

# VE281

## Data Structures and Algorithms

### Priority Queues and Heaps

#### Learning Objectives:

- Know what a priority queue is
- Know what a min heap is
- Know how a min heap performs enqueue and extractMin operations
- Know how to efficiently initialize a min heap

# Outline

- Priority Queue
- Min Heap and Its Operations
- Min Heap Initialization and Application

# Priority Queues

- Two kinds of priority queues:
  - Min priority queue.
  - Max priority queue.
- We will focus on **min priority queue**.
  - The max priority queue is similar.

# What Is Min Priority Queue?

- A collection of items.
- Each item has a key (or “**priority**”).
- Support the following operations:
  - **isEmpty**  $O(1)$
  - **size**  $O(1)$
  - **enqueue**: put an item into the priority queue.  $O(1)$
  - **dequeueMin**: remove element with **min** key.  $O(n)$
  - **getMin**: get item with **min** key.  $O(n)$

# Applications of Priority Queue

- Banking services
  - VIP customer who arrives later gets served first.
- Network bandwidth management
  - The prioritized traffic, such as real-time data, is forwarded with the least delay once it reaches the network router.
- Discrete event simulation
  - One event happening triggers a few others, which are put into a queue.
  - Simulating in the order of the **beginning time** of the events.

# Min Priority Queue: Implementation

- A collection of items.
- Each item has a key (or “**priority**”).
- Support the following operations:
  - **isEmpty**  $O(1)$
  - **size**  $O(1)$
  - **enqueue**: put an item into the priority queue.  $O(1)$
  - **dequeueMin**: remove element with **min** key.  $O(n)$
  - **getMin**: get item with **min** key.  $O(n)$

What's the time complexity for an unsorted array-based implementation?

# Priority Queue Implemented with Heap

- Priority queues are most commonly implemented using **Binary Heaps** (will be shown soon).
- Complexity of the operation using heap implementation:
  - **isEmpty**, **size**, and **getMin** are  $O(1)$  time complexity in the worst case.
  - **enqueue** and **dequeueMin** are  $O(\log n)$  time complexity in the worst case, where  $n$  is the size of the priority queue.

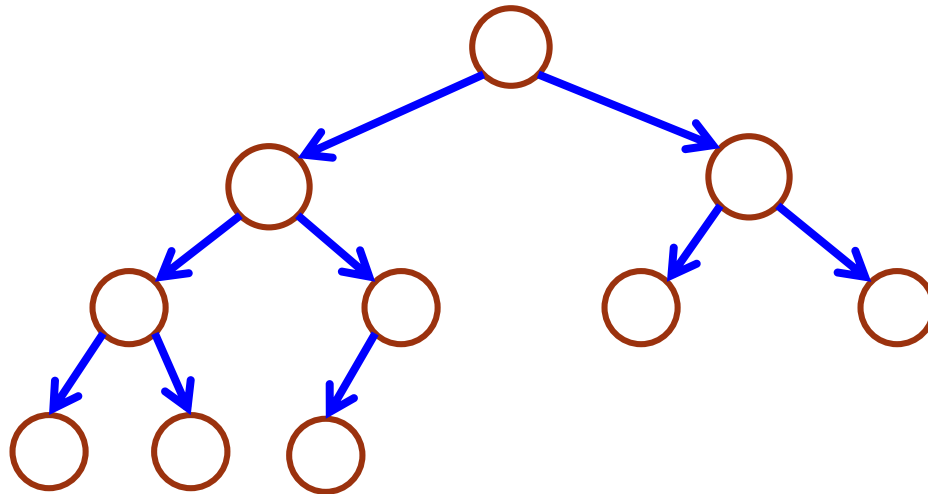
# Outline

- Priority Queue
- **Min Heap and Its Operations**
- Min Heap Initialization and Application



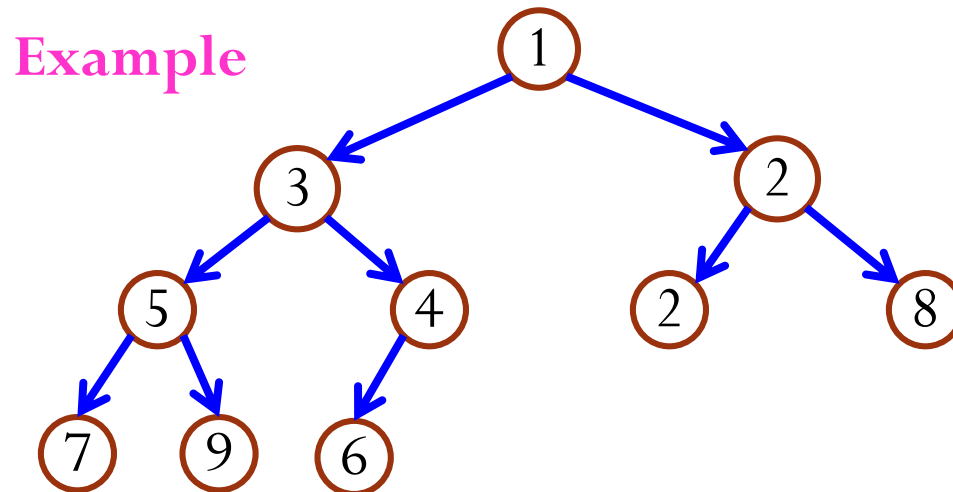
# Binary Heap

- A **binary heap** is a **complete binary tree**.

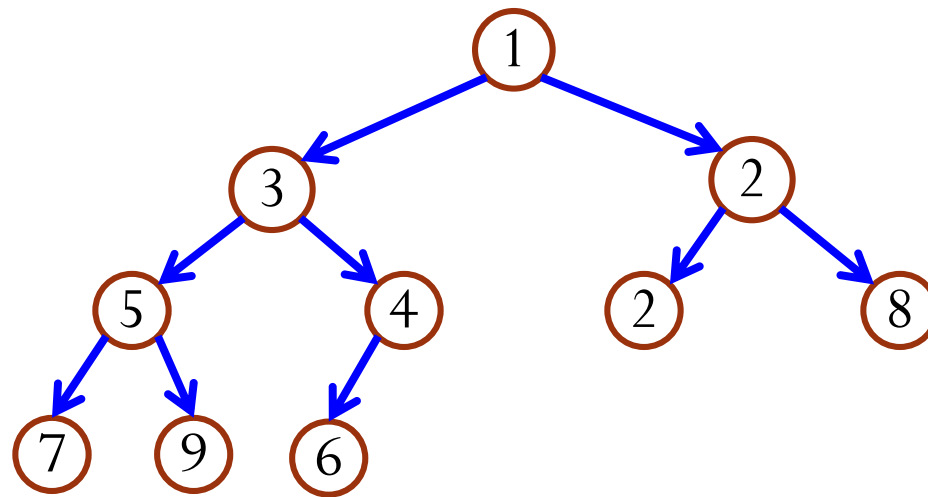


# Min Heap

- A min heap is
  - a **binary heap**, and
  - a tree where for **any** node  $v$ , the key of  $v$  is smaller than or equal to ( $\leq$ ) the keys of any **descendants** of  $v$ .
- Property: The key of the root of **any** subtree is always the smallest among all the keys in that subtree.



# Min Heap



- However, the keys of nodes **across** subtrees have no required relationship.
  - Different from binary search trees, which we will show later.

# ? What's the Height of a Heap of $n$ Nodes?

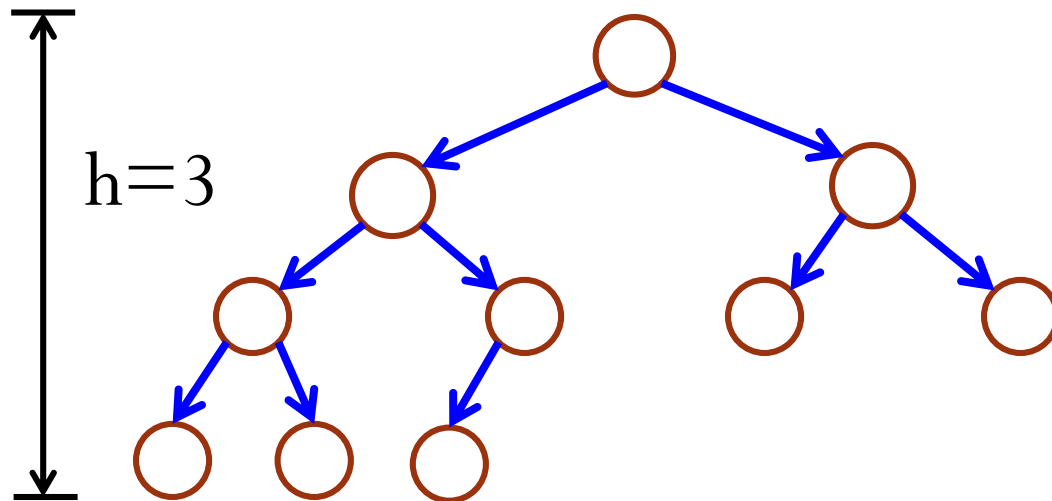
Select **all** the correct answers:

A.  $\lfloor \log_2(n + 1) \rfloor - 1$

B.  $\lfloor \log_2 n \rfloor - 1$

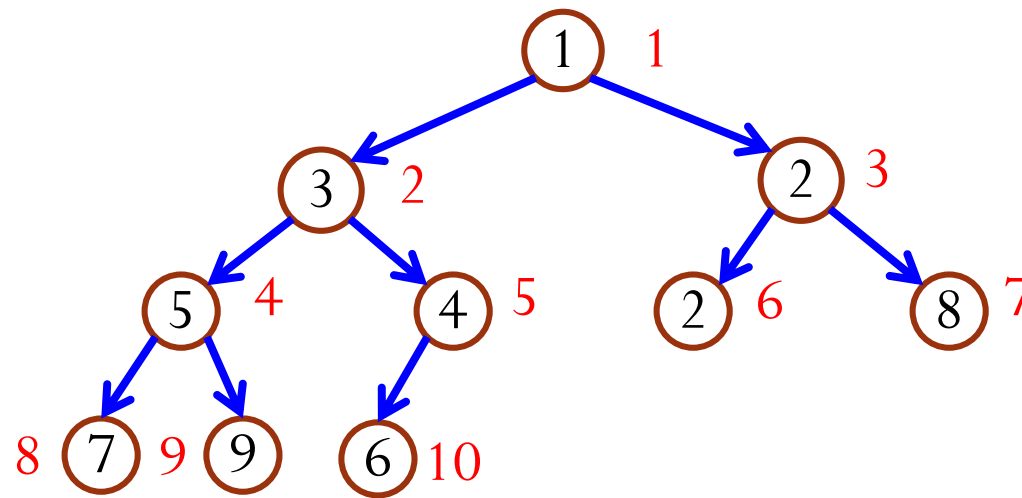
C.  $\lfloor \log_2(n + 1) \rfloor - 1$

D.  $\lfloor \log_2 n \rfloor$



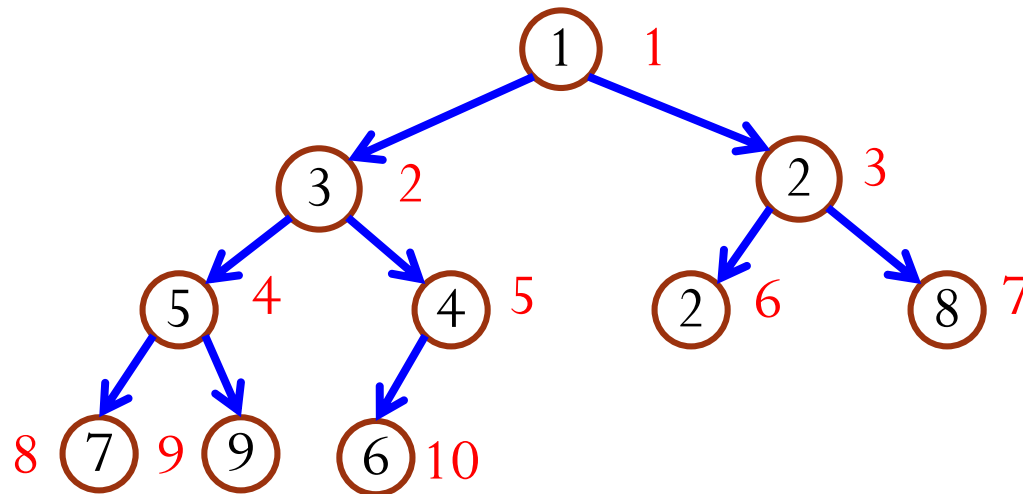
# Binary Heap Implementation as an Array

- Store the elements in an array in the order produced by a level-order traversal.
- The first element is stored at index 1.



1	3	2	5	4	2	8	7	9	6
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

# Index Relation



Index relation allows us to move up and down a heap easily.

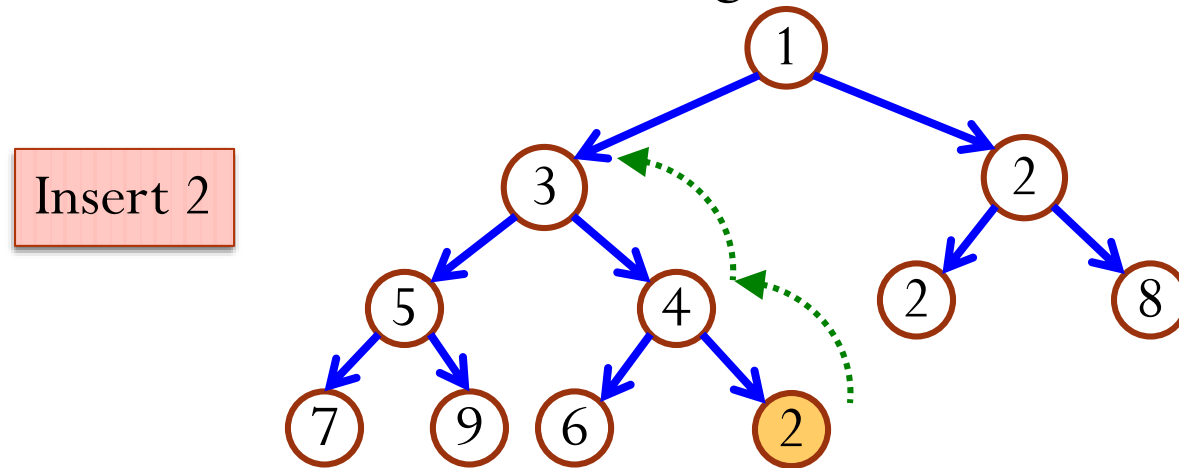
- A node at index  $i$  ( $i \neq 1$ ) has its parent at index  $\lfloor i/2 \rfloor$ .
- Assume the number of nodes is  $n$ . A node at index  $i$  ( $2i \leq n$ ) has its left child at  $2i$ .
  - If  $2i > n$ , it has no left child.
- A node at index  $i$  ( $2i + 1 \leq n$ ) has its right child at  $2i + 1$ .
  - If  $2i + 1 > n$ , it has no right child.

# Min Heap Implementation

- We also have a **size** variable to keep the number of nodes in the heap.
  - The heap elements are stored in `heap[1]`, `heap[2]`, ..., `heap[size]`.
- Operations
  - **isEmpty**: `return size==0;`
  - **size**: `return size;`
  - **getMin**: `return heap[1];`

# Procedure of enqueue

- Insert **newItem** as the rightmost leaf of the tree.



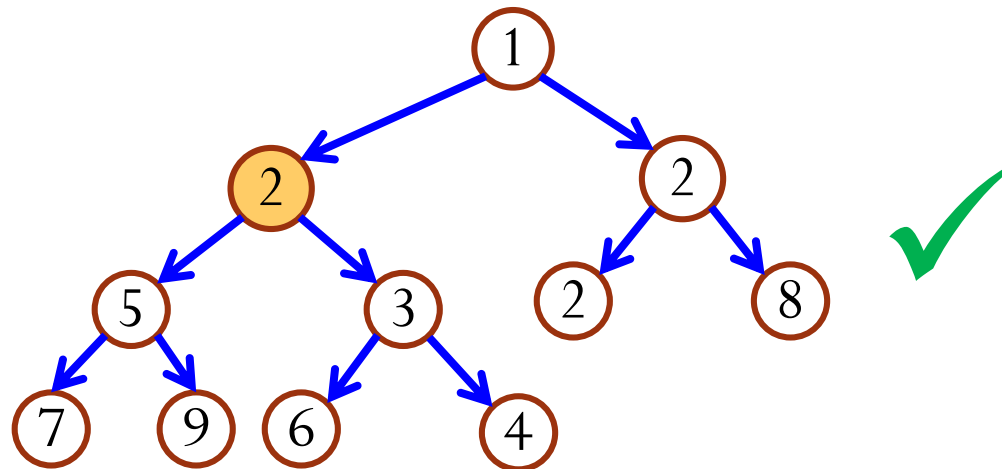
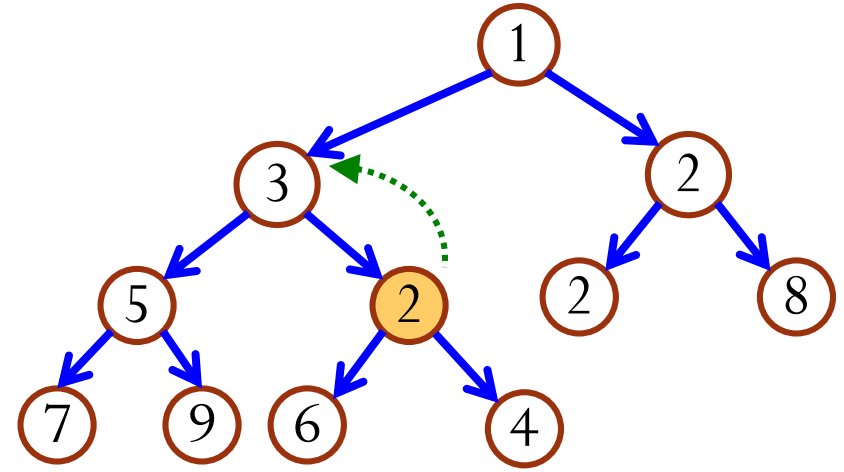
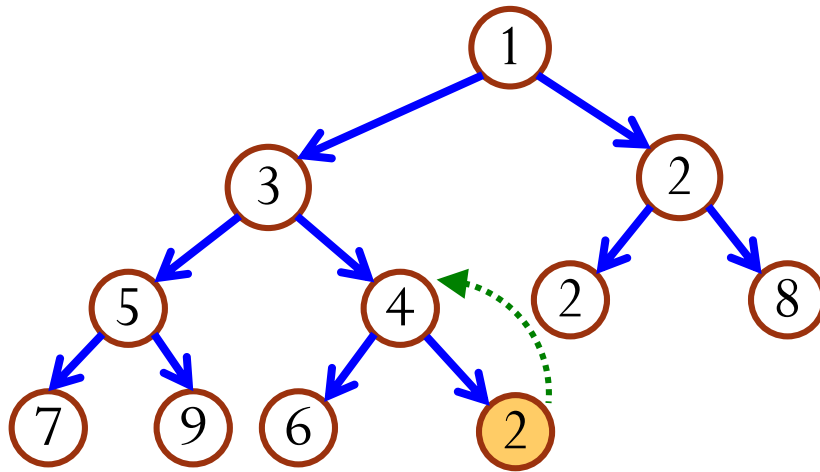
**heap[++size] = newItem;**

- The tree may no longer be a heap at this point!
- **Percolate up** **newItem** to an appropriate spot in the heap to restore the heap property.



# Percolate Up

## Illustration



# Percolate Up

## Code

```
void minHeap::percolateUp(int id) {  
    while(id > 1 && heap[id/2] > heap[id]) {  
        swap(heap[id], heap[id/2]);  
        id = id/2;  
    }  
}
```

- Pass index (**id**) of array element that needs to be percolated up.
- Swap the given node with its parent and move up to parent until:
  - we reach the root at position 1, or
  - the parent has a smaller or equal key.

# enqueue

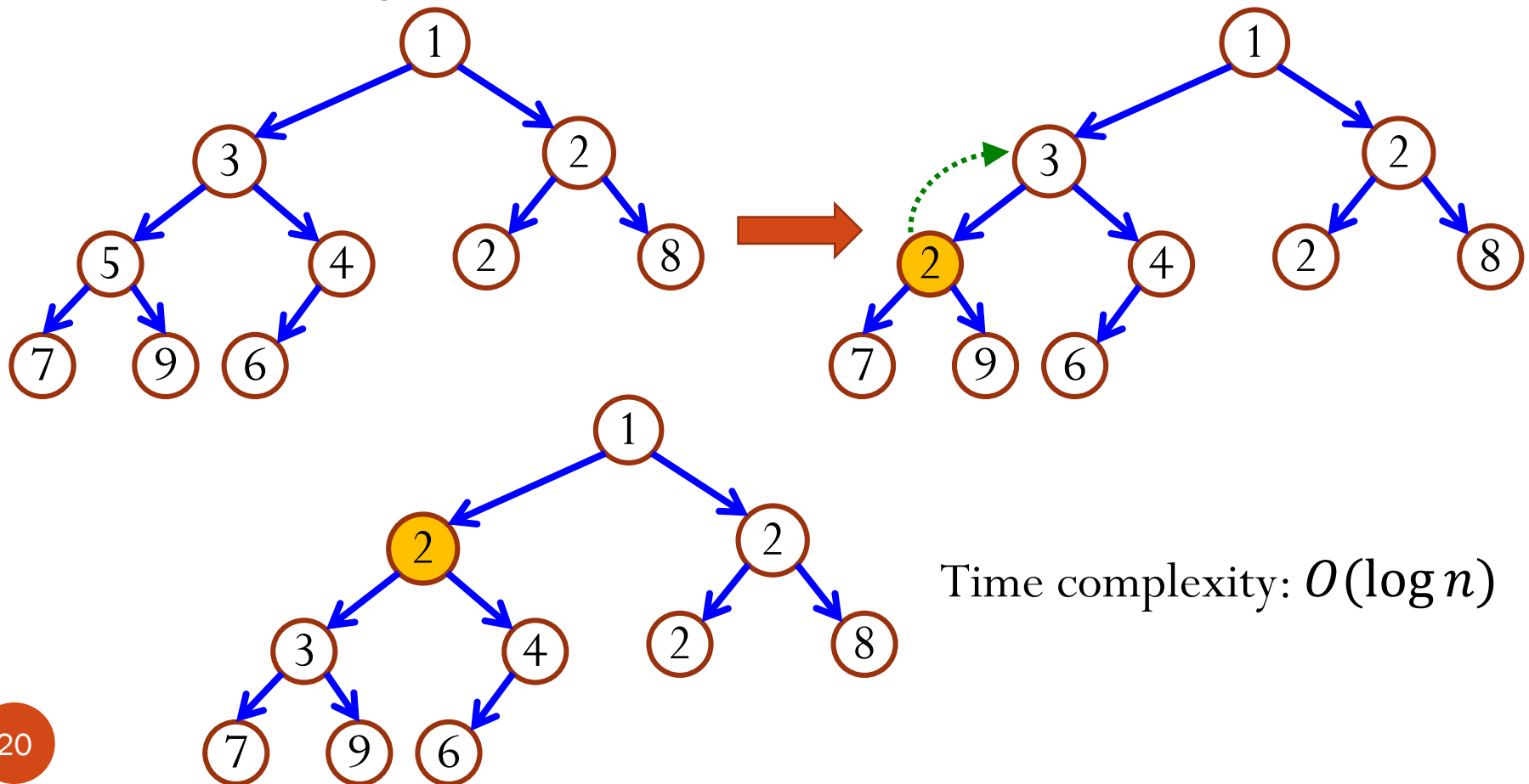
Code

```
void minHeap::enqueue(Item newItem) {  
    heap[++size] = newItem;  
    percolateUp(size) ;  
}
```

- What is the time complexity?
  - $O(\log n)$

# Aside: Decrease Key

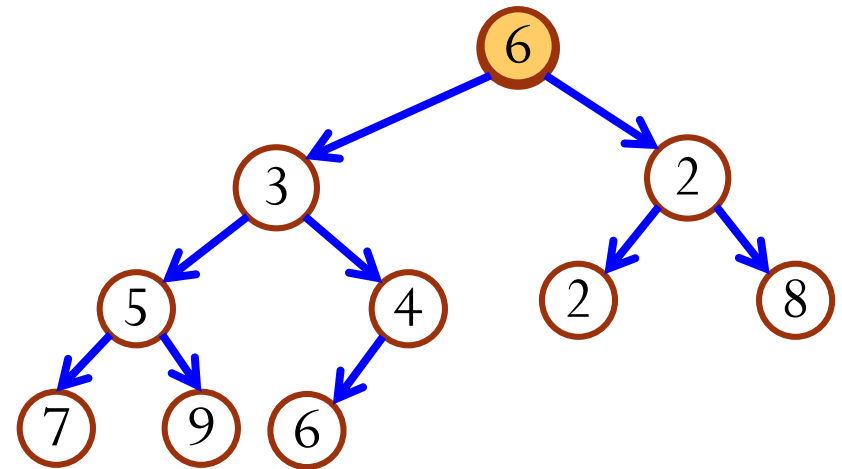
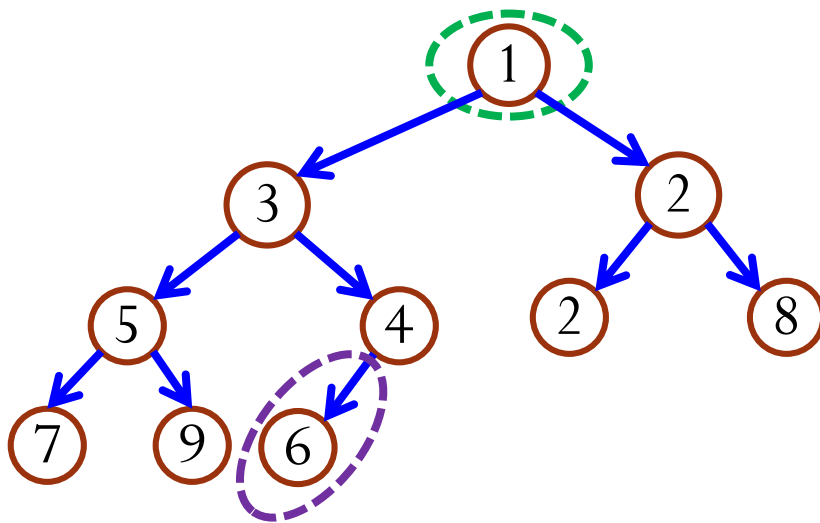
- Percolating-up can also be exploited to implement the decreasing-key operation



# Procedure of **dequeueMin**

- The min item is at the root. Save that item to be returned.
- Move the item in the rightmost leaf of the tree to the root.

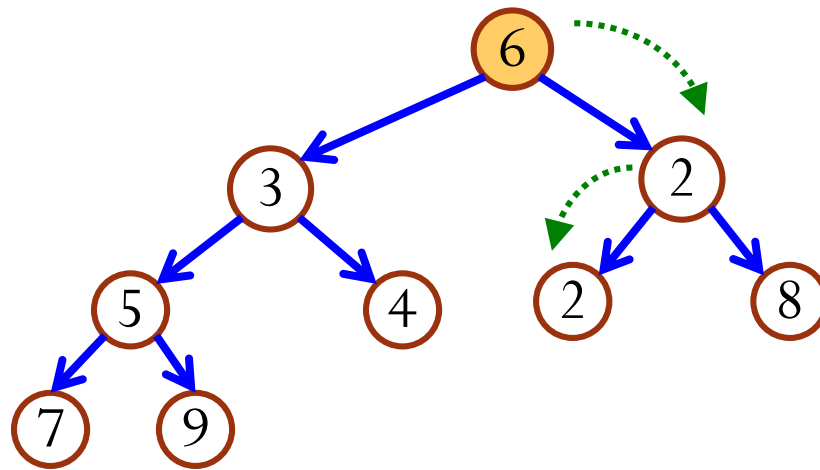
**swap(heap[1], heap[size--]);**



- The tree may no longer be a heap at this point!

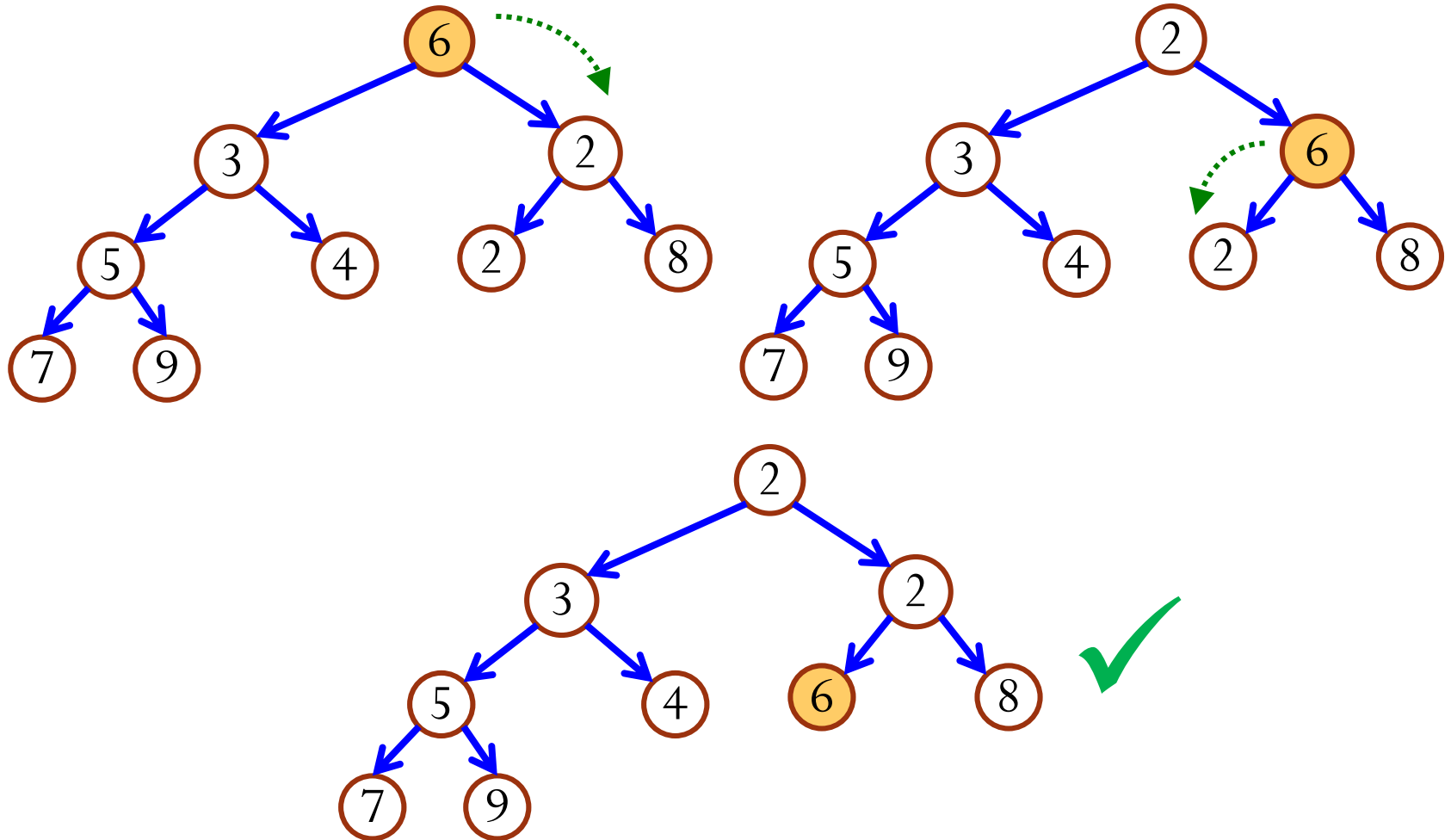
# Procedure of dequeueMin

- **Percolate down** the recently moved item at the root to its proper place to restore heap property.
  - For each subtree, if the root has a **larger** search key than **either of its children**, swap the item in the root with that of the **smaller** child.



# Percolate Down

Illustration



# Percolate Down

## Code

```
void minHeap::percolateDown(int id) {
    for(j = 2*id; j <= size; j = 2*j) {
        if(j < size && heap[j] > heap[j+1]) j++;
        if(heap[id] <= heap[j]) break;  find the smaller child
        swap(heap[id], heap[j]);
        id = j;
    }
}
```

- Pass index (**id**) of array element that needs to be percolated down.
- Swap the key in the given node with the smallest key among the node's children, moving down to that child, until:
  - we reach a leaf node, or
  - both children have larger (or equal) key



# dequeueMin

Code

```
Item minHeap::dequeueMin() {  
    swap(heap[1], heap[size--]);  
    percolateDown(1);  
    return heap[size+1];  
}
```

- What is the time complexity?
  - $O(\log n)$

# Outline

- Priority Queue
- Min Heap and Its Operations
- Min Heap Initialization and Application

# Initializing a Min Heap

- How do we initialize a min heap from a set of items?
- Simple solution: insert each entry one by one.
  - The worst case time complexity for inserting the  $k$ -th item is  $O(\log k)$ , so creating a heap in this way is  $O(n \log n)$ .
- Instead, we can do better by putting the entries into a **complete** binary tree and running **percolate down** intelligently.

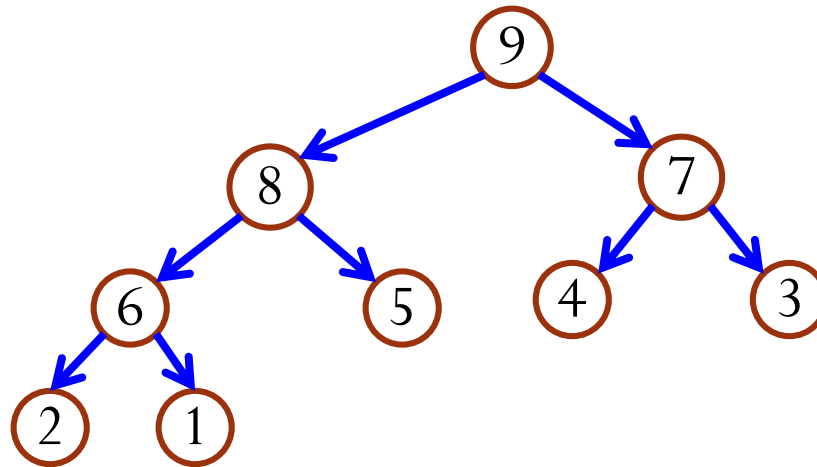
# Initializing a Min Heap

- Put all the items into a complete binary tree.
  - Implemented using an array.
- Starting at the rightmost array position **that has a child**, percolate down all nodes in **reverse** level-order.
  - The rightmost array position **that has a child** is **size/2**.
- Procedure:  
For  **$i = \text{size}/2$**  down to **1**  
    **percolateDown(i) ;**

# Initializing a Min Heap

## Illustration

- Input items: 9, 8, 7, 6, 5, 4, 3, 2, 1
- First step: put all the items into a complete binary tree.

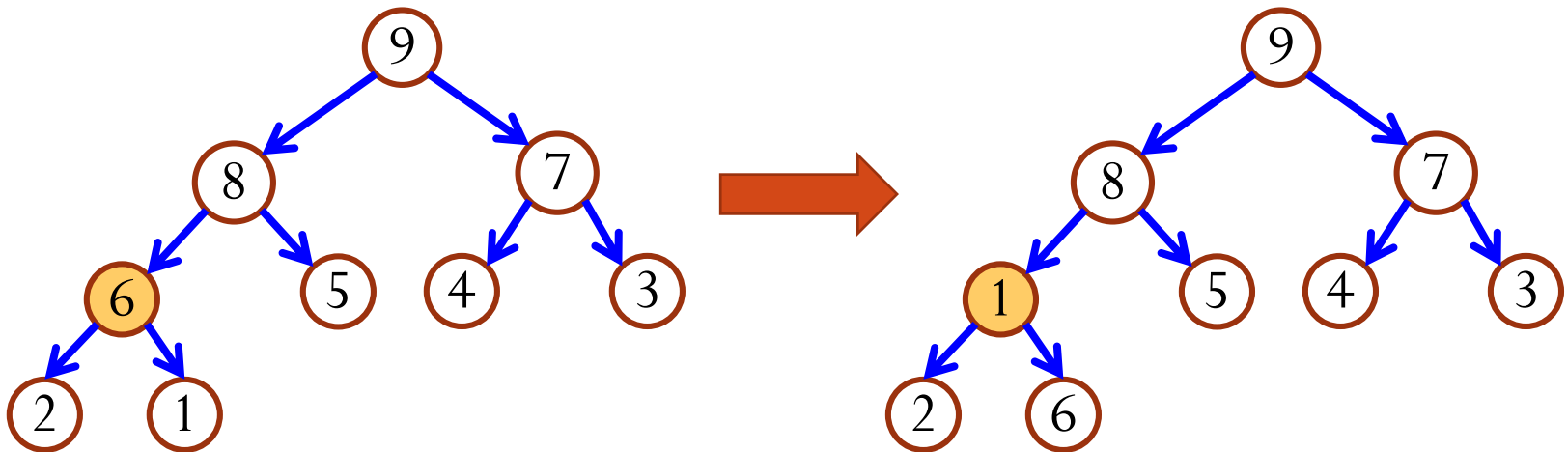


# Initializing a Min Heap

## Illustration

- Starting at the rightmost array position **that has a child**, percolate down all nodes in **reverse** level-order.

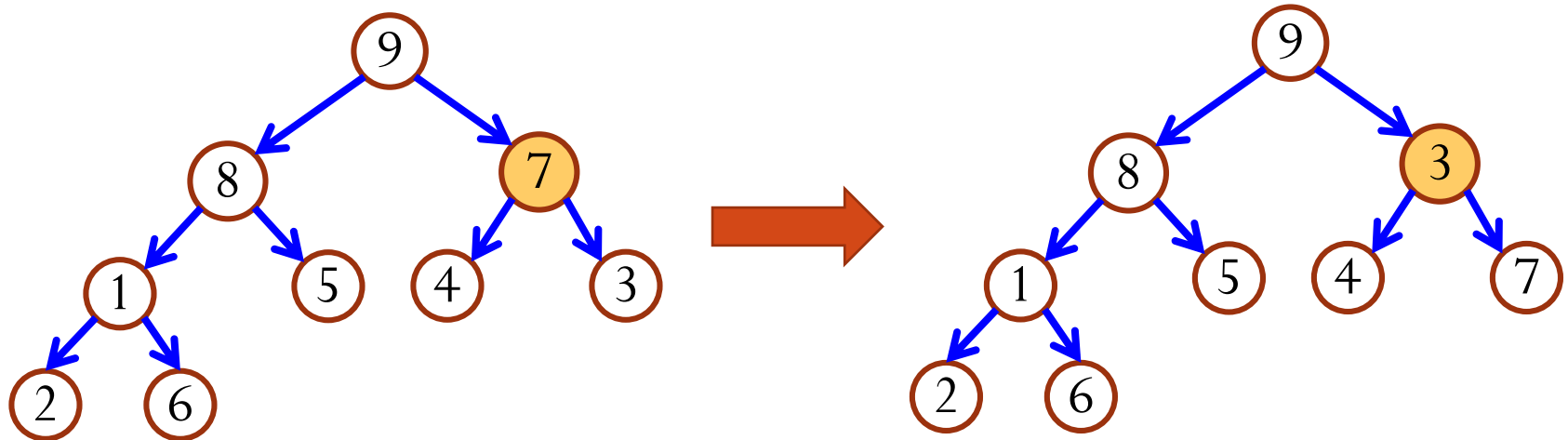
Node at index  $9/2 = 4$



# Initializing a Min Heap

## Illustration

Node at index 3

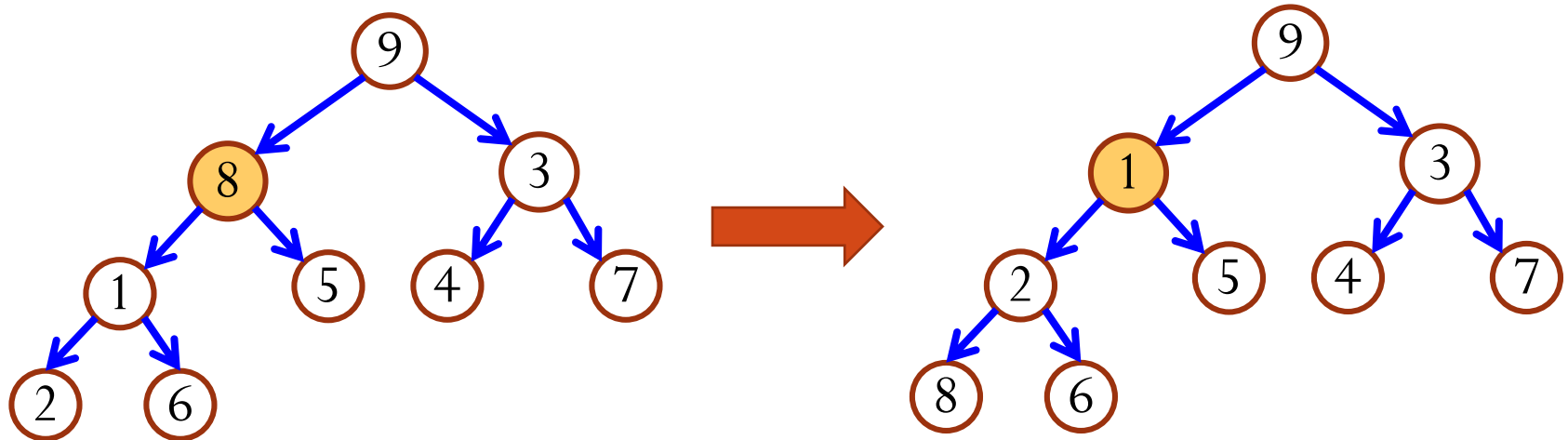


Move to next lower array position.

# Initializing a Min Heap

## Illustration

Node at index 2



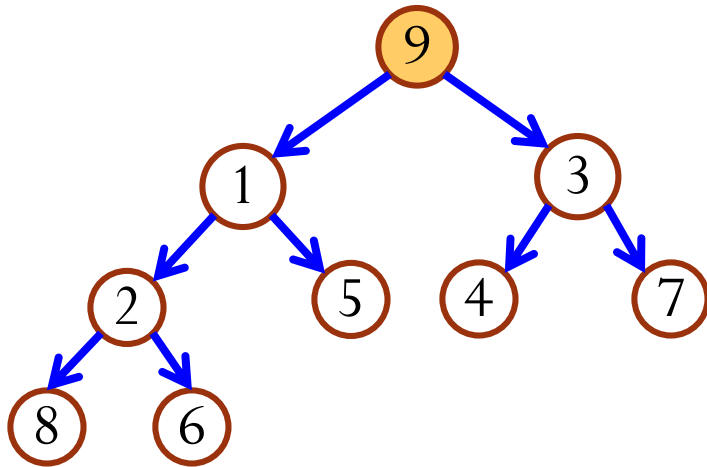
Move to next lower array position.



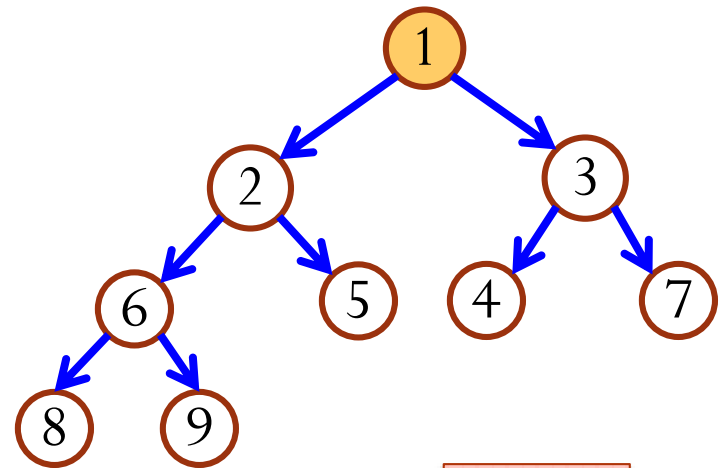
# Initializing a Min Heap

## Illustration

Node at index 1

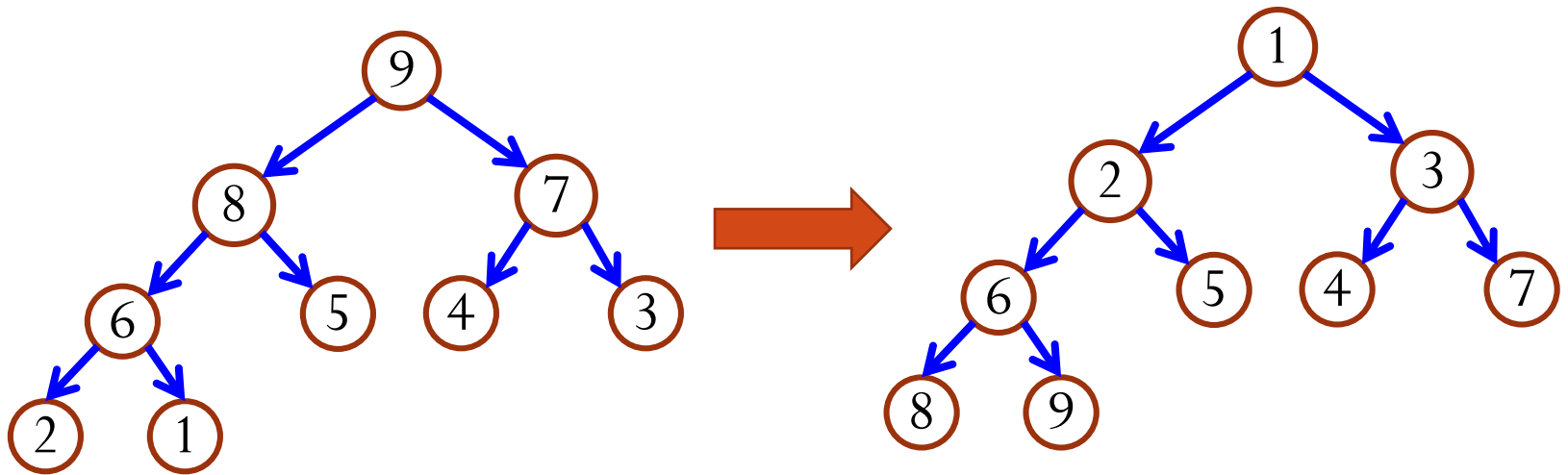


Exercise: What's the result?



Done!

# Time Complexity Analysis



- Suppose: the **height** of the heap is  $h$ .
- Note: Number of nodes at level  $k$  ( $0 \leq k \leq h$ ) is  $\leq 2^k$ .
- Note: The worst case time complexity of percolating down a node at level  $k$  is  $O(h - k)$ .

# Time Complexity Analysis

$$T(h) \leq \sum_{k=0}^{h-1} 2^k O(h-k) = O\left(\sum_{k=0}^{h-1} 2^k (h-k)\right)$$

- What is  $S(h) = \sum_{k=0}^{h-1} 2^k (h-k)$ ?

$$S(h) = 2^0 h + 2^1 (h-1) + 2^2 (h-2) + \dots + 2^{h-1} \cdot 1$$

$$2S(h) = 2^1 h + 2^2 (h-1) + \dots + 2^{h-1} \cdot 2 + 2^h \cdot 1$$

$$2S(h) = \quad 2^1 h \quad + 2^2 (h-1) + \dots + 2^{h-1} \cdot 2 + 2^h \cdot 1$$

$$S(h) = 2S(h) - S(h) = 2^1 + 2^2 + \dots + 2^h - h = 2^{h+1} - 2 - h$$

# Time Complexity Analysis

$$T(h) \leq O(2^{h+1} - 2 - h)$$

- For a complete binary tree, we have

$$h = \lceil \log_2(n + 1) \rceil - 1 \leq \log_2(n + 1)$$

where  $n$  is the number of nodes.

- Therefore, the algorithm for initializing a min heap with  $n$  nodes has worst case time complexity  $T(n) = O(n)$ .
  - Better than the way to enqueue entry one by one.

# Application of Heap: Sorting

- Procedure:

1. Initialize a min heap with all the elements to be sorted

Complexity:  $O(n)$

2. Repeatedly call **dequeueMin** to extract elements out of the heap.

Complexity:  $O(n \log n)$

- The resulting elements are sorted by their keys.
- What is the time complexity?  $O(n \log n)$
- This is known as **heap sort**.

# Application: Median Maintenance

- Input: a sequence of numbers  $x_1, x_2, \dots, x_n$ , one-by-one
- Output: at each time step  $i$ , the median of  $x_1, x_2, \dots, x_i$
- Problem: how to do this with  $O(\log i)$  time at each step  $i$ ?
- Hint: using two heaps, one min heap and one max heap
- Key idea: maintain the smallest half ( $\lceil \frac{n}{2} \rceil$ ) in max heap and the largest half ( $\lfloor \frac{n}{2} \rfloor$ ) in the min heap
- Question: How do you get the median (i.e., the  $\lceil \frac{n}{2} \rceil$ -th smallest item)?
  - Answer: get max from the max heap

# How to Insert a New Item?

- Key problem: maintain the **invariant** that the smallest half ( $\lceil \frac{n}{2} \rceil$ ) in max heap and the largest half ( $\lfloor \frac{n}{2} \rfloor$ ) in the min heap
  - To maintain balance between the two heaps
- If  $n$  (before insertion) is even
  - If new item  $\leq \min(\text{minHeap})$ , insert it into maxHeap
  - Else (new item  $> \min(\text{minHeap})$ ), first extract min value from minHeap, then insert that value in maxHeap, and finally insert new item into minHeap
- If  $n$  (before insertion) is odd
  - If new item  $\geq \max(\text{maxHeap})$ , insert it into minHeap
  - Else (new item  $< \max(\text{maxHeap})$ ), first extract max value from maxHeap, then insert that value in minHeap, and finally insert new item into maxHeap

Time complexity is  $O(\log i)$