VE281

Data Structures and Algorithms

Linear Time Selection

Learning Objective:

- Understand randomized selection algorithm
- Understand deterministic selection algorithm
- Know how to analyze their runtime complexity

Outline

- Randomized selection algorithm
- Deterministic selection algorithm

The Selection Problem

- Input: array A with n distinct numbers and a number i
  - "**Distinct**" for simplicity
- Output: i-th smallest element in the array
  - Assume index starts from 1
- Example: $A = (6, 3, 5, 4, 2)$, $i = 3$
  - Should return 4
- Special cases
  - $i = 1$: the smallest item. Runtime: $O(n)$
  - $i = n$: the largest item. Runtime: $O(n)$
  - $i = n/2$: the median

Solution: Reduction to Sorting

- Step 1: Do merge sort
- Step 2: output the i-th element of the sorted array
- Time complexity is $O(n \log n)$

- Can we do better?
  - This essentially asks whether selection is **fundamentally easier** than sorting
  - Answer: Yes!
  - We will show an $O(n)$ time randomized algorithm by modifying quick sort
  - Also will show an $O(n)$ time deterministic algorithm (However, not as practical as the randomized algorithm)

Recall: Partitioning in Quick Sort

- Pick a pivot
- Put all elements < pivot to the left of pivot
- Put all elements $\geq$ pivot to the right of pivot
- Move pivot to its correct place in the array

6	2	8	5	11	10	4	1	9	7	3

pivot

4	2	3	5	1	6	10	11	9	7	8

Basic Idea

- Suppose we are looking for 6^{th} smallest item in an array of length 12. We do partition.
  - Suppose the pivot is at position 4. Then we only need to focus on the sub-array right of the pivot and look for the 2^{nd} item in the array
  - Suppose the pivot is at position 8. Then we only need to focus on the sub-array left of the pivot and look for the 6^{th} item in the array
  - In both cases, recurse!

Randomized Selection

```
Rselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
  if(n == 1) return A[1];
  Choose pivot p from A uniformly at random;
  Partition A using pivot p;
  Let j be the index of p;
  if(j == i) return p;
  if(j > i) return Rselect(1st part of A, j-1, i);
  else return Rselect(2nd part of A, n-j, i-j);
}
```

Which Statements Are Correct?

Given **a fixed input array**, consider the runtime of the randomized selection algorithm to choose the i-th smallest element

- **A.** The runtime depends on the pivot sequence
  ø(c*(N+(N-1)+(N-2)+…+(N-N)))=ø(N^2)
- **B.** When $i = n/2$, the worst-case runtime is $\Theta(n^2)$
- **C.** When $i = n/2$, the worst case happens when the pivot sequence is the sorted version of the input array
- **D.** For any given i, the best-case runtime is $\Theta(1)$ θ(N)

Average Runtime of Rselect

- Theorem: for every input array of length n, the average runtime of Rselect is $O(n)$
  - Holds for every input data (no assumption on data)
  - "Average" is over random pivot choices made by the algorithm

Average Runtime Analysis

- Note: Rselect uses $\leq cn$ operations outside of recursive call (from partitioning)
- Observation: the length of the array the algorithm works on decreases
- Definition: We say Rselect is in phase j if current array size is between $(\frac{3}{4})^{j+1}n$ and $(\frac{3}{4})^{j}n$
- X_j denote the number of recursive calls in phase j
- $runtime \leq \sum_j X_j \cdot c \cdot (\frac{3}{4})^j \, n$

We need to further get $E[X_j]$

$$E[runtime] \leq E\left[\sum_j X_j \cdot c \cdot \left(\frac{3}{4}\right)^j n\right] = cn \sum_j \left(\frac{3}{4}\right)^j E[X_j]$$

Average Runtime Analysis

- **Claim**: If Rselect chooses a pivot so that the **left sub-array**'s size is am, where $a \in [\frac{1}{4}, \frac{3}{4}]$ and m is the old length, then the current phase ends
  - Because new sub-array length is at most 75% of the old length
  - **"Good pivot"**
- What is the probability of $a \in [\frac{1}{4}, \frac{3}{4}]$ (i.e., good pivot)?
  - Answer: 0.5
- Claim: $E[X_j] \leq$ Expected number of times you need to get a good pivot
  - Same as the expected number of times you flip a fair coin to get a "head". (Heads: good pivot; tails: bad pivot)

Coin Flipping Analysis

- Let N be the number of coin flips until you get heads
  - N is a geometric random variable: $P(N = k) = \frac{1}{2^k}, k = 1,2, \ldots$

#flips when 1st is head

#flips when 1st is tail

- $E[N] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + E[N]) \Rightarrow E[N] = 2$

Prob. 1st flip is head

Prob. 1st flip is tail

Therefore, $E[X_j] \leq E[N] = 2$

Average Runtime Analysis

$$E[runtime] \le E\left[\sum_j X_j \cdot c \cdot \left(\frac{3}{4}\right)^j n\right]$$

$$= cn \sum_j \left(\frac{3}{4}\right)^j E[X_j] \;\; \le 2cn \sum_j \left(\frac{3}{4}\right)^j \le 2cn \frac{1}{1 - \frac{3}{4}}$$

$$= 8cn = O(n)$$

Outline

- Randomized selection algorithm

- Deterministic selection algorithm

A Good Pivot

- Best pivot: the median
  - But, this is a circular problem
- Goal: find pivot guaranteed to be good enough
- Idea: use “median of medians”

A Deterministic ChoosePivot

`ChoosePivot(A, n)`

- A subroutine called by the deterministic selection algorithm
- Steps:

1. Break A into n/5 groups of size 5 each
2. Sort each group (e.g., use insertion sort)
3. Copy n/5 medians into new array C
4. Recursively compute median of C
   - By calling the deterministic selection algorithm!
5. Return the median of C as pivot

Deterministic Selection Algorithm

```
Dselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
  if(n == 1) return A[1];
  Break A into groups of 5, sort each group;
  C = n/5 medians;
  p = Dselect(C, n/5, n/10);          ChoosePivot
  Partition A using pivot p;
  Let j be the index of p;
  if(j == i) return p;
  if(j > i) return Dselect(1st part of A, j-1, i);
  else return Dselect(2nd part of A, n-j, i-j);
}
```

Same as Rselect

The function has two recursive calls

Runtime of Dselect

- Theorem: For every input array of length n, Dselect runs in $O(n)$ time

- Warning: not as good as Rselect in practice
  - Worse constants cn 的c更大
  - Not-in-place更多memory

What's the Runtime of Step 2?

```
Dselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
1 if(n == 1) return A[1];
2 Break A into groups of 5, sort each group;
3 C = n/5 medians;
4 p = Dselect(C, n/5, n/10);
5 Partition A using pivot p;
6 Let j be the index of p;
7 if(j == i) return p;
8 if(j > i) return Dselect(1st part of A, j-1, i);
9 else return Dselect(2nd part of A, n-j, i-j);
}
```

A. $\Theta(n)$ **B.** $\Theta(n^2)$

C. $\Theta(n \log n)$ **D.** $\Theta(n \log \log n)$

Runtime of Dselect

Assume the runtime is $T(n)$

```
Dselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
if(n == 1) return A[1];
Break A into groups of 5, sort each group; Θ(n)
C = n/5 medians;  Θ(n)
p = Dselect(C, n/5, n/10); T(n/5)
Partition A using pivot p; Θ(n)
Let j be the index of p;
if(j == i) return p;
if(j > i) return Dselect(1st part of A, j-1, i);
else return Dselect(2nd part of A, n-j, i-j);     T(?)
}
```

Recurrence

- There exists a positive constant c such that
  - $T(1) \le c$
  - $T(n) \le cn + T\left(\frac{n}{5}\right) + T(?)$

- The next question is what is the size of the array of the second recursive call

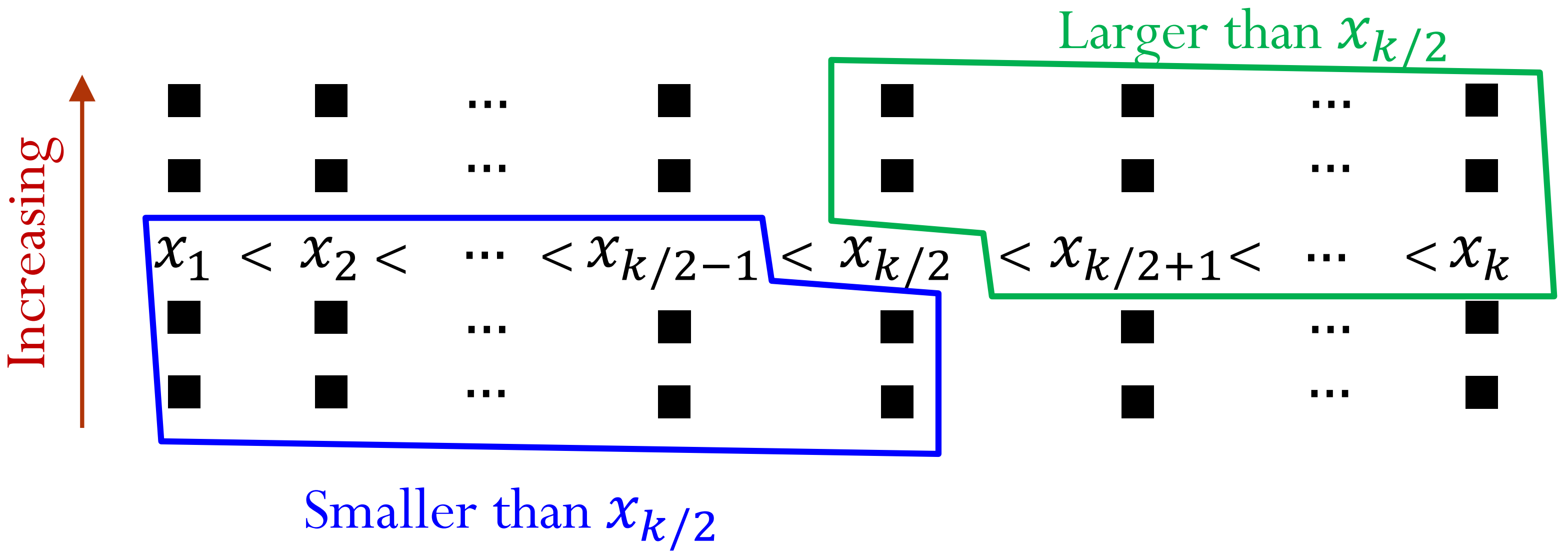
Lemma on Size

- Lemma: 2nd recursive call guaranteed to be on an array of size $\leq 0.7n$ (roughly)

- (Rough) proof:
  - Let $k = n/5$: number of groups
  - Let x_i be the i-th smallest of the k medians
  - Thus, the pivot is $x_{k/2}$
  - Goal
    - >=30% of input array smaller than $x_{k/2}$
    - >=30% of input array larger than $x_{k/2}$

Proof of Lemma

- Imagine we layout elements of A in a 2-D grid



- At least ~ (3/5)*(1/2) = 30% elements smaller than $x_{k/2}$
- At least ~ 30% elements larger than $x_{k/2}$
- <u>Result</u>: Number of elements < $x_{k/2}$ is in between 30% and 70%. The same for number of elements > $x_{k/2}$

Example

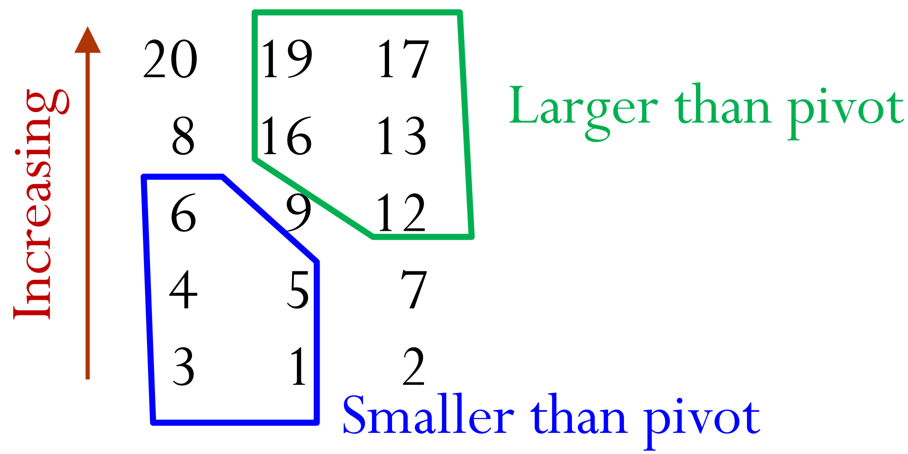
- Input:

  7,2,17,12,13,8,20,4,6,3,19,1,9,5,16

- After sorting each group of 5 elements

  2,7,12,13,17,3,4,6,8,20,1,5,9,16,19

  pivot



Recurrence

- There exists a positive constant c such that
  - $T(1) \leq c$
  - $T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$
- Note: different-sized sub-problems. Cannot use master method!
- How can we solve this?
  - Strategy: Hope and check
- Hope: there is a constant a (independent of n) such that $T(n) \leq an$ for all $n > 1$
  - Then $T(n) = O(n)$
- We choose $a = 10c$

Proof $T(n) = O(n)$

- Claim: suppose there exists a positive constant c such that
  1. $T(1) \le c$
  2. $T(n) \le cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$

  Then $T(n) \le 10cn$
- Proof by induction
  - Base case: $T(1) \le 10c$
  - Inductive step: inductive hypothesis $T(k) \le 10ck, \forall k < n$.

    Then

    $$T(n) \le cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \le cn + 2cn + 7cn = 10cn$$

Dselect runs in linear time