Midterm 2 Review - CH4&CH5

Chapter 4 Solution of Electrostatic Problems

Poisson's and Laplace's Equations

$$\boldsymbol{E} = -\nabla V$$
$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon}$$

Poisson's Equation (in a homogeneous medium, ϵ is a constant over space)

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Laplace's Equation (in a simple medium, $\rho = 0$)

$$\nabla^2 V = 0$$

Uniqueness of Electrostatic Solutions

A solution of Poisson's equation that satisfies the given boundary conditions is a unique solution.

Methods of Images

Replacing boundaries by appropriate image charges in lieu of a formal solution of Poisson's or Laplace's equation.

1. Point Charge and Grounded Conduction Places
   I. Remove the conductor;
   II. Replace with an image point charge $-Q$ at $y = -d$;
   Note: For $y < 0$ region, $\boldsymbol{E} = 0, V = 0$
2. Line Charge and Parallel Conducting Cylinder
   I. Image is a parallel line charge inside the cylinder;
   II. The line charge should be on OP;
   III. Cylinder surface is an equi-potential surface;
   IV.
   $$\rho_i = \rho_l, d_i = \frac{a^2}{d}$$
3. Two Parallel Conducting Cylinders
   I.
   $$b^2 = c_1^2 - a_1^2 = c_2^2 - a_2^2$$
   II.
   $$c_1 + c_2 = D$$
4. Point Charge and Grounded Conducting Sphere
   I. Image charge is a negative point charge inside the sphere and on the line OQ;
   II.
   $$Q_i = -\frac{a}{d}Q, d_i = \frac{a^2}{d}$$

5. Charged Sphere and Grounded Plane
   I. Both sphere and plane are equi-potential surfaces;
   II. Point charge and plane;
   III. Point charge and sphere;
   IV. Repeat II. and III.

$$Q_0 \Rightarrow -Q_0, -Q_0 \Rightarrow Q_1 = \frac{a}{2c}Q_0, d_1 = \frac{a^2}{2c} \cdots$$

See the examples on the textbook.

Boundary-Value Problems in Cartesian Coordinates

$$V(x,y,z) = X(x)Y(y)Z(z)$$
$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$
$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0$$
$$k_x^2 + k_y^2 + k_z^2 = 0$$

TABLE 4–1
Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

Boundary-Value Problems in Cylindrical Coordinates

$$V(r,\phi) = R(r)\Phi(\phi)$$
$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0$$

case 1:

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi$$
$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - n^2 R(r) = 0$$
$$R(r) = A_r r^n + B_r r^{-n}$$

If r can be zero, $B_r = 0$; if r can be infinity, $A_r = 0$.
case 2:

$$\Phi(\phi) = A_0\phi + B_0$$

$R(r) = C_0 \ln r + D_0$ and $V(r) = C_1 \ln r + C_2$.

Boundary-Value Problems in Spherical Coordinates

Assume ϕ is independent.

$$V(R,\theta) = \Gamma(R)\Theta(\theta)$$

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right] + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] = 0$$

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right] = k^2$$

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] = -k^2$$

$$\Gamma_n(R) = A_nR^n + B_nR^{-(n+1)}$$

where $n(n+1) = k^2$, $n = 0, 1, 2, \ldots$

$$\Theta_n(\theta) = P_n(\cos\theta)$$

TABLE 4–2
Several Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

Steady Electric Currents

Current Density and Ohm's Law

$$\Delta Q = Nq\boldsymbol{u}\cdot\boldsymbol{a}_n\Delta s\Delta t$$

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\boldsymbol{u}\cdot\Delta\boldsymbol{s}$$

$$\boldsymbol{J} = Nq\boldsymbol{u} = \rho\boldsymbol{u}$$

$$\Delta I = \boldsymbol{J}\cdot\Delta\boldsymbol{s}$$

$$I = \int_S \boldsymbol{J}\cdot d\boldsymbol{s}$$

If there are more than one kind of charge carriers,

$$\boldsymbol{J} = \sum_i N_iq_i\boldsymbol{u}_i$$

For electrons,

$$\boldsymbol{u} = -\mu_e\boldsymbol{E}$$

μ_e is the electron mobility $(m^2/V\cdot s)$

$$\boldsymbol{J} = -Ne\mu_e\boldsymbol{E} = -\rho_e\mu_e\boldsymbol{E}$$

σ is conductivity, it is equal to $-\rho_e\mu_e$.

$$\boldsymbol{J} = \sigma\boldsymbol{E}$$

For semiconductors, $\sigma = -\rho_e\mu_e + \rho_h\mu_h$

$$V_{12} = RI$$
$$R = \frac{l}{\sigma S}$$

Electromotive Force and Kirchhoff's Voltage Law

In a closed circuit, to maintain a steady current, there must be non-conservative field.

$$\mathcal{V} = \int_2^1 \boldsymbol{E}_i \cdot d\boldsymbol{l} = -\int_2^1 \boldsymbol{E} \cdot d\boldsymbol{l}$$
$$\mathcal{V} = V_{12} = V_1 - V_2$$

KVL:

$$\sum_j \mathcal{V}_j = \sum_k R_k I_k$$

Equation of Continuity and Kirchhoff's Current Law

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial\rho}{\partial t} = 0$$

KCL:

$$\sum_j I_j = 0$$

$$\sigma\nabla \cdot \boldsymbol{E} = -\frac{\partial\rho}{\partial t} = \frac{\sigma}{\epsilon}\rho$$
$$\rho = \rho_0 e^{-(\sigma/\epsilon)t}$$

Relaxation time: time for ρ_0 to decay to $1/e$.

$$\tau = \frac{\epsilon}{\sigma}$$

Power Dissipation and Joule's Law

$$dP = \boldsymbol{E} \cdot \boldsymbol{J} dv$$

Power density is $\boldsymbol{E} \cdot \boldsymbol{J}$

$$P = \int_V \boldsymbol{E} \cdot \boldsymbol{J} dv$$

In a conductor with constant cross section,

$$P = VI$$

Boundary Conditions for Current Density

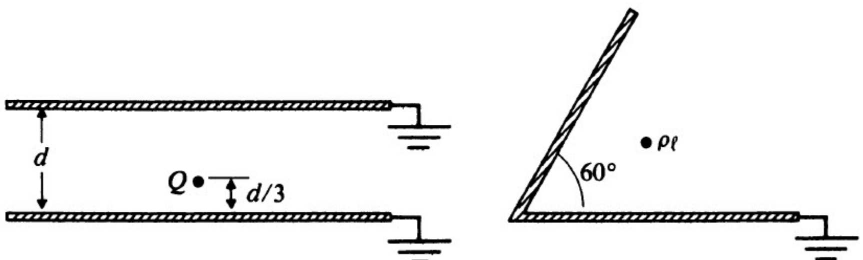
$$J_{1n} = J_{2n}$$
$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

Resistance Calculations

$$RC = \frac{C}{G} = \frac{\epsilon}{\sigma}$$

Governing Equations for Steady Current Density	
Differential Form	**Integral Form**
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$

Exercise



(a) Point charge between grounded parallel planes. **(b) Line charge between grounded intersecting plane.**

1. A point charge Q located between two large, grounded, parallel conducting plane as shown in figure.

2. An infinite line charge ρ_l located midway between two large, intersecting conducting planes forming a 60-degree angle, as shown in figure.

3. A point charge Q is located inside and at distance d between the center of a grounded spherical conducting shell of radius b where $b > d$. Use the method of images to determine the potential distribution inside the shell and the charge density ρ_s induced on the inner surface of the shell.

4. Two conducting spheres of equal radius a are maintained at potentials V_0 and 0, respectively. Their centers are separated by a distance D.
   a) Find the image charges and their locations that can electrically replace the two spheres.
   b) Find the capacitance between the two spheres.

5. A d-c voltage of 6V applied to the ends of 1km of a conducting wire of 0.5mm radius results in a current of 1/6A. Find
   a) the conductivity of the wire
   b) the electric field intensity in the wire
   c) the power dissipated in the wire
   d) the electron drift velocity, assuming electron mobility in the wire to be $1.4 \times 10^{-3} m^2/V \cdot s$

6. A d-v voltage V_0 is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ϵ_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ϵ_2 and conductivity σ_2 in the region $c < r < b$. Determine
   a) the current density in each region

b) the surface charge densities on the inner and outer conductors and at the interface between the two dielectrics.