Imperial College London

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Signals and Control II

Signals

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Update Notes

13^{th} Jan, 2021

1. *Section 3.5.3*: Corrected typos: missing comma in $< f_1(t), f_2(t) >$.
2. *Section 5.3.8*: Corrected math typos in the derivation: missing $d\tau$ in line 1, $d\tau \to d\alpha$ in line 3.
3. *Section 5.3.8*: Corrected typos: missing ω_0 in $\cos(\omega_0 t)$.
4. *Section 5.6*: Added the formula for finding the phase angle. Added a plot for $sign(x)$.

27^{th} Dec, 2020

1. *Section 5.3.2*: Added the derivation of time shifting property.
2. *Section 5.5*: Corrected math typos, rearranged the example and improved the readability.
3. *Section 5.5.1*: Added a description of LTI systems.
4. *Section 5.6*: Refined the description of magnitude and phase spectra. An example is adopted from the textbook.

Contents

1 Definition, Classification and Properties of Signals

1.1 Definition of signals

- Signals describe physical phenomena as patterns of variations of some form.
- Mathematically, signals are **functions** of one or more independent variables.
- For example, a signal $s(t)$ can be a function of the continuous independent variable time $t \in [\alpha, \beta]$ A two-dimensional signal $f(x, y)$ can be a function of two spatial coordinates x, y.

1.2 Continuous and Discrete-time Signals

- Signals can be a function of the **continuous time** variable, in which case we will use the notation $x(t)$ with $t \in \mathbb{R}$,
  or of the **discrete time** variable, in which case we will use the notation $x[n]$ with $n \in \mathbb{Z}$. (FIG.1)

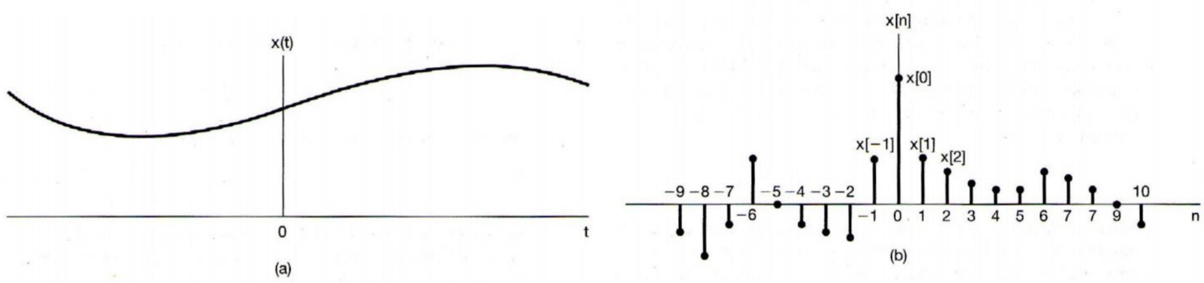


FIG. 1: *Continuous signal $x(t)$ and discrete signal $x[n]$*

- Discrete-time signals are often (but not necessarily) a sampling of continuous-time signals.
  $$x[n] = x_c(nT) \quad -\infty < n < +\infty$$
  T is sampling period.
- A discrete-time signal can be represented as a sequence of numbers, or, a vector.

1.2.1 Digital Signals

- When we speak of a digital signal we often mean one that has been **sampled** (captured at regular points in time) and **quantisised**.
- When one refers to a 12-bit signal, they are referring to the number of amplitude quantisation levels.
- Sampling a continuous signal may be done **without losing any information** from the original signal. Conversely, quantisation always implies **losing information**. (FIG.2)

We focus on the signals of one independent variable!

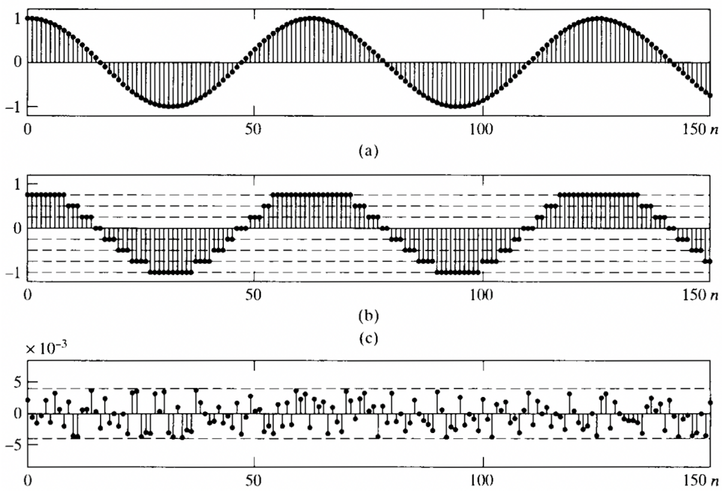


FIG. 2: *Sampling and quantisation*

1.3 Deterministic and Stochastic Signals

- **Deterministic**: a signal that **can be predicted** exactly (an analytical formulation exists).
  - Example: $x(t) = \sin(2\pi t)$
- **Stochastic**: a signal that **cannot be predicted** exactly before it has "occurred"; any signal that conveys information to us when we observe it.
  - Example: Thermal noise across a resistor, EEG traces, etc.
- We can often meaningfully describe the statistical properties of stochastic signals by building a model of their generation (stochastic processes).

We will mainly deal with deterministic signals in this course!

1.4 Periodic Signals

- A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which $x(t) = x(t + T)$ for all values of t (similar definition for discrete-time signals).
- A periodic signal has the property that it is unchanged by a time shift of T, we will say that $x(t)$ is periodic with period T. (FIG.3)

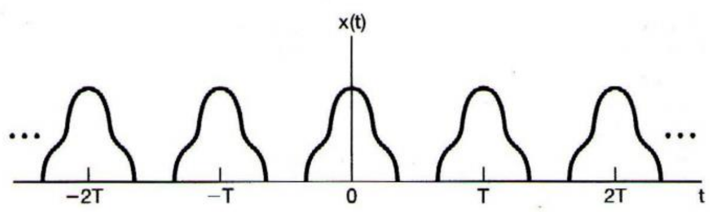


FIG. 3: *A periodic signal with period T*

1.5 Signal Energy and Power

For a continuous-time signal $x(t)$ for $t_1 \leq t \leq t_2$ and for a discrete-time signal $x[n]$ for $n_1 \leq n \leq n_2$, energy and power can be represented as follows:

$$\text{Energy(continuous time)} = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

$$\text{Power(continuous time)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

$$\text{Energy(discrete time)} = \sum_{n_1}^{n_2} |x[n]|^2$$

$$\text{Power(discrete time)} = \frac{1}{n_2 - n_1} \sum_{n_1}^{n_2} |x[n]|^2$$

We get the conclusion above from the calculation for electrical power and energy. Let $v(t)$ and $i(t)$ represent the voltage and current across the resistor of resistance R.

- The instantaneous power across the resistor is the product $v(t)i(t)$, which is proportional to $v^2(t)$.
- The total energy

$$\int_{t_1}^{t_2} \frac{1}{R} v^2(t) \, dt$$

- The average power

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R} v^2(t) \, dt$$

Similar properties can be applied to any continuous-time signals and discrete-time signals.

- Often, the signals are directly related to physical quantities capturing power and energy in a physical form.
- These properties are important characteristics of signals, even if in some cases do not reflect physical energy or power.

1.5.1 Energy and Power of a Generic Signal

Extend the range to: $-\infty < t < +\infty$ or $-\infty < n < +\infty$

- In continuous time:

$$\text{Energy} = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

- In discrete time:

$$\text{Energy} = \sum_{-\infty}^{+\infty} |x[n]|^2$$

$$\text{Power} = \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$$

We will use the mathematical definitions above, regardless of the direct physical meaning of each term!

2 Types of Signals

2.1 Periodic Complex Exponential Signals in Continuous-time Domain

$$x(t) = e^{j\omega_0 t}$$

- Periodic, period $T = \frac{2\pi}{|\omega_0|}$.
- The signal $x(t) = e^{-j\omega_o t}$ has the same period.
- The complex exponential defined above is closely related to the sinusoidal signal:

$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

  which has the same period $T = \frac{2\pi}{|\omega_0|}$.

- The complex exponentials and sinusoidal signals have **infinite energy** and **finite power**.
  - Example: for the signal $x(t) = A\cos(2\pi\omega_0 t + \phi)$ with the period T_1,

$$\text{Power} = \frac{A^2}{2}$$

2.2 Periodic Complex Exponential Signals in Discrete-time Domain

$$x[n] = e^{j\omega_0 n}$$

And the sinusoidal signal becomes

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

- $n \in \mathbb{Z}$. Thus, $x[n]$ is the same signal for $\omega_0 + 2\pi k$ with $k \in \mathbb{Z}$. The frequency of oscillation in discrete time exponentials does not increase monotonically but is limited to 2π.
- $x[n]$ is not always periodic.

2.3 The Unit Impulse in Discrete-time Domain

In discrete-time domain, the **unit impulse** is the simplest signal: (FIG.4)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

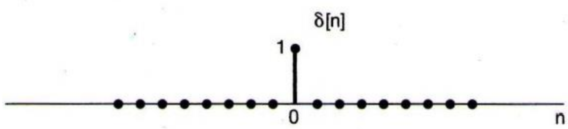


FIG. 4: *The unit impulse*

The discrete-time signal impulse **delayed by the integer** k is as follows: (FIG.5)

$$\delta[n-k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

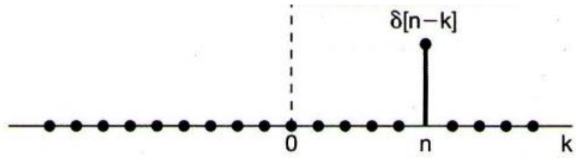


FIG. 5: *The unit impulse delayed by the integer* k

- For any discrete-time signal $x[n]$, directly from the definition of $\delta[n]$, we have

$$x[n]\ \delta[n-k] = x[k]\ \delta[n-k]$$

  This implies: any signal multiplied by the unit impulse is zeroed for all time samples, apart from the integer time where the unit impulse is centered.

- From the property above, we have:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]\delta[n-k] &= \sum_{n=-\infty}^{\infty} x[k]\ \delta[n-k] \\ &= x[k] \underbrace{\sum_{n=\infty}^{\infty} \delta[n-k]}_{1,\text{ when } k=n} \\ &= x[k] \end{aligned}$$

- Any arbitrary discrete-time signal can be expressed as the sum of **scaled** and **delayed** impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

2.4 The Unit Step in Discrete-time Domain

Unit step in discrete-time domain is defined as: (FIG.6)

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = \sum_{k=0}^{+\infty} \delta[n-k]$$

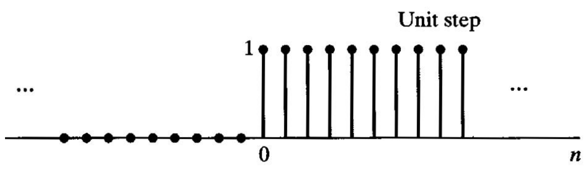


FIG. 6: *Unit step in discrete-time domain*

- The unit impulse is the discrete time derivative of the unit step.

$$\delta[n] = u[n] - u[n-1]$$

2.5 The Unit Step in Continuous-time Domain

Unit step in continuous-time domain is defined as: (FIG.7)

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

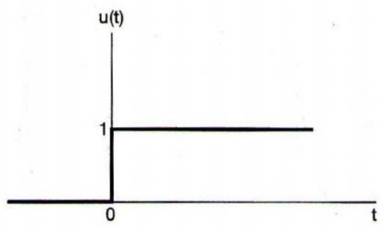


FIG. 7: *Unit step in continuous-time domain*

- The continuous-time unit step function has a discontinuity in $t = 0$, it is **not differentiable**. To solve the issue, we define:

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

  - The area of $\delta_\Delta(t)$ is equal to 1 at any value of Δ. (FIG.8, RIGHT)
  - As $\Delta \to 0$, $u_\Delta(t) \to u(t)$. (gradient $\to 0$) (FIG.8, LEFT)
  - As $\Delta \to 0$, the impulse $\delta_\Delta(t)$ becomes of shorter duration and higher amplitude: $\delta(t) = \lim_{\Delta\to 0} \delta_\Delta(t)$

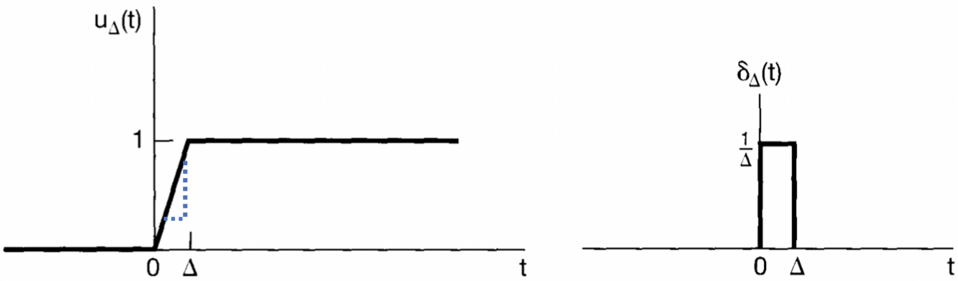


FIG. 8: *Continuous approximation to unit step*

- From the definition above, the following property holds: (FIG.9, LEFT)

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

- For continuous-time, the unit impulse in the continuous-time domain shifted by the time delay is $\delta(t - \sigma)$.(FIG.9, RIGHT)

FIG. 9: *Unit impulse shifted by the time delay*

- In continuous-time domain: (similar to discrete-time domain)

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

As $\delta_\Delta(t) \to \delta(t)$: better approximation

$$x(t)\delta(t) = x(0)\delta(t)$$

More generally: with time shifting

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

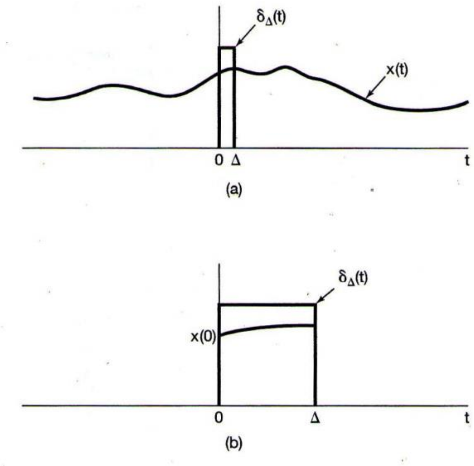


Fig. 10

- We also obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt &= \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt \\ &= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0) \\ &= x(t_0)\end{aligned}$$

- This implies that any arbitrary continuous-time signal $x(t)$ can be represent as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

2.5.1 Convolution

We define the following transformation between two signals (Convolution):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)d\tau = x(t) * h(t)$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k] = x[n] * h[n]$$

For any continuous-time signal and any discrete-time signal:

$$x(t) = x(t) * \delta(t)$$

$$x[n] = x[n] * \delta[n]$$

by extension for arbitrary delays:

$$x(t-t_0) = x(t) * \delta(t-t_0)$$

$$x[n-k] = x[n] * \delta[n-k]$$

SUMMARY OF PROPERTIES OF THE UNIT IMPULSE

- For **multiplication**:

$$x(0) \cdot \delta(t) = x(t) \cdot \delta(t)$$
$$x[0] \cdot \delta[n] = x[n] \cdot \delta[n]$$

$$x(t_0) \cdot \delta(t - t_0) = x(t) \cdot \delta(t - t_0)$$
$$x[k] \cdot \delta[n - k] = x[n] \cdot \delta[n - k]$$

- For **convolution**:

$$x(t) = x(t) * \delta(t)$$
$$x[n] = x[n] * \delta[n]$$

$$x(t - t_0) = x(t) * \delta(t - t_0)$$
$$x[n - k] = x[n] * \delta[n - k]$$

3 Simple Operations on Signals

3.1 Transformations of the Time Variable

- Time delay
- Time reversal
- Time scaling

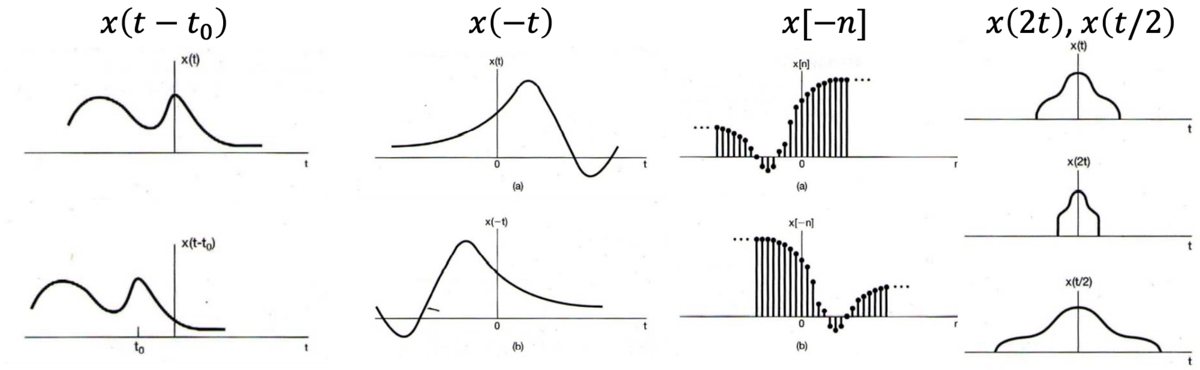


FIG. 11: *Transformations of the time variable*

3.2 Amplitude Transformation

$$y(x) = Ax(t) + B$$

where A, B are constants.

3.3 Linear Combination

- In continuous-time domain:

$$y(t) = a_1x_1(t) + a_2x_2(t) + ... + a_Nx_N(t)$$

- In discrete-time domain:

$$y[n] = a_1x_1[n] + a_2x_2[n] + ... + a_Nx_N[n]$$

where $a_i \in \mathbb{C}$, $i = 1, 2, ..., N$ are real or complex numbers.

3.4 Multiplication

- In continuous-time domain:

$$y(t) = x_1(t) \cdot x_2(t)$$

- In discrete-time domain:

$$y[n] = x_1[n] \cdot x_2[n]$$

This implies **instantaneous multiplication** for each time instant or each discrete time sample.

3.5 Scalar Products and Norms

3.5.1 Scalar Product and Norm of Vectors

- The **scalar product** between two 3-D vectors $\vec{A}$ and $\vec{B}$: projection of $\vec{A}$ on $\vec{B}$.

$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z = |\vec{A}| \cdot |\vec{B}| \cos(\phi)$$

- For vectors, the **Euclidean norm**, or **norm-2**, is the length of vectors in Euclidean space:

$$||\vec{A}||_2 = \sqrt{\vec{A} \cdot \vec{A}} = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

3.5.2 Scalar Product and Norm of Discrete-time Signals

- **Scalar product** for discrete-time signals:

$$\langle x_1[n], x_2[n] \rangle = \sum_{n=-\infty}^{\infty} x_1[n] x_2^*[n]$$

- **Norm-2** for discrete-time signals:

$$||x[n]||_2 = \sqrt{\langle x[n], x[n] \rangle} = \sqrt{\sum_{-\infty}^{\infty} |x[n]|^2} = \left(\sum_{-\infty}^{\infty} |x[n]|^2 \right)^{\frac{1}{2}}$$

  norm-2 is the square root of the **energy** of the signal

- **Norm-p** for discrete-time signals:

$$||x[n]||_p \ = \ \left(\sum_{-\infty}^{\infty} |x[n]|^p \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty$$

  – $p = 1$:

$$||x[n]||_1 \ = \ \sum_{-\infty}^{\infty} |x[n]|$$

  – $p \to \infty$: infinity norm / maximum norm:

$$||x[n]||_\infty \ = \ \max_n |x[n]|$$

- Norms are measures of the signal "strength". Each norm is a different way of measuring signal strength.
  e.g. the norm-2 is associated to the energy.

- The space of signals with a finite norm$-p$ is called L^p space.

3.5.3 Scalar Product and Norm for Continuous-time Signals

- **Scalar product** for continuous-time signals:

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{+\infty} x_1(t)x_2^*(t)dt$$

- **Norm-2** for continuous-time signals:

$$||x(t)||_2 = \sqrt{\langle x(t), x(t)\rangle} = \sqrt{\int_{-\infty}^{+\infty} |x(t)|^2 dt} = \left(\int_{-\infty}^{\infty} |x(t)|^2\right)^{\frac{1}{2}}$$

- **Norm-p** for continuous-time signals:

$$||x(t)||_p = \left(\int_{-\infty}^{+\infty} |x(t)|^p dt\right)^{\frac{1}{p}} \quad ; \quad ||x(t)||_\infty = \max_n |x(t)|$$

3.6 Characterising Similarity/Difference Between Signals

3.6.1 Measuring the Similarity

The scalar product is a measure of the similarity between two signals:

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| cos(\phi)$$

Normalize the scalar product to the lengths of the vectors: if $\cos(\phi) = 1$, the two vectors have the same direction.

$$\cos(\phi) = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \cdot |\vec{B}|}$$

The **normalized scalar product** for vectors indicates how similar the two vectors are.

- Normalized scalar product for vectors between continuous-time signals:

$$\frac{\langle x_1(t), x_2(t)\rangle}{||x_1(t)||_2 \; ||x_2(t)||_2}$$

- Normalized scalar product for vectors between discrete-time signals:

$$\frac{\langle x_1[n], x_2[n]\rangle}{||x_1[n]||_2 \; ||x_2[n]||_2}$$

3.6.2 Normalized Cross-correlation Function $f_c(\theta)$

- In practical conditions, the signals are corrupted by noise.

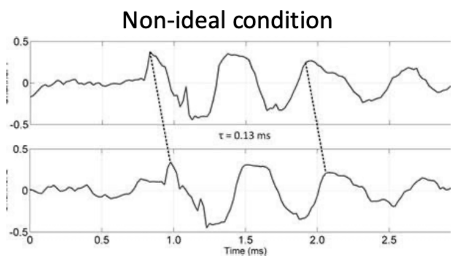


FIG. 12: *Signals under non-ideal condition*

- A possible estimate of the delay between the two signals is the time interval by which we need **to shift one of the signals so that it is maximally similar to the other**.

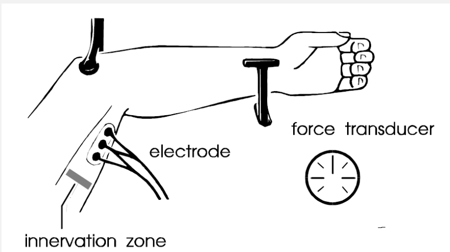
Normalized cross-correlation function between $x_1(t)$ and $x_2(t)$:

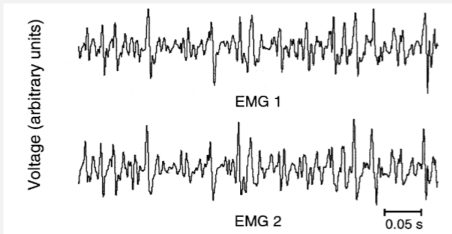
$$f_c(\theta) = \frac{\langle x_1(t), x_2(t-\theta)\rangle}{||x_1(t)||_2 \; ||x_2(t)||_2}$$

$$= \frac{\int_{-\infty}^{\infty} x_1(t)x_2(t-\theta)dt}{\sqrt{\int_{-\infty}^{+\infty}|x_1(t)|^2dt} \cdot \sqrt{\int_{-\infty}^{+\infty}|x_2(t-\theta)|^2dt}}$$

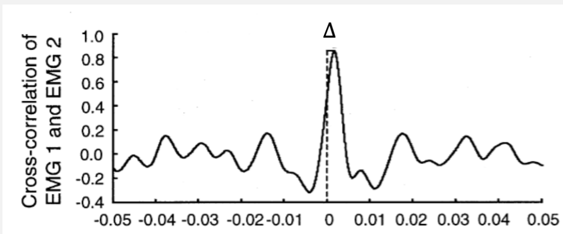
- The θ value corresponding to the maximum of $f(\theta)$ is an estimate of the delay between the two signals.

<u>Example: Measure muscle fiber conduction velocity</u>





By using cross-correlation function, we are able to find the time delay between two EMG signals. By measuring the distance between two electrodes, we can calculate the conduction velocity.



3.6.3 Measuring the Difference

- An alternative way to measure the similarity between two signals is to compute the **strength of their difference** (i.e., norm).

- For example, using the norm-2 as measure of strength, we can define the **mean squared error (MSE)** between signals :

$$MSE(\theta) = ||x_1(t) - x_2(t-\theta)||_2^2$$
$$= \int_{-\infty}^{+\infty} |x_1(t) - x_2(t-\theta)|^2 dt$$

  - $MSE(\theta)$ is a function of the shift θ.
  - Minimum value of θ best estimates the delay.
  - It is the energy of the error signal.

4 Fourier Series

4.1 Orthonormal Functions

- **Orthonormal functions** has the following property:

$$\langle \phi_i(t), \phi_k(t) \rangle = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$$

  The N signals $\{\phi_i(t)\}_{i=1\ldots N}$ with this property is referred to as an **orthonormal set of signals**.

  Think the orthogonal unitary vectors defining coordinate axes (i, j, k) in Euclidean space!

- Consider a generic signal that can be described by a **linear combination** of a set of orthonormal signals:

$$x(t) = \sum_{i=1}^{N} a_i \phi_i(t)$$

  where a_i are unknown complex or real numbers.

- The coefficients a_i can be determined by projecting the signal into each function $\{\phi_i(t)\}_{i=1\ldots N}$:

$$\begin{aligned} \langle x(t), \phi_k(t) \rangle &= \langle \sum_{i=1}^{N} a_i \phi_i(t), \phi_k(t) \rangle \\ &= \int_{-\infty}^{\infty} \sum_{i=1}^{N} a_i \ \phi_i(t) \ \phi_k^*(t) dt \\ &= \sum_{i=1}^{N} a_i \int_{-\infty}^{\infty} \phi_i(t) \ \phi_k^*(t) dt \\ &= a_k \end{aligned}$$

  - Scalar product is a linear operator.
  - The coefficients provide all the information in the signal: if we know a_k, we know the signal.
  - If the signal $x(t)$ belongs to a larger space, the projected signal will be an *approximation* of the original signal with minimum MSE.

EXAMPLE: HAAR BASIS FUNCTION

Given the signal:

$$\phi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} < t \leq 1 \\ 0, & otherwise \end{cases}$$

The following signals define a set of orthonormal basis functions:

$$\phi_{rk} = \phi(2^r t - k) \text{ for } r = 0, 1, 2... \text{ and } k = 0, 1, 2, ..., 2^r - 1$$

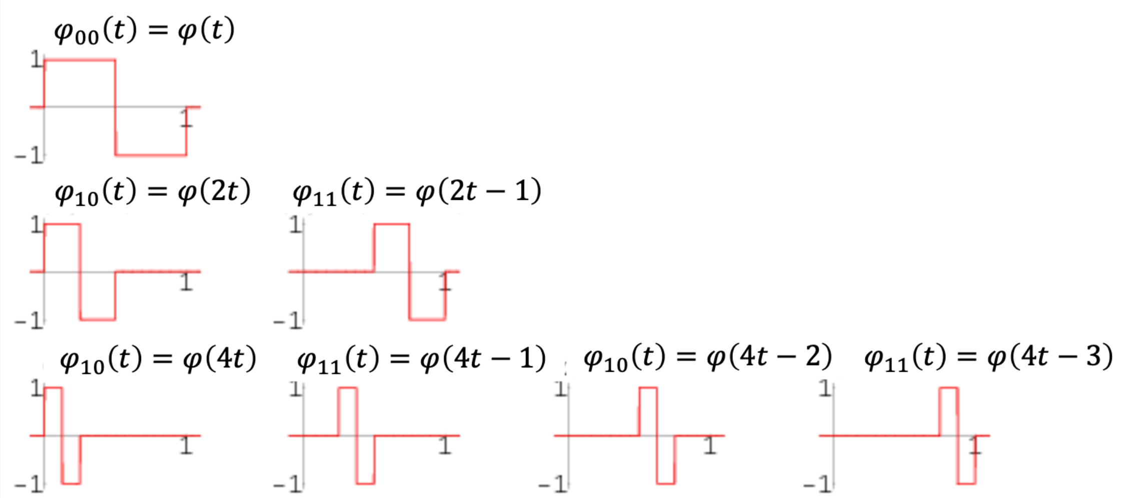


FIG. 13: *Haar basis functions*

4.2 Fourier Basis Functions

Fourier basis functions are:

$$\phi_i(t) = \frac{1}{\sqrt{T}} e^{j\omega_i t} = \frac{1}{\sqrt{T}} e^{j\frac{2\pi i}{T} t}$$

t is defined in the time interval $[0, T]$.

These functions have the following properties:

- periodic, $period = \frac{T}{i}$
- period is a function of the fundamental period T
- frequencies are the multiples of fundamental frequency $\frac{1}{T}$
- orthonormal, when $0 \leq t \leq T$:
  - $i \neq k$,

$$\langle \phi_i(t), \phi_k(t) \rangle = \int_0^T \phi_i(t)\ \phi_k^*(t) dt = \frac{1}{T} \int_0^T e^{j\frac{2\pi i}{T}t} e^{-j\frac{2\pi k}{T}t} dt = 0$$

  - $i = k$,

$$\langle \phi_i(t), \phi_k(t) \rangle = \int_0^T |\phi_i(t)|^2 dt = \frac{1}{T} \int_0^T dt = 1$$

4.3 Fourier Series

Fourier series: can be used to represent any periodic signal with finite energy in a single period.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{j\frac{2\pi k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-j\frac{2\pi k}{T}t}$$

DERIVATION

If we assume:

$$\begin{aligned} x(t) &= \sum_{i=-\infty}^{+\infty} a_i \; \phi_i(t) \\ &= \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{+\infty} a_k \; e^{j\frac{2\pi k}{T}t} \end{aligned}$$

with the coefficients a_k :

$$\begin{aligned} a_k = \langle x(t), \phi_k(t) \rangle &= \int_0^T x(t) \; \phi_k^*(t)dt \\ &= \frac{1}{\sqrt{T}} \int_0^T x(t) \; e^{-j\frac{2\pi k}{T}t}dt \end{aligned}$$

An equivalent expression is the **Fourier series**

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \; e^{j\frac{2\pi k}{T}t} \quad \text{with } c_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi k}{T}t}dt$$

- The infinite set of orthonormal functions of the Fourier series describes any periodic signal with finite energy in a single period.

Fourier series with a finite number of terms:

$$x(t) \approx x_N(t) = \sum_{k=-N}^{+N} c_k e^{j\frac{2\pi k}{T}t}$$

Error of approximation decreases as the number of terms increases,

$$E_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} c_k \; e^{j\frac{2\pi k}{T}t}$$

$$\lim_{N\to\infty} \int_0^T |E_N(t)|^2 dt = 0, \quad \text{if} \quad \int_0^T |x(t)|^2 dt < \infty$$

Fourier series converges as the number of terms increases:

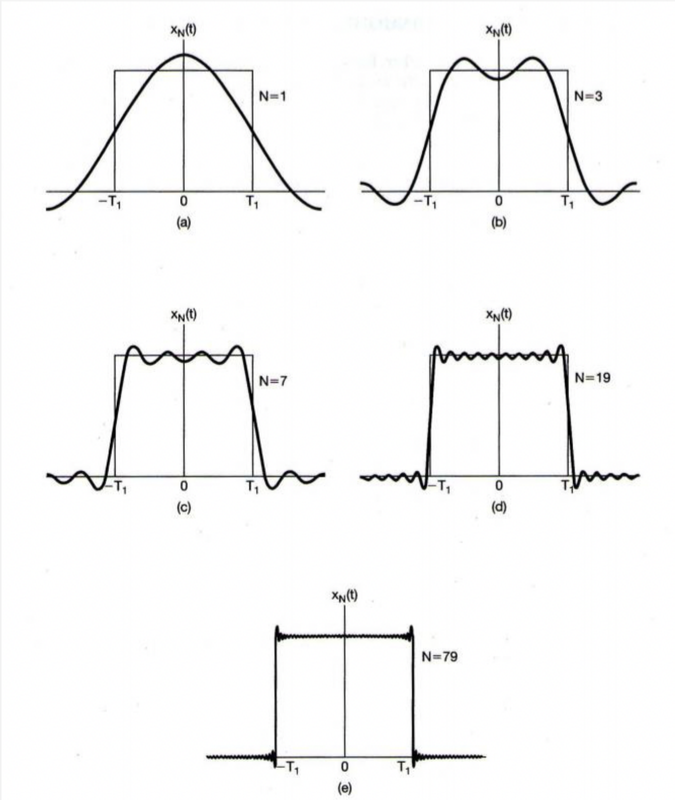


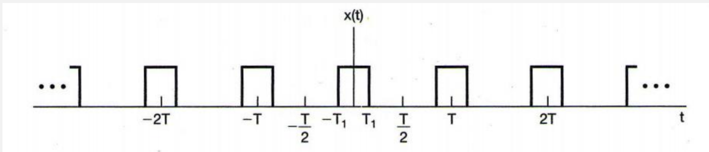
FIG. 14: *Example of convergence of the Fourier series of a periodic square wave*

5 Fourier Transform

5.1 From Fourier Series to Fourier Transform

To find a representation of any finite energy signal, not necessarily periodic: set the periodical of a period signal to *infinity*.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$



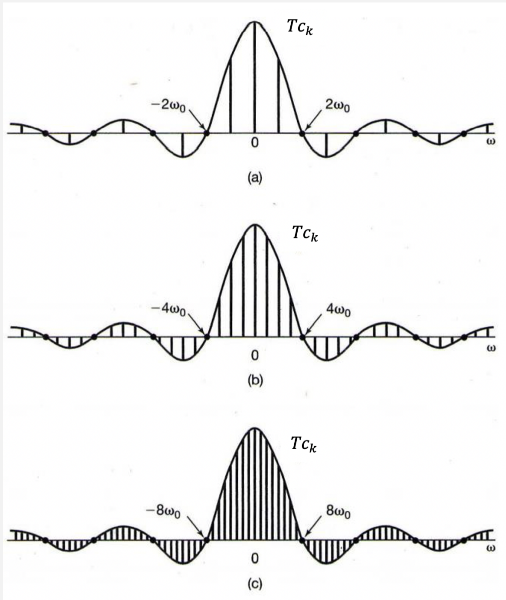
The Fourier series of the periodic signal $x(t)$ above is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T}t}$$

with

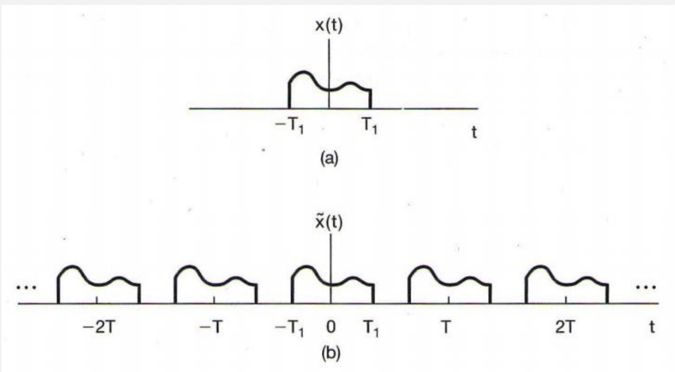
$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-j\frac{2\pi k}{T}t} \\ &= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \\ &= \frac{1}{T}\left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0} \end{aligned}$$

where $\omega_0 = \frac{2\pi}{T}$ is the frequency of the first harmonic.



Plot c_k against ω. As T increases, the frequency becomes smaller, the points on the plot become closer.

Generalize the example above:
The signal $x(t)$ defined in the interval $t \in [-T_1, T_1]$. We build the corresponding periodic signal $\tilde{x}(t)$, which equals to $x(t)$ in one period. As $T \to \infty$, $\tilde{x}(t) \to x(t)$.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \; e^{j\frac{2\pi k}{T}t}$$

with

$$\begin{aligned} c_k &= \frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} \tilde{x}(t) e^{-j\frac{2\pi k}{T}t} dt \\ &= \frac{1}{T}\underbrace{\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt}_{X(k\omega_0)} \\ &= \frac{1}{T} X(k\omega_0) \end{aligned}$$

$x(t) \to X(\omega)$ **is known as Fourier transform.**

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} c_k \; e^{j\frac{2\pi k}{T}t} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(k\omega_0) e^{j\frac{2\pi k}{T}t} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) \; e^{jk\omega_0 t} \; \omega_0 \end{aligned}$$

As $T \to \infty$, $\tilde{x}(t) \to x(t)$, $\omega_0 \to 0$:

$$\tilde{x}(t) = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \; e^{j\omega t} \; d\omega \tag{1}$$

$X(\omega) \to x(t)$ **is known as inverse Fourier transform.**

5.2 The Continuous-time Fourier Transform

Fourier transform:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \; e^{-j\omega t} dt$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \; e^{j\omega t} \; d\omega$$

- Fourier transform is a mathematical transformation employed to transform signals between time domain and frequency domain.
- Fourier transform is a reversible operation.
- **Fourier transform is a linear transformation**: it is defined by an integral

- For each value of ω, the Fourier transform is a complex number representing the projection of the signal on the complex exponential function $e^{j\omega t}$.

$$X(\omega) = \langle x(t), e^{j\omega t} \rangle$$

- **The Fourier transform exists for signals in the L^2 space.** These signals can be expressed as the combination of functions $e^{j\omega t}$.

Compare Fourier series and Fouririer transform:
Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \ e^{j\frac{2\pi k}{T}t} \quad with \quad c_k = \frac{1}{T}\int_0^T x(t) \ e^{-j\frac{2\pi k}{T}t}$$

Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \ e^{j\omega t} \ d\omega$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \ e^{-j\omega t} \ dt$$

The Fourier series represent *periodic* signals with *discrete* frequencies; the Fourier transform represents *non-periodic* signals with *continuous* frequencies.

5.3 Properties of Fourier Transform

5.3.1 Linearity

$$\mathcal{FT}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(\omega) + a_2 X_2(\omega)$$

Derivation
Given

$$X_1(\omega) = \int_{-\infty}^{+\infty} x_1(t) \ e^{-j\omega t} \ dt \quad \text{and} \quad X_2(\omega) = \int_{-\infty}^{+\infty} x_2(t) \ e^{-j\omega t} \ dt$$

$$\begin{aligned}
\mathcal{FT}\{a_1 x_1(t) + a_2 x_2(t)\} &= \int_{-\infty}^{+\infty} \big(a_1 x_1(t) + a_2 x_2(t)\big) \ e^{-j\omega t} \ dt \\
&= a_1 \int_{-\infty}^{+\infty} x_1(t) e^{-j\omega t} \ dt + a_2 \int_{-\infty}^{+\infty} x_2(t) e^{-j\omega t} \ dt \\
&= a_1 X_1(\omega) + a_2 X_2(\omega)
\end{aligned}$$

5.3.2 Time shifting

$$\mathcal{FT}\{x(t-t_0)\} = e^{-j\omega t_0} \mathcal{FT}\{x(t)\} = e^{-j\omega t_0} \ X(\omega)$$

Derivation

$$\mathcal{FT}\{x(t-t_0)\} = \int_{-\infty}^{+\infty} x(t-t_0)\ e^{-j\omega t}\ dt$$

Let $r = t - t_0$:

$$\begin{aligned}\mathcal{FT}\{x(t-t_0)\} &= \int_{-\infty}^{+\infty} x(r)\ e^{-j\omega(r+t_0)}\ dr \\ &= \int_{-\infty}^{+\infty} x(r)\ e^{-j\omega r}\ e^{-j\omega t_0} dr \\ &= e^{-j\omega t_0} \int_{-\infty}^{+\infty} x(r)\ e^{-j\omega r} dr\end{aligned}$$

Let $r = t$:

$$\begin{aligned}\mathcal{FT}\{x(t-t_0)\} &= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{+\infty} x(t)\ e^{-j\omega t} dt}_{\text{Fourier Transform}} \\ &= e^{-j\omega t_0} \mathcal{FT}\{x(t)\} \\ &= e^{-j\omega t_0}\ X(\omega)\end{aligned}$$

5.3.3 Conjugation

$$\mathcal{FT}\{x^*(t)\} = X^*(-\omega)$$

if $x(t)$ is real, $X(-\omega) = X^*(\omega)$.

<u>Derivation</u>

The Fourier transform is:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\ e^{-j\omega t}\ dt$$

Take the complex conjugate:

$$X^*(\omega) = \int_{-\infty}^{+\infty} x^*(t)\ e^{j\omega t}\ dt$$

Change $\omega \to -\omega$:

$$\begin{aligned}X^*(-\omega) &= \int_{-\infty}^{+\infty} x^*(t)\ e^{-j\omega t}\ dt \\ &= \mathcal{FT}\{x^*(t)\}\end{aligned}$$

5.3.4 Dual property

$$\text{if } x(t) \xleftrightarrow{\mathcal{FT}} X(\omega), \quad \text{then } X(t) \xleftrightarrow{\mathcal{FT}} 2\pi x(-\omega)$$

<u>Example:</u>

- $\delta(t) \overset{\mathcal{FT}}{\longleftrightarrow} 1, \quad 1 \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi\delta(\omega)$[1]
- $\delta(t + t_0) \overset{\mathcal{FT}}{\longleftrightarrow} e^{j\omega t_0}, \quad e^{j\omega t_0} \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$

DERIVATION
The Fourier transform is:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\ e^{-j\omega t}\ dt$$

Change $\omega \to t$, to avoid confusion, also change $t \to u$:

$$X(t) = \int_{-\infty}^{+\infty} x(u)\ e^{-jtu}\ du$$

The inverse Fourier transform of $x(-\omega)$ is:

$$\mathcal{FT}^{-1}\{x(-\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x(-\omega)e^{j\omega t}d\omega$$

Change $-\omega \to u$,

$$\mathcal{FT}^{-1}\{x(u)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x(u)e^{-jut}du$$

This yields the dual property:

$$X(t) \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi x(-\omega)$$

5.3.5 Time scaling

$$\mathcal{FT}\{x(at)\} = \frac{1}{|a|}X(\frac{\omega}{a})$$

a is a non-zero real number

From the above property,

$$\mathcal{FT}\{x(-t)\} = X(-\omega)$$

DERIVATION
Fourier transform of $x(at)$ is:

$$\mathcal{FT}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt$$

Replace $at \to u$, then $t = \frac{u}{a}, dt = \frac{1}{a}du$

$$\begin{aligned}\mathcal{FT}\{x(u)\} &= \frac{1}{|a|}\int_{-\infty}^{+\infty} x(u)e^{-j\frac{\omega}{a}u}du \\ &= \frac{1}{|a|}X(\frac{\omega}{a})\end{aligned}$$

[1]Note that $\delta(-\omega) = \delta(\omega)$

5.3.6 Parseval's relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

the function $|X(\omega)|^2$ is termed the **energy-density spectrum** of the signal.

DERIVATION

Start from the L.H.S:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |x(t)|^2 dt &= \int_{-\infty}^{+\infty} x(t)x^*(t)dt \\
&= \int_{-\infty}^{+\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)e^{-j\omega t}d\omega\right)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)\left(\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)X(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega
\end{aligned}$$

5.3.7 Differentiation in time

$$\mathcal{FT}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(\omega)$$

$$\mathcal{FT}\left\{\frac{d^n x(t)}{dt^n}\right\} = (j\omega)^n X(\omega)$$

5.3.8 Convolution

$$\mathcal{FT}\{x(t) * h(t)\} = X(\omega)H(\omega)$$

DERIVATION

$$\mathcal{FT}\{x(t) * h(t)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)e^{-j\omega t}d\tau dt$$

By the change of variables $t - \tau = \alpha$,

$$\begin{aligned}
\mathcal{FT}\{x(t) * h(t)\} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(\alpha)e^{-j\omega(\tau+\alpha)}d\tau d\alpha \\
&= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{+\infty} h(\alpha)e^{-j\omega\alpha}d\alpha \\
&= X(\omega)H(\omega)
\end{aligned}$$

Convolution in the time domain is equivalent to multiplication in the Fourier domain.

5.4 More basic Fourier transforms

5.4.1 *Impulse*

$$x(t) = \delta(t - T) \quad X(\omega) = e^{-j\omega T}$$

In particular, for $T = 0$, $X(\omega) = 1$.

5.4.2 *Complex exponential*

$$x(t) = e^{j\omega_0 t} \quad X(\omega) = 2\pi\delta(\omega - \omega_0)$$
$$x(t) = e^{-j\omega_0 t} \quad X(\omega) = 2\pi\delta(\omega + \omega_0)$$

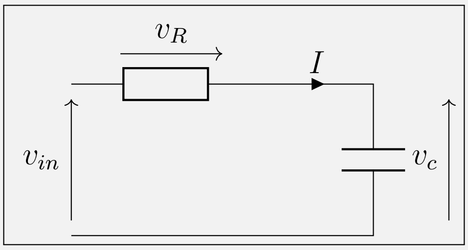
5.4.3 *Cosine*

$$x(t) = A\cos(\omega_0 t) \quad X(\omega) = A\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

5.4.4 *Sine*

$$x(t) = A\sin(\omega_0 t) \quad X(\omega) = \frac{A\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

5.5 Example of application of properties of the Fourier transform



For the circuit above, the input/output relation is characterized by the following differential equation:

$$v_c + RC\frac{dv_c}{dt} = v_{in}$$

By taking Fourier transform (using linearity property and differentiation in time property):

$$V_c(\omega) + j\omega RCV_c(\omega) = V_{in}(\omega) \quad \rightarrow \quad V_c(\omega) = \frac{1}{1 + j\omega RC}V_{in}(\omega)$$

- The Fourier transforms are coefficients indicating the weights of each complex exponential signal $e^{-j\omega t}$ composing the signals.
- The above relation therefore tells us how the circuit processes the input signal by changing the weights of the coefficients.

- In the frequency domain, the circuit acts as a factor that multiplies each coefficient in a frequency-dependant way.

Let $H(\omega) = \frac{1}{1+j\omega RC}$, this represents the **frequency response** of the circuit.

$$V_c(\omega) = H(\omega)V_{in}(\omega)$$

Let the signal input $v_{in}(t) = e^{j\omega_0 t}$, when applying the dual property:

$$V_{in}(\omega) = 2\pi\delta(\omega - \omega_0)$$

Therefore:

$$V_c(\omega) = \frac{2\pi}{1+j\omega RC}\delta(\omega-\omega_0) = \frac{2\pi}{1+j\omega_0 RC}\delta(\omega-\omega_0) = H(\omega_0)2\pi\delta(\omega-\omega_0)$$

Take inverse Fourier transform:

$$v_c(t) = H(\omega_0)e^{-j\omega_0 t}$$

Since $H(\omega_0)$ is a complex number,

$$H(\omega_0) = \underbrace{|H(\omega_0)|}_{maginitude} \cdot \underbrace{e^{j\angle H(\omega_0)}}_{phase\ spectra} \quad \rightarrow \quad \boxed{v_c(t) = |H(\omega_0)|e^{j(\omega_0 t + \angle H(\omega_0))}}$$

The frequency response of the circuit changes the magnitude and the phase of the complex exponential, NOT the frequency.

If the input is an impulse signal $v_{in}(t) = \delta(t)$, the response of the circuit in the frequency domain is:

$$V_c(\omega) = H(\omega)V_{in}(\omega) = H(\omega), \quad \text{since } V_{in}(\omega) = 1$$

And the response to the impulse in the time domain is:

$$\mathcal{FT}^{-1}\{H(\omega)\} = \frac{1}{RC}e^{\frac{t}{RC}}u(t) = \frac{1}{\tau}e^{\frac{-t}{\tau}}u(t) = h(t)$$

From the example above, there is an input-output relation in the frequency domain.

$$Y(\omega) = H(\omega)X(\omega)$$

5.5.1 LTI systems

The relation $Y(\omega) = H(\omega)X(\omega)$ holds for a larger class of systems: LTI systems.

For any continuous-time signals:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

Assume there is a system $T\{\cdot\}$:

- associates an output $y(t)$ to an input $x(t) : y(t) = T\{x(t)\}$.
- linear, acts on the signal in the same way over time.

For any signal $x(t)$:

$$y(t) = T\{x(t)\} = T\left\{ \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \right\} = \int_{-\infty}^{+\infty} x(\tau)T\left\{ \delta(t-\tau)d\tau \right\}d\tau$$

If $h(t) = T\{\delta(\tau)\}$ (response of the system to the **impulse response**):

$$T\{\delta(t-\tau)\} = h(t-\tau), \text{then} : y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

The output of a **linear, time-invariant (LTI) system** is the convolution of the input with the impulse response, i.e. the system is fully defined by the impulse response.

5.6 Magnitude and Phase Spectra

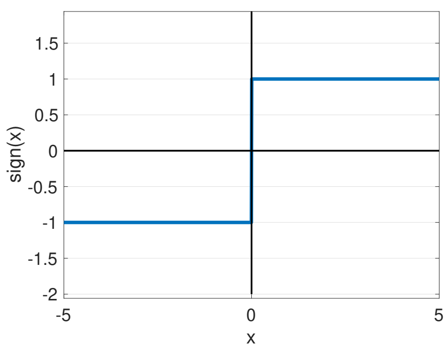
In general, $X(\omega)$ is a **complex** function of ω:

$$X(\omega) = a(\omega) + \mathbf{j}b(\omega) = |X(\omega)|e^{j\angle X(\omega)}$$

- $|X(\omega)|$ is **magnitude**, it describes the basic frequency content of a signal, i.e. the relative magnitudes of the complex exponentials that make up $x(t)$.
- $\angle X(\omega)$ is **phase angle**, it determines the different look of signals, even if the magnitude remains unchanged.

$$\angle X(\omega) = arctan\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] + \pi\left[\frac{1 - sign(Re\{X(\omega)\})}{2}\right]$$

where $sign(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$



EXAMPLE[a]
A ship encounters the superposition of three wave trains, each of which can be

modeled as a sinusoidal signal.

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$

With fixed magnitudes for these sinusoids, **the amplitude of their sum may be quite small or very large, depending on the relative phases.** The implications of phase for the ship, therefore, are quite significant.

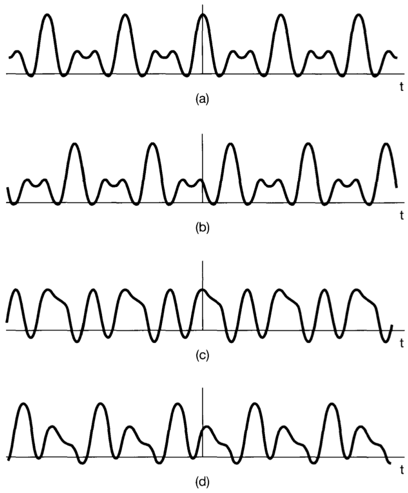


Figure 6.1 The signal $x(t)$ given in eq. (6.3) for several different choices of the phase angles ϕ_1, ϕ_2, and ϕ_3: (a) $\phi_1 = \phi_2 = \phi_3 = 0$; (b) $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad; (c) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad; (d) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad.

[a]Example adopted from **Signals and Systems, 2nd Edition**, P424

Specifically, for the circuit example above:

$$|H(\omega_0)| = \frac{1}{\sqrt{1+\omega^2(RC)^2}} = \frac{1}{\sqrt{1+(\frac{\omega}{\omega_c})^2}}$$

$$\angle H(\omega_0) = -tan^{-1}(\omega RC) = -tan^{-1}(\frac{\omega}{\omega_c})$$

5.7 Fourier Transform of Periodic Signals

If $x(t)$ is a periodic signal:

$$\mathcal{FT}\{x(t)\} = 2\pi \sum_{k=-\infty}^{+\infty} c_k \delta(\omega - k\omega_0)$$

where

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-j\frac{2\pi k}{T}t} dt$$

<u>DERIVATION</u>

Fourier series of a periodic signal $x(t)$ with period T is:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

By applying linearity property:

$$\begin{aligned} X(\omega) &= \mathcal{FT}\left\{\sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi k}{T}t}\right\} \\ &= \sum_{k=-\infty}^{+\infty} c_k \; \mathcal{FT}\{e^{jk\omega_0 t}\} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} c_k \delta(\omega - k\omega_0) \end{aligned}$$

5.7.1 Fourier transform of a train of impulses

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \xrightarrow{\mathcal{FT}} X(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

where

$$\omega_0 = \frac{2\pi}{T}$$

<u>DERIVATION</u>

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - T)$$

From the definition above, $x(t)$ is periodic with period T:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta(t)e^{-j\frac{2\pi k}{T}t}dt = \frac{1}{T}$$

Thus:

$$X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_0) \quad \text{with } \omega_0 = \frac{2\pi}{T}$$

6 Sampling Theorem

To obtain a discrete-time signal from a continuous-time signal, we need a **C/D converter**.

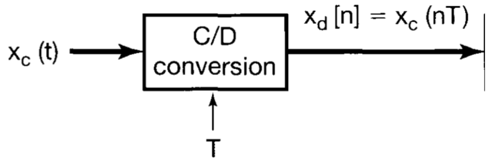


FIG. 15: *C/D converter*

The mathematical expression for a C/D converter is:

$$x[n] = x_c(nT) \quad -\infty < n < +\infty$$

where T is sampling period, $f_s = \frac{1}{T}$ is sampling frequency.

- In general, the C/D transformation cannot be inverted.
- Infinite continuous signals can reproduce a given sequence of samples,

An ideal C/D converter applies T property, so that the sampling can be done without loosing information.

6.1 Sampling Process

Impulse train modulator $s(t)$ is:

$$s(t) = \sum_{-\infty}^{+\infty} \delta(t - nT)$$

The sampled signal $x_s(t)$ is obtained by multiplying the impulse train modulator by the continuous-time signal $x_c(t)$:

$$\begin{aligned} x_s(t) &= x_c(t)\ s(t) \\ &= \sum_{n=-\infty}^{+\infty} x_c(t)\delta(t - nT) \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT) \end{aligned}$$

Sampled signal, $x_s(t)$, is still defined in continuous-time, but it contains all information in the sampled discrete-time domain.

Apply the Fourier transform to $x_s(t)$:

$$
\begin{aligned}
X_s(\omega) &= \mathcal{FT}\{x_c(t)\} \cdot \mathcal{FT}\{s(t)\} \\
&= \frac{1}{2\pi} X_c(\omega) * \mathcal{FT}\{s(t)\} \\
&= \frac{1}{T} X_c(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s) \\
&= \boxed{\frac{1}{T} \sum_{n=-\infty}^{+\infty} X_c(\omega - k\omega_s)}
\end{aligned}
$$

where sampling frequency $\omega_s = \omega_0 = \frac{2\pi}{T}$.

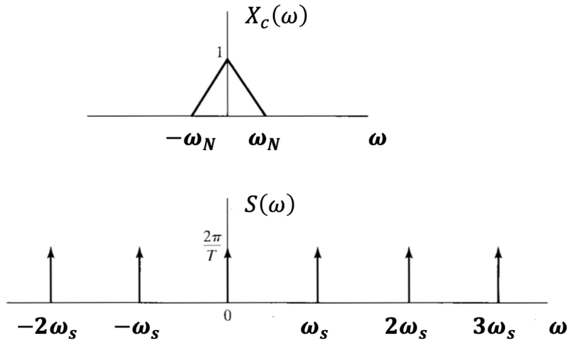


FIG. 16: *(a)Original signal;(b) Sampled signal*

For sampled signals: ω_N is the signal bandwidth

- if $\omega_s \geq 2\omega_N$, the replicas in the periodization do not overlap
- if $\omega_s < 2\omega_N$, the replicas overlap, also known as aliasing[2].

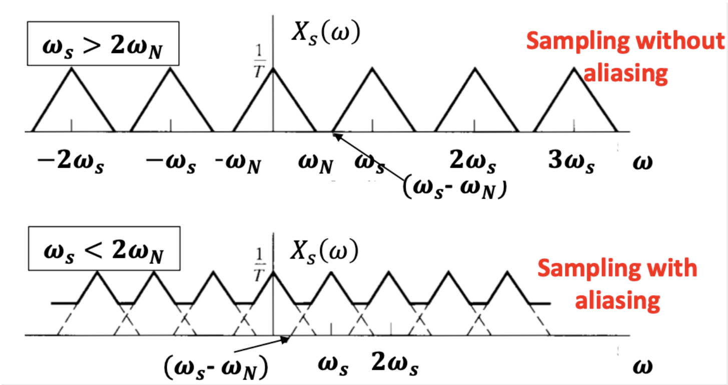


FIG. 17: *(c)* $\omega_s \geq 2\omega_N$*; (d) aliasing:* $\omega_s < 2\omega_N$

6.2 Reconstruction Process

Ideal low-pass filters can be used to reconstruct the signals.

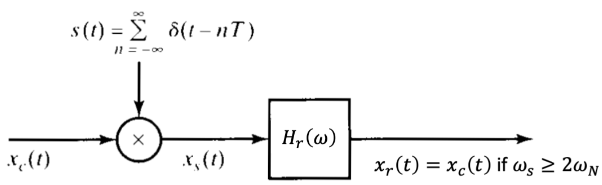


FIG. 18: *Low-pass filter system*

[2] In Latin, "alias" means "others"

1. Multiply the sampled signals by the function $H_r(\omega)$

$$X_c(\omega) = X_s(\omega) \cdot H_r(\omega)$$

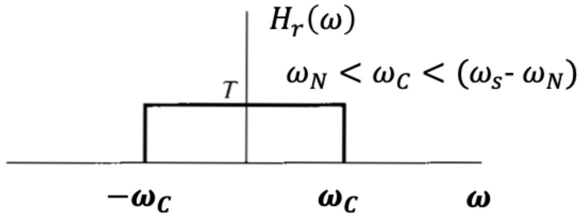


FIG. 19: $H_r(\omega)$

2. Due to the convolution property:

$$X_c(\omega) = \mathcal{FT}\{x_s(t) * h_r(t)\}$$

3. Apply inverse Fourier transform:

$$\begin{aligned} x_c(t) &= x_s(t) * h_r(t) \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT) * h_r(t) \\ &= \boxed{\sum_{n=-\infty}^{+\infty} x_c(nT) h_r(t-nT)} \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT) \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \end{aligned}$$

with

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

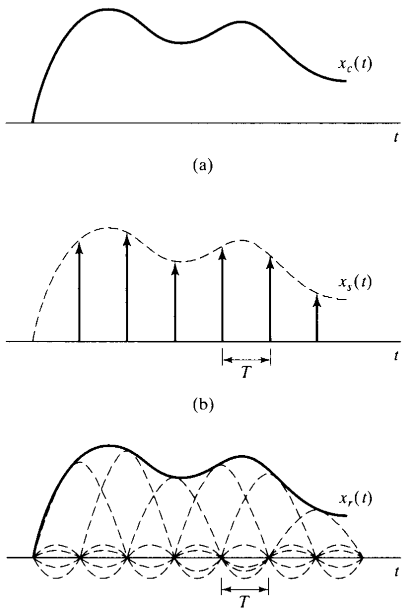


FIG. 20: *Reconstruction*

6.3 Nyquist-Shannon Sampling Theorem

Nyquist-Shannon sampling theorem provides the condition under which the C/D transformation **can be inverted** without loosing information.

Let $x_c(t)$ be a band-limited signal with $X_c(\omega) = 0,\ |\omega| \geq \omega_N$. Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, if

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_N$$

where $2\omega_N$ is the minimal sampling rate and referred to as the **Nyquist rate**.

$x_c(t)$ can be reconstructed by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter. The resulting output signal will be exactly equal $x_c(t)$.