

VE281

Data Structures and Algorithms

Red-black Trees

Learning Objectives:

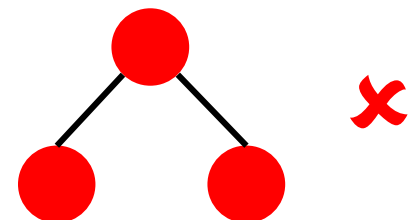
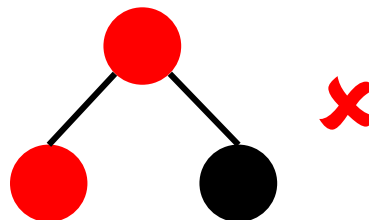
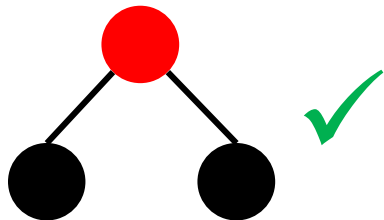
- Know what a red-black tree is and its properties
- Know how to do insertion for a red-black tree

Outline

- Red-black Trees: Basics
- Red-black Trees: Insertion

Red-Black Tree

- A binary search tree. The data structure requires an extra one-bit color field in each node.
- Property
 1. Every node is either red or black.
 2. **Root rule**: The root is black.
 3. **Red rule**: Red node can **only have** black children.
 - Can't have two consecutive red nodes on a path.

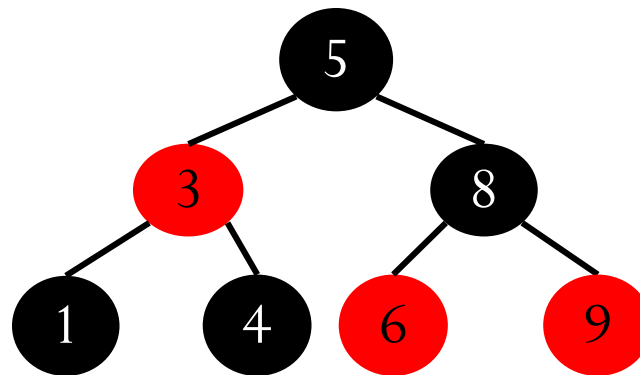


4. **Path rule**: **Every** path from a node x to NULL must have the **same number** of black nodes (including x itself).

Red-Black Tree Example

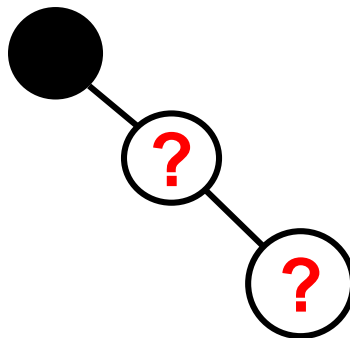
- Property

1. A binary search tree
2. Every node is either red or black.
3. **Root rule:** The root is black.
4. **Red rule:** Red node can **only have** black children.
5. **Path rule:** **Every** path from a node x to NULL must have the **same number** of **black** nodes (including x itself).



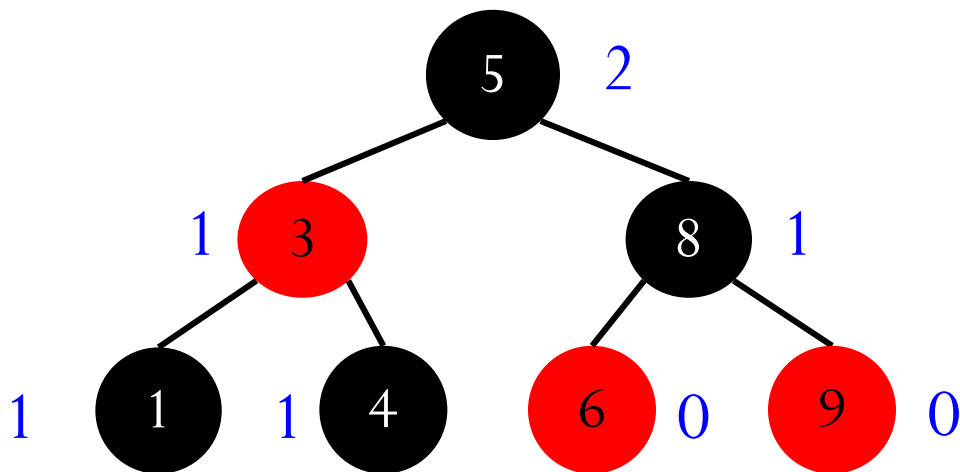
Counter Example

- Property
 1. A binary search tree
 2. Every node is either red or black.
 3. **Root rule**: The root is black.
 4. **Red rule**: Red node can **only have** black children.
 5. **Path rule**: **Every** path from a node x to NULL must have the **same number** of black nodes (including x itself).
- **Claim**: a chain of length 3 cannot be a red-black tree



Black Height

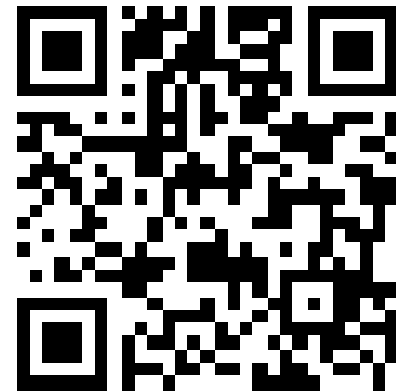
- **Black height** of a node x is the number of black nodes on the path from x to NULL, **including** x itself.





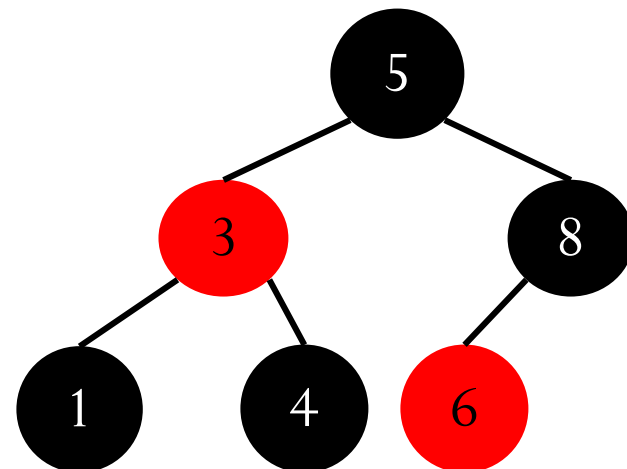
Which Statements Are Correct?

- A. It is possible for a **red** node to have a single child.
- B. It is possible for a **black** node to have a single child.
- C. It is possible for a node to have two children of different colors.
- D. It is possible for a node to have two children and the node and its children are all of the same color.



Implication of the Rules

- If a **red** node has **at least one** child, it must have **two children** and they must be **black**.
 - Why?
 - A red node's child can only be black.
 - If has only one black child, then violate the **path rule**.
- If a black node has **only one** child, that child must be a **red leaf**.
 - Why?
 - Can't be black.
 - Must be a leaf.



Height Guarantee

- **Claim**: every red-black tree with n nodes has height $\leq 2 \log_2(n + 1)$.
- Proof:
 - In a binary tree with n nodes, there is a root-NULL path with **at most** $\log_2(n + 1)$ nodes. (why?)
 - **Thus**: # black nodes on that path $\leq \log_2(n + 1)$.
 - By **path rule**: every root-NULL path has $\leq \log_2(n + 1)$ **black nodes**.
 - By **red rule**: every root-NULL path has $\leq 2 \log_2(n + 1)$ **total nodes**.

Q.E.D.

Operations on Red-Black Trees

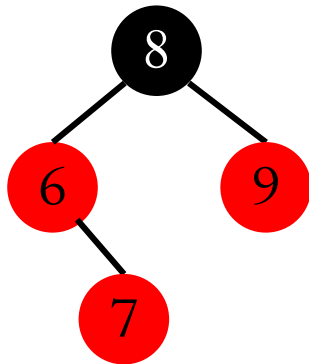
- All **query operations** (e.g., search, min, max, succ, pred) work just like those on general BST.
 - They run in $O(\log n)$ time on a red-black trees with n nodes in the **worst case**.
- The **modifying** operations “insertion” and “removal” must maintain the red-black tree properties.
 - They are complex.

Outline

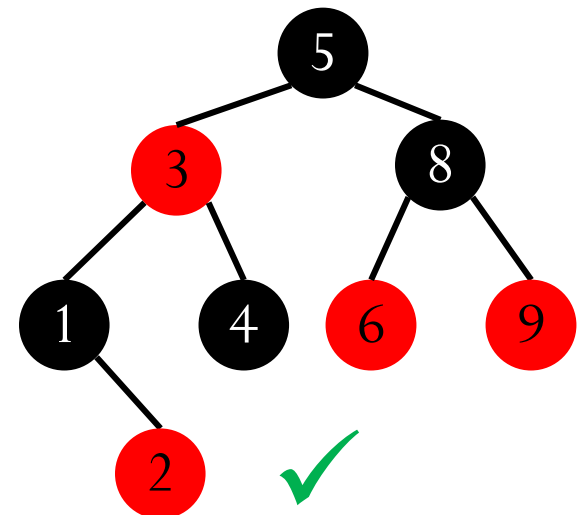
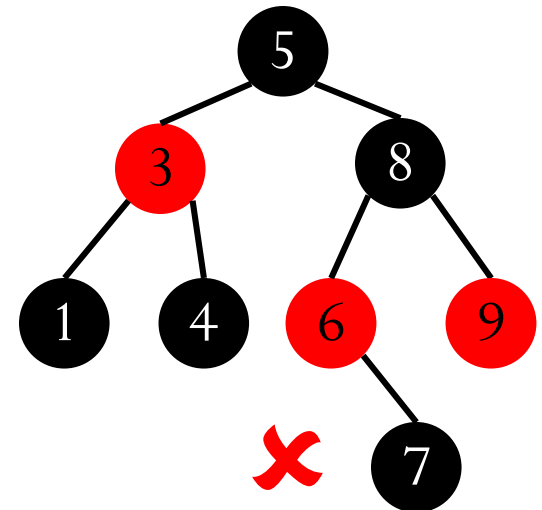
- Red-black Trees: Basics
- Red-black Trees: Insertion

Insertion

- New node is always a **leaf**.
 - However, it can't be **black**!
 - Otherwise, violate path rule.
 - Therefore the new leaf must be **red**.
- If parent is black, done (trivial case).
- If parent is red, violate the **red rule**!

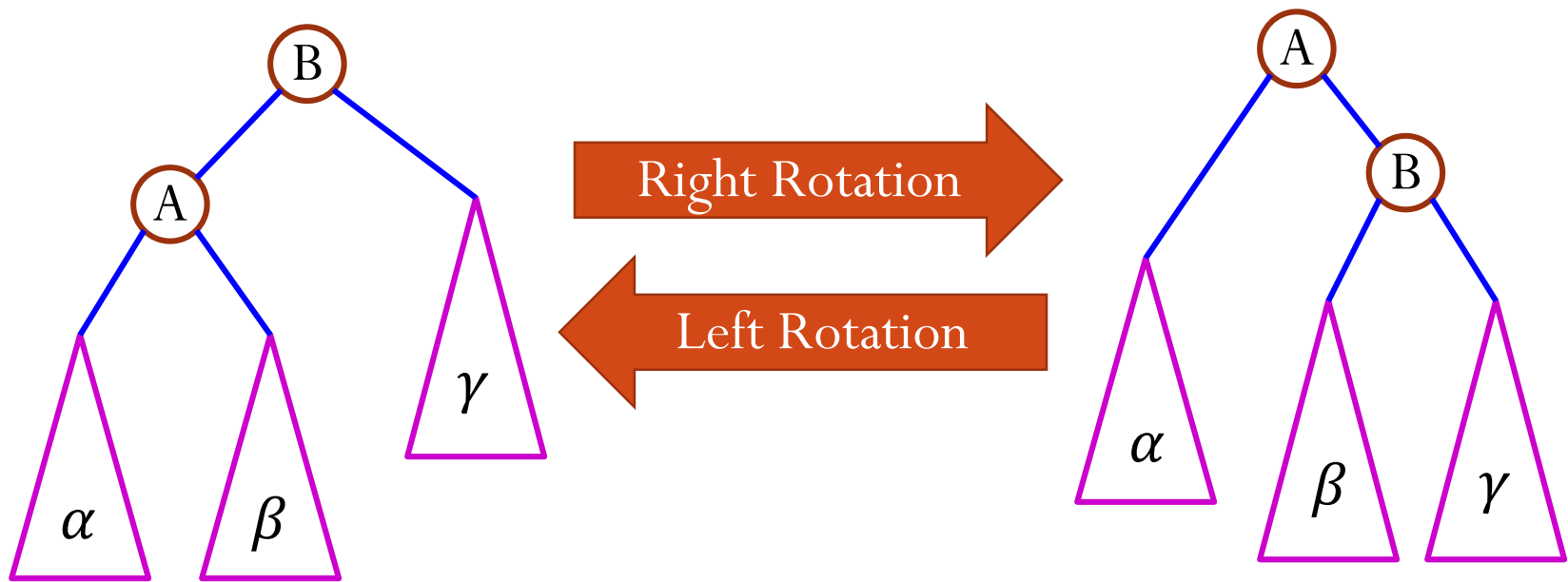


We have to do some work...

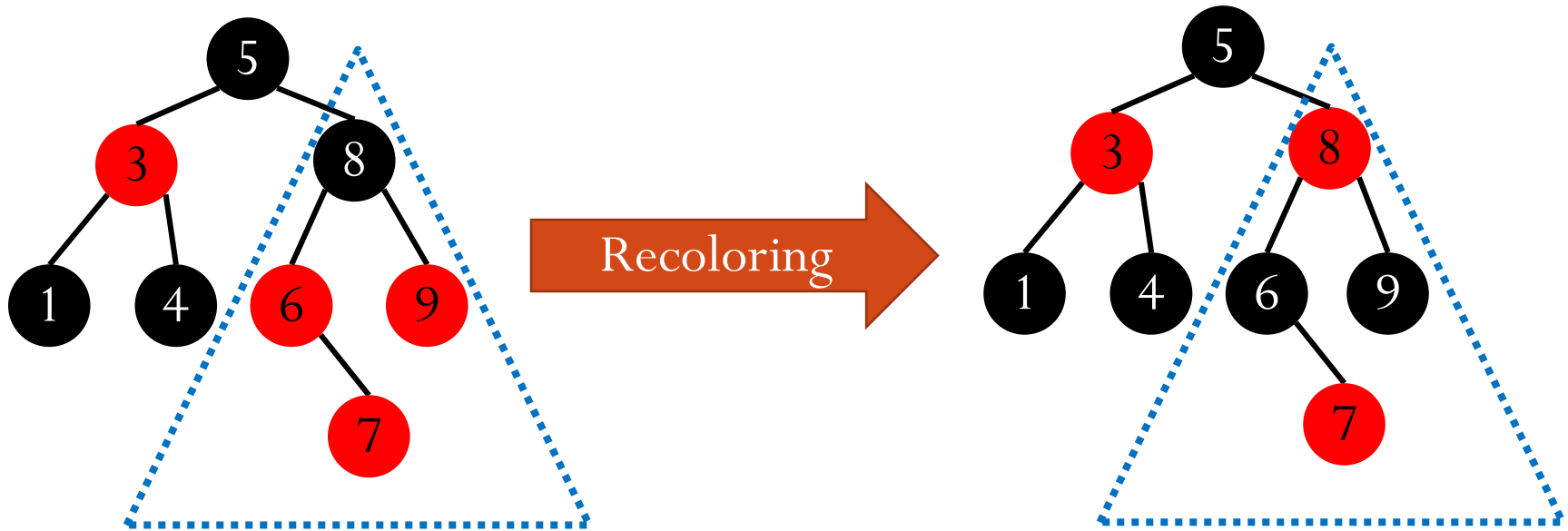


Modification: Rotation

- Maintain the binary search tree property.
- Can be done in $O(1)$ time.



Modification: Recoloring

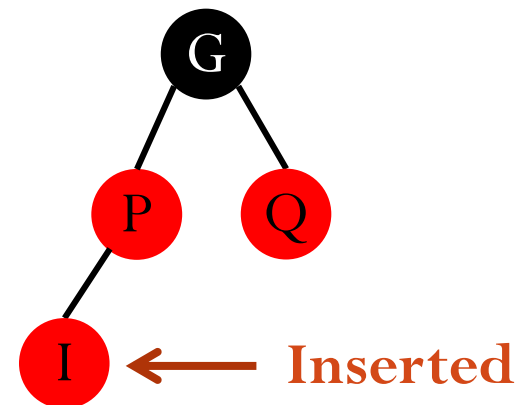


Insertion: Sketch

- Insert x as a **leaf**.
- Color x **red**.
 - Only **red rule** may be violated.
- Move the violation **up the tree** by recoloring/rotation.
 - At some point, the violation will be fixed.

Violation at Leaf

- Note: only **red rule** may be violated by inserting a (red) node as a leaf.
- When violating, its **parent** is **red** and its **grandparent** is **black**.
- Denote: the inserted node as “I”, its parent as “P”, its grandparent as “G”.





Which Statements Are Correct?

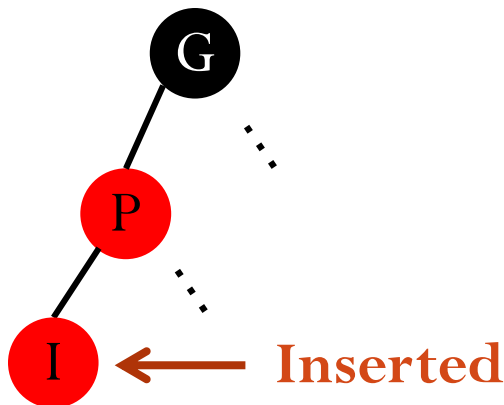
- Suppose there is a violation at the leaf. Suppose the parent of the inserted node is “P”. Select all the correct statements.

A. P could be a non-leaf in the original tree.

B. P could have a sibling.

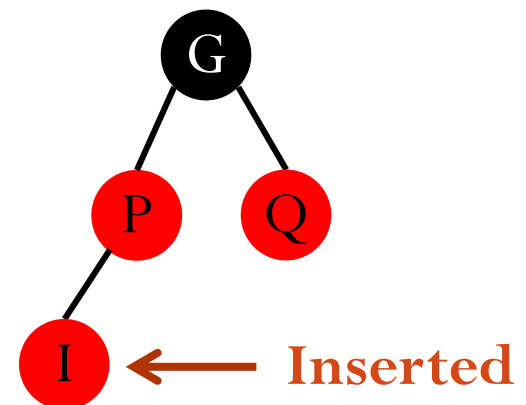
C. P cannot have a sibling.

D. P could have a sibling and that sibling must be a leaf node.



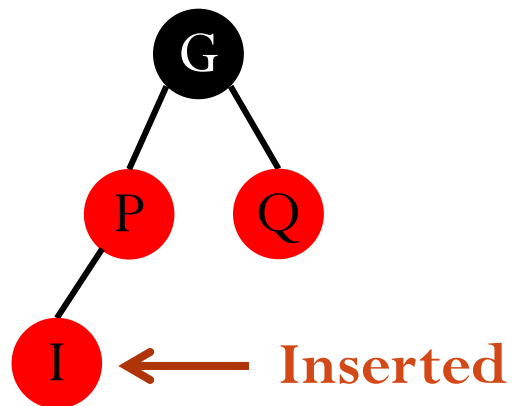
Violation at Leaf

- **Note**: only **red rule** may be violated by inserting a (red) node as a leaf.
- When violating, its **parent** is **red** and its **grandparent** is **black**.
- **Denote**: the inserted node as “I”, its parent as “P”, its grandparent as “G”.
- **Claim**: in the old tree, “P” is a leaf, i.e., has no children.



Violation at Leaf

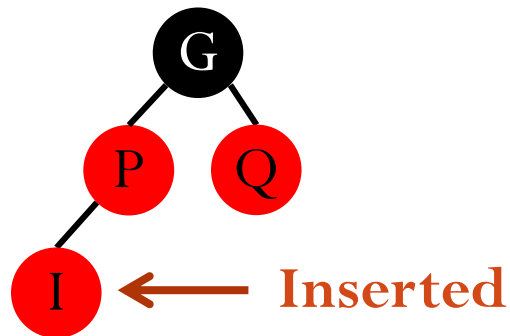
- **Assume**: the parent “P” is the **left child** of the grandparent “G”.
 - The “right child” case is **symmetric**.
- **Denote**: the right child of the grandparent to be Q.
- **Claim**: Q is either a red leaf or a NULL.
 - Why?



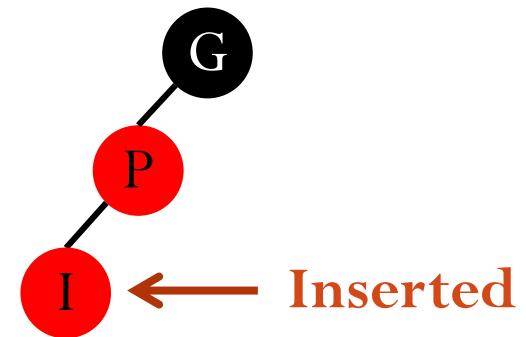
Violation at Leaf

- Three cases:

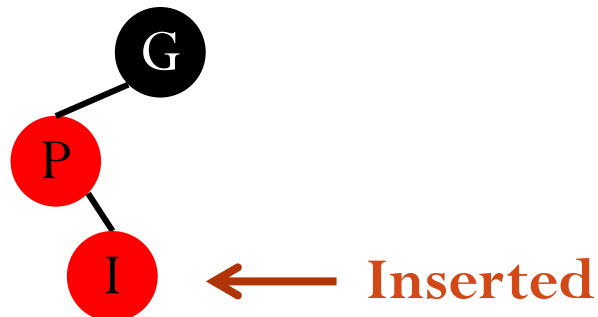
1. Q is a **red leaf**.



2. Q is empty; I is P's **left** child.

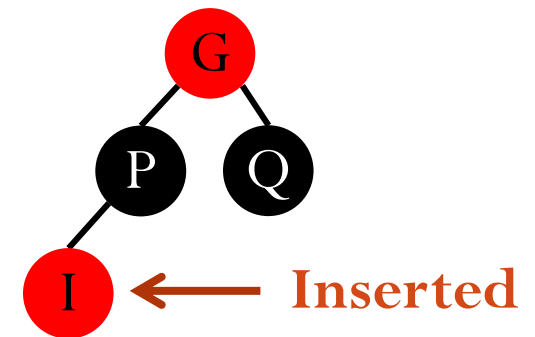
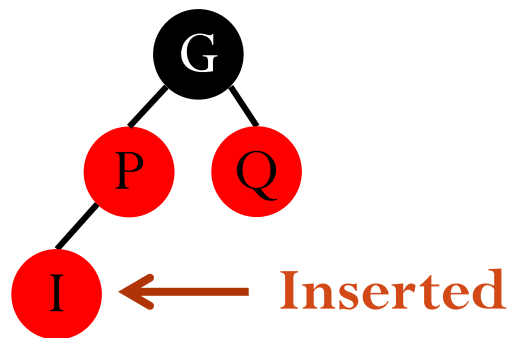


3. Q is empty; I is P's **right** child.



Violation at Leaf

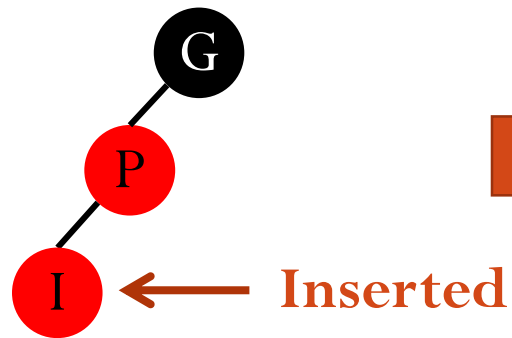
- Case 1: Q is a **red leaf**.



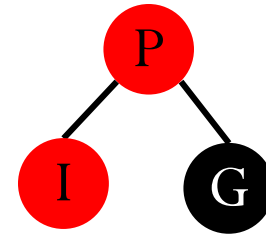
May **recurse**, since G's parent may be red.

Violation at Leaf

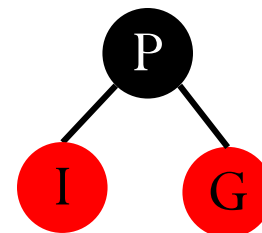
- Case 2: Q is empty; I is P's **left** child.



Right Rotation



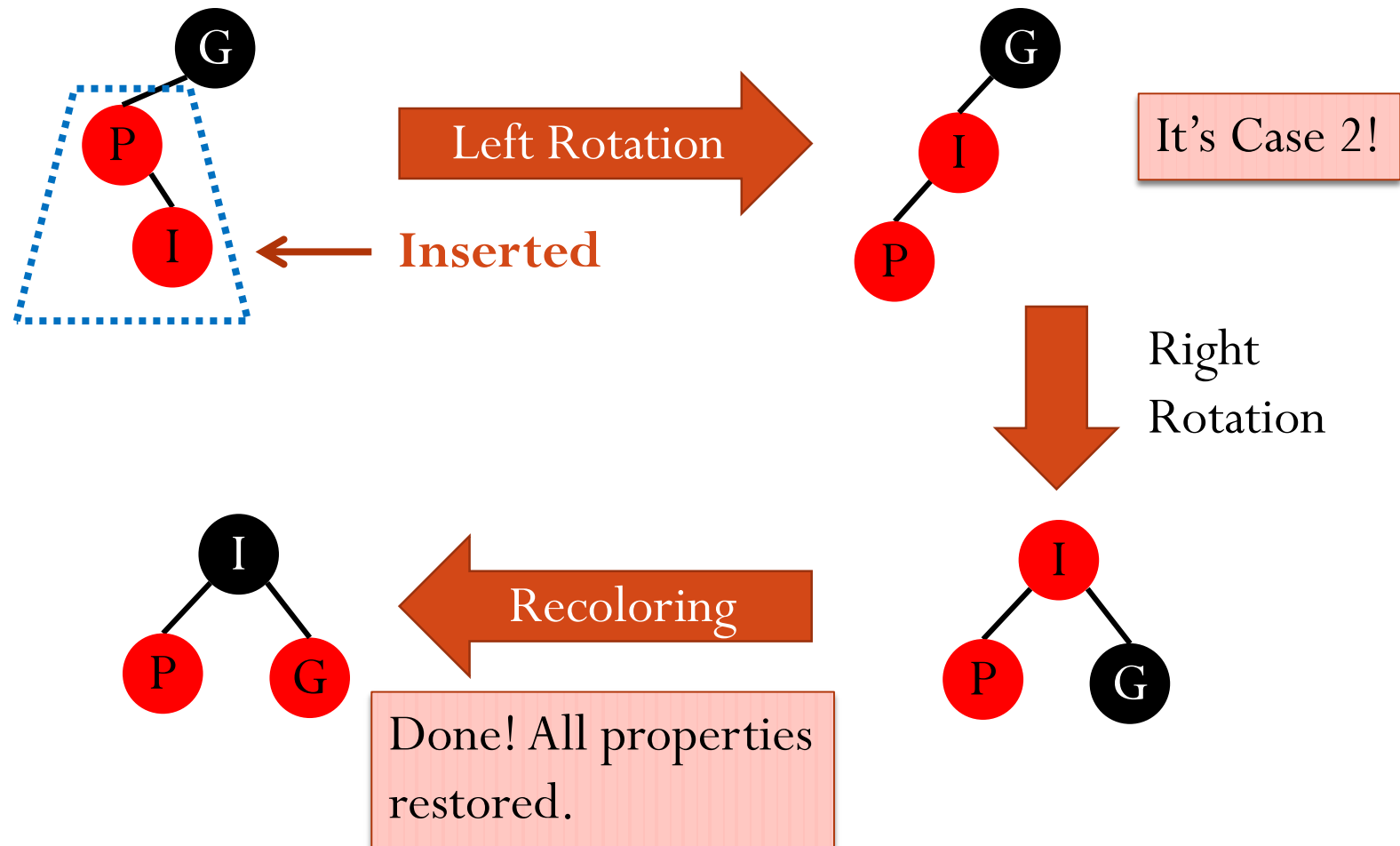
Recoloring



Done! All properties restored. (Why?)

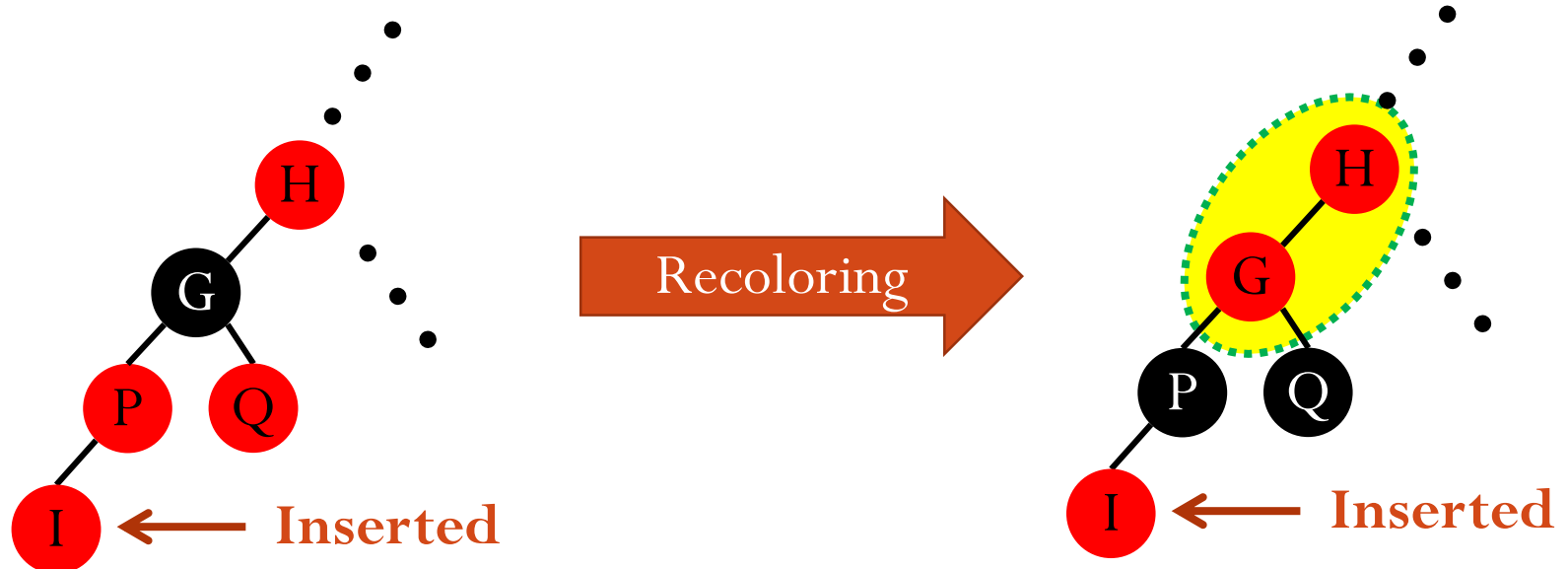
Violation at Leaf

- Case 3: Q is empty; I is P's **right** child.



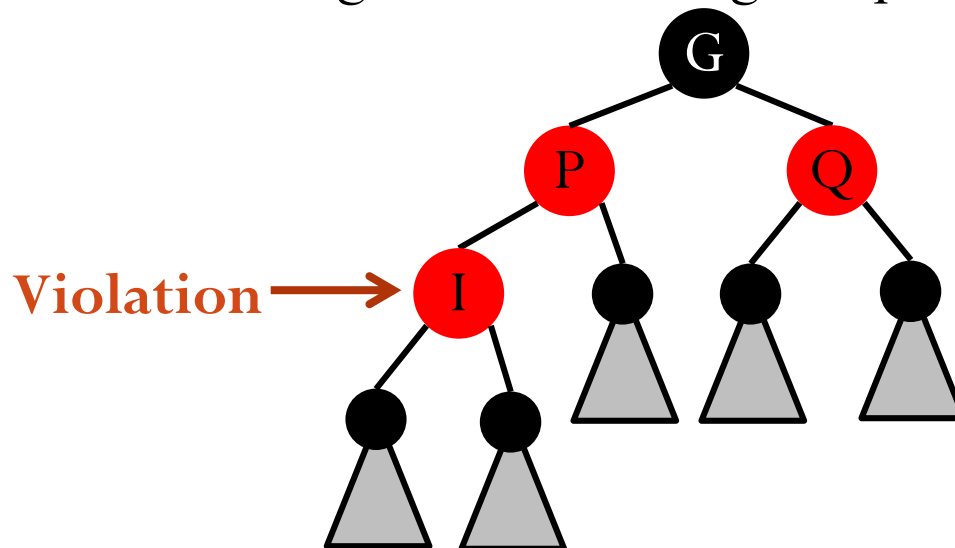
Violation at Leaf: Summary

- For Case 2 (Q is empty; I is P's **left** child) and Case 3 (Q is empty; I is P's **right** child), **we're done**.
- For Case 1 (Q is a **red leaf**), we may recurse.
 - Violation of **red rule**.



Violation at Internal Nodes

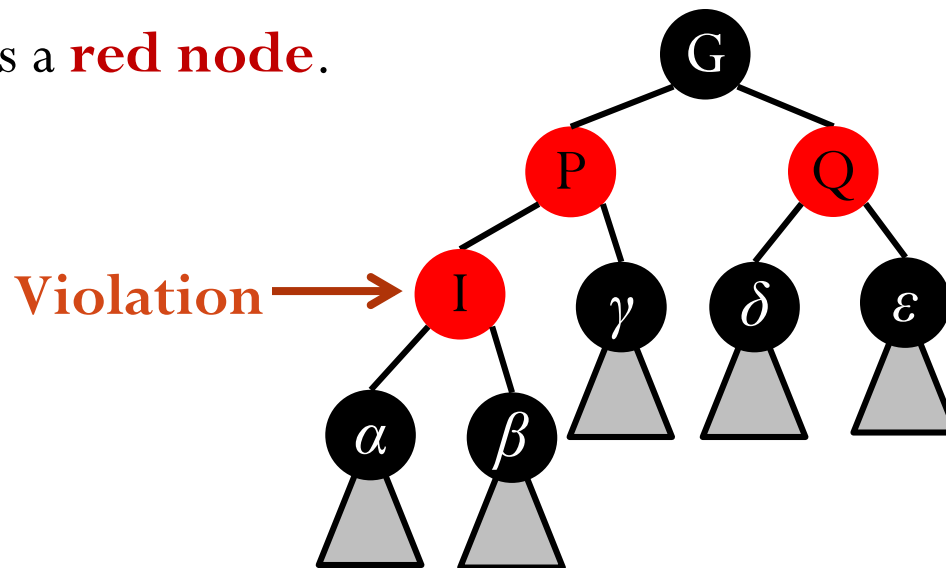
- Caused by **moving the violation up** the tree.
- When violating, its **parent** is **red** and its **grandparent** is **black**.
- **Assume**: the parent “P” is the **left child** of the grandparent “G”. (The “right child” case is **symmetric**.)
- **Denote**: the right child of the grandparent to be Q.



Violation at Internal Nodes

- Three Cases:

1. Q is a **red node**.

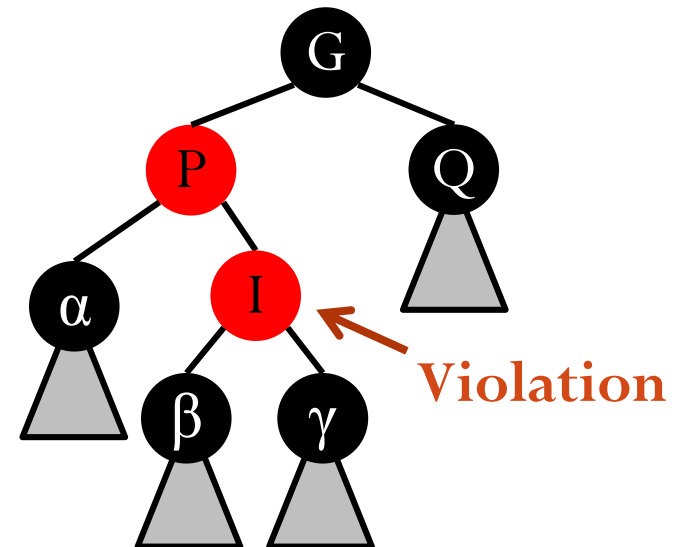
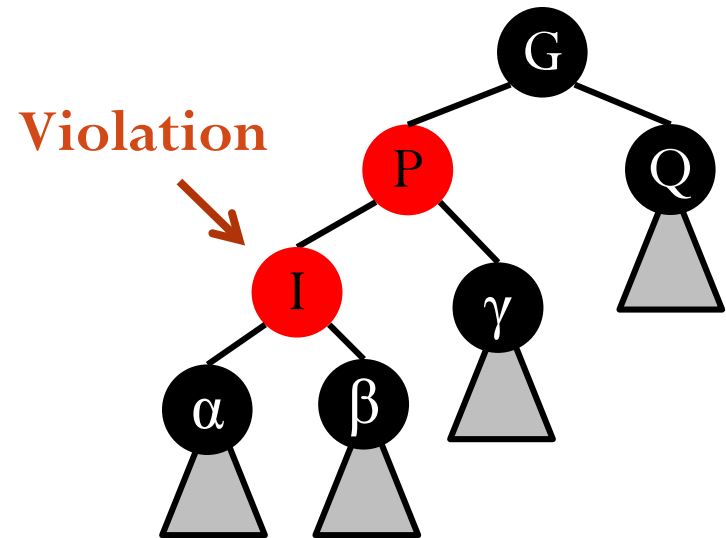


- Claim:

- $\alpha, \beta, \gamma, \delta, \epsilon$ are trees with **black root**.
- $\alpha, \beta, \gamma, \delta, \epsilon$ have the same **black height**.

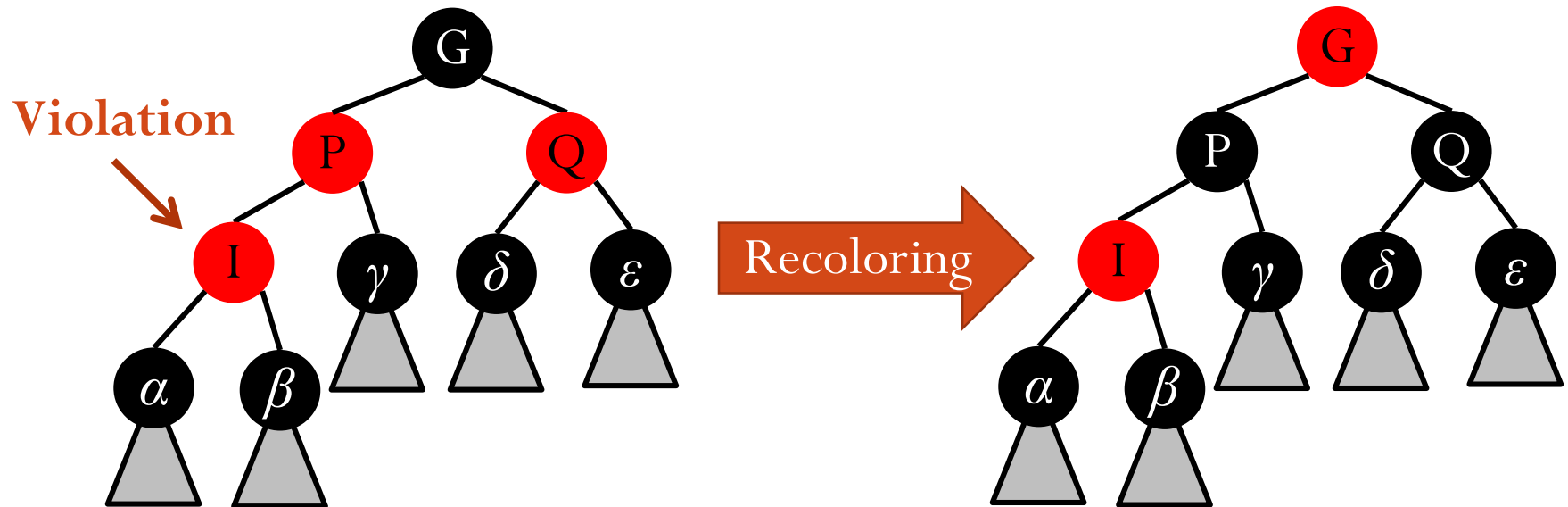
Violation at Internal Nodes

- Three Cases:
 2. Q is a **black node**; I is P's **left** child.
 3. Q is a **black node**; I is P's **right** child.
- **Claim** for Case 2 and 3:
 - α , β , γ , Q are trees with **black root**.
 - α , β , γ , Q have the **same black height**.



Violation at Internal Nodes

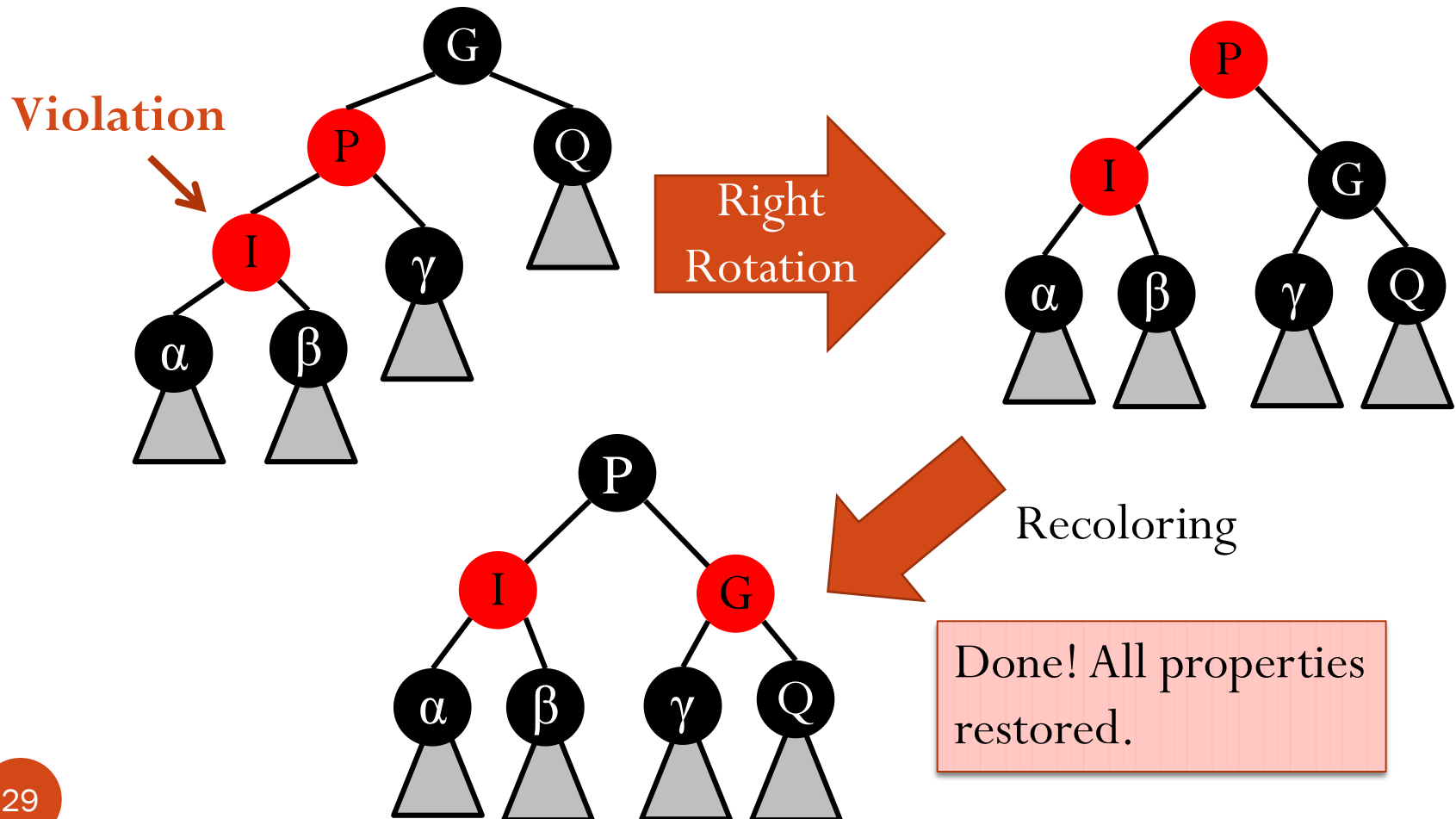
- Case 1: Q is a **red node**.



May **recurse**, since G's parent may be red.

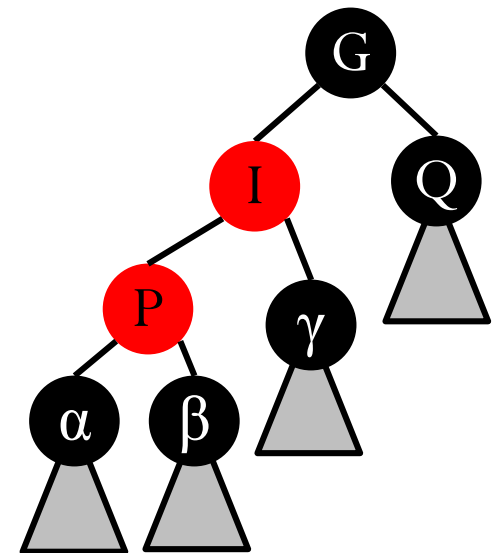
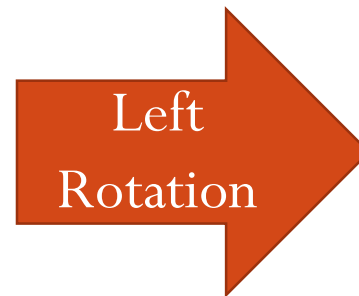
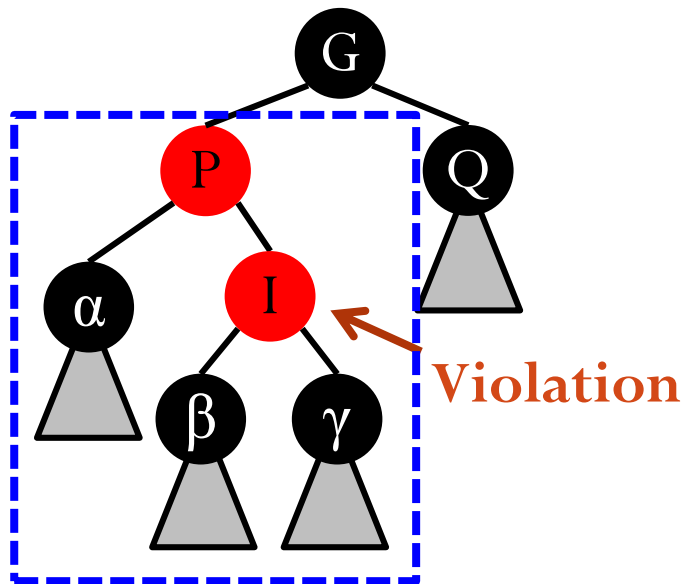
Violation at Internal Nodes

- Case 2: Q is a **black node**; I is P's **left** child.



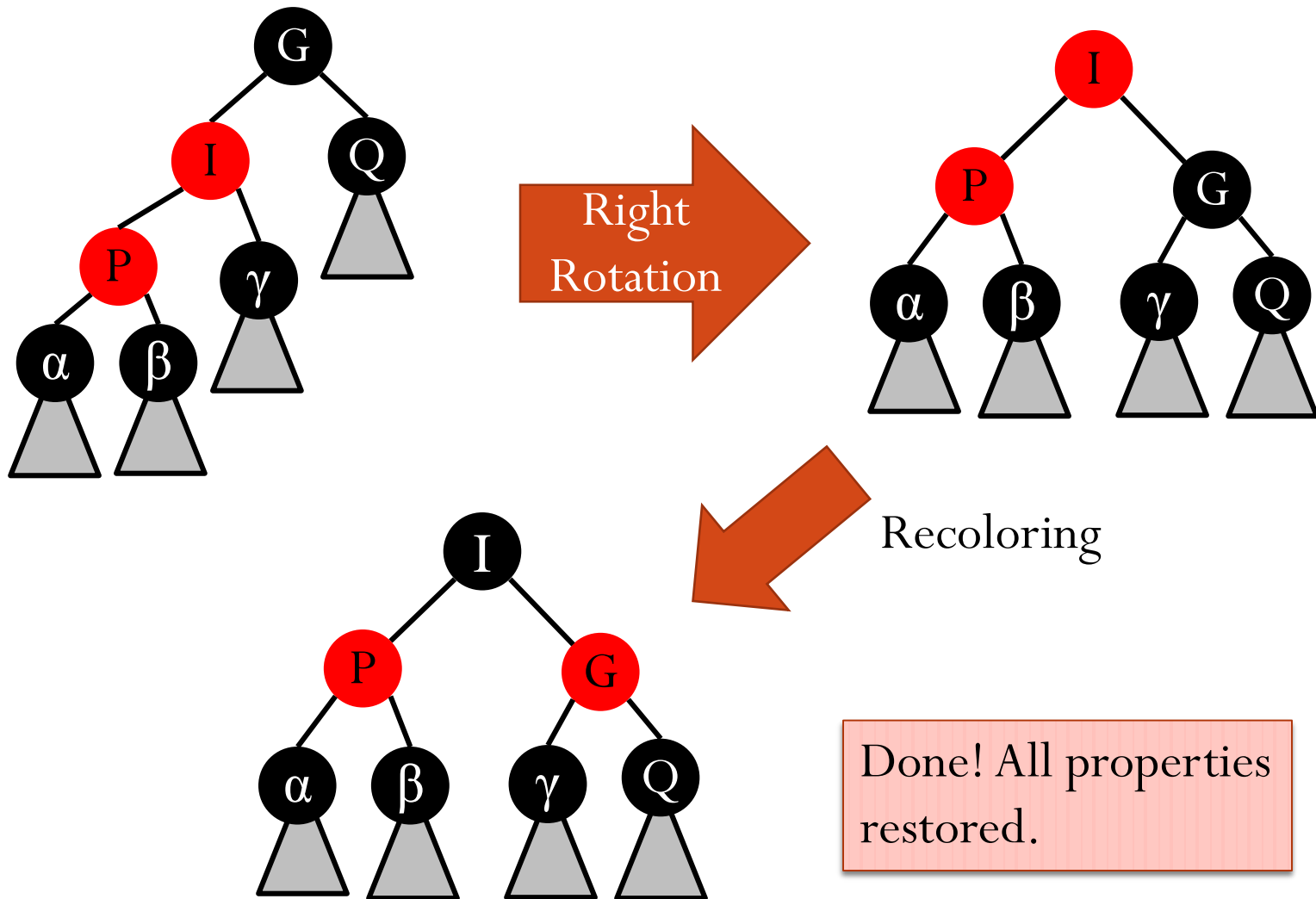
Violation at Internal Nodes

- Case 3: Q is a **black node**; I is P's **right** child.



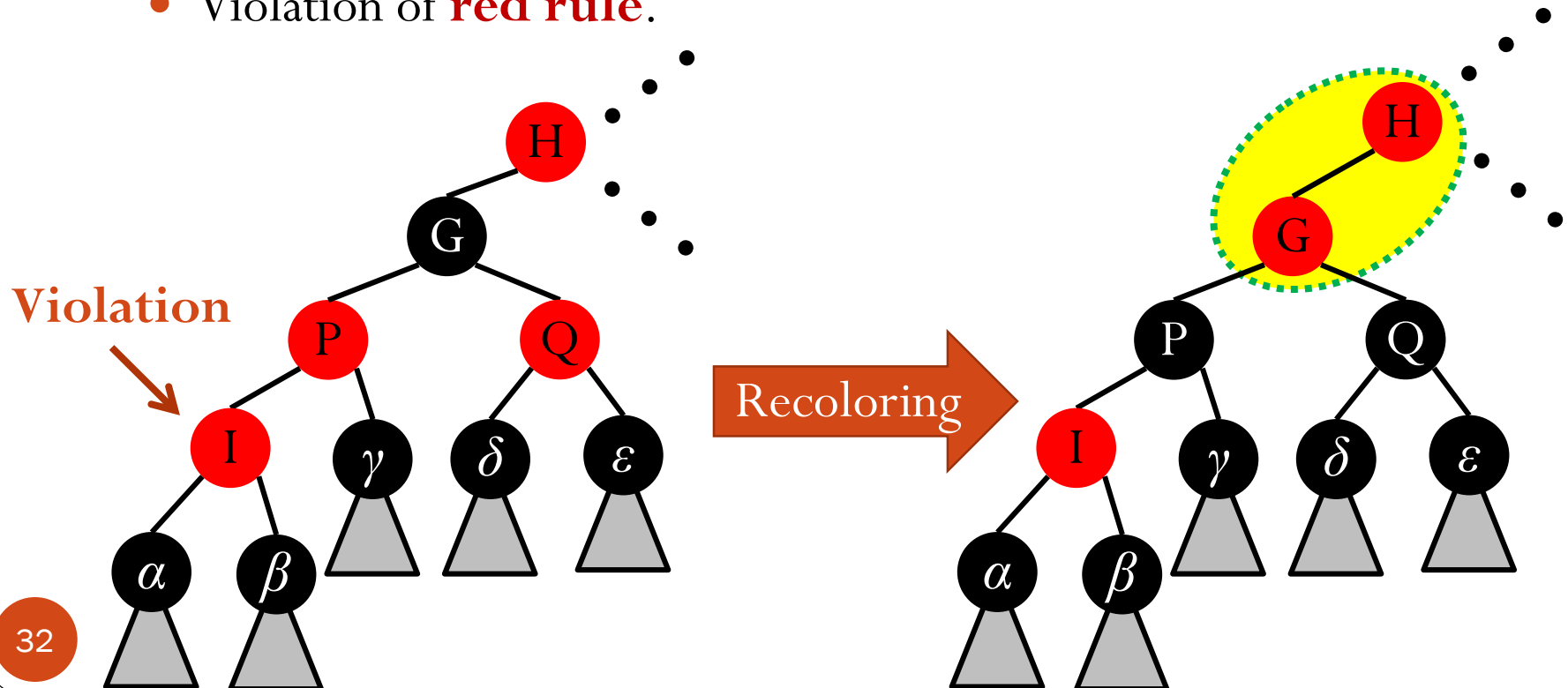
It's Case 2!

Violation at Internal Nodes: Case 3 (cont.)



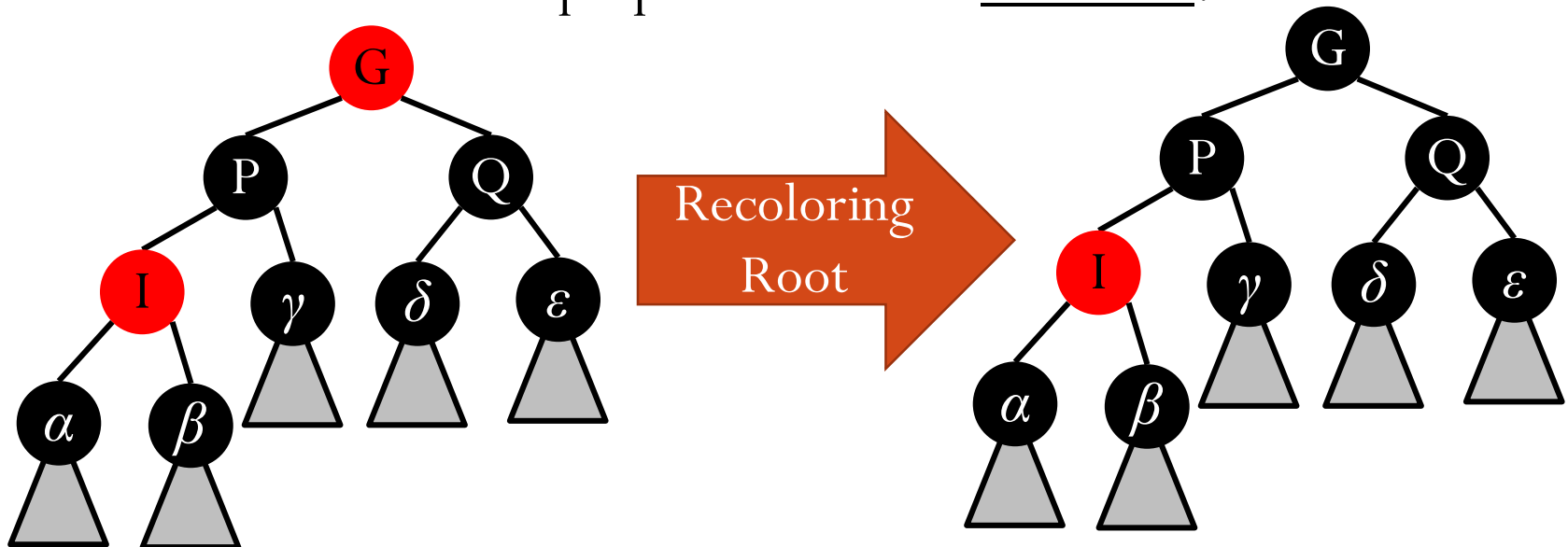
Violation at Internal Nodes: Summary

- For Case 2 (Q is a **black node**; I is P's **left** child) and Case 3 (Q is a **black node**; I is P's **right** child), **we're done**.
- For Case 1 (Q is a **red node**), we may recurse.
 - Violation of **red rule**.



Final Step: Violation Fix at the Root

- By **moving the violation up** the tree ...
 - ... the root may become **red**.
- Final step: set root to be **black**.
 - All red-black tree properties are now restored.

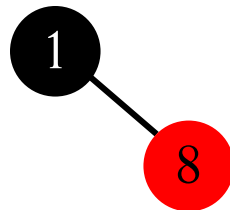


Example

- Insert 1

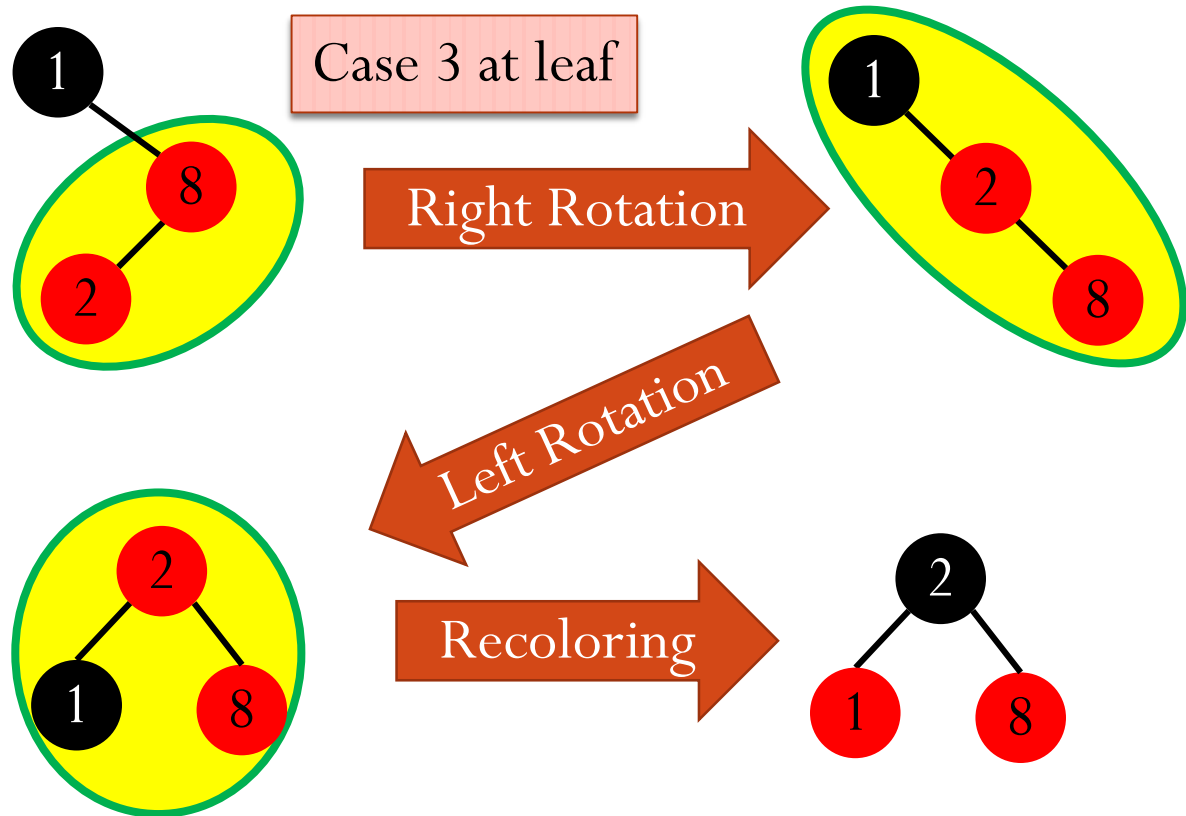


- Insert 8



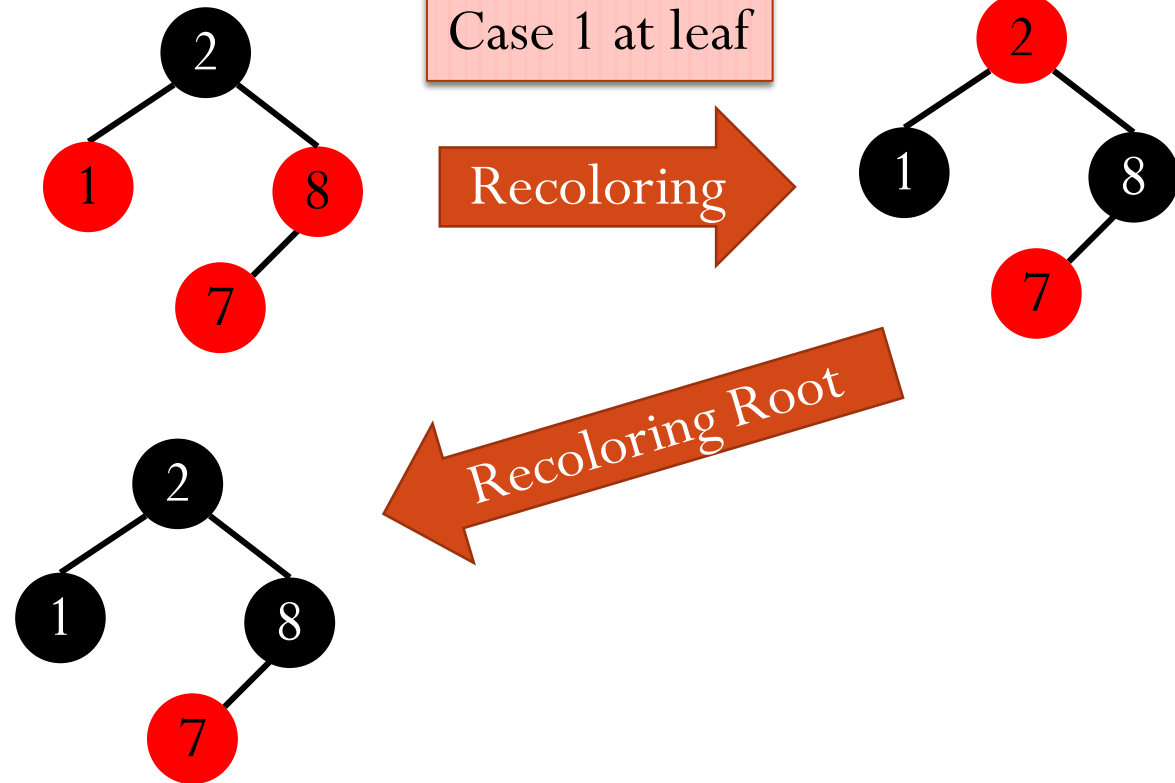
Example (cont.)

- Insert 2



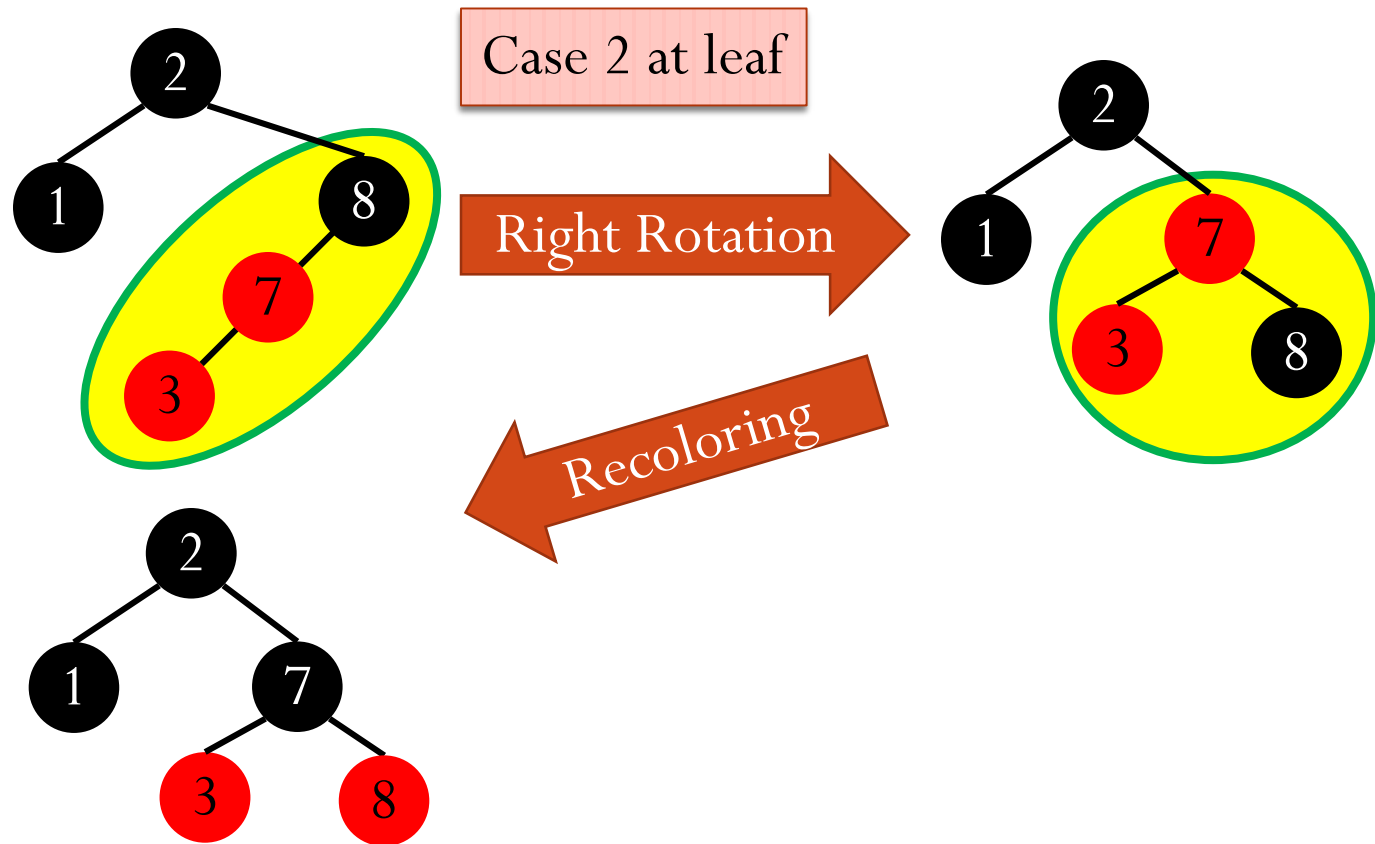
Example (cont.)

- Insert 7



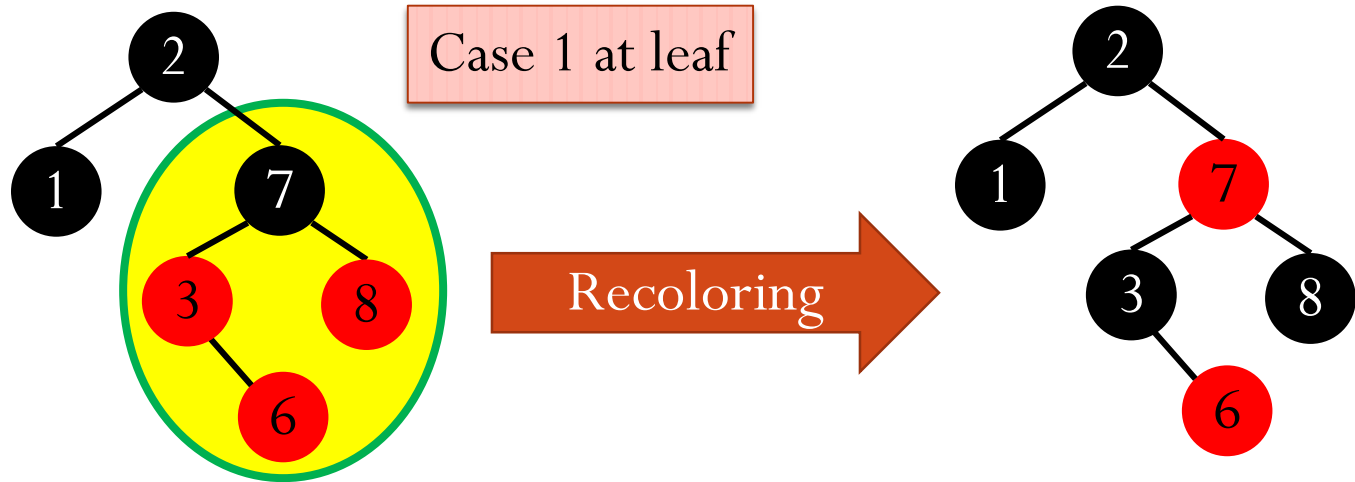
Example (cont.)

- Insert 3



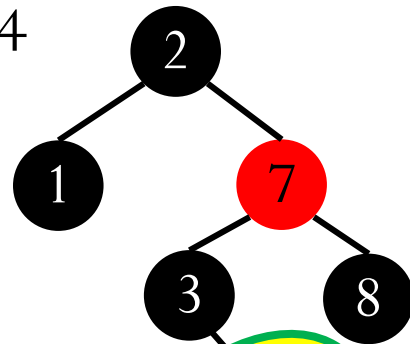
Example (cont.)

- Insert 6



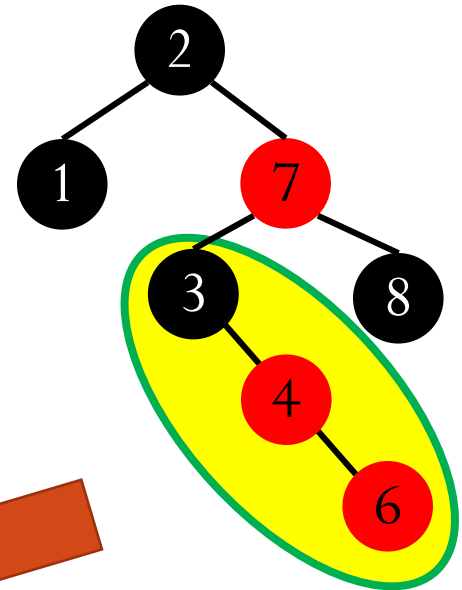
Example (cont.)

- Insert 4

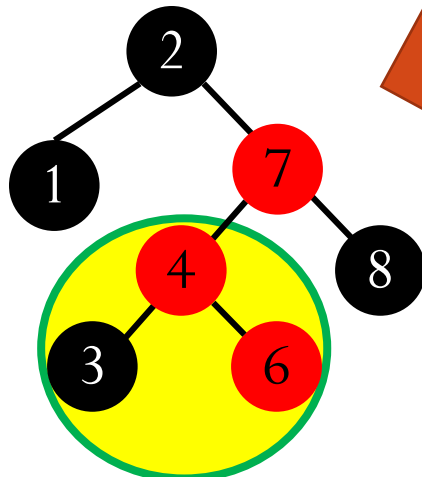


Case 3 at leaf

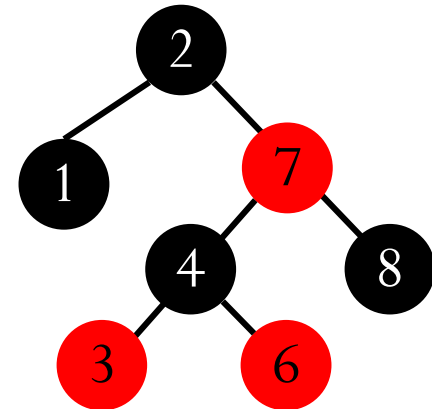
Right Rotation



Left Rotation

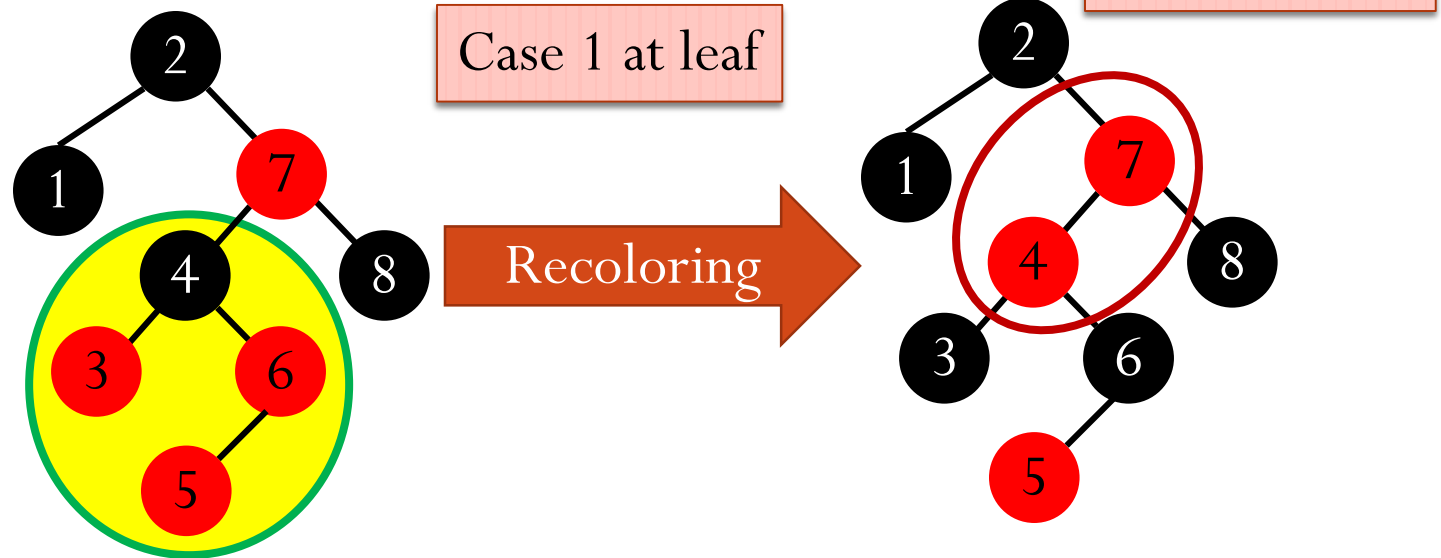


Recoloring



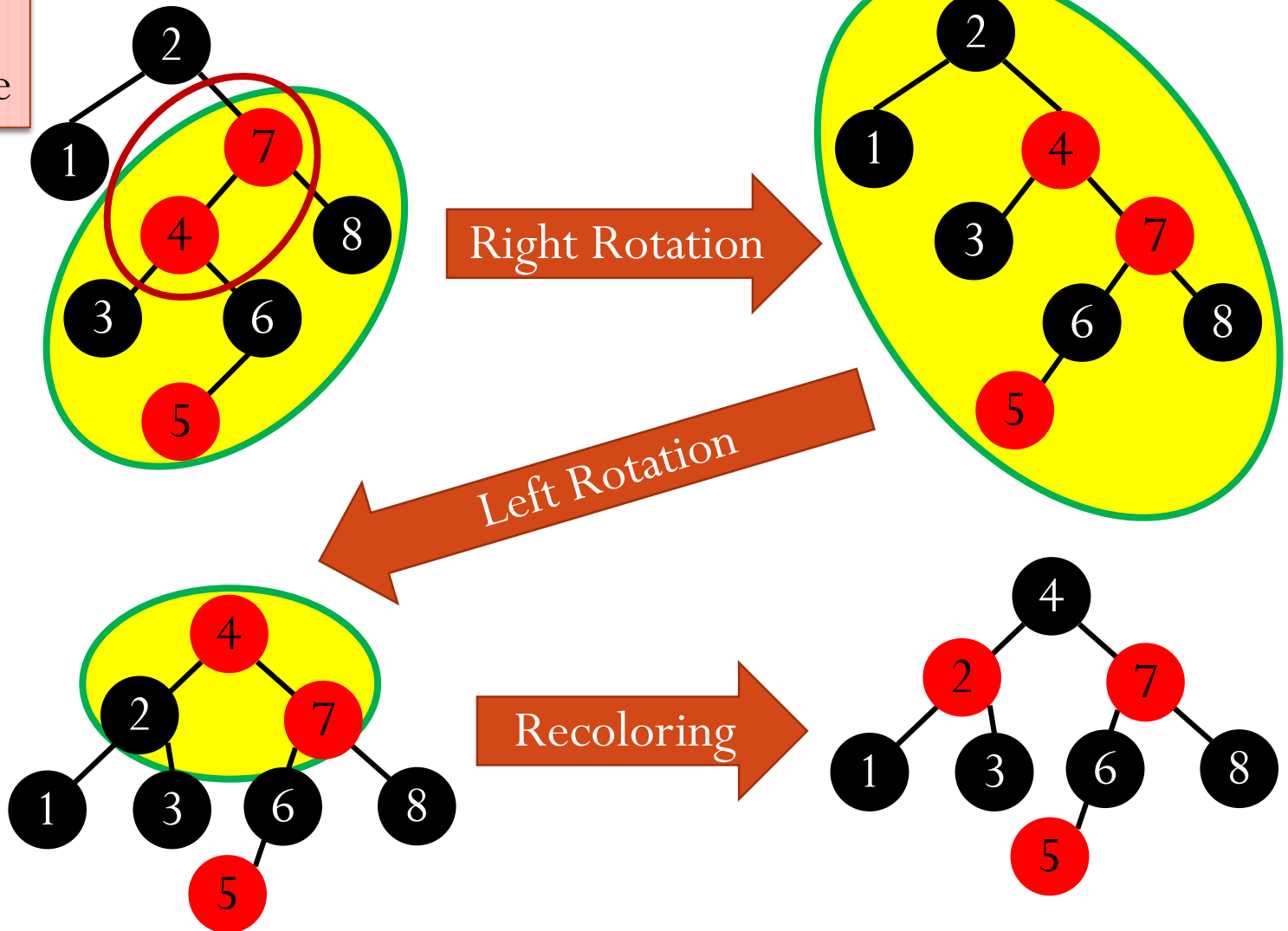
Example (cont.)

- Insert 5



Example (cont.)

Case 3 at
internal node



Runtime Complexity

- Number of rotations required
 - For case 1, only need to recolor, **no** rotation.
 - For case 2 or 3, perform 1 or 2 rotations and terminate.
 - **Thus**: # rotations = $O(1)$.
- Number of recoloring required
 - Worst case: $O(\log n)$
- Runtime complexity is $O(\log n)$.