

VE281

Data Structures and Algorithms

Comparison Sort

Learning Objectives:

- Know the difference between comparison sort and non-comparison sort
- Know the procedures of merge sort and quick sort
- Know the master theorem
- Know different characteristics of sorting algorithms, such as time complexity, stableness, etc.

Outline

- Sorting Basics
- Merge Sort
- Quick Sort
- Comparison Sort Summary

Sorting

- Given array A of size N , reorder A so that its elements are in order.
 - "In order" with respect to a consistent comparison function, such as " \leq " or " \geq ".
- Sorting order
 - Ascending order
 - Descending order
- Unless otherwise specified, we consider sorting in ascending order.

Characteristics of Sorting Algorithms

- Average-case time complexity
- Worst-case time complexity
- Space usage: **in place** or not?
 - **in place**: requires $O(1)$ additional memory
 - Don't forget the stack space used in recursive calls
 - **In place is better**
 - Why? The data can fit into cache, not main memory
 - Real example: quick sort versus merge sort. Both have average-case time complexity of $O(n \log n)$. Quick sort is faster, due to in place

Characteristics of Sorting Algorithms

- **Stability**: whether the algorithm maintains the relative order of records with equal keys

$(4, b), (3, e), (3, b), (5, b) \longrightarrow (3, e), (3, b), (4, b), (5, b)$

Sort on the first number

Stable!

- Usually there is a secondary key whose ordering you want to keep. Stable sort is thus useful for sorting over multiple keys
- Example: sort complex numbers $a+bi$
 - Ordering rule: first compare a ; when there is a tie, compare b
 - One sorting method: first sort b , then sort a

$3+5i, 2+6i, 3+4i, 5+2i$

Sort on b

$5+2i, 3+4i, 3+5i, 2+6i$

... sort on a

$2+6i, 3+4i, 3+5i, 5+2i$

Stability is important!

Types of Sorting Algorithms

- Sorting algorithms can be classified as **comparison sort** and **non-comparison sort**.
- **Comparison sort**: each item is compared against others to determine its order.
- **Non-comparison sort**: each item is put into predefined “bins” independent of the other items presented.
 - No comparison with other items needed.
 - It is also known as **distribution-based sort**.

Types of Sorting Algorithms

- General types of comparison sort
 - Insertion-based: insertion sort
 - Selection-based: selection sort, heap sort
 - Exchange-based: bubble sort, quick sort
 - Merging-based: merge sort
- Non-comparison sort:
counting sort, bucket sort, radix sort

Insertion Sort

- **A[0]** alone is a sorted array.
- For **i=1** to **N-1**
 - **Insert A[i]** into the appropriate location in the sorted array **A[0], ..., A[i-1]**, so that **A[0], ..., A[i]** is sorted.
 - To do so, save **A[i]** in a temporary variable **t**, shift sorted elements greater than **t** right, and then insert **t** in the gap.
- Time complexity? $O(N^2)$
- In place? Yes. $O(1)$ additional memory.
- Stable?
 - Yes, because elements are visited in order and equal elements are inserted after its equals.

Insertion Sort

Best Case Time Complexity

- For **$i=1$** to **$N-1$**
 - **Insert $A[i]$** into the appropriate location in the sorted array **$A[0], \dots, A[i-1]$** , so that **$A[0], \dots, A[i]$** is sorted.
- The **best case** time complexity is $O(N)$.
 - It happens when the array is already sorted.
 - For other sorting algorithms we will talk, their best case time complexity is $\Omega(N \log N)$.

Selection Sort

- For **$i=0$** to **$N-2$**
 - Find the smallest item in the array **$A[i]$** , ... , **$A[N-1]$** .
Then, swap that item with **$A[i]$** .
- Finding the smallest item requires **linear scan**.



Which Statements Are Correct for Selection Sort?

For **i=0** to **N-2**

Find the smallest item in the array **A[i]** , ... , **A[N-1]** . Then, swap that item with **A[i]** .

- **A.** Its worse-case time complexity is $O(N^2)$
- **B.** Its best-case time complexity is $\Omega(N^2)$
- **C.** It is not in-place
- **D.** It is stable



Bubble Sort

For $i=N-2$ **downto** 0

For $j=0$ **to** i

If $A[j]>A[j+1]$ **swap** $A[j]$ **and** $A[j+1]$

- Compares two adjacent items and swap them to keep them in ascending order.
 - From the beginning to the end. The last item will be the largest.
- Time complexity? $O(N^2)$
- In place? Yes.
- Stable?
 - Yes, because equal elements will not be swapped.

Two Problems with Simple Sorts

- They learn only one piece of information per comparison and hence might compare every pair of elements.
 - Contrast with binary search: learns $N/2$ pieces of information with first comparison.
- They often move elements one place at a time (bubble sort and insertion sort), even if the element is “far” from its **final place**.
 - Contrast with selection sort, which moves each element exactly to its final place.
- Fast sorts attack these two problems.
 - Two famous ones: **merge sort** and **quick sort**.

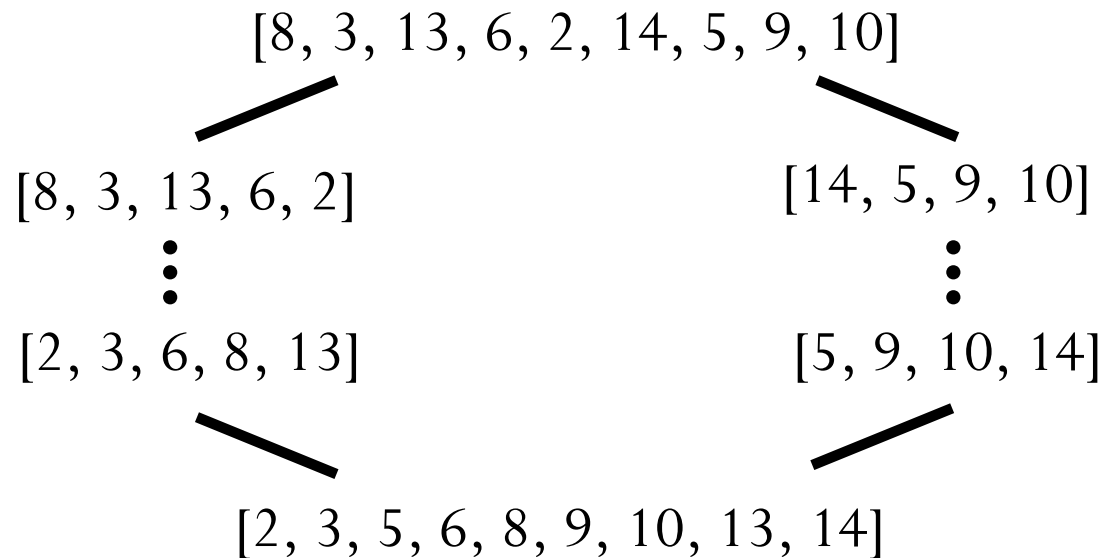
Outline

- Sorting Basics
- **Merge Sort**
- Quick Sort
- Comparison Sort Summary

Merge Sort

Algorithm

- Spilt array into two (roughly) equal subarrays.
- Merge sort each subarray recursively.
 - The two subarrays will be sorted.
- Merge the two sorted subarrays into a sorted array.



Merge Sort

Pseudo-code

```
void mergesort(int *a, int left, int
    right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```


Merge Two Sorted Arrays

- For example, merge $A = (2, 5, 6)$ and $B = (1, 3, 8, 9, 10)$.
- Compare the smallest element in the two arrays A and B and move the smaller one to an additional array C .
- Repeat until one of the arrays becomes empty.
- Then append the other array at the end of array C .

Merge Two Sorted Arrays

Implementation

- We actually do not “remove” element from arrays A and B.
 - We just keep a pointer indicating the smallest element in each array.
 - We “remove” element by incrementing that pointer.

```
i = j = k = 0;
while(i < sizeA && j < sizeB) {
    if(A[i] <= B[j]) C[k++] = A[i++];
    else C[k++] = B[j++];
}
if(i == sizeA) append(C, B);
else append(C, A);
```

Time complexity?

Time complexity is $O(\text{sizeA} + \text{sizeB})$

Merge Sort

Time Complexity

```
void mergesort(int *a, int left, int
    right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);  $T(N/2)$ 
    mergesort(a, mid+1, right);  $T(N/2)$ 
    merge(a, left, mid, right);  $O(N)$ 
}
```

- Let $T(N)$ be the time required to merge sort N elements.
- Merge two sorted arrays with total size N takes $O(N)$.

Recursive relation: $T(N) = 2T(N/2) + O(N)$

How to solve the recurrence?

Solve Recurrence: Master Method

- A “black box” for solving recurrence.
- However, there is an important assumption: all sub-problems have roughly **equal** sizes.
 - E.g., merge sort
 - Not apply to unbalanced division.

Solve Recurrence: Master Method

- Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$
 - Base case: $T(n) \leq \text{constant}$ for all sufficiently small n .
 - a = number of recursive calls (integer ≥ 1)
 - b = input size shrinkage factor (integer > 1)
 - $O(n^d)$: the runtime of merging solutions. d is real value ≥ 0 .
 - a, b, d are independent of n .
- Claim:

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

base doesn't matter

base matters!

Example of Merge Sort

Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

Claim: $T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$

- $a = 2, b = 2, d = 1 \Rightarrow b^d = a$
- $T(n) = O(n \log n)$



What are a , b , d for Binary Search?

Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$

Claim: $T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$

A. $a = 2, b = 2, d = 0$ B. $a = 1, b = 2, d = 0$

C. $a = 2, b = 2, d = 1$ D. $a = 1, b = 2, d = 1$



Merge Sort

Characteristics

- Not in-place
 - For efficient merging two sorted arrays, we need an auxiliary $O(N)$ space.
 - Recursion needs up to $O(\log N)$ stack space.
- Stable if **merge () maintains** the relative order of equal keys.

Divide-and-Conquer Approach

- Merge sort uses the **divide-and-conquer** approach.
- Recursively **breaking** down a problem into two or more sub-problems of the same (or related) type, until these become simple enough to be solved directly.
 - For merge sort, split an array into two and sort them respectively.
- The solutions to the sub-problems are then **combined** to give a solution to the original problem.
 - For merge sort, merge two sorted arrays.

Outline

- Sorting Basics
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Quick Sort

Algorithm

Another divide-and-conquer approach to sort

- Choose an array element as **pivot**.
 - Put all elements $<$ pivot to the left of pivot.
 - Put all elements \geq pivot to the right of pivot.
 - Move pivot to its correct place in the array.
 - Sort left and right subarrays recursively (not including pivot).
- } **partition()**

```
void quicksort(int *a, int left,
               int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

Choice of Pivot

- If your input is random, you can choose the **first** element.
 - But this is very bad for presorted input.
- A better strategy: **randomly** pick an element from the array as pivot.
 - **Claim:** **for any input**, the average running time is $O(n \log n)$.
 - **Note:** average is over random choice of pivots made by the algorithm, **not** on the input.

Partitioning the Array

- Once pivot is chosen, swap pivot to the beginning of the array.
- When another array B is available, scan original array A from left to right.
 - Put elements $<$ pivot at the left end of B.
 - Put elements \geq pivot at the right end of B.
 - The pivot is put at the remaining position of B.
 - Copy B back to A.

A

6	2	8	5	11	10	4	1	9	7	3
---	---	---	---	----	----	---	---	---	---	---

B

2	5	4	1	3	6	7	9	10	11	8
---	---	---	---	---	---	---	---	----	----	---

In-Place Partitioning the Array

1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **$i=1$** and **$j=N-1$** .
3. Increment **i** until we find element **$A[i] \geq \text{pivot}$** .
 - **$A[i]$** is the leftmost item \geq pivot.
4. Decrement **j** until we find element **$A[j] < \text{pivot}$** .
 - **$A[j]$** is the rightmost item $<$ pivot.
5. If **$i < j$** , swap **$A[i]$** with **$A[j]$** . Go back to step 3.
6. Otherwise, swap the first element (pivot) with **$A[j]$** .

In-Place Partitioning the Array

Example

i j

A

6	2	8	5	11	10	4	1	9	7	3
---	---	---	---	----	----	---	---	---	---	---

A

6	2	3	5	11	10	4	1	9	7	8
---	---	---	---	----	----	---	---	---	---	---

A

6	2	3	5	1	10	4	11	9	7	8
---	---	---	---	---	----	---	----	---	---	---

A

6	2	3	5	1	4	10	11	9	7	8
---	---	---	---	---	---	----	----	---	---	---

- Now, $j < i$, swap the first element (pivot) with $A[j]$.

A

4	2	3	5	1	6	10	11	9	7	8
---	---	---	---	---	---	----	----	---	---	---

In-Place Partitioning the Array

Time Complexity

1. Once pivot is chosen, swap pivot to the beginning of the array.
 2. Start counters **$i=1$** and **$j=N-1$** .
 3. Increment **i** until we find element **$A[i] \geq \text{pivot}$** .
 4. Decrement **j** until we find element **$A[j] < \text{pivot}$** .
 5. If **$i < j$** , swap **$A[i]$** with **$A[j]$** . Go back to step 3.
 6. Otherwise, swap the first element (pivot) with **$A[j]$** .
- Scan the entire array no more than twice.
 - Time complexity is $O(N)$, where N is the size of the array.

Quick Sort

Time Complexity

```
void quicksort(int *a, int left,
               int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right); O(N)
    quicksort(a, left, pivotat-1); T(LeftSz)
    quicksort(a, pivotat+1, right); T(RightSz)
}
```

- Let $T(N)$ be the time needed to sort N elements.
 - $T(0) = c$, where c is a constant.
- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- $LeftSz + RightSz = N - 1$

Quick Sort

Worst Case Time Complexity

- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- Worst case happens when each time the pivot is the smallest item or the largest item

- $T(N) = T(N - 1) + T(0) + O(N)$

$$\leq T(N - 1) + T(0) + dN$$

$$\leq T(N - 2) + 2T(0) + d(N - 1) + dN$$

...

$$\leq T(0) + NT(0) + d + 2d + \dots + d(N - 1) + dN$$

$$= O(N^2)$$

Quick Sort

Best Case Time Complexity

- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- Best case happens when each time the pivot divides the array into two equal-sized ones.
 - $T(N) = T((N - 1)/2) + T((N - 1)/2) + O(N)$
 - The recursive relation is similar to that of merge sort.
 - $T(N) = O(N \log N)$

Quick Sort

Average Time Complexity

- Average time complexity of quick sort can be proved to be $O(N \log N)$.
 - Assume **randomly** pick an element from the array as pivot.
 - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.
 - The claim holds for any input.

Quick Sort

Other Characteristics

- In-place?
 - In-place partitioning.
 - Worst case needs $O(N)$ stack space.
 - Average case needs $O(\log N)$ stack space.
 - “Weakly” in-place.
- Not stable.

Quick Sort

Summary

- Like merge sort, quick sort is a divide-and-conquer algorithm.
- Merge sort: easy division, complex combination.
- Quick sort: complex division (partition with pivot step), easy combination.
- Insertion sort is faster than quick sort for small arrays.
 - Terminate quick sort when array size is below a threshold. Do insertion sort on subarrays.

Outline

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Comparison Sorts

Summary

	Worst Case Time	Average Case Time	In Place	Stable
Insertion	$O(N^2)$	$O(N^2)$	Yes	Yes
Selection	$O(N^2)$	$O(N^2)$	Yes	No
Bubble	$O(N^2)$	$O(N^2)$	Yes	Yes
Merge Sort	$O(N \log N)$	$O(N \log N)$	No	Yes
Quick Sort	$O(N^2)$	$O(N \log N)$	Weakly	No

For comparison sort, is $O(N \log N)$ the best we can do in the **worst case**?

Comparison Sorts

Worst Case Time Complexity

- Theorem: A sorting algorithm that is based on pairwise comparisons must use $\Omega(N \log N)$ operations to sort in the worst case.