HOL Theorem Proving and Formal Probability (3)

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Recall: σ-algebra and measure space

$(X, \mathcal{A})$ is *σ-algebra* if (under *subset-class* condition: $A \in \mathcal{A} \Rightarrow A \subseteq X$):

- $\emptyset \in \mathcal{A}$ (Σ_1)
- $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$ (or $\overline{A} \in \mathcal{A}$) (Σ_2)
- $A_i \in \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$ (Σ_3)

$(X, \mathcal{A}, \mu)$ is *pre-measure space* (*measure space* when $(X, \mathcal{A})$ is *σ-algebra*) if:

- $\mu(\emptyset) = 0$ (M_1)
- $A \in \mathcal{A} \Rightarrow 0 \leqslant \mu(A)$ (M_2)
- if $A_i \in \mathcal{A}$ are *pairwise disjoint* and $\dot{\bigcup}_{i \in \mathbb{N}} A_i \in \mathcal{A}$, then (M_3)

$$\mu(\dot{\bigcup_{i \in \mathbb{N}}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$$

Recall: σ-algebra and measure space (HOL defs)

$$\vdash \texttt{subset\_class}\ sp\ sts \iff \forall x.\ x \in sts \Rightarrow x \subseteq sp \quad \texttt{[subset\_class\_def]}$$

$$\begin{array}{l}
\vdash \texttt{sigma_algebra}\ p \iff \\
\quad \texttt{subset_class (space}\ p\texttt{) (subsets}\ p\texttt{)} \wedge \\
\quad \emptyset \in \texttt{subsets}\ p \wedge \\
\quad (\forall s.\ s \in \texttt{subsets}\ p \Rightarrow \texttt{space}\ p\ \texttt{DIFF}\ s \in \texttt{subsets}\ p) \wedge \\
\quad \forall c.\ \texttt{countable}\ c \wedge c \subseteq \texttt{subsets}\ p \Rightarrow \\
\qquad \bigcup c \in \texttt{subsets}\ p \qquad \texttt{[SIGMA_ALGEBRA]}
\end{array}$$

$$\begin{array}{l}
\vdash \texttt{positive}\ m \iff \\
\quad \texttt{measure}\ m\ \emptyset = 0 \wedge \\
\quad \forall s.\ s \in \texttt{measurable_sets}\ m \Rightarrow 0 \leq \texttt{measure}\ m\ s \qquad \texttt{[positive_def]}
\end{array}$$

$$\begin{array}{l}
\vdash \texttt{countably_additive}\ m \iff \\
\quad \forall f.\ f \in (\mathcal{U}(\texttt{:num}) \to \texttt{measurable_sets}\ m) \wedge \\
\qquad (\forall i\ j.\ i \neq j \Rightarrow \texttt{DISJOINT}\ (f\ i)\ (f\ j)) \wedge \\
\qquad \bigcup\ (\texttt{IMAGE}\ f\ \mathcal{U}(\texttt{:num})) \in \texttt{measurable_sets}\ m \Rightarrow \\
\qquad \texttt{measure}\ m\ (\bigcup\ (\texttt{IMAGE}\ f\ \mathcal{U}(\texttt{:num}))) = \\
\qquad \texttt{suminf}\ (\texttt{measure}\ m \circ f) \qquad \texttt{[countably_additive_def]}
\end{array}$$

$$\begin{array}{l}
\vdash \texttt{premeasure}\ m \iff \\
\quad \texttt{positive}\ m \wedge \texttt{countably_additive}\ m \qquad \texttt{[premeasure_def]}
\end{array}$$

$$\begin{array}{l}
\vdash \texttt{measure_space}\ m \iff \\
\quad \texttt{sigma_algebra (measurable_space}\ m\texttt{)} \wedge \\
\quad \texttt{positive}\ m \wedge \texttt{countably_additive}\ m \qquad \texttt{[measure_space_def]}
\end{array}$$

Beyond trivial σ-algebras: σ-generator

It's hard to construct explicitly non-trivial σ-algebras, e.g. sets of reals. Instead, σ-algebra can be generated from any family of sets.

Let $(X, \mathcal{G})$ be a family of sets (as a *generator*), $\sigma(X, \mathcal{G})$ is the *smallest σ-algebra* containing $(X, \mathcal{G})$:

$$\sigma(X, \mathcal{G}) := (X, \bigcap_{\substack{\mathcal{G} \subseteq \mathcal{A} \\ (X,\mathcal{A})\ \sigma\text{-alg.}}} \mathcal{A})$$

$\vdash$ sigma sp sts =
  (sp,$\bigcap$ { s | $sts \subseteq s$ $\land$ sigma_algebra (sp,s)})   [sigma_def]

$\vdash$ sigma_algebra a $\Rightarrow$
  sigma (space a) (subsets a) = a [SIGMA_STABLE]

$\vdash$ $a \subseteq b$ $\Rightarrow$
  subsets (sigma sp a) $\subseteq$ subsets (sigma sp b)   [SIGMA_MONOTONE]

$\vdash$ $a \subseteq$ subsets (sigma sp a)   [SIGMA_SUBSET_SUBSETS]

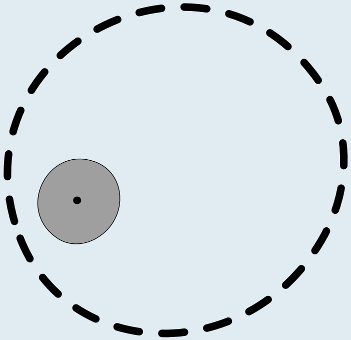
$\vdash$ sigma_algebra b $\land$ $a \subseteq$ subsets b $\Rightarrow$
  subsets (sigma (space b) a) $\subseteq$ subsets b [SIGMA_SUBSET]

Borel σ-algebra generated from $\mathbb{R}$

$(\mathbb{R}, \mathcal{B}) := \sigma(\mathbb{R}, \{s | s \text{ is } \textit{open}\})$ is called the *Borel* σ-algebra from $\mathbb{R}$.

$\vdash$ `borel = sigma` $\mathcal{U}$`(:real)` $\{s$ `|` `open` $s\}$ [real_borel.borel]

In a metric space, an *open set* is a set that, along with every point P, contains all points that are sufficiently near to P. The open interval (a, b) is open: for any point c such that $a < c < b$, there exists ϵ such that $(c - \epsilon, c + \epsilon) \subseteq (a, b)$.



Open sets in higher dimensions:

Furthermore,

- A set A is *closed* if it's complementation $X \setminus A$ is open;
- There exists sets neither open nor closed.

Topology: Open Sets (HOL defs)

A long chain of HOL definitions:

$\vdash$ $\texttt{open = open\_in euclidean}$ [real_topology.euclidean_open_def]

$\vdash$ $\texttt{euclidean = mtop mr1}$ [real_topology.euclidean_def]

$\vdash$ $\texttt{mr1 = metric}\ (\lambda\,(x,y).\ \texttt{abs}\ (y - x))$ [metric.mr1]

$$
\begin{array}{l}
\vdash \texttt{mtop}\ m\ \texttt{=} \\
\quad \texttt{topology} \\
\qquad (\lambda\, S'. \\
\qquad\qquad \forall x.\ S'\ x \Rightarrow \\
\qquad\qquad\qquad \exists e.\ 0 < e \wedge \\
\qquad\qquad\qquad\qquad \forall y.\ \texttt{dist}\ m\ (x,y) < e \Rightarrow S'\ y)
\end{array}
$$

[metric.mtop]

$\vdash$ $\exists\, rep.\ \texttt{TYPE\_DEFINITION ismet}\ rep$ [metric.metric_TY_DEF]

$$
\begin{array}{l}
\vdash \texttt{ismet}\ m \iff \\
\quad (\forall x\ y.\ m\ (x,y) = 0 \iff x = y) \wedge \\
\quad \forall x\ y\ z.\ m\ (y,z) \leq m\ (x,y) + m\ (x,z)
\end{array}
$$

[metric.ismet]

$\vdash$ $\exists\, rep.\ \texttt{TYPE\_DEFINITION istopology}\ rep$ [topology.topology_TY_DEF]

$$
\begin{array}{l}
\vdash \texttt{istopology}\ L \iff \\
\quad \emptyset \in L \wedge (\forall s\ t.\ s \in L \wedge t \in L \Rightarrow s \cap t \in L) \wedge \\
\quad \forall k.\ k \subseteq L \Rightarrow \bigcup k \in L
\end{array}
$$

[topology.istopology]

Alternative definitions of Borel σ-algebra

- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{(a, b)\,|a, b \in \mathbb{R}\})$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{[a, b]\,|a, b \in \mathbb{R}\})$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{(a, b]\,|a, b \in \mathbb{R}\})$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{[a, b)\,|a, b \in \mathbb{R}\})$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{(a, +\infty)\,|a \in \mathbb{R}\})$ where $(a, +\infty)$ denotes $\{x|a < x\}$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{[a, +\infty)\,|a \in \mathbb{R}\})$ where $[a, +\infty)$ denotes $\{x|a \leqslant x\}$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{(-\infty, b)\,|b \in \mathbb{R}\})$ where $(-\infty, b)$ denotes $\{x|x < b\}$;
- $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{(-\infty, b]\,|b \in \mathbb{R}\})$ where $(-\infty, b)$ denotes $\{x|x \leqslant b\}$;

Also true if $a, b \in \mathbb{Q}$ instead of $\mathbb{R}$ in above alternative definitions, e.g. $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \{(a, b)\,|a, b \in \mathbb{Q}\}$;.

Proving alternative Borel definitions (1)

Starting with $(a, b) \in \mathcal{B}$, then:

- $(a, +\infty) = \bigcup_{n\in\mathbb{N}}(a, n)$;
- $[a, +\infty) = \bigcap_{n\in\mathbb{N}}(a - \frac{1}{n}, +\infty)$;
- $(-\infty, c] = \overline{(c, +\infty)}$;
- $(-\infty, c) = \bigcup_{n\in\mathbb{N}}(-\infty, c - \frac{1}{n}]$ (or just $\overline{[c, +\infty)}$);
- $[a, b] = [a, +\infty) \cap (-\infty, b]$ (similar for $(a, b]$ and $[a, b)$);

Furthermore, the singleton set $\{c\}$ ($c \in \mathbb{R}$) is in $\mathcal{B}$:

$$\{c\} = (-\infty, c] \cap [c, +\infty)$$

Then, a real number is the limit of a (countable) sequence of rational numbers (to be continued).

Proving alternative Borel definitions (2)

Proof goals for $\sigma(\mathbb{R}, \{(a,b) \,|\, a, b \in \mathbb{Q}\}) = \sigma(\mathbb{R}, \{s | s \text{ is } \textit{open}\})$:

- $\sigma(\mathbb{R}, \{(a,b) \,|\, a, b \in \mathbb{Q}\}) \subseteq \sigma(\mathbb{R}, \{s | s \text{ is } \textit{open}\})$ [goal 1]
- $\sigma(\mathbb{R}, \{s | s \text{ is } \textit{open}\}) \subseteq \sigma(\mathbb{R}, \{(a,b) \,|\, a, b \in \mathbb{Q}\})$ [goal 2]

Proof outline of Goal 1 (easy):

1. it suffices to prove: $\{(a,b) \,|\, a, b \in \mathbb{Q}\} \subseteq \{s | s \text{ is } \textit{open}\}$
2. it suffices to prove: $\forall a, b \in \mathbb{Q}.\, (a, b)$ is open (immediate).

Proof outline of Goal 2 (hard):

1. $\forall s$ (open). $s = \bigcup_{(a,b) \subseteq s} (a,b) = \bigcup_{\substack{(p,q) \subseteq s \\ p,q \in \mathbb{Q}}} (p,q)$ (a countable union!)
2. it suffices to prove: $(p,q) \in \sigma(\mathbb{R}, \{(a,b) \,|\, a, b \in \mathbb{Q}\})$
3. it suffices to prove: $(p,q) \in \{(a,b) \,|\, a, b \in \mathbb{Q}\}$ (immediate)

Borel σ-algebra generated from $\overline{\mathbb{R}}$

$(\overline{\mathbb{R}}, \overline{\mathcal{B}})$ can be generated in the following ways:

- $(\overline{\mathbb{R}}, \overline{\mathcal{B}}) = \sigma(\overline{\mathbb{R}}, \{(a, +\infty] \,|a \in \mathbb{R}$ or $\mathbb{Q}\})$;
- $(\overline{\mathbb{R}}, \overline{\mathcal{B}}) = \sigma(\overline{\mathbb{R}}, \{[a, +\infty] \,|a \in \mathbb{R}$ or $\mathbb{Q}\})$;
- $(\overline{\mathbb{R}}, \overline{\mathcal{B}}) = \sigma(\overline{\mathbb{R}}, \{[-\infty, b) \,|b \in \mathbb{R}$ or $\mathbb{Q}\})$;
- $(\overline{\mathbb{R}}, \overline{\mathcal{B}}) = \sigma(\overline{\mathbb{R}}, \{[-\infty, b] \,|b \in \mathbb{R}$ or $\mathbb{Q}\})$.

It can be proved that (a, b), $[a, b]$ etc. and singletons $\{+\infty\}$ and $\{-\infty\}$ are all in $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$. Alternatively $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$ can be defined by $(\mathbb{R}, \mathcal{B})$:

$$B^* \in \overline{\mathcal{B}} \Longleftrightarrow B^* = B \cup S \wedge B \in \mathcal{B} \wedge S \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$$

On the other hand, $\mathcal{B} = \mathbb{R} \cap \overline{\mathcal{B}} := \{A \cap \mathbb{R} \,|\, A \in \overline{\mathcal{B}}\}$.

Borel σ-algebra $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$: formal version

$$
\begin{aligned}
&\vdash \texttt{Borel} = \\
&\quad (\mathcal{U}(\texttt{:extreal}), \\
&\quad\ \{B' \mid \\
&\quad\quad \exists B\ S. \\
&\quad\quad\quad B' = \texttt{IMAGE Normal}\ B \cup S \land \\
&\quad\quad\quad B \in \texttt{subsets borel} \land \\
&\quad\quad\quad S \in \{\emptyset;\ \{-\infty\};\ \{+\infty\};\ \{-\infty;\ +\infty\}\}\}) \qquad [\texttt{borelTheory.Borel}] \\
&\vdash \texttt{Borel} = \\
&\quad \texttt{sigma}\ \mathcal{U}(\texttt{:extreal}) \\
&\quad\quad (\texttt{IMAGE}\ (\lambda a.\ \{x \mid x < \texttt{Normal}\ a\})\ \mathcal{U}(\texttt{:real})) \qquad [\texttt{Borel\_def}] \\
&\vdash \texttt{Borel} = \\
&\quad \texttt{sigma}\ \mathcal{U}(\texttt{:extreal}) \\
&\quad\quad (\texttt{IMAGE}\ (\lambda a.\ \{x \mid \texttt{Normal}\ a \le x\})\ \mathcal{U}(\texttt{:real})) \qquad [\texttt{Borel\_eq\_ge}] \\
&\vdash \texttt{Borel} = \\
&\quad \texttt{sigma}\ \mathcal{U}(\texttt{:extreal}) \\
&\quad\quad (\texttt{IMAGE}\ (\lambda a.\ \{x \mid \texttt{Normal}\ a < x\})\ \mathcal{U}(\texttt{:real})) \qquad [\texttt{Borel\_eq\_gr}] \\
&\vdash \texttt{Borel} = \\
&\quad \texttt{sigma}\ \mathcal{U}(\texttt{:extreal}) \\
&\quad\quad (\texttt{IMAGE}\ (\lambda a.\ \{x \mid x \le \texttt{Normal}\ a\})\ \mathcal{U}(\texttt{:real})) \qquad [\texttt{Borel\_eq\_le}]
\end{aligned}
$$

$$
\begin{aligned}
&\vdash \texttt{borel} = \\
&\quad (\mathcal{U}(\texttt{:real}),\texttt{IMAGE real\_set (subsets Borel)}) \qquad [\texttt{borel\_eq\_real\_set}] \\
&\vdash \texttt{real\_set}\ s = \{\texttt{real}\ x \mid x \ne +\infty \land x \ne -\infty \land x \in s\}
\end{aligned}
$$

Constructing the Borel measure space (1D)

- The σ-algebra $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ is too big for assigning a non-trivial measure.
- Goal is to construct $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ such that $\mu((a,b)) = b - a$ ($a \leqslant b$), the "household" measure.
- Main difficulity: it's hard to directly define a measure function on $\mathcal{B}(\mathbb{R})$.
- Idea: first define a pre-measure on a generator (a *semi-ring*), then extend the pre-measure to a measure on the σ-algebra generated from it.

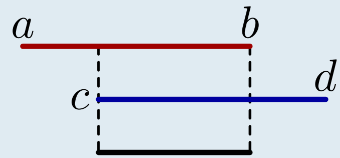
Semi-ring $(X, \mathcal{S})$ is a system of sets such that:

- $\emptyset \in \mathcal{S}$ $\quad (S_1)$
- $S, T \in \mathcal{S} \Rightarrow S \cap T \in \mathcal{S}$ $\quad (S_2)$
- for $S, T \in \mathcal{S}$ there exist finitely many disjoint $S_1, S_2, \ldots, S_M \in \mathcal{S}$ such that $S \setminus T = \dot{\bigcup}_{i=1}^{M} S_i$ $\quad (S_3)$

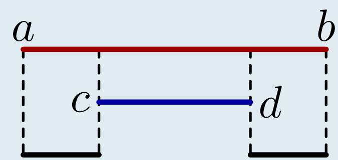
Semi-ring of half-open intervals

$(\mathbb{R}, \mathcal{S}) := (\mathbb{R}, \{[a, b) \,|\, a, b \in \mathbb{R}, a \leqslant b\})$ is indeed a semi-ring:

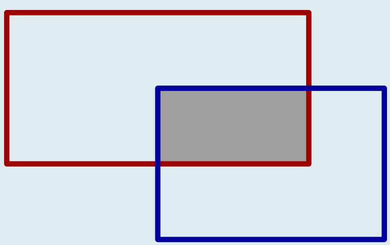
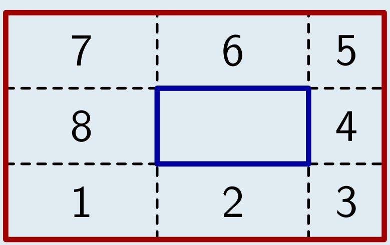
- S_1: $a = b \Rightarrow [a, b) = \emptyset \in \mathcal{S}$.



- S_2: $[a, b) \cap [c, d) = [c, b)$ $(a < c < b < d)$



- S_3: $[a, b) \setminus [c, d) = [a, c) \cup [d, b)$ $(a < c < d < b)$

In higher dimensions:  , and 

Existence of Measure: Carathéodory's Theorem

From $m_0 := (\mathbb{R}, \mathcal{S}, \lambda_0)$ to $m := (\mathbb{R}, \mathcal{B}, \lambda)$:

$$
\begin{array}{l}
\vdash \texttt{semiring (measurable\_space } m_0\texttt{)} \land \texttt{premeasure } m_0 \Rightarrow \\
\quad \exists\, m.\ (\forall\, s.\ s \in \texttt{measurable\_sets } m_0 \Rightarrow \\
\qquad\qquad \texttt{measure } m\ s \texttt{ = measure } m_0\ s) \land \\
\qquad \texttt{measurable\_space } m \texttt{ =} \\
\qquad \texttt{sigma (m\_space } m_0\texttt{) (measurable\_sets } m_0\texttt{)} \land \\
\qquad \texttt{measure\_space } m \qquad\qquad \texttt{[measureTheory.CARATHEODORY\_SEMIRING]}
\end{array}
$$

What we have now:

- Semi-ring $(\mathbb{R}, \mathcal{S}) := (\mathbb{R}, \{[a, b)\,|a, b \in \mathbb{R}, a \leqslant b\})$;
- Borel σ-algebra: $(\mathbb{R}, \mathcal{B}) = \sigma(\mathbb{R}, \mathcal{S})$;
- A pre-measure: $\forall a, b \in \mathbb{R}.\, a \leqslant b \Rightarrow \lambda_0([a, b)) = b - a \in \overline{\mathbb{R}}$, or equivalently

$$\lambda_0(s) := \text{if } s = \emptyset \text{ then } 0 \text{ else } \sup(s) - \inf(s)$$

It remains to show that $(\mathbb{R}, \mathcal{S}, \lambda_0)$ is a pre-measure space, i.e., positive (easy) and countably additive (hard).

Uniqueness of Measure

The generated $(\mathbb{R}, \mathcal{B}, \lambda)$ is unique: if $(\mathbb{R}, \mathcal{B}, \lambda')$ is another measure space asserted by Carathéodory's Theorem, then we have

$$\forall s \in \mathcal{B}.\, \lambda(s) = \lambda'(s)$$

[UNIQUENESS_OF_MEASURE]

$\vdash$ `subset_class` sp sts $\wedge$
  $(\forall s\ t.\ s \in sts \wedge t \in sts \Rightarrow s \cap t \in sts)$ $\wedge$
  `sigma_finite` (sp, sts, u) $\wedge$
  `measure_space` $(sp,$`subsets (sigma` sp $sts), u)$ $\wedge$
  `measure_space` $(sp,$`subsets (sigma` sp $sts), v)$ $\wedge$
  $(\forall s.\ s \in sts \Rightarrow u\ s = v\ s)$ $\Rightarrow$
  $\forall s.\ s \in$ `subsets (sigma` sp $sts)$ $\Rightarrow u\ s = v\ s$

$\vdash$ `sigma_finite` $m \iff$
  $\exists f.\ f \in (\mathcal{U}($`:num`$) \to$ `measurable_sets` $m)$ $\wedge$
    $(\forall n.\ f\ n \subseteq f\ ($`SUC` $n))$ $\wedge$
    $\bigcup$ `(IMAGE` f $\mathcal{U}($`:num`$))$ = `m_space` m $\wedge$
    $\forall n.$ `measure` $m\ (f\ n) < +\infty$ [sigma_finite_def]

$(\mathbb{R}, \mathcal{S}, \lambda_0)$ is indeed σ-finite: $f_n = [-n, n)$ is an exhausting sequence.

Countable Additivity of $(\mathbb{R}, \mathcal{S}, \lambda_0)$

Let $I_n = [a_n, b_n)$ be mutually disjoint intervals such that

$$\dot{\bigcup_{n \in \mathbb{N}}} I_n = [a, b)$$

The goal is to show:

$$\lambda_0(\dot{\bigcup_{n \in \mathbb{N}}} I_n) = \sum_{n \in \mathbb{N}} \lambda_0(I_n)$$

This proof needs the Heine-Borel Theorem (completeness of real numbers):

[real_topologyTheory.COMPACT_EQ_HEINE_BOREL]

$\vdash$ compact $s \iff$

$\quad \forall f.\ (\forall t.\ t \in f \Rightarrow \texttt{open}\ t) \land s \subseteq \bigcup f \Rightarrow$

$\qquad \exists f'.\ f' \subseteq f \land \texttt{FINITE}\ f' \land s \subseteq \bigcup f'$

[real_topologyTheory.COMPACT_EQ_BOUNDED_CLOSED]

$\vdash$ compact $s \iff$ bounded $s \land$ closed s

Formalisations in (real_)borelTheory

$\vdash$ right_open_interval a b = $\{x$ | $a \le x \land x < b\}$
$\vdash$ right_open_intervals =
  $(\mathcal{U}$(:real),$\{$right_open_interval a b | T$\})$
$\vdash$ semiring right_open_intervals
$\vdash$ $a \le b \Rightarrow$
  lambda0 (right_open_interval a b) =
  Normal ($b - a$) [lambda0_def]
$\vdash$ premeasure lborel0 [lborel0_premeasure]

Overload lambda = ‘‘measure lborel‘‘
$\vdash$ ($\forall s$. $s \in$ subsets right_open_intervals $\Rightarrow$
    lambda s = lambda0 s) $\land$
  measurable_space lborel = borel $\land$
  measure_space lborel [lborel_def]

[lambda_open_interval]
$\vdash$ $a \le b \Rightarrow$
  lambda (interval (a,b)) = Normal ($b - a$)
[lambda_closed_interval]
$\vdash$ $a \le b \Rightarrow$
  lambda (interval [(a,b)]) = Normal ($b - a$)
[lambda_sing]
$\vdash$ lambda $\{c\}$ = 0