RC for Mid-term 2
Chapter 6

Fundamental Postulates of Magnetostatics in Free Space

Postulates of Magnetostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

- The integral form of **B**: $\boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0,}$

  The total outward magnetic flux through any closed surface is zero.

  – **No magnetic flow sources**

  – The magnetic flux lines always close upon themselves

Ampere's circuital law: the circulation of the magnetic flux density in free space around any closed is equal to μ_0 times the total current flowing through the surface bounded by the path.

Vector Magnetic Potential

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad (\mathrm{T}).}$$

where **A**: **vector** magnetic potential (Wb/m)

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}.$$

Definition of Laplacian of **A**

Vector's Poisson's equation

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

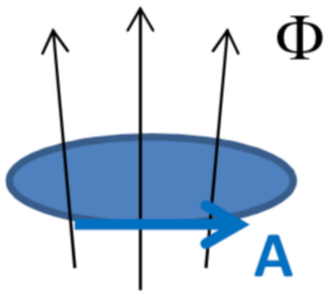
$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv' \qquad (\mathrm{Wb/m}).}$$

Vector Magnetic Potential

- Relation of Magnetic Flux Φ and Magnetic Vector Potential A

$$\Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell} \qquad \text{(Wb).}$$

Physical significance of **A**: line integral of **A** around any closed path = the total Φ passing through the area closed by the path



The Biot-Savart Law and Applications

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \qquad \text{(T)}.$$

Biot-Savart law: **B** due to a current element Idl’

The Magnetic Dipole

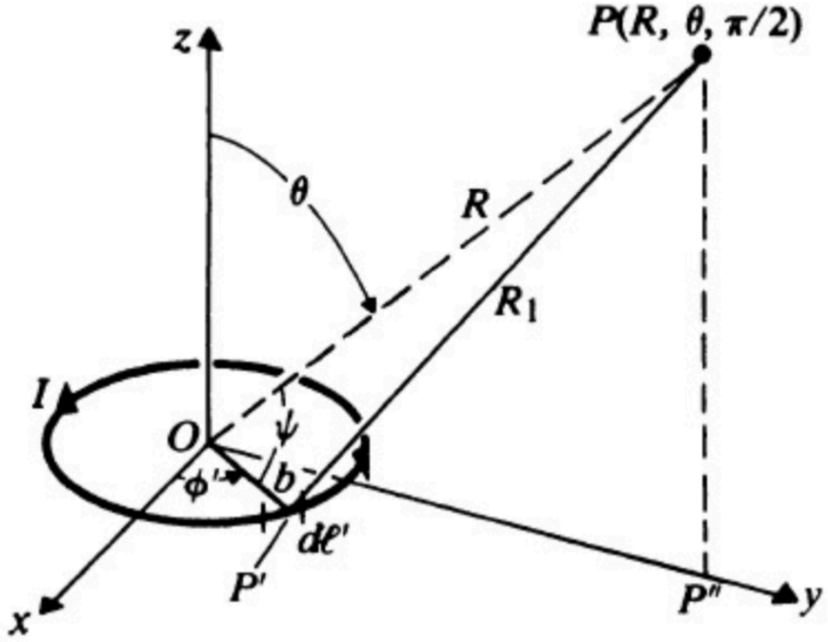


FIGURE 6–8
A small circular loop carrying current I

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin\theta.$$

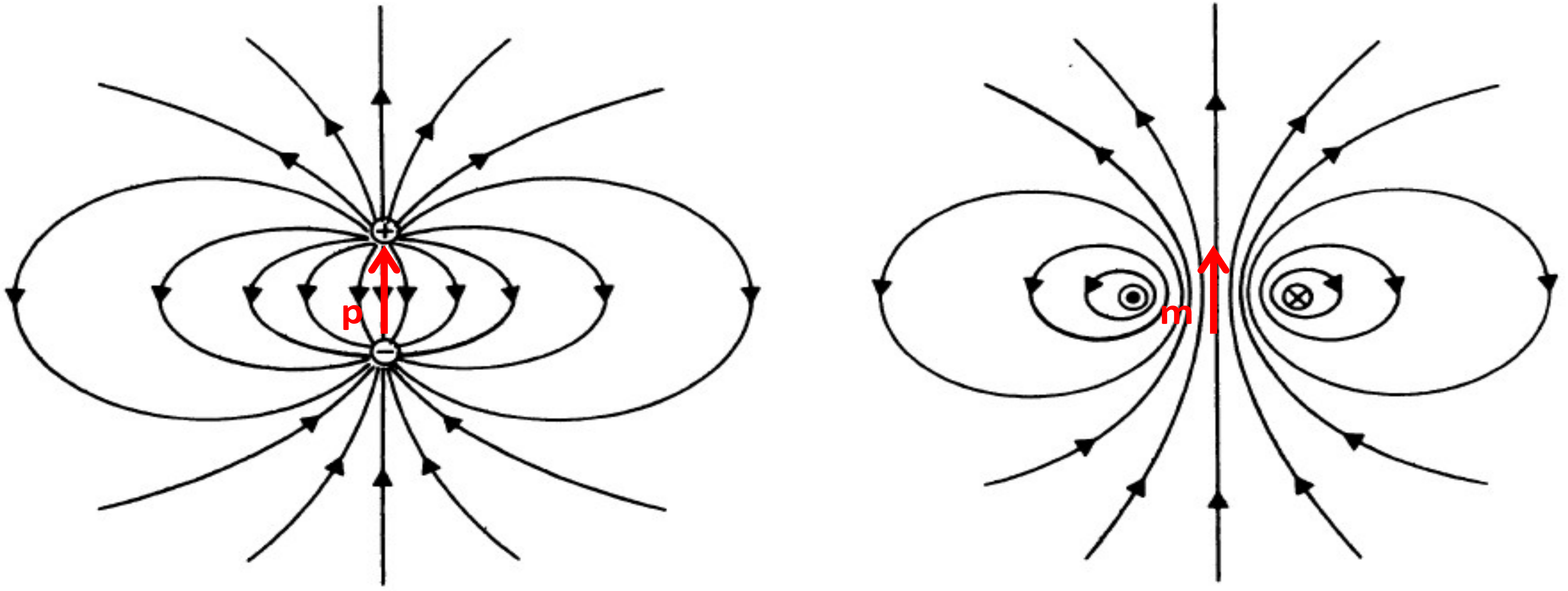
$$\mathbf{B} = \frac{\mu_0 I b^2}{4R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta),$$

$$\boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(Wb/m)},}$$

where $\mathbf{m} = \mathbf{a}_z I\pi b^2 = \mathbf{a}_z IS = \mathbf{a}_z m \qquad (\mathrm{A\cdot m^2})$

Defined as **magnetic dipole moment**

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \text{(T).}}$$



(a) Electric dipole. (b) Magnetic dipole.

Electric dipole moment **p**

Magnetic dipole moment **m**

p->m
$1/\varepsilon_0 \to \mu_0$
$\bullet \to \times$

Right-hand rule: **m** along thumb and **direction of current** along fingers

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad \text{(V)},$$

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(Wb/m)},$$

We call a small current-carrying loop a **magnetic dipole**

Scalar Magnetic Potential

$$\mathbf{B} = -\mu_0 \nabla V_m,$$

V_m: scalar magnetic potential (A)

Magnetization and Equivalent Current Densities

Let $\mathbf{m}_k$: magnetic dipole moment of an atom

Define magnetization vector **M**

n: number density

nΔv=N: total #

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{m}_k}{\Delta v} \qquad (\mathrm{A/m}),$$

M: density of total magnetic dipoles

$$d\mathbf{m} = \mathbf{M}\, dv'$$

$$\boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\mathrm{A/m^2})}$$

$$\boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (\mathrm{A/m}).}$$

Inductance & Inductor

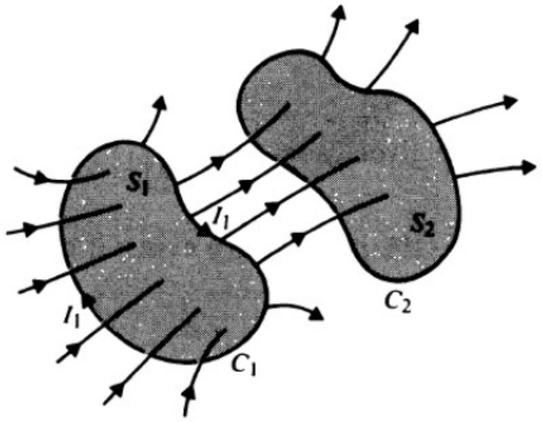


FIGURE 6–22
Two magnetically coupled loops.

I_1 ➔ Φ_1 ➔ part of Φ_1 (Φ_{12}) passes through S_2

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \qquad \text{(Wb)}.$$

** Pay attention!*

General Expression

$$\Lambda_{12} = L_{12} I_1 \qquad \text{(Wb)}$$

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1} \qquad \text{(H)}.}$$

where

$$\Lambda_{12} = N_2 \Phi_{12} \qquad \text{(Wb)},$$

$$\boxed{\Phi \cong BS,}$$

➔ $\Phi \sim S$

Two cases

For linear media, μ is a constant

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \quad \text{(T)}.$$

B ~ I

$$\Phi_{12} = L_{12} I_1,$$

$$L_{12} = \frac{\Lambda_{12}}{I_1} \quad \text{(H)}.$$

For linear media only

For nonlinear media, μ is a function of I

B(I)

Φ_{12}(I)

$$L_{12} = \frac{d\Lambda_{12}}{dI_1} \quad \text{(H)}.$$

In general

Self-incidence

$$L_{11} = \frac{\Lambda_{11}}{I_1} \quad \text{(H)},$$

For linear media only

$$L_{11} = \frac{d\Lambda_{11}}{dI_1} \quad \text{(H)}.$$

In general

*Another way to calculate the self-incidence (see later)

$$L = \frac{2W_m}{I^2} \quad \text{(H)}.$$

Inductor

-just follow the procedure in the lecture slides

The procedure to determine self-inductance of an inductor: **From I to Λ**

- 1. Choose an appropriate coordinate system
- 2. Assume I in the conducting wire
- 3. Find B from I by Ampere's circuital law (for symmetric case) or Biot-Savart law (otherwise)

- 4. Find the flux linkage with each turn, Φ, from **B**

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s},$$

- 5. Find the total flux linkage Λ

$$\Lambda = N\Phi$$

- 6. Find L by $L = \Lambda / I$

The procedure to determine mutual-inductance L_{12}: slight modification

$I_1 \rightarrow B_1 \rightarrow$ **Φ_{12} by integrating B_1 over S_2** $\rightarrow$ $\Lambda_{12} = N_2\Phi_{12} \rightarrow L_{12} = \Lambda_{12}/I_1$

Magnetic Energy

In terms of field quantities

$$\boxed{W_m = \frac{1}{2}\int_{V'} \mathbf{A}\cdot\mathbf{J}\,dv' \qquad \text{(J)},} \longrightarrow \boxed{W_m = \frac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}\,dv' \qquad \text{(J).}}$$

V': the volume of the loop or the linear medium in which J exists

For Linear media, H=B/μ

$$\boxed{W_m = \frac{1}{2}\int_{V'} \frac{B^2}{\mu}\,dv' \qquad \text{(J)}}$$

$$\boxed{W_m = \frac{1}{2}\int_{V'} \mu H^2\,dv' \qquad \text{(J).}}$$

** Pay attention!*

Magnetic energy density

$$w_m = \tfrac{1}{2}\,\mathbf{H} \cdot \mathbf{B} \qquad (\mathrm{J/m^3})$$

$$w_m = \frac{B^2}{2\mu} \qquad (\mathrm{J/m^3})$$

$$w_m = \tfrac{1}{2}\mu H^2 \qquad (\mathrm{J/m^3}).$$

Magnetic Forces & Torques

A magnetic force $\mathbf{F_m}$ on a moving charge q

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}.$$

Two ways of expression

1. In terms of stored magnetic energy
2. In terms of mutual incidence

Forces & Torques on Current-Carrying Conductors

General expression

$$\vec{\mathbf{F}}_{12}=\frac{\mu_0}{4\pi}I_2I_1\oint_{C_2}\oint_{C_1}\frac{d\ell_2\times(d\ell_1\times\mathbf{a}_{R_{12}})}{R_{12}^2}.$$

When solving the problems, we do not always directly substitute the variables in the since sometimes it is too complex. However some simple judgements considering some symmetrical systems can make the problem much easier. (*see eg. 6-21, 6-22*) And for those asymmetrical cases, we may use the methods discussed laster.

A Circular Circuit Carrying Currents

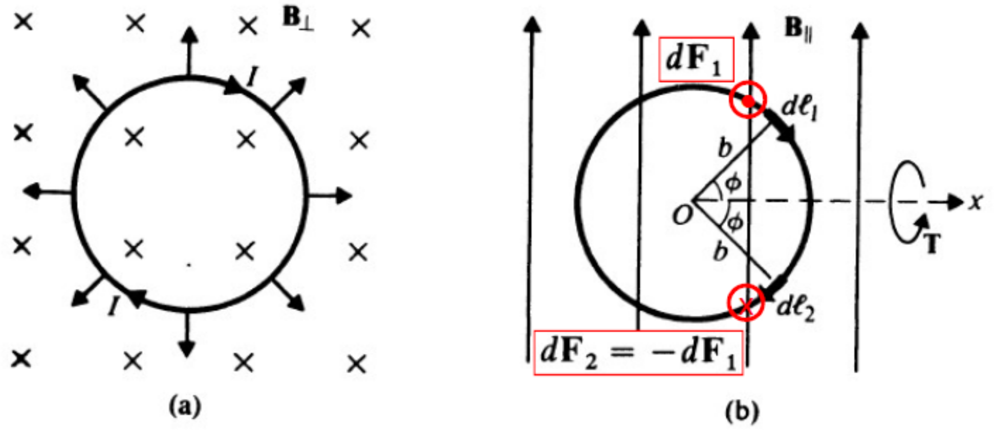


FIGURE 6–30
A circular loop in a uniform magnetic field $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$.

By definition of magnetic dipole moment **m**

$$\boxed{\mathbf{m} = \mathbf{a}_n I(\pi b^2) = \mathbf{a}_n I S,}$$

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\mathrm{N\cdot m}).}$$

**When does the equation hold?*

Forces & Torques in Terms of Stored Magnetic energy

Two cases

1. System of Circuits with **Constant Flux Linkages**

2. System of Circuits with **Constant Currents**

Case 1: System of Circuits with **Constant Flux Linkages**

$$\mathbf{F}_\Phi \cdot d\ell = -dW_m = -(\nabla W_m)\cdot d\ell,$$

$$\boxed{\mathbf{F}_\Phi = -\nabla W_m \qquad \text{(N)}.}$$

In Cartesian coordinate

$$(F_\Phi)_x = -\frac{\partial W_m}{\partial x},$$

$$(F_\Phi)_y = -\frac{\partial W_m}{\partial y},$$

$$(F_\Phi)_z = -\frac{\partial W_m}{\partial z}.$$

For the case with a virtual rotation dϕ about an axis (say z axis)

$$\boxed{(T_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \qquad \text{(N·m)},}$$

Case 1: System of Circuits with **Currents**

$$\boxed{\mathbf{F}_I = \nabla W_m \qquad \text{(N)},}$$

Similar to case 1 except for a sign change

For the case with a virtual rotation dϕ about an axis (say z axis)

$$\boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad \text{(N·m)}.}$$

Forces & Torques in Terms of Mutual Inductance

- Method of virtual displacement (dl) for constant currents is powerful to determine the **F** and **T** between rigid-carrying circuits.
- The magnetic energy of two circuits with currents I_1 and I_2:

$$W_m = \tfrac{1}{2}L_1 I_1^2 + L_{12} I_1 I_2 + \tfrac{1}{2}L_2 I_2^2.$$

$$\boxed{\mathbf{F}_I = \nabla W_m \qquad \text{(N)},}$$

$$\boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad (\text{N}\cdot\text{m}).}$$

L_1 and L_2 remain constants given a virtual displacement dl

$$\boxed{\mathbf{F}_I = I_1 I_2 (\nabla L_{12}) \qquad \text{(N)}.}$$

$$\boxed{(T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi} \qquad (\text{N}\cdot\text{m}).}$$