Midterm 1 Review Chapter 2

1 Products of Vector

1.1 Scalar or Dot Product

$$\vec{A} \cdot \vec{B} = |A||B| \cos \theta_{AB}$$

1.2 Vector or Cross Product

$$\vec{A} \times \vec{B} = |A||B| \sin \theta_{AB} \hat{C}$$

where $\hat{C} = \hat{A} \times \hat{B}$.

1.3 Product of Three Vectors

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C} \cdot (\vec{A} \cdot \vec{B})$$

2 Orthogonal Coordinate Systems

General Expression

$$d\boldsymbol{l} = \boldsymbol{a}_{u_1} dl_1 + \boldsymbol{a}_{u2} dl_2 + \boldsymbol{a}_{u_3} dl_3$$
$$dl_i = h_i du_i$$

where h_i is metric coefficient.

$$d\boldsymbol{s} = \boldsymbol{a}_n ds$$

where $\boldsymbol{a}_n$ is the unit vector perpendicular to the surface.

$$ds_1 = h_2 h_3 du_2 du_3$$
$$ds_2 = h_1 h_3 du_1 du_3$$
$$ds_3 = h_1 h_2 du_1 du_2$$
$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

2.1 Cartesian Coordinates

Base vectors are $\boldsymbol{a}_x$, $\boldsymbol{a}_y$, $\boldsymbol{a}_z$.

$$h_1 = 1$$
$$h_2 = 1$$
$$h_3 = 1$$

2.2 Cylindrical Coordinates

Base vectors are $\boldsymbol{a}_r$, $\boldsymbol{a}_\phi$, $\boldsymbol{a}_z$.

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = 1$$

2.3 Spherical Coordinates

Base vectors are $\boldsymbol{a}_R$, $\boldsymbol{a}_\theta$, $\boldsymbol{a}_\phi$.

$$h_1 = 1$$
$$h_2 = R$$
$$h_3 = R\sin\theta$$

Pay attention to θ and ϕ.

2.4 Coordinate Transformation

Cartesian to Cylindrical

$$r = \sqrt{x^2+y^2}, \phi = \arctan\frac{y}{x}, z = z$$

Cylindrical to Cartesian

$$x = r\cos\phi, y = r\sin\phi, z = z$$

Cartesian to Spherical

$$R = \sqrt{x^2+y^2+z^2}, \theta = \arctan\frac{\sqrt{x^2+y^2}}{z}, \phi = \arctan\frac{y}{x}$$

Spherical to Cartesian

$$x = R\sin\theta\cos\phi, y = R\sin\theta\sin\phi, z = R\cos\theta$$

3 Integrals Containing Vector Functions

$$\int_V \boldsymbol{F}dv$$
$$\int_C V d\boldsymbol{l} = \boldsymbol{a}_x\int_C V dx + \boldsymbol{a}_y\int_C V dy + \boldsymbol{a}_z\int_C V dz$$
$$\int_C \boldsymbol{F}\cdot d\boldsymbol{l}$$
$$\int_S \boldsymbol{A}\cdot d\boldsymbol{s} = \int_S \boldsymbol{A}\cdot\boldsymbol{a}_n ds$$

If S is a closed surface, $\boldsymbol{a}_n$ is in the outward direction.
If S is an open surface, $\boldsymbol{a}_n$ is decided by right-hand rule.
See Ex 2-13, 2-14 and 2-15 in textbook.

4 Gradient of a Scalar Field

$$\nabla V = \boldsymbol{a}_{u_1}\frac{\partial V}{h_1\partial u_1} + \boldsymbol{a}_{u_2}\frac{\partial V}{h_2\partial u_2} + \boldsymbol{a}_{u_3}\frac{\partial V}{h_3\partial u_3}$$
$$\nabla \equiv \left(\boldsymbol{a}_{u_1}\frac{\partial}{h_1\partial u_1} + \boldsymbol{a}_{u_2}\frac{\partial}{h_2\partial u_2} + \boldsymbol{a}_{u_3}\frac{\partial}{h_3\partial u_3}\right)$$

4.1 Integrating Gradients Along on Open Path

$$W = \int_L \nabla V \cdot d\boldsymbol{l} = \int_L \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = V(r_f) - V(r_i)$$

A gradient function over any closed integral vanishes. For example electric potential in a static electric field.

5 Divergence of a Vector Field

Flux: A flow, $\vec{A} \cdot d\vec{S}$.
Flux density: $\frac{\vec{A} \cdot \hat{n} dS}{dS}$. Divergence is the net outward flux of a per unit volume as the volume about the point tends to zero.

$$div\vec{A} = \lim \frac{\oint \vec{A} \cdot d\vec{S}}{\Delta V}$$

$$div\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

6 Divergence Theorem

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume (Flux density $\times$ volume=Flux).

$$\int_V \nabla \cdot \boldsymbol{A} dv = \oint_S \boldsymbol{A} \cdot d\boldsymbol{s}$$

7 Curl of a Vector Field

Circulation: $\oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$.
Curl is the circulation per unit area.

$$\nabla \times \boldsymbol{A} = \lim \frac{1}{\Delta s}\left(\boldsymbol{a}_n \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}\right)$$

$$\nabla \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \boldsymbol{a}_{u_1} h_1 & \boldsymbol{a}_{u_2} h_2 & \boldsymbol{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

8 Stoke's Theorem

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface (Circulation density $\times$ area=Circulation).

$$\int_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$$

9 Two Null Identities

$$\nabla \times (\nabla V) = 0$$

If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.

$$\nabla \cdot (\nabla \times A) = 0$$

If a vector field is divergenceless, then it can be expressed as the curl of another vector field.

10 Combination of Vector Operators: Laplacian

$$\nabla \cdot \nabla V = \nabla^2 V$$

In Cartesian,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

In Cylindrical,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In Spherical,

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

11 Helmholtz Theorem

- Solenoidal and irrotational:
  $\nabla \cdot \boldsymbol{F} = 0$ and $\nabla \times \boldsymbol{F} = 0$
  Example: a static electric field in a charge-free region
- Solenoidal but not irrotational:
  $\nabla \cdot \boldsymbol{F} = 0$ and $\nabla \times \boldsymbol{F} \neq 0$
  Example: a steady magnetic field in a current-carrying conductor
- Irrotational but not solenoidal:
  $\nabla \times \boldsymbol{F} = 0$ and $\nabla \cdot \boldsymbol{F} \neq 0$
  Example: a static electric field in a charged region
- Neither solenoidal nor irrotational:
  $\nabla \cdot \boldsymbol{F} \neq 0$ and $\nabla \times \boldsymbol{F} \neq 0$
  Example: an electric field in a charged medium with a time-varying magnetic field

Helmholtz Theorem: A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.
See Ex2-23 in textbook.

$$\nabla\left(f(\vec{r})g(\vec{r})\right) = f\nabla g + g\nabla f$$

$$\nabla \cdot \left(f(\vec{r})\vec{G}(\vec{r})\right) = f\,\nabla \cdot \vec{G} + \vec{G} \cdot \nabla f$$

$$\nabla \times \left(f(\vec{r})\vec{G}(\vec{r})\right) = f\,\nabla \times \vec{G} + \nabla f \times \vec{G}$$

$$\nabla \cdot \left(\vec{F}(\vec{r}) \times \vec{G}(\vec{r})\right) = \vec{G} \cdot \nabla \times \vec{F} - \vec{F} \cdot \nabla \times \vec{G}$$

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla \cdot (\psi\mathbf{A}) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\psi$$

$$\nabla \times (\psi\mathbf{A}) = \psi(\nabla \times \mathbf{A}) + \nabla\psi \times \mathbf{A}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla\psi) = \mathbf{0}$$

$\nabla \cdot (\nabla\psi) = \nabla^2\psi$ (scalar Laplacian)

$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2\mathbf{A}$ (vector Laplacian)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

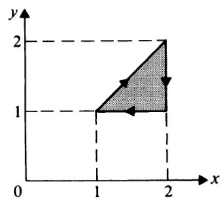

FIGURE 2–36
Graph for Problem P.2–34.

Practice

1

A vector field $\boldsymbol{D} = \boldsymbol{a}_R(\cos^2\phi)/R^3$ exists in the region between two spherical shells defined by $R = 1$ and $R = 2$. Evaluate
a). $\oint \boldsymbol{D} \cdot d\boldsymbol{s}$
b). $\int \nabla \cdot \boldsymbol{D} dv$

2

Assume the vector function $\boldsymbol{A} = \boldsymbol{a}_x 3x^2y^3 - \boldsymbol{a}_y x^3y^2$.
a) Find $\oint \boldsymbol{A} \cdot d\boldsymbol{l}$ around the triangular contour shown in Fig. 2-36.
b) Evaluate $\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s}$ over the triangular area.
c) Can $\boldsymbol{A}$ be expressed as the gradient of a scalar? Explain.

Solution

1

a)

When $R = 1$,

$$d\boldsymbol{s} = -\boldsymbol{a}_R(R^2 \sin\theta) d\theta d\phi$$

When $R = 2$

$$d\boldsymbol{s} = \boldsymbol{a}_R(R^2 \sin\theta) d\theta d\phi$$

$$\begin{aligned}
\oint \boldsymbol{D} \cdot d\boldsymbol{s} &= \int_{R=2} \frac{\cos^2\phi}{R} \sin\theta d\theta d\phi - \int_{R=1} \frac{\cos^2\phi}{R} \sin\theta d\phi \\
&= -\frac{1}{2} \int_0^{2\pi} \cos^2\phi d\phi \int_0^{\pi} \sin\theta d\theta \\
&= -\pi
\end{aligned}$$

b)

$$\begin{aligned}
\nabla \cdot \boldsymbol{D} &= \frac{1}{R^2} \frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta} \frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta} \frac{\partial A_\phi}{\partial\phi} \\
&= \frac{\partial}{\partial R}\left(\frac{\cos^2\phi}{R^3}\right) + 0 + 0 \\
&= -\frac{3\cos^2\phi}{R^4}
\end{aligned}$$

$$dv = R^2 \sin\theta dR d\theta d\phi$$

$$\begin{aligned}
\int \nabla \cdot \boldsymbol{D} dv &= \int_0^{\pi} \int_0^{2\pi} \int_1^2 -\frac{3\cos^2\phi \sin\theta}{R^2} dR d\phi d\theta \\
&= -3 \cdot 2 \cdot \pi \cdot \frac{1}{2} \\
&= -3\pi
\end{aligned}$$

2

a)

$$d\boldsymbol{l} = \boldsymbol{a}_x dx + \boldsymbol{a}_y dy$$

$$\boldsymbol{A} \cdot d\boldsymbol{l} = 3x^2y^3 dx - x^3y^2 dy$$

$$\oint \boldsymbol{A} \cdot d\boldsymbol{l} = \int_A^B \boldsymbol{A} \cdot d\boldsymbol{l} + \int_B^C \boldsymbol{A} \cdot d\boldsymbol{l} + \int_C^A \boldsymbol{A} \cdot d\boldsymbol{l}$$

From A to B, $y = x$

$$\int_A^B \boldsymbol{A} \cdot d\boldsymbol{l} = \int_A^B 3x^2y^3 dx - x^3y^2 dy = \int_1^2 2x^5 dx = 21$$

From B to C, $x = 2$

$$\int_B^C \boldsymbol{A} \cdot d\boldsymbol{l} = \int_2^1 3(2^3)y^2(0) - 2^3y^2 dy = \int_2^1 -8y^2 dy = \frac{56}{3}$$

From C to A, $y = 1$

$$\int_C^A \boldsymbol{A} \cdot d\boldsymbol{l} = \int_2^1 3x^2 dx = -7$$

$$\oint \boldsymbol{A} \cdot d\boldsymbol{l} = \frac{98}{3}$$

b)

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^3 & -x^3y^2 & 0 \end{vmatrix} = -\boldsymbol{a}_z \cdot 12x^2y^2$$

$$d\boldsymbol{s} = -\boldsymbol{a}_z dxdy$$

$$\begin{aligned}
\int_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} &= \int_S 12x^2y^2 dxdy \\
&= 12 \int_1^2 x^2 dx \int_1^x y^2 dy \\
&= 12 \int_1^2 x^2 \left(\frac{x^2}{3} - \frac{1}{3} \right) dx \\
&= 4 \int_1^2 (x^5 - x^2) dx \\
&= 4 \left(\frac{2^6 - 1^6}{6} - \frac{2^3 - 1^3}{3} \right) \\
&= \frac{98}{3}
\end{aligned}$$

c)

Assume that $\boldsymbol{A} = \nabla V$.

$$\frac{\partial V}{\partial x} = 3x^2y^3 \Rightarrow V = x^3y^3 + C_1$$

$$\frac{\partial V}{\partial y} = -x^3y^2 \Rightarrow V = -\frac{x^3y^3}{3} + C_2$$

The functions are not same so V is not a conservative field. And there is no single scalar function to describe a non-conservative field.
Thus, the given vector $\boldsymbol{A}$ cannot be expressed as the gradient of a scalar.