Some famous formula for pulsar timing array (PTA)

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BACKGROUND

We consider a general metric

$$g_{ab} = \eta_{ab} + h_{ab}, \tag{1}$$

where

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

is the Minkowski metric, and h_{ab} is the perturbation of the spacetime or gravitational wave (GW). A metric perturbation in a spatial transverse and traceless gauge has a plane wave expansion given by

$$h_{ab}(t, \vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int d\hat{\Omega}\, e^{i2\pi f(t-\hat{\Omega}\cdot\vec{x})} h_A(f, \hat{\Omega}) e^A_{ij}(\hat{\Omega}), \tag{3}$$

where f is the frequency of the GWs, $\vec{k} = 2\pi f \hat{\Omega}$ is the wave vector, $\hat{\Omega}$ is a unit vector that points along the direction of travel of the waves, $i, j = x, y, z$ are spatial indices, and the index $A = (+, \times, b, l, x, y)$ labels polarization of GWs. The polarization tensors $e^A_{ij}(\hat{\Omega})$ are defined as

$$e^+_{ij} = \hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j, \tag{4}$$

$$e^\times_{ij} = \hat{m}_i\hat{n}_j + \hat{n}_i\hat{m}_j, \tag{5}$$

$$e^b_{ij} = \hat{m}_i\hat{m}_j + \hat{n}_i\hat{n}_j, \tag{6}$$

$$e^l_{ij} = \hat{\Omega}_i\hat{\Omega}_j, \tag{7}$$

$$e^x_{ij} = \hat{m}_i\hat{\Omega}_j + \hat{\Omega}_i\hat{m}_j, \tag{8}$$

$$e^y_{ij} = \hat{n}_i\hat{\Omega}_j + \hat{\Omega}_i\hat{n}_j, \tag{9}$$

where

$$\hat{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{10}$$

$$\hat{m} = (\sin\phi, -\cos\phi, 0), \tag{11}$$

$$\hat{n} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta). \tag{12}$$

It is easy to verify that $\hat{\Omega}$, $\hat{m}$ and $\hat{n}$ are perpendicular to each other. Furthermore, the polarization tensors are normalized as

$$e^A_{ij} e^{Bij} = 2\delta^{AB}. \tag{13}$$

Now consider the metric perturbation from a single gravitational wave traveling along the z-axis so that $\hat{\Omega} = \hat{z} = (0, 0, 1)$, $\theta = 0$ and $\phi = \pi/2$. In this case $\hat{m} = \hat{x} = (1, 0, 0)$ and $\hat{n} = \hat{y} = (0, 1, 0)$. Besides,

$$e^+_{ij}(\hat{z}) = \hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j \tag{14}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1, 0, 0) - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0, 1, 0) \tag{15}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

and

$$e^{\times}_{ij}(\hat{z}) = \hat{m}_i\hat{n}_j + \hat{n}_i\hat{m}_j \tag{17}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix}(0,1,0) + \begin{pmatrix}0\\1\\0\end{pmatrix}(1,0,0) \tag{18}$$

$$= \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}. \tag{19}$$

The metric perturbation is given explicitly by

$$h_{ij}(t-z) \equiv h_{ij}(t, \hat{\Omega} = \hat{z}) \tag{20}$$

$$= \sum_A \int_{-\infty}^{\infty} df e^{i2\pi f(t-z)} h_A(f,\hat{z}) e^A_{ij}(\hat{z}) \tag{21}$$

$$= \sum_A h_A(f, t-z) e^A_{ij}(\hat{z}) \tag{22}$$

$$= \sum_A h_A e^A_{ij}(\hat{z}) = h_+ e^+_{ij}(\hat{z}) + h_\times e^\times_{ij}(\hat{z}) \tag{23}$$

$$= \begin{pmatrix}h_+&0&0\\0&-h_+&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&h_\times&0\\h_\times&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}h_+&h_\times&0\\h_\times&-h_+&0\\0&0&0\end{pmatrix}. \tag{24}$$

since

$$\hat{\Omega}\cdot\vec{x} = \hat{z}\cdot\vec{x} = z, \tag{25}$$

and we have defined

$$h_A = h_A(f, t-z) \equiv \int_{-\infty}^{\infty} df e^{i2\pi f(t-z)} h_A(f,\hat{z}). \tag{26}$$

Therefore the physical metric due to the perturbation is given by

$$g_{ab} = \eta_{ab} + h_{ab}(t-z) = \begin{pmatrix}-1&0&0&0\\0&1+h_+&h_\times&0\\0&h_\times&1-h_+&0\\0&0&0&1\end{pmatrix}. \tag{27}$$

CORRELATION

The antenna patterns are defined as

$$F^A(\hat{\Omega}) = e^A_{ij}(\hat{\Omega})\frac{\hat{p}^i\hat{p}^j}{2(1+\hat{\Omega}\cdot\hat{p})}. \tag{28}$$

The overlap function for two pulsars are

$$\Gamma_{ab}(|f|) = \sum_A \Gamma^A_{ab}(|f|), \tag{29}$$

where

$$\Gamma^A_{ab}(|f|) = \frac{3}{4\pi}\int d\hat{\Omega}\left(e^{2\pi i f L_a(1+\hat{\Omega}\cdot\hat{p}_a)} - 1\right)\left(e^{2\pi i f L_b(1+\hat{\Omega}\cdot\hat{p}_b)} - 1\right) F^A_a(\hat{\Omega}) F^A_b(\hat{\Omega}). \tag{30}$$

We now define

$$\Gamma_{ab}^{TT}(|f|) = \Gamma_{ab}^{+}(|f|) + \Gamma_{ab}^{\times}(|f|), \tag{31}$$
$$\Gamma_{ab}^{ST}(|f|) = \Gamma_{ab}^{b}(|f|), \tag{32}$$
$$\Gamma_{ab}^{VL}(|f|) = \Gamma_{ab}^{x}(|f|) + \Gamma_{ab}^{y}(|f|), \tag{33}$$
$$\Gamma_{ab}^{SL}(|f|) = \Gamma_{ab}^{l}(|f|). \tag{34}$$

Then the cross-spectral density is

$$S_{ab}(f) = \frac{1}{24\pi^2 f^3} \frac{1+\kappa^2}{1+\kappa^2\left(\frac{f}{f_{yr}}\right)^{-\frac{2}{3}}} \left[\Gamma_{ab}^{TT} A_{TT}^2 \left(\frac{f}{f_{yr}}\right)^{-\frac{4}{3}} + \left(\Gamma_{ab}^{ST} A_{ST}^2 + \Gamma_{ab}^{VL} A_{VL}^2 + \Gamma_{ab}^{SL} A_{SL}^2\right) \left(\frac{f}{f_{yr}}\right)^{-2}\right] \tag{35}$$
$$= \Gamma_{ab}^{TT} S_{TT} + \Gamma_{ab}^{ST} S_{ST} + \Gamma_{ab}^{VL} S_{VL} + \Gamma_{ab}^{SL} S_{SL}, \tag{36}$$

where

$$S_{TT} = \frac{1}{24\pi^2 f^3} \frac{1+\kappa^2}{1+\kappa^2\left(\frac{f}{f_{yr}}\right)^{-\frac{2}{3}}} A_{TT}^2 \left(\frac{f}{f_{yr}}\right)^{-\frac{4}{3}}, \tag{37}$$
$$S_{ST} = \frac{1}{24\pi^2 f^3} \frac{1+\kappa^2}{1+\kappa^2\left(\frac{f}{f_{yr}}\right)^{-\frac{2}{3}}} A_{ST}^2 \left(\frac{f}{f_{yr}}\right)^{-2}, \tag{38}$$
$$S_{VL} = \frac{1}{24\pi^2 f^3} \frac{1+\kappa^2}{1+\kappa^2\left(\frac{f}{f_{yr}}\right)^{-\frac{2}{3}}} A_{VL}^2 \left(\frac{f}{f_{yr}}\right)^{-2}, \tag{39}$$
$$S_{SL} = \frac{1}{24\pi^2 f^3} \frac{1+\kappa^2}{1+\kappa^2\left(\frac{f}{f_{yr}}\right)^{-\frac{2}{3}}} A_{SL}^2 \left(\frac{f}{f_{yr}}\right)^{-2}. \tag{40}$$