VE230 RC2

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Maxwell's Equations

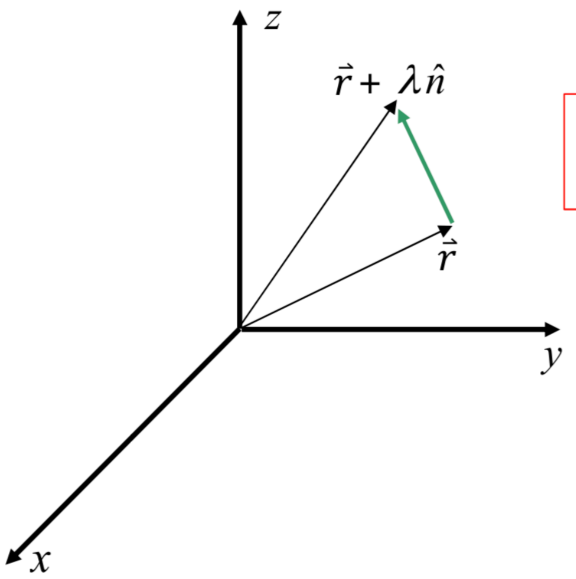
$$\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0} \longleftrightarrow \oiint_S \vec{E} \cdot \hat{n} dS = \frac{Q}{\varepsilon_0}$$

$$\nabla \times \vec{E} = 0 \longleftrightarrow \oint_L \vec{E} \cdot \hat{t} dl = 0$$

Vector Calculus

Directional derivative $$\frac{Du(\vec{r})}{Dn} = \lim_{\lambda \to 0} \frac{u(\vec{r} + \lambda \hat{n}) - u(\vec{r}')}{\lambda}$$

Gradient $$\nabla u(\vec{r}) \cdot \hat{n} = \frac{Du(\vec{r})}{Dn} = \lim_{\lambda \to 0} \frac{u(\vec{r} + \lambda \hat{n}) - u(\vec{r})}{\lambda}$$



Derivative along a direction **n** with a distance λ

Vector Calculus

$$|\vec{r}+\lambda\hat{n}| = \sqrt{r^2 + 2\lambda\hat{n}\cdot\vec{r} + \lambda^2} = r\sqrt{1 + 2\frac{\lambda}{r}\hat{n}\cdot\hat{r} + \frac{\lambda^2}{r^2}} \approx r\left(1 + \frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)$$

$$when \quad \lambda << r$$

Useful examples:

$$\nabla r = \hat{r}$$

When r is not 0:

$$\nabla\frac{1}{r} = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3} \qquad \nabla\frac{1}{r^n} = -n\frac{\hat{r}}{r^{n+1}} = -n\frac{\vec{r}}{r^{n+2}} \qquad \nabla\ln r = \frac{\hat{r}}{r} = \frac{\vec{r}}{r^2}$$

The Laplacian of the function 1/r

When r is not 0: $\nabla^2 \frac{1}{r} = \nabla \cdot \nabla \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3}$ $\quad \nabla \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3}\nabla \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3} = \frac{3}{r^3} - 3\vec{r} \cdot \frac{\vec{r}}{r^5} = \boxed{0}$

When r is 0: $\nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = -4\pi\delta(\vec{r})$

Or alternatively, we can say that the function: $\varphi = \frac{1}{4\pi r}$

Is a solution to the differential equation: $\nabla^2 \varphi = -\delta(\vec{r})$

This result is of fundamental importance in the subject of electrostatic and magnetostatic fields !!!!!

Gauss's Law and Applications

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.}$$

Gauss's law: The total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

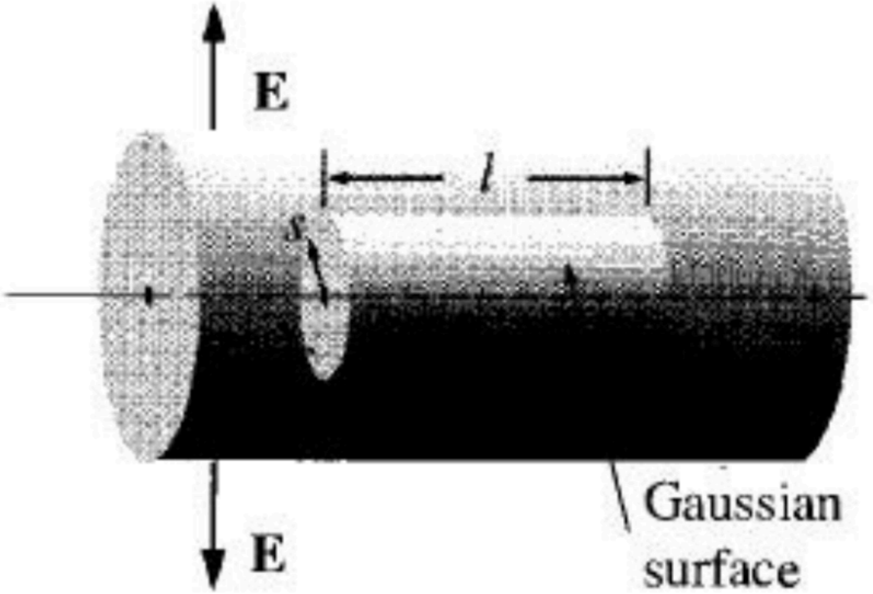
S: can be any hypothetical closed surface

Useful:

- **The normal component of E is constant over an enclosed surface**
- **A high degree of symmetry in the charge distribution or in the electrical field**

Practice

A long cylinder (Fig. 2.21) carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.



$$\mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$$

Electric Potential

Define a scalar **electric potential V** such that $\boxed{\mathbf{E} = -\nabla V}$

Physical Meaning: Work done against the field in moving a unit charge from point p_1 to p_2 another

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(J/C or V).}$$

Electric Potential Difference

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(V).}}$$

In going against the E field the electric potential V **increases**

Electric Potential

$$\nabla V$$

Its direction is ⊥ constant-V surfaces

$$\boxed{\mathbf{E} = -\nabla V}$$

E ⊥ constant-V surfaces

Field lines
Streamlines

Equipotential lines
Equipotential surfaces

Electric Potential due to a Charge

- V(*R*) of **a point charge at origin**

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(V)}.}$$

With reference point at infinity

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \right) \cdot (\mathbf{a}_R \, dR),$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \qquad \text{(V)}.}$$

- Potential difference between any two points

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right).$$

A charge is moved from P_1 to P_2 (against the E field if V_{21}>0)

Electric Potential due to n Discrete Point Charges

$$V = \frac{q}{4\pi\epsilon_0 R} \quad \text{(V)}.$$

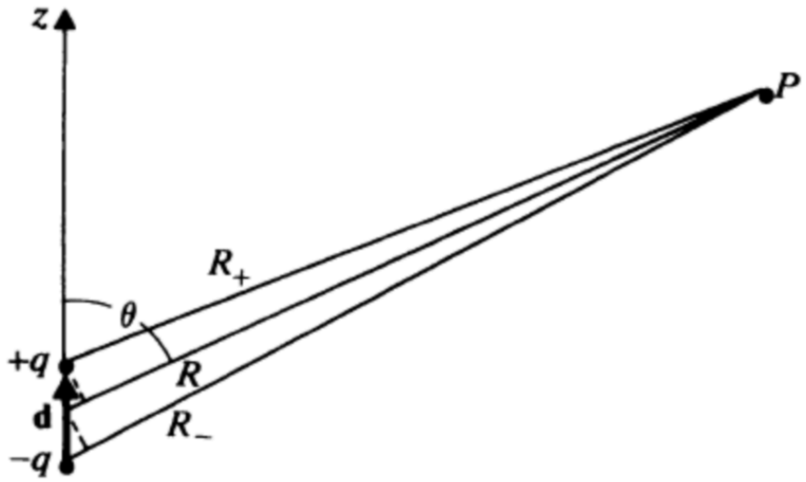
Reference point at infinity

R ➔ ***R*** – ***R'*** (Charges located at ***R'***)

$\sum$

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \quad \text{(V)}.$$

Electric Potential of an Electric Dipole



$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

If $d \ll R$

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right).$$

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad (\mathrm{V}),}$$

where $\mathbf{p} = q\mathbf{d}$.

$$\boxed{\mathbf{d}\cdot\mathbf{a}_R = d\cos\theta}$$

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2} \Rightarrow \mathbf{E} = -\nabla V = -\mathbf{a}_R\frac{\partial V}{\partial R} - \mathbf{a}_\theta\frac{\partial V}{R\,\partial\theta}$$

$$= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta).$$

At θ=0 (Field/Potential along z axis)

	CHARGE	DIPOLE
POTENTIAL	$V \approx \dfrac{q}{4\pi\varepsilon_0 r}$	$V \approx \dfrac{p}{4\pi\varepsilon_0 r^2}$
FIELD	$E \approx \dfrac{q}{4\pi\varepsilon_0 r^2}$	$E \approx \dfrac{p}{4\pi\varepsilon_0 r^3}$

Conductors in Static Electric Field

Inside a Conductor (Under Static Conditions)
$\rho = 0$
$\mathbf{E} = 0$

Boundary Conditions at a Conductor/Free Space Interface
$E_t = 0$
$E_n = \dfrac{\rho_s}{\epsilon_0}$

Polarization

- Macroscopic effect
- Polarization vector:

N = nΔv,
where N is the Total # in a volume (Δv);
n is the number density

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \qquad (\mathrm{C/m^2}),$$

P: volume density of electric dipole moment **p**

$$d\mathbf{p} = \mathbf{P}\, dv'$$

- Derivation for dielectrics

$$\boxed{V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad (\mathrm{V}),} \Rightarrow dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\, dv'.$$

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\boldsymbol{\nabla} \cdot \mathbf{P}.}$$

Polarization charge densities, or bound-charge densities

Practice

Derive $\mathbf{\nabla} \cdot \mathbf{D} = \rho$

Dielectric

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \underline{\rho_p}).$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho.$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\text{C/m}^2).}$$

Where **D**: electric flux density, electric displacement

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\text{C/m}^3),}$$

$$\int_V \nabla \cdot \mathbf{D}\, dv = \int_V \rho\, dv$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (\text{C}).}$$

Integral form

Equations of Electrostatics in Any Medium

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\text{C/m}^3),}$$

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

Electric Susceptibility and Relative Permittivity

- Electric susceptibility
  - For linear and isotropic medium,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E},$$

χ_e dimensionless quantity called ***electric susceptibility***

- Relative permittivity (dielectric constant)

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).}$$

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon_0(1 + \chi_e)\mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}), \end{aligned}}$$

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

ε: absolute permittivity (or simply permittivity)