Coordinate system, Scalar versus Vector fields Operators

- Position of the problem: Coordinate system
- Vector calculus: Basics
- Application to the scalar and vector **fields**

The ultimate goal of manipulating scalar and vector field is to give a **mathematical view** of electricity and magnetism

Explain of Coulomb – Poisson – Biot&Savart – Ampere – Gauss and Maxwell's laws

Position of the problem: Coordinate system

What do we need to evaluate scalar and vector fields at some point in space?

- Localize the source (charge or current distribution) in space
- Define a reference frame
- Localize the point where the effect of the source is to be evaluated
- Make use of **symmetry** if necessary to simplify the problem

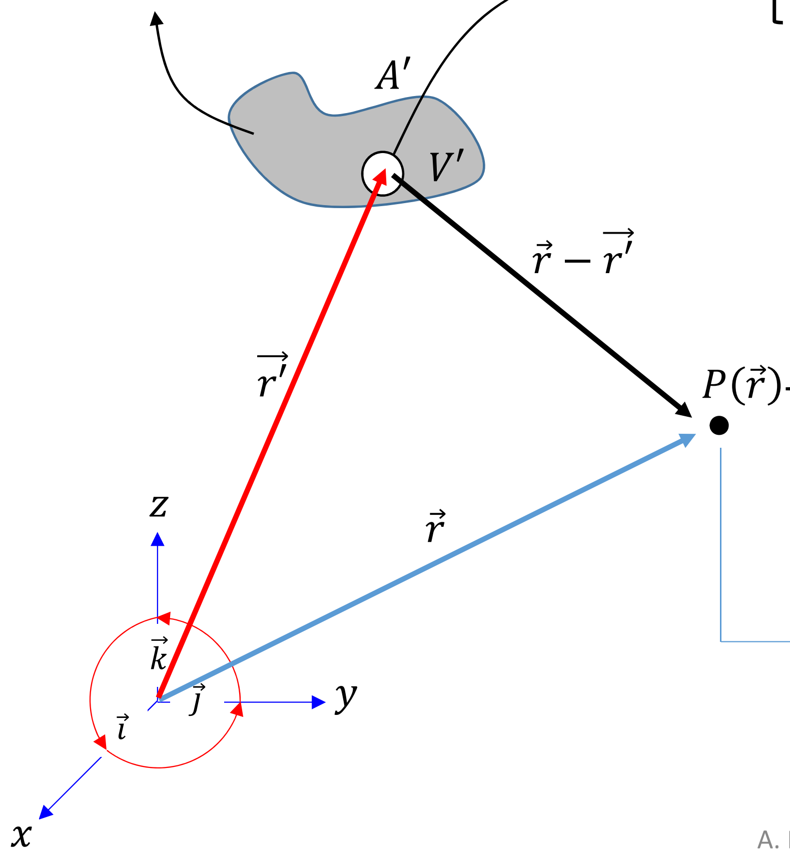
Make a choice between

Cartesian **cylindrical** **spherical** representation

Reference frame: localization in space

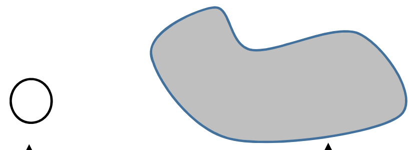
Source
- Charges immobile
- Charges in motion (current)
- Magnet

dV', $\vec{n}dA'$, $d\vec{l'}$



Effects

$\varphi(\vec{r})$
$\vec{E}(\vec{r})$
$\vec{B}(\vec{r})$
$\vec{A}(\vec{r})$



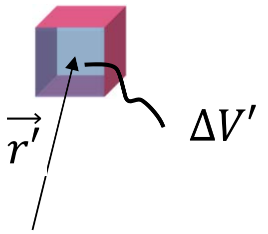
From the **element** to the **whole**
(Superposition principle)

$$\int_{V'} Source(\vec{\boldsymbol{r'}}).\,dV'$$
$$\int_{A'} Source(\vec{\boldsymbol{r'}}).\,\vec{n}dA'$$
$$\int_{l'} Source(\vec{\boldsymbol{r'}}).\,d\vec{l'}$$

Operators are acting on **$\vec{r}$ NOT on $\vec{r'}$**

Operators:
- Gradient $\vec{\nabla}\varphi(\vec{r})$ = Vector
- Divergence $\vec{\nabla}.\,\vec{E}(\vec{r})$ = Scalar
- Curl $\vec{\nabla}\times\vec{E}(\vec{r})$ = Vector

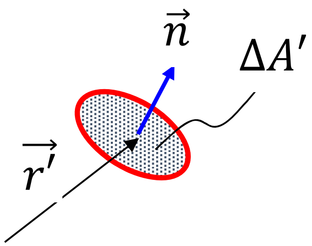
Charge and current distribution



$$\rho\left(\vec{r'}\right) = \lim_{\Delta V' \to 0} \frac{Q_{\Delta V'}}{\Delta V'}$$

$\Delta V'$ = **<u>volume</u>** element around $\vec{r'}$

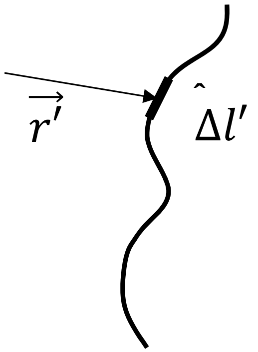
$Q_{\Delta V'}$ = total charge enclosed in $\Delta V'$



$$\sigma\left(\vec{r'}\right) = \lim_{\Delta A' \to 0} \frac{Q_{\Delta A'}}{\Delta A'}$$

$\Delta A'$ = **<u>surface</u>** element around $\vec{r'}$

$Q_{\Delta A'}$ = total charge distributed on $\Delta A'$



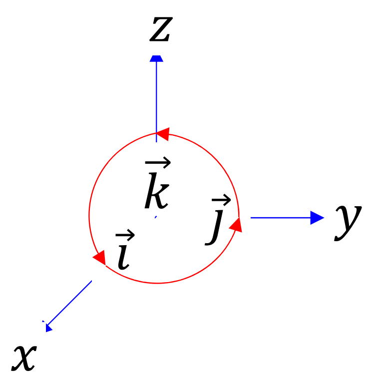
$$\lambda\left(\vec{r'}\right) = \lim_{\Delta l' \to 0} \frac{Q_{\Delta l'}}{\Delta l'}$$

$\Delta l'$ = **<u>line</u>** element around $\vec{r'}$

$Q_{\Delta A'}$ =total charge distributed along $\Delta l'$

Reference frame

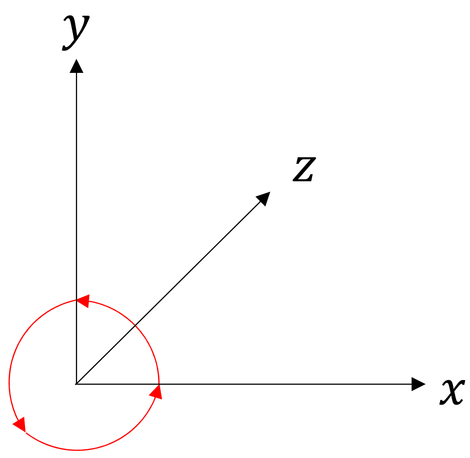
Rule



$\vec{\imath} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{\imath}$

$\boxed{\vec{k} \times \vec{\imath} = \vec{j}}$

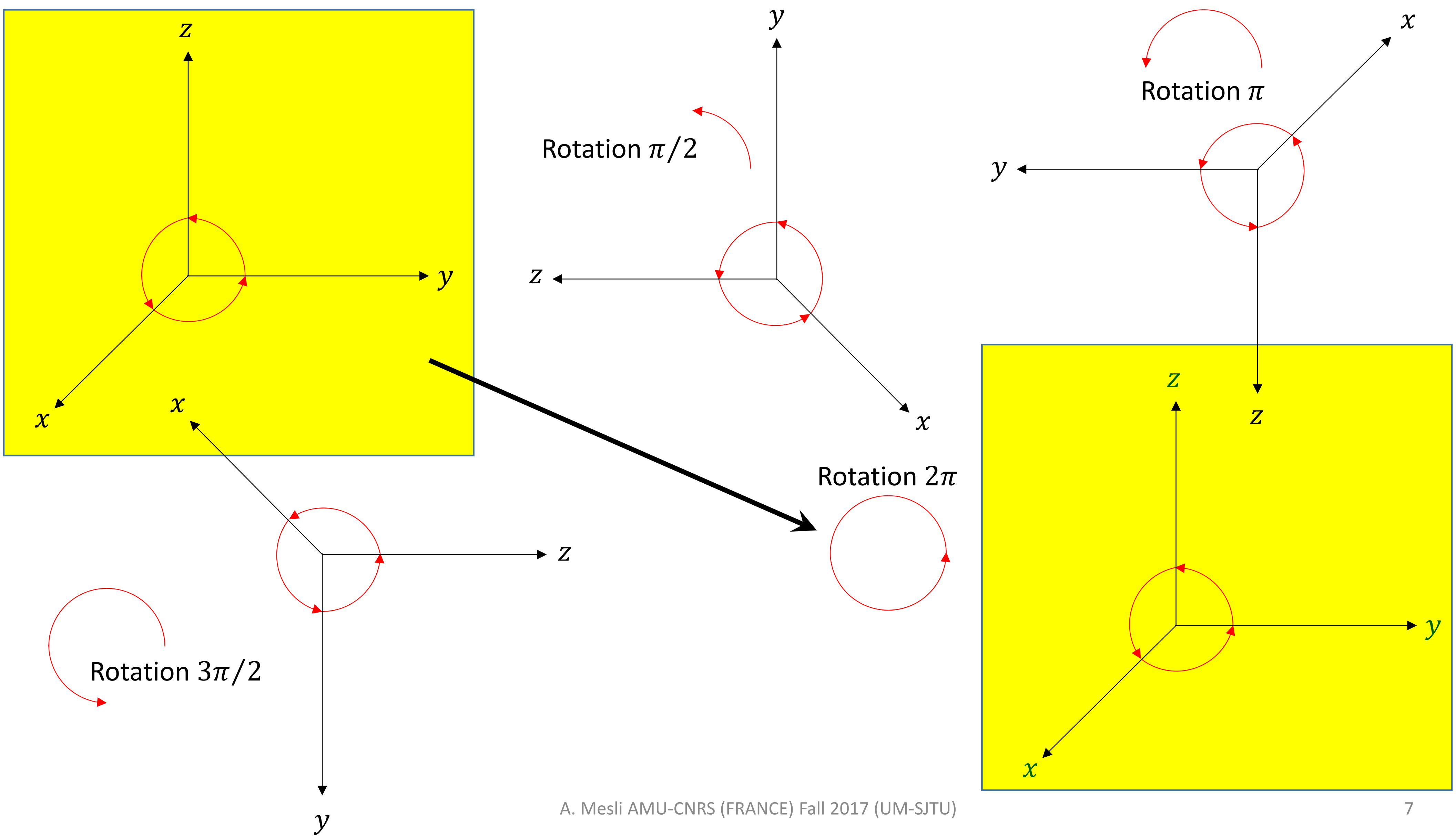


Forbidden

* Question #1: Why is this frame forbidden?

Answer to *Question 1 According to frame $\vec{\imath} \times \vec{k} = \vec{j}$

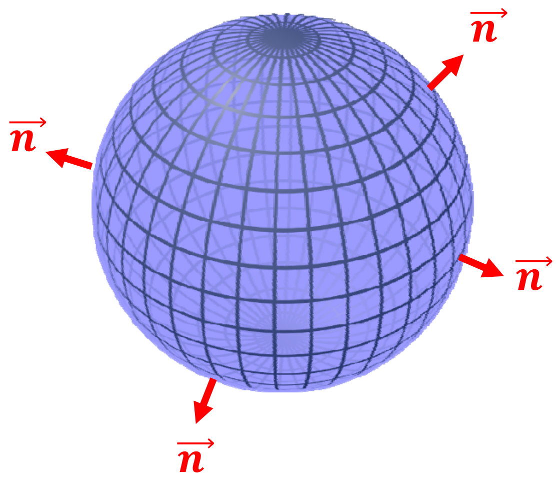
According to the rule $\vec{k} \times \vec{\imath} = \vec{j}$ ⇒ $\vec{\imath} \times \vec{k} = -\vec{j}$



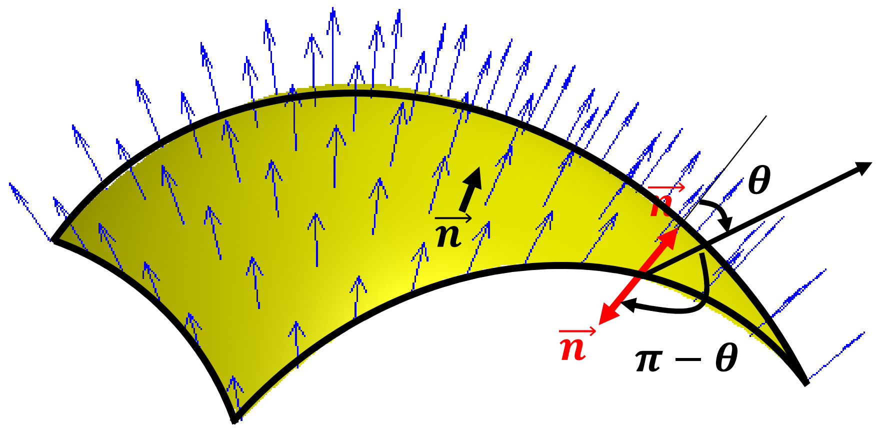
z
y
x
Rotation $\pi/2$
y
z
x
Rotation π
x
y
z
x
z
y
Rotation 2π
Rotation $3\pi/2$
z
x
y

Closed versus open surface

Closed surface



Open surface



$$cos\theta = -\cos(\pi - \theta)$$

Unit vector is always directed outwards

Both orientations of the unit vector are valid

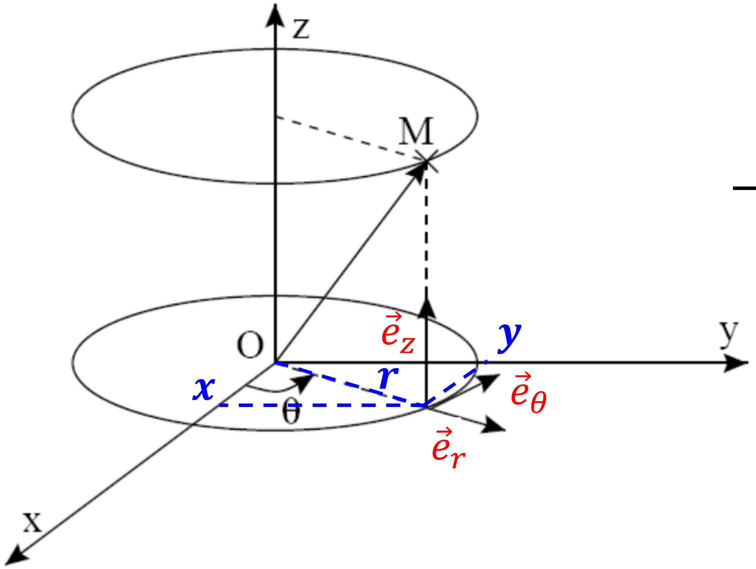
Closed surface defines a volume

Open surface defines a closed path

Divergence and Gauss theorem

Curl and Stokes theorem

Surface and volume element in cylindrical coordinates

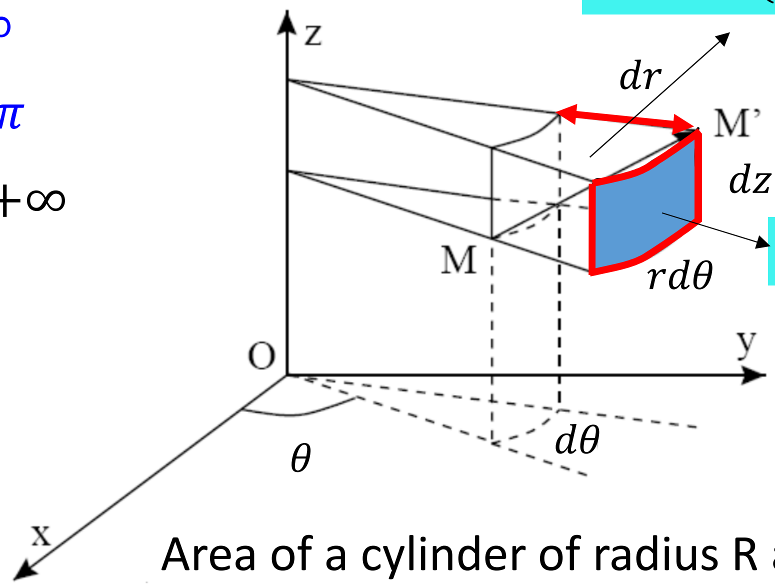


$0 < r < \infty$

$0 < \theta < 2\pi$

$-\infty < z < +\infty$

$dV = dr(rd\theta)dz$



$dA = (rd\theta)dz$

$x = rcos\theta$
$y = rsin\theta$

Polar coordinates

$z = z$

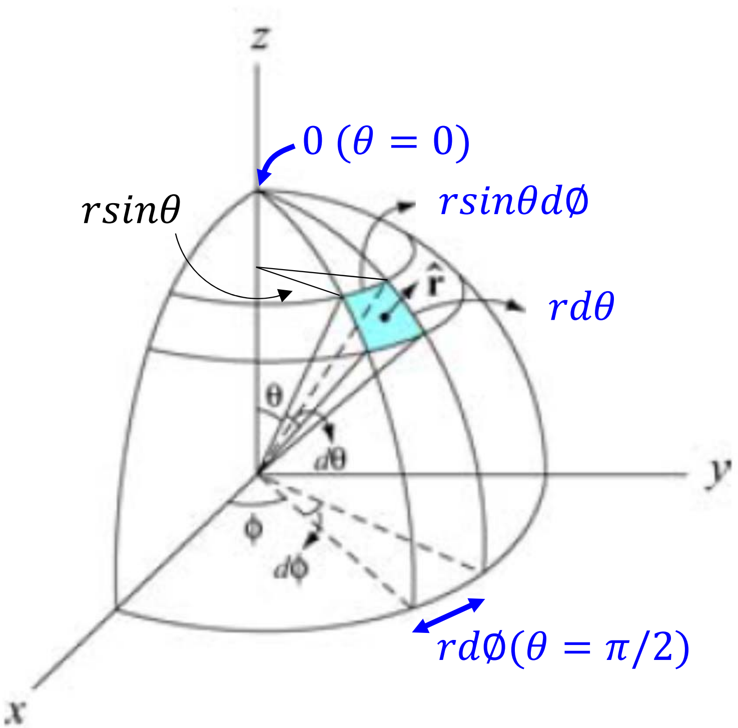
Area of a cylinder of radius R and height H

$$dA = (Rd\theta)dz \Rightarrow A = \int_0^{2\pi} Rd\theta \int_0^H dz = 2\pi RH$$

Volume of a cylinder of radius R and height H

$$dV = dr(rd\theta)dz \Rightarrow V = \int_0^R rdr \int_0^{2\pi} d\theta \int_0^H dz = \pi R^2 H$$

Surface and volume element in spherical coordinates



$$0 < \theta < \pi$$
$$0 < \emptyset < 2\pi$$

$$x = rsin\theta cos\phi$$
$$y = rsin\theta sin\phi$$
$$z = rcos\theta$$

$$dA = (rsin\theta d\varphi)(rd\theta)$$

$$dA = r^2 sin\theta d\theta d\varphi$$

$$dV = (dA)dr$$
$$dV = r^2 sin\theta dr d\theta d\varphi$$

Area of a sphere of radius R

$$A = R^2 \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi R^2$$

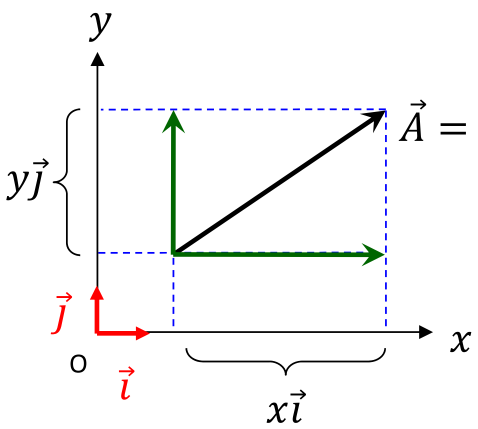
Volume of a sphere of radius R

$$V = \int_0^R r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi$$
$$= \frac{4}{3}\pi R^3$$

Vector calculus: Basics

Vector and representation

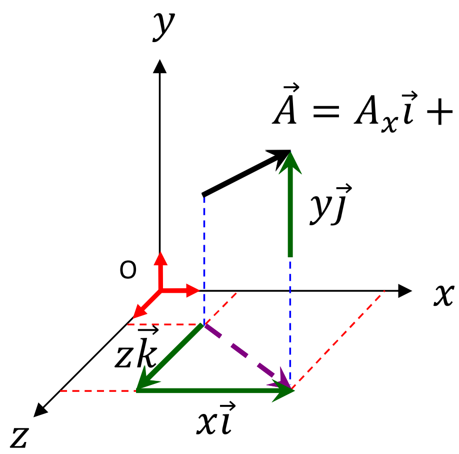
Representation in 2D **2 components**



$$\vec{A} = A_x\vec{i} + A_y\vec{j}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

Representation in 3D **3 components**



$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{j}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors of magnitude 1 in Cartesian frame in direction along x, y, z axes

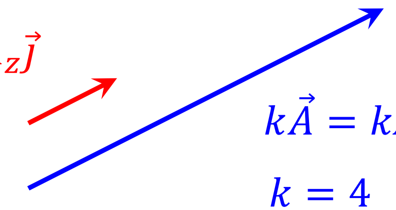
$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

A_x, A_y, A_z are the components along the three axes

$|\vec{A}|$**= Magnitude** of the vector

Multiplying a vector by a scalar

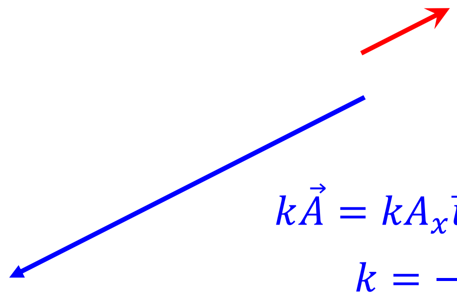
$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}$



$k\vec{A} = kA_x\vec{\imath} + kA_y\vec{\jmath} + kA_z\vec{\jmath}$

$k = 4$

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}$



$k\vec{A} = kA_x\vec{\imath} + kA_y\vec{\jmath} + kA_z\vec{\jmath}$

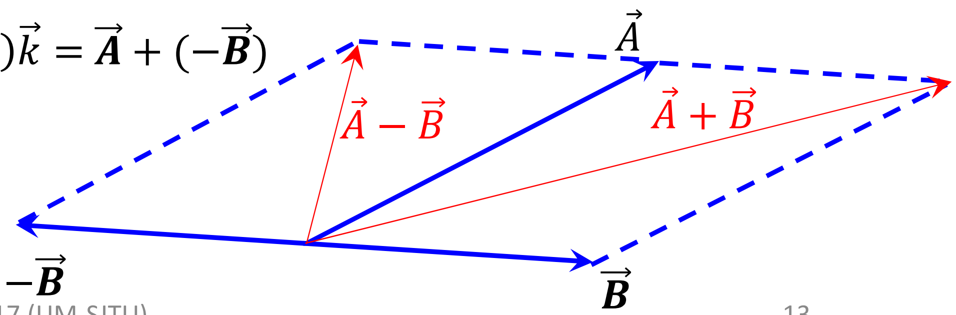
$k = -4$

Vectors $\vec{A}$ and $k\vec{A}$ are co-linear (parallel)

Addition and subtraction of vectors

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}$ $\quad \vec{A} + \vec{B} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k}$

$\vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{\jmath}$ $\quad \vec{A} - \vec{B} = (A_x - B_x)\vec{\imath} + (A_y - B_y)\vec{\jmath} + (A_z - B_z)\vec{k} = \vec{\boldsymbol{A}} + (-\vec{\boldsymbol{B}})$



Scalar product

Cartesian vs polar coordinates

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath}$ ⟶ Cartesian

$\vec{A} = |\vec{A}|cos\theta_A\vec{\imath} + |\vec{A}|sin\theta_A\vec{\jmath}$ ⟶ Polar

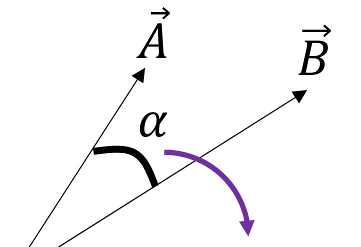
$\vec{B} = B_x\vec{\imath} + B_y\vec{\jmath}$

$\vec{B} = |\vec{B}|cos\theta_B\vec{\imath} + |\vec{B}|sin\theta_B\vec{\jmath}$

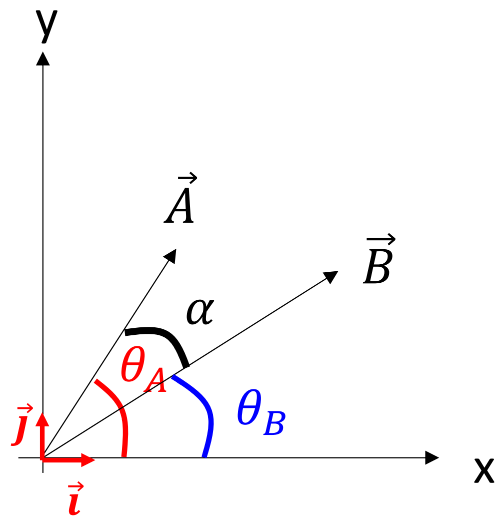
$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|cos\theta_A cos\theta_B + |\vec{A}||\vec{B}|sin\theta_A sin\theta_B$$

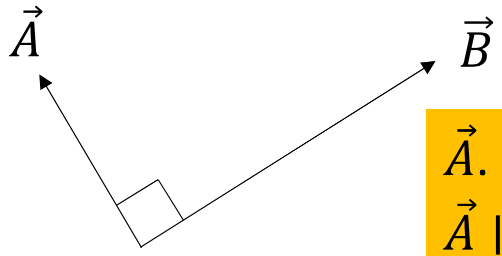
$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|[cos\theta_A cos\theta_B + sin\theta_A sin\theta_B]$$

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|[cos(\theta_A - \theta_B)] = AB\cos\alpha$$



$\boldsymbol{\alpha = \theta_A - \theta_B}$





$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\alpha$

$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A}.\vec{B} = |\vec{A}||\vec{B}|$

$\vec{A} \perp \vec{B} \Longrightarrow \vec{A}.\vec{B} = 0$

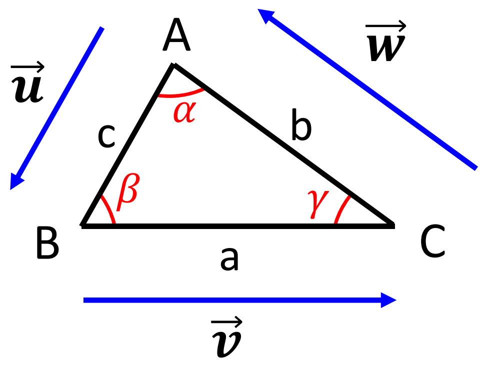
$\boldsymbol{cos(\theta_A - \theta_B) = cos\theta_A cos\theta_B + sin\theta_A sin\theta_B}$

Example: Al-Kashi's theorem

Given a triangle ABC, demonstrate on the basis of vector calculus Al-Kashi's theorem which states that:

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

Where a is for BC, b for AC and c for AB and α is the angle $\widehat{BAC}$ etc…



Vectors turning counterclockwise

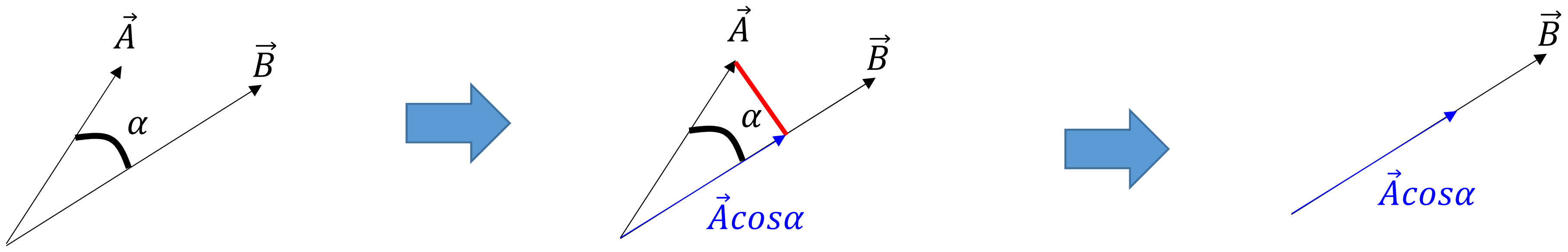
$$\vec{u} + \vec{v} = -\vec{w}$$

$$a^2 = \vec{v}.\vec{v} = (-\vec{w} - \vec{u})^2 = w^2 + v^2 + 2uwcos(\pi + \alpha)$$

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

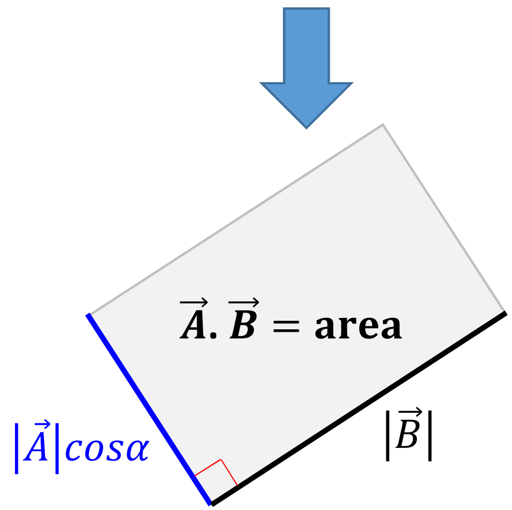
Pythagoras's theorem ⇨ $\alpha = \pi/2$ ⇨ $a^2 = b^2 + c^2$

Geometrical interpretation of Scalar product



The scalar product defines an area

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$
$$\vec{A} \perp \vec{B} \Longrightarrow \vec{A}.\vec{B} = 0$$
$$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A}.\vec{B} = |\vec{A}||\vec{B}|$$



The area may involve other dimensions than surface **!**

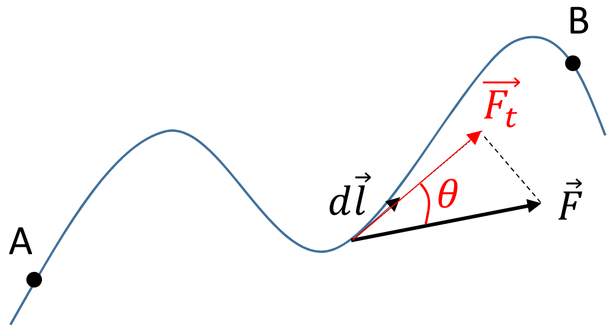
Example of scalar product in physics

*Question #2 What major physical quantity is expressed by scalar product?

Answer to *Question #2 Work done by a force on a particle moving along a path AB

$$dW = \vec{F}.d\vec{l} = F cos\theta dl$$

$$W = \int_A^B F cos\theta dl = \int_A^B F_t dl$$



Here the area has a new dimension: Energy

Line integral

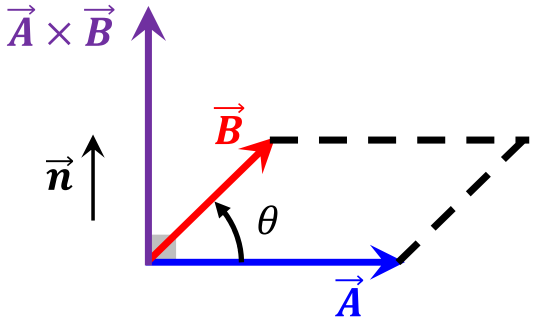
Close the path

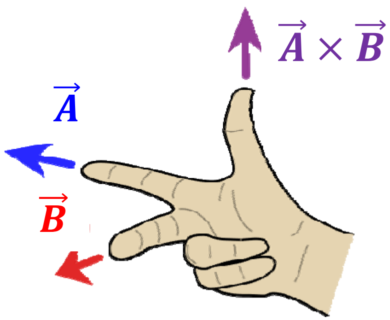
Circulation

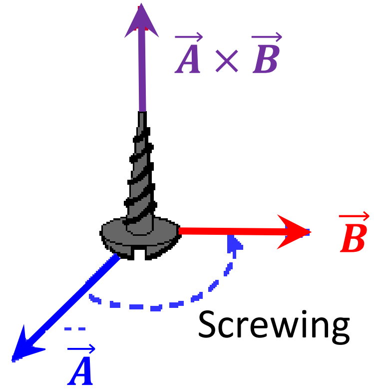
*Question #3 What is the value of the circulation in this case?

Answer to *Question #3 **0**: force is conservative, dW is exact differential

Cross product







Right hand rule

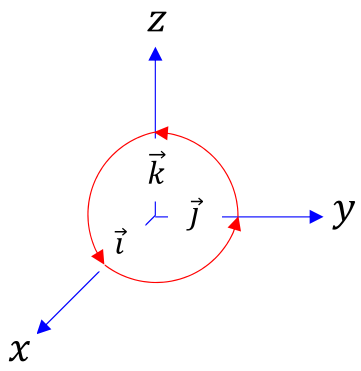
Screwing and unscrewing rule

- Screwing ⇨ $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$
- Unscrewing ⇨ $\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}} = -\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = |\vec{A}||\vec{B}|sin\theta.\vec{\boldsymbol{n}}$$

The vector "cross product" is **<u>always</u>** perpendicular to the plan containing the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$

Rules



Scalar product

$$\vec{i}.\vec{i} = 1$$

$$\vec{j}.\vec{j} = 1$$

$$\vec{k}.\vec{k} = 1$$

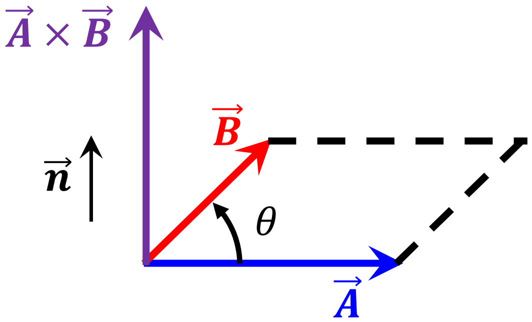
$$\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$$

Cross product

$$\boxed{\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \end{aligned}}$$

$$\boxed{\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}}$$

Vector product



Cartesian notation

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath}+A_z\vec{k}$

$\vec{B} = B_x\vec{\imath} + B_y\vec{\jmath}+B_z\vec{k}$

+

$$\vec{A} \times \vec{B} = \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} = \begin{pmatrix} A_y & A_z \\ B_y & B_z \end{pmatrix}\vec{\imath} - \begin{pmatrix} A_x & A_z \\ B_x & B_z \end{pmatrix}\vec{\jmath} + \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}\vec{k}$$

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\vec{\imath} - (A_xB_z - A_zB_x)\vec{\jmath} + (A_xB_y - A_yB_x)\vec{k}$$

Scalar versus Vector product

Scalar product

$\vec{A}.\vec{B} = \vec{B}.\vec{A}$

$\vec{A} \perp \vec{B} \Longrightarrow \vec{A}.\vec{B} = 0$

$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A}.\vec{B} = |\vec{A}||\vec{B}|$

$\boldsymbol{Cos}$ is involved

Cross product

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

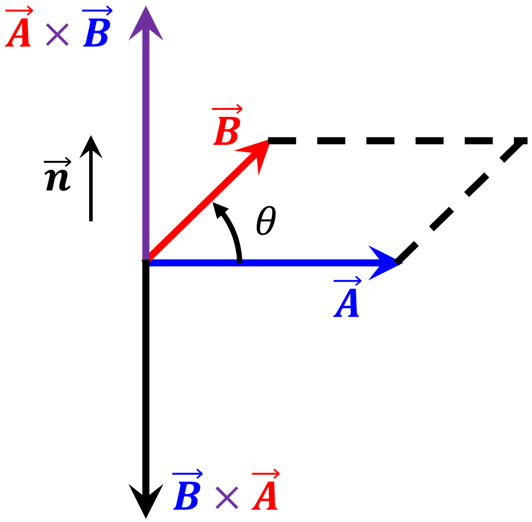
$\vec{A} \perp \vec{B} \Longrightarrow |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|$

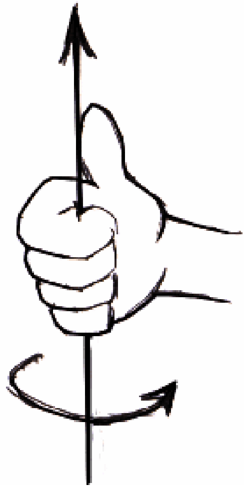
$\vec{A} \parallel \vec{B} \Longrightarrow \vec{A} \times \vec{B} = \vec{0}$

$\boldsymbol{Sin}$ is involved

$$\vec{A}\times\vec{A} = \vec{0} \qquad (\theta = 0 \text{ then } \sin(\theta) = 0)$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

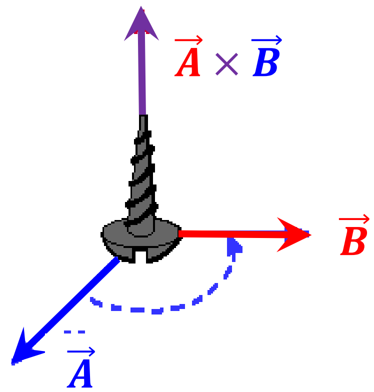




$$\boxed{\vec{A}.\left(\vec{A}\times\vec{B}\right) = 0}$$

Scalar product

$$\boxed{\vec{B}.\left(\vec{A}\times\vec{B}\right) = 0}$$



A property that did not escape to Maxwell's attention !

« Chance favors prepared minds » (Louis Pasteur)

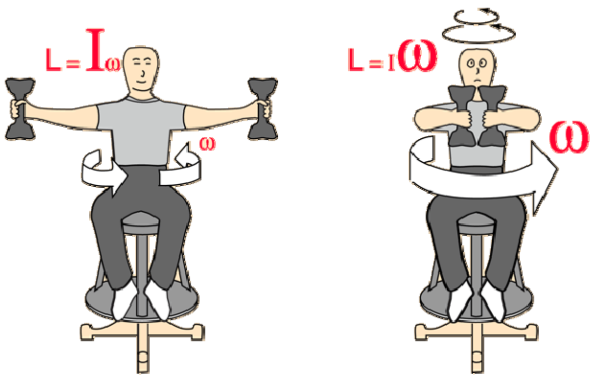
****Question #3

Give from one to four examples in physics involving cross product:
Two from classical mechanics and two from Vp260

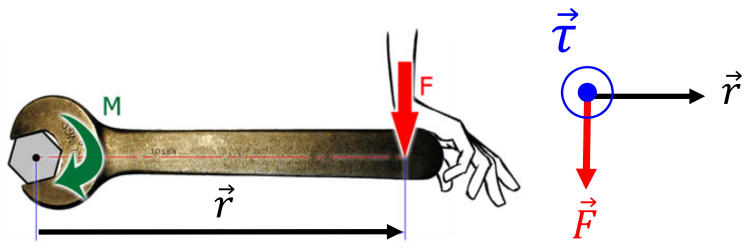
Answer to ****Question #3

From classical mechanics

Angular Momentum: $\vec{L} = \vec{r} \times m\vec{v}$



Torque : $\vec{\tau} = \vec{r} \times \vec{F}$

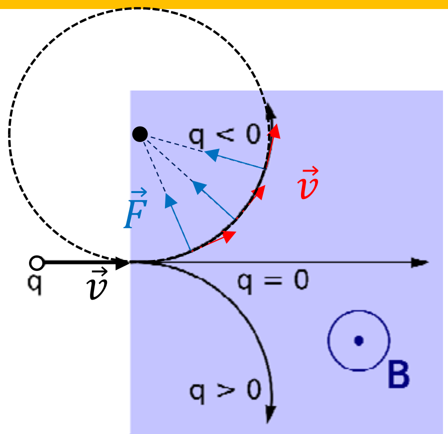


Example of cross product

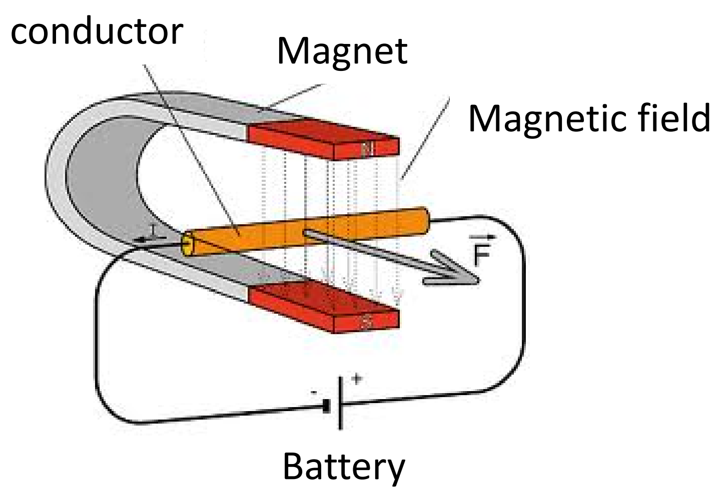
Answer to ****Question #3

From magnetostatic

Lorentz force: $\vec{F} = q\vec{v} \times \vec{B}$



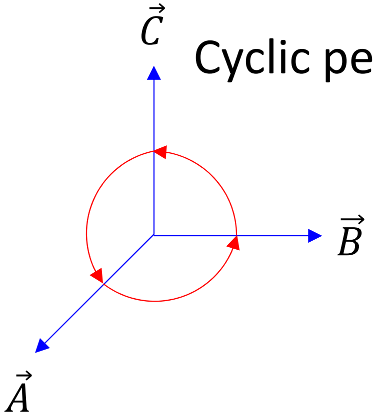
Laplace force: $\vec{F} = i\vec{l} \times \vec{B}$



Some geometrical interpretation of Cross product

Scalar triple product

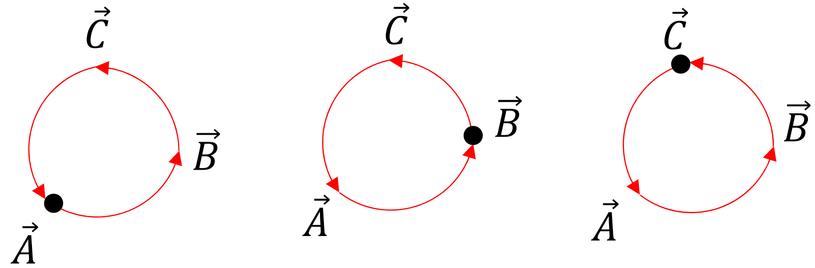
$$\vec{A}.(\vec{B} \times \vec{C})$$



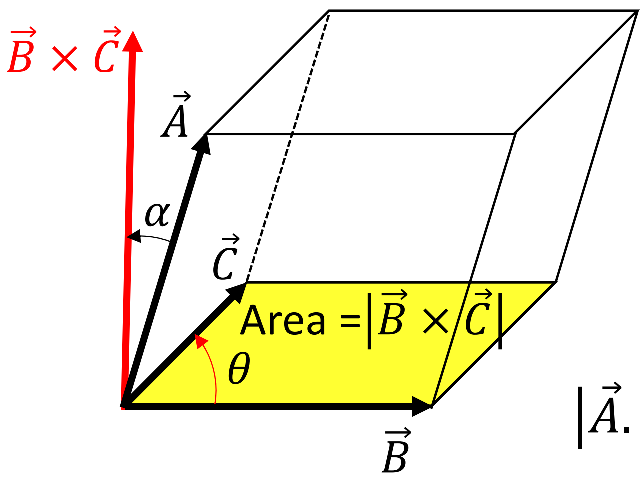
Cyclic permutation

$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$$

Commutativity of dot product



Counterclockwise

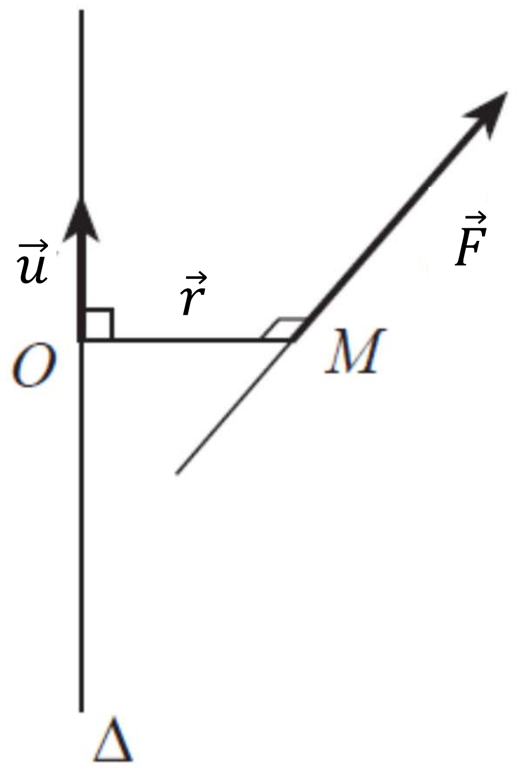


$$\vec{A}.(\vec{B} \times \vec{C}) = \begin{pmatrix} A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \\ C_X & C_Y & C_Z \end{pmatrix} = V$$

If $\alpha = 0$ and $\theta = \pi/2$ → Cube

$|\vec{A}.(\vec{B} \times \vec{C})| = |\vec{A}||\vec{B}||\vec{C}|(\cos\alpha)(sin\theta\ )$ = volume of the parallelepiped

Example of a Scalar triple product



Moment of a force about an axis

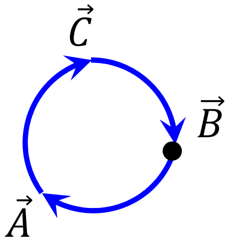
$$\Gamma_{\Delta} = \vec{u}.\left(\vec{r} \times \vec{F}\right)$$

Cross triple product

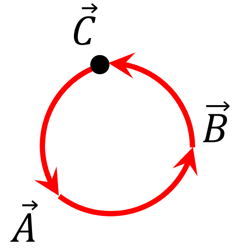
These are NOT scalar product !

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$$

Scalar times Vector = Vector



clockwise



counterclockwise

Caution ! *When dealing with vector fields and operators,* $\vec{C}(\vec{A}.\vec{B})$ *may lose physical meaning then we use* ***<u>commutativity</u>*** $\underbrace{(\vec{A}.\vec{B})}_{a}\vec{C}$

$$\boxed{\Rightarrow a\vec{C}}$$

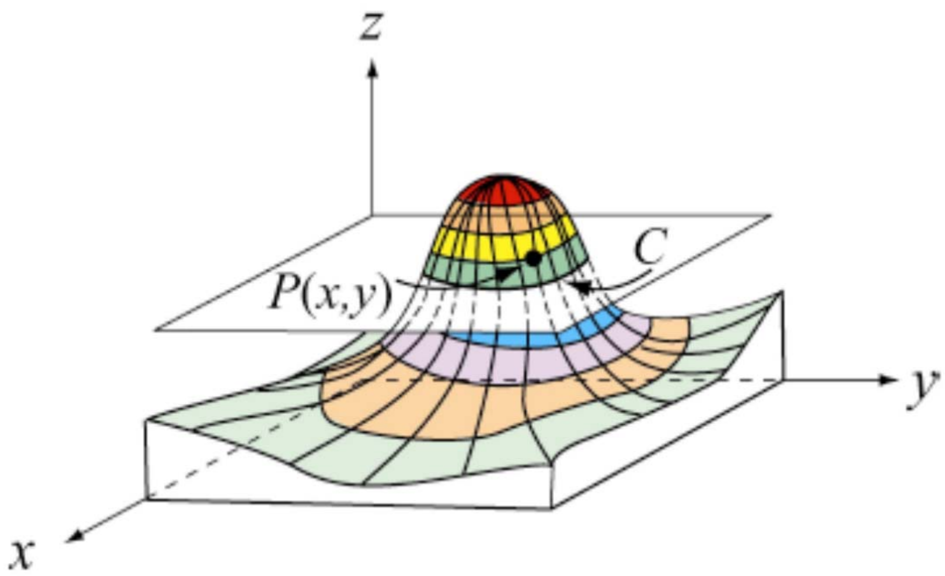
Manipulating scalar and vector fields

The central master piece in Electromagnetism

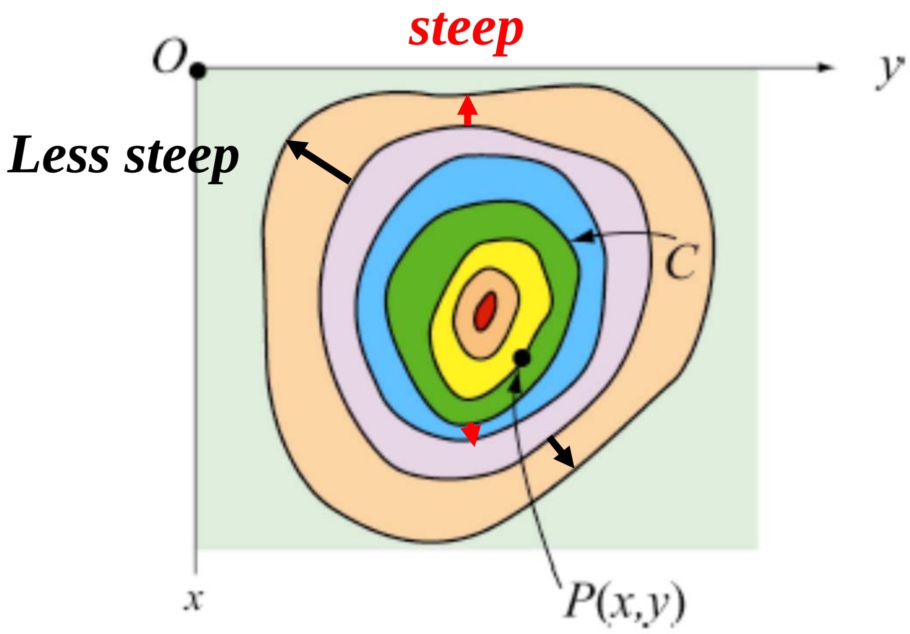
$$\vec{\nabla}$$

The Nabla or Del operator

Topographic map of a mountain: Hiking in the mountain



side view of a mountain

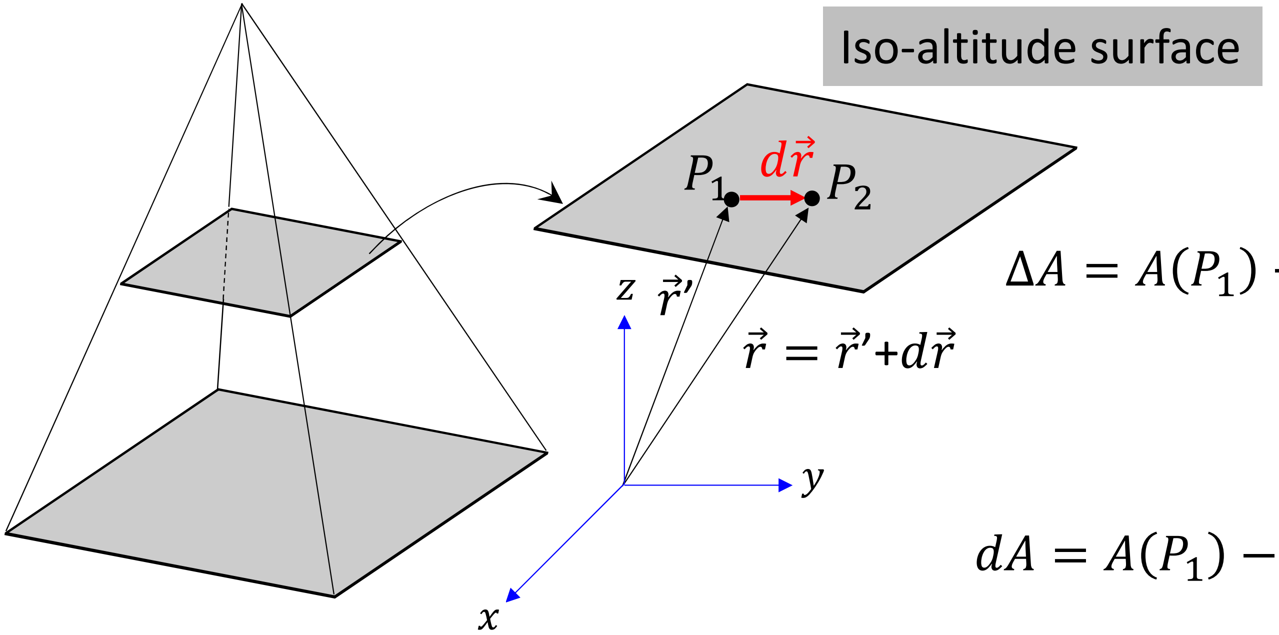


top view of iso-altitude surfaces

Where the arrow is small, the path is steep

How can we link the steepness to the iso-altitude surfaces?

Another way of looking at the gradient



$$P_1 \qquad \rightarrow \qquad P_2$$

$$A(x, y, z) \rightarrow A(x + \Delta x, y + \Delta y, z + \Delta z)$$

$$\Delta A = A(P_1) - A(P_2) = \frac{\partial A}{\partial x}\Delta x + \frac{\partial A}{\partial y}\Delta y + \frac{\partial A}{\partial z}\Delta z = \mathbf{0}$$

$$\Delta x, \Delta y, \Delta z \rightarrow 0$$

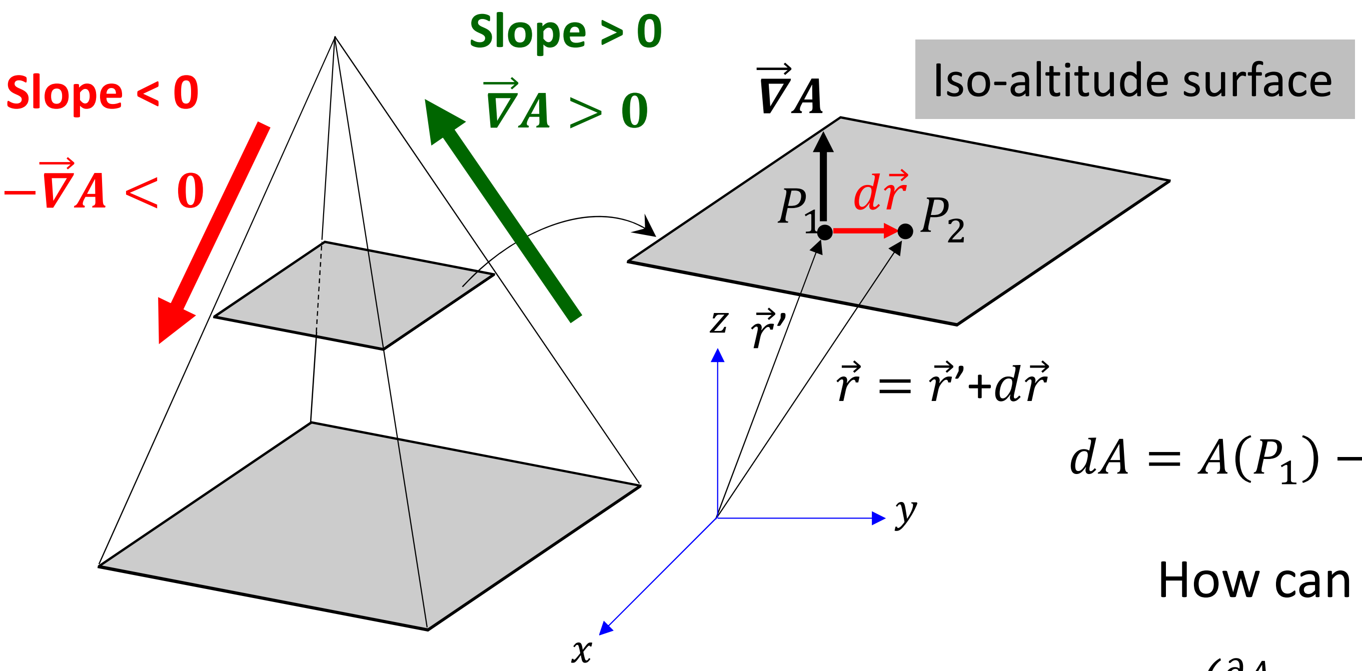
$$dA = A(P_1) - A(P_2) = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz = \mathbf{0}$$

What do dx, dy and dz represent?

The components of the vector displacement $d\vec{r}$

$$\boldsymbol{d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}}$$

Another way of looking at the gradient



$$P_1 \quad \rightarrow \quad P_2$$

$$A(x, y, z) \rightarrow A(x + dx, y + dy, z + dz)$$

$$dA = A(P_1) - A(P_2) = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz = \mathbf{0}$$

How can we transform this sum of products?

$$dA = \left(\frac{\partial A}{\partial x}\vec{\imath} + \frac{\partial A}{\partial y}\vec{\jmath} + \frac{\partial A}{\partial z}\vec{k}\right).\left(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}\right) = \mathbf{0}$$

$$dA = \vec{\nabla} A . d\vec{r} = \mathbf{0} \quad \Rightarrow \quad \vec{\nabla} \boldsymbol{A} \perp \boldsymbol{d\vec{r}}$$

The gradient of a scalar field $f(x, y, z)$ at a given point $P(x, y, z)$, $\vec{\nabla} f(x, y, z)$, is a vector perpendicular to the iso-surface $f(x, y, z) = C$ at point $P(x, y, z)$

- If the iso-surface is flat, the vector is perpendicular everywhere to this surface

- If the iso-surface is curved, the vector is perpendicular to surface at the specified point $P(x, y, z)$

The gradient is a local property

Water flows naturally from top to bottom $\longrightarrow$ $\vec{W}(x,y,z,t) = -K_w\vec{\nabla}A(x,y,z,t)$

$A(x,y,z,t)$ = Iso-altitude surface

Heat flows naturally from hot to cold $\longrightarrow$ $\vec{h}(x,y,z,t) = -K_h\vec{\nabla}T(x,y,z,t)$

$T(x,y,z,t)$ = Iso-thermal surface

Electric field also “flows” from one $+q$ to $-q$ $\longrightarrow$ $\vec{E}(x,y,z,t) = -\vec{\nabla}\varphi(x,y,z,t)$

$\varphi(x,y,z,t)$ = Iso-potential surface

The same equation governs electromagnetism, water flow, heat flow etc…

Slide 29_ A_Lecture 0_General

*Question #4 What is for a point charge the shape of the iso-potential surface ?

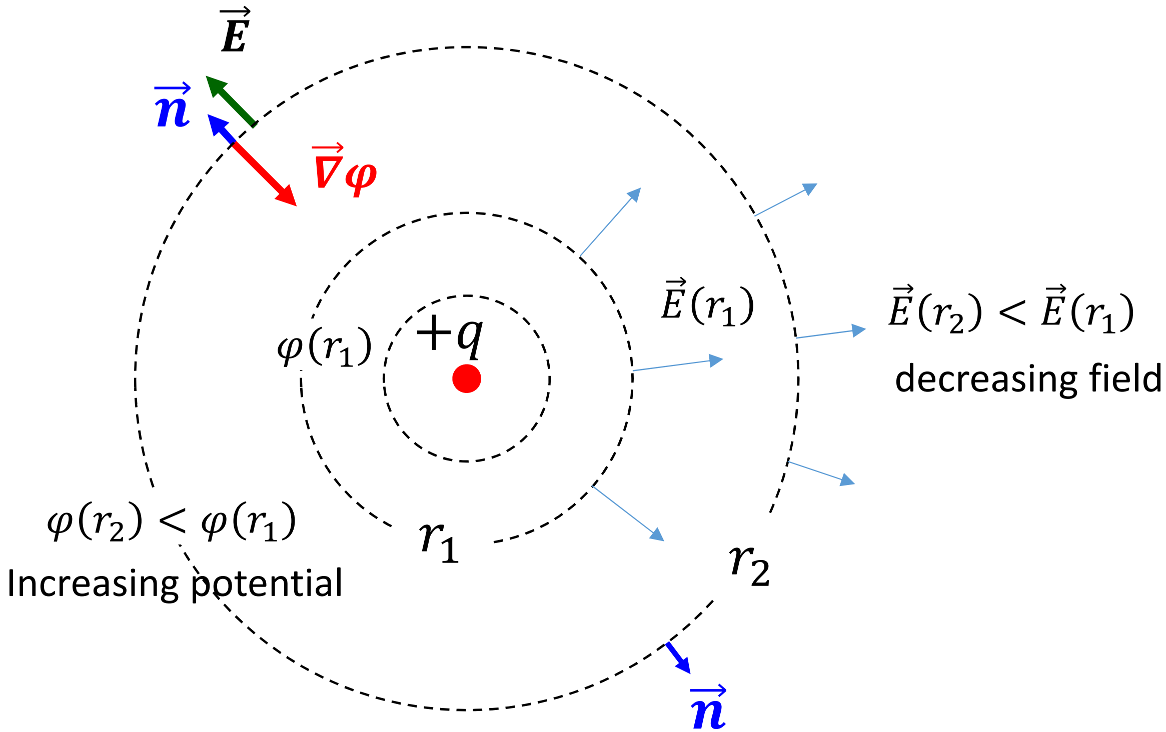
Answer to *Question #4 Spheres centered on the charge

$\vec{E} \parallel \vec{n} \perp$ iso-potential surface

$$\vec{E}(x,y,z,t) = -\vec{\nabla}\varphi(x,y,z,t)$$

$\varphi(x,y,z,t) =$ cte

Iso-potential surface



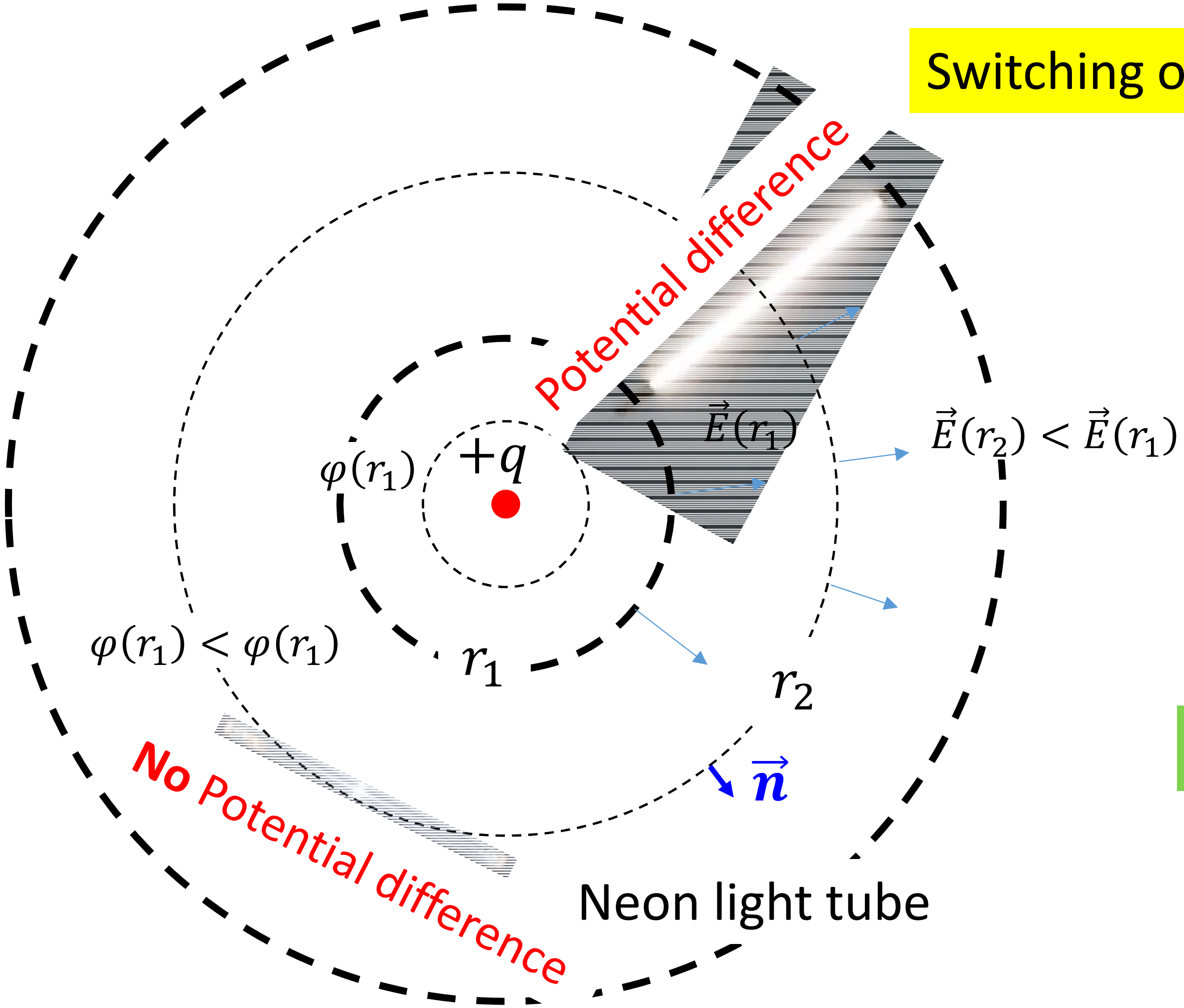
Obtained from
$\vec{E}(x,y,z,t) = -\vec{\nabla}\varphi(x,y,z,t)$
If $\vec{E}(x,y,z,t)$ is known

First part of the first Maxwell's equation

$$\vec{E} = -\vec{\nabla}\varphi$$

Valid in Electrostatic only

Switching on the light without any closed circuit !



***Question #5

Does it flash or remains ON ?

Answer to ***Question #5

Flashes, because soon equilibrium is reached ⇔ No more discharge

The Nabla or Del operator

$$\vec{\boldsymbol{\nabla}} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

This operator alone means nothing, just as…

$\sqrt{}$ means nothing

Both operators must have something to act on

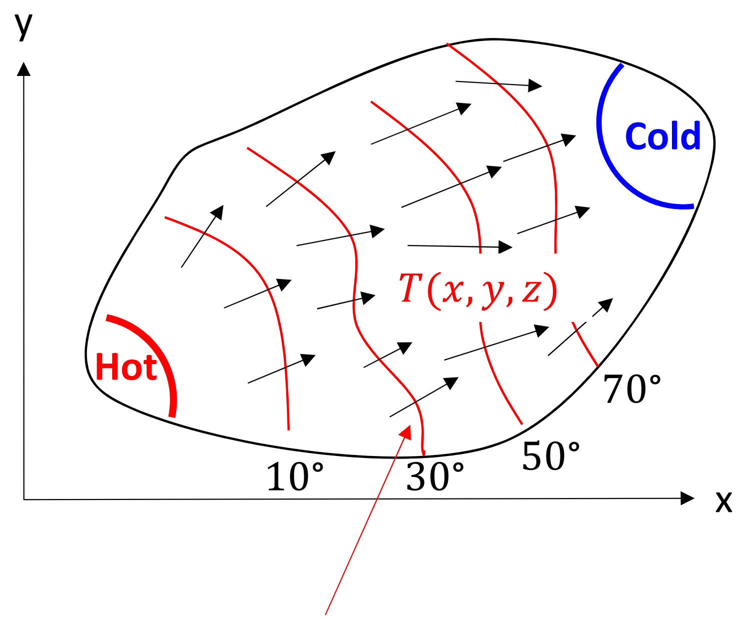
The best way to visualize the scalar and vector fields is via heat propagation

How do we get the flux?

Scalar field: Temperature in 3D space

*Question #6

Does temperature move or change from one position in space to another?



Isotherm = isothermal line (2D) or surface (3D)

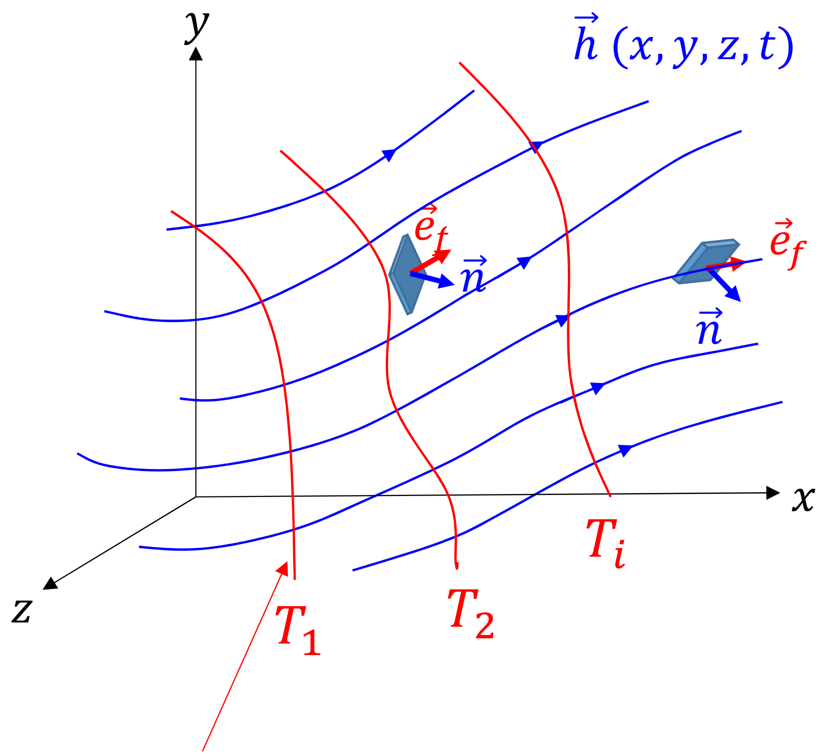
*Answer to question #6

$T(x,y,z)$ does not move: **<u>It changes</u>**

- Temperature $T(x,y,z)$ is a scalar field
- If it depends on time ⇨ $T(x,y,z,t)$

Defining the concept of Flux

Vector field: Heat flow in 3D space



Heat $\vec{\boldsymbol{h}}\,(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$ is a vector field. It flows in 3D space

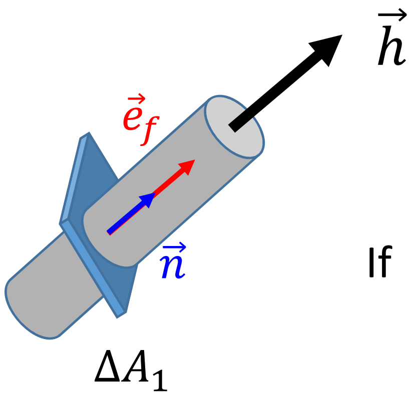
It gives rise to a scalar field: temperature in 3D space $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$

$\left|\vec{\boldsymbol{h}}\,(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})\right|$ = How much heat is flowing
= Thermal energy/time/area

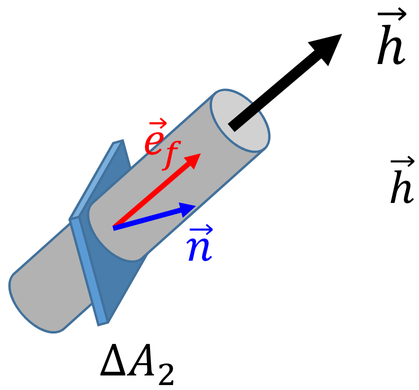
$\vec{\boldsymbol{h}}\,(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = (\text{scalar quantity})\vec{e}_f$ **Heat flow**

Thermal energy/time/area

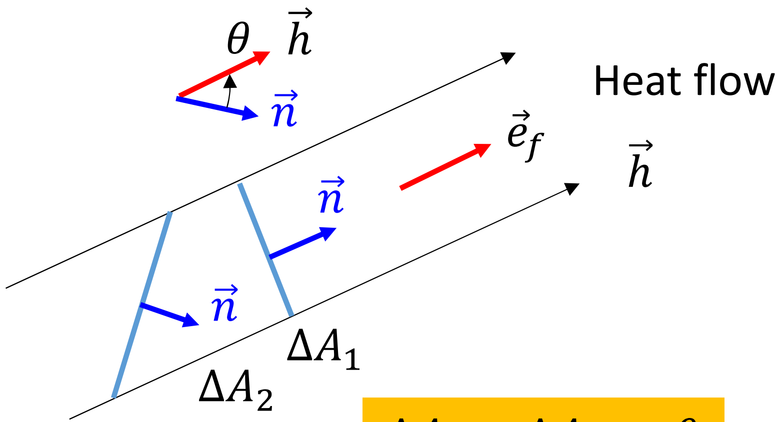
- Need to define a unit vector along the flow $\vec{e}_f$ at a given location in space
- Need to define a unit vector $\vec{n}$ for the area under consideration at a given location in space



If $\vec{n} = \vec{e}_f \rightarrow \vec{h} \perp \vec{n}\Delta A_1$

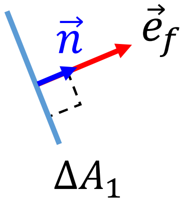


$\vec{h} \parallel \vec{e}_f$



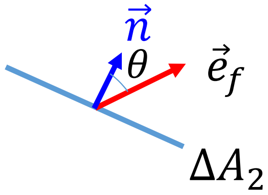
$$\boxed{\Delta A_1 = \Delta A_2 cos\theta}$$

Definition of heat flow $\vec{h}$



$$\vec{\boldsymbol{h}}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) = \frac{P}{\Delta A_1}\vec{e}_f = |\vec{h}\,|\vec{e}_f$$

P = Thermal energy/time



What is the flow per unit area through ΔA_2 ?

?

$$\frac{P}{\Delta A_2} = \frac{P}{\Delta A_1}cos\theta = \vec{h}.\vec{n} = |\vec{h}_\perp|$$

Scalar product !

$$\Delta A_1 = \Delta A_2 cos\theta \qquad |\vec{h}\,| = |\vec{h}_\perp| \text{ if } \theta = 0 \iff \vec{e}_f = \vec{n}$$

$$\boxed{Flux = \int_A \vec{h}.\vec{n}dA}$$

or

$$\boxed{Flux = \int_A |\vec{h}_\perp|\, dA}$$

Applying Del operator $\vec{\nabla}$ to heat flow

Gradient of a <u>scalar field</u> $T(x,y,z)$

<u>Differentiate</u> a stationary scalar field $T(x,y,z)$

$T(x,y,z) = \text{C} \Leftrightarrow$ Isotherm or isothermal surface

$$\underbrace{\Delta T(x,y,z)}_{\text{scalar}} = \underbrace{\frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z}_{\text{dot product of two vectors}}$$

$$\Downarrow$$

$$\underbrace{\left(\frac{\partial T}{\partial x}\vec{\imath} + \frac{\partial T}{\partial y}\vec{\jmath} + \frac{\partial T}{\partial z}\vec{k}\right)}_{\text{Vector}}.\underbrace{\left(\Delta x\vec{\imath} + \Delta y\vec{\jmath} + \Delta z\vec{k}\right)}_{\text{Vector displacement}}$$

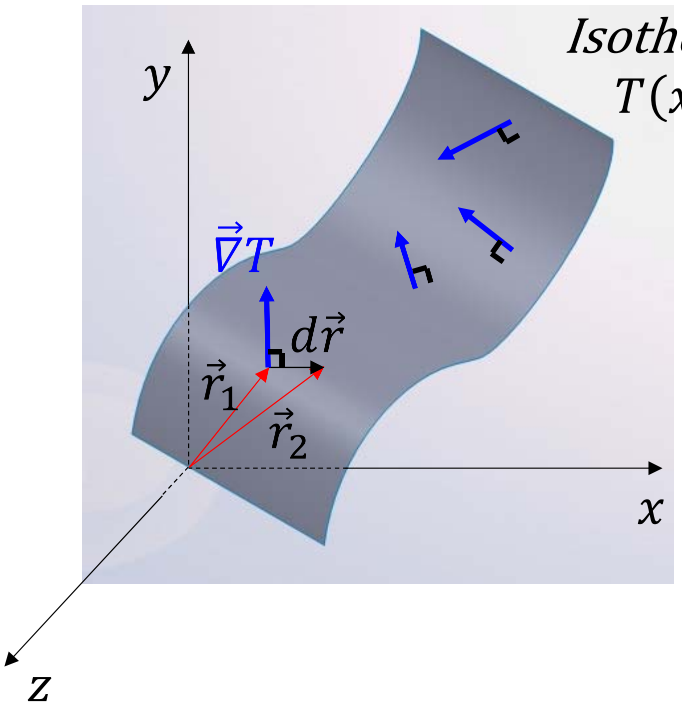
$$\Downarrow$$

$$\underbrace{\left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right)}_{\text{Vector Operator } \vec{\nabla}} T(x,y,z)$$

Reminder

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right)$$

Is **NOT** a vector as long as it does not operate on a scalar !



Isothermal surface

$T(x,y,z) = T_0 \quad \Rightarrow \quad \Delta T(x,y,z) = T(x,y,z) - T_0 = 0$

Temperature is everywhere the same on this surface
(Surface is not necessarily flat !)

$$\Delta T(x,y,z) = \underbrace{\left(\frac{\partial T}{\partial x}\vec{\imath} + \frac{\partial T}{\partial y}\vec{\jmath} + \frac{\partial T}{\partial z}\vec{k}\right)}.\underbrace{\left(\Delta x\vec{\imath} + \Delta y\vec{\jmath} + \Delta z\vec{k}\right)} = 0$$

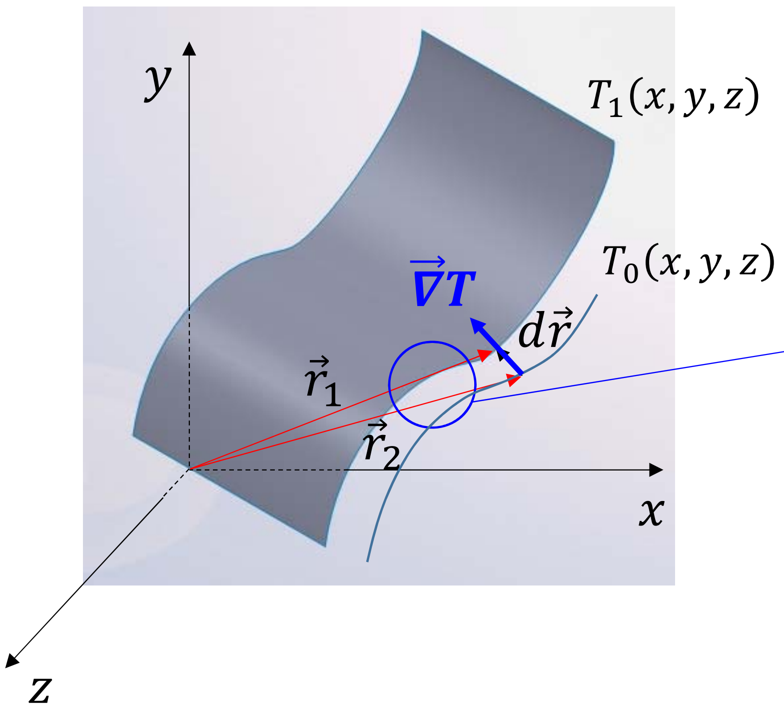
$$\Delta T(x,y,z) = \vec{\nabla}T.d\vec{r} = \left|\vec{\nabla}T\right|.\left|d\vec{r}\right|.cos(\theta) = 0$$

!

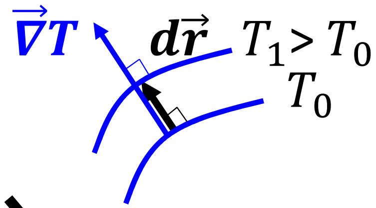
$\vec{r}_1$ and $\vec{r}_2$ indicate two close points
on the isothermal surface

$\theta = \pi/2$ $\vec{\nabla}T$ and $d\vec{r}$ are $\perp$ everywhere on the isothermal surface
- $d\vec{r}$ is tangent to the isothermal surface at (x,y,z)
- $\vec{\nabla}T$ is $\perp$ to the isothermal surface **at $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$**

Isothermal surfaces T_1 and T_0



$\vec{r}_1$ and $\vec{r}_2$ indicate two close points
<u>on two close isothermal surfaces</u>



$$\Delta T(x,y,z) = \vec{\nabla}T.d\vec{r} = \nabla T.dr.cos(\theta)$$
$$= \nabla T.dr = Max > 0$$
$$\theta = 0$$

Natural flow of heat $\vec{h} \propto -\vec{\nabla}T$

$$\Delta T > 0, dr > 0, \Rightarrow \vec{\nabla}T \parallel d\vec{r}, \theta = 0$$

The gradient is oriented towards the increasing field

$\vec{\nabla}$ is a vector operator: What can we do with it?

$\vec{E} = -\vec{\nabla}\varphi$

$f(x, y, z)$ is a scalar $\quad \vec{\nabla}.f$

Gradient
(Vector)

$\vec{\boldsymbol{\nabla}}.\vec{E} = \vec{\boldsymbol{\nabla}}.(-\vec{\nabla}\varphi)$

$\vec{\boldsymbol{\nabla}} \times \vec{E} = \vec{\boldsymbol{\nabla}} \times (-\vec{\nabla}\varphi)$

$\vec{A}(x, y, z)$ is a vector

$\vec{\nabla}.\vec{A}$

Divergence
(Scalar)

$\vec{\nabla} \times \vec{A}$

Curl
(Vector)

<u>All these things are local properties</u>: They are defined at a specific point $P(x, y, z)$

Gradient, Divergence and Curl might be $> \mathbf{0}, < \mathbf{0}$ $\boldsymbol{or} = \mathbf{0}$

Del operator $\vec{\boldsymbol{\nabla}}$ transforms a <u>scalar field</u> into a <u>vector field</u>

Gradient of a <u>scalar field</u> $T(x,y,z), \varphi(x,y,z)$ etc… ⇔ $\vec{\nabla}T, \vec{\nabla}\varphi$, etc… ($\vec{\nabla}$ = vector **<u>operator</u>** on a scalar field)

Del operator $\vec{\boldsymbol{\nabla}}$ transforms a <u>vector field</u> into a <u>scalar field</u>: New concept, **Divergence**

Divergence of a <u>vector field</u> $\vec{A}(x,y,z)$ ⇔ $\vec{\nabla}.\vec{A}$ (= **<u>“scalar product”</u>**: If $\vec{A}$ is a vector, $\vec{\nabla}$ is **NOT**)

- What is the divergence of a vector and its physical meaning ?

Divergence of a <u>vector field</u> $\vec{A}(x,y,z)$ at a given point $P(x,y,z)$

$$\vec{\nabla}.\vec{A} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).\left(A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}\right)$$

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = F(x,y,z)$$

Divergence of a vector field is a scalar field

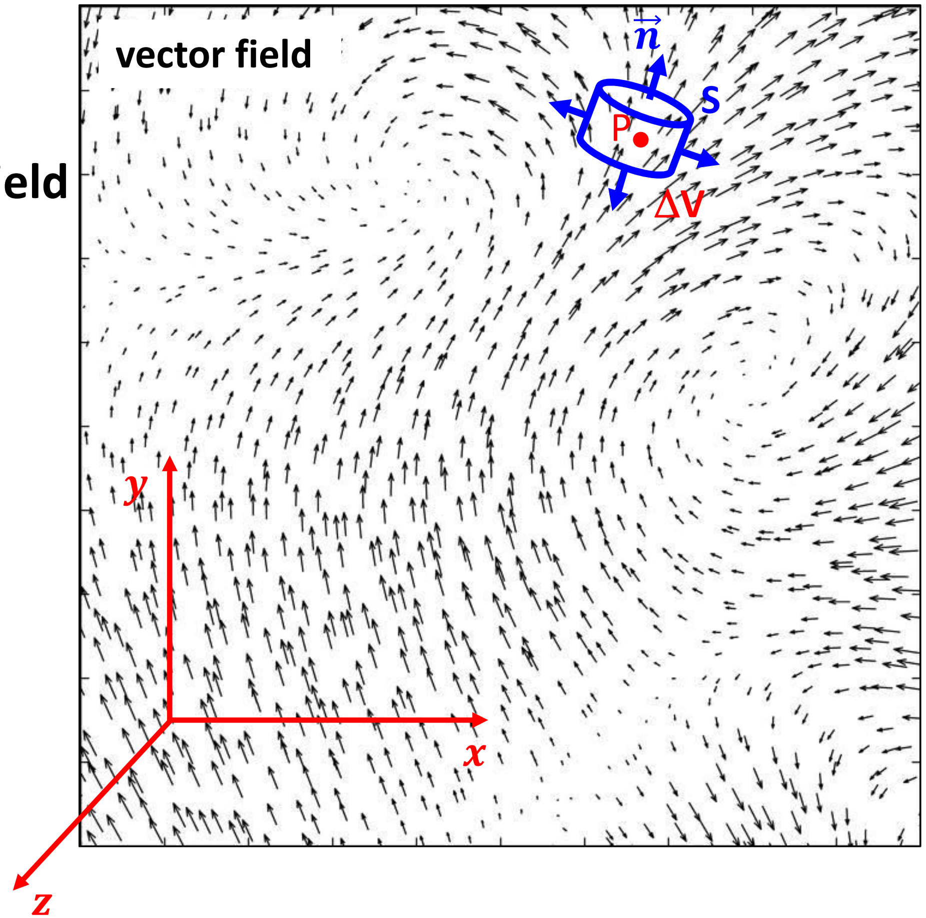
Measures the change of the components of the field vector while moving from

$$x \rightarrow x + dx$$
$$y \rightarrow y + dy$$
$$z \rightarrow z + dz$$

Divergence of a <u>vector field</u> $\vec{A}(x, y, z)$ at a given point $P(x, y, z)$

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = F(x, y, z)$$

The **operator** acts on a **vector field** and produces a **scalar field**



The components of $\vec{A}(x, y, z)$ are in general all function of the 3 variables to which **time** may be added

$\vec{A}_x(x, y, z)$

$\vec{A}_y(x, y, z)$

$\vec{A}_z(x, y, z)$

$\frac{\partial A_x}{\partial x}$ rate of change of $\vec{A}_x(x, y, z)$ at $P(x, y, z)$ respective to x, etc…

$F(x, y, z)$ = $\sum$ of spatial change of the components

Caution !

$$\left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) T(x, y, z) = \vec{\nabla} T$$

The vector operator $\vec{\nabla}$ is acting on the scalar $T(x, y, z)$

$$T(x, y, z)\left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) = T\vec{\nabla}$$

Here the operator $\vec{\nabla}$ is acting on nothing, so $T\vec{\nabla}$ is still just an operator

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = F(x, y, z)$$

The operator $\vec{\nabla}$ is acting on the vector$\vec{A}(x, y, z)$

$$\left(A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{\jmath}\right)\left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) = \vec{A}\vec{\nabla}$$

Here too the operator $\vec{\nabla}$ is acting on nothing.

- $\vec{A}\vec{\nabla}$ could lead to a vector $\vec{A}\left(\vec{\nabla}.\vec{B}\right)$ when acting on vector $\vec{B}$
- $\vec{A}\vec{\nabla}$ could lead to a scalar $\vec{A}.\left(\vec{\nabla}T\right)$ when acting on a scalar T

$\boldsymbol{div\,\vec{A}} = \overrightarrow{\nabla}.\overrightarrow{A} = \boldsymbol{F(x,y,z)}$

is a natural 3D function which may take up all kind of values (> 0, < 0 or = 0)

If $\vec{A}$ is uniform all components of $\vec{A}$ are constant => $\boldsymbol{div\,\vec{A} = 0}$

BUT ! **If** $\boldsymbol{div\,\vec{A} = 0}$ does it mean that the vector is constant everywhere ? **NO!** Magnetic field

Here is another example: $\vec{A} = 2xy\vec{\imath} - y^2\vec{\jmath}$

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = 0$$

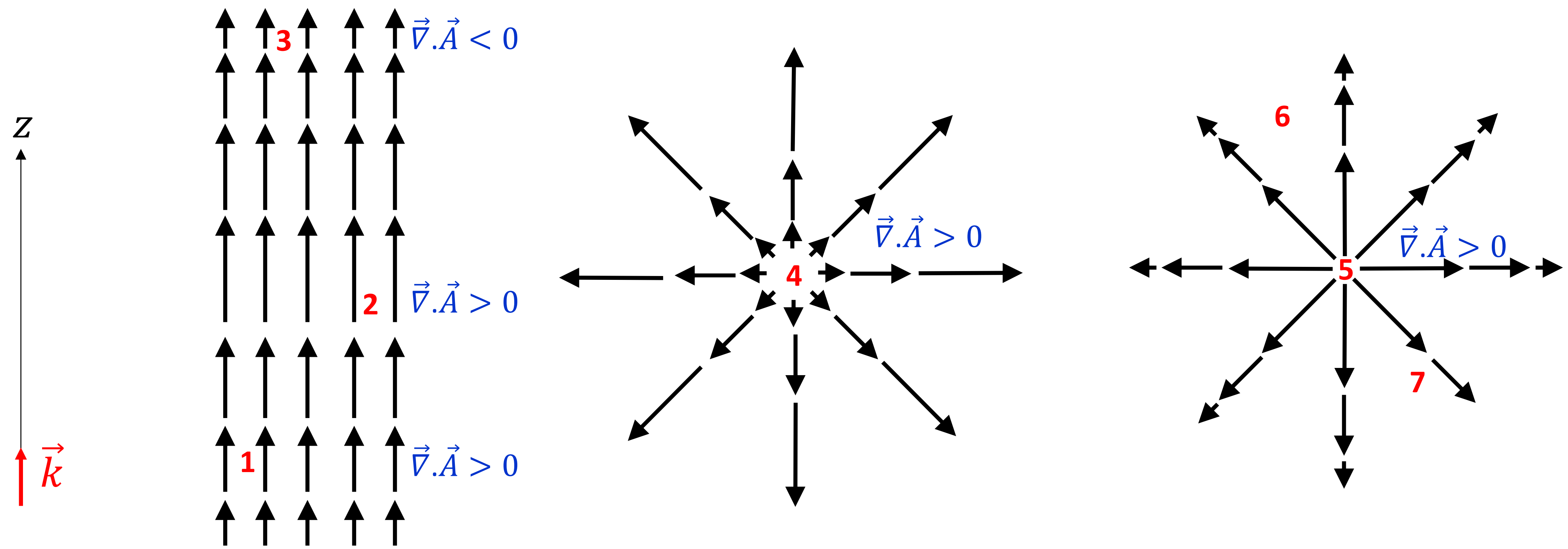
In general $\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$

$$\left\{\begin{array}{l} \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = -\frac{\partial A_z}{\partial z} \\ \frac{\partial A_x}{\partial x} + \frac{\partial A_z}{\partial z} = -\frac{\partial A_y}{\partial y} \\ \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = -\frac{\partial A_x}{\partial x} \end{array}\right.$$

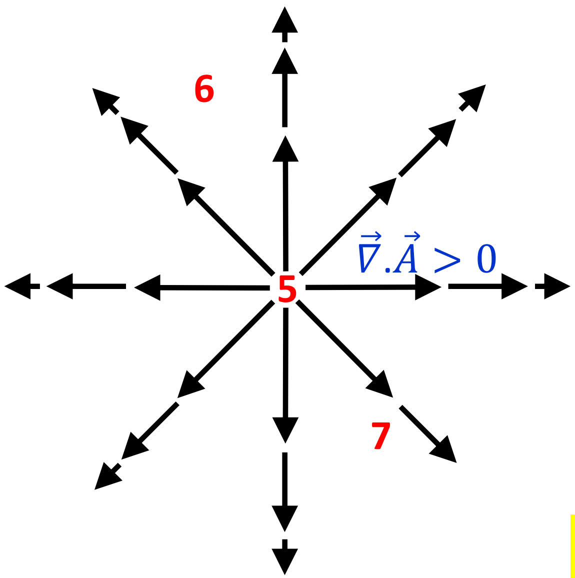
$\rightarrow \boldsymbol{div\,\vec{A} = 0}$ ⇨ **Infinite possibilities**

Physical meaning of the divergence

Divergence expresses **the local change** of the components of a vector field



$\vec{A} \propto sin(\pi z)\vec{k}$ $\quad$ $\vec{A} \propto r^2\vec{u}_r$ $\quad$ $\vec{A} \propto 1/r^2\vec{u}_r$



For points **6** and **7** the situation is less obvious

- The flow lines are clearly spreading from these points
- **BUT** they are getting shorter away from the center

Trade-off between spreading and intensity of the field **locally**

$\vec{A} \propto 1/r^2 \vec{u}_r$
Coulomb's law

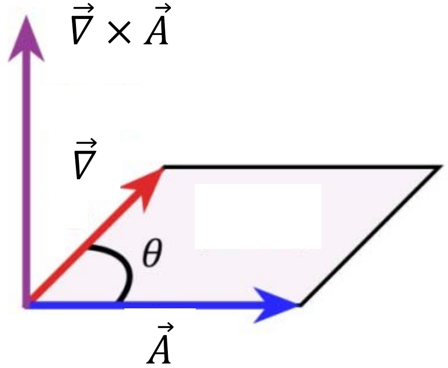
Perfect compensation between **spreading** and **intensity** of the field at any point except around the source **(5)**

$\vec{\nabla}.\vec{A} = 0$ everywhere in space EXCEPT around the origin (source of the field)

Del operator $\vec{\boldsymbol{\nabla}}$ may transform a vector field into another vector field: New concept, **Curl**

Curl of a vector field $\vec{A}(x,y,z)$ at a given point $P(x,y,z)$

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\vec{i}-\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\vec{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\vec{k}$$



The curl vector is always perpendicular to vector $\vec{A}$ on which operates $\vec{\nabla}$ and to the vector operator $\vec{\nabla}$

Let $\vec{U}$ and $\vec{V}$ be two vectors

$$\vec{U}.(\vec{U}\times\vec{V}) = 0 \quad \Leftrightarrow \quad \vec{U} \perp (\vec{U}\times\vec{V})$$

See slide #22

$$\Downarrow$$

$$\vec{U} \to \vec{\nabla}$$

$$\Downarrow$$

$$\vec{\nabla}.(\vec{\nabla} \times \vec{V}) = 0$$

Theorem 1

Theorem 2

$$\boxed{\vec{\nabla}\times\vec{V} = \vec{0} \quad \Rightarrow \quad \vec{V} = \vec{\nabla} f}$$

$$\boxed{\overrightarrow{W} = \vec{\nabla} \times \vec{V}}$$

In Electrostatic

In Magnetostatic $\overrightarrow{W} = \vec{B}$

$$\vec{\nabla}\times\vec{E} = \overrightarrow{0} \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\varphi$$

$\vec{E}$ is the conservative electrostatic field

φ is the scalar potential field

$$\vec{\nabla}.\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$\vec{B}$ is the non conservative magnetostatic field

$\vec{A}$ is the vector potential field

First part of the first Maxwell's equation

Applying Gradient $\vec{E} = -\vec{\nabla}\varphi$

Applying Divergence $\vec{\boldsymbol{\nabla}}.\vec{E} = \vec{\boldsymbol{\nabla}}.\left(-\vec{\nabla}\varphi\right)$

$\Rightarrow \quad \vec{\boldsymbol{\nabla}}.\vec{E} = -\nabla^2\varphi$

Towards Poisson and Laplace equations

Second Maxwell's equation

Applying Curl $\vec{\nabla} \times \vec{E} = \vec{0}$

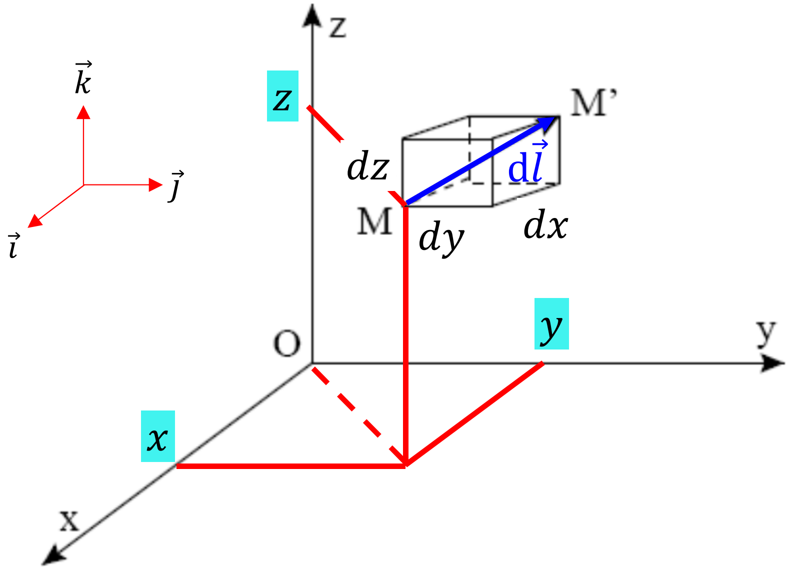
Valid in Electrostatic only

Fourth Maxwell's equation

$\vec{\nabla}.\vec{B} = 0$ $\quad\quad$ $\vec{B} = \vec{\nabla} \times \vec{A}$

Valid ALWAYS

Gradient in **Cartesian** coordinates



Differential

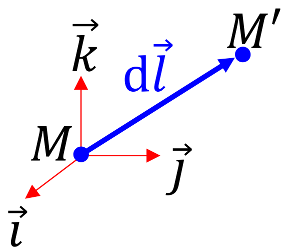
$$\psi(x, y, z) \Rightarrow d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz$$

Via the Gradient

$$d\psi = \vec{\nabla}\psi.\mathrm{d}\vec{l} = \vec{\nabla}\psi.(dx.\vec{\imath} + dy.\vec{\jmath} + dz.\vec{k})$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial x}\vec{\imath} + \frac{\partial\psi}{\partial y}\vec{\jmath} + \frac{\partial\psi}{\partial z}\vec{k}$$

From M to M'



$$\mathrm{d}\vec{l} = dx.\vec{\imath} + dy.\vec{\jmath} + dz.\vec{k}$$

Linear combination

Gradient operator

$$\vec{\nabla} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

Divergence and Curl in **Cartesian** coordinates

$$\vec{\nabla}.\vec{A} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}\right)$$

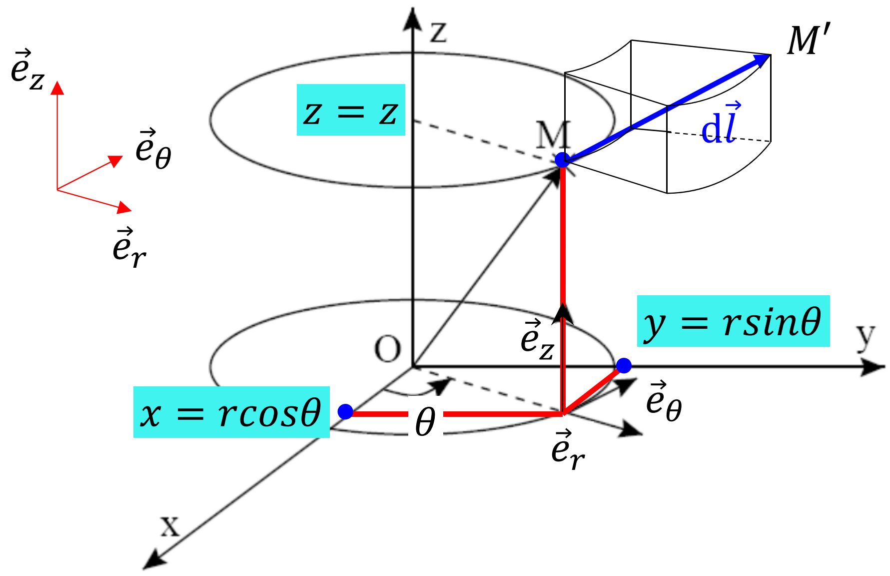
$$\boxed{\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

In the same manner we get the curl
$$\vec{\nabla} \times \vec{A} = \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{pmatrix}$$

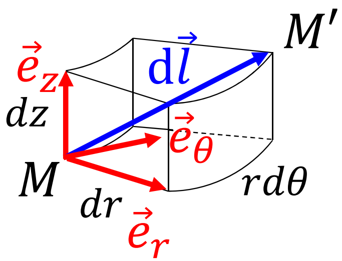
$$\boxed{\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{\jmath} + \frac{1}{r}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}}$$

For cylindrical and spherical symmetry things become **little tricky**

Gradient in **cylindrical** coordinates



From M to M’



$$\mathrm{d}\vec{l} = dr.\vec{e}_r + \boldsymbol{r}d\theta.\vec{e}_\theta + dz.\vec{e}_z$$

Linear combination

Differential

$$\psi(r,\theta,z) \Rightarrow d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\theta}d\theta + \frac{\partial\psi}{\partial z}dz$$

Via the Gradient

$$d\psi = \vec{\nabla}\psi.\mathrm{d}\vec{l} = \vec{\nabla}\psi.(dr.\vec{e}_r + \boldsymbol{r}d\theta.\vec{e}_\theta + dz.\vec{e}_z)$$

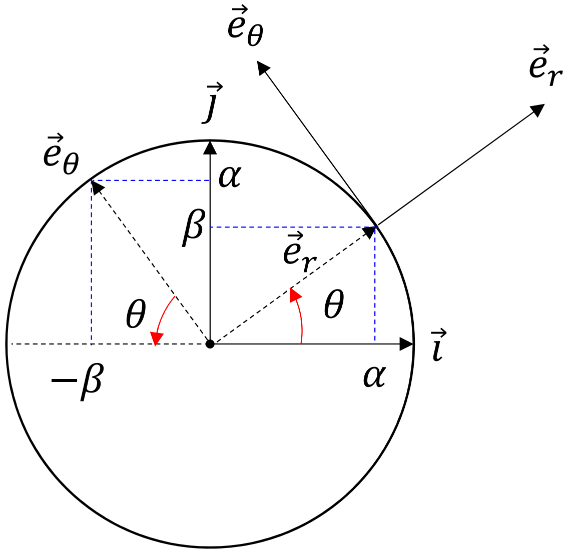
$$d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial\psi}{\partial\theta}\boldsymbol{r}d\theta + \frac{\partial\psi}{\partial z}dz$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\vec{e}_r + \frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial\psi}{\partial\theta}\vec{e}_\theta + \frac{\partial\psi}{\partial z}\vec{e}_z$$

Gradient operator

$$\vec{\nabla} = \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\right)$$

Polar coordinates



$$|\vec{e}_r| = |\vec{e}_\theta| = |\vec{\imath}| = |\vec{\jmath}| = 1$$

$$\vec{e}_r = \alpha\vec{\imath} + \beta\vec{\jmath}$$

$$\alpha = cos\theta$$

$$\beta = sin\theta$$

$$\vec{e}_r = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$$

$$\vec{e}_\theta = -sin\theta\vec{\imath} + cos\theta\vec{\jmath}$$

$$\vec{e}_z = \vec{e}_z$$

$$\begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

$$\boxed{\begin{array}{ll} \dfrac{\partial \vec{e}_r}{\partial r} = \vec{0} & \dfrac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta \\[2ex] \dfrac{\partial \vec{e}_\theta}{\partial r} = \vec{0} & \dfrac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r \end{array}}$$

Divergence and Curl in **cylindrical** coordinates

$$\vec{\nabla}.\vec{A} = \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\right).(A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_z\vec{e}_z)$$

$$\frac{\partial\vec{e}_r}{\partial r} = \vec{0} \quad \frac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta$$

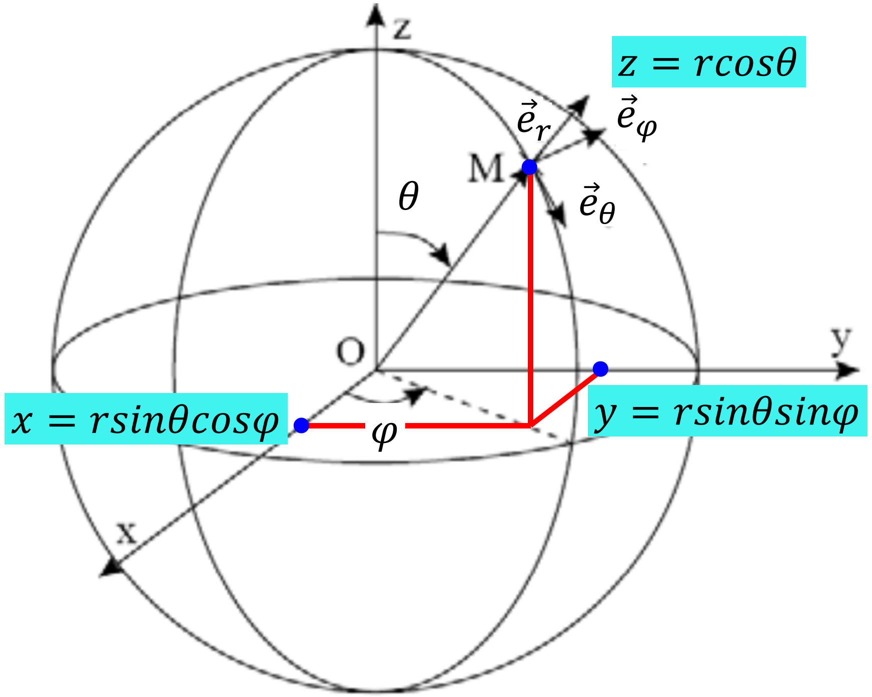
$$\frac{\partial\vec{e}_\theta}{\partial r} = \vec{0} \quad \frac{\partial\vec{e}_\theta}{\partial\theta} = -\vec{e}_r$$

$$\boxed{\vec{\nabla}.\vec{A} = \frac{1}{r}\frac{\partial rA_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}}$$

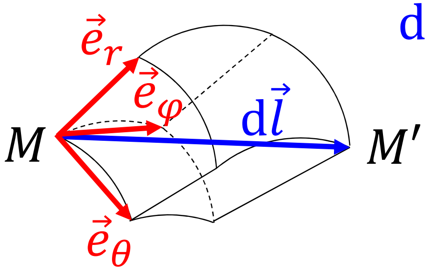
In the same manner we get the curl $\vec{\nabla}\times\vec{A} = \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\right)\times(A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_z\vec{e}_z)$

$$\boxed{\vec{\nabla}\times\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{e}_z}$$

Gradient in **spherical** coordinates



From M to M'



$$\mathrm{d}\vec{l} = dr.\vec{e}_r + \boldsymbol{r}d\theta.\vec{e}_\theta + \boldsymbol{r\sin\theta}d\varphi.\vec{e}_\varphi$$

Linear combination

Gradient

$$\psi(r,\theta,\varphi) \Rightarrow d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\theta}d\theta + \frac{\partial\psi}{\partial\varphi}d\varphi$$

$$d\psi = \vec{\nabla}\psi.\mathrm{d}\vec{l} = \vec{\nabla}\psi.(dr.\vec{e}_r + \boldsymbol{r}d\theta.\vec{e}_\theta + \boldsymbol{r\sin\theta}d\varphi.\vec{e}_\varphi)$$

$$d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial\psi}{\partial\theta}\boldsymbol{r}d\theta + \frac{\boldsymbol{1}}{\boldsymbol{r\sin\theta}}\frac{\partial\psi}{\partial\varphi}\boldsymbol{r\sin\theta}d\varphi$$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\vec{e}_r + \frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial\psi}{\partial\theta}\vec{e}_\theta + \frac{\boldsymbol{1}}{\boldsymbol{r\sin\theta}}\frac{\partial\psi}{\partial\varphi}\vec{e}_\varphi$$

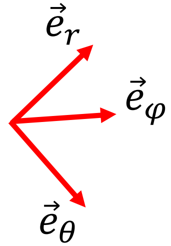
Gradient operator

$$\vec{\nabla} = \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\vec{e}_r = sin\theta cos\varphi\vec{\imath} + sin\theta sin\varphi\vec{\jmath} + cos\theta\vec{k}$$

$$\vec{e}_\theta = cos\theta\cos\varphi\vec{\imath} + cos\theta sin\varphi\vec{\jmath} - \sin\theta\vec{k}$$

$$\vec{e}_\varphi = -sin\varphi\vec{\imath} + cos\varphi\vec{\jmath}$$



$$\frac{\partial\vec{e}_r}{\partial r} = \vec{0} \quad \frac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta \quad \frac{\partial\vec{e}_r}{\partial\varphi} = \vec{e}_\varphi sin\theta$$

$$\frac{\partial\vec{e}_\theta}{\partial r} = \vec{0} \quad \frac{\partial\vec{e}_\theta}{\partial\theta} = -\vec{e}_r \quad \frac{\partial\vec{e}_\theta}{\partial\varphi} = \vec{e}_\varphi cos\theta$$

$$\frac{\partial\vec{e}_\varphi}{\partial r} = \vec{0} \quad \frac{\partial\vec{e}_\varphi}{\partial\theta} = \vec{0} \quad \frac{\partial\vec{e}_\varphi}{\partial\varphi} = -(sin\theta\vec{e}_r + cos\theta\vec{e}_\theta)$$

Divergence and Curl in **spherical** coordinates

$$\frac{\partial \vec{e}_r}{\partial r} = \vec{0} \qquad \frac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta \qquad \frac{\partial \vec{e}_r}{\partial \varphi} = \vec{e}_\varphi sin\theta$$

$$\frac{\partial \vec{e}_\theta}{\partial r} = \vec{0} \qquad \frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r \qquad \frac{\partial \vec{e}_\theta}{\partial \varphi} = \vec{e}_\varphi cos\theta$$

$$\frac{\partial \vec{e}_\varphi}{\partial r} = \vec{0} \qquad \frac{\partial \vec{e}_\varphi}{\partial \theta} = \vec{0} \qquad \frac{\partial \vec{e}_\varphi}{\partial \varphi} = -(sin\theta\vec{e}_r + cos\theta\vec{e}_\theta)$$

$$\vec{\nabla}.\vec{A} = \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{rsin\theta}\frac{\partial}{\partial \varphi}\right).\left(A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\varphi\vec{e}_\varphi\right)$$

$$\boxed{\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{rsin\theta}\frac{\partial sin\theta A_\theta}{\partial \theta} + \frac{1}{rsin\theta}\frac{\partial A_\varphi}{\partial \varphi}}$$

In the same manner we get the curl

$$\vec{\nabla} \times \vec{A} = \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{rsin\theta}\frac{\partial}{\partial \varphi}\right) \times \left(A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\varphi\vec{e}_\varphi\right)$$

$$\boxed{\vec{\nabla} \times \vec{A} = \frac{1}{rsin\theta}\left(\frac{\partial sin\theta A_\varphi}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial r A_\varphi}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{e}_\varphi}$$

Second derivatives of scalar and vector fields

2nd 1st

1) $\vec{\nabla}.(\vec{\nabla}\psi)$ $Div(Grad\psi)$ $= \nabla^2\psi$ (∇^2= Laplacian)

2) $\vec{\nabla}\times(\vec{\nabla}\psi)$ $Curl(Grad\psi)$ $= 0$

"Curl of two parallel vectors is 0"

3) $\vec{\nabla}(\vec{\nabla}.\vec{A})$ $Grad(Div\vec{A})$ Vector field

4) $\vec{\nabla}.(\vec{\nabla}\times\vec{A})$ $Div(Curl\vec{A})$ $= 0$

"Scalar product of two perpendicular vectors is 0"

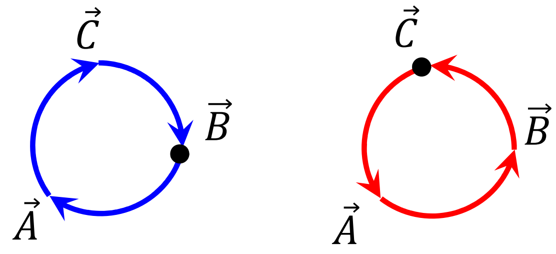
5) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$ $Curl(Curl\vec{A})$

Second derivatives of scalar and vector fields

5) $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)$ $\quad Curl(Curl\vec{A})$

Cross triple product

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A}.\vec{C}\right) - \vec{C}\left(\vec{A}.\vec{B}\right)$$



clockwise counterclockwise

Caution ! *When dealing with vector fields and operators,*
$\vec{C}\left(\vec{A}.\vec{B}\right)$ *may lose physical meaning then we use commutativity* $\left(\vec{A}.\vec{B}\right)\vec{C}$

Curl(Curl $\vec{A}$)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{A}(\vec{\nabla}.\vec{\nabla}) \qquad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - (\vec{\nabla}.\vec{\nabla})\vec{A}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - (\vec{\nabla}.\vec{\nabla})\vec{A}$$

5) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ $\qquad$ $Curl(Curl\vec{A})$

$$\Downarrow$$

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}}$$

Laplacian

Acting on a scalar field ψ

$$\vec{\nabla}.(\vec{\nabla}\psi) = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right)\left(\frac{\partial \psi}{\partial x}\vec{\imath} + \frac{\partial \psi}{\partial y}\vec{\jmath} + \frac{\partial \psi}{\partial z}\vec{k}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Laplacian

In Electrostatic only! $\vec{\boldsymbol{\nabla}}.\vec{E} = \vec{\boldsymbol{\nabla}}.\left(-\vec{\nabla}\varphi\right) = -\boldsymbol{\nabla}^2\varphi$ Poisson Equation

Laplacian

Acting on a vector field $\vec{A}$ Here the Laplacian is acting on **each** component of the vector field

$$\nabla^2\vec{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(A_x\vec{i} + A_y\vec{j} + A_z\vec{j}\right)$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\vec{i} + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\vec{j} + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\vec{k}$$

$$\nabla^2\vec{A} = \underbrace{\left(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right)\vec{\imath}}_{\left(\nabla^2\vec{A}\right)_x = \nabla^2 A_x\vec{\imath}} + \underbrace{\left(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right)\vec{\jmath}}_{\left(\nabla^2\vec{A}\right)_y = \nabla^2 A_y\vec{\jmath}} + \underbrace{\left(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}\right)\vec{k}}_{\left(\nabla^2\vec{A}\right)_z = \nabla^2 A_z\vec{k}}$$

In cylindrical or spherical frame $\left(\nabla^2\vec{A}\right)_r \neq (\nabla^2 A_r)\vec{e}_r$ **!**

$$\vec{\nabla}\times\vec{\nabla}\psi = \vec{0} \quad \Rightarrow \quad \vec{\nabla}\varphi\times\vec{\nabla}\psi \overset{?}{=} \vec{0}$$

<u>Answer to </u>**?** : $\vec{\nabla}\varphi$ and $\vec{\nabla}\psi$ do not have necessarily the same direction

The direction of a gradient depends on the function

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

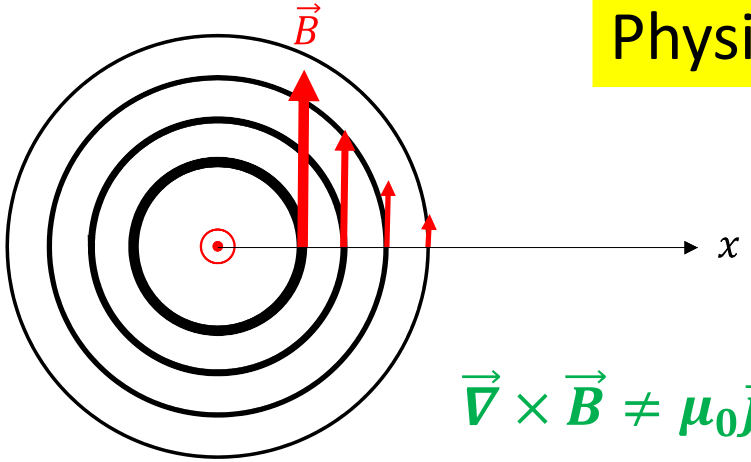
What happens if $\vec{B} = \vec{0}$? Curl free

$$\underbrace{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}_{\substack{||\\0}}\vec{i} + \underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{\substack{||\\0}}\vec{j} + \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{\substack{||\\0}}\vec{k} = \vec{0}$$

It does not mean that $\vec{A} = \vec{0}$

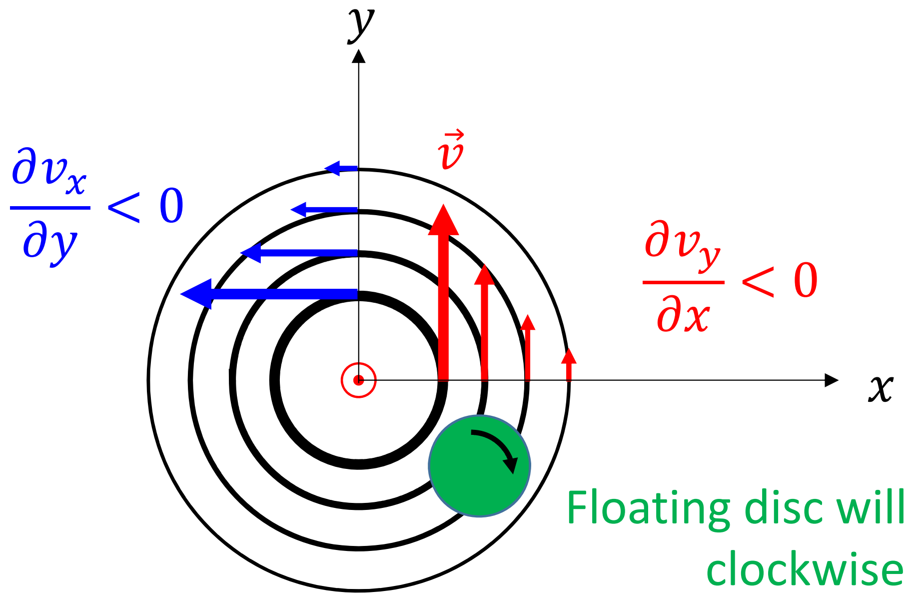
Example: Field outside a solenoid $\vec{B} = \vec{0}$ **BUT** not $\vec{A}$

Physical interpretation of the Curl



$$\vec{\nabla} \times \vec{B} \neq \mu_0 \vec{j}$$

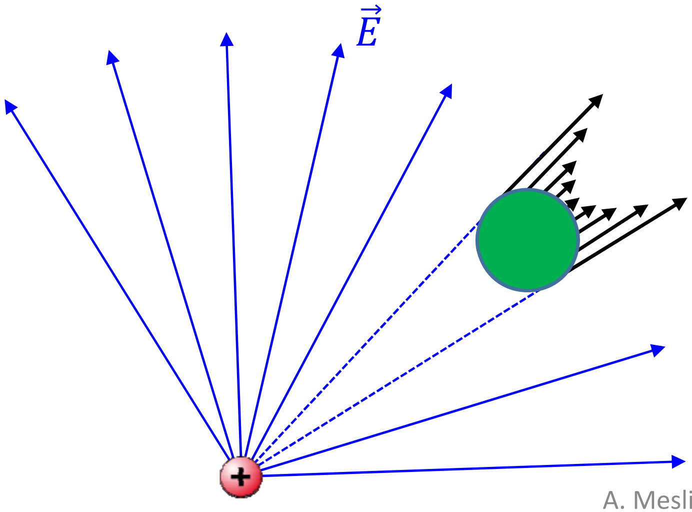
Magnetic field generated
by a current carrying wire



Floating disc will rotate
clockwise

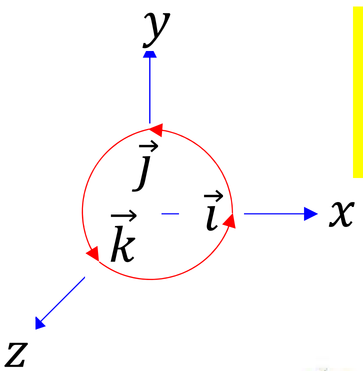
Rotating fluid around a sink with
velocity decreasing along axis x

$$\vec{\nabla} \times \vec{v} \neq \mathbf{0}$$



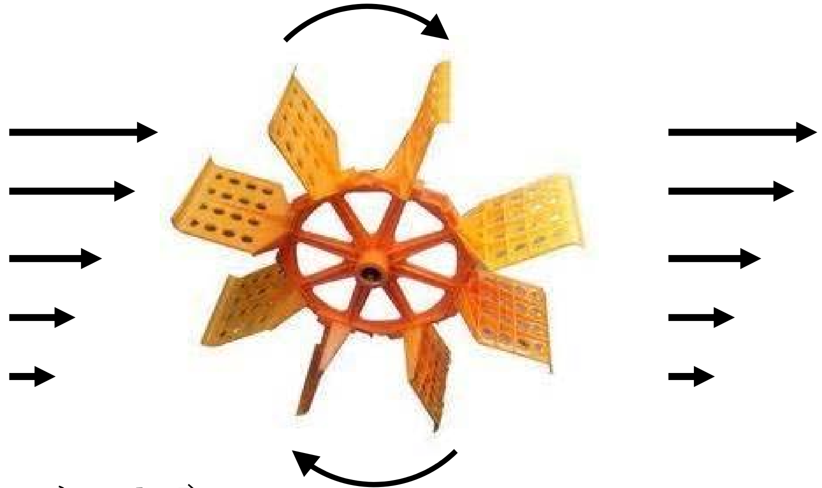
No rotation, only translation along radius

$$\vec{\nabla} \times \vec{E} = \mathbf{0}$$



Another illustration of the curl
Circulation of a paddle wheel in an unperturbed fluid

Does the paddle wheel rotate in each of these two situations?



$\vec{A}(x,y) = a\vec{i}$
$a > 0$

$\vec{\nabla}.\vec{A}(x,y) = 0$

$\vec{A}(x,y) = by\vec{i}$
$b > 0$

No circulation (rotation)

Circulation (rotation)

$\vec{\nabla}.\vec{A}(x,y) = 0$ $\quad$ $\vec{\nabla} \times \vec{A}(x,y) \neq 0$

The space is filled with actions at distance

From single scalar... to the concept of... **scalar field**
From single vector... to the concept of... **vector field**

Differential and Integral equations

Differential

- Gradient
- Divergence
- Curl

Look at a **specific location** in space and at a **given time**

Integral

- Flux
- Circulation
- Curl and divergence free

Look over a **surface or Volume** at a **given time**

Differential form

- First derivation of the fields ⇨ Vector operator $\vec{\nabla}$
  - <u>Gradient</u> $\vec{\nabla}.\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t})$
  - <u>Divergence</u> $\vec{\nabla}.\vec{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t})$
  - <u>Curl</u> $\vec{\nabla}\times\vec{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t})$

- Second derivation of the fields ⇨ operator ∇^2 (Laplacian)

Integral form

- Surface integral: flux and Gauss theorem

- Line integral of a vector field

- Circulation of a vector field and Stokes theorem

Consequence on divergence and curl concepts

Integral form

Symbol conventions

$\int$	Line integral	$\oint$	Closed loop
$\iint$	Open surface	$\oiint$	Closed surface
$\iiint$	Volume		

Integral form

Gauss's theorem

- Relation between flux through a closed surface and divergence through a volume enclosed by that surface

$$\Leftrightarrow \quad \oiint \vec{V}.\vec{n}dA = \iiint \vec{\nabla}.\vec{V}dV$$

Closed surface *Volume enclosed by the surface*

Stoke's theorem

- Relation between the circulation of a field vector (line integral of its tangential component) around a closed loop and the flux of (the normal component) through the surface bounded by the loop

$$\Leftrightarrow \quad \oint \vec{V}.d\vec{l} = \iint (\vec{\nabla} \times \vec{V}).\vec{n}dA$$

Closed path (loop) *Open surface*

What happens if the surfaces is closed ?

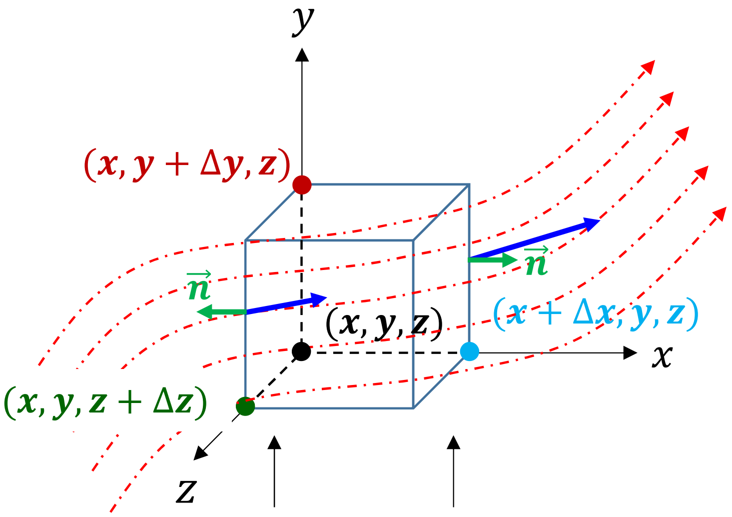
Gauss's theorem

$$\oiint \vec{V}.\vec{n}\,dA = \iiint \vec{\nabla}.\vec{V}\,dV$$

Closed surface

Volume enclosed by the surface

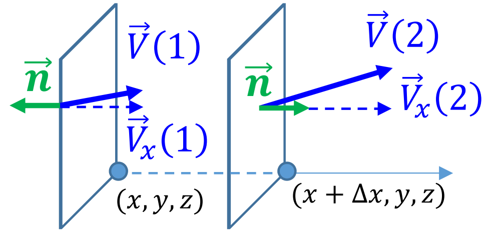
Flux of vector field $\vec{V}$ through a closed surface and its relation to divergence: **Gauss theorem**



Side 1 Side 2

Areas $\Delta A = \Delta y\ \Delta z$

$$Flux = \int \vec{V}.d\vec{A}$$



Normal component

$$\Delta A = \Delta y\ \Delta z$$

Total flux out of opposite sides

$$-V_x(1)\ \Delta y\ \Delta z + V_x(2)\ \Delta y\ \Delta z = \left(V_x(2) - V_x(1)\right)\ \Delta y\ \Delta z$$

$$\frac{[V_x(2)\ - V_x(1)]}{\Delta x} \Delta x \Delta y \Delta z = \frac{\partial V_x}{\partial x} \Delta x \Delta y \Delta z$$

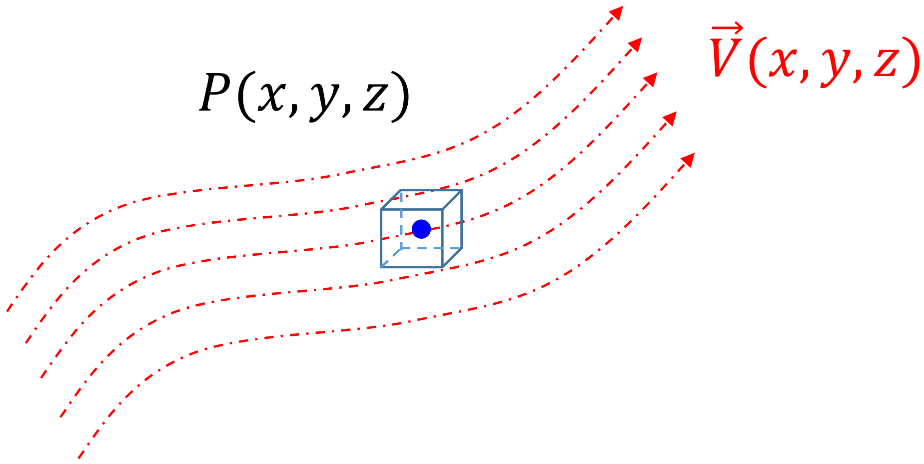
Total flux out of the elementary box

$$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) \Delta x \Delta y \Delta z$$

Gauss theorem

$$\oiint \vec{V}.d\vec{A} = \iiint \vec{\nabla}.\vec{V}\, dV$$

Closed surface *Volume enclosed by the surface*



The divergence of vector field $\vec{V}(x, y, z)$ at point $P(x, y, z)$ is the flux or "**NET** outgoing flow" of $\vec{V}(x, y, z)$ per unit volume in the neighborhood of $P(x, y, z)$

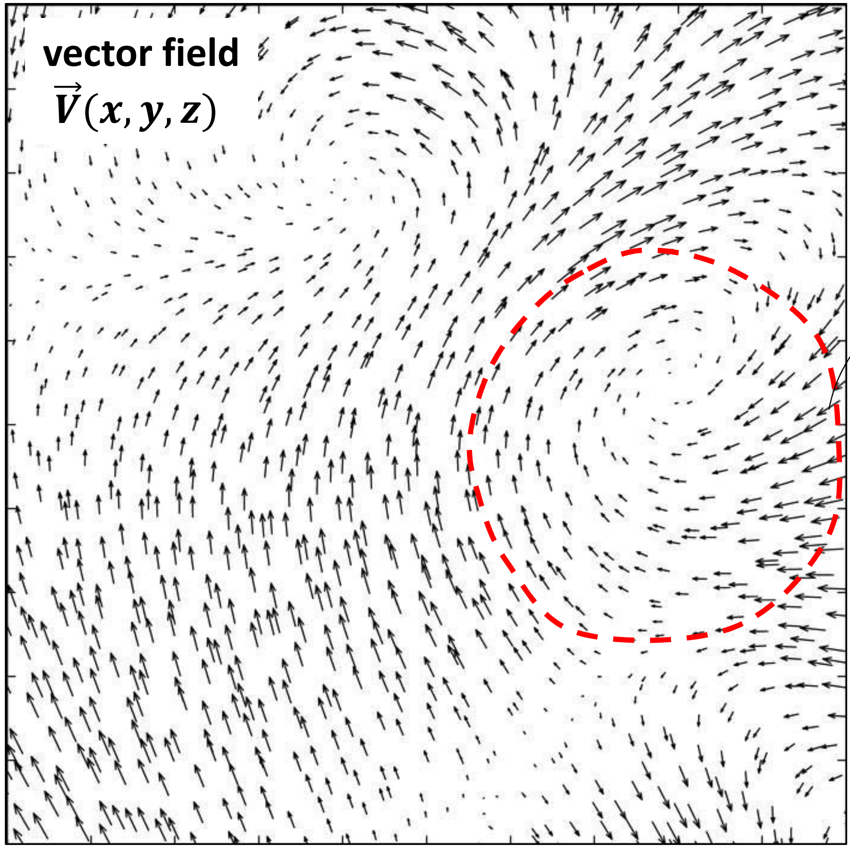
Stoke’s theorem

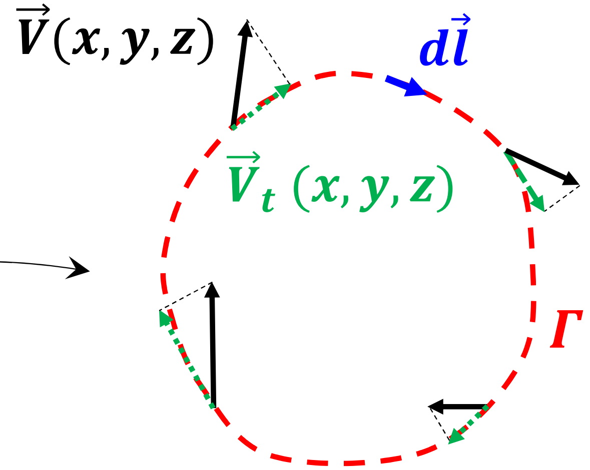
$$\oint \vec{V}.d\vec{l} = \iint (\vec{\nabla} \times \vec{V}).\vec{n}dA$$

Closed path (loop)

Open surface

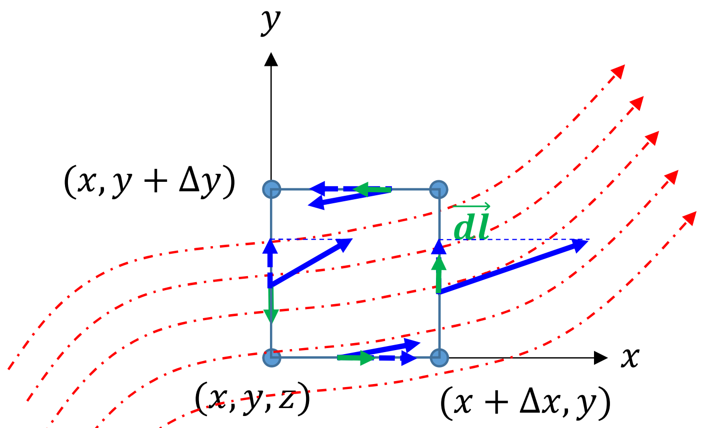
Circulation around a closed loop and its relation to curl: Stokes theorem



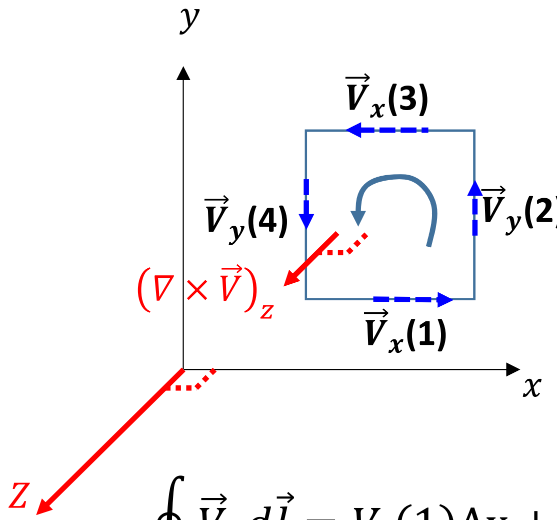


Circulation around the closed path (Γ) = loop

$$\oint_{\Gamma} \vec{V}.d\vec{l} = \oint_{\Gamma} V_t\,.\,dl$$



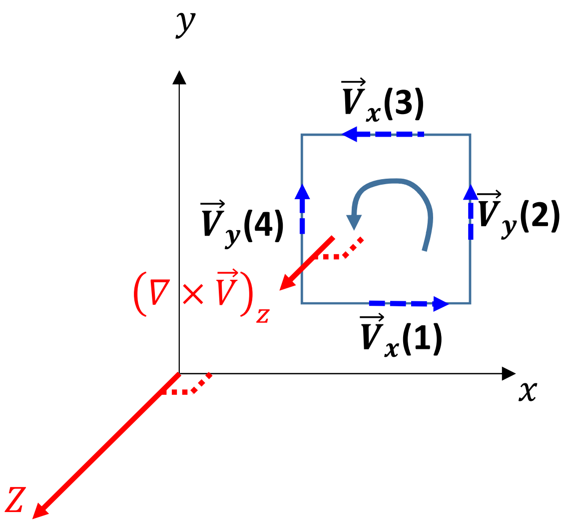
Area $\Delta A = \Delta x\ \Delta y$

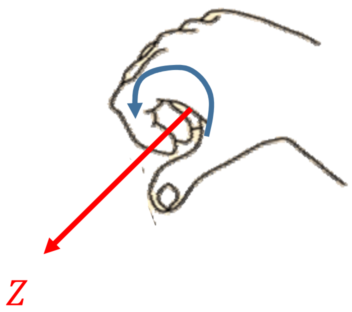


$$\oint \vec{V}.d\vec{l} = V_x(1)\Delta\mathrm{x} + V_y(2)\Delta\mathrm{y} - V_x(3)\Delta\mathrm{x} - V_y(4)\Delta\mathrm{y}$$

$$= \underbrace{[V_x(1) - V_x(3)]}_{-\frac{\partial V_x}{\partial y}\Delta y}\Delta\mathrm{x} + \underbrace{[V_y(2) - V_y(4)]}_{+\frac{\partial V_y}{\partial x}\Delta x}\Delta\mathrm{y}$$

$$\boxed{\oint \vec{V}.d\vec{l} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\Delta x\Delta y = (\nabla \times \vec{V})_z\ \Delta x\Delta y}$$



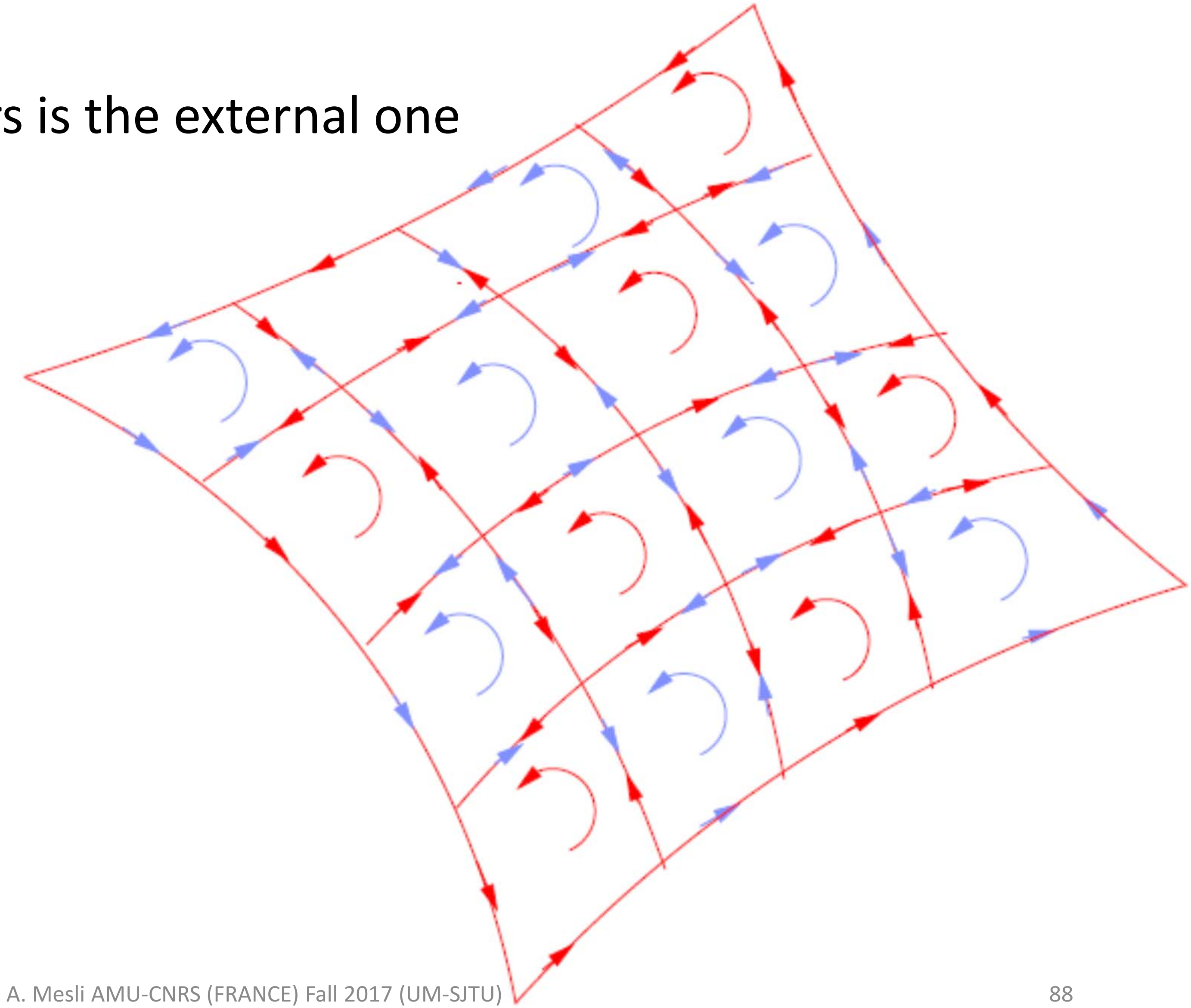


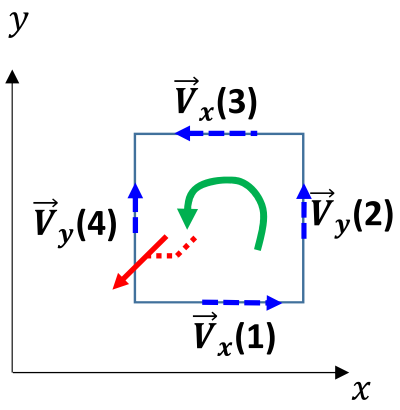
$(\nabla \times \vec{V})_z$ = The normal component to the surface $\Delta x \Delta y$

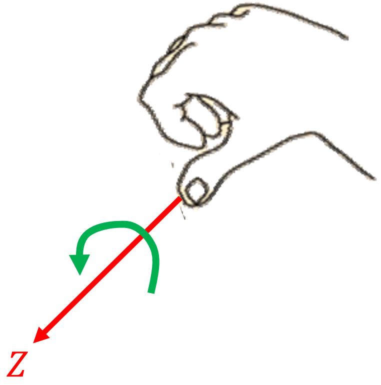
A circulation of a vector around a closed path (loop) gives a vector ***perpendicular*** *to the surface bounded by the loop*

$$\oint_{loop} \vec{V}.d\vec{l} = \iint_{Open\ surface} (\vec{\nabla} \times \vec{V}).\vec{n}\, dA$$

The only path that matters is the external one







A circulation of a vector around a closed loop gives a vector <u>*perpendicular*</u> *to the surface bounded by the loop*

Stokes Theorem

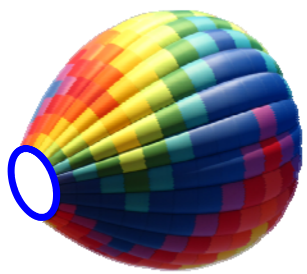
$$\oint \vec{V}.d\vec{l} = \iint (\vec{\nabla} \times \vec{V}).\vec{n}\, dA$$

Closed path (loop)

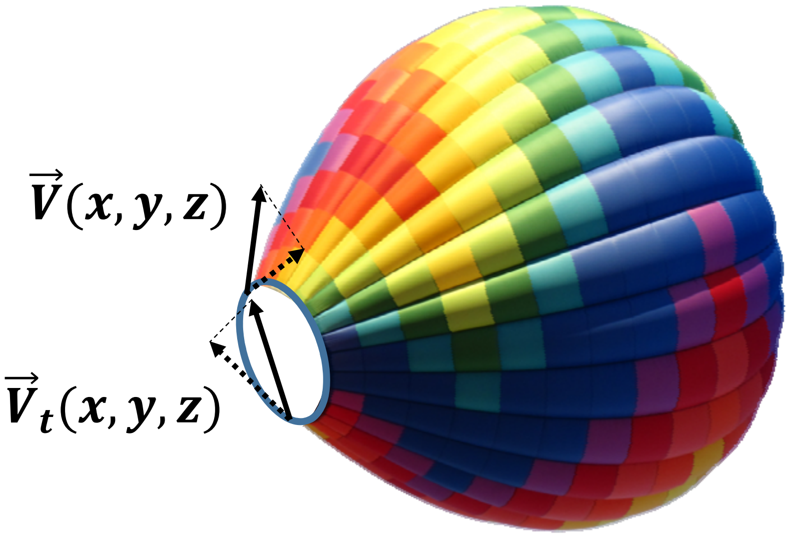
Any surface Bounded by the closed loop

Could be the surface of a balloon

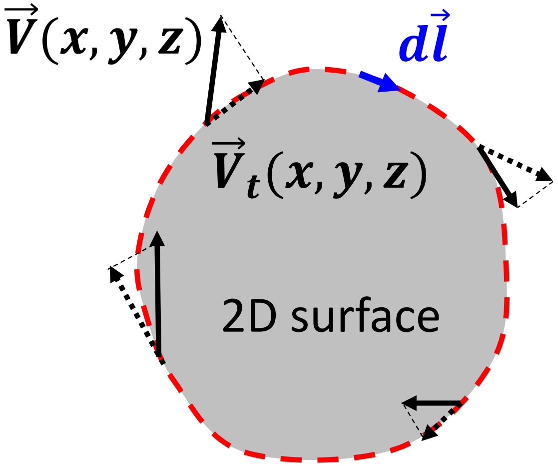
Closed path (loop)



What happens if the loop shrinks to zero ? **Div and Curl free field**



3D surface



$$\oint \vec{V}.d\vec{l} = \oint V_t\,.\,dl$$

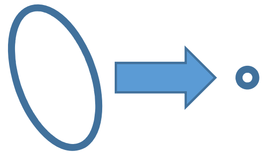
$$\oint \vec{V}.d\vec{l} = 0$$

$$f(2) - f(1) = \int_1^2 \vec{V}.d\vec{l} = 0$$

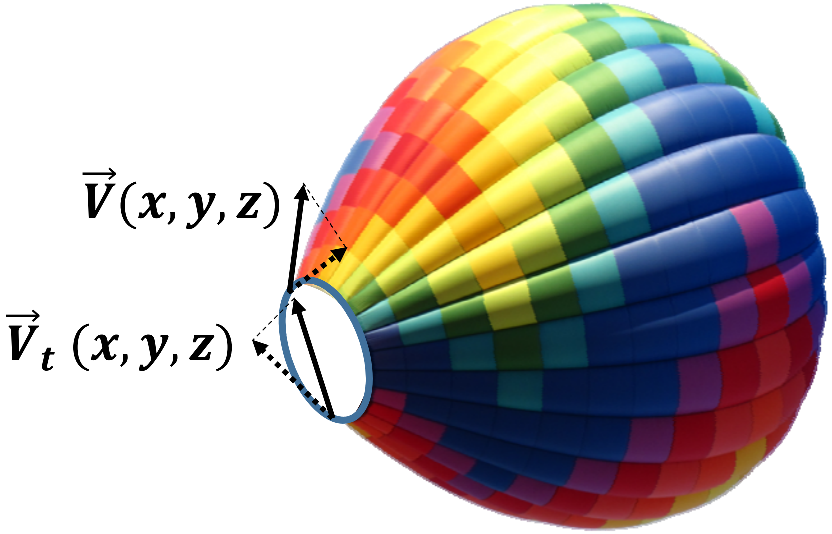
Exact differential

$$f(1) = f(2)$$

The loop shrinks to zero



The surface of the balloon closes



3D surface

$$\oint \vec{V}.d\vec{l} = \iint (\vec{\nabla} \times \vec{V}).\vec{n}\, dA = \mathbf{0}$$

$$\oiint (\vec{\nabla} \times \vec{V}).\vec{n} dA = 0$$

Closed surface

Gauss theorem $\iiint \vec{\nabla}.(\vec{\nabla} \times \vec{V}) dV = \mathbf{0}$

$\vec{\nabla}.(\vec{\nabla} \times \vec{V}) = \mathbf{0}$ Already demonstrated **(slide #57)**

Div free field

Some interesting consequences

$$f(2) - f(1) = \int_1^2 \vec{\nabla} f . d\vec{l}$$

If the loop is closed then $\Rightarrow$ $\int_1^2 \vec{\nabla} f . d\vec{l} = \mathbf{0}$

$\Downarrow$

If $\vec{V} = \vec{\nabla} f$

Stokes theorem

$$\oint \vec{V} . d\vec{l} = \oint \vec{\nabla} f . d\vec{l} = \iint \underbrace{(\vec{\nabla} \times \vec{\nabla} f)}_{\mathbf{0}} . \vec{n}\, dA = \mathbf{0}$$

Curl free field

Already demonstrated **(slide #57)**

Strategy solving problems

Refer to slide #4

Tool box

Operators
- Gradient $\vec{\nabla}\varphi(\vec{r},t)$
- Divergence $\vec{\nabla}.\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$
- Curl $\vec{\nabla}\times\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$

Effects
- Induction
- Polarization
- Equilibrium vs steady state

Concepts
- Scalar vs vector fields
- Lines field

Theorems
- Gauss theorem
- Stokes theorem

Laws and principles
- Lorentz force
- Work and Energy Conservation
- Momentum conservation
- Symmetry
- Superposition principle
- Charge conservation

Source of electromagnetism

$\rho(\vec{r},t) - \sigma(\vec{r},t) - \lambda(\vec{r},t) - \vec{J}(\vec{r},t)$

Evaluate
- At a given point in space P, $(\vec{r},t)$
- In a defined volume – surface – segment

$\varphi(\vec{r},t) - \vec{E}(\vec{r},t) - \vec{B}(\vec{r},t) - \vec{A}(\vec{r},t)$

Interesting questions for those who want to go a bit more into vector calculus

Q1

Any three number can be components of a vector **only if**, when we rotate the coordinate system, they transform among themselves in the correct way. Considering coordinate frames (x, y, z) and $(x', y', z' = z)$ rotated by θ, find the proper transformation relationships

Q2

Let a particle be at position A in a two dimensional frame (x, y) . Its coordinates are expressed in polar system. Knowing r and θ helps determining the trajectory of the particle.

1. Draw the reference frame in which you show the position of the particle, and draw the unit Cartesian and polar vectors
1. Express the position in coordinate frame and extract the velocity and acceleration
2. Deduce the radial and circumferential acceleration
3. Finally calculate the velocity and acceleration amplitudes