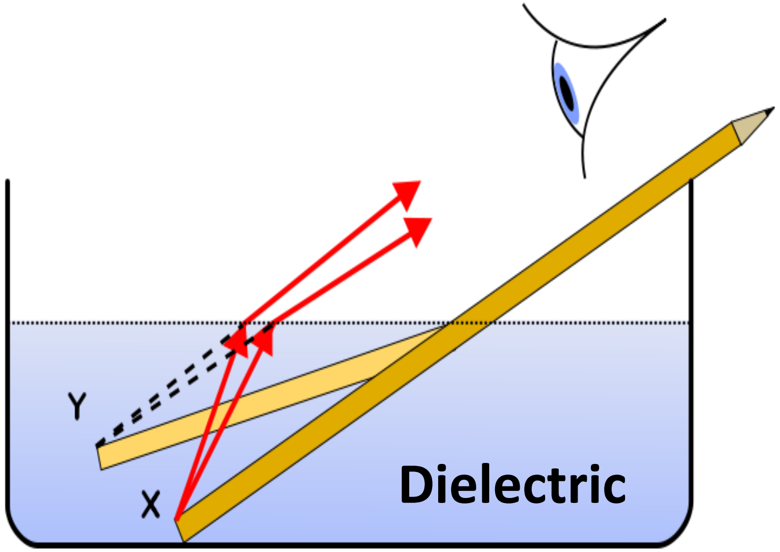
Dielectrics

Consequence of the behavior of the "electric field" at the interface between two different media.

<u>It illustrates the boundary conditions</u>



- How the applied field affects the dielectric ?
- How the dielectric reacts to the field inside, outside and at the interface ?

Quantities in Electrostatic

- Free Charge and free charge density (linear, surface, volume)
- Bound charges (linear, surface, volume)
- Capacitance
- Conductor
- Dielectric
- Permittivity of materials ε versus dielectric constants ε_r
- Electric susceptibility χ
- Permittivity of dielectric= $\varepsilon_0(1+\chi) = \varepsilon_0\varepsilon_r$

Scalar

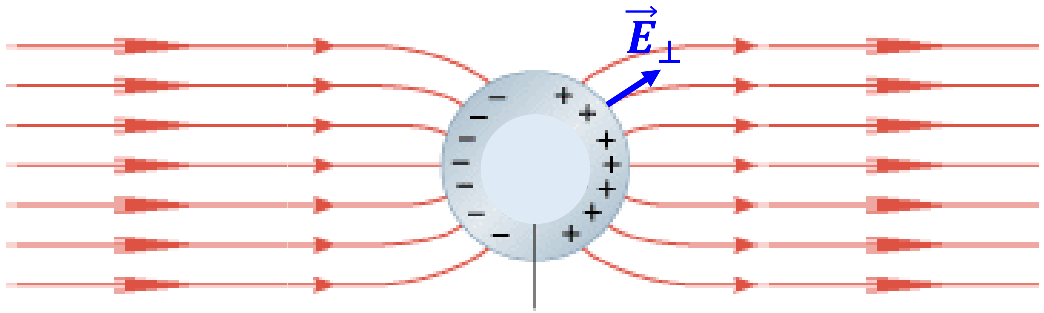
- Electric field
- Electric Force
- Dipole moment
- Polarization
- Displacement

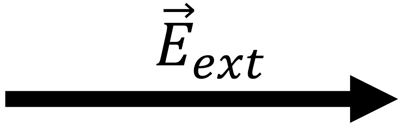
Vector field

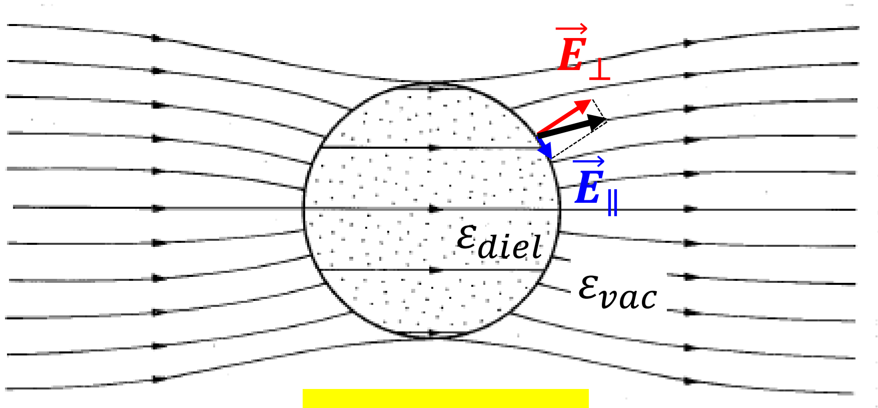
- Work done
- Electric Potential
- Potential Energy

Scalar field

Differences and similarities between conductor and dielectric in the presence of electric field







$\boldsymbol{\varepsilon_{diel} > \varepsilon_{vac}}$

Conductor: **Induction**

- Charge free to move
- At the surface $\vec{E}_{\perp}$ ⇨ $\boxed{\vec{E}_{\parallel} = \vec{0}}$
- Inside $\vec{E}_{ind}$ opposes $\vec{E}_{ext}$
- Total compensation $\vec{E}_{net}\ (inside) = \vec{0}$

Dielectric: **Polarization**

- Charge **NOT** free to move
- At the surface $(\vec{E}_{\perp}\ \&\ \vec{E}_{\parallel}) \neq \vec{0}$
- Inside $\vec{E}_{ind}$ opposes $\vec{E}_{ext}$
- Partial compensation $\vec{E}_{net}\ (inside) \neq \vec{0}$

Both materials are affected by the electric field and both affect its shape

What is a dielectric ?

- A dielectric has the ability to get **polarized** by an external applied field. This field <u>induces electric dipoles</u> inside the dielectric

- Polarization occurs in both **polar** an **nonpolar** materials

- Although any kind of substance is **polarizable** to some extent, the effect of <u>polarization is important only in insulating materials</u>

Consequence

The <u>induction</u> of electric dipoles within the dielectric modifies the electric field pattern both **inside** and **outside** the material

Perfect conductor: **infinite conductivity**

Polarization = induction of “dipoles” made of free charges

Perfect dielectric: **zero conductivity**

Polarization = induction of dipoles made of bound charges

Concept of Permittivity of a medium

From ε_0(vacuum) to ε (medium) $= \varepsilon_r \varepsilon_0$

From Slides 48 and 49 in Lectures 3&4_Introduction I_2

- Electric permittivity ε describes how an electric field affects **AND** is affected by a medium. It is determined by the ability of a material to polarize in response to an applied field, and thereby to cancel, partially, the field inside the material

In vacuum, Coulomb's law contains the constant $\boxed{\dfrac{1}{4\pi\varepsilon_0}}$

in which ε_0 is defined as the permittivity of vacuum

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{Coulomb's law} \qquad F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

Conductor versus dielectric: Induction versus polarization

There is a fundamental difference between induction and polarization by an external field

In both cases an external field generates dipoles

Induction applies to conductors:

It results in the generation of opposite charges residing at the surface

The field inside the conductor = 0 ⇨ ε goes from ε_0 (vacuum) to $\varepsilon_r\varepsilon_0$ where $\boldsymbol{\varepsilon_r} = \infty$

The dipoles are macroscopic, dimension of the conductor. **No dipoles inside**

It is $\boldsymbol{\varepsilon_r} = \infty$ that makes a perfect conductor = perfect reflector

Forbids the electrostatic field to penetrate inside

Polarization applies to dielectrics (insulators):

It results in the generation of dipoles due to **BOUND** charges induced locally

The field inside the dielectric $\neq 0$ ⇨ ε goes from ε_0 (vacuum) to $\varepsilon_r\varepsilon_0$ where ε_r = finite

The dipoles are microscopic. There are **dipoles inside**

ε(permittivity of the dielectric) $= \varepsilon_r\varepsilon_0$

ε_r is the relative permittivity or dielectric constant (No dimension)

$\varepsilon_r = 1$ in vacuum

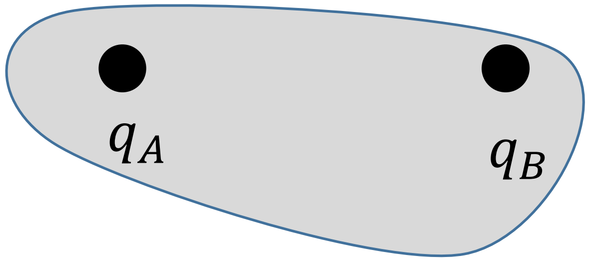
ε_r indicates the strength of the polarizability of a material

⇩

Concept of screening

What is the meaning of the dielectric constant ε_r

$\boldsymbol{\varepsilon_r}$ is a material property that affect the coulomb force between two point charges



$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{r^2}$$

In vaccuum

$$F = \frac{1}{4\pi\boldsymbol{\varepsilon_r}\varepsilon_0}\frac{q_A q_B}{r^2}$$

Force is lowered in a dielectric

Another point of view: As if q_A is reduced to $\boldsymbol{q'_A} = q_A/\boldsymbol{\varepsilon_r}$

$$F = \frac{1}{4\pi\boldsymbol{\varepsilon_r}\varepsilon_0}\frac{q_A q_B}{r^2} \Rightarrow \boxed{F = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{q'_A} q_B}{r^2}}$$

Screening effect

Materials	Dielectric constants
Vacuum	1
Air	1.005364
Glass, pyrex 7740	5
Mica	5.4
Muscle	58
Skin	33-44
Tongue	38
Water, liquid, 0 °C	87.9 (made of permanent dipoles)
Polyethylene	2.26

Why do we use the generalized permittivity concept: $\boldsymbol{\varepsilon_0 \varepsilon_r}$

- The concepts of polarizability and dipole moment distribution are introduced to relate <u>microscopic</u> phenomena to the <u>macroscopic fields</u>

- The introduction of ***permittivity*** eliminates the need to explicitly consider microscopic effects… *in simple case of linear, homogeneous and isotropic medium*

From the macroscopic point of view

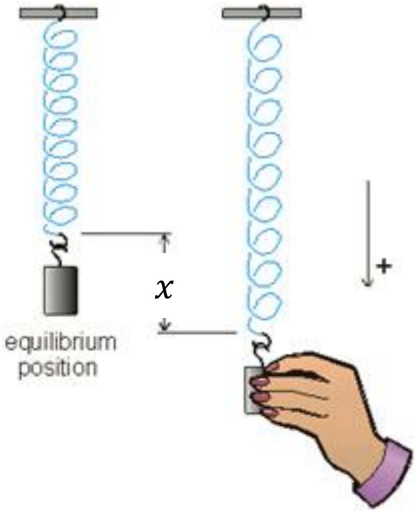
knowing the ***permittivity*** of a dielectric is all what we need to describe its interaction with the external field

Three important properties to characterize a Dielectric

Linearity → **Effect ∝ Cause**

Homogeneity → **Equally linear at any point in space**

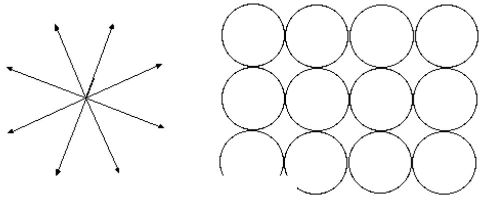
Isotropy → **Equally linear along any direction**



Hook's law

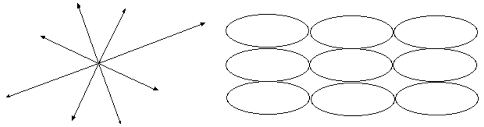
$$\boxed{x = \frac{F}{k}}$$

Small displacement



Isotropy

Effect **Cause**



Anisotropy

Effect **Cause**

Main questions regarding dielectrics

- What is the mechanism behind polarization of a dielectric ?

  Polar versus non polar material

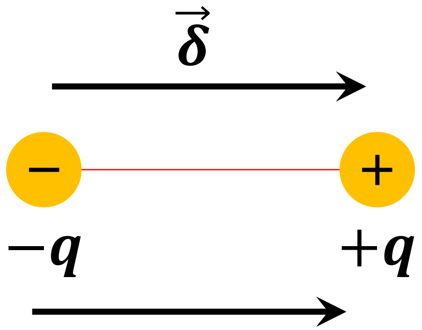
- What are the consequences of polarization?
  - Concept of bulk and surface **bound charges**
  - Boundary conditions

- Characteristics of the internal electric field at the macroscopic and microscopic level?

Restriction to Linear – homogenous – isotropic dielectric

What is polarization and what is polarizability ? Electronic polarization

Fictive dipole in vacuum



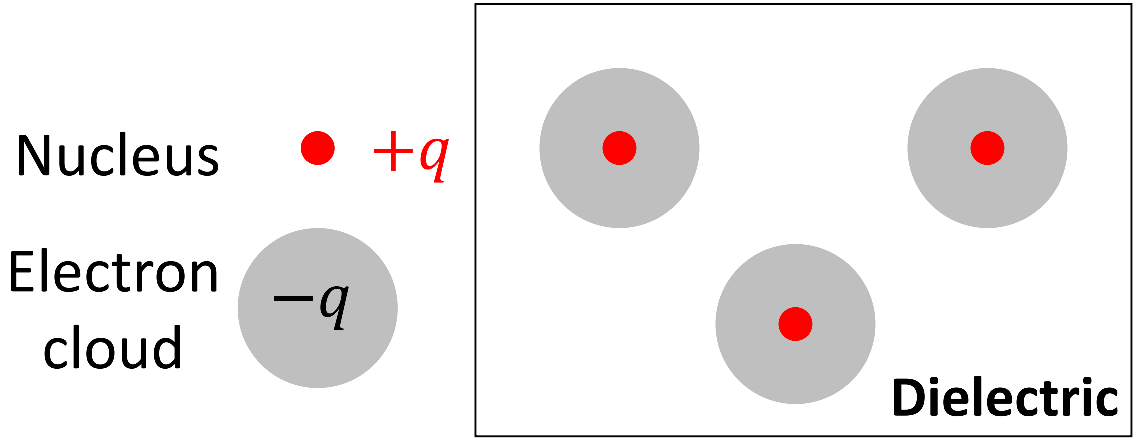
Dipole moment $\vec{\boldsymbol{p}} = \boldsymbol{q}\vec{\boldsymbol{\delta}}$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{e}_r}{r^2} \quad \vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})$$

$\vec{e}_r$ unit vector along $\vec{r}$

Polarizability: The ability of a material to become polarized in the presence of $\vec{\boldsymbol{E}}_{\boldsymbol{ext}}$

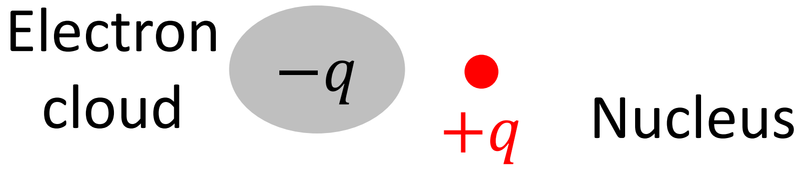
$\vec{\boldsymbol{E}}_{\boldsymbol{ext}} = \vec{0}$

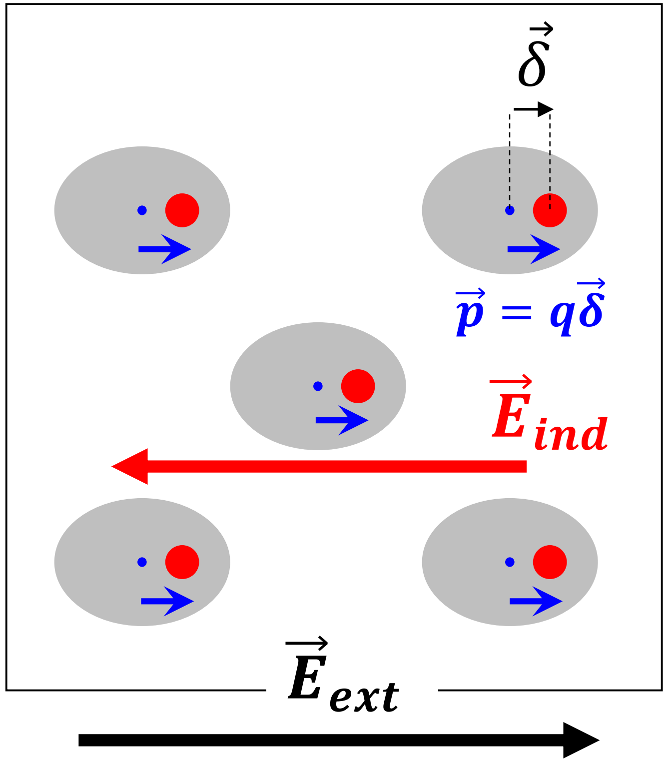


The centers of gravity of the nucleus and the electron cloud coincide

$\vec{E}_{ext}$ distorts the electronic cloud. It becomes elongated along the direction of the field

- The electronic cloud is pushed back
- The nucleus is pushed forward in the direction of the field





$\vec{p} = q\vec{\delta}$ What causes the separation $\vec{\delta}$?

The local $\vec{E}_{loc}$ field

$\vec{p} = \alpha_e \vec{E}_{loc}$ $[p] = \mathrm{Cm}$ $[\alpha_e] = \mathrm{F}m^2$

α_e = atomic (electronic) polarizability

$\vec{E}_{ind}$ = $\vec{E}_{pol}$ macroscopic internal field which tends to oppose the external field

Three different fields are taking place in the dielectric

$$\vec{\boldsymbol{p}} = \boldsymbol{\alpha_e} \vec{\boldsymbol{E}}_{\boldsymbol{loc}}$$

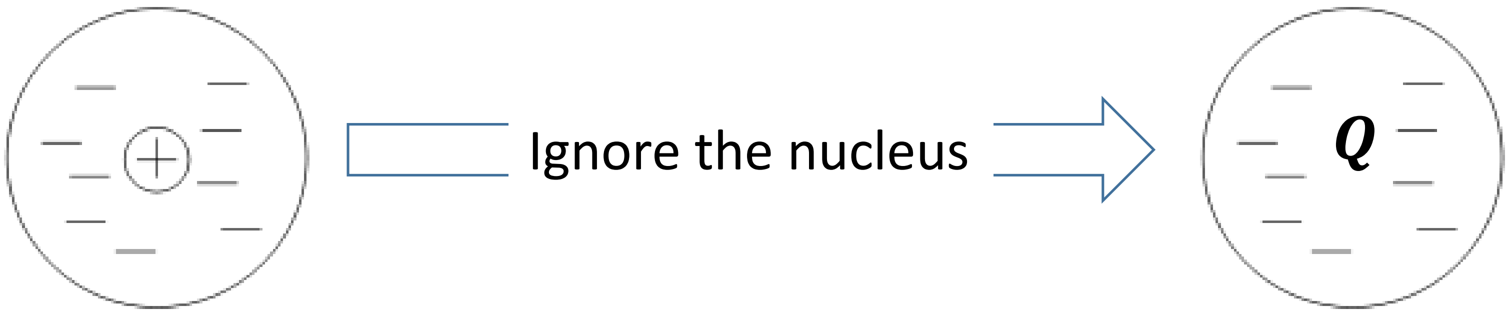
- If the dipole were isolated, the **local** electric field would simply be the applied macroscopic field $\vec{E}_{ext}$

- However, the large number of of N neighboring dipoles also contribute to the polarizing electric field.

- The electric field changes drastically from point to point within a small volume containing many dipoles.

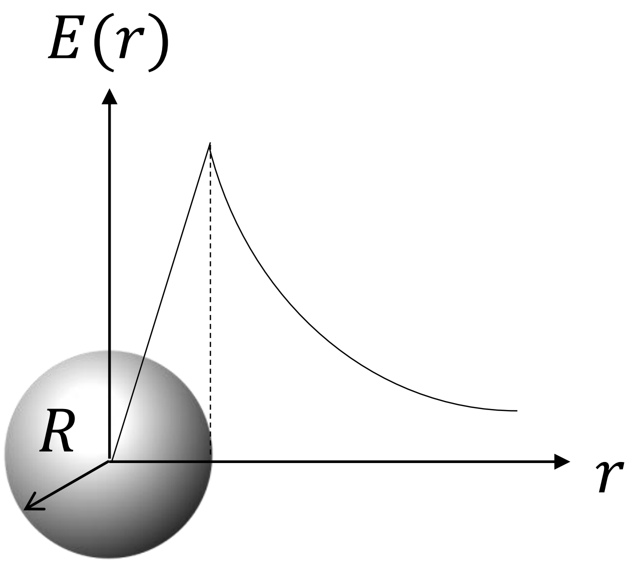
- By <u>superposition principle</u>, the macroscopic resulting field inside the dielectric will then be equal to the average over this small volume.

Electronic polarization of an atom in free space



Spherical electronic cloud

Slide #93 in E_Lectures 8&9_Electrostatics_Gauss law

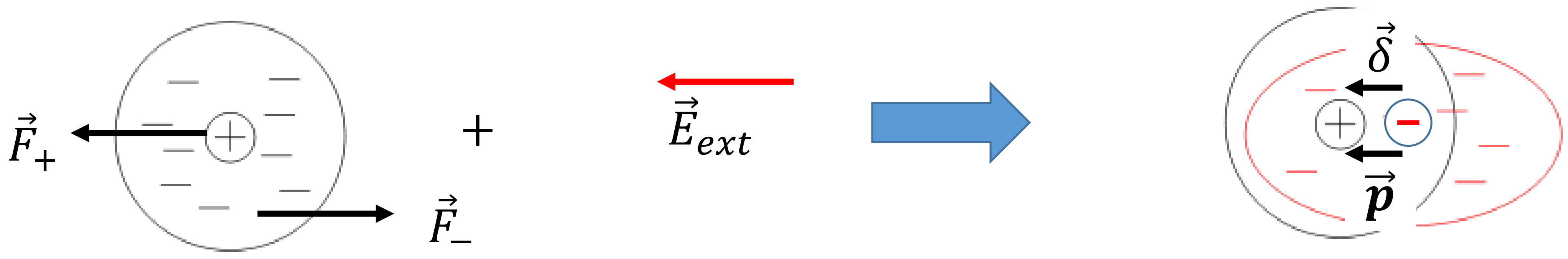


Charge $Q = \rho(\frac{4\pi}{3}R^3)$

Using Gaussian surface inside the charged sphere

$\boxed{\boldsymbol{r < R}}$ ⇨ $E(r)4\pi r^2 = -\frac{Q}{\varepsilon_0}$, $\quad Q = \rho(\frac{4\pi}{3}r^3)$

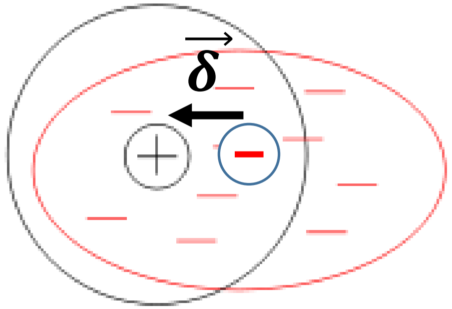
⇨ $E(r) = \frac{\rho}{3\varepsilon_0}r$ ⇨ $\boxed{E(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3}r}$



Question: When does the separation reaches its maximum?

Answer: When the external force is balanced by the attraction between the nucleus and the electronic cloud

Crude model: electronic polarization



Electronic cloud = charged sphere $E(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3}r$

Force exerted on the nucleus inside the cloud at distance δ from the center

$$F_{cloud} = QE(\delta) = -\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^3}\delta$$

Equilibrium position is reached when $\vec{F}_{cloud} + \vec{F}_{ext} = \vec{0}$ $\Rightarrow$ $-\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^3}\delta + QE_{ext} = 0$

Equilibrium distance $\delta = 4\pi\varepsilon_0 R^3 \frac{E_{ext}}{Q}$

Induced dipole $\vec{\boldsymbol{p}} = \boldsymbol{Q}\vec{\boldsymbol{\delta}}$ $\quad p = Q\delta = 4\pi\varepsilon_0 R^3 E_{ext}$

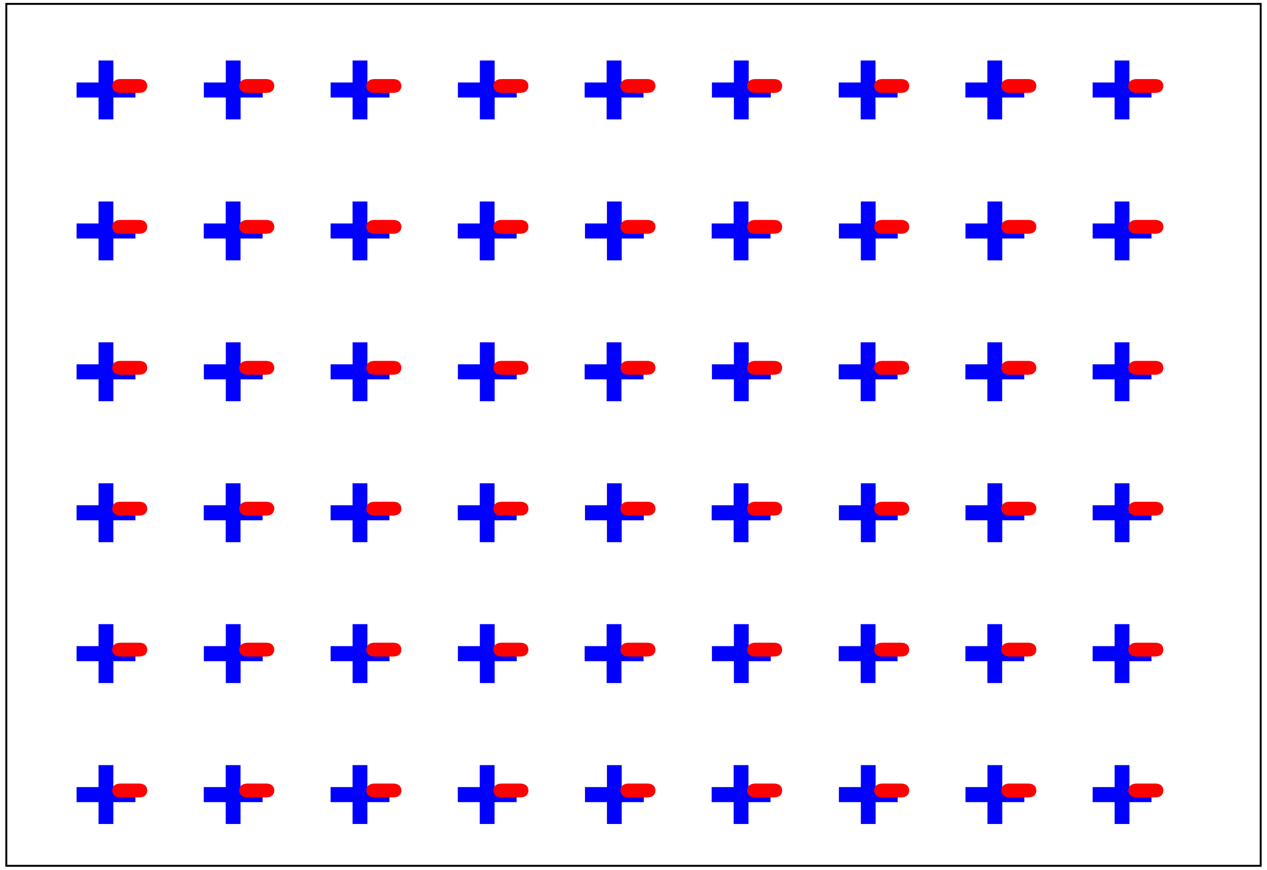
$\vec{p} = \alpha_e \vec{E}_{loc}$ $\quad \vec{E}_{loc} = \vec{E}_{ext}$

Electronic polarizability of atoms

$$\boldsymbol{\alpha_e} = \frac{p}{E_{ext}} = 4\pi\varepsilon_0 R^3$$

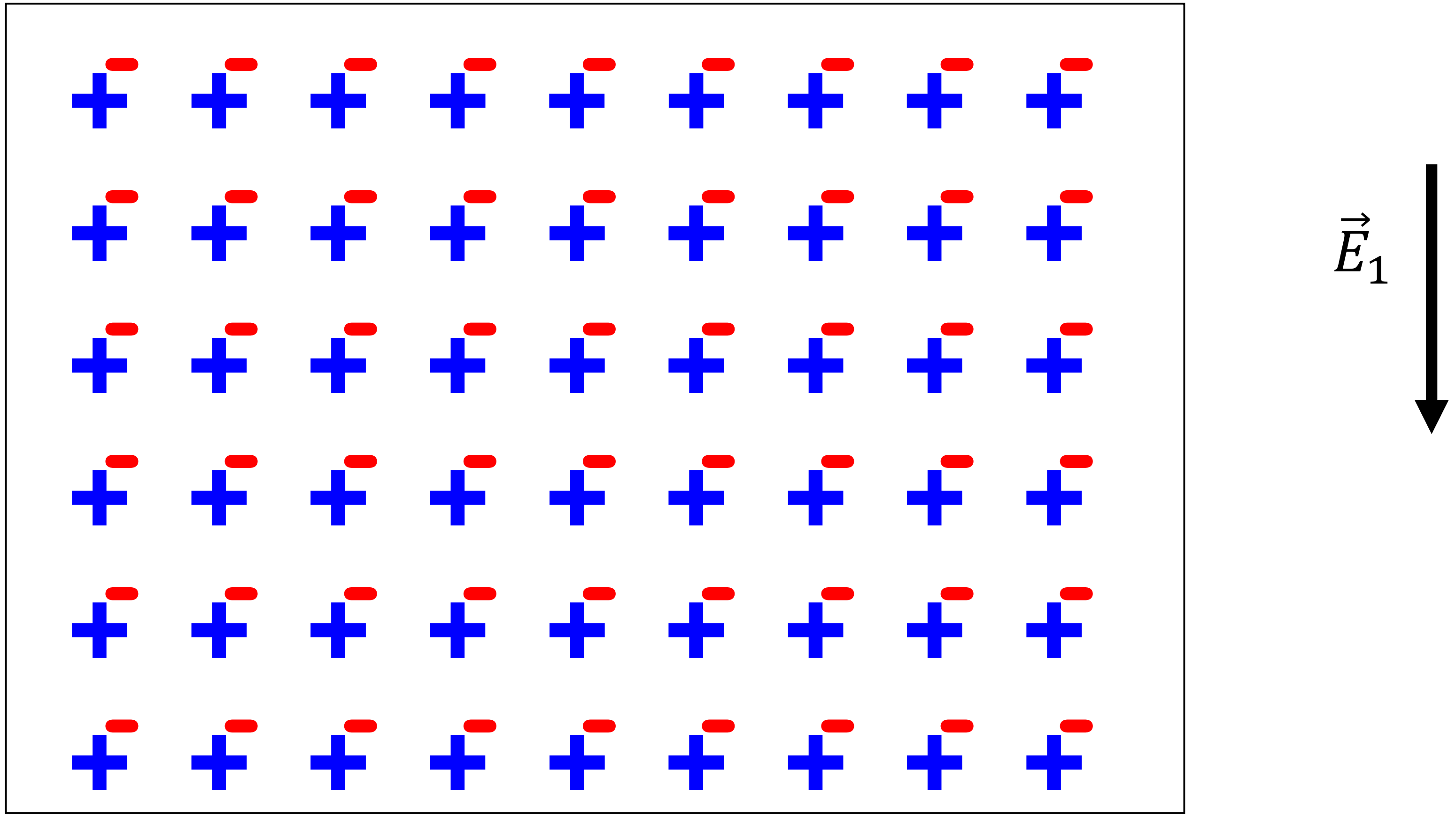
Mechanism of polarization of a <u>simple linear dielectric</u>

External field $\vec{E}_{ext} = \vec{0}$



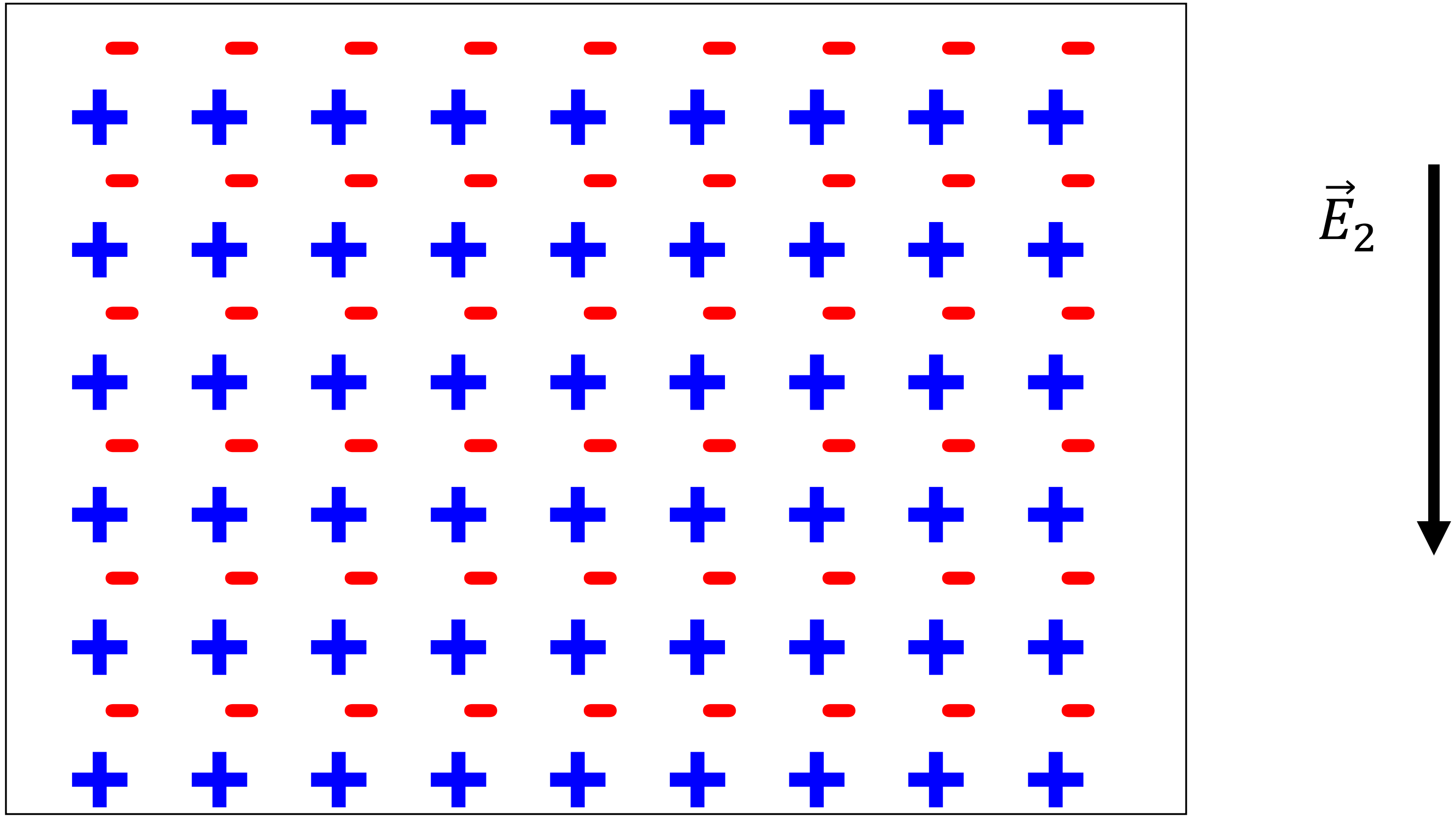
External field $\vec{E}_1 \neq \vec{0}$

$E_1 > 0$



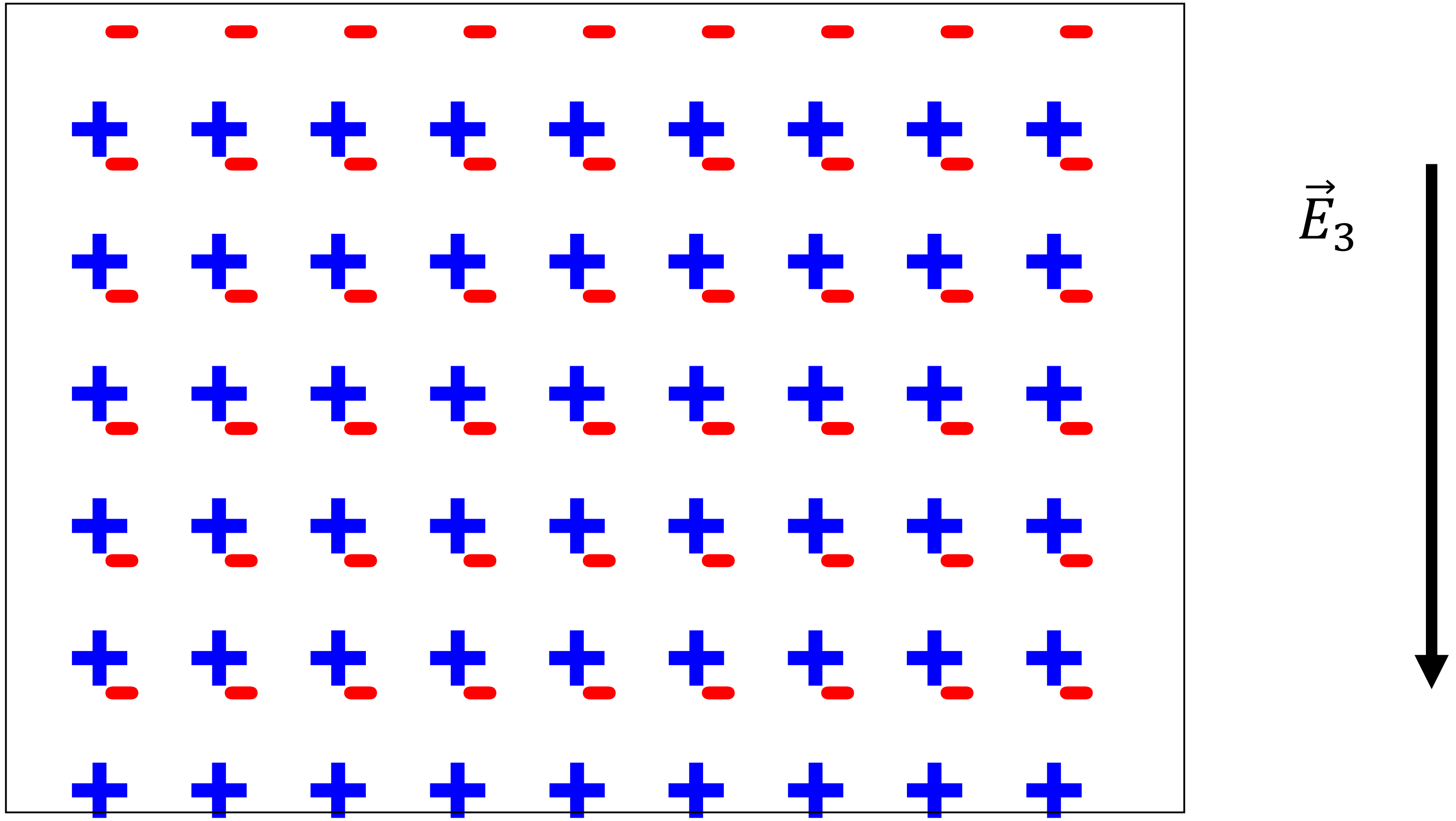
External field $\vec{E}_2 \neq \vec{0}$

$E_2 > E_1$



External field $\vec{E}_3 \neq \vec{0}$

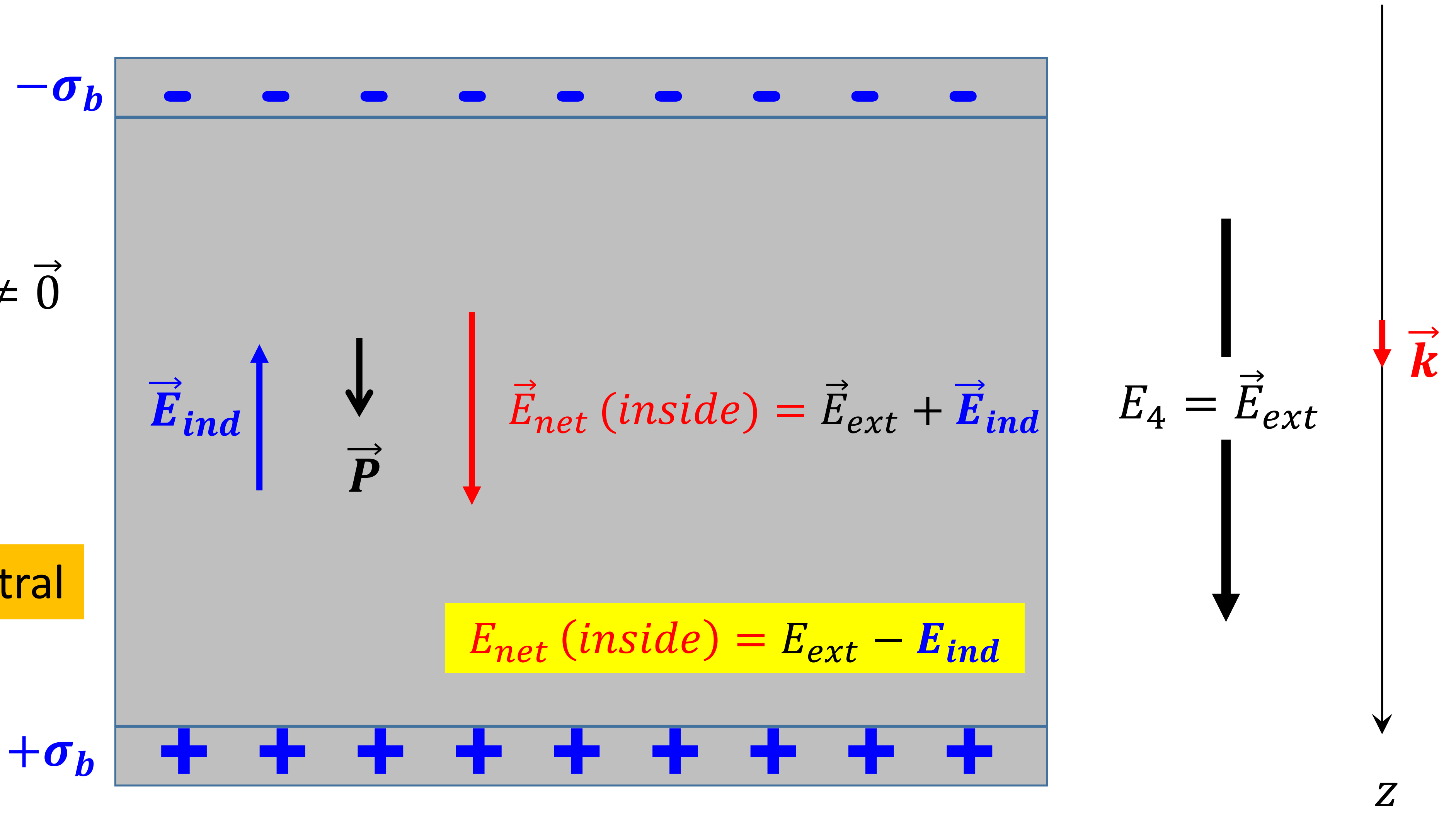
$E_3 > E_2$



External field $\vec{E}_4 \neq \vec{0}$

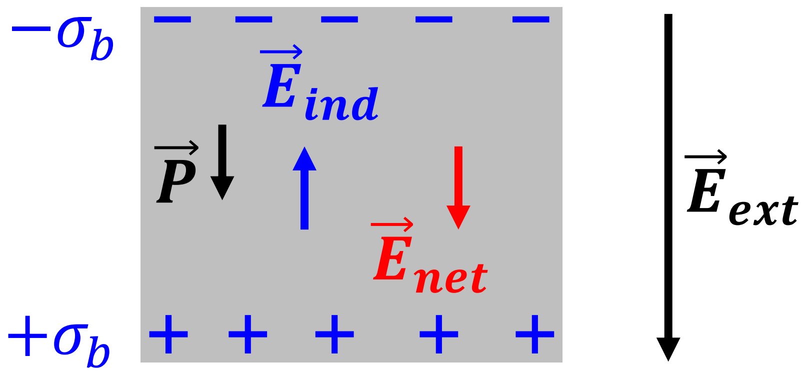
$E_4 > E_3$

Dielectric still neutral



Polarization of the dielectric = induction of **bound charges** at the surface

Electronic polarization



$\vec{E}_{net}(inside) < \vec{E}_{ext}$

Linear – Homogeneous - Isotropic

$$\vec{P} \propto \vec{E}_{ext}$$

$$\vec{p} \Rightarrow \varphi_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\vec{e}_r}{r^2}$$

$$\Rightarrow \varphi_{net}(\vec{r}) = \int \varphi_{dipole}(\vec{r})dV \quad \Rightarrow \vec{E}_{net}(\vec{r}) = -\vec{\nabla}\varphi_{net}(\vec{r})$$

$\varphi_{net}(\vec{r})$ inside and $\varphi_{ext}(\vec{r})$ ⇨ Continuity at the boundary
⇨ the potential MUST be continuous

There are mainly two other types of polarizations

Ionic Polarizability:

Materials made of two or more types of ions will develop a polarization under the action of an external feld. Negative ions are attracted by the field and positive ions are repelled. The materials becomes ordered by pair

Orientational Polarizability:

Under the external field, permanent dipoles, which are otherwise randomly oriented in the absence of the field are aligned in the direction of the field

Two types of dielectrics

<u>Permanent dipoles do not exist</u>

Need external electric field to create dipoles

<u>Permanent dipoles exist</u>

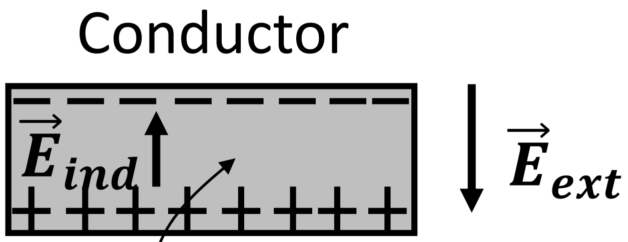
Non polar materials

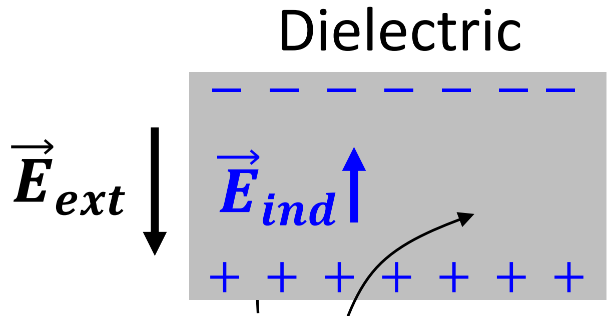
Competition of T and $\vec{E}_{ext}$

- **Temperature** tends to randomize the dipoles. <u>No preferred orientation</u>
- **Electric field** tends to orient the dipoles. <u>Alignment of existing dipoles</u>

Polar materials

- **Electric field** strengthen the alignment which <u>pre-exists</u>





$$\vec{E}_{net}\,(inside) = \vec{E}_{ext} + \vec{E}_{ind} = \vec{0}$$

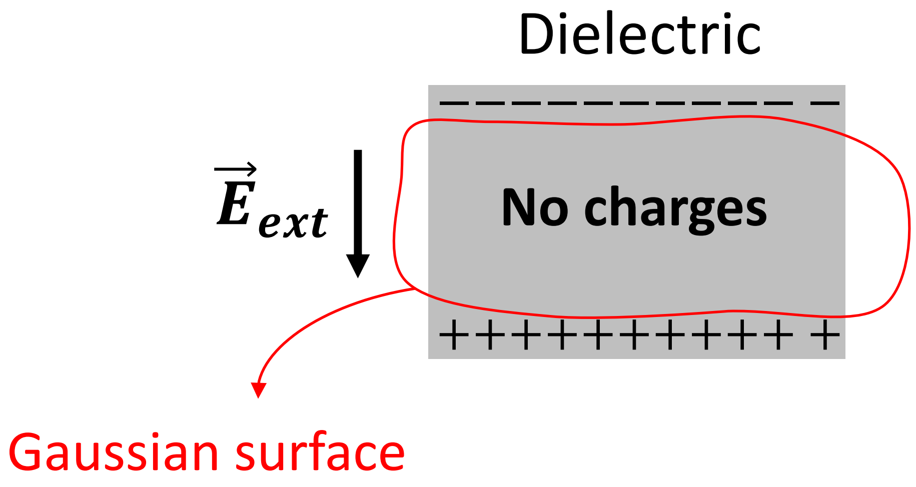
$$\vec{E}_{net}\,(inside) = \vec{E}_{ext} + \vec{E}_{ind} \neq \vec{0}$$

Charge neutrality
Total net charge = 0 in both cases

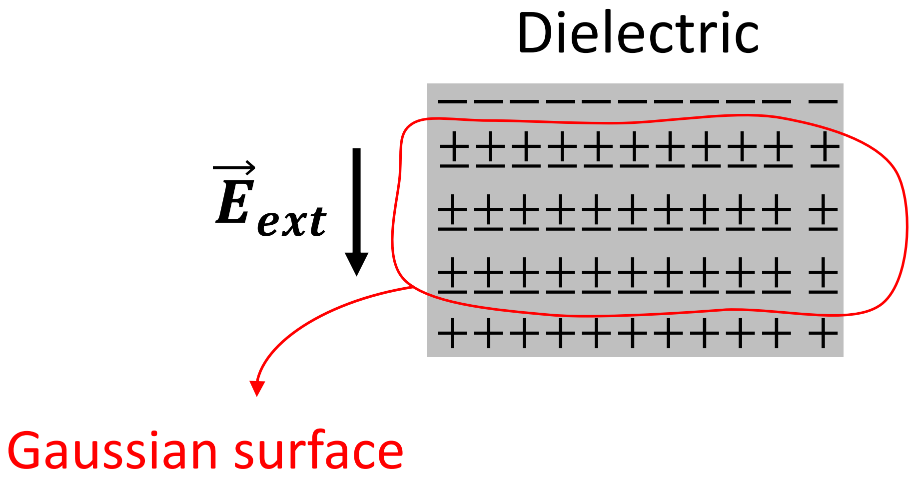
Electrostatically stable because:

- Presence of external field $\vec{E}_{ext}$
- Surface barrier confines charges inside

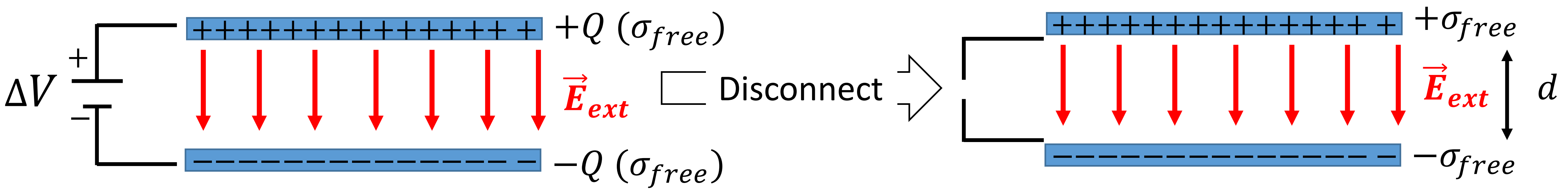
In conductor charge = 0



In dielectric net charge = 0 ⇨ there are dipoles



The parallel plates (capacitor) produce a uniform field



$\vec{E}_{ext} = -\vec{\nabla}V \Rightarrow \Delta V = (V_+ - V_-) = \int_0^d \vec{E}_{ext} d\vec{l} = E_{ext}d \Rightarrow \Delta V = E_{ext}d$

From Gauss law $E_{ext} = \dfrac{\sigma_{free}}{\varepsilon_0} \Rightarrow \Delta V = \dfrac{\sigma_{free}}{\varepsilon_0}d \Rightarrow \sigma_{free} = \dfrac{\Delta V \varepsilon_0}{d}$

E_{ext} is due to free charges

From now on $\boldsymbol{E_{ext} = E_{free}}$

$Q = \sigma_{free}A \Rightarrow Q = \dfrac{A\varepsilon_0}{d}\Delta V = C_0\Delta V \Rightarrow \boxed{C_0 = \dfrac{A\varepsilon_0}{d}}$

In vacuum

$$Q = \frac{A\varepsilon_0}{d}\Delta V$$

Charge Q put on the plates depends

- On the area of the plates A
- On the potential difference ΔV
- On the distance between the plates d ⇨ …less trivial

But once the plates are disconnected

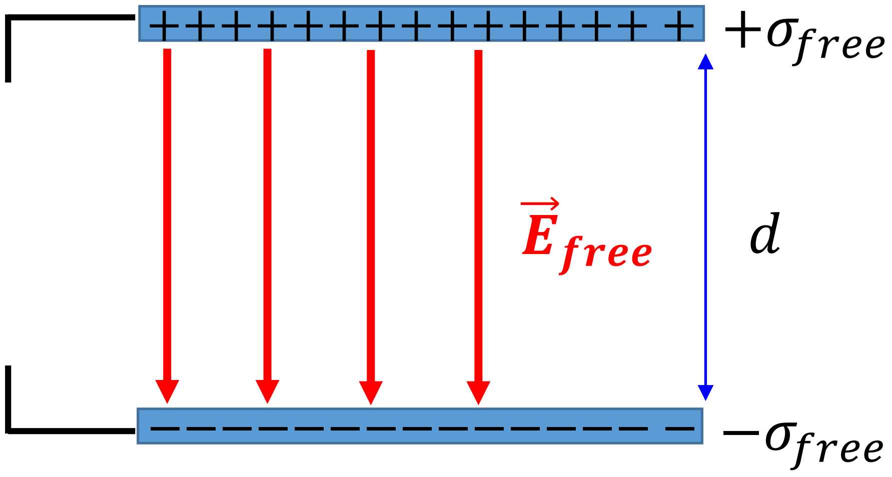
$$\boldsymbol{E_{free} = \frac{\sigma_{free}}{\varepsilon_0}}$$

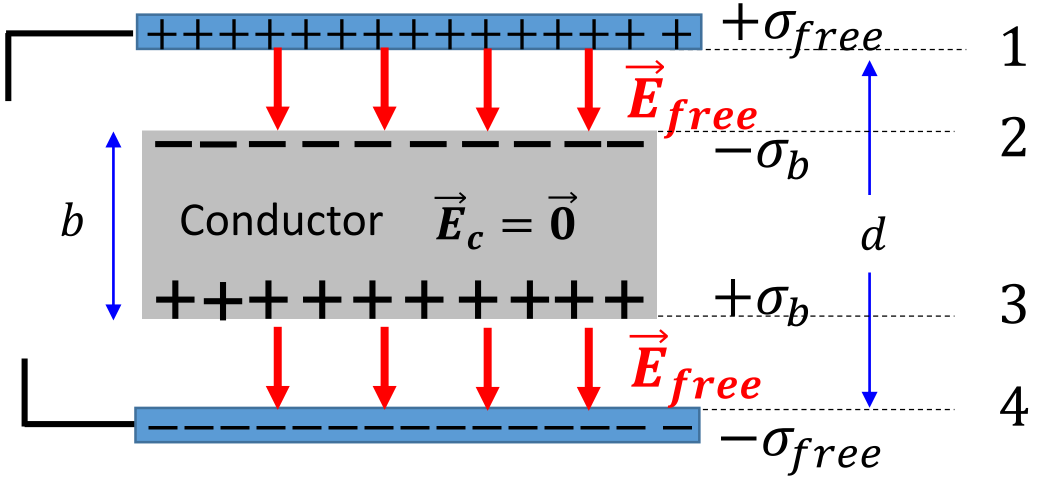
Charges are trapped and frozen in the plates $\boldsymbol{E_{free} = Cte}$

$\Delta V = E_{free} d$

If d changes ⇨ ΔV change

Space filled with a conductor





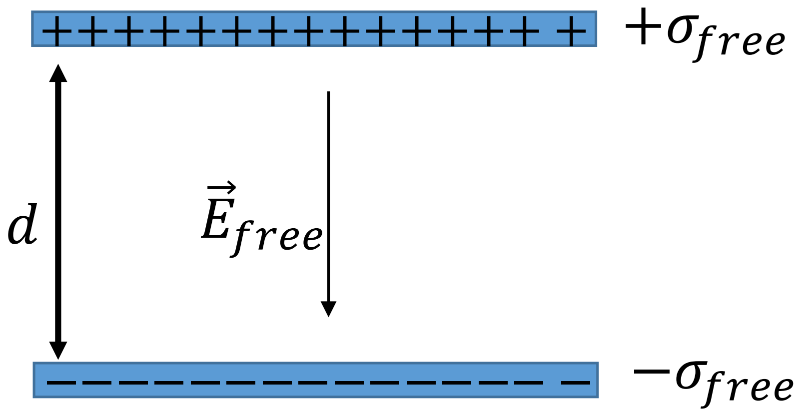
$$\vec{E}_{free} = -\vec{\nabla}V \Rightarrow \Delta V = \int_0^d \vec{E}_{free} d\vec{l} = \int_1^2 \vec{E}_{free} d\vec{l} + \underbrace{\int_2^3 \vec{E}_c d\vec{l}}_{=0} + \int_3^4 \vec{E}_{free} d\vec{l} = \boldsymbol{E(d-b)}$$

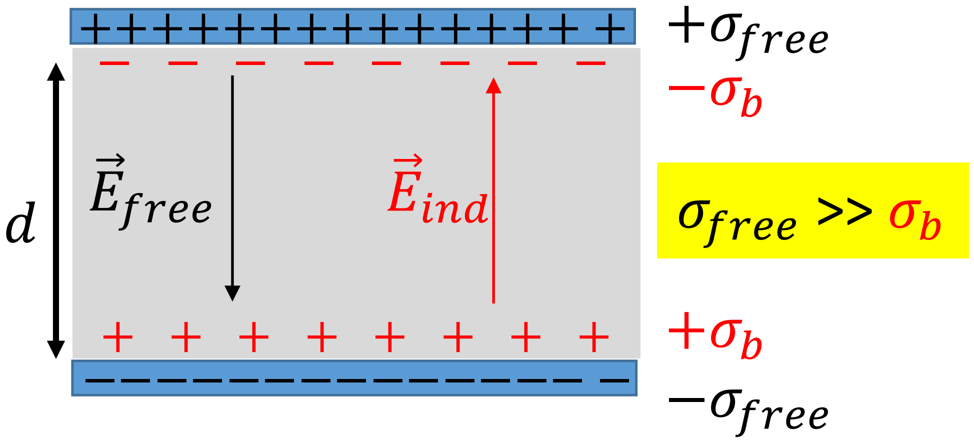
$$\Delta V = E(d-b) = \frac{\sigma_{free}}{\varepsilon_0}(d-b)$$

$$\boxed{C_{cond} = \frac{A\varepsilon_0}{d} \frac{1}{1-b/d}}$$

If $b = d$ ⇨ shortening

Inserting a **linear** dielectric





Voltage supply has been removed

A dielectric has been inserted

$$\vec{E} = \vec{E}_{net} = \vec{E}_{free}$$

$$E = E_{net} = E_{free}$$

$$E_{free} = \frac{\sigma_{free}}{\varepsilon_0}$$

In a conductor $E_{net} = 0$
E_{ind} compensates completely E_{free}

$$\vec{E} = \vec{E}_{net} = \vec{E}_{free} + \vec{E}_{ind}$$

$$E = E_{net} = E_{free} - E_{ind}$$

$$E_{ind} = \frac{\sigma_b}{\varepsilon_0}$$

Postulate:

$$\sigma_b = \beta\sigma_{free} \qquad E_{ind} = \beta E_{free} \qquad E_{net} = (1-\beta)E_{free} = \frac{E_{free}}{\boldsymbol{\varepsilon_r}} \qquad \boldsymbol{\varepsilon_r} = \frac{1}{1-\beta} > 1$$

$$\beta < 1$$

If $\beta = 1$ ⇨ $\sigma_b = \sigma_{free}$ $E_{net} = 0$ ⇨ Replacing the dielectric by a conductor (Field inside the conductor $= 0, \boldsymbol{\varepsilon_r} = \infty$)

$$E = E_{net} = \frac{E_{free}}{\varepsilon_r}$$

Potential inside the dielectric $V_{ind} = E_{net}(dielectric)d = \frac{E_{free}}{\boldsymbol{\varepsilon_r}}d$

Potential before inserting the dielectric $V_{\boldsymbol{free}} = E_{net}(vacuum)d = E_{free}d$

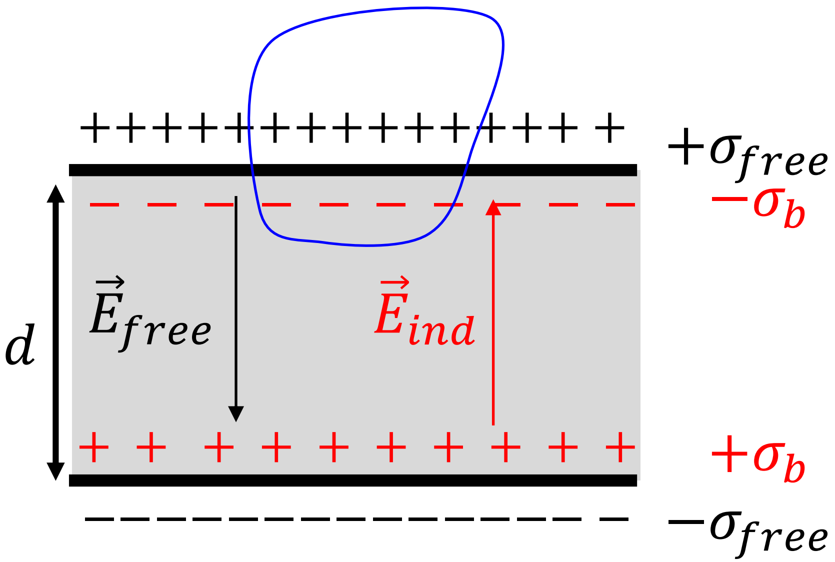
Faraday's observation

$$\frac{V_{ind}}{V_{free}} = \frac{1}{\varepsilon_r}$$

The potential drops by the factor $\frac{1}{\varepsilon_r}$ after inserting the dielectric

Faraday could measure the dielectric constants for a variety of materials

What about the Gauss law: Is it still valid in a dielectric?



Net charge inside the Gaussian surface $Q_{inside} = Q_{free}^{+} - Q_{ind}^{-}$

$$Q_{ind}^{-} = \sigma_b \times A = \beta\sigma_{free} \times A$$

$$\boxed{Q_{inside} = (1-\beta)\sigma_{free} \times A = \frac{Q_{free}}{\varepsilon_r}}$$

E_{net} inside dielectric: from now on

Gauss law in dielectric $\oint_{surface} \vec{E}.d\vec{A} = \frac{1}{\varepsilon_0}\sum Q_{inside} = \frac{\sum Q_{free}}{\varepsilon_0\varepsilon_r}$

Gauss law in **vacuum** $\oint_{surface} \vec{E}.d\vec{A} = \frac{\sum Q_{free}}{\varepsilon_0}$

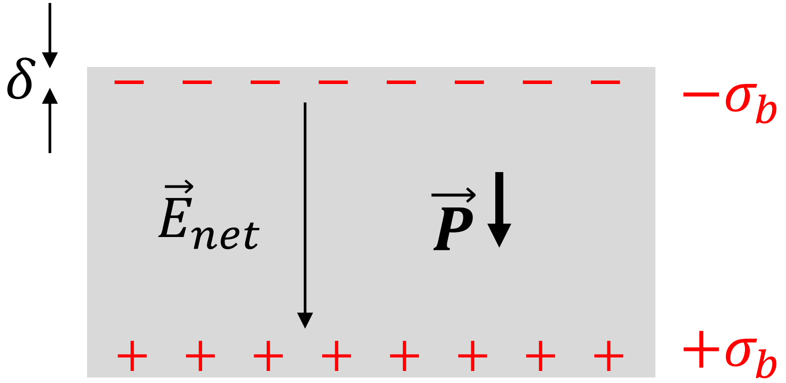
Gauss law in **dielectric** $\oint_{surface} \vec{E}.d\vec{A} = \frac{\sum Q_{free}}{\varepsilon_0 \boldsymbol{\varepsilon_r}}$

Space filled with a dielectric

$$E_{net} = \frac{E_{free}}{\varepsilon_r} \qquad Q_{net} = \frac{Q_{free}}{\varepsilon_r} \qquad \boxed{C_{diel} = \boldsymbol{\varepsilon_r} \frac{A\varepsilon_0}{d}}$$

$\boldsymbol{\varepsilon_r > 1}$

$\boldsymbol{\varepsilon_r}$ = property of the dielectric



$$\sigma_b = \frac{Q_b}{A} = \frac{Q_b}{V/\delta} = \left(\frac{Q_b}{V}\right)\delta = (Nq)\delta$$

$$\vec{P} = N\vec{p} = (Nq)\vec{\delta}$$

Dielectric:

- Linear
- Homogeneous
- Isotropic

$$\vec{P} \parallel \vec{\delta} \qquad P = (Nq)\delta \qquad \boxed{\sigma_b = P}$$

Space filled with a dielectric: $\boldsymbol{E}$ inside the dielectric

Linear, homogeneous and isotropic dielectric $P \propto E$

$= 0$ if $E = 0$

$= 0$ if vacuum

$$P = \chi\varepsilon_0 E$$

$\chi =$ Electric susceptibility

$$E = \frac{\sigma_{free}}{\varepsilon_0} - \frac{\sigma_b}{\varepsilon_0} = E_{free} - \frac{\sigma_b}{\varepsilon_0} = E_{free} - \chi E$$

E_{net}

$$E = \frac{E_{free}}{1+\chi}$$

$$(1+\chi) = \varepsilon_r$$

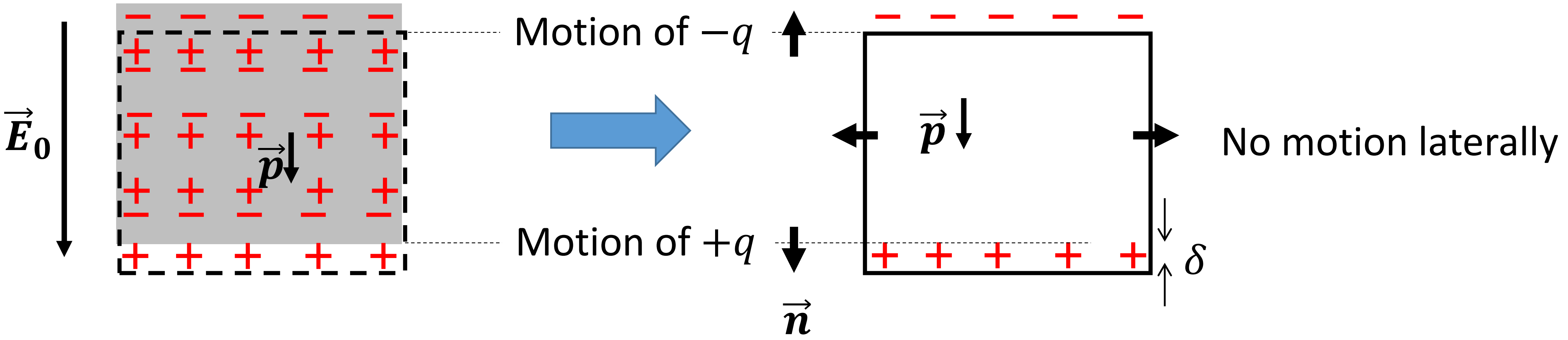
$\varepsilon_r =$ dielectric constant

$$\Delta V = Ed = \frac{E_{free}}{1+\chi} d \Rightarrow C_{diel} = \frac{A(1+\chi)\varepsilon_0}{d} \Rightarrow C_0 = \frac{A\varepsilon_0}{d}$$

See previous slide

$\chi = 0$ if vacuum

Is $\sigma_b = P$ a coincidence or does it have a physical meaning?



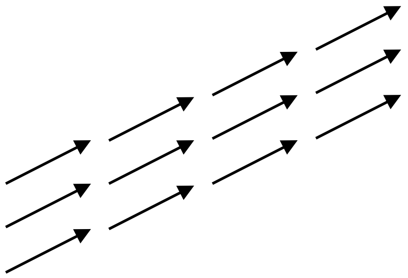
$$\sigma_b = \vec{P}.\vec{n}$$

Polarization is a **VECTOR**:

1) It is considered **UNIFORM** when

$$|\vec{P}| = \text{constant}$$

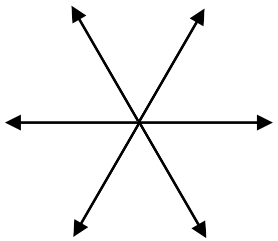
$\vec{P}$ has a unique direction



2) It is considered **NONUNIFORM** when

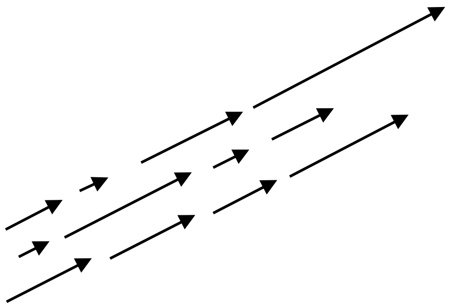
$$|\vec{P}| = \text{constant}$$

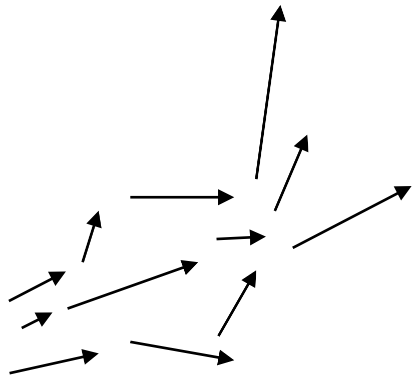
$\vec{P}$ changes direction



$|\vec{P}|$ changes

$\vec{P}$ has unique direction





$|\vec{P}|$ changes

$\vec{P}$ changes direction

Uniform versus non-uniform polarization

No polarization:

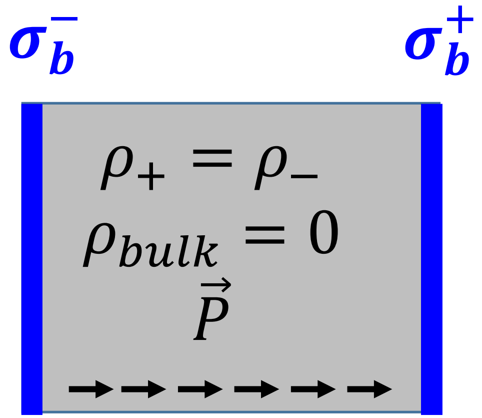
As the dielectric is neutral, charge density is zero since there are equal amount of positive (ρ_+) and negative (ρ_-): Charge conservation

$$\boxed{\begin{gathered} \rho_+ = \rho_- \\ \rho_{bulk} = 0 \\ \vec{P} = \vec{0} \end{gathered}}$$

Uniform versus non-uniform polarization

Uniform polarization:

The negative and positive charges are shifted by the same amount everywhere in the dielectric. Positive and negative **surface** charges appear with equal amount. In the bulk we still have zero **NET** charge

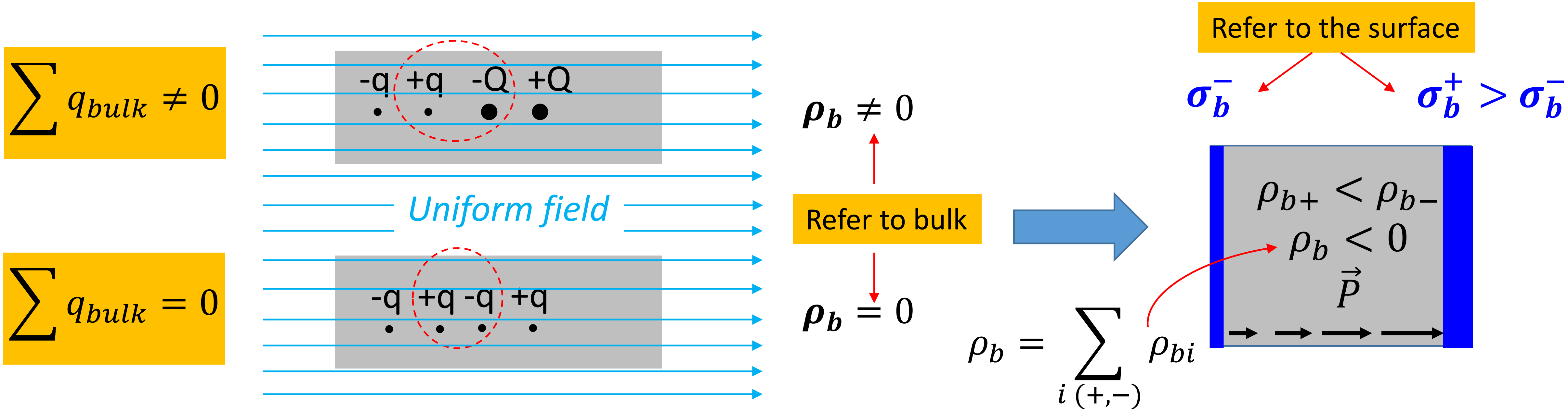


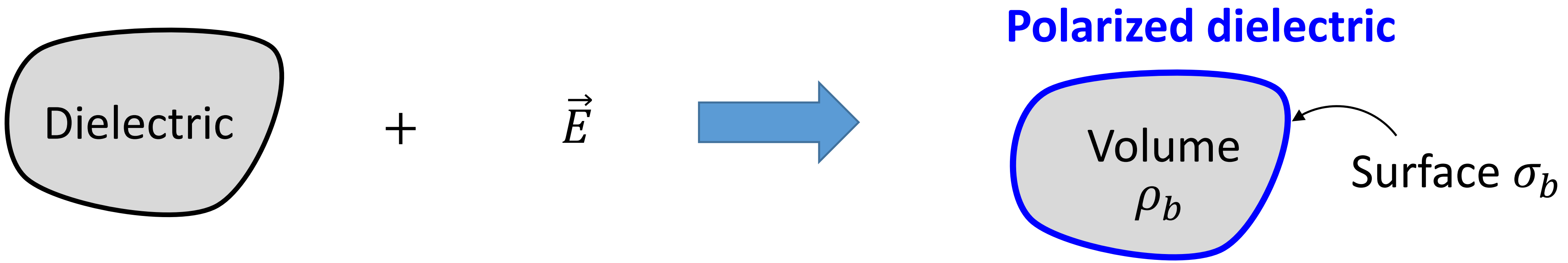
Uniform versus non-uniform polarization

Non-uniform polarization:

In the illustrated example where the polarization increases to the right, positive charges are stretched out and displaced to the right. The positive surface charge density is thus greater on the right than the negative surface charge density accumulating on the left.

A negative charge density develops in the bulk as a consequence of charge conservations law





Polarization induces **BOUND** charges $\Rightarrow$ $\sum$ **BOUND** charges $= 0$ Charge conservation

Neutrality of the dielectric

$$\int \sigma_b dA \begin{cases} = 0 \Rightarrow |+\sigma_b| = |-\sigma_b| \Rightarrow \int \rho_b dV = 0 \Rightarrow \rho_b = 0 \\ > 0 \quad \int \rho_b dV < 0 \\ < 0 \quad \int \rho_b dV > 0 \end{cases}$$

Polarization in the bulk

Charge conservation

$$\oint_A \boldsymbol{\sigma_b} dA = -\int_V \boldsymbol{\rho_P} dV = \underbrace{\oint_A \vec{\boldsymbol{P}}.\vec{\boldsymbol{n}} dA = \int_V \vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{P}}\, dV}_{\text{Gauss theorem}}$$

$$\rho_b = -\vec{\nabla}.\vec{P}$$

If polarization ($\vec{P}$) is uniform $\Rightarrow$ $\rho_b = 0$ $\Rightarrow$ $+\sigma_b$ and $-\sigma_b$ are induced at the surface with

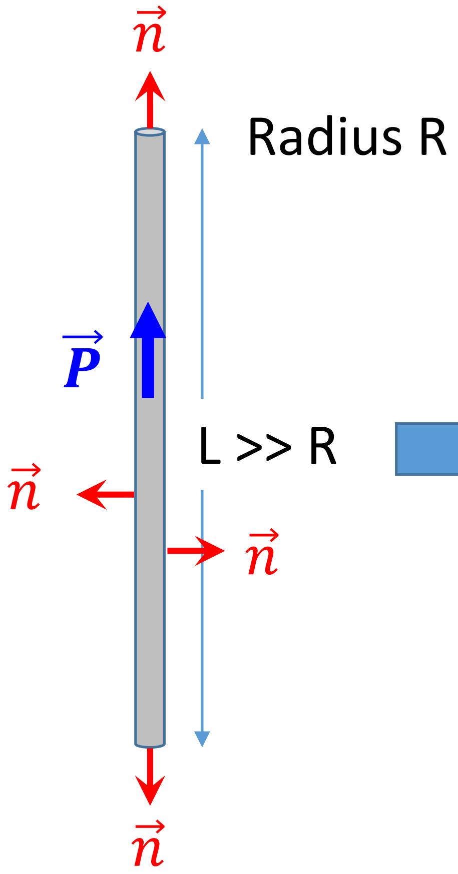
$$\int \sigma_b dA = 0$$

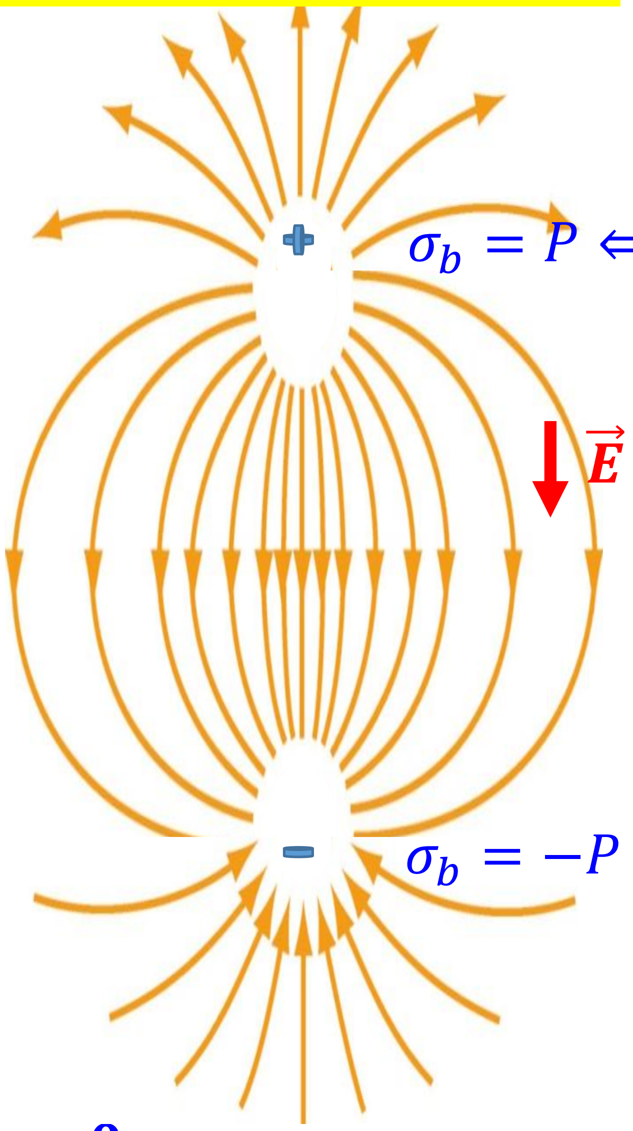
Whether the polarization is uniform or not the total polarization charge is always zero because of:

$$\sigma_b = P \qquad \rho_b = -\vec{\nabla}.\vec{P}$$

Charge conservation

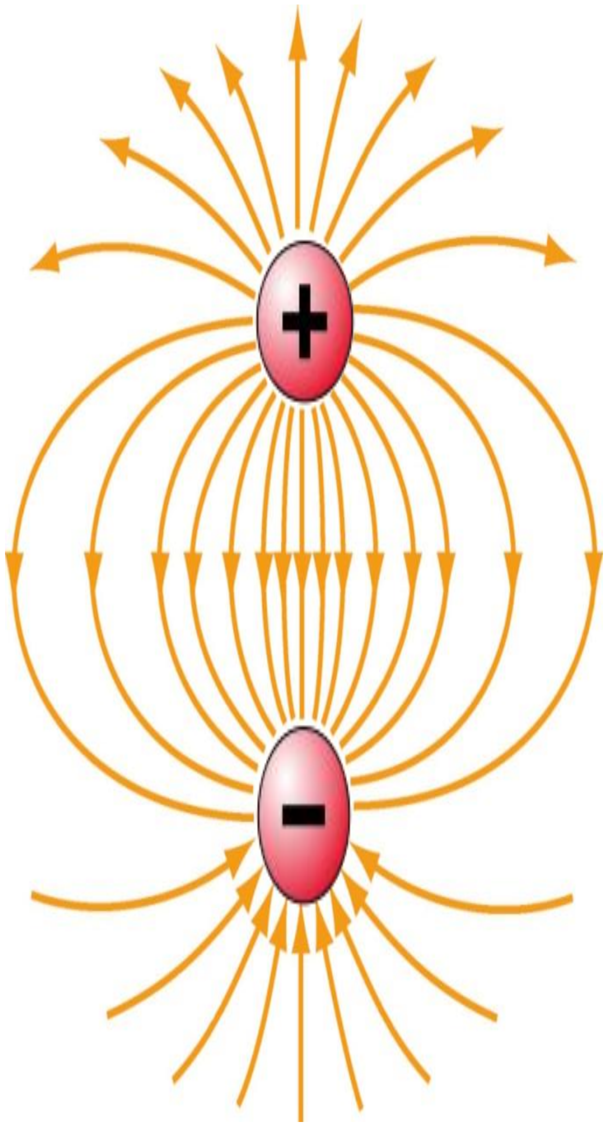
Example 1: Uniform polarization of a dielectric rod





$\sigma_b = P \Leftrightarrow Q = P\pi R^2$

$\sigma_b = -P \Leftrightarrow Q = -P\pi R^2$

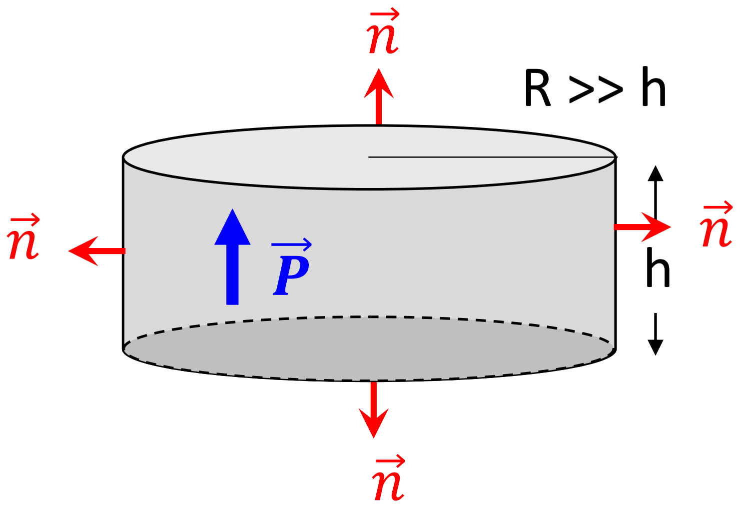


$\vec{\nabla}.\vec{P} = 0$

No bound charge in the bulk $\rho_b = 0$

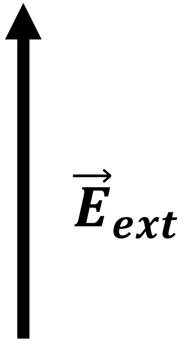
Similarity with to two opposite point charges

Example 2: Uniform polarization of a solid cylindrical dielectric

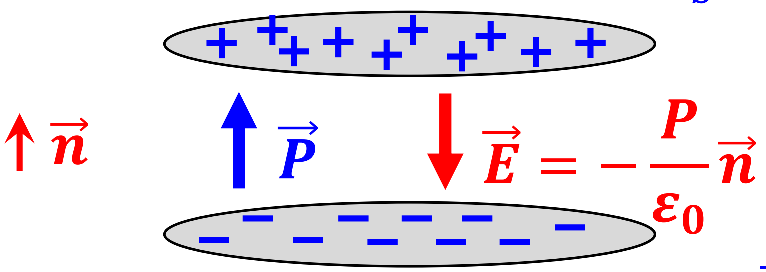


$\vec{\nabla}.\vec{P} = 0$

No bound charge in the bulk $\rho_b = 0$



$\sigma_b = \vec{P}.\vec{n} = P$



Dipole field lines

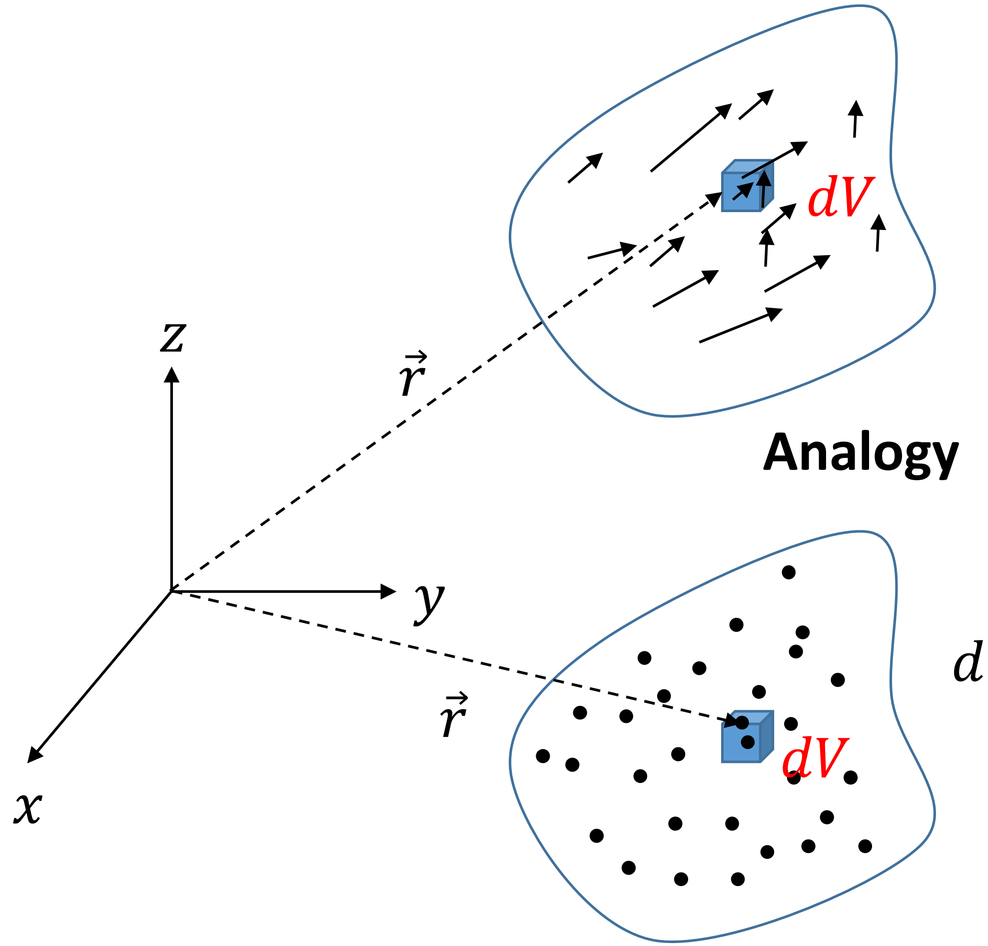
$\sigma_b = \vec{P}.\vec{n} = -P$

Similar to a parallel plate capacitor

Why as for a conductor, a dielectric distorts an initially uniform electric field ?

From superposition principle to bound bulk and surface polarizing charges

Slide #4 in Lectures 5-7_Coordinate system_Scalar versus Vector fields_Operators



$d\vec{p} = \vec{P}(\vec{r})dV$

Polarization density

$dq = \rho(\vec{r})dV$

Charge density

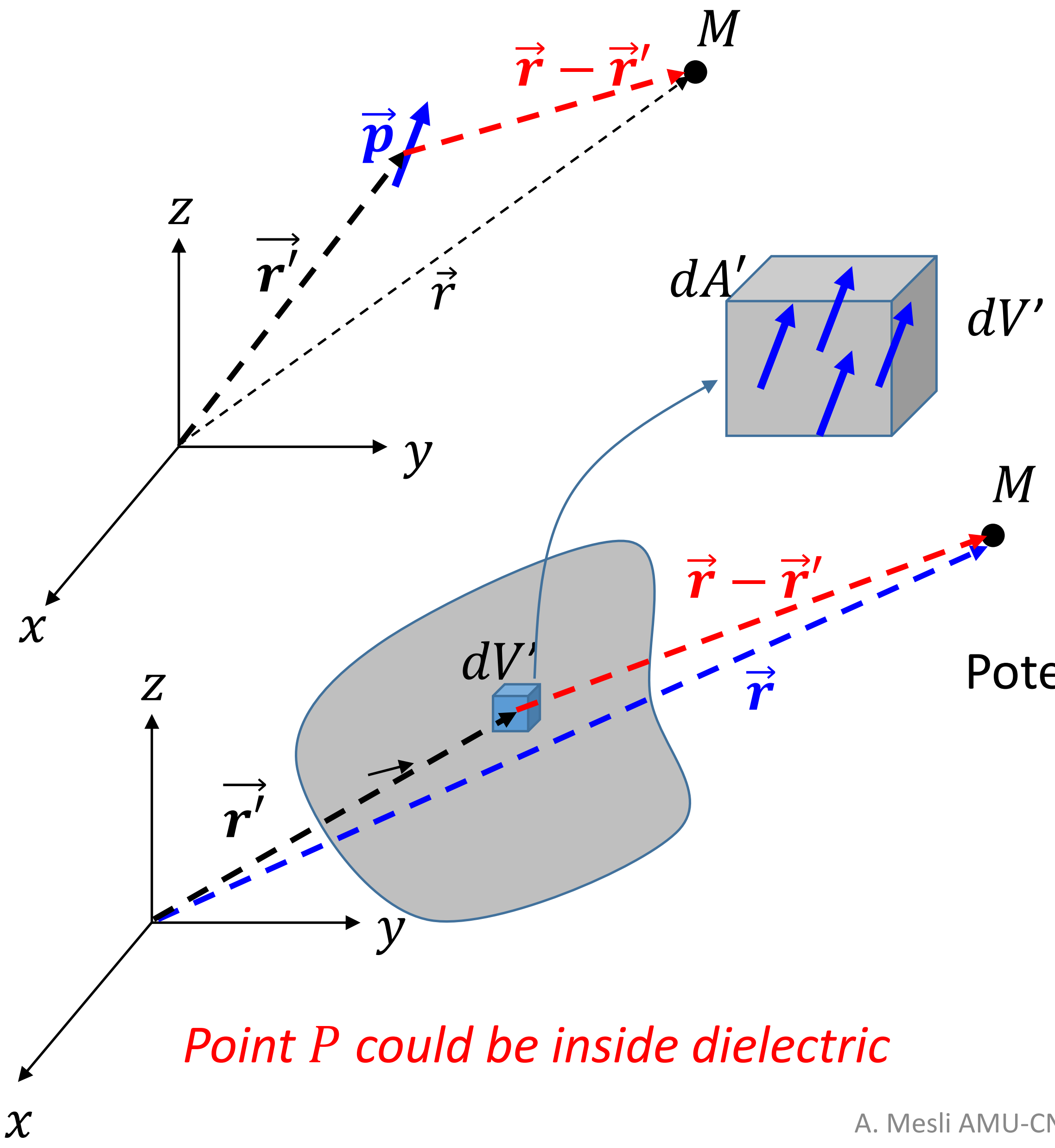
$\vec{\boldsymbol{p}}$ is a **microscopic** property

$\vec{\boldsymbol{P}}(\vec{\boldsymbol{r}})$ is **macroscopic** property

We consider continuous distribution of dipoles

Provided we remain far from the dielectric

From single dipole to a distribution of dipoles



Point P could be inside dielectric

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}.(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\mathbf{d}\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{d}\vec{p}.(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \qquad \mathbf{d}\vec{p} = \vec{P}(\vec{r}')dV'$$

Potential at M by the whole dipole distribution

$$\boxed{\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{P}(\vec{r}').(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'} \qquad \boxed{\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})}$$

Potential created by a polarized dielectric

The variable is $\vec{r}'$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}').(\vec{r}-\vec{r}'\,)}{|\vec{r}-\vec{r}'|^3}dV'$$

$$\frac{(\vec{r}-\vec{r}'\,)}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \underbrace{\vec{P}(\vec{r}')}_{\vec{A}}.\vec{\nabla}'\underbrace{\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}_{f}dV'$$

Divergence Gradient

$$\underbrace{\vec{\nabla}.(f\vec{A})}_{\text{scalar}} = \underbrace{f(\vec{\nabla}.\vec{A})}_{\text{scalar}} + \underbrace{\vec{A}.\vec{\nabla}f}_{\text{scalar}}$$

See D_Lectures 5-7_Coordinate system_Scalarversus Vector fields_Operators

$$\rho_b(\vec{r}') = -\vec{\nabla}'.\vec{P}(\vec{r}')$$

Potential created by a polarized dielectric

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \underbrace{\vec{P}(\vec{r}')}_{\vec{A}}.\vec{\nabla}'\Bigg(\underbrace{\frac{1}{|\vec{r}-\vec{r}'|}}_{f}\Bigg)dV' \qquad \vec{A}.\vec{\nabla}f$$

$$+$$

$$\vec{\nabla}.(f\vec{A}) = f(\vec{\nabla}.\vec{A}) + \vec{A}.\vec{\nabla}f$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{\nabla}'.\left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)dV' - \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\vec{\nabla}'\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV'$$

Gauss's Theorem

$$\rho_b(\vec{r}')$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{A} \frac{1}{|\vec{r}-\vec{r}'|}\sigma_b(\vec{r}')\,dA' + \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{-\vec{\nabla}'.\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV'$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{A} \frac{1}{|\vec{r}-\vec{r}'\,|}\sigma_b(\vec{r}')\,dA' + \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{-\vec{\nabla}'.\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'\,|}dV'$$

Including <u>bound charges</u> in all space

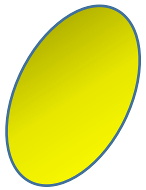
$$\varphi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\vec{\nabla}'.\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'\,|}dV'$$

Including <u>**ALL charges**</u> in the whole space

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{dV'}{|\vec{r}-\vec{r}'\,|}\underbrace{\left[\rho_{free}(\vec{r}') - \vec{\nabla}'.\vec{P}(\vec{r}')\right]}_{\boldsymbol{\rho_{free}} + \boldsymbol{\rho_b}}$$

Initially

Space = $\vec{\boldsymbol{E}}_{\boldsymbol{ext}}$ + dielectric $\leftrightarrow$



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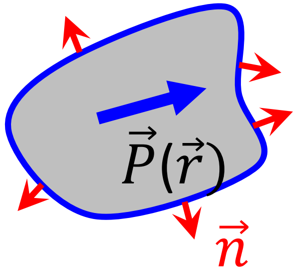
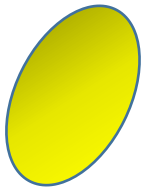


Free charges $\rho_{free}(r)$

Primary source of field

Now: From polarization to charge distribution

Space =



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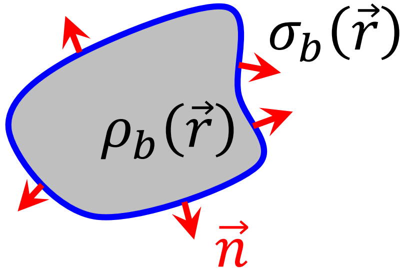


$\leftrightarrow$

Free charges $\rho_{free}(\vec{r})$

Primary source of field

Bound charges

- $\sigma_b(\vec{r}\,') = \vec{P}(\vec{r}\,').\vec{n}$
- $\rho_b(\vec{r}\,') = -\vec{\nabla}'.\vec{P}(\vec{r}\,')$

Secondary source of field

Superposition principle applies to three charge distributions

- $\rho_{free}(\vec{r})$
- $\sigma_b(\vec{r}\,')$
- $\rho_b(\vec{r}\,')$

Consequences of polarization: Strategy solving problems

Electric field + Dielectric $\Rightarrow$ Surface and bulk **BOUND** charges with a **feedback** on the external field

Step 1

Determine the polarization configuration for:

- A given configuration of the applied field $\vec{E}_{ext}(\vec{r})$
- A given shape of the dielectric

Step 2

Remove the applied field and consider the polarized dielectric as a new source:

- Of electric field $\vec{E}_{diel}(\vec{r})$
- Of potential $\varphi_{diel}(\vec{r})$

$\vec{E}_{net}$ and φ_{net}

produced both inside and outside the dielectric

Consequences of polarization: Strategy solving problems

Electric field + Dielectric ⇨ Surface and bulk **BOUND** charges with a **feedback** on the external field

Step 3

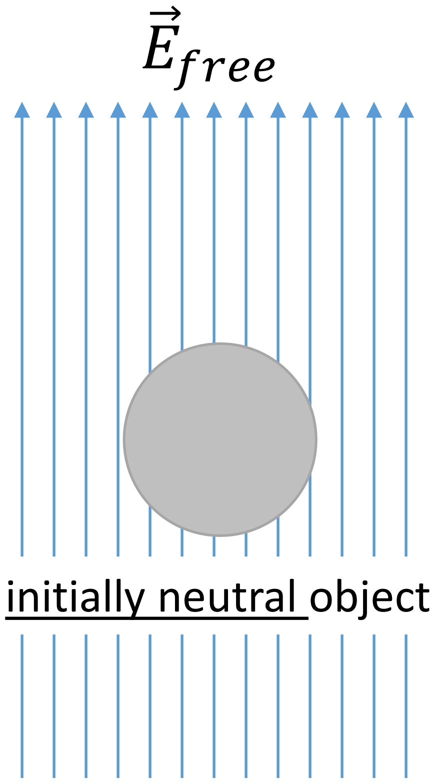
- By superposition principle determine the final distribution of the field = $\sum\left[\vec{E}_{ext}(\vec{r}) + \vec{E}_{diel}(\vec{r})\right]$ in the presence of both the applied field and the one raising from the polarized dielectric

Step 4

- Look at the boundary between the dielectric and the surrounding

Identify – Evaluate – Execute

Case of a dielectric sphere inserted in a uniform electric field: Uniformly polarized dielectric sphere



1) Once in the external field $\vec{\boldsymbol{E}}_{\boldsymbol{free}}$, bound surface and / or bulk charges are induced in the initially neutral object

2) A new field $\vec{\boldsymbol{E}}$ is now generated by the object itself (dipoles):
Resulting field $\vec{\boldsymbol{E}}_{\boldsymbol{free}} + \vec{\boldsymbol{E}}_{\boldsymbol{diel}}$

Consequence:

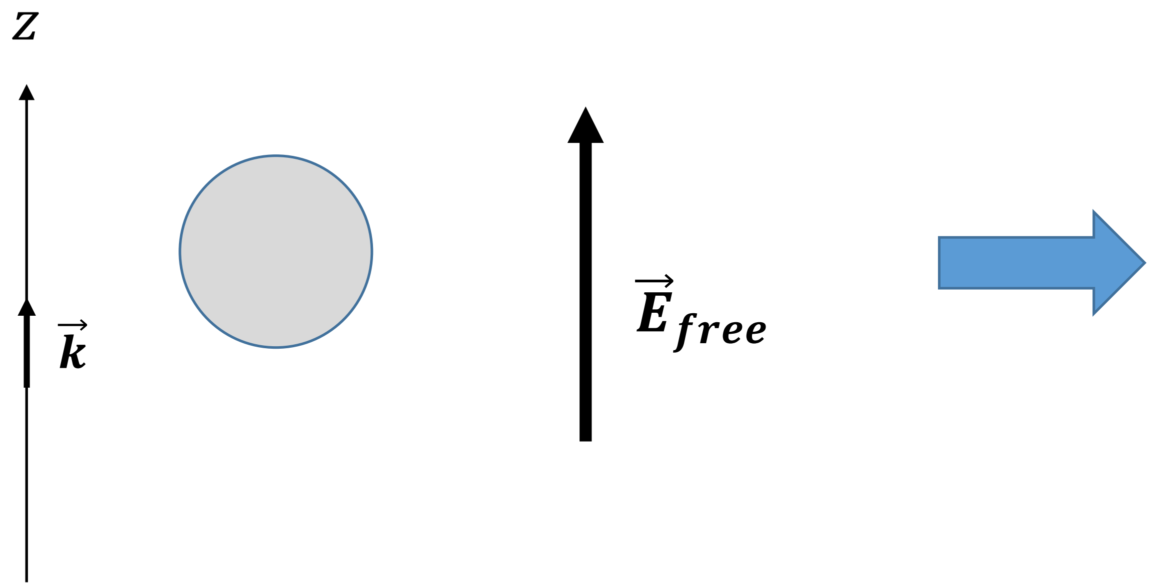
Far away from the sphere, the external field ($\vec{\boldsymbol{E}}_{\boldsymbol{free}}$) is not disturbed, but close to the objet the field is distorted

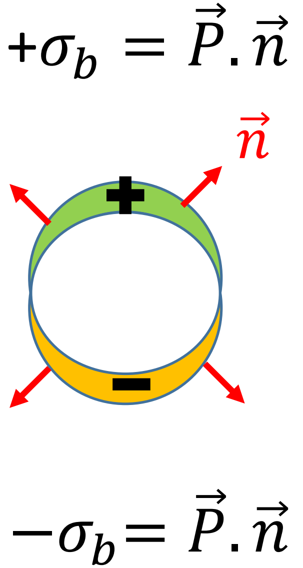
Within any polarizable body placed into an electric field, polarization charge density is induced which, in turn, modifies the electric field

Dipole formation in a dielectric: **Field inside**

A dielectric sphere of radius $\boldsymbol{R}$ is <u>uniformly</u> polarized along the z axis: $\boldsymbol{\vec{P} = P_0 \vec{k}}$

$$\rho_b(\vec{r}\,') = -\vec{\nabla}'.\vec{P}(\vec{r}\,') = 0$$

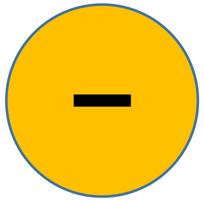




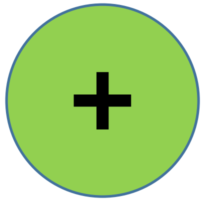
How can we treat this problem?

Dipole formation in a dielectric: **Field inside**

A dielectric sphere of radius $\boldsymbol{R}$ is uniformly polarized along the z axis: $\boldsymbol{\vec{P} = P_0\vec{k}}$



$\rho_- = -\rho$



$\rho_+ = \rho$

$$\rho_- = \frac{Q_-}{\frac{4}{3}\pi R^3}$$

$$\rho_+ = \frac{Q_+}{\frac{4}{3}\pi R^3}$$

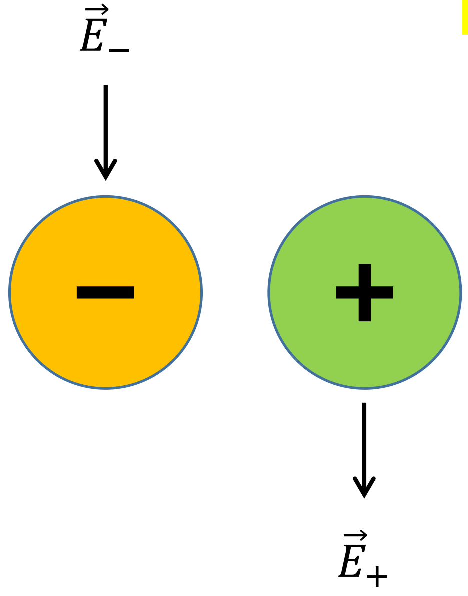
$$\vec{E}_-(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_-}{R^3}\vec{r}$$

$$\vec{E}_+(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_+}{R^3}\vec{r}$$

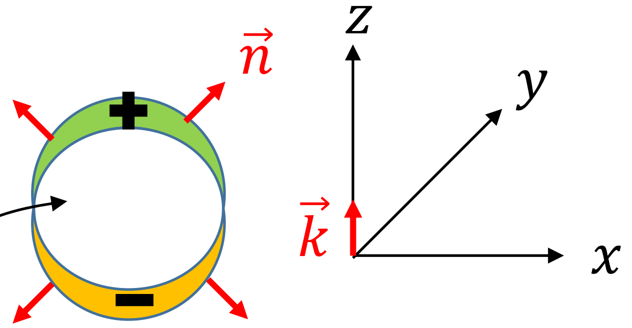
$$\vec{E}_-(\vec{r}) = \frac{\rho_-}{3\varepsilon_0}\vec{r} = \boxed{-\frac{\rho}{3\varepsilon_0}\vec{r}}$$

$$\vec{E}_+(\vec{r}) = \frac{\rho_+}{3\varepsilon_0}\vec{r} = \boxed{\frac{\rho}{3\varepsilon_0}\vec{r}}$$

Dipole formation in a dielectric: **Field inside**

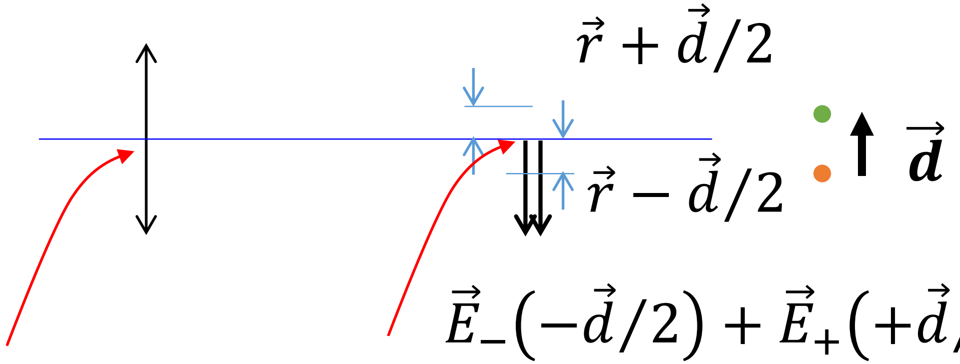


Mixed and Slightly shifted
in opposite z direction



Overlapping volume

Central line of the overlapping volume



$\vec{E}_-(-\vec{d}/2)+\vec{E}_+(+\vec{d}/2)$

$\vec{E}=\vec{0}$
Before shift

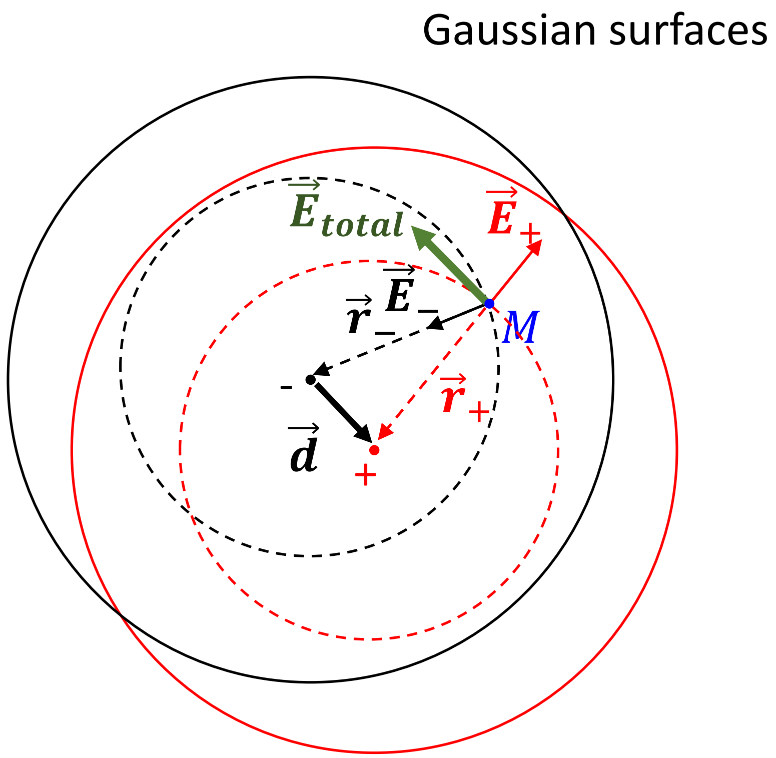
$$\boxed{\vec{E}=-\frac{1}{3\varepsilon_0}\rho d\vec{k}}$$

Easy to demonstrate that this field is the same in the entire overlapping volume

The same result can be obtained in a much easier way

Make use of <u>superposition principle</u> and <u>Gauss' law</u>

$$\vec{r}_- + \vec{d} = \vec{r}_+ \qquad \vec{E}_+ = -\frac{\rho}{3\varepsilon_0}\vec{r}_+ \qquad \vec{E}_- = \frac{\rho}{3\varepsilon_0}\vec{r}_-$$



$$\vec{E}_{total} = -\frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = -\frac{\rho}{3\varepsilon_0}\vec{d}$$

Constant everywhere in the entire **overlapping** volume !

And **ONLY** in the overlapping volume: Why?

If $\vec{d} = d\vec{k}$

$$\boxed{\vec{E}_{total} = -\frac{1}{3\varepsilon_0}\rho d\vec{\boldsymbol{k}}}$$

Identical to result in previous slide !

Dipole formation in a dielectric: **Field inside**

$$\boxed{\vec{\boldsymbol{E}} = -\frac{1}{3\varepsilon_0}\rho d\vec{k}}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \qquad \vec{p} = Q\vec{d}$$

$$\vec{\boldsymbol{E}} = -\frac{1}{3\varepsilon_0}\left(\frac{\vec{p}}{\frac{4}{3}\pi R^3}\right) = -\frac{\vec{P}}{3\varepsilon_0} = -\frac{P_0}{3\varepsilon_0}\vec{\boldsymbol{k}}$$

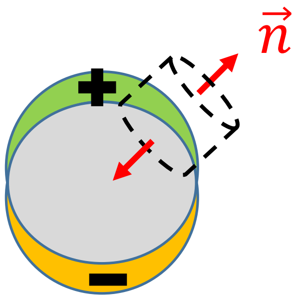
$$\vec{\boldsymbol{P}} = \boldsymbol{P_0}\vec{\boldsymbol{k}}$$

Spherical coordinate

$$\begin{bmatrix} P_r \\ P_\theta \\ P_\phi \end{bmatrix} = \begin{bmatrix} sin\theta cos\phi & sin\theta sin\phi & cos\theta \\ cos\theta cos\phi & cos\theta sin\phi & -sin\theta \\ -sin\theta & cos\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ P_0 \end{bmatrix}$$

$$\vec{P} = P_0 cos\theta \vec{e}_r - P_0 sin\theta \vec{e}_\theta$$

See D_Lectures 5-7_Coordinate system_Scalarversus Vector fields_Operators



Gauss theorem through the pill box

$$\sigma_b = \vec{P}.\vec{n} - \underbrace{\vec{P}_{vac}.\,\vec{n}}_{\substack{\parallel \\ \mathbf{0}}} \qquad \sigma_b = \vec{P}_{diel}.\vec{n}$$

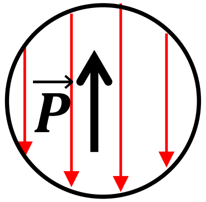
$$\vec{n} \perp \vec{e}_\theta$$

$$\vec{n} \parallel \vec{e}_r$$

$$\sigma_b = P_0 cos\theta$$

 # field lines reduced in the dielectric ⇨ field inside less strong than field outside

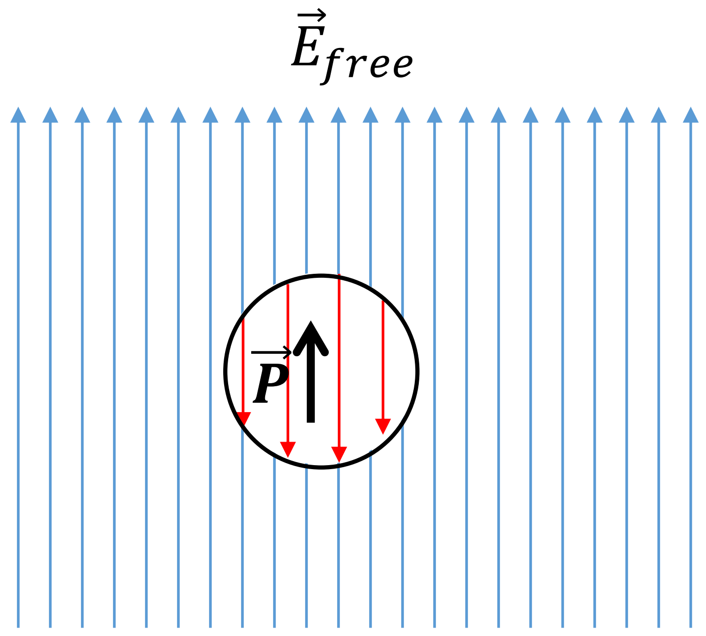
$\vec{P} = P_0 \vec{k}$



$\left|\vec{E}_{ind}\right| < \left|\vec{E}_{free}\right|$

This field is restricted to the interior of the dielectric sphere and arises exclusively from the contribution of the surface bound charges.
<u>It does not include the external applied field used to create dipoles</u>.

The aim of this dipole **induced** field is to reduce the impact of the applied field inside the sphere.



In the presence of the external applied field, the field **NET** $\boldsymbol{\vec{E}}$ **inside** is:

$$\vec{E} = \vec{E}_{free} + \boldsymbol{\vec{E}_{ind}}$$

$$\boxed{\boldsymbol{\vec{E}_{ind}} = -\frac{P_0}{3\varepsilon_0}\boldsymbol{\vec{k}}}$$

$$E = E_{free} - E_{ind}$$

$$E_{ind} = \frac{P_0}{3\varepsilon_0} \qquad E = E_{free} - E_{ind} \qquad E = E_{free} - \frac{P_0}{3\varepsilon_0}$$

$$P_0 = \varepsilon_0 \chi E \qquad \chi = \varepsilon_r - 1 \qquad \boxed{E = \frac{3}{\varepsilon_r + 2} E_{free}}$$

If no dielectric $\varepsilon_r = 1 \quad \rightarrow \quad E = E_{free}$

Polarization is proportional to the field inside ($\boldsymbol{\vec{E}}$)

Dipole formation in a dielectric: **Field inside**

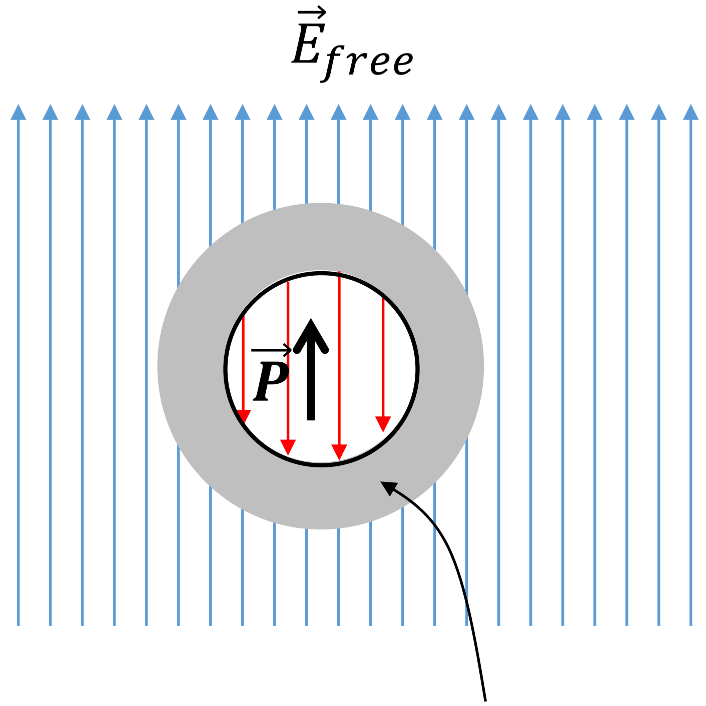
$$\vec{E} = \frac{3}{\varepsilon_r + 2}\vec{E}_{free}$$

For $r \gg \ R$ Vacuum $\varepsilon_r = 1$ $\vec{E} = \vec{E}_{free}$

The field inside is reduced due to **screening**

$$\vec{P} = \varepsilon_0 \chi E = \frac{3\varepsilon_0(\varepsilon_r - 1)}{\varepsilon_r + 2}\vec{E}_{free}$$

Vacuum $\varepsilon_r = 1$ $\vec{P} = \vec{0}$



Area where the field is most distorted

Dipole formation in a dielectric: **Field outside**

The field outside is identical to that generated by a single dipole moment $\vec{p} = \vec{P}.\frac{4}{3}\pi R^3$ $\vec{P} = P_0\vec{k}$

From dipole lecture $\varphi(r > R) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}.\vec{e}_r}{r^2} = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{4}{3}\pi R^3\right)\dfrac{P_0\vec{k}.\vec{e}_r}{r^2}$ $\Rightarrow$ $\boxed{\varphi(r > R) = \dfrac{R^3 P_0 cos\theta}{3\varepsilon_0 r^2}}$

$\vec{E} = -\vec{\nabla}\varphi(r)$ Field outside the dielectric sphere due to polarization

in spherical coordinate

$$E_r(r) = -\frac{\partial\varphi(r)}{\partial r} = \frac{2R^3 P_0 cos\theta}{3\varepsilon_0}\frac{1}{r^3}$$

$$E_\theta(r) = -\frac{1}{r}\frac{\partial\varphi(r)}{\partial\theta} = \frac{R^3 P_0 sin\theta}{3\varepsilon_0}\frac{1}{r^3}$$

$$E_\varphi(r) = 0$$

Dipole formation in a dielectric: **Total Field outside**

Superposition principle → **Total Field outside = Sum of field due to dipole and external field generating that dipole**

External field due to polarization in spherical coordinate

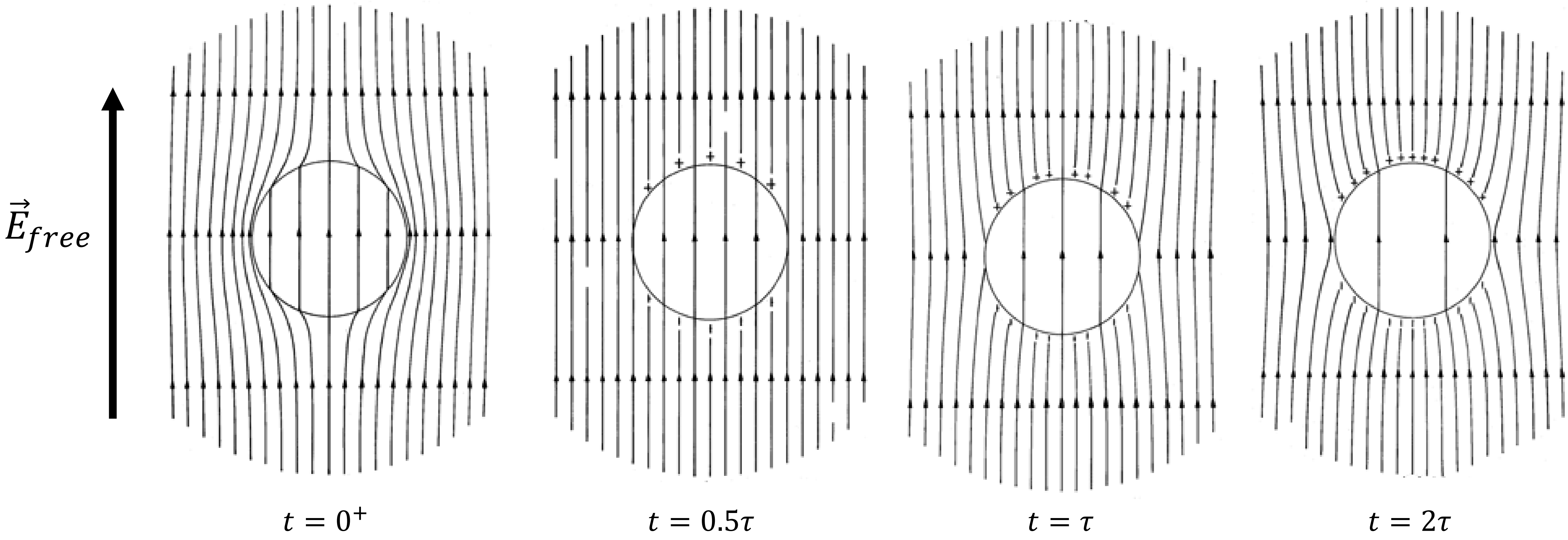
External applied field ($E_0 = E_{free}$) in spherical coordinate

$$E_r(r) = -\frac{\partial\varphi(r)}{\partial r} = \frac{2R^3 P_0 cos\theta}{3\varepsilon_0}\frac{1}{r^3} \quad + \quad E_{0r}(r) = E_0 cos\theta$$

$$E_\theta(r) = -\frac{1}{r}\frac{\partial\varphi(r)}{\partial\theta} = \frac{R^3 P_0 sin\theta}{3\varepsilon_0}\frac{1}{r^3} \quad + \quad E_{0\theta}(r) = -E_0 sin\theta$$

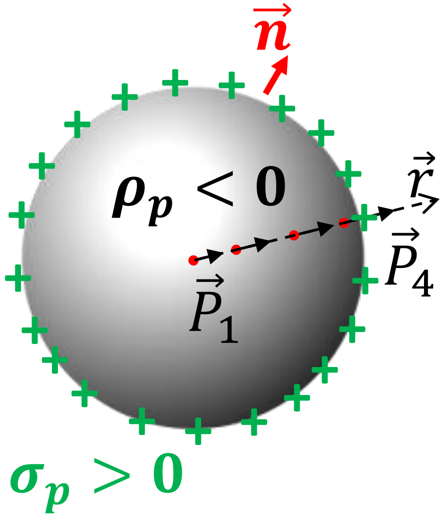
$$E_\varphi(r) = 0 \quad + \quad E_\varphi(r) = 0$$

Time evolution of the distribution of the filed lines around a dielectric after polarization has been initiated



The field strength inside is decreasing

Dielectric sphere radially polarized: **Field outside $r > R$**



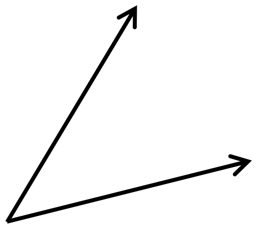
$\vec{P}(\vec{r}) = \beta\vec{r}$

β is constant

Polarization not uniform !

Polarization is a **VECTOR**

$\vec{P}(\vec{r}) = \beta\vec{r}$

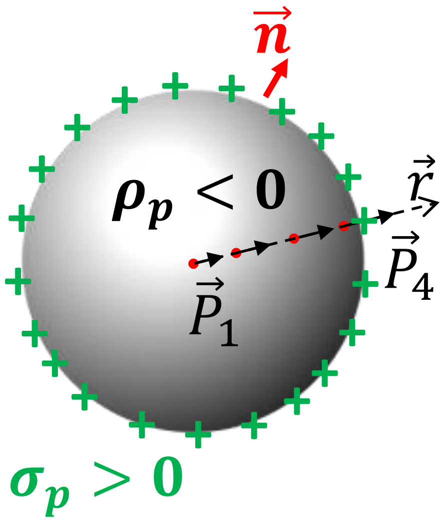


These two vectors are different

Both ρ_b and σ_b $exist$

$\sigma_b > 0 \quad \Rightarrow \quad \rho_b < 0$

Dielectric sphere radially polarized: **Field outside $r > R$**



$$\vec{P}(\vec{r}) = \beta\vec{r}$$

β is constant

$$\sigma_b = \vec{P}.\vec{n} = \beta R$$

$$\rho_b = -\vec{\nabla}.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\beta r) = -3\beta$$

$$Q_s = 4\pi R^2\sigma_b = 4\pi\beta R^3$$

$$Q_v = \frac{4\pi}{3}R^3\rho_b = -4\pi\beta R^3$$

Equivalent to **conducting** sphere
Positively charged on the surface

Equivalent to **non conducting** sphere
Negatively charged in the volume

See E_Lectures 8&9_Electrostatics_Gauss law

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

$$\vec{E}(r) = \frac{\beta R^3}{\varepsilon_0}\frac{\vec{e}_r}{r^2}$$

$$\vec{E}(r) = -\frac{\beta R^3}{\varepsilon_0}\frac{\vec{e}_r}{r^2}$$

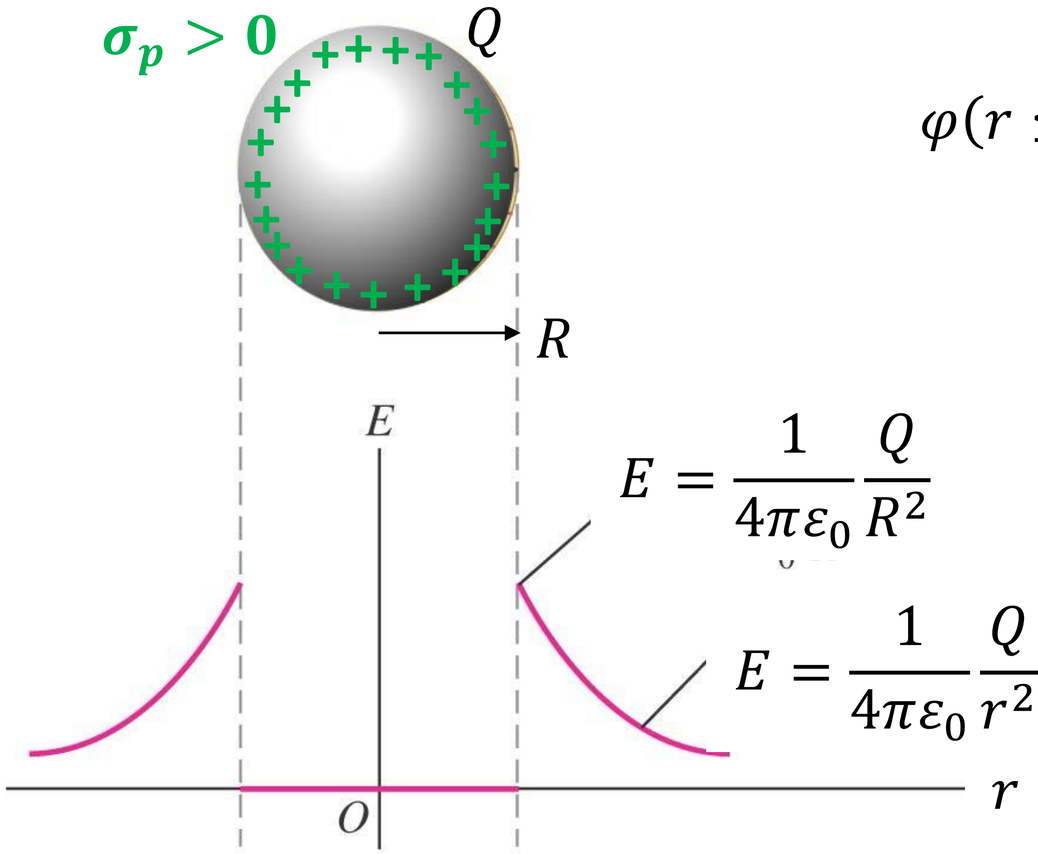
Outside the sphere

Superposition principle: **field outside = 0**

Dielectric sphere radially polarized: **Field inside $r < R$**

Gauss law inside the sphere

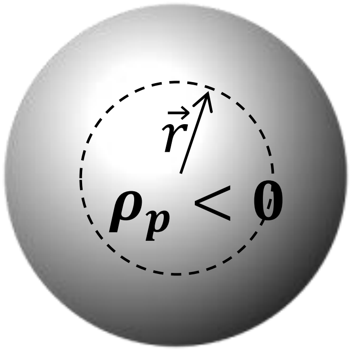
Surface polarization:



field inside = 0

$$\varphi(r \le R) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Volume polarization



$$\vec{E} = -\vec{\nabla}.\varphi(r) \qquad \vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{e}_r$$

$$Q = \frac{4\pi}{3}r^3\rho_b \qquad \vec{E}(r) = -\frac{\beta r}{\varepsilon_0}\vec{e}_r$$

Superposition principle: **field inside** $\boldsymbol{\vec{E}(r) = -\frac{\beta r}{\varepsilon_0}\vec{e}_r}$

Concept of electric displacement $\vec{D}$

What do the first Maxwell equations $\boxed{\vec{\nabla}.\vec{E} = \frac{\rho_{free}}{\varepsilon_0}}$ and $\boxed{\vec{\nabla} \times \vec{E} = 0}$ Become in a dielectric ?

$$\underbrace{\vec{\nabla}.\vec{E} = \frac{\rho_{free}}{\varepsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{E} = 0}_{\text{vacuum}}$$

If there are bound charges due to polarization $\vec{P}$

$$\rho_{free} \rightarrow \rho_{free} + \rho_b$$

With $\rho_b = -\vec{\nabla}.\vec{P}$

Concept of electric displacement $\vec{D}$

$$\vec{\nabla}.\vec{E} = \frac{\rho_{free} + \rho_b}{\varepsilon_0} \Rightarrow \vec{\nabla}.\vec{E} = \frac{\rho_0 - \vec{\nabla}.\vec{P}}{\varepsilon_0} \Rightarrow \vec{\nabla}.\left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right) = \frac{\rho_{free}}{\varepsilon_0}$$

$$\vec{P} = \chi\varepsilon_0\overrightarrow{E}$$

$$(1+\chi) = \varepsilon_r$$

$$\Rightarrow \boxed{\vec{\nabla}.\left(\varepsilon_r\vec{E}\right) = \frac{\rho_{freee}}{\varepsilon_0}}$$

If the dielectric is linear homogeneous and isotropic

$$\boxed{\begin{aligned}\vec{\nabla}.\vec{E} &= \frac{\rho_{free}}{\boldsymbol{\varepsilon_r}\varepsilon_0}\\ \vec{\nabla}\times\vec{E} &= 0\end{aligned}}$$

Relative permittivity

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$$

$\varepsilon \rightarrow$ Permittivity of dielectric

Another way of looking at things

$$\vec{\nabla}.\left(\vec{E}+\frac{\vec{P}}{\varepsilon_0}\right)=\frac{\rho_{free}}{\varepsilon_0}$$

We do not know about the polarization, then we invent a new field:

The electric displacement $\vec{D}$

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$$

$$\boxed{\begin{aligned}\vec{\nabla}.\vec{D}&=\rho_{free}\\ \vec{\nabla}\times\vec{E}&=0\end{aligned}}$$

To solve the system we need a third equation linking $\vec{D}$ to $\vec{E}$

$$P=\chi\varepsilon_0 E$$

If polarization is linear

$$\boxed{\vec{D}=\varepsilon_r\varepsilon_0\vec{E}}$$

This proportionality may break down if $\vec{E}$ is too large

Analogy with Hook's law in mechanics

$$\vec{P}=\vec{D}-\varepsilon_0\vec{E}$$

$$\boxed{\vec{P}=(\varepsilon_r-1)\varepsilon_0\vec{E}}$$

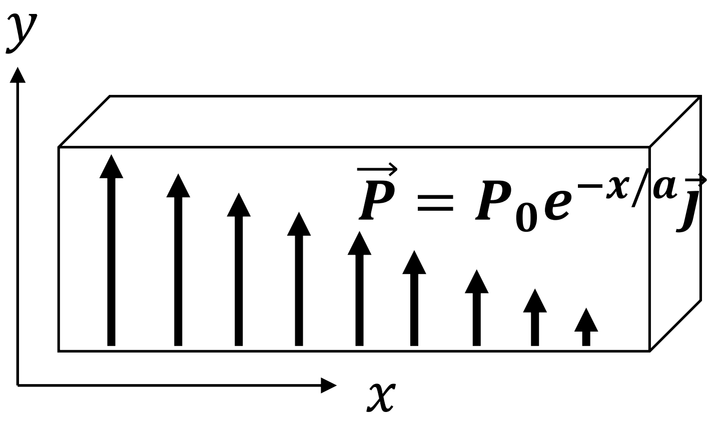
In vacuum $\varepsilon_r=1$ $\vec{P}=\vec{0}$

Why not using $\vec{\nabla}.\vec{D} = \rho_{free}$, $\vec{\nabla} \times \vec{D} = 0$ Instead of using $\vec{\nabla}.\vec{D} = \rho_{free}$, $\vec{\nabla} \times \vec{E} = 0$

Could equation $\vec{\nabla} \times \vec{D} = 0$ hold ?

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{\nabla} \times \vec{D} = \varepsilon_0 \cancel{\vec{\nabla} \times \vec{E}} + \vec{\nabla} \times \vec{P} \Rightarrow \boxed{\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}}$$

electrostatic



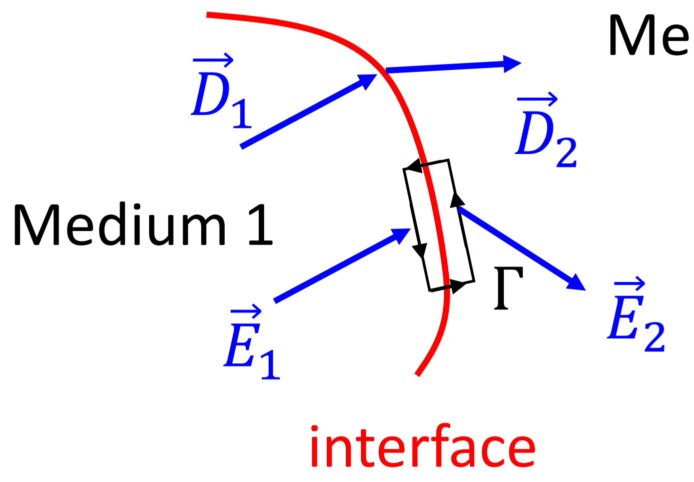
$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & P_0 e^{-x/a} & 0 \end{vmatrix} = -\frac{P_0}{a} e^{-x/a} \vec{k}$$

Clearly $\vec{\nabla} \times \vec{P}$ and thus $\vec{\nabla} \times \vec{D}$ cannot be zero !

Helmholtz's theorem: To know a vector field quantity we need to calculate it **divergence** and **Curl**

Looking at the boundary

Boundary conditions for electrostatic fields



In electrostatic we have three functions

- Potential $\varphi(\vec{r})$
- Electric field $\vec{E}(\vec{r})$
- Electric displacement $\vec{D}(\vec{r})$

How these quantities change in crossing any interface ?

Potential $\varphi(\vec{r})$ MUST be continuous

Boundary conditions

- Potential is **continuous** through the interface $\varphi_1(\vec{r}) = \varphi_2(\vec{r})$ Otherwise, we may have
  - Two values of the field at a single point in space
  - The field $\vec{E} = -\vec{\nabla}.\varphi(r)$ may become infinite

- Stokes theorem

$$\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \oint_{\Gamma} \vec{E} d\vec{l} = \oint_{\Gamma} E_{\parallel} dl = 0$$

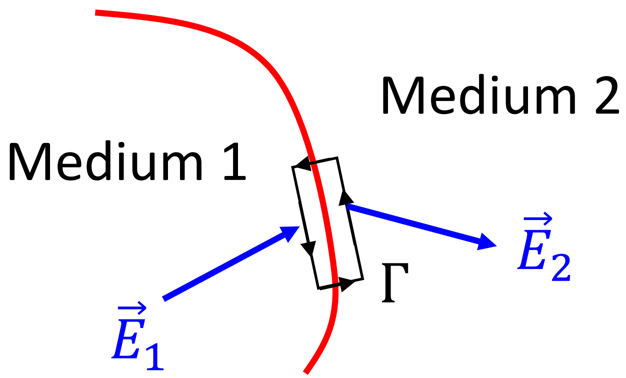
Tangential component of $\vec{E}$ ($E_{\parallel}$) is **always continuous** along any path

- Gauss theorem

$$\vec{\nabla}.\vec{D} = \rho_{free} \Rightarrow \oint \vec{D} d\vec{A} = \oint \vec{D}.\vec{n} dA = \oint D_{\perp} dA = Q_{free}$$

Normal component of $\vec{D}$ ($D_{\perp}$) **may be** discontinuous

Boundary conditions for electrostatic fields



For conductor/vacuum

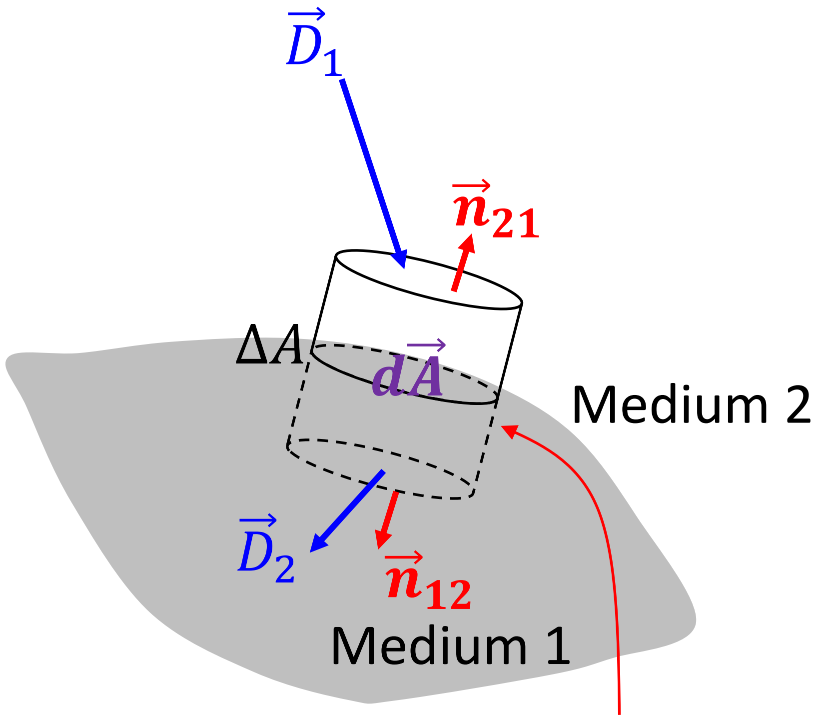
- Inside the conductor: $\rho_{bulk} = 0$, $E = 0$
- At the interface conductor /vacuum: $E_t = 0$, $E = E_\perp = \dfrac{\sigma_{free}}{\varepsilon_0}$

For conductor/vacuum

- Applying Stokes theorem to path Γ for the electric field crossing the interface: $E_{t1} = E_{t2}$
- If one of the medium is a conductor: $E_{t1} = E_{t2} = 0$
- If both media are dielectrics: $\dfrac{D_{t1}}{\varepsilon_1} = \dfrac{D_{t2}}{\varepsilon_2}$ With $\varepsilon_i = \varepsilon_{ir}\varepsilon_0$

What about the normal component $D_\perp$?

Boundary condition for normal component of electric displacement



Gaussian surface = Pillbox

Gauss law $\oint \vec{D}.d\vec{A} = (\vec{D}_1.\vec{n}_{21} + \vec{D}_2.\vec{n}_{12})\Delta A$

$= \vec{n}_{21}(\vec{D}_1 - \vec{D}_2)\Delta A =$ Charge inside the Gaussian surface

The pillbox being reduced to infinitesimal dimensions, the charge Q is the interface charge Q_s between the two media

$$\vec{n}_{21}(\vec{D}_1 - \vec{D}_2)\Delta A = \sigma_s \Delta A$$

$$D_{1\perp} - D_{2\perp} = \sigma_s$$

$\sigma_s$introduces a discontinuity of the normal component of the electric displacement

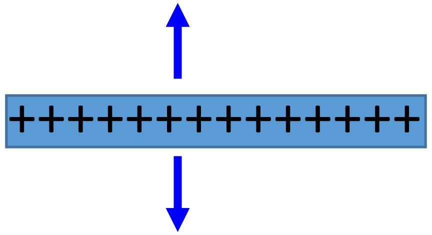
$$D_{1\perp} - D_{2\perp} = \sigma_s$$

If medium 2 is a conductor $D_{1\perp} = \varepsilon_1 E_{1\perp} = \sigma_{free}$ In the conductor $E_2 = 0$ $\Rightarrow D_2 = D_{2\perp} = 0$

And if medium 1 is vacuum $E_{1\perp} = \dfrac{\sigma_{free}}{\varepsilon_0}$

Caution ! *If the conductor is a plate or sheet then we have three media*

medium 2: vacuum



medium 3: conductor

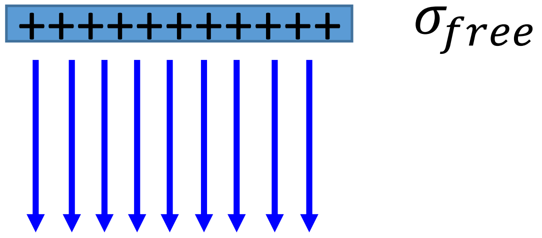
medium 1: vacuum

$$E_{1\perp} = \frac{\sigma_{free}}{\mathbf{2}\varepsilon_0}$$

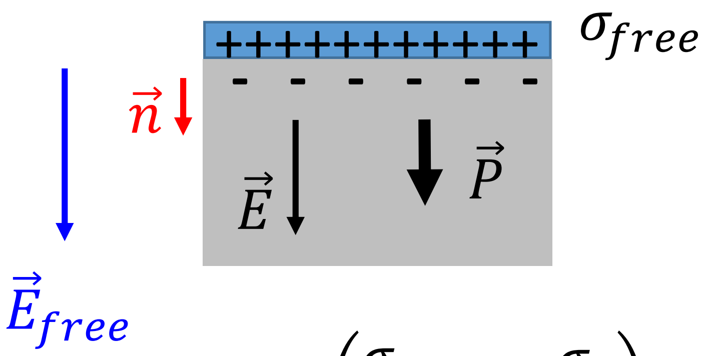
Continuity of normal component of $\vec{D}$ if $\sigma_{free} = 0$…

… which is impossible if there is polarization, whether uniform or not

Charged conductor in contact with a dielectric



$$\vec{E}_{free} = \frac{\sigma_{free}}{\varepsilon_0}$$



$$\sigma_b = \vec{P}.\vec{n}$$

$$\vec{E} = \left(\frac{\sigma_{free}}{\varepsilon_0} - \frac{\sigma_b}{\varepsilon_0}\right)\vec{n}$$

This $\vec{E}$ is **NOT** the induced filed **BUT** the **NET** field

$$\vec{P} = \sigma_b \vec{n}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(\frac{\sigma_{free}}{\varepsilon_0} - \frac{\sigma_b}{\varepsilon_0}\right)\vec{n} + \sigma_b \vec{n}$$

$$\vec{D} = \sigma_{free}\vec{n}$$

- Lines of $\vec{D}$ begin and end **ONLY** at free charges
- Lines of $\vec{E}$ begin and end on either free or bound charges