HOL Theorem Proving and Formal Probability (7)

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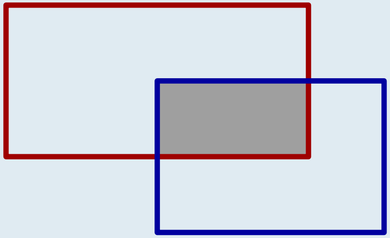
The Need of Product Measure Space

How to construct a *household* measure space for 2-dimensional Euclidean space? such that:

$$\lambda^2([a_1, b_1) \times [a_2, b_2)) = (b_1 - a_1) \cdot (b_2 - a_2)$$

Using Carathéodory's Extension Theorem, one needs to show that:

- The system of sets $\mathcal{S} = \{[a_1, b_1) \times [a_2, b_2) \mid a_1, b_1, a_2, b_2 \in \mathbb{R}\}$ is a semiring.



- λ^2 is a premeasure (in particular *countable additive*) over $\mathcal{S}$ (very hard).

Using product measure, one can easily construct λ^2 from λ.

Cross and Product of Sets (of Sets)

Some definitions:

- The *cross* of two sets: $A \times B := \{(a,b) \mid a \in A \wedge b \in B\}$;
- The *product* of two systems of sets:

$$\mathcal{A} \times \mathcal{B} := \{A \times B \mid A \in \mathcal{A} \wedge B \in \mathcal{B}\}$$

But what is a pair (a,b)? In HOL4, there is the official `pairTheory` but in practice other representations of pairs may be needed.

```
[martingaleTheory.pair_operation_def]
 ⊢ pair_operation cons car cdr ⟺
   (∀ a b. car (cons a b) = a ∧ cdr (cons a b) = b) ∧
   ∀ a b c d. cons a b = cons c d ⟺ a = c ∧ b = d

[martingaleTheory.general_cross_def]
 ⊢ general_cross cons A B =
   {cons a b | a ∈ A ∧ b ∈ B}
[martingaleTheory.general_prod_def]
 ⊢ general_prod cons A B =
   {general_cross cons a b | a ∈ A ∧ b ∈ B}
```

Sample applications of pair operations

- `pairTheory`:

$\vdash$ `pair_operation (,) FST SND`

- `listTheory`:

$\vdash$ `pair_operation` $(\lambda x\ y.\ [x;\ y])$ (`EL` 0) (`EL` 1)

- `fcpTheory`:

$\vdash$ `FCP_CONCAT` $(a\ {:}\alpha[\beta])\ (b\ {:}\alpha[\gamma])$ =
(`FCP`$(i$ `:num`).
  if $i <$ `dimindex` $({:}\gamma)$ **then** b ' i
  else a ' $(i\ -$ `dimindex` $({:}\gamma)))$
$\vdash$ `FCP_FST` $(v\ {:}\alpha[\beta + \gamma])$ =
(`FCP`$(i$ `:num`). v ' $(i\ +$ `dimindex` $({:}\gamma)))$
$\vdash$ `FCP_SND` $(v\ {:}\alpha[\beta + \gamma])$ = (`FCP`$(i$ `:num`). v ' $i)$

$\vdash$ `FINITE` $\mathcal{U}({:}\beta) \wedge$ `FINITE` $\mathcal{U}({:}\gamma) \Rightarrow$
`pair_operation FCP_CONCAT FCP_FST FCP_SND`

Product σ-algebra

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, the σ-algebra $\mathcal{A} \otimes \mathcal{B} := \sigma(\mathcal{A} \times \mathcal{B})$ is called a *product σ-algebra*, and $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ is the product of measurable spaces.

```
[sigma_algebraTheory.prod_sigma_def]
 ⊢ a × b =
   sigma (space a × space b)
     (prod_sets (subsets a) (subsets b))

[martingaleTheory.general_sigma_def]
 ⊢ general_sigma cons A B =
   sigma (general_cross cons (space A) (space B))
     (general_prod cons (subsets A) (subsets B))
```

The product of two semirings is still a semiring:

```
[sigma_algebra.SEMIRING_PROD_SETS]
 ⊢ semiring a ∧ semiring b ⇒
   semiring
     (space a × space b,
      prod_sets (subsets a) (subsets b))
```

Product σ-algebra by generators

If $\mathcal{A} = \sigma(\mathcal{F})$ and $\mathcal{B} = \sigma(\mathcal{G})$ and $\mathcal{F}$ and $\mathcal{G}$ contain *exhausting sequences*, then

$$\sigma(\mathcal{F} \times \mathcal{G}) = \sigma(\mathcal{A} \times \mathcal{B}) := \mathcal{A} \otimes \mathcal{B}$$

```
[martingaleTheory.PROD_SIGMA_OF_GENERATOR]
 ⊢ subset_class X E ∧ subset_class Y G ∧
   has_exhausting_sequence (X,E) ∧
   has_exhausting_sequence (Y,G) ⇒
   (X,E) × (Y,G) = sigma X E × sigma Y G
```

Therefore,

$$\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \sigma(\{(a_1, b_1] \mid a_1, b_1 \in \mathbb{R}\} \times \{(a_2, b_2] \mid a_2, b_2 \in \mathbb{R}\})$$

$$= \sigma(\{[a_1, b_1) \times [a_2, b_2) \mid a_1, b_1, a_2, b_2 \in \mathbb{R}\})$$

NOTE: One such exhausting sequence is $([-n, n))_{n \in \mathbb{N}}$.

Uniqueness of Product Measure (1)

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces and assume that $\mathcal{A} = \sigma(\mathcal{F})$ and $\mathcal{B} = \sigma(\mathcal{G})$. If

- $\mathcal{F}, \mathcal{G}$ are $\cap$-stable;
- $\mathcal{F}, \mathcal{G}$ contain exhausting sequences $F_k \uparrow X$ and $G_k \uparrow Y$ with $\mu(F_k) < \infty$ and $\nu(G_n) < \infty$ for all $k, n \in \mathbb{N}$,

Then there is at most one measure ρ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ satisfying

$$\rho(F \times G) = \mu(F)\,\nu(G) \qquad \forall F \in \mathcal{F}, G \in \mathcal{G}.$$

Uniqueness of Product Measure (2)

[martingaleTheory.UNIQUENESS_OF_PROD_MEASURE]

$\vdash$ subset_class X E $\land$ subset_class Y G $\land$
sigma_finite (X,E,u) $\land$ sigma_finite (Y,G,v) $\land$
$(\forall s\ t.\ s \in E \land t \in E \Rightarrow s \cap t \in E)$ $\land$
$(\forall s\ t.\ s \in G \land t \in G \Rightarrow s \cap t \in G)$ $\land$
A = sigma X E $\land$ B = sigma Y G $\land$
measure_space $(X,$subsets $A,u)$ $\land$
measure_space $(Y,$subsets $B,v)$ $\land$
measure_space $(X \times Y,$subsets $(A \times B),m)$ $\land$
measure_space $(X \times Y,$subsets $(A \times B),m')$ $\land$
$(\forall s\ t.\ s \in E \land t \in G \Rightarrow m\ (s \times t) = u\ s \times v\ t)$ $\land$
$(\forall s\ t.\ s \in E \land t \in G \Rightarrow m'\ (s \times t) = u\ s \times v\ t)$ $\Rightarrow$
$\forall x.\ x \in$ subsets $(A \times B) \Rightarrow m\ x = m'\ x$

[martingaleTheory.UNIQUENESS_OF_PROD_MEASURE']

$\vdash$ sigma_finite_measure_space (X,A,u) $\land$
sigma_finite_measure_space (Y,B,v) $\land$
measure_space $(X \times Y,$subsets $((X,A) \times (Y,B)),m)$ $\land$
measure_space
$(X \times Y,$subsets $((X,A) \times (Y,B)),m')$ $\land$
$(\forall s\ t.\ s \in A \land t \in B \Rightarrow m\ (s \times t) = u\ s \times v\ t)$ $\land$
$(\forall s\ t.\ s \in A \land t \in B \Rightarrow m'\ (s \times t) = u\ s \times v\ t)$ $\Rightarrow$
$\forall x.\ x \in$ subsets $((X,A) \times (Y,B)) \Rightarrow m\ x = m'\ x$

Existence of Product Measure (1)

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. The set function

$$\rho\colon \mathcal{A} \times \mathcal{B} \to [0, \infty], \quad \rho(A \times B) := \mu(A)\,\nu(B),$$

extends uniquely to a σ-finite measure on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ such that

$$\rho(E) = \int \left(\int \mathbb{1}_E(x, y)\,\mu(dx) \right) \nu(dy) = \int \left(\int \mathbb{1}_E(x, y)\,\nu(dy) \right) \mu(dx)\,.$$

holds for all $E \in \mathcal{A} \otimes \mathcal{B}$. In particular, the functions

$$x \mapsto \mathbb{1}_E(x, y),\ y \mapsto \mathbb{1}_E(x, y),\ x \mapsto \int \mathbb{1}_E(x, y)\,\nu(dy),\ y \mapsto \int \mathbb{1}_E(x, y)\,\mu(dx)$$

are , resp. $\mathcal{B}$-measurable for every fixed $y \in Y$, resp. $x \in X$.

(2) EXISTENCE_OF_PROD_MEASURE

$$
\begin{array}{l}
\vdash \texttt{sigma\_finite\_measure\_space}\ (X,A,u)\ \wedge \\
\quad \texttt{sigma\_finite\_measure\_space}\ (Y,B,v)\ \wedge \\
\quad (\forall s\ t.\ s \in A \wedge t \in B \Rightarrow m_0\ (s \times t) = u\ s \times v\ t) \Rightarrow \\
\quad (\forall s.\ s \in \texttt{subsets}\ ((X,A) \times (Y,B)) \Rightarrow \\
\qquad (\forall x.\ x \in X \Rightarrow \\
\qquad\quad (\lambda y.\ \mathbb{1}\ s\ (x,y)) \in \\
\qquad\quad \texttt{Borel\_measurable}\ (Y,B))\ \wedge \\
\qquad (\forall y.\ y \in Y \Rightarrow \\
\qquad\quad (\lambda x.\ \mathbb{1}\ s\ (x,y)) \in \\
\qquad\quad \texttt{Borel\_measurable}\ (X,A))\ \wedge \\
\qquad (\lambda y.\ \int^+ (X,A,u)\ (\lambda x.\ \mathbb{1}\ s\ (x,y))) \in \\
\qquad \texttt{Borel\_measurable}\ (Y,B)\ \wedge \\
\qquad (\lambda x.\ \int^+ (Y,B,v)\ (\lambda y.\ \mathbb{1}\ s\ (x,y))) \in \\
\qquad \texttt{Borel\_measurable}\ (X,A))\ \wedge \\
\quad \exists m.\ \texttt{sigma\_finite\_measure\_space} \\
\qquad (X \times Y, \texttt{subsets}\ ((X,A) \times (Y,B)), m)\ \wedge \\
\qquad (\forall s.\ s \in \texttt{prod\_sets}\ A\ B \Rightarrow m\ s = m_0\ s)\ \wedge \\
\qquad \forall s.\ s \in \texttt{subsets}\ ((X,A) \times (Y,B)) \Rightarrow \\
\qquad\quad m\ s = \\
\qquad\quad \int^+ (Y,B,v) \\
\qquad\qquad (\lambda y.\ \int^+ (X,A,u)\ (\lambda x.\ \mathbb{1}\ s\ (x,y)))\ \wedge \\
\qquad\quad m\ s = \\
\qquad\quad \int^+ (X,A,u) \\
\qquad\qquad (\lambda x.\ \int^+ (Y,B,v)\ (\lambda y.\ \mathbb{1}\ s\ (x,y)))
\end{array}
$$

Application of Existence of Product Measure

```
[martingaleTheory.lborel_2d_def]
 ⊢ sigma_finite_measure_space lborel_2d ∧
   m_space lborel_2d = 𝒰(:real) × 𝒰(:real) ∧
   measurable_sets lborel_2d =
   subsets
     ((𝒰(:real),subsets borel) ×
      (𝒰(:real),subsets borel)) ∧
   (∀s t.
      s ∈ subsets borel ∧ t ∈ subsets borel ⇒
      measure lborel_2d (s × t) =
      lambda s × lambda t) ∧
   ∀s. s ∈ measurable_sets lborel_2d ⇒
       (∀x. (λy. 𝟙 s (x,y)) ∈
            Borel_measurable borel) ∧
       (∀y. (λx. 𝟙 s (x,y)) ∈
            Borel_measurable borel) ∧
       (λy. ∫⁺ lborel (λx. 𝟙 s (x,y))) ∈
       Borel_measurable borel ∧
       (λx. ∫⁺ lborel (λy. 𝟙 s (x,y))) ∈
       Borel_measurable borel ∧
       measure lborel_2d s =
       ∫⁺ lborel (λy. ∫⁺ lborel (λx. 𝟙 s (x,y))) ∧
       measure lborel_2d s =
       ∫⁺ lborel (λx. ∫⁺ lborel (λy. 𝟙 s (x,y)))
```

Lebesgue Integration in Product Measure Space

Theorem (Tonelli): Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $u\colon X \times Y \to [0, \infty]$ be $\mathcal{A} \otimes \mathcal{B}$-measurable. Then

- $x \mapsto u(x, y)$ is $\mathcal{A}$-measurable for fixed y, resp.
- $x \mapsto \int_Y^+ u(x, y)\,\nu(dy)$ is $\mathcal{A}$-measurable, resp.
- The following formula holds:

$$\int_{X\times Y}^+ ud(\mu \times \nu) = \int_Y^+ \left(\int_X^+ u(x, y)\,\mu(dx)\right)\nu(dy) = \int_Y^+ \left(\int_X^+ u(x, y)\,\nu(dy)\right)\mu(dx)$$

Tonelli Theorem in HOL4

[martingaleTheory.TONELLI]

$\vdash$ `sigma_finite_measure_space` (X,A,u) $\wedge$
  `sigma_finite_measure_space` (Y,B,v) $\wedge$
  $f \in$ `Borel_measurable` $((X,A) \times (Y,B))$ $\wedge$
  $(\forall s.\ s \in X \times Y \Rightarrow 0 \le f\ s) \Rightarrow$
  $(\forall y.\ y \in Y \Rightarrow$
      $(\lambda x.\ f\ (x,y)) \in$ `Borel_measurable` $(X,A))$ $\wedge$
  $(\forall x.\ x \in X \Rightarrow$
      $(\lambda y.\ f\ (x,y)) \in$ `Borel_measurable` $(Y,B))$ $\wedge$
  $(\lambda x.\ \int^+ (Y,B,v)\ (\lambda y.\ f\ (x,y))) \in$
  `Borel_measurable` (X,A) $\wedge$
  $(\lambda y.\ \int^+ (X,A,u)\ (\lambda x.\ f\ (x,y))) \in$
  `Borel_measurable` (Y,B) $\wedge$
  $\int^+ ((X,A,u) \times (Y,B,v))\ f =$
  $\int^+ (Y,B,v)\ (\lambda y.\ \int^+ (X,A,u)\ (\lambda x.\ f\ (x,y)))$ $\wedge$
  $\int^+ ((X,A,u) \times (Y,B,v))\ f =$
  $\int^+ (X,A,u)\ (\lambda x.\ \int^+ (Y,B,v)\ (\lambda y.\ f\ (x,y)))$

Fubini Theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $u\colon X \times Y \to [0, \infty]$ be $\mathcal{A} \otimes \mathcal{B}$-measurable. If at least one of the three integrals

$$\int_{X \times Y} |u| d(\mu \times \nu), \int_Y \int_X |u(x, y)|\, \mu(dx)\, \nu(dy), \int_Y \int_X |u(x, y)|\, \nu(dy)\, \mu(dx)$$

is finite, then all three integrals are finite, $u \in \mathcal{L}^1(\mu \times \nu)$, and

- $x \mapsto u(x, y)$ is in $\mathcal{L}^1(\mu)$ for ν-a.e. $y \in Y$, etc.
- $x \mapsto \int_Y u(x, y)\, \nu(dy)$ is in $\mathcal{L}^1(\mu)$, etc.
- The following formula holds:

$$\int_{X \times Y} u d(\mu \times \nu) = \int_Y \int_X u(x, y)\, \mu(dx)\, \nu(dy) = \int_Y \int_X u(x, y)\, \nu(dy)\, \mu(dx)\,.$$

Fubini Theorem in HOL4

```
⊢ sigma_finite_measure_space (X,A,u) ∧
  sigma_finite_measure_space (Y,B,v) ∧
  f ∈ Borel_measurable ((X,A) × (Y,B)) ∧
  (∫⁺ ((X,A,u) × (Y,B,v)) (abs ∘ f) ≠ +∞ ∨
   ∫⁺ (Y,B,v)
     (λy. ∫⁺ (X,A,u) (λx. (abs ∘ f) (x,y))) ≠ +∞ ∨
   ∫⁺ (X,A,u)
     (λx. ∫⁺ (Y,B,v) (λy. (abs ∘ f) (x,y))) ≠ +∞) ⇒
  ∫⁺ ((X,A,u) × (Y,B,v)) (abs ∘ f) ≠ +∞ ∧
  ∫⁺ (Y,B,v) (λy. ∫⁺ (X,A,u) (λx. (abs ∘ f) (x,y))) ≠
  +∞ ∧
  ∫⁺ (X,A,u) (λx. ∫⁺ (Y,B,v) (λy. (abs ∘ f) (x,y))) ≠
  +∞ ∧ integrable ((X,A,u) × (Y,B,v)) f ∧
  (AE y::(Y,B,v).
     integrable (X,A,u) (λx. f (x,y))) ∧
  (AE x::(X,A,u).
     integrable (Y,B,v) (λy. f (x,y))) ∧
  integrable (X,A,u)
    (λx. ∫ (Y,B,v) (λy. f (x,y))) ∧
  integrable (Y,B,v)
    (λy. ∫ (X,A,u) (λx. f (x,y))) ∧
  ∫ ((X,A,u) × (Y,B,v)) f =
  ∫ (Y,B,v) (λy. ∫ (X,A,u) (λx. f (x,y))) ∧
  ∫ ((X,A,u) × (Y,B,v)) f =
  ∫ (X,A,u) (λx. ∫ (Y,B,v) (λy. f (x,y)))
```

Application of Fubini Theorem (1)

If X and Y are independent r.v.'s and both have finite expectations, then

$$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$$

Proof. Consider the random vector (X, Y) and let the probability measure induced by it be $\mu^2(dx, dy) := \mu(dx) \times \mu(dy)$. Then we have:

$$\mathbb{E}(XY) = \int_\Omega XY d\mathcal{P} = \int_{\mathbb{R}^2} xy\mu^2(dx, dy)$$

By independence, the last integral is equal to

$$\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} xy\mu(dx) \Big) \mu(dy) = \int_{\mathbb{R}} x\mu(dx) \int_{\mathbb{R}} x\mu(dy) = \mathbb{E}(X)\,\mathbb{E}(Y).$$

Application of Fubini Theorem (2)

The previous theorem formalized in HOL4's `probabilityTheory`: (A key lemma used in the proof of Laws of Large Numbers for IID r.v.'s)

```
[probabilityTheory.indep_vars_expectation]
 ⊢ prob_space p ∧ real_random_variable X p ∧
   real_random_variable Y p ∧
   indep_vars p X Y Borel Borel ∧ integrable p X ∧
   integrable p Y ⇒
   expectation p (λx. X x × Y x) =
   expectation p X × expectation p Y
```

```
[martingaleTheory.Cauchy_Schwarz_inequality]
 ⊢ measure_space m ∧ u ∈ L2_space m ∧
   v ∈ L2_space m ⇒
   integrable m (λx. u x × v x) ∧
   ∫ m (λx. abs (u x × v x)) ≤
   seminorm 2 m u × seminorm 2 m v
```

Try replaying its formal proof to understand it in *full* details! (A chance that you can *never* have in conventional math courses.) `QED.`