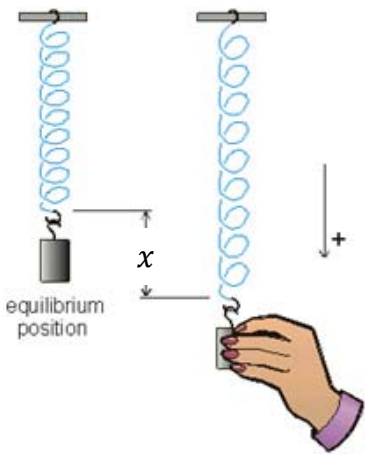


**Cause  $\propto$  Effect**

Hook's law

$$F = kx$$

**Small displacement**



**Homogeneity**



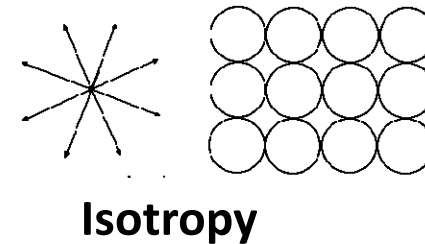
**Equally linear at any point in space**



**Isotropy**

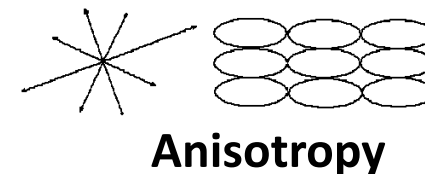


**Equally linear along any direction**



**Isotropy**

**Effect**  **Cause**



**Anisotropy**

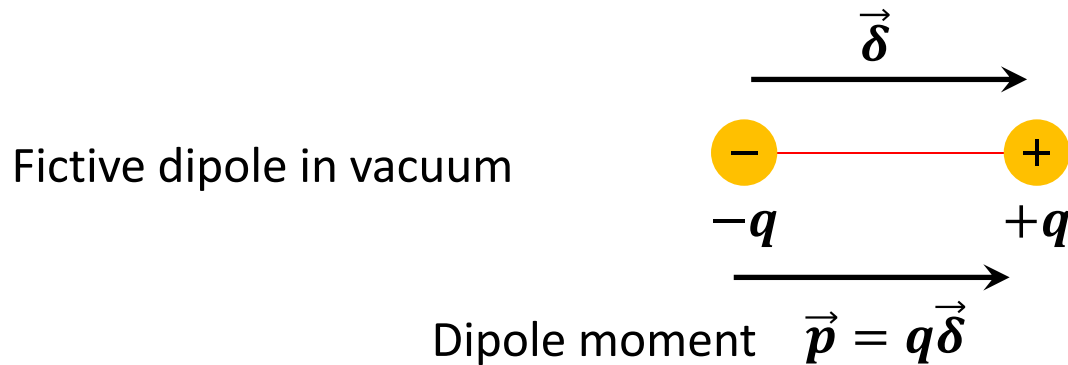
**Effect**  **Cause**

## Main questions regarding dielectrics

- What is the mechanism behind polarization of a dielectric ?
  - Polar versus non polar material
- What are the consequences of polarization?
  - Concept of **bulk** and **surface bound charges**
  - Boundary conditions
- Characteristics of the internal electric field at the macroscopic and microscopic level?

Restriction to Linear – homogenous – isotropic dielectric

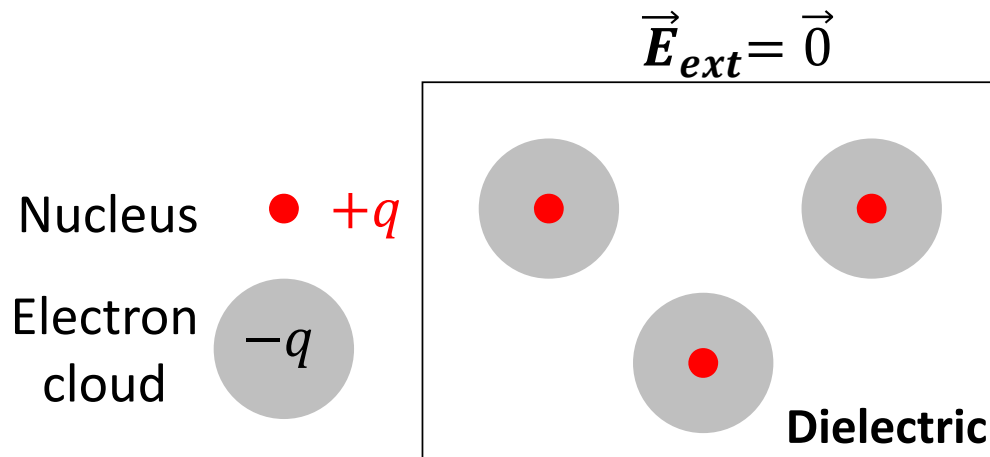
# What is polarization and what is polarizability ? Electronic polarization



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{e}_r}{r^2} \quad \vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

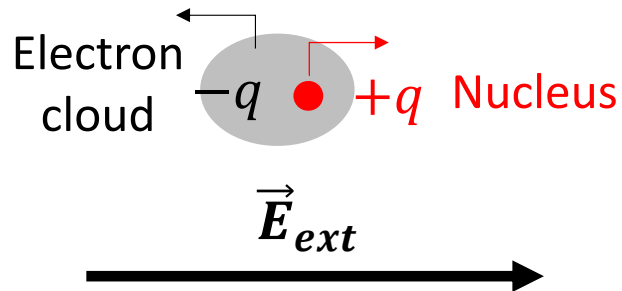
$\vec{e}_r$  unit vector along  $\vec{r}$

**Polarizability:** The ability of a material to become polarized in the presence of  $\vec{E}_{ext}$



The centers of gravity of the nucleus and the electron cloud coincide

$\vec{E}_{ext}$  distorts the electronic cloud. It becomes elongated along the direction of the field



- The electronic cloud is pushed back
- The nucleus is pushed forward in the direction of the field

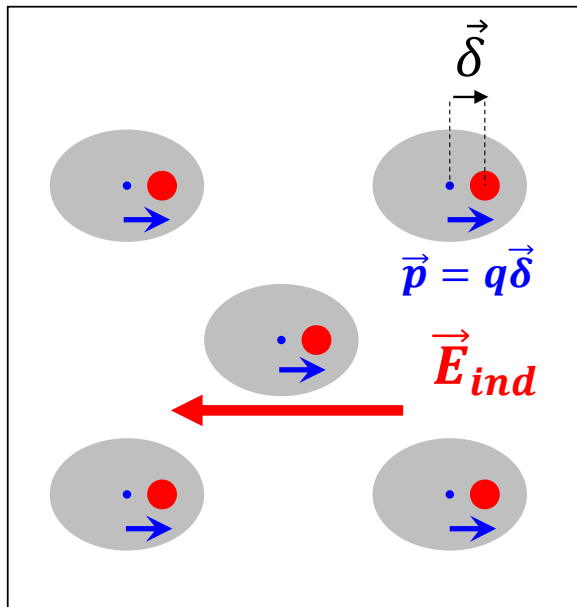
$$\vec{p} = q\vec{\delta} \quad \text{What causes the separation } \vec{\delta} ?$$

The local  $\vec{E}_{loc}$  field

$$\vec{p} = \alpha_e \vec{E}_{loc} \quad [p] = \text{Cm} \quad [\alpha_e] = \text{Fm}^2$$

$\alpha_e$  = atomic (electronic) polarizability

$\vec{E}_{ind} = \vec{E}_{pol}$  macroscopic internal field which tends to oppose the external field



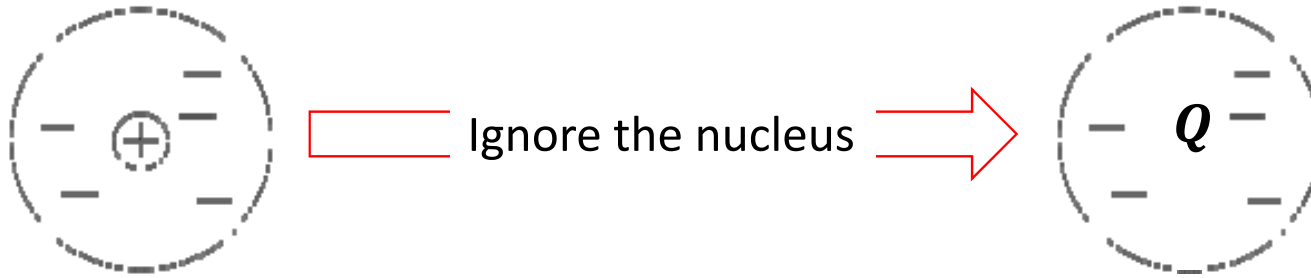
Three different fields are taking place in the dielectric

$$\vec{p} = \alpha_e \vec{E}_{loc}$$

- If the dipole were isolated, the **local** electric field would simply be the applied macroscopic field  $\vec{E}_{ext}$
- However, the large number of  $N$  neighboring dipoles also contribute to the polarizing electric field.
- The electric field changes drastically from point to point within a small volume containing many dipoles.
- By superposition principle, the macroscopic resulting field inside the dielectric will then be equal to the average over this small volume.



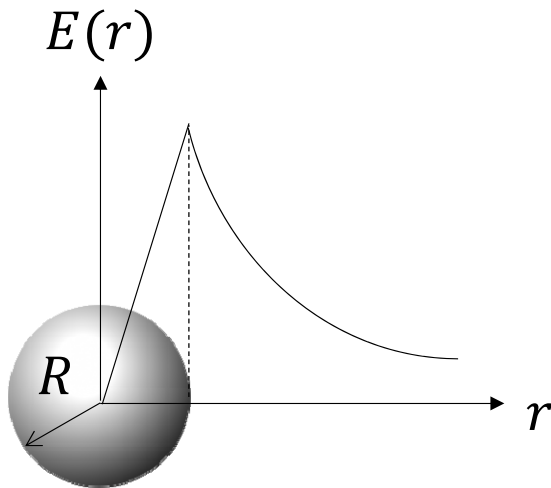
## Electronic polarization of an atom in free space: crude approach



Spherical electronic cloud

Slide #80 in E\_Lectures 8&9 Gauss law in Electrostatics

Using Gaussian surface inside the charged sphere

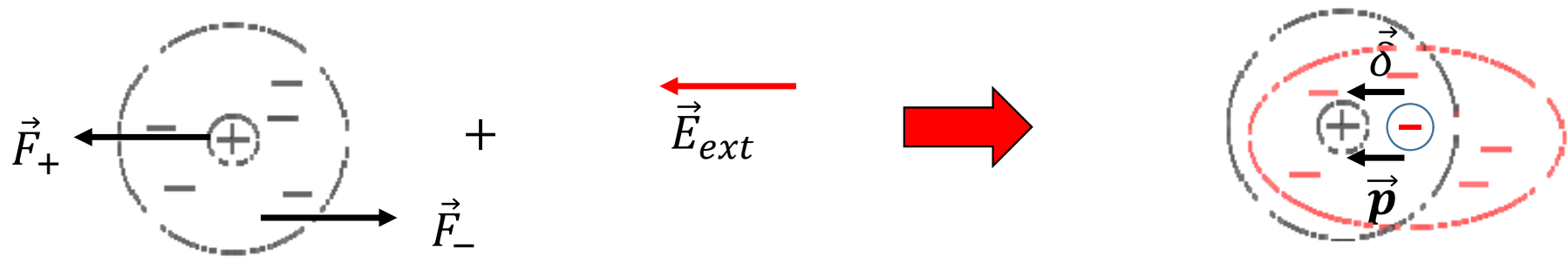


$$\text{Charge } Q = \rho \left( \frac{4\pi}{3} R^3 \right)$$

$$\boxed{r < R} \Rightarrow E(r) 4\pi r^2 = -\frac{Q}{\epsilon_0}, \quad Q = \rho \left( \frac{4\pi}{3} r^3 \right)$$

$$\Rightarrow E(r) = \frac{\rho}{3\epsilon_0} r \quad \Rightarrow$$

$$\boxed{E(r) = -\frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r}$$



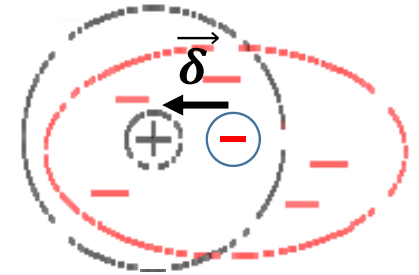
**Question:** When does the separation reach its maximum?

**Answer:** When the external force is balanced by the attraction between the nucleus and the electronic cloud

*In a conductor the separation = dimension of the conductor (electron is free)*

## Crude model: electronic polarization

Electronic cloud = charged sphere  $E(r) = -\frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r$



Force exerted on the nucleus (charge  $Q$ )  
inside the cloud at distance  $\delta$  from the center

$$F_{cloud} = QE(\delta) = -\frac{1}{4\pi\epsilon_0} \frac{Q^2}{R^3} \delta$$

Equilibrium position is reached when  $\vec{F}_{cloud} + \vec{F}_{ext} = \vec{0} \quad \Rightarrow \quad -\frac{1}{4\pi\epsilon_0} \frac{Q^2}{R^3} \delta + QE_{ext} = 0$

Equilibrium distance  $\delta = 4\pi\epsilon_0 R^3 \frac{E_{ext}}{Q}$

Induced dipole  $\vec{p} = Q\vec{\delta} \quad p = Q\delta = 4\pi\epsilon_0 R^3 E_{ext}$

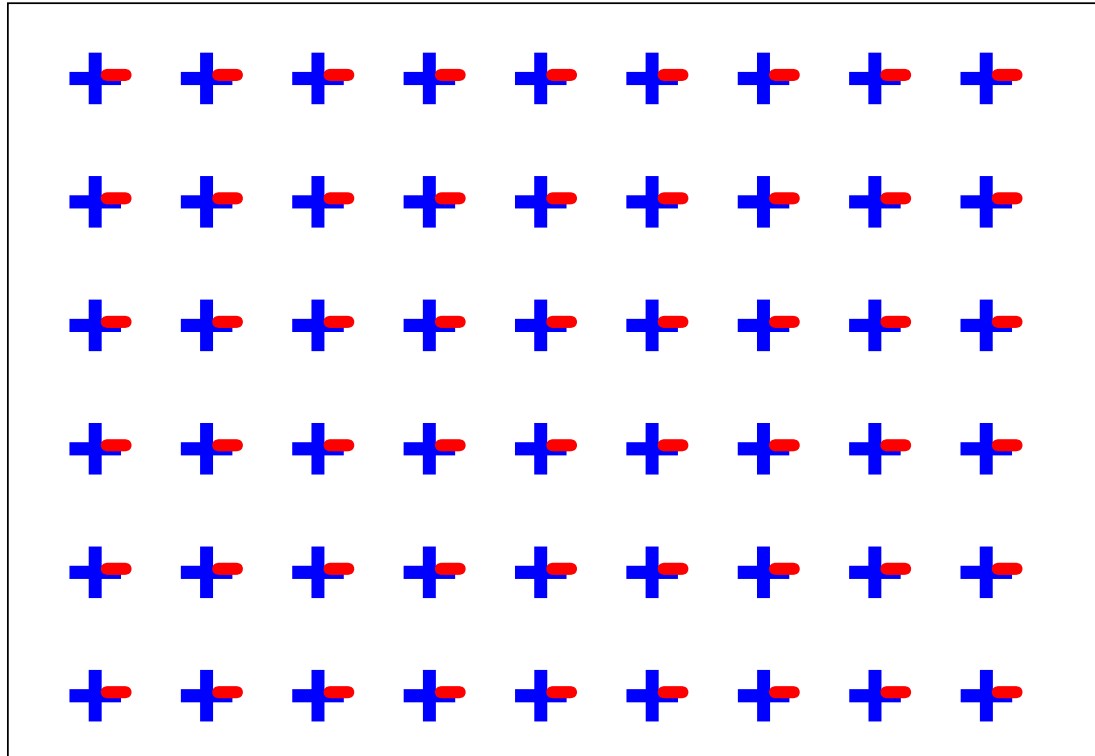
$$\vec{p} = \alpha_e \vec{E}_{loc} \quad \vec{E}_{loc} = \vec{E}_{ext}$$

Electronic polarizability of atoms

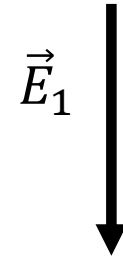
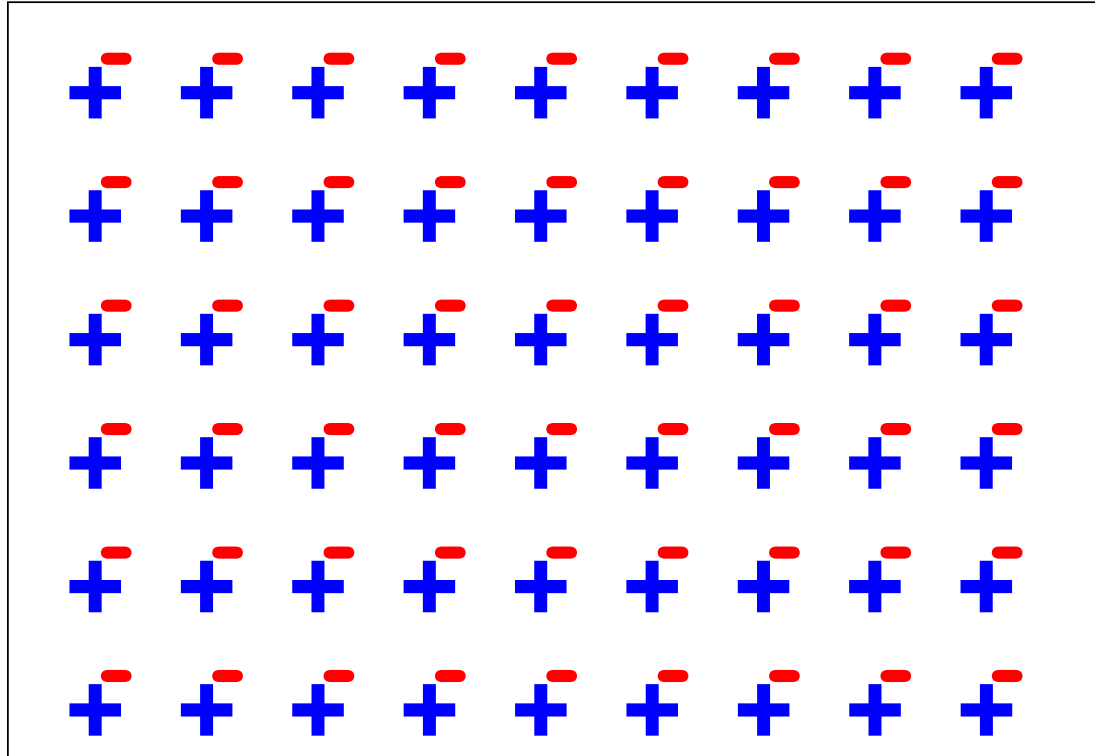
$$\alpha_e = \frac{p}{E_{ext}} = 4\pi\epsilon_0 R^3$$

## Mechanism of polarization of a simple linear dielectric

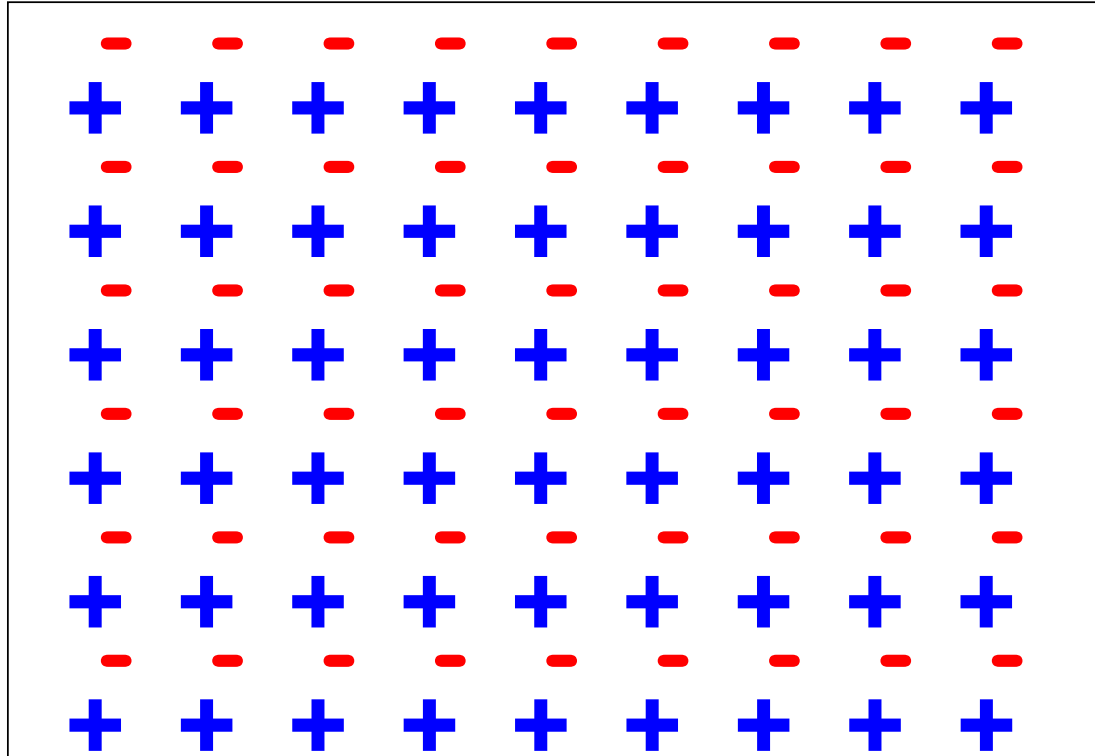
External field  $\vec{E}_{ext} = \vec{0}$



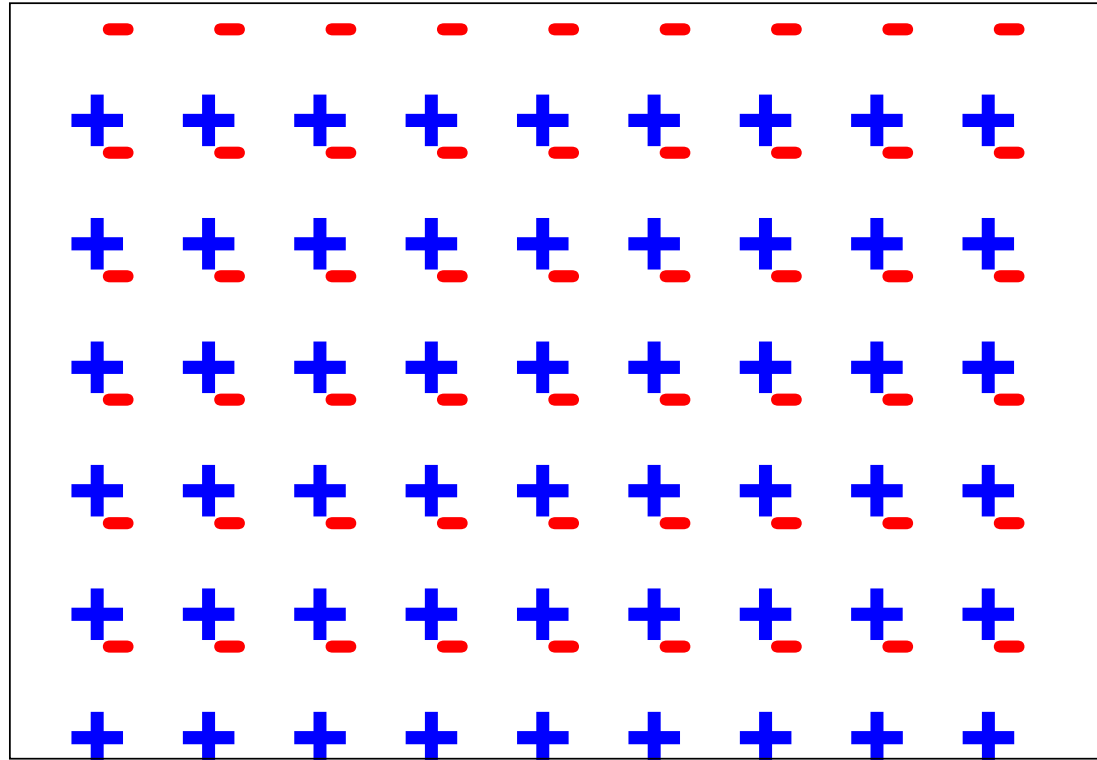
External field  $\vec{E}_1 \neq \vec{0}$   
 $E_1 > 0$



External field  $\vec{E}_2 \neq \vec{0}$   
 $E_2 > E_1$



External field  $\vec{E}_3 \neq \vec{0}$   
 $E_3 > E_2$

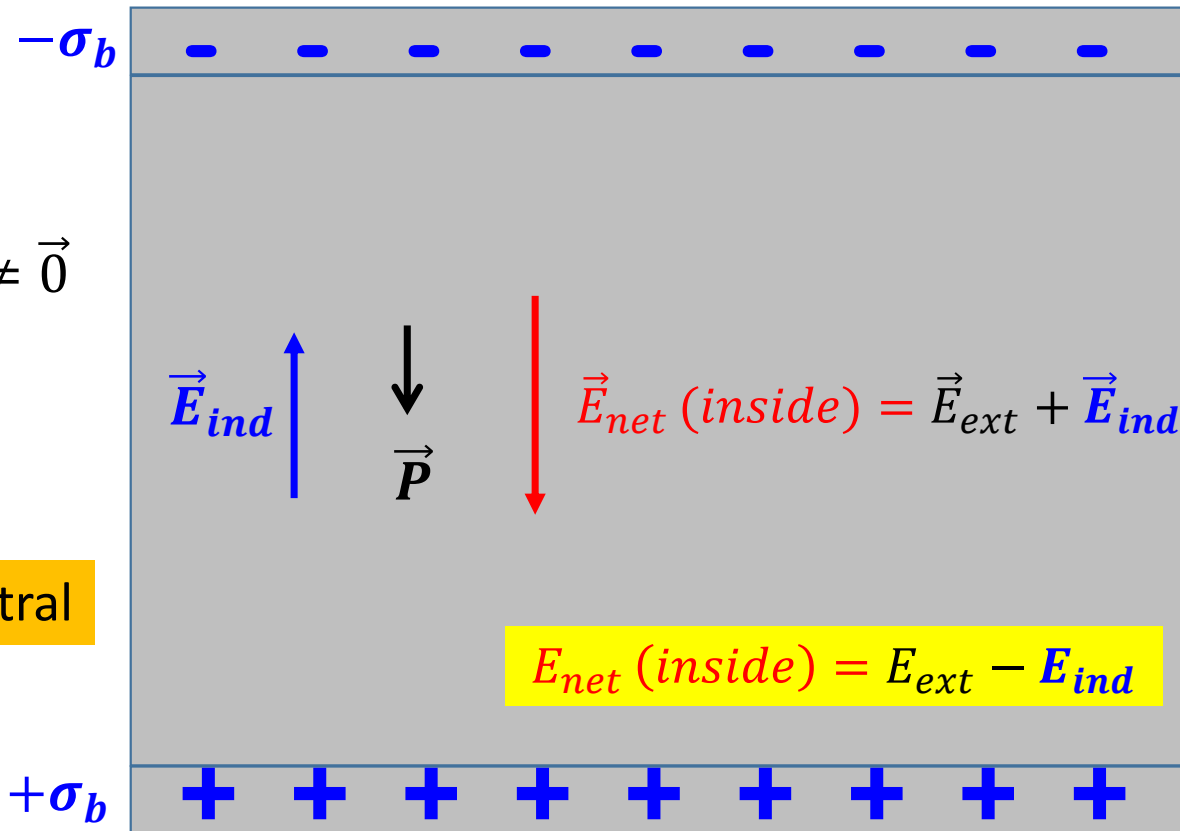


$\vec{E}_3$



External field  $\vec{E}_4 \neq \vec{0}$   
 $E_4 > E_3$

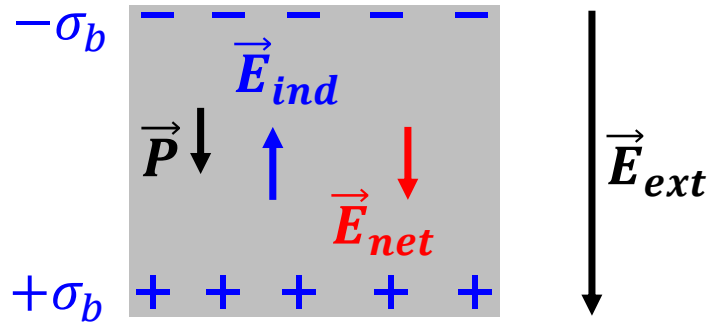
Dielectric still neutral



Polarization of the dielectric = induction of bound charges at the surface



## Electronic polarization



Linear – Homogeneous - Isotropic

$$\vec{P} \propto \vec{E}_{ext}$$

$$\vec{E}_{net}(inside) < \vec{E}_{ext}$$

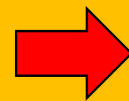
$$\vec{p} \Rightarrow V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{e}_r}{r^2}$$

$$\Rightarrow V_{net}(\vec{r}) = \int V_{dipole}(\vec{r}) dV$$

$$\Rightarrow \vec{E}_{net}(\vec{r}) = -\vec{\nabla} V_{net}(\vec{r})$$

Condition of derivability

$V_{net}(\vec{r})$  inside and  $V_{ext}(\vec{r})$



Continuity at the boundary

$\Rightarrow$  the potential MUST be continuous

There are mainly two other types of polarizations

**Ionic Polarizability:**

Materials made of two or more types of ions will develop a polarization under the action of an external field. Negative ions are attracted by the field and positive ions are repelled. The materials becomes ordered by pair

**Orientalional Polarizability:**

Under the external field, permanent dipoles, which are otherwise randomly oriented in the absence of the field are aligned in the direction of the field  
Liquid crystals, water etc...

## Two types of dielectrics

Permanent dipoles do not exist



Need external electric field to create dipoles

Permanent dipoles exist

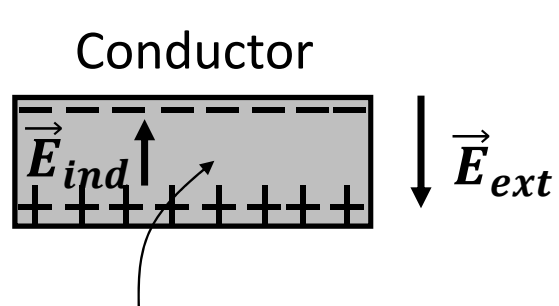
Non polar materials

Polar materials

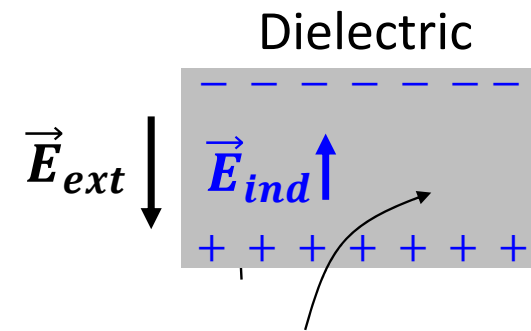
Competition of  $T$  and  $\vec{E}_{ext}$

- **Temperature** tends to randomize the dipoles  
No preferred orientation
- **Electric field** tends to orient the dipoles.  
Alignment of existing dipoles

- **Electric field** strengthens the alignment which pre-exists



$$\vec{E}_{net} (inside) = \vec{E}_{ext} + \vec{E}_{ind} = \vec{0}$$



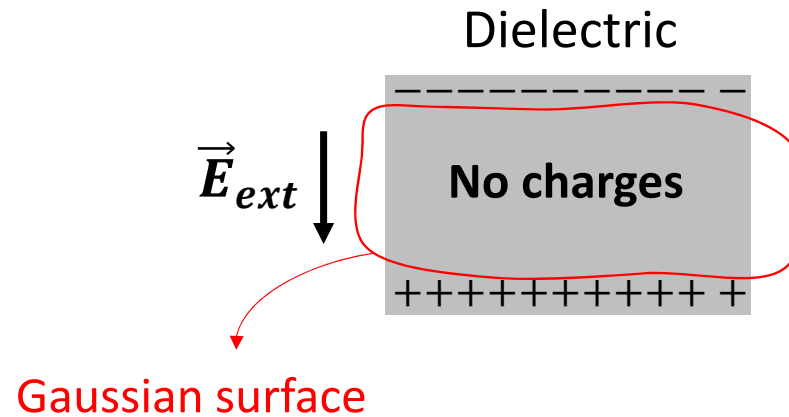
$$\vec{E}_{net} (inside) = \vec{E}_{ext} + \vec{E}_{ind} \neq \vec{0}$$

Charge neutrality  
Total net charge = 0 in both cases

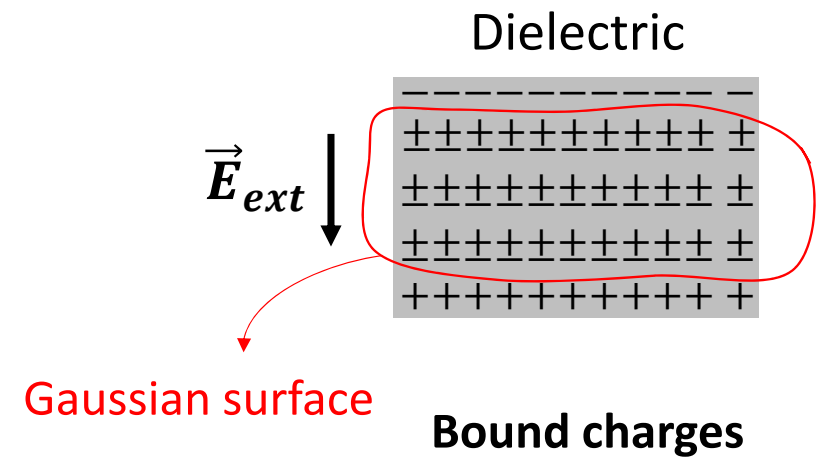
Electrostatically stable because:

- Presence of external field  $\vec{E}_{ext}$
- Surface barrier confines charges inside

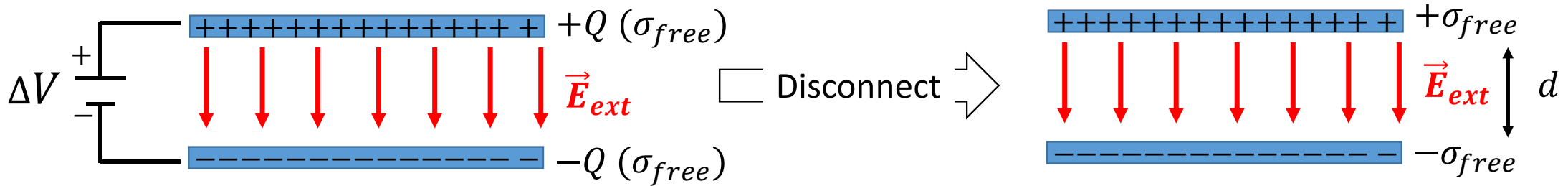
In conductor net charge = 0



In dielectric net charge = 0  $\Rightarrow$  there are dipoles



## The parallel plates (capacitor) produce a uniform field



$$\vec{E}_{ext} = -\vec{\nabla}V \quad \Rightarrow \quad \Delta V = (V_+ - V_-) = \int_0^d \vec{E}_{ext} d\vec{l} = E_{ext}d \quad \Rightarrow \quad \Delta V = E_{ext}d$$

From Gauss law

$$E_{ext} = \frac{\sigma_{free}}{\epsilon_0} \quad \Rightarrow \quad \Delta V = \frac{\sigma_{free}}{\epsilon_0} d \quad \Rightarrow \quad \sigma_{free} = \frac{\Delta V \epsilon_0}{d}$$

$E_{ext}$  is due to free charges  
From now on  $\mathbf{E}_{ext} = \mathbf{E}_{free}$

$$Q = \sigma_{free}A \quad \Rightarrow \quad Q = \frac{A\epsilon_0}{d} \Delta V = C_0 \Delta V \quad \Rightarrow \quad C_0 = \frac{A\epsilon_0}{d}$$

Capacitance

$$C_0 = \frac{A\epsilon_0}{d}$$

In vacuum

$$Q = \frac{A\epsilon_0}{d} \Delta V$$

Charge  $Q$  **added** on the plates depends

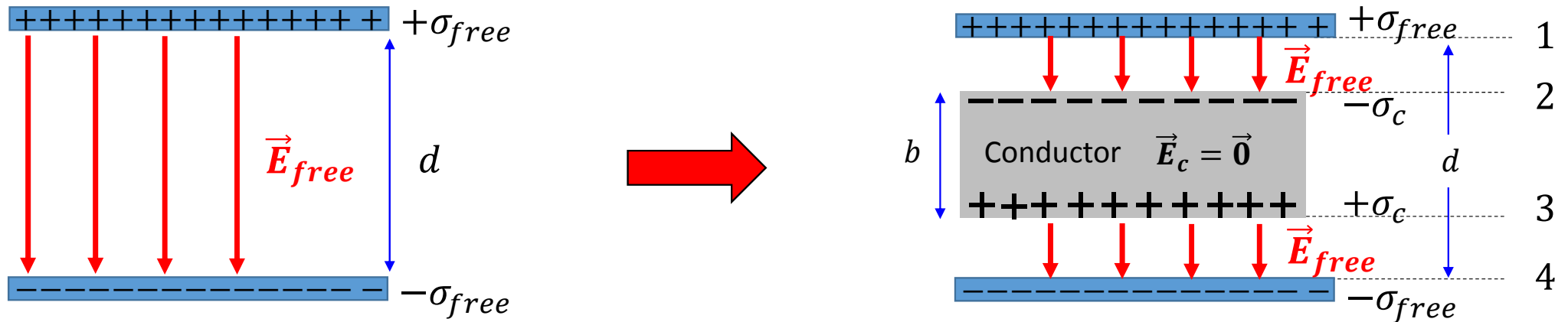
- On the area of the plates  $A$
- On the potential difference  $\Delta V$
- On the distance between the plates  $d \Rightarrow \dots$  less trivial

**But once the plates are disconnected**

$$E_{free} = \frac{\sigma_{free}}{\epsilon_0}$$

**Charges are trapped and frozen in the plates  $E_{free} = Cte$**

## Space filled with a conductor



$$\vec{E}_{free} = -\vec{\nabla}V \quad \Rightarrow \quad \Delta V = \int_0^d \vec{E}_{free} d\vec{l} = \int_1^2 \vec{E}_{free} d\vec{l} + \int_2^3 \vec{E}_c d\vec{l} + \int_3^4 \vec{E}_{free} d\vec{l} = E(d - b)$$

$\parallel$   
 $\bigcirc$

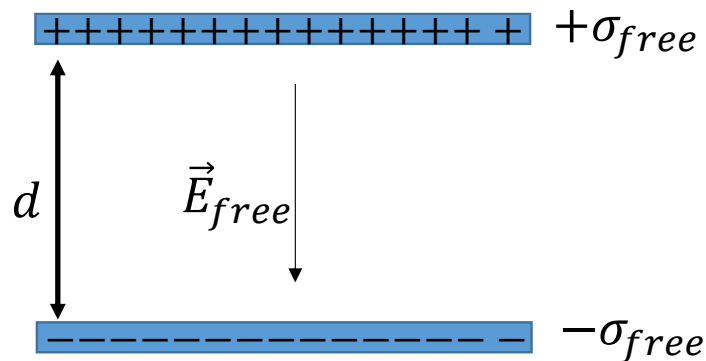
$$\Delta V = E(d - b) = \frac{\sigma_{free}}{\epsilon_0} (d - b)$$

$$C_{cond} = \frac{A\epsilon_0}{d} \frac{1}{1 - b/d}$$

If  $b = d \Rightarrow$  shortening



## Inserting a linear dielectric



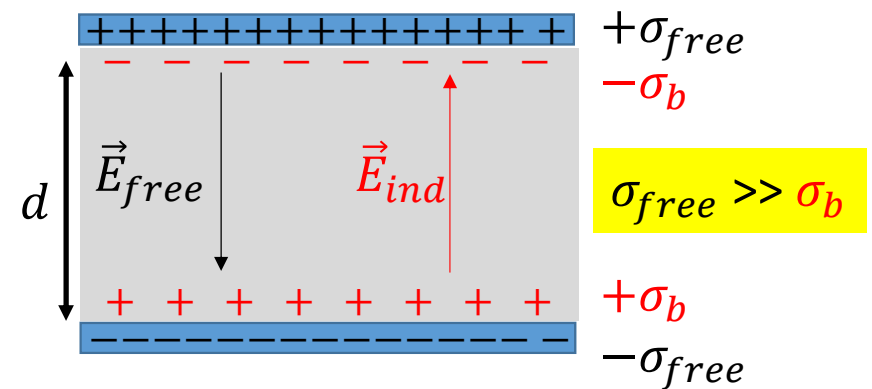
Voltage supply has been removed

$$\vec{E} = \vec{E}_{net} = \vec{E}_{free}$$

$$E = E_{net} = E_{free}$$

$$E_{free} = \frac{\sigma_{free}}{\epsilon_0}$$

In a conductor  $E_{net} = 0$   
 $E_{ind}$  compensates completely  $E_{free}$



A dielectric has been inserted

$$\vec{E} = \vec{E}_{net} = \vec{E}_{free} + \vec{E}_{ind}$$

$$E = E_{net} = E_{free} - E_{ind}$$

$$E_{ind} = \frac{\sigma_b}{\epsilon_0}$$

**Postulate:**

$$\sigma_b = \beta \sigma_{free} \quad E_{ind} = \beta E_{free} \quad E_{net} = (1 - \beta) E_{free} = \frac{E_{free}}{\epsilon_r} \quad \epsilon_r = \frac{1}{1 - \beta} > 1$$

$$\beta < 1$$

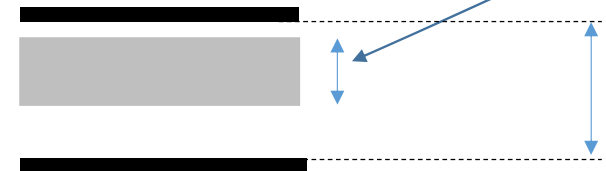
If  $\beta = 1 \Rightarrow \sigma_b = \sigma_{free} \quad E_{net} = 0 \Rightarrow$  Replacing the dielectric by a conductor  
(Field inside the conductor = 0,  $\epsilon_r = \infty$ )

$$E = E_{net} = \frac{E_{free}}{\epsilon_r} \quad \text{Potential inside the dielectric} \quad V_{ind} = E_{net}(\text{dielectric})d = \frac{E_{free}}{\epsilon_r} d$$

Potential before inserting the dielectric

$$V_{free} = E_{net}(\text{vacuum})d = E_{free} d$$

These two  $d$ 's may happen to be different: Example



## Faraday's observation

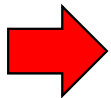
$$\frac{V_{ind}}{V_{free}} = \frac{1}{\epsilon_r}$$

When the dielectric fills the total space between the plates !

The potential drops by the factor  $\frac{1}{\epsilon_r}$  after inserting the dielectric

Faraday could measure the dielectric constants for a variety of materials

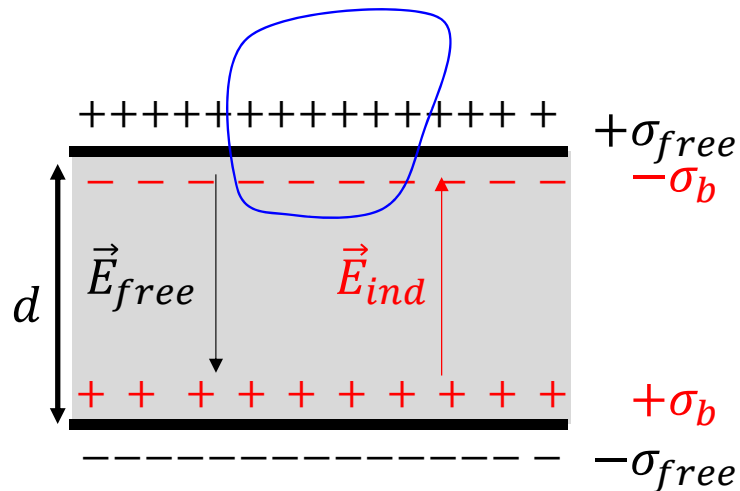
$$Q = \frac{A\epsilon_r\epsilon_0}{d} \Delta V$$



For the same voltage inserting a dielectric increases the total charge on the plate

# What about the Gauss law: Is it still valid after inserting a dielectric?

Gaussian surface



Net charge inside the Gaussian surface  $Q_{inside} = Q_{free}^+ - Q_{ind}^-$

$$Q_{ind}^- = \sigma_b \times A = \beta \sigma_{free} \times A$$

$$Q_{inside} = (1 - \beta) \sigma_{free} \times A = \frac{Q_{free}}{\epsilon_r}$$

Gauss law in dielectric  $\oint_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \sum Q_{inside} = \frac{\sum Q_{free}}{\epsilon_0 \epsilon_r}$

$E_{net}$  inside dielectric: from now on

Gauss law in **vacuum**  $\oint_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{\sum Q_{free}}{\epsilon_0}$

Gauss law in **dielectric**  $\oint_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{\sum Q_{free}}{\epsilon_0 \epsilon_r}$

## Space filled with a dielectric

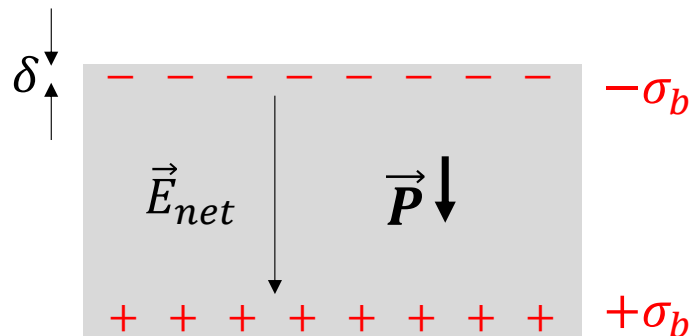
$$E_{net} = \frac{E_{free}}{\epsilon_r}$$

$$Q_{net} = \frac{Q_{free}}{\epsilon_r}$$

$$C_{diel} = \epsilon_r \frac{A\epsilon_0}{d}$$

$$\epsilon_r > 1$$

$\epsilon_r$  = property of the dielectric



$$\sigma_b = \frac{Q_b}{A} = \frac{Q_b}{V/\delta} = \left( \frac{Q_b}{V} \right) \delta = (Nq)\delta$$

$$\vec{P} = N\vec{p} = (Nq)\vec{\delta}$$

Dielectric:

- Linear
- Homogeneous
- Isotropic



$$\vec{P} \parallel \vec{\delta}$$



$$P = (Nq)\delta$$

$$\sigma_b = P$$

## Space filled with a dielectric: $E$ inside the dielectric

Linear, homogeneous and isotropic dielectric  $P \propto E$   $\begin{cases} = 0 \text{ if } E = 0 \\ = 0 \text{ if vacuum} \end{cases}$

$$P = \chi \epsilon_0 E$$

$\chi$  = Electric susceptibility

$$E = \frac{\sigma_{free}}{\epsilon_0} - \frac{\sigma_b}{\epsilon_0} = E_{free} - \frac{\sigma_b}{\epsilon_0} = E_{free} - \chi E$$

$$E = \frac{E_{free}}{1 + \chi}$$

$$(1 + \chi) = \epsilon_r$$

$\epsilon_r$  = dielectric constant

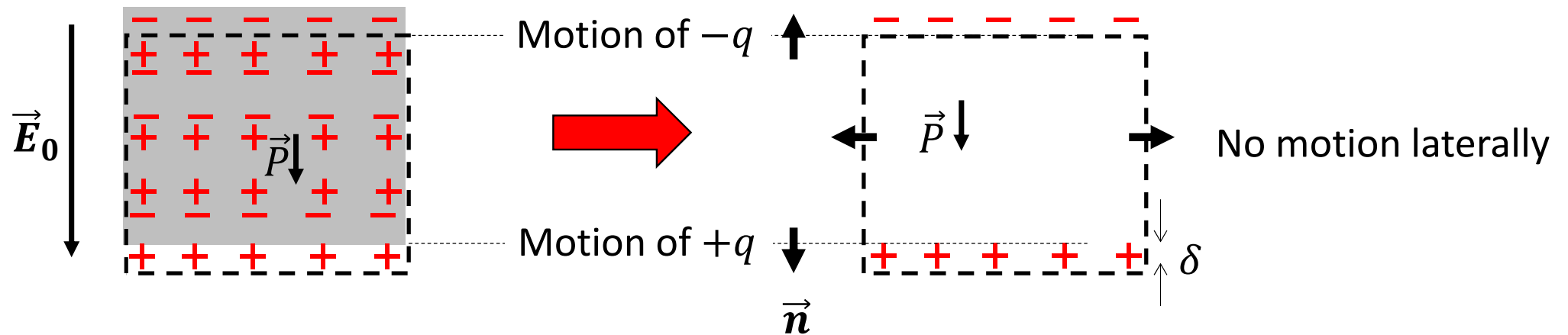
$E_{net}$

$$\Delta V = E d = \frac{E_{free}}{1 + \chi} d \quad \longrightarrow \quad C_{diel} = \frac{A(1 + \chi)\epsilon_0}{d} \quad \longrightarrow \quad C_0 = \frac{A\epsilon_0}{d}$$

See previous slide

$\chi = 0$  if vacuum

Is  $\sigma_b = P$  a coincidence or does it have a physical meaning?



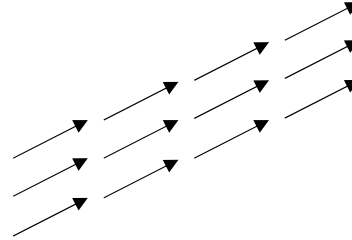
$$\sigma_b = \vec{P} \cdot \vec{n}$$

## Polarization is a **VECTOR**

1) It is considered **UNIFORM** when

$$|\vec{P}| = \text{constant}$$

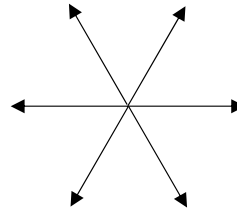
$\vec{P}$  has a unique direction



2) It is considered **NONUNIFORM** when

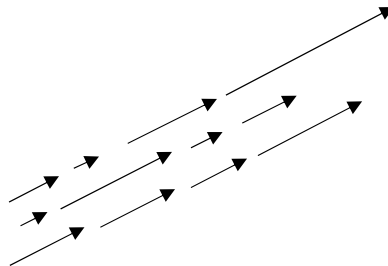
$$|\vec{P}| = \text{constant}$$

$\vec{P}$  changes direction

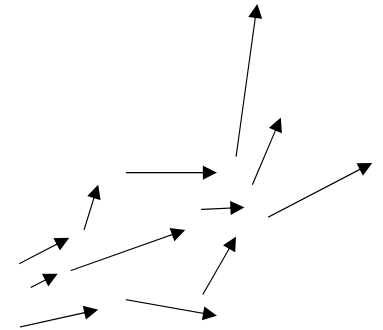


$$|\vec{P}| \text{ changes}$$

$\vec{P}$  has unique direction



$|\vec{P}|$  changes  
 $\vec{P}$  changes direction





## Uniform versus non-uniform polarization

### No polarization:

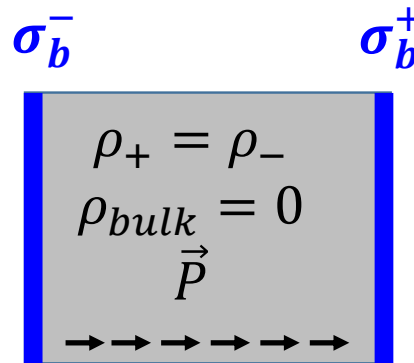
As the dielectric is neutral, charge density is zero since there are equal amount of positive ( $\rho_+$ ) and negative ( $\rho_-$ ): **Charge conservation**

$$\begin{aligned}\rho_+ &= \rho_- \\ \rho_{bulk} &= 0 \\ \vec{P} &= \vec{0}\end{aligned}$$

## Uniform versus non-uniform polarization

### Uniform polarization:

The negative and positive charges are shifted by the same amount everywhere in the dielectric. Positive and negative surface charges appear with equal amount. In the bulk we still have zero **NET** charge



# Uniform versus non-uniform polarization

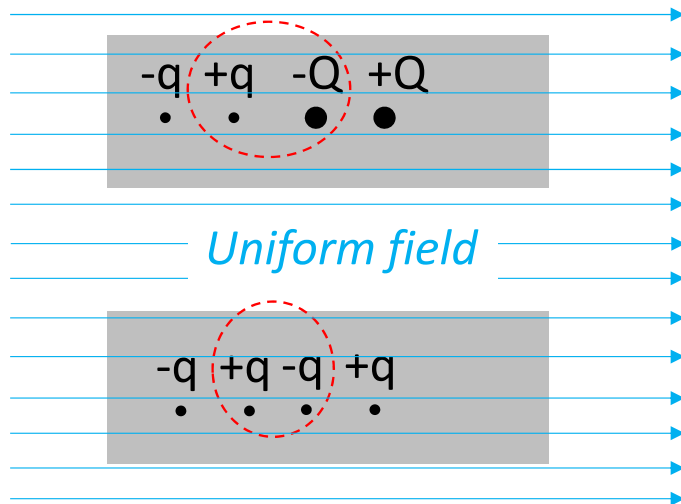
## Non-uniform polarization:

In the illustrated example where the polarization increases to the right, positive charges are stretched out and displaced to the right. The positive surface charge density is thus greater on the right than the negative surface charge density accumulating on the left.

A negative charge density develops in the bulk as a consequence of charge conservations law

$$\sum q_{bulk} \neq 0$$

$$\sum q_{bulk} = 0$$



$$\rho_b \neq 0$$

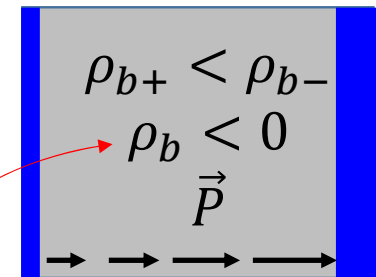
Refer to bulk

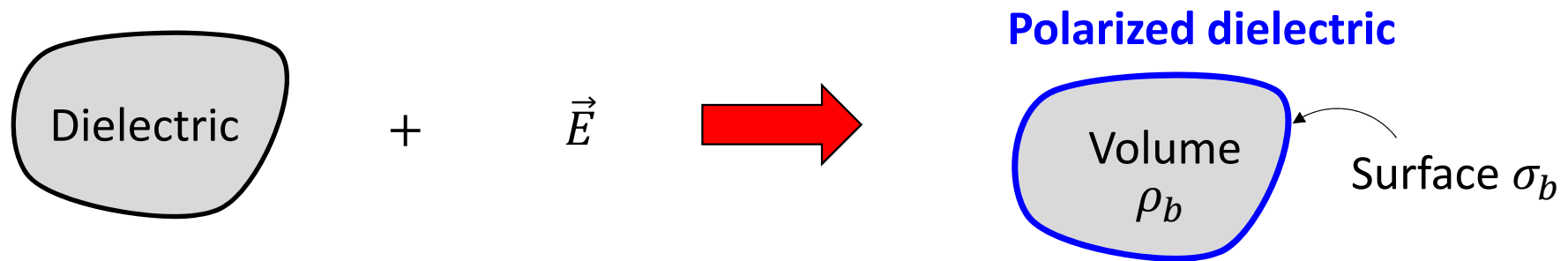
$$\rho_b = 0$$

$$\rho_b = \sum_{i(+,-)} \rho_{bi}$$

Refer to the surface

$$\sigma_b^- \quad \sigma_b^+ > \sigma_b^-$$





Polarization induces **BOUND** charges →  $\sum \text{BOUND charges} = 0$  Charge conservation

Neutrality of the dielectric

$$\int \sigma_b dA \left[ \begin{array}{l} = 0 \\ > 0 \\ < 0 \end{array} \right. \left. \begin{array}{l} \int \rho_b dV < 0 \\ \int \rho_b dV > 0 \end{array} \right] \left. \begin{array}{l} \Rightarrow |+\sigma_b| = |-\sigma_b| \\ \Rightarrow \int \rho_b dV = 0 \\ \Rightarrow \rho_b = 0 \end{array} \right\} \text{Polarization in the bulk}$$

## Charge conservation

Gauss theorem

$$\oint_A \sigma_b dA = - \int_V \rho_P dV = \oint_A \vec{P} \cdot \vec{n} dA = \int_V \vec{\nabla} \cdot \vec{P} dV$$

➔  $\rho_b = -\vec{\nabla} \cdot \vec{P}$

If polarization ( $\vec{P}$ ) is uniform

➔  $\rho_b = 0$

➔  $+\sigma_b$  and  $-\sigma_b$  are induced at the surface with

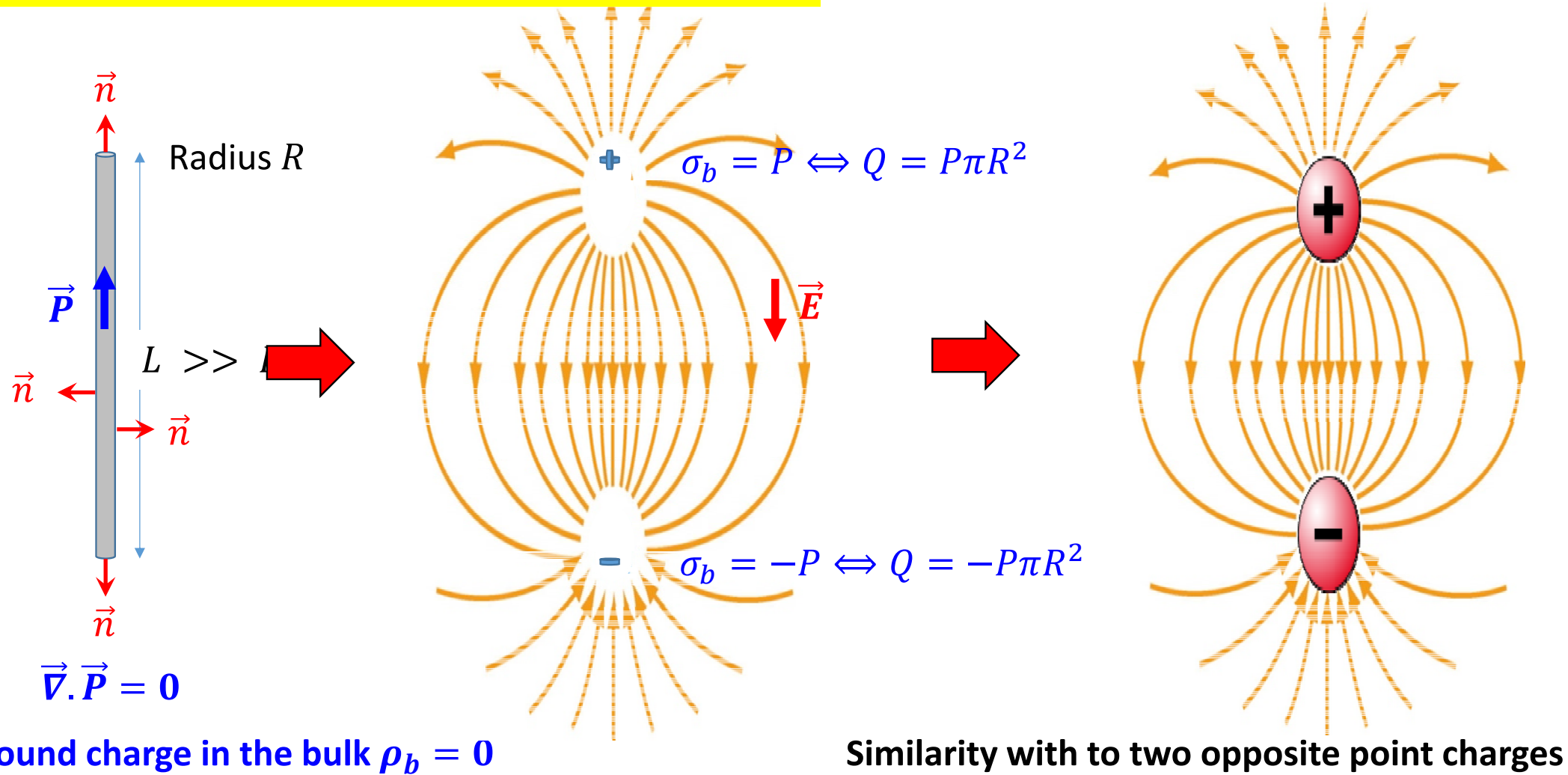
$$\int \sigma_b dA = 0$$

Whether the polarization is **uniform or NOT** the total polarization charge is always zero because of:

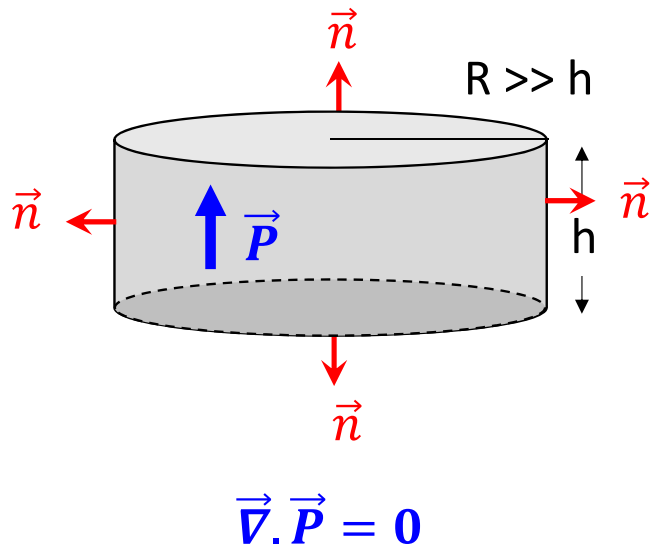
$$\sigma_b = P \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$

Charge conservation

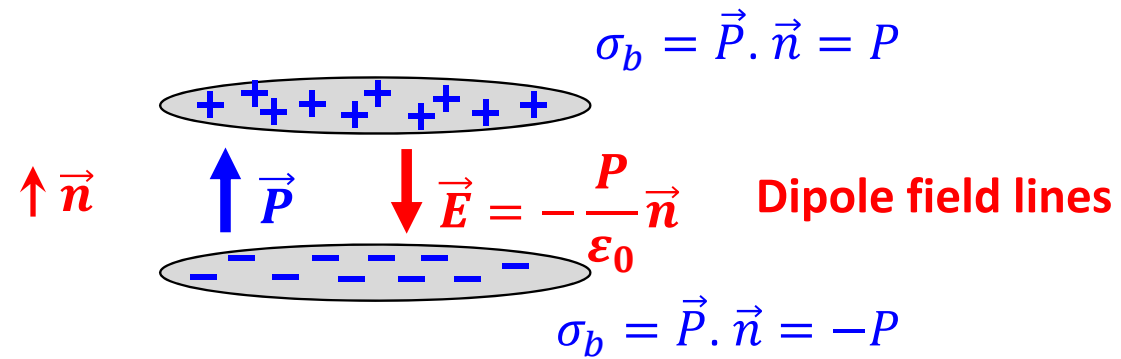
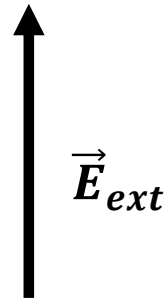
## Example 1: Uniform polarization of a dielectric rod



## Example 2: Uniform polarization of a solid cylindrical dielectric



No bound charge in the bulk  $\rho_b = 0$



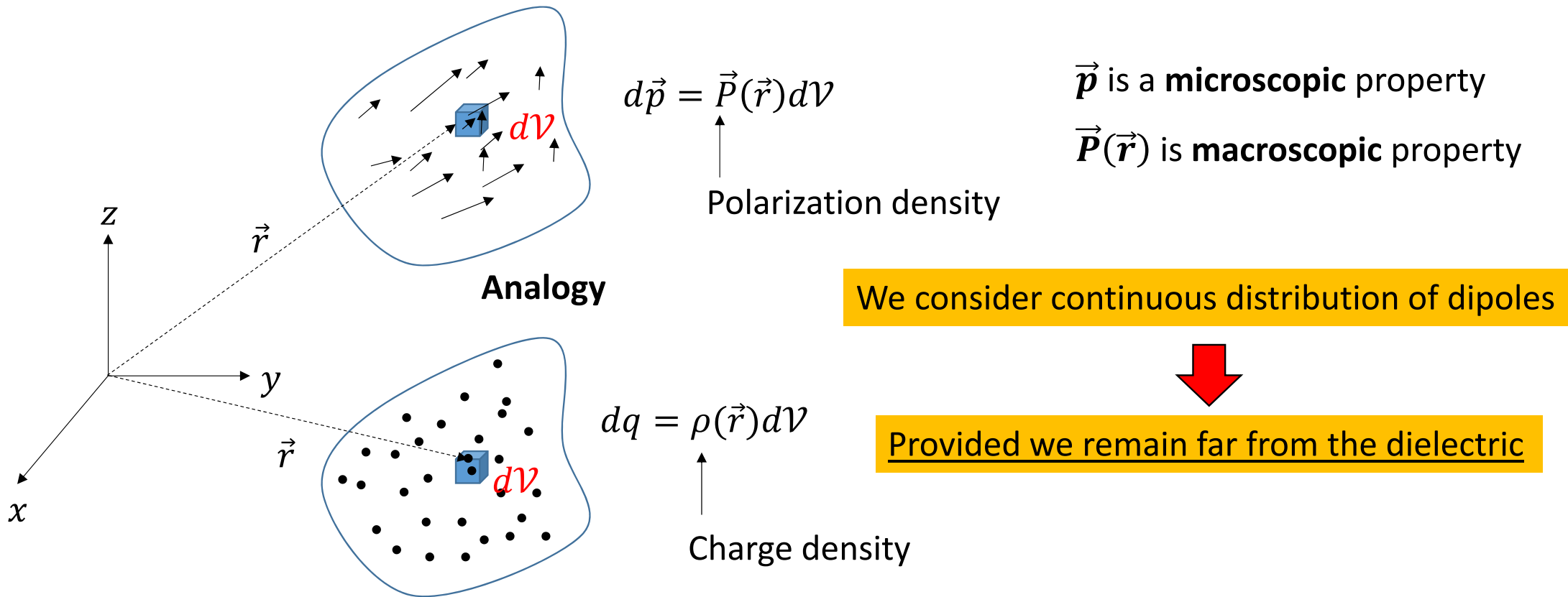
Similar to a parallel plate capacitor

Why as for a conductor, a dielectric distorts an initially uniform electric field ?

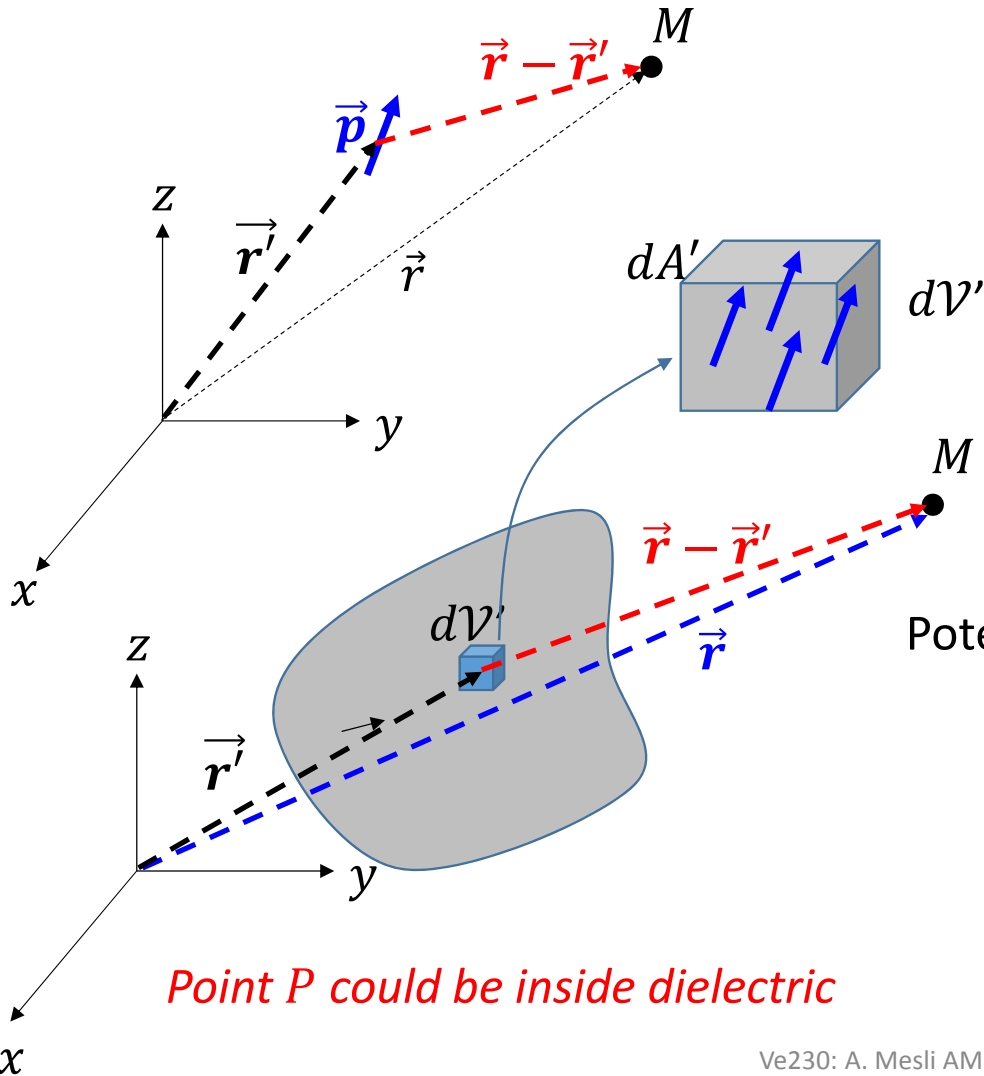


# From superposition principle to bound bulk and surface polarizing charges

Slide #4 in Lectures 5-7\_Coordinate system\_Scalar versus Vector fields\_Operators



## From single dipole to a distribution of dipoles



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$



$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{d\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$d\vec{p} = \vec{P}(\vec{r}') dV'$$



Potential at  $M$  by the whole dipole distribution



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

## Potential created by a polarized dielectric

The variable is  $\vec{r}'$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

$$\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = -\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} = \vec{\nabla}' \left( \frac{1}{|\vec{r} - \vec{r}'|} \right)$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \underbrace{\vec{P}(\vec{r}')}_{\vec{A}} \cdot \underbrace{\vec{\nabla}' \left( \frac{1}{|\vec{r} - \vec{r}'|} \right)}_f dV'$$

**→**

Divergence

$\vec{\nabla} \cdot (f \vec{A})$

scalar

Gradient

$f(\vec{\nabla} \cdot \vec{A})$

scalar

$+ \vec{A} \cdot \vec{\nabla} f$

scalar

See D\_Lectures 5-7\_Coordinate system\_Scalar versus Vector fields\_Operators

$$\rho_b(\vec{r}') = -\vec{\nabla}' \cdot \vec{P}(\vec{r}')$$

## Potential created by a polarized dielectric

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \underbrace{\vec{P}(\vec{r}')}_{\vec{A}} \cdot \underbrace{\vec{\nabla}'}_{f} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) dV' \quad + \quad \vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \underbrace{\vec{A} \cdot \vec{\nabla} f}_{\vec{A} \cdot \vec{\nabla} f}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}'} \vec{\nabla}' \cdot \left( \frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) dV' - \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}'} \frac{\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$



Gauss's Theorem

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_A \frac{1}{|\vec{r} - \vec{r}'|} \sigma_b(\vec{r}') dA' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}'} \frac{-\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

$\rho_b(\vec{r}')$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_A \frac{1}{|\vec{r} - \vec{r}'|} \sigma_b(\vec{r}') dA' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}'} \frac{-\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

Including bound charges in all space

$$V(\vec{r}) = - \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

Including **ALL charges** in the whole space

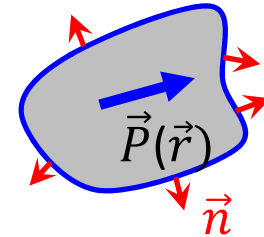
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{dV'}{|\vec{r} - \vec{r}'|} \underbrace{[\rho_{free}(\vec{r}') - \vec{\nabla}' \cdot \vec{P}(\vec{r}')]_{\rho_{free} + \rho_b}}$$

Initially

$$\text{Space} = \vec{E}_{ext} + \text{dielectric} \leftrightarrow$$



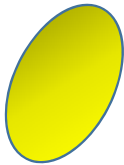
+



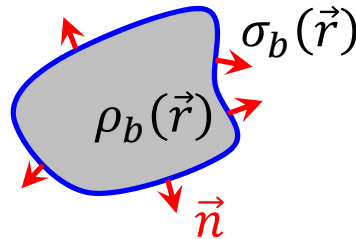
Free charges  $\rho_{free}(r)$   
Primary source of field

Now: From polarization to charge distribution

Space =



+



$\leftrightarrow$

Free charges  $\rho_{free}(\vec{r})$   
Primary source of field

Bound charges

- $\sigma_b(\vec{r}') = \vec{P}(\vec{r}') \cdot \vec{n}$
- $\rho_b(\vec{r}') = -\vec{\nabla}' \cdot \vec{P}(\vec{r}')$

Secondary source of field

Superposition principle applies to the three charge distributions

- $\rho_{free}(\vec{r})$
- $\sigma_b(\vec{r}')$
- $\rho_b(\vec{r}')$

## Consequences of polarization: Strategy solving problems

Electric field + Dielectric  Surface and bulk **BOUND** charges with a **feedback** on the external field

### Step 1

Determine the polarization configuration for:

- A given configuration of the applied field  $\vec{E}_{ext}(\vec{r})$
- A given shape of the dielectric

### Step 2

**Remove** the applied field and consider the polarized dielectric as a new source:

- Of electric field  $\vec{E}_{diel}(\vec{r})$
  - Of potential  $V_{diel}(\vec{r})$
- }  $\vec{E}_{net}$  and  $V_{net}$

produced both inside and outside the dielectric

## Consequences of polarization: Strategy solving problems

Electric field + Dielectric  Surface and bulk **BOUND** charges with a **feedback** on the external field

### Step 3

- By superposition principle determine the final distribution of the field =  $\sum [\vec{E}_{ext}(\vec{r}) + \vec{E}_{diel}(\vec{r})]$  in the presence of both the applied field and the one raising from the polarized dielectric

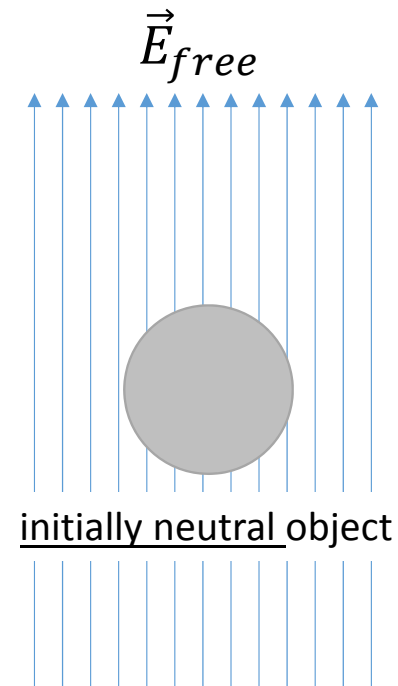
### Step 4

- Look at the boundary between the dielectric and the surrounding

**Identify – Evaluate – Execute**



## Case of a dielectric sphere inserted in a uniform electric field: Uniformly polarized dielectric sphere



1) Once in the external field  $\vec{E}_{free}$ , bound surface and / or bulk charges are induced in the initially neutral object

2) A new field  $\vec{E}$  is now generated by the object itself (dipoles):  
**Resulting field**  $\vec{E}_{free} + \vec{E}_{diel}$

### Consequence:

Far away from the sphere, the external field ( $\vec{E}_{free}$ ) is not disturbed, but close to the object the field is distorted

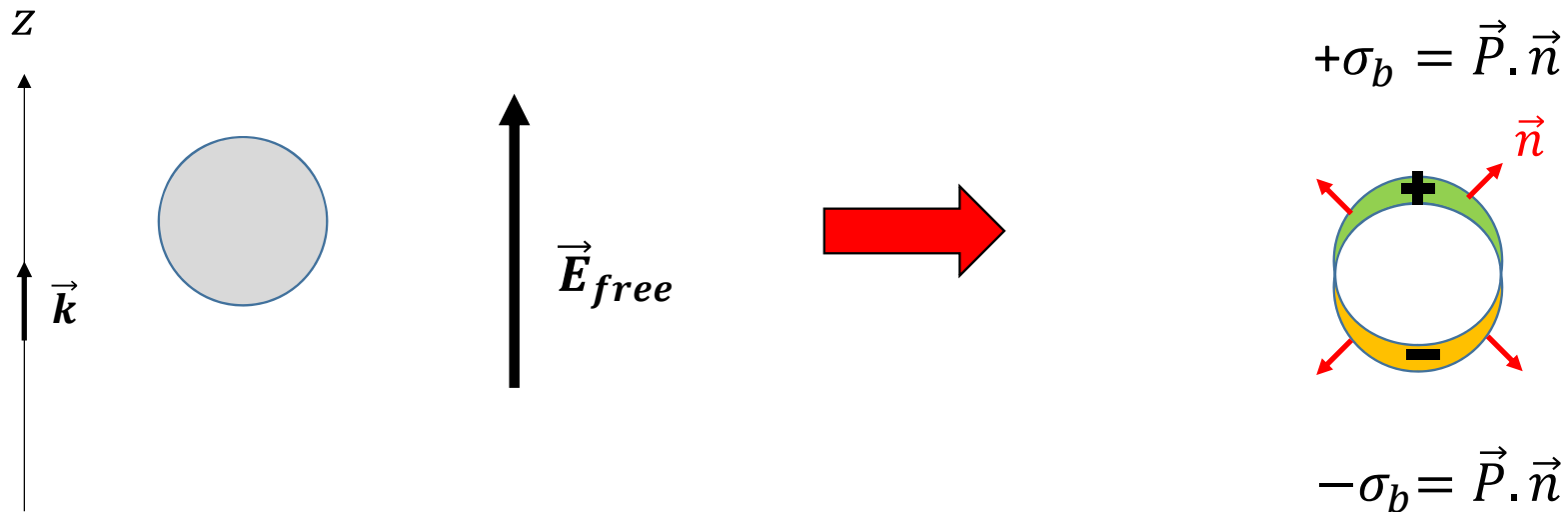
Within any polarizable body placed into an electric field, polarization charge density is induced which, in turn, modifies the electric field

## Dipole formation in a dielectric: Field inside

A dielectric sphere of radius  $R$  is uniformly polarized along the z axis:

$$\vec{P} = P_0 \vec{k}$$

$$\rho_b(\vec{r}') = -\vec{\nabla}' \cdot \vec{P}(\vec{r}') = 0$$

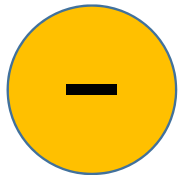


How can we treat this problem?

## Dipole formation in a dielectric: **Field inside**

A dielectric sphere of radius  $R$  is uniformly polarized along the z axis:

$$\vec{P} = P_0 \vec{k}$$

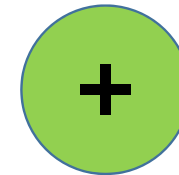


$$\rho_- = -\rho$$

$$\rho_- = \frac{Q_-}{\frac{4}{3}\pi R^3}$$

$$\vec{E}_-(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_-}{R^3} \vec{r}$$

$$\vec{E}_-(\vec{r}) = \frac{\rho_-}{3\epsilon_0} \vec{r} = -\frac{\rho}{3\epsilon_0} \vec{r}$$



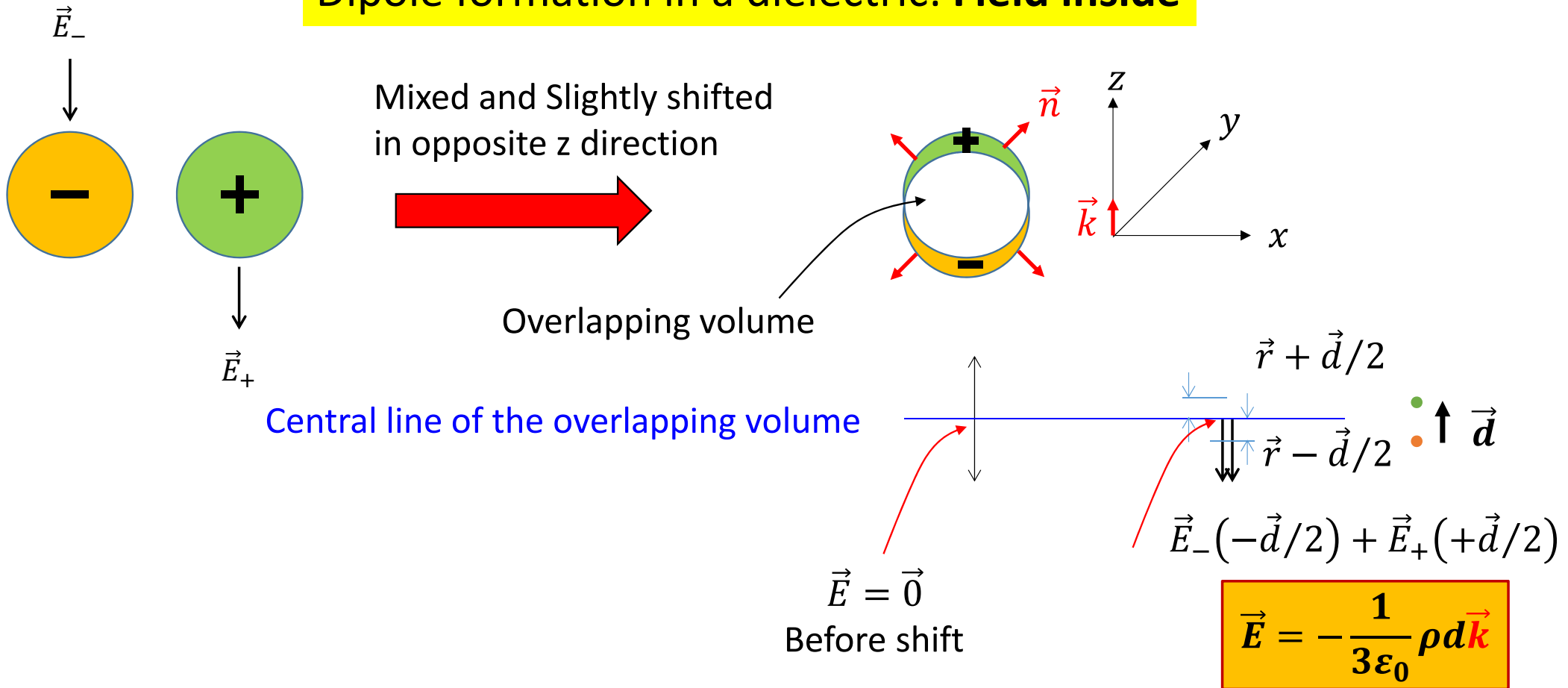
$$\rho_+ = \rho$$

$$\rho_+ = \frac{Q_+}{\frac{4}{3}\pi R^3}$$

$$\vec{E}_+(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_+}{R^3} \vec{r}$$

$$\vec{E}_+(\vec{r}) = \frac{\rho_+}{3\epsilon_0} \vec{r} = \frac{\rho}{3\epsilon_0} \vec{r}$$

## Dipole formation in a dielectric: Field inside



Easy to demonstrate that this field is the same in the entire overlapping volume

The same result can be obtained in a much easier way

Make use of superposition principle and Gauss' law

$$\vec{r}_- + \vec{d} = \vec{r}_+ \quad \vec{E}_+ = -\frac{\rho}{3\epsilon_0} \vec{r}_+ \quad \vec{E}_- = \frac{\rho}{3\epsilon_0} \vec{r}_-$$

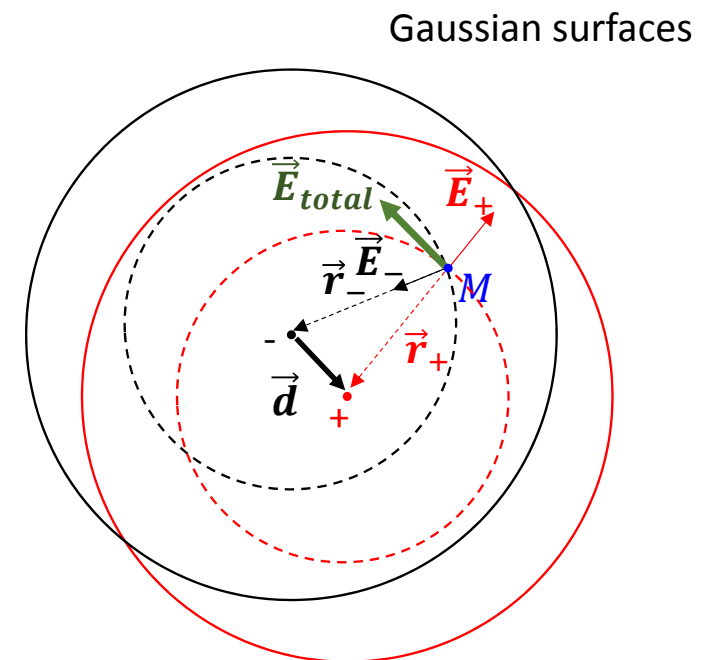
$$\vec{E}_{total} = -\frac{\rho}{3\epsilon_0} (\vec{r}_+ - \vec{r}_-) = -\frac{\rho}{3\epsilon_0} \vec{d}$$

Constant everywhere in the entire **overlapping** volume !  
And **ONLY** in the overlapping volume: Why?

If  $\vec{d} = d\vec{k}$

$$\vec{E}_{total} = -\frac{1}{3\epsilon_0} \rho d \vec{k}$$

Identical to result in previous slide !



## Dipole formation in a dielectric: **Field inside**

$$\vec{E} = -\frac{1}{3\epsilon_0}\rho d\vec{k}$$

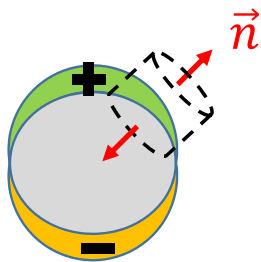
$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \quad \vec{p} = Q\vec{d}$$

$$\vec{E} = -\frac{1}{3\epsilon_0}\left(\frac{\vec{p}}{\frac{4}{3}\pi R^3}\right) = -\frac{\vec{P}}{3\epsilon_0} = -\frac{P_0}{3\epsilon_0}\vec{k}$$

$$\vec{P} = P_0\vec{k}$$

Spherical coordinate  $\rightarrow$  
$$\begin{bmatrix} P_r \\ P_\theta \\ P_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ P_0 \end{bmatrix}$$

$$\vec{P} = P_0\cos\theta\vec{e}_r - P_0\sin\theta\vec{e}_\theta$$



See D\_Lectures 5-7\_Coordinate system\_Scalarversus Vector fields\_Operators

Gauss theorem through the pill box

$$\sigma_b = \vec{P} \cdot \vec{n} - \vec{P}_{vac} \cdot \vec{n} \quad \sigma_b = \vec{P}_{diel} \cdot \vec{n}$$

$\parallel$

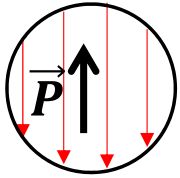
$$\vec{n} \perp \vec{e}_\theta$$

$\mathbf{0}$

$$\vec{n} \parallel \vec{e}_r$$

$$\sigma_b = P_0\cos\theta$$

$\vec{P} = P_0 \vec{k}$  *# field lines reduced in the dielectric  $\Rightarrow$  field inside less strong than field outside*

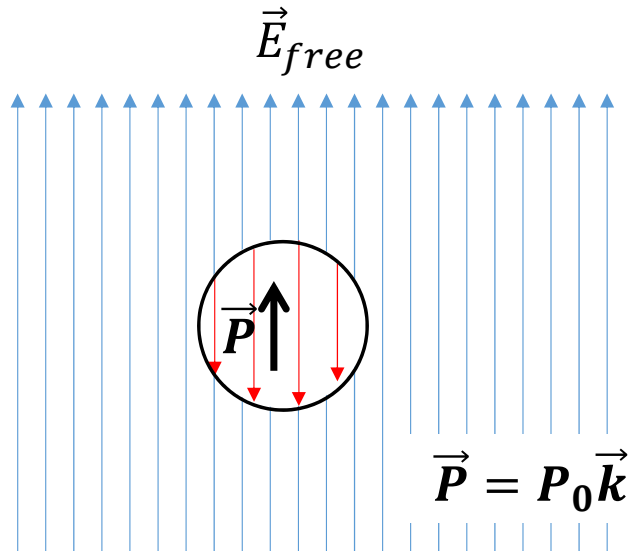


$$|\vec{E}_{ind}| < |\vec{E}_{free}|$$

This field is restricted to the interior of the dielectric sphere and arises exclusively from the contribution of the surface bound charges.  
It does not include the external applied field used to create dipoles.



The aim of this dipole **induced** field is to reduce the impact of the applied field inside the sphere.



In the presence of the external applied field, the field **NET  $\vec{E}$  inside** is:

$$\vec{E} = \vec{E}_{free} + \vec{E}_{ind}$$

$$\vec{E}_{ind} = -\frac{\vec{P}}{3\epsilon_0} = -\frac{P_0}{3\epsilon_0} \vec{k}$$

$$E = E_{free} - E_{ind}$$

$$E_{ind} = \frac{P_0}{3\epsilon_0}$$

$$E = E_{free} - E_{ind}$$

$$E = E_{free} - \frac{P_0}{3\epsilon_0}$$

$$P_0 = \epsilon_0 \chi E \quad \chi = \epsilon_r - 1$$

$$E = \frac{3}{\epsilon_r + 2} E_{free}$$

$$\text{If no dielectric } \epsilon_r = 1 \rightarrow E = E_{free}$$

*Polarization is proportional to the field inside ( $\vec{E}$ )*



## Dipole formation in a dielectric: **Field inside**

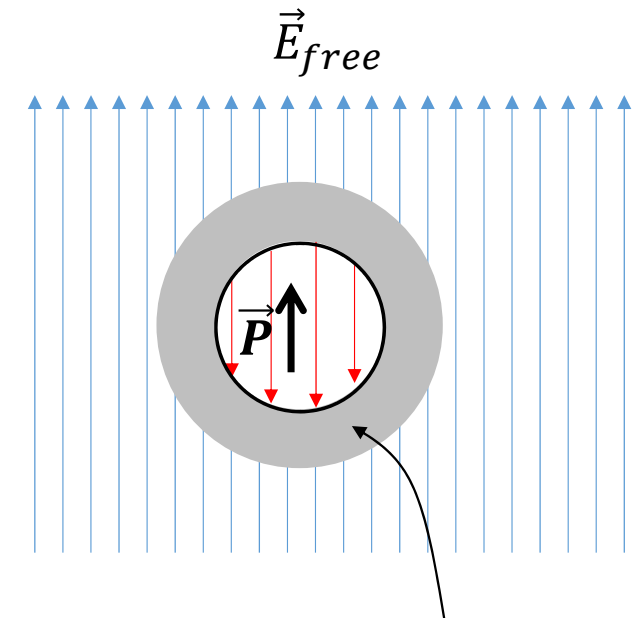
$$\vec{E} = \frac{3}{\epsilon_r + 2} \vec{E}_{free}$$

For  $r \gg R$  Vacuum  $\epsilon_r = 1$   $\vec{E} = \vec{E}_{free}$

The field inside is reduced due to screening

$$\vec{P} = \epsilon_0 \chi E = \frac{3\epsilon_0(\epsilon_r - 1)}{\epsilon_r + 2} \vec{E}_{free}$$

Vacuum  $\epsilon_r = 1$   $\vec{P} = \vec{0}$



Area where the field is most distorted

## Dipole formation in a dielectric: **Field outside**

The field outside is identical to that generated by a single dipole moment  $\vec{p} = \vec{P} \cdot \frac{4}{3}\pi R^3$   $\vec{P} = P_0 \vec{k}$

From dipole lecture  $V(r > R) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{e}_r}{r^2} = \frac{1}{4\pi\epsilon_0} \left( \frac{4}{3}\pi R^3 \right) \frac{P_0 \vec{k} \cdot \vec{e}_r}{r^2} \rightarrow V(r > R) = \frac{R^3 P_0 \cos\theta}{3\epsilon_0 r^2}$

$\vec{E} = -\vec{\nabla}V(r)$  Field outside the dielectric sphere due to polarization

in spherical coordinate

$$E_r(r) = -\frac{\partial V(r)}{\partial r} = \frac{2R^3 P_0 \cos\theta}{3\epsilon_0} \frac{1}{r^3}$$

$$E_\theta(r) = -\frac{1}{r} \frac{\partial V(r)}{\partial \theta} = \frac{R^3 P_0 \sin\theta}{3\epsilon_0} \frac{1}{r^3}$$

$$E_\phi(r) = 0$$

**Go to F\_Lecture 10&11\_Conductors and Dipoles**

## Dipole formation in a dielectric: **Total Field outside**

Superposition principle



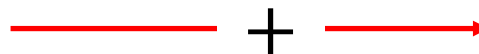
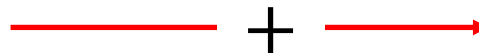
**Total Field outside = Sum of field due to dipole and external field generating that dipole**

External field due to polarization  
in spherical coordinate

$$E_r(r) = -\frac{\partial\varphi(r)}{\partial r} = \frac{2R^3P_0\cos\theta}{3\varepsilon_0} \frac{1}{r^3}$$

$$E_\theta(r) = -\frac{1}{r} \frac{\partial\varphi(r)}{\partial\theta} = \frac{R^3P_0\sin\theta}{3\varepsilon_0} \frac{1}{r^3}$$

$$E_\phi(r) = 0$$



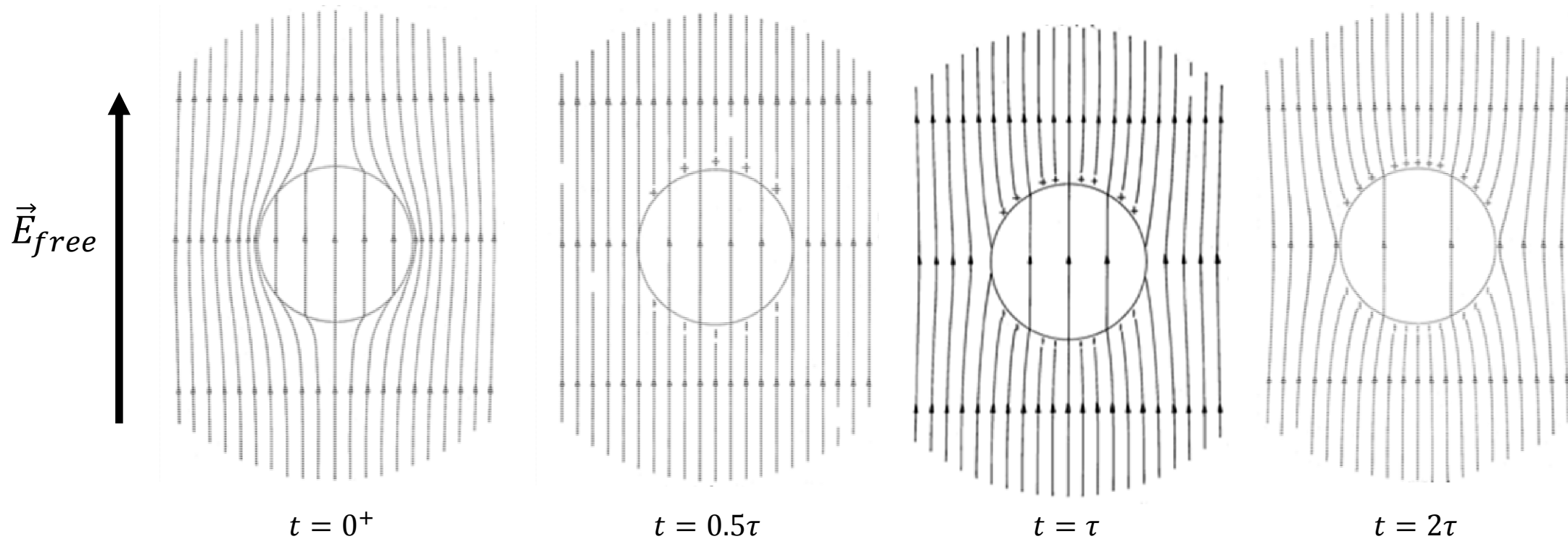
External applied field ( $E_0 = E_{free}$ )  
in spherical coordinate

$$E_{0r}(r) = E_0\cos\theta$$

$$E_{0\theta}(r) = -E_0\sin\theta$$

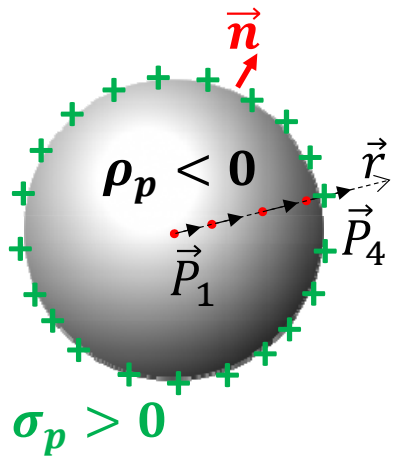
$$E_\phi(r) = 0$$

## Time evolution of the distribution of the field lines around a dielectric after polarization has been initiated



The field strength inside is decreasing

## Dielectric sphere radially polarized: **Field outside $r > R$**



$$\vec{P}(\vec{r}) = \beta \vec{r}$$

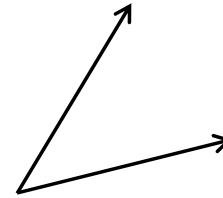
$\beta$  is constant



Polarization not uniform !

Polarization is a **VECTOR**

$$\vec{P}(\vec{r}) = \beta \vec{r}$$



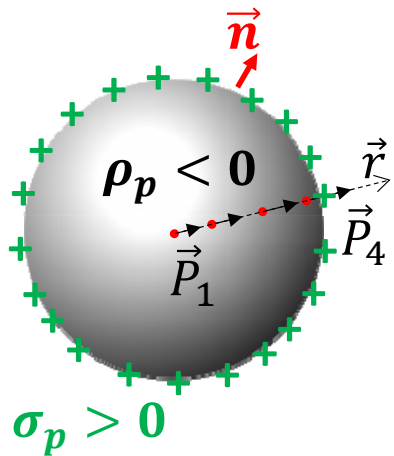
These two vectors are different



Both  $\rho_b$  and  $\sigma_b$  exist

$$\sigma_b > 0 \Rightarrow \rho_b < 0$$

## Dielectric sphere radially polarized: **Field outside $r > R$**



$$\vec{P}(\vec{r}) = \beta \vec{r}$$

$\beta$  is constant

$$\sigma_b = \vec{P} \cdot \vec{n} = \beta R$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \beta r) = -3\beta$$

$$Q_s = 4\pi R^2 \sigma_b = 4\pi \beta R^3$$

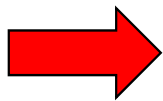
$$Q_v = \frac{4\pi}{3} R^3 \rho_b = -4\pi \beta R^3$$

Equivalent to **conducting** sphere  
Positively charged on the surface

Equivalent to **non conducting** sphere  
Negatively charged in the volume

See E\_Lectures 8&9\_Electrostatics\_Gauss law

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$



$$\vec{E}(r) = \frac{\beta R^3}{\epsilon_0} \frac{\vec{e}_r}{r^2}$$

$$\vec{E}(r) = -\frac{\beta R^3}{\epsilon_0} \frac{\vec{e}_r}{r^2}$$

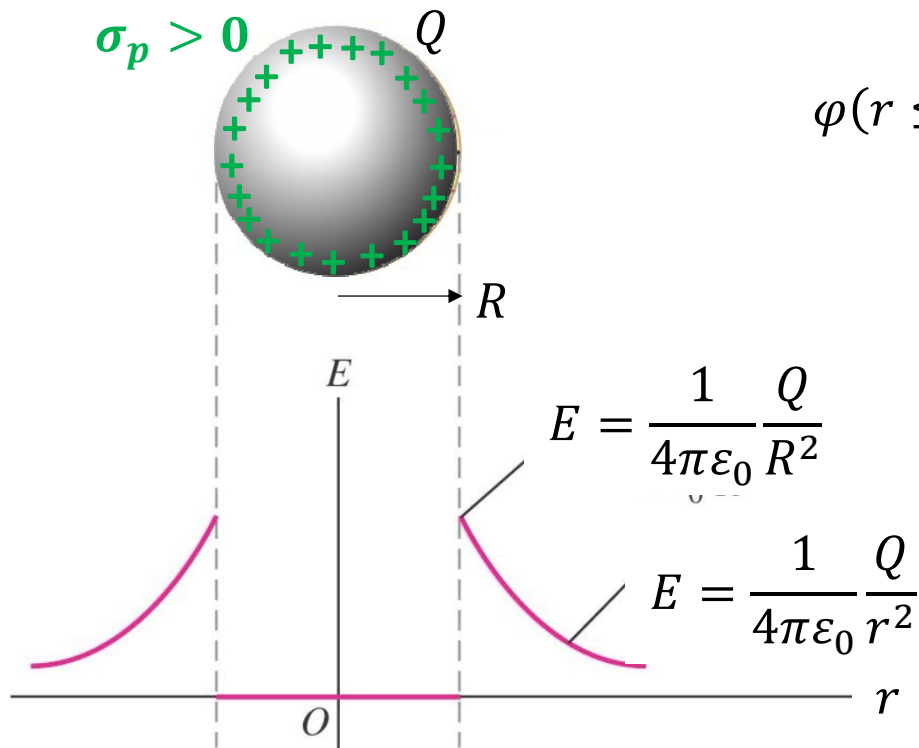
Outside the sphere

Superposition principle: **field outside = 0**

# Dielectric sphere radially polarized: **Field inside $r < R$**

Gauss law inside the sphere

Surface polarization:



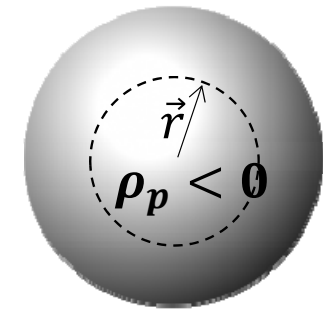
field inside = 0

$$\varphi(r \leq R) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

$$\vec{E} = -\vec{\nabla} \cdot \varphi(r)$$

$$Q = \frac{4\pi}{3} r^3 \rho_b$$

Volume polarization



$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \vec{e}_r$$

$$\vec{E}(r) = -\frac{\beta r}{\epsilon_0} \vec{e}_r$$

Superposition principle: **field inside**  $\vec{E}(r) = -\frac{\beta r}{\epsilon_0} \vec{e}_r$

## Concept of electric displacement $\vec{D}$

What do the first Maxwell equations  $\vec{\nabla} \cdot \vec{E} = \frac{\rho_{free}}{\epsilon_0}$  and  $\vec{\nabla} \times \vec{E} = 0$  Become in a dielectric ?

vacuum

If there are bound charges due to polarization  $\vec{P}$

$$\rho_{free} \rightarrow \rho_{free} + \rho_b$$

$$\text{With } \rho_b = -\vec{\nabla} \cdot \vec{P}$$



## Concept of electric displacement $\vec{D}$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{free} + \rho_b}{\epsilon_0} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho_0 - \vec{\nabla} \cdot \vec{P}}{\epsilon_0} \quad \Rightarrow \quad \vec{\nabla} \cdot \left( \vec{E} + \frac{\vec{P}}{\epsilon_0} \right) = \frac{\rho_{free}}{\epsilon_0}$$

$$\left. \begin{aligned} \vec{P} &= \chi \epsilon_0 \vec{E} \\ (1 + \chi) &= \epsilon_r \end{aligned} \right\} \quad \Rightarrow \quad \vec{\nabla} \cdot (\epsilon_r \vec{E}) = \frac{\rho_{free}}{\epsilon_0}$$

If the dielectric is linear  
homogeneous and isotropic

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{free}}{\epsilon_r \epsilon_0}$$

$$\vec{\nabla} \times \vec{E} = 0$$

Relative permittivity

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} \quad \rightarrow \quad \text{Permittivity of dielectric}$$

## Another way of looking at things

$$\vec{\nabla} \cdot \left( \vec{E} + \frac{\vec{P}}{\epsilon_0} \right) = \frac{\rho_{free}}{\epsilon_0}$$

We do not know about the polarization, then we invent a new field:

**The electric displacement  $\vec{D}$**

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho_{free} \\ \vec{\nabla} \times \vec{E} &= 0 \end{aligned}$$

To solve the system we need a third equation linking  $\vec{D}$  to  $\vec{E}$

$$\vec{P} = \chi \epsilon_0 \vec{E}$$

If polarization is linear

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}$$

This proportionality may break down if  $\vec{E}$  is too large

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E}$$

Analogy with Hook's law in mechanics

$$\vec{P} = (\epsilon_r - 1) \epsilon_0 \vec{E}$$

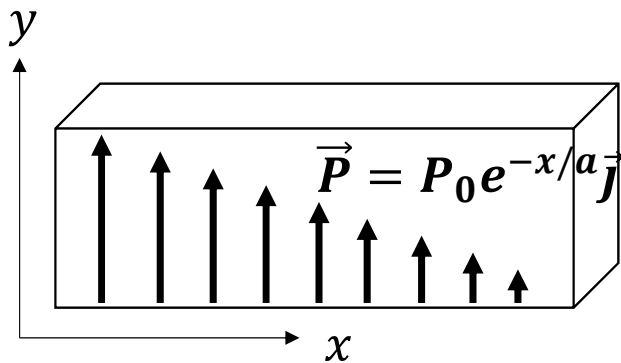
In vacuum  $\epsilon_r = 1$   $\vec{P} = \vec{0}$

Why not using  $\vec{\nabla} \cdot \vec{D} = \rho_{free}$   $\vec{\nabla} \times \vec{D} = 0$  Instead of using  $\vec{\nabla} \cdot \vec{D} = \rho_{free}$   $\vec{\nabla} \times \vec{E} = 0$

Could equation  $\vec{\nabla} \times \vec{D} = 0$  hold ?

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \longrightarrow \quad \vec{\nabla} \times \vec{D} = \epsilon_0 \cancel{\vec{\nabla} \times \vec{E}} + \vec{\nabla} \times \vec{P} \quad \longrightarrow \quad \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$

electrostatic



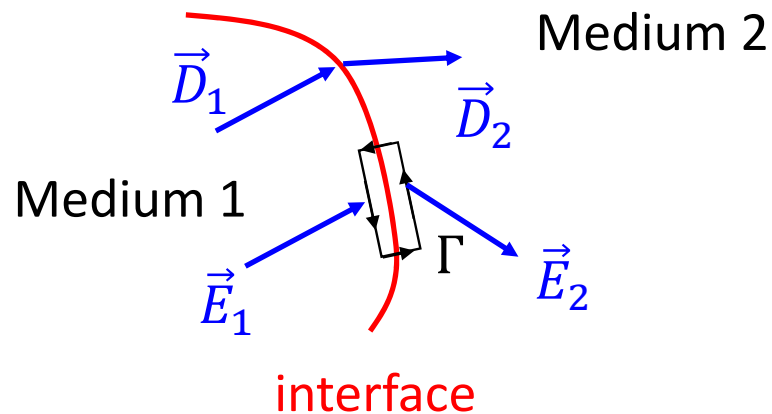
$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & P_0 e^{-x/a} & 0 \end{vmatrix} = -\frac{P_0}{a} e^{-x/a} \vec{k}$$

Clearly  $\vec{\nabla} \times \vec{P}$  and thus  $\vec{\nabla} \times \vec{D}$  cannot be zero !

**Helmholtz's theorem:** To know a vector field quantity we need to calculate its **divergence** and **Curl**

## Looking at the boundary

## Boundary conditions for electrostatic fields



In electrostatic we have three functions

- Potential  $V(\vec{r})$
- Electric field  $\vec{E}(\vec{r})$
- Electric displacement  $\vec{D}(\vec{r})$

How these quantities change in crossing any interface ?

Potential  $V(\vec{r})$  MUST be continuous

## Boundary conditions

- Potential is **continuous** through the interface  $V_1(\vec{r}) = V_2(\vec{r})$  Otherwise, we may have
  - Two values of the field at a single point in space
  - The field  $\vec{E} = -\vec{\nabla}V(r)$  may become infinite


- Stokes theorem

$$\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \oint_{\Gamma} \vec{E} d\vec{l} = \oint_{\Gamma} E_{\parallel} dl = 0$$

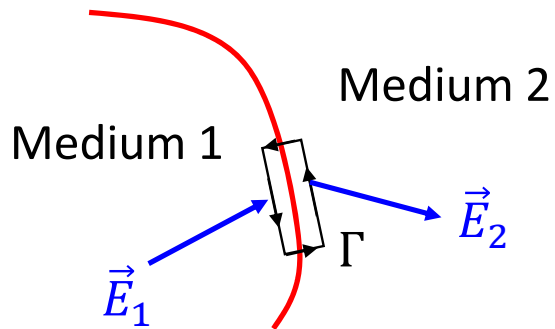
 Tangential component of  $\vec{E}$  ( $E_{\parallel}$ ) is **always continuous** along any path

- Gauss theorem

$$\vec{\nabla} \cdot \vec{D} = \rho_{free} \Rightarrow \oint \vec{D} d\vec{A} = \oint \vec{D} \cdot \vec{n} dA = \oint D_{\perp} dA = Q_{free}$$

 Normal component of  $\vec{D}$  ( $D_{\perp}$ ) **may be** discontinuous

# Boundary conditions for electrostatic fields



For conductor/vacuum

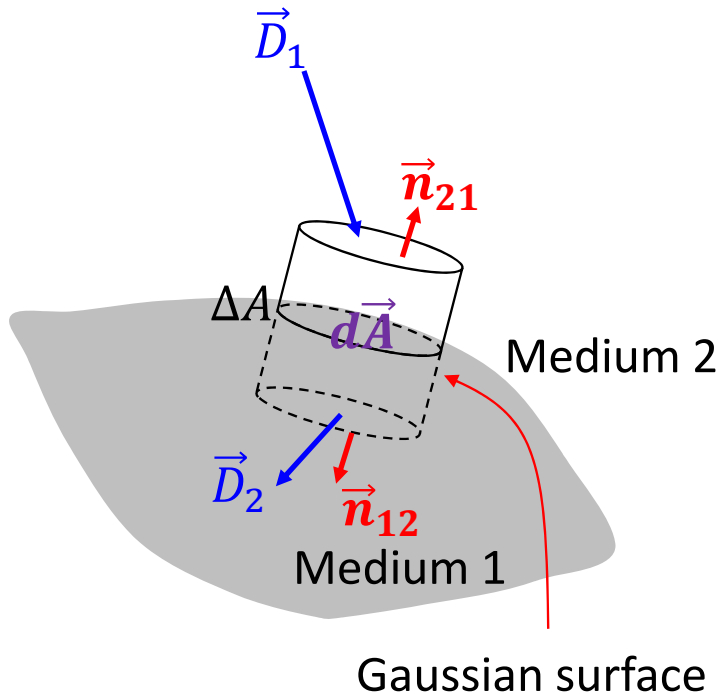
Inside the conductor	$\rho_{bulk} = 0$
	$E = 0$
At the interface conductor /vacuum	$E_t = 0$
	$E = E_{\perp} = \frac{\sigma_{free}}{\epsilon_0}$

For conductor/vacuum

Applying Stokes theorem to path $\Gamma$ for the electric field crossing the interface	$E_{t1} = E_{t2}$
	If one of the medium is a conductor $E_{t1} = E_{t2} = 0$
	If both media are dielectrics $\frac{D_{t1}}{\epsilon_1} = \frac{D_{t2}}{\epsilon_2}$ With $\epsilon_i = \epsilon_{ir}\epsilon_0$

What about the normal component  $D_{\perp}$  ?

## Boundary condition for normal component of electric displacement



$$\begin{aligned} \text{Gauss law } \oint \vec{D} \cdot d\vec{A} &= (\vec{D}_1 \cdot \vec{n}_{21} + \vec{D}_2 \cdot \vec{n}_{12}) \Delta A \\ &= \vec{n}_{21} (\vec{D}_1 - \vec{D}_2) \Delta A = \text{Charge inside the} \\ &\quad \text{Gaussian surface} \end{aligned}$$

The pillbox being reduced to infinitesimal dimensions, the charge  $Q$  is the interface charge  $Q_s$  between the two media

$$\vec{n}_{21} (\vec{D}_1 - \vec{D}_2) \Delta A = \sigma_s \Delta A$$

$$D_{1\perp} - D_{2\perp} = \sigma_s$$

$\sigma_s$  introduces a discontinuity of the normal component of the electric displacement



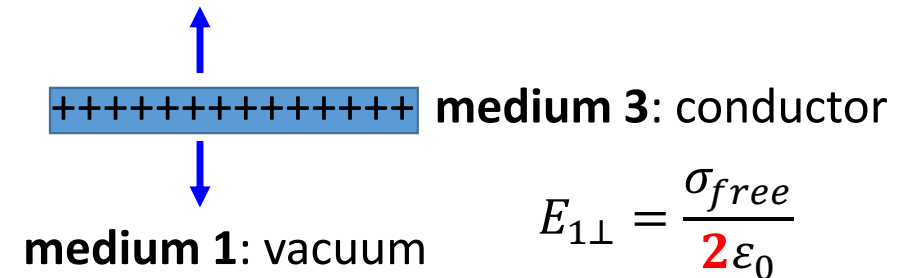
$$D_{1\perp} - D_{2\perp} = \sigma_s$$

If medium 2 is a conductor  $D_{1\perp} = \varepsilon_1 E_{1\perp} = \sigma_{free}$  In the conductor  $E_2 = 0 \Rightarrow D_2 = D_{2\perp} = 0$

And if medium 1 is vacuum  $E_{1\perp} = \frac{\sigma_{free}}{\varepsilon_0}$

**Caution !** If the conductor is a plate or sheet then we have three media

medium 2: vacuum

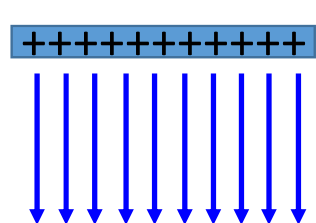


$$E_{1\perp} = \frac{\sigma_{free}}{2\varepsilon_0}$$

**Continuity** of normal component of  $\vec{D}$  if  $\sigma_{free} = 0...$

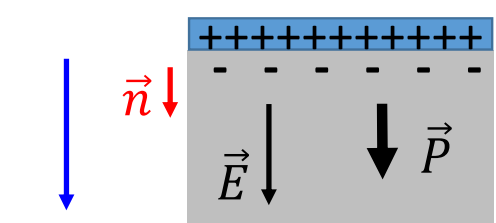
... which is impossible if there is polarization, whether uniform or not

## Charged conductor in contact with a dielectric



$\sigma_{free}$

$\vec{E}_{free} = \frac{\sigma_{free}}{\epsilon_0}$



$\vec{n}$

$\vec{E}_{free}$

$\sigma_{free}$

$\sigma_b = \vec{P} \cdot \vec{n}$

$\vec{E}$

$\vec{P}$

$$\vec{E} = \left( \frac{\sigma_{free}}{\epsilon_0} - \frac{\sigma_b}{\epsilon_0} \right) \vec{n}$$

This  $\vec{E}$  is **NOT** the induced field **BUT** the **NET** field

$$\vec{P} = \sigma_b \vec{n}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \left( \frac{\sigma_{free}}{\epsilon_0} - \frac{\sigma_b}{\epsilon_0} \right) \vec{n} + \sigma_b \vec{n}$$

$$\vec{D} = \sigma_{free} \vec{n}$$

- Lines of  $\vec{D}$  begin and end **ONLY** at free charges
- Lines of  $\vec{E}$  begin and end on either free or bound charges