VE281

Data Structures and Algorithms

Dynamic Programming

Learning Objectives:

- Understand the basic idea of dynamic programming
- Know under what situation dynamic programming could be applied

Outline

- Dynamic Programming
  - Motivation
  - Example: Matrix-Chain Multiplication
  - Summary

Limitation of Divide and Conquer

- Recursively solving subproblems can result in the same computations being repeated when the subproblems **overlap.**

- For example: computing the **Fibonacci sequence**

$$f_0 = 0;\ \ f_1 = 1;\ \ f_n = f_{n-1} + f_{n-2}, n \geq 2$$

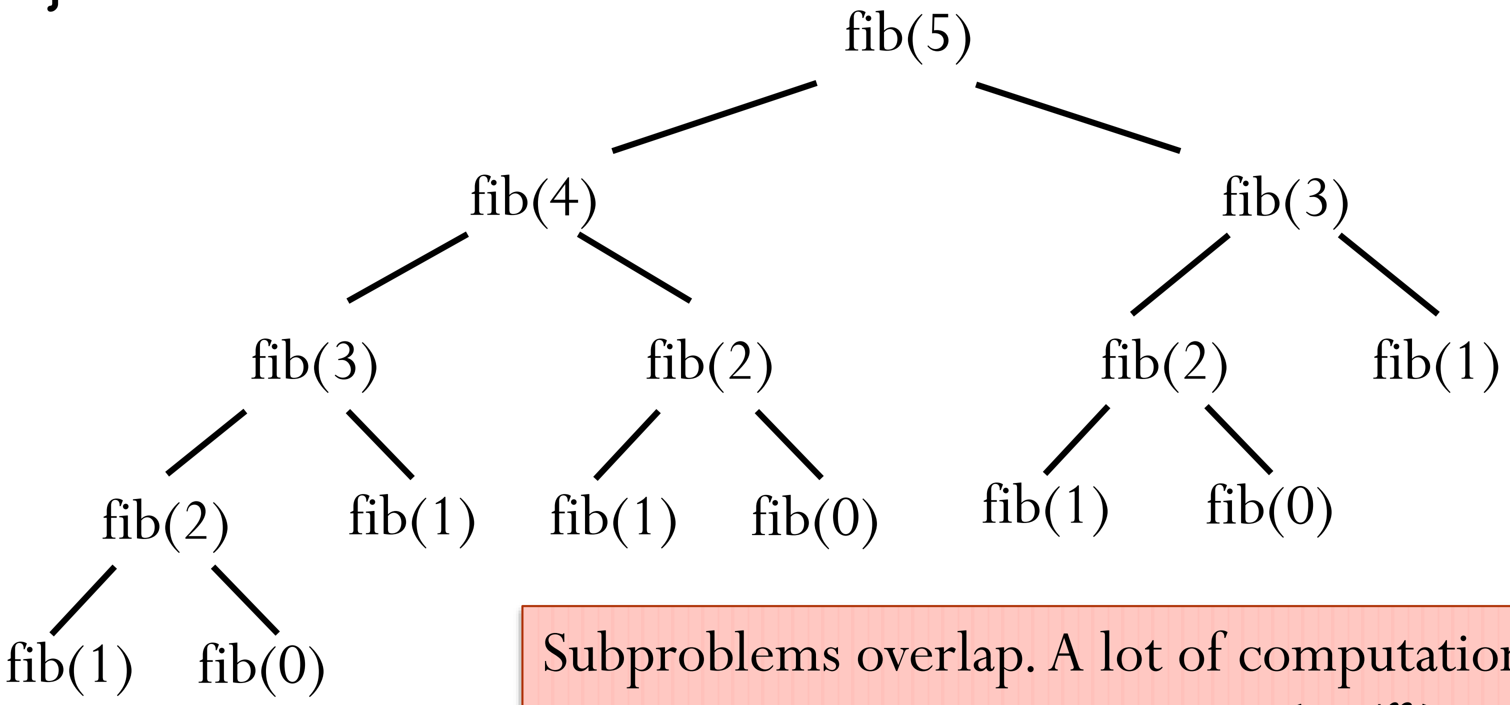
- Divide and conquer approach:

```
int fib(int n) {
  if(n <= 1) return n;
  return fib(n-1)+fib(n-2);
}
```

Fibonacci Sequence

Divide and Conquer Solution

```
int fib(int n) {
  if(n <= 1) return n;
  return fib(n-1)+fib(n-2);
}
```



Subproblems overlap. A lot of computation is wasted. Time complexity is $\Omega(1.5^n)$.

Fibonacci Sequence

Iterative Solution

- We can also compute the Fibonacci sequence in iterative way:

```
int fib(int n) {
   f[0] = 0; f[1] = 1;
   for(i = 2 to n)
      f[i] = f[i-1]+f[i-2];
   return f[n];
}
```

- Time complexity is $\Theta(n)$.

Dynamic Programming

- Used when a problem can be divided into subproblems that **overlap**.
  - Solve each subproblem **once** and store the solution in a table.
  - If a subproblem is encountered **again**, simply look up its solution in the table.
  - Reconstruct the solution to the original problem from the solutions to the subproblems.
- The more overlap the better, as this reduces the number of subproblems.
- Dynamic programming can be applied to solve **optimization problem**.

Optimization Problem

- Many problems we encounter are **optimization problems**:
  - A problem in which some function (called the **objective function**) is to be optimized (usually minimized or maximized) subject to some **constraints**.

- The solutions that satisfy the constraints are called **feasible solutions**.
- The number of feasible solutions is typically very large.
- We obtain the optimal solution by **searching** the feasible solution space.

Optimization Problem

Example

- Minimum spanning tree.
    - Objective function: the sum of all edge weights.
    - Constraints: the subgraph must be a spanning tree.

Outline

- Dynamic Programming
  - Motivation
  - Example: Matrix-Chain Multiplication
  - Summary

Matrix-Chain Multiplication

- What is the cost of multiplying two matrices A and B?
  - Suppose A is a $p \times q$ matrix and B is a $q \times r$ matrix.
  - Since the time to compute $C = AB$ is dominated by the number of **scalar multiplications**, we use the number of scalar multiplications as the complexity measure.

- $C_{ij} = \sum_{k=1}^{q} A_{ik} B_{kj}$.
  - We need q scalar multiplications to calculate C_{ij}.
  - C is of size $p \times r$.
- The number of scalar multiplications is pqr.

Matrix-Chain Multiplication

- Now how would you compute the multiplication of three matrices $A \times B \times C$?
  - Suppose A is of size 100×1, B is of size 1×100, and C is of size 100×1.

- If we multiply as $(A \times B) \times C$, the number of scalar multiplications is 20000.
- If we multiply as $A \times (B \times C)$, the number of scalar multiplications is 200.

Matrix-Chain Multiplication

- If we want to multiply a chain of matrices $A_1 \times A_2 \times \cdots \times A_n$, where A_i is of size $p_{i-1} \times p_i$, what is the best order of multiplication to minimize the number of scalar multiplications?

- This is an optimization problem.
- It can be proved that number of different orders on n matrices is $\Omega(4^n/n^{1.5})$.
- Instead of **enumerating** all of the orders, can we do better to solve the optimization problem?

Matrix-Chain Multiplication

- For simplicity, define the problem of finding the optimal order to multiply $A_i \times A_{i+1} \times \cdots \times A_j$ as Q_{ij}. The minimal number of scalar multiplications is m_{ij}.
  - We ultimately want to solve Q_{1n}.

Matrix-Chain Multiplication

- Suppose in the optimal order for $A_i \times \cdots \times A_j$, the **last** multiplication is $(A_i \times \cdots \times A_k) \times (A_{k+1} \times \cdots \times A_j)$.
- Then the order of computing $A_i \times \cdots \times A_k$ in the **optimal** order of computing $A_i \times \cdots \times A_j$ must be an **optimal** order to compute $A_i \times \cdots \times A_k$.
  - Why?
    - If not, then we copy and paste the better order ➔ we have a better order for computing $A_i \times \cdots \times A_j$!
    - Similar conclusion for computing $A_{k+1} \times \cdots \times A_j$.

- If we know k, we can divide the problem Q_{ij} into two smaller instances: Q_{ik} and $Q_{(k+1)j}$.

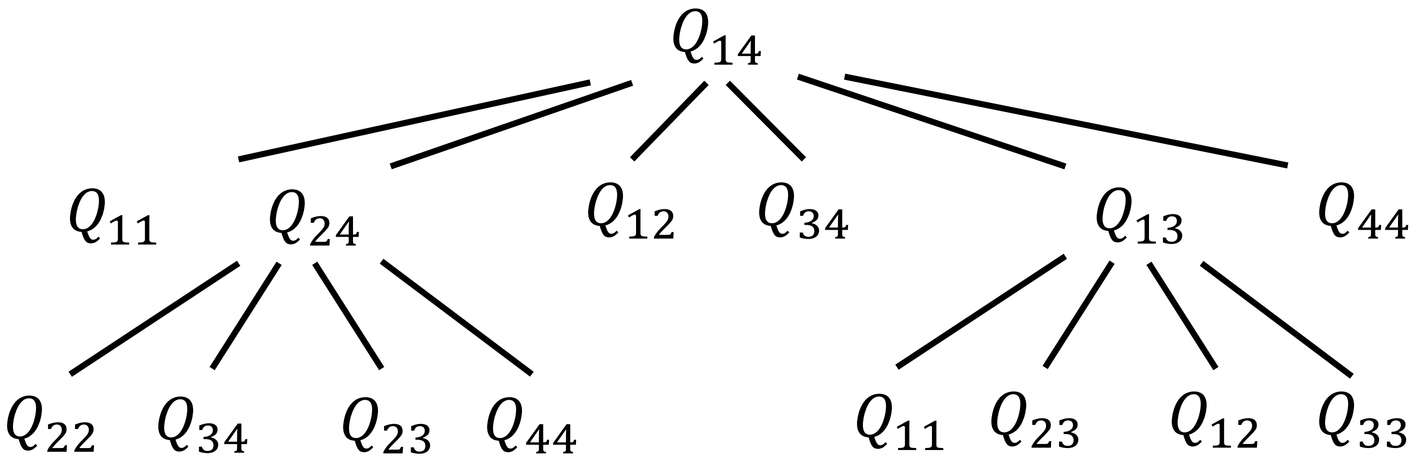
Matrix-Chain Multiplication

- Assume we have known the minimum number of scalar multiplications for Q_{ik} and $Q_{(k+1)j}$ as m_{ik} and $m_{(k+1)j}$.
  - Then $m_{ij} = m_{ik} + m_{(k+1)j} + p_{i-1}p_kp_j$.

- **However, we don't know *k*!** We need to consider all possible divisions, i.e., all $i \leq k \leq j-1$.
- Thus, in order to solve Q_{ij}, we need to consider all subproblems Q_{ik} and $Q_{(k+1)j}$, for all $i \leq k \leq j-1$.
  - $m_{ij} = \min_{i \leq k \leq j-1} (m_{ik} + m_{(k+1)j} + p_{i-1}p_kp_j)$

Matrix-Chain Multiplication

- In summary, we can divide the problem into subproblems of the same form.



Many subproblems are overlapped.

Matrix-Chain Multiplication

- The straightforward recursive algorithm has exponential time complexity.
  - However, it will encounter each subproblem many times in different branches of the tree.

- The total number of different subproblems is not exponential.
  - They are Q_{ij}, for $1 \leq i \leq j \leq n$.
  - The total number is $n(n+1)/2$.
- Instead, we use a **tabular**, **bottom-up** approach.

Matrix-Chain Multiplication

Bottom-up Approach

- Apply the recursive relation:

$$m_{ij} = \min_{i \le k \le j-1} (m_{ik} + m_{(k+1)j} + p_{i-1} p_k p_j)$$

- Initial situation $m_{11} = m_{22} = \cdots = m_{nn} = 0$.
- In the first round, we compute $m_{12}, m_{23}, \dots, m_{(n-1)n}$.
- In the second round, we compute $m_{13}, m_{24}, \dots, m_{(n-2)n}$.
- So on and so forth. In the l-th round, we compute $m_{1(l+1)}, m_{2(l+2)}, \dots, m_{(n-l)n}$.
- Finally, we compute m_{1n}.
- To obtain the multiplication order, we also record the partition k which gives the minimal m_{ij} as s_{ij}.

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}

j

		1	2	3	4
	1	0			
i	2	–	0		
	3	–	–	0	
	4	–	–	–	0

s_{ij}

j

		1	2	3	4
	1	–			
i	2	–	–		
	3	–	–	–	
	4	–	–	–	–

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4$.
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20$.

m_{ij}		1	2	3	4
	1	0			
i	2	–	0		
	3	–	–	0	
	4	–	–	–	0

s_{ij}		1	2	3	4
	1	–			
i	2	–	–		
	3	–	–	–	
	4	–	–	–	–

$$\begin{aligned} m_{i(i+1)} &= m_{ii} + m_{(i+1)(i+1)} + p_{i-1}p_ip_{i+1} \\ &= p_{i-1}p_ip_{i+1} \end{aligned}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}

j

i	1	2	3	4
1	0	100		
2	–	0	10	
3	–	–	0	200
4	–	–	–	0

s_{ij}

j

i	1	2	3	4
1	–	1		
2	–	–	2	
3	–	–	–	3
4	–	–	–	–

$$\begin{aligned} m_{i(i+1)} &= m_{ii} + m_{(i+1)(i+1)} + p_{i-1}p_ip_{i+1} \\ &= p_{i-1}p_ip_{i+1} \end{aligned}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
i	1	0	100		
	2	–	0	10	
	3	–	–	0	200
	4	–	–	–	0

s_{ij}			j		
		1	2	3	4
i	1	–	1		
	2	–	–	2	
	3	–	–	–	3
	4	–	–	–	–

$$m_{i(i+2)} = \min\{m_{ii} + m_{(i+1)(i+2)} + p_{i-1}p_ip_{i+2},$$

$$m_{i(i+1)} + m_{(i+2)(i+2)} + p_{i-1}p_{i+1}p_{i+2}\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij} j

i

	1	2	3	4
1	0	100		
2	–	0	10	
3	–	–	0	200
4	–	–	–	0

s_{ij} j

i

	1	2	3	4
1	–	1		
2	–	–	2	
3	–	–	–	3
4	–	–	–	–

$$\begin{aligned} m_{13} = \min\{ & m_{11} + m_{23} + p_0 p_1 p_3, \\ & m_{12} + m_{33} + p_0 p_2 p_3\} = \min\{20, 200\} \end{aligned}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij} j

i	1	2	3	4
1	0	100	20	
2	–	0	10	
3	–	–	0	200
4	–	–	–	0

s_{ij} j

i	1	2	3	4
1	–	1	1	
2	–	–	2	
3	–	–	–	3
4	–	–	–	–

$$m_{13} = \min\{m_{11} + m_{23} + p_0p_1p_3,$$

$$m_{12} + m_{33} + p_0p_2p_3\} \ = \min\{20, 200\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

j

m_{ij}		1	2	3	4
	1	0	100	20	
i	2	–	0	10	
	3	–	–	0	200
	4	–	–	–	0

j

s_{ij}		1	2	3	4
	1	–	1	1	
i	2	–	–	2	
	3	–	–	–	3
	4	–	–	–	–

$$m_{24} = \min\{m_{22} + m_{34} + p_1 p_2 p_4,$$

$$m_{23} + m_{44} + p_1 p_3 p_4\} \ = \min\{400, 30\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij} j

i	1	2	3	4
1	0	100	20	
2	–	0	10	30
3	–	–	0	200
4	–	–	–	0

s_{ij} j

i	1	2	3	4
1	–	1	1	
2	–	–	2	3
3	–	–	–	3
4	–	–	–	–

$$m_{24} = \min\{m_{22} + m_{34} + p_1p_2p_4,$$

$$m_{23} + m_{44} + p_1p_3p_4\} \;= \min\{400, 30\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100	20	
i	2	−	0	10	30
	3	−	−	0	200
	4	−	−	−	0

s_{ij}			j		
		1	2	3	4
	1	−	1	1	
i	2	−	−	2	3
	3	−	−	−	3
	4	−	−	−	−

$$m_{i(i+3)} = \min_{i \le k \le i+2} \left(m_{ik} + m_{(k+1)(i+3)} + p_{i-1} p_k p_{(i+3)} \right)$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100	20	
i	2	–	0	10	30
	3	–	–	0	200
	4	–	–	–	0

s_{ij}			j		
		1	2	3	4
	1	–	1	1	
i	2	–	–	2	3
	3	–	–	–	3
	4	–	–	–	–

$$m_{14} = \min_{1 \le k \le 3} (m_{1k} + m_{(k+1)4} + p_0 p_k p_4)$$
$$= \min\{230, 2300, 220\}$$

Matrix-Chain Multiplication

Example

- $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
- $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

m_{ij}			j		
		1	2	3	4
	1	0	100	20	220
i	2	–	0	10	30
	3	–	–	0	200
	4	–	–	–	0

Optimal Value

s_{ij}			j		
		1	2	3	4
	1	–	1	1	3
i	2	–	–	2	3
	3	–	–	–	3
	4	–	–	–	–

$$m_{14} = \min_{1 \le k \le 3} (m_{1k} + m_{(k+1)4} + p_0 p_k p_4)$$
$$= \min\{230, 2300, 220\}$$

Matrix-Chain Multiplication

Constructing an Optimal Order

- We can construct an optimal order based on the records s_{ij}.

```
Print_Order(s, i, j) {
   if(i == j) cout << "$A_i$";
   else {
      cout << "(";
      Print_Order(s, i, $s_{ij}$);
      cout << "*";
      Print_Order(s, $s_{ij}$+1, j);
      cout << ")";
   }
}
```

- Initial call is `Print_Order(s, 1, n);`

Matrix-Chain Multiplication

Example

- Construct an optimal order
  - $n = 4, A_1 \times A_2 \times A_3 \times A_4.$
  - $p_0 = 10, p_1 = 1, p_2 = 10, p_3 = 1, p_4 = 20.$

s_{ij}

j

i	1	2	3	4
1	–	1	1	3
2	–	–	2	3
3	–	–	–	3
4	–	–	–	–

$s_{14} = 3$ ⇒ $A_1 \times A_2 \times A_3 \times A_4 = (A_1 \times A_2 \times A_3) \times A_4$

$s_{13} = 1$ ⇒ $A_1 \times A_2 \times A_3 = A_1 \times (A_2 \times A_3)$

$s_{23} = 2$ ⇒ $A_2 \times A_3 = A_2 \times A_3$

$$A_1 \times A_2 \times A_3 \times A_4 = (A_1 \times (A_2 \times A_3)) \times A_4$$

Matrix-Chain Multiplication

Time Complexity

- Get the minimum number of scalar multiplications:
  - We need to obtain all m_{ij} and s_{ij}, for $1 \leq i \leq j \leq n$.
    - $O(n^2)$ records
    - Each m_{ij} is the minimum of $O(n)$ terms.
  - Total time complexity is $O(n^3)$.

- Obtain the optimal order:
  - $O(n)$

Matrix-Chain Multiplication

Summary

- Matrix-chain multiplication is an optimization problem.

- The solution is based on **dynamic programming**.
  - The original problem can be divided into same subproblems that **overlap**.
  - Each subproblem is solved once and stored in a table.
  - If a subproblem is encountered again, simply look up its solution in the table.
  - Reconstruct the solution to the original problem from the solutions to the subproblems.

Outline

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Dynamic Programming for Optimization

- There are two key ingredients that an optimization problem must have in order for dynamic programming to apply:
  - Optimal substructure;
  - Overlapping subproblems.

Optimal Substructure

- An optimal solution to the problem contains **within** it **optimal solutions to subproblems**.
  - In matrix-chain multiplication, the optimal order on calculating $A_i \times \cdots \times A_j$ that splits the product between A_k and A_{k+1} contains within it optimal solutions to the problem of ordering $A_i \times \cdots \times A_k$ and $A_{k+1} \times \cdots \times A_j$.

- You can show optimal substructure property by supposing that each of the subproblem solutions is not optimal and then deriving a contradiction.

Overlapping Subproblems

- A recursive algorithm for the problem solves the same subproblems **over and over**, rather than always generating new subproblems.
  - E.g., subproblems of matrix-chain multiplication overlap.
  - In contrast, a problem for which a divide-and-conquer approach is suitable usually generates **brand-new** problems at each step of the recursion.
- Dynamic-programming algorithms take advantage of overlapping subproblems by
  - solving each subproblem once …
  - … and then storing the solution in a table where it can be looked up when needed.

Designing a Dynamic-Programming Algorithm

1. Characterize **the structure** of an optimal solution.
   - Usually, we need to define a **general** problem.
2. **Recursively** define the value of an optimal solution.
3. Compute the value of an optimal solution, typically in a **bottom-up** fashion.
4. Construct an optimal solution from computed information.