Demo 9

This demo entry is used to test out further polynomials.

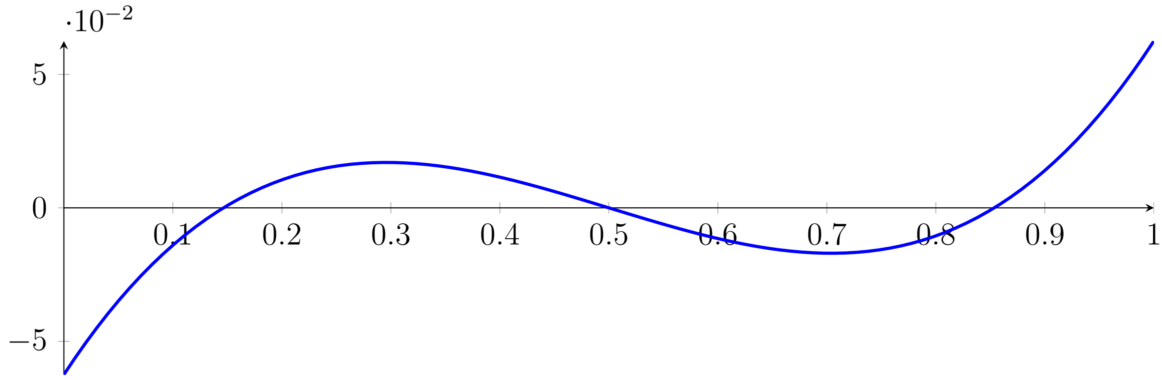
Contents

1 Running `BezClip` on p3 with epsilon 6

$$1X^3 - 1.5X^2 + 0.625X - 0.0625$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

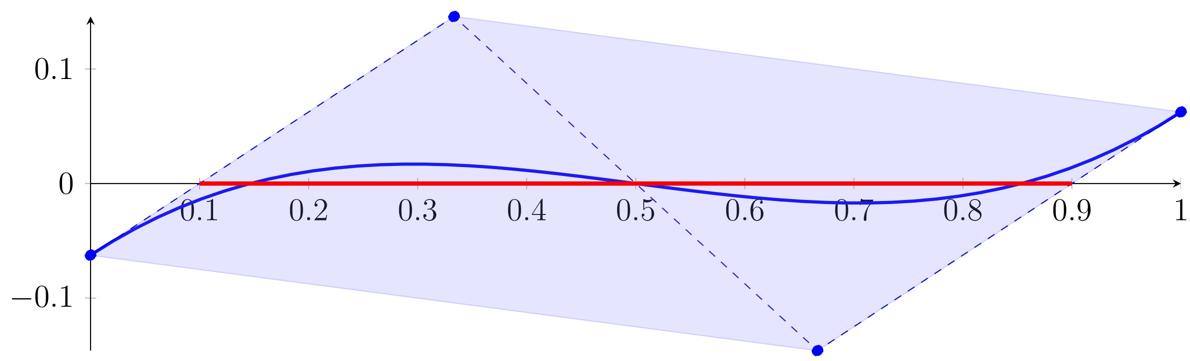
$$p = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$



1.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1X^3 - 1.5X^2 + 0.625X - 0.0625 \\ &= -0.0625B_{0,3}(X) + 0.145833B_{1,3}(X) - 0.145833B_{2,3}(X) + 0.0625B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.1, 0.9\}$$

Intersection intervals with the x axis:

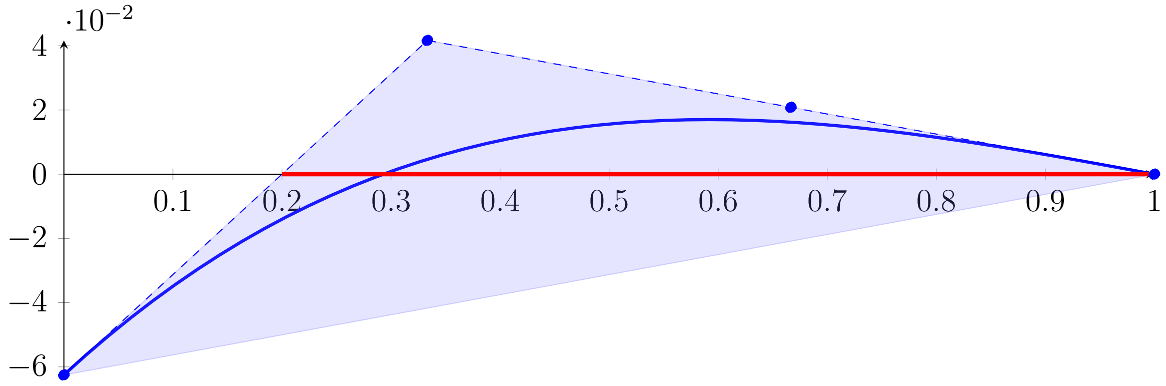
$$[0.1, 0.9]$$

Longest intersection interval: 0.8
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

1.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.125X^3 - 0.375X^2 + 0.3125X - 0.0625 \\ &= -0.0625B_{0,3}(X) + 0.0416667B_{1,3}(X) + 0.0208333B_{2,3}(X) + 1.01644 \cdot 10^{-20}B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.2, 1\}$$

Intersection intervals with the x axis:

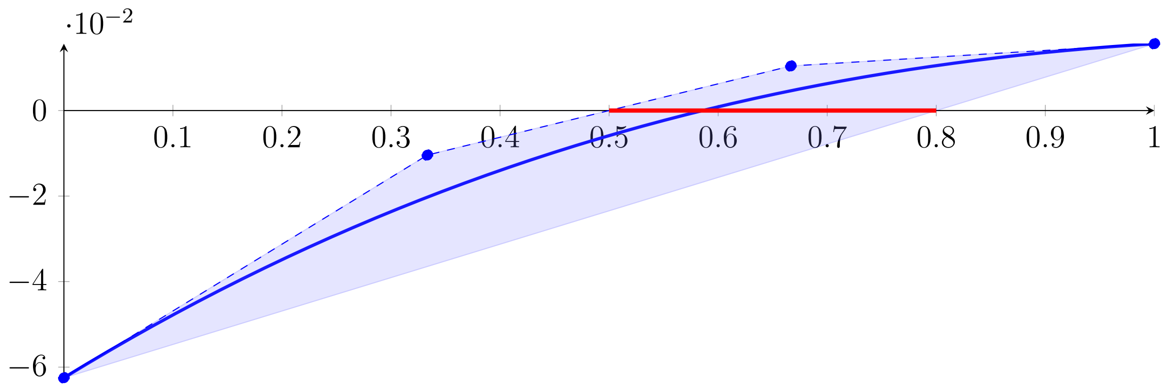
$$[0.2, 1]$$

Longest intersection interval: 0.8
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

1.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.015625X^3 - 0.09375X^2 + 0.15625X - 0.0625 \\ &= -0.0625B_{0,3}(X) - 0.0104167B_{1,3}(X) + 0.0104167B_{2,3}(X) + 0.015625B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.5, 0.8\}$$

Intersection intervals with the x axis:

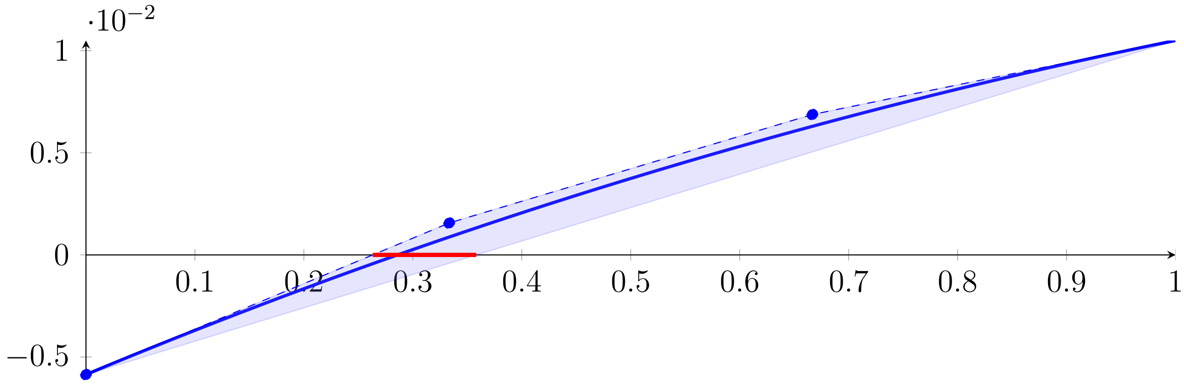
$$[0.5, 0.8]$$

Longest intersection interval: 0.3
$\Longrightarrow$ Selective recursion: interval 1: $[0.125, 0.2]$,

1.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.125, 0.2]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000421875X^3 - 0.00632813X^2 + 0.0222656X - 0.00585938 \\ &= -0.00585938B_{0,3}(X) + 0.0015625B_{1,3}(X) + 0.006875B_{2,3}(X) + 0.0105B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.263158, 0.358166\}$$

Intersection intervals with the x axis:

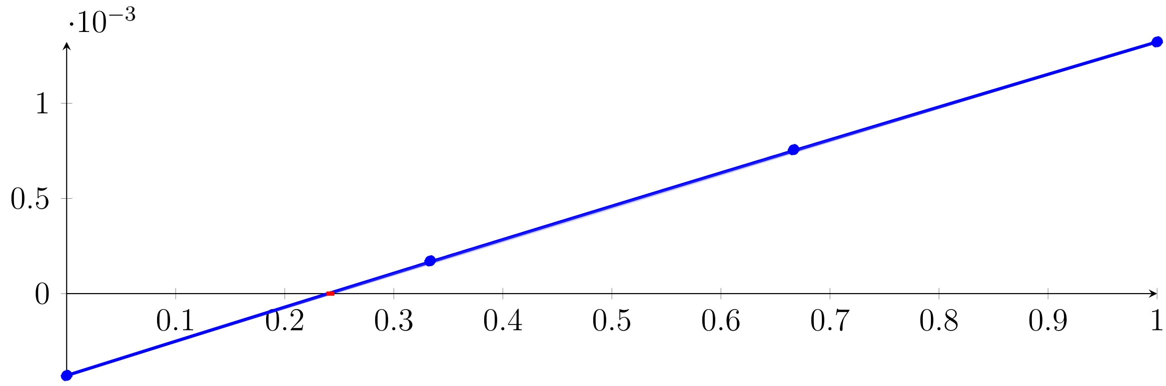
$$[0.263158, 0.358166]$$

Longest intersection interval: 0.0950083
$\Longrightarrow$ Selective recursion: interval 1: $[0.144737, 0.151862]$,

1.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.144737, 0.151862]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.618 \cdot 10^{-07}X^3 - 5.41149 \cdot 10^{-05}X^2 + 0.00180731X - 0.000430547 \\ &= -0.000430547B_{0,3}(X) + 0.00017189B_{1,3}(X) + 0.000756289B_{2,3}(X) + 0.00132301B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.238225, 0.245528\}$$

Intersection intervals with the x axis:

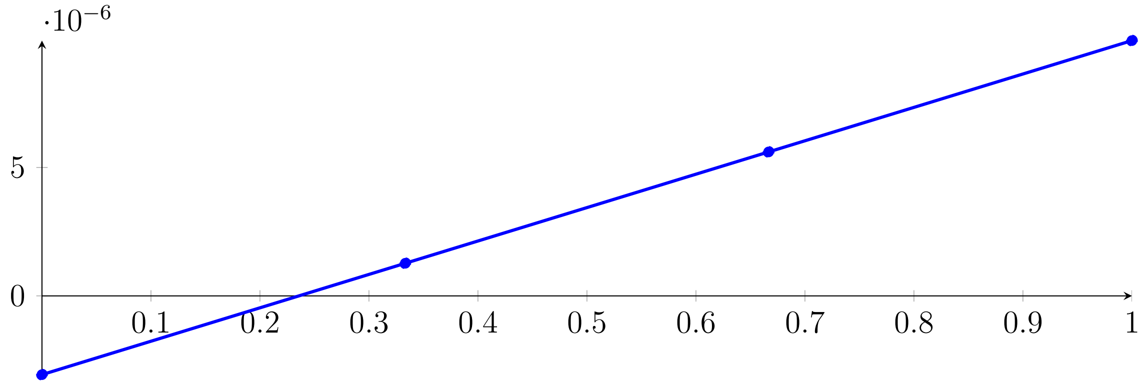
$$[0.238225, 0.245528]$$

Longest intersection interval: 0.00730249
$\Longrightarrow$ Selective recursion: interval 1: $[0.146434, 0.146486]$,

1.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.146434, 0.146486]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 1.4089\cdot 10^{-13}X^3 - 2.87196\cdot 10^{-09}X^2 + 1.30101\cdot 10^{-05}X - 3.0662\cdot 10^{-06}$$
$$= -3.0662\cdot 10^{-06}B_{0,3}(X) + 1.27048\cdot 10^{-06}B_{1,3}(X) + 5.60621\cdot 10^{-06}B_{2,3}(X) + 9.94098\cdot 10^{-06}B_{3,3}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.235679, 0.235731\}$$

Intersection intervals with the x axis:

$$[0.235679, 0.235731]$$

Longest intersection interval: $5.2035\cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.146447, 0.146447]$,

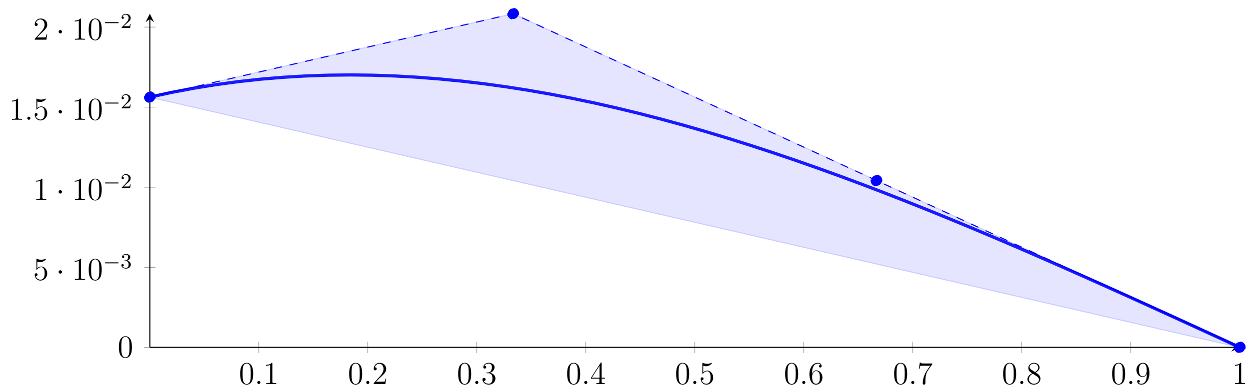
1.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.146447, 0.146447]$

Found root in interval $[0.146447, 0.146447]$ at recursion depth 7!

1.8 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.015625X^3 - 0.046875X^2 + 0.015625X + 0.015625$$
$$= 0.015625B_{0,3}(X) + 0.0208333B_{1,3}(X) + 0.0104167B_{2,3}(X) + 1.01644\cdot 10^{-20}B_{3,3}(X)$$



Intersection of the convex hull with the x axis:

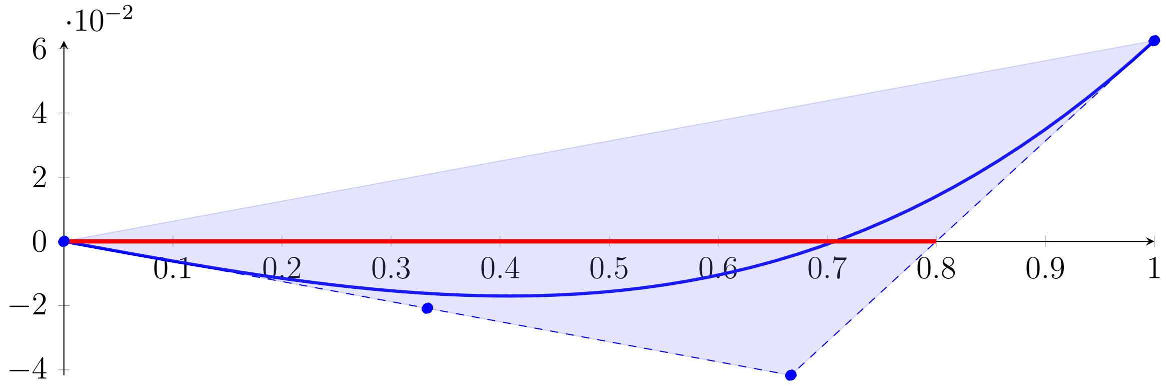
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

1.9 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.125X^3 - 1.01644 \cdot 10^{-20}X^2 - 0.0625X + 1.01644 \cdot 10^{-20} \\ &= 1.01644 \cdot 10^{-20}B_{0,3}(X) - 0.0208333B_{1,3}(X) - 0.0416667B_{2,3}(X) + 0.0625B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{-3.68629e-17, 0.8\}$$

Intersection intervals with the x axis:

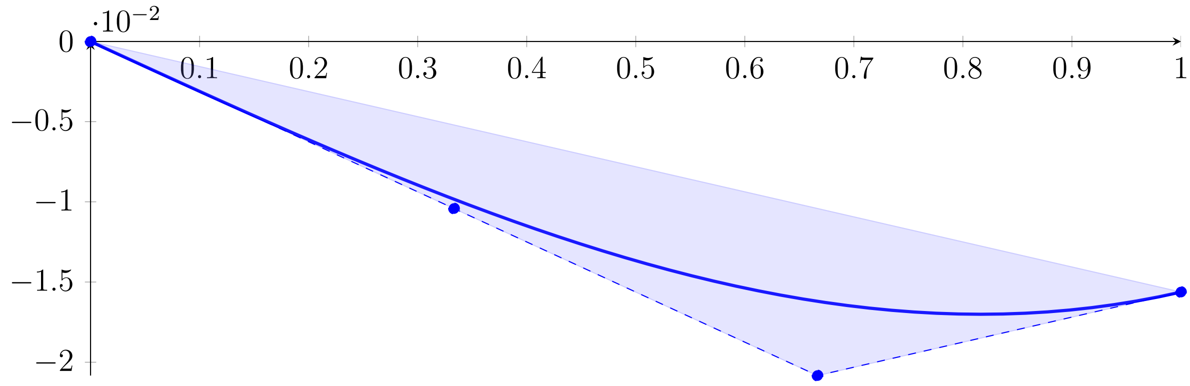
$$[-3.68629e-17, 0.8]$$

Longest intersection interval: 0.8
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

1.10 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.015625X^3 - 0.03125X + 1.01644 \cdot 10^{-20} \\ &= 1.01644 \cdot 10^{-20}B_{0,3}(X) - 0.0104167B_{1,3}(X) - 0.0208333B_{2,3}(X) - 0.015625B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{-3.67002e-17, 6.50521e-19\}$$

Intersection intervals with the x axis:

$$[-3.67002e-17, 6.50521e-19]$$

Longest intersection interval: $3.73508 \cdot 10^{-17}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

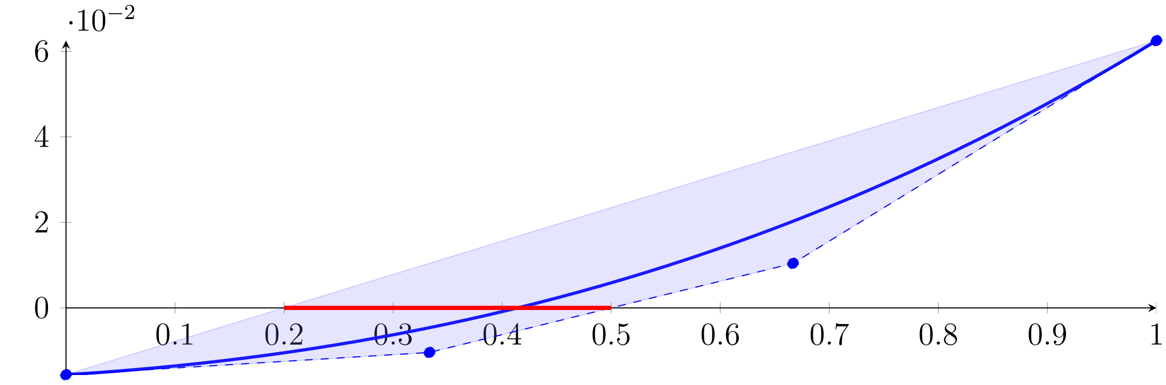
1.11 Recursion Branch 1 2 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 4!

1.12 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.015625X^3 + 0.046875X^2 + 0.015625X - 0.015625 \\ &= -0.015625B_{0,3}(X) - 0.0104167B_{1,3}(X) + 0.0104167B_{2,3}(X) + 0.0625B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.2, 0.5\}$$

Intersection intervals with the x axis:

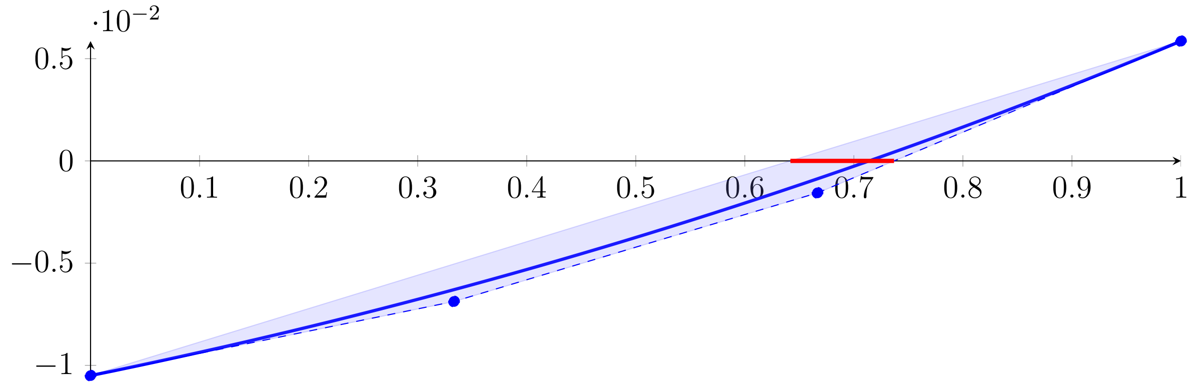
$$[0.2, 0.5]$$

Longest intersection interval: 0.3
$\Longrightarrow$ Selective recursion: interval 1: $[0.8, 0.875]$,

1.13 Recursion Branch 1 2 2 1 in Interval 1: $[0.8, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000421875X^3 + 0.0050625X^2 + 0.010875X - 0.0105 \\ &= -0.0105B_{0,3}(X) - 0.006875B_{1,3}(X) - 0.0015625B_{2,3}(X) + 0.00585937B_{3,3}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.641834, 0.736842\}$$

Intersection intervals with the x axis:

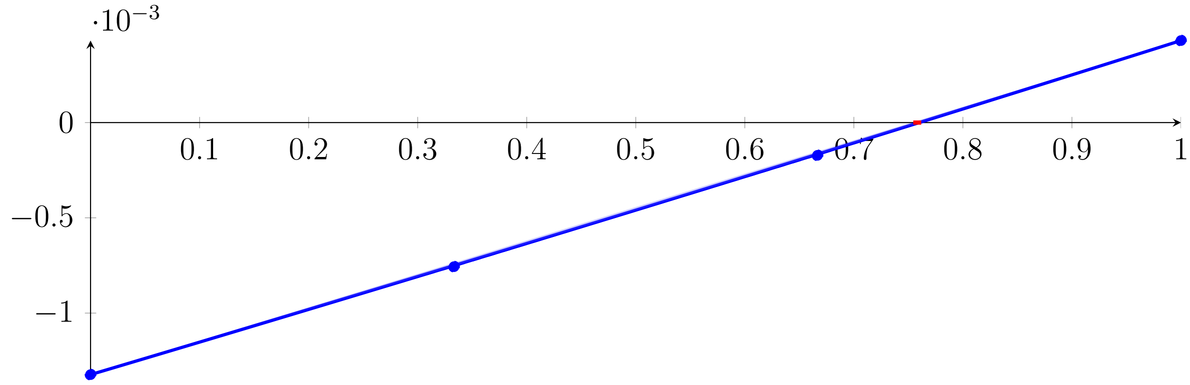
$$[0.641834, 0.736842]$$

Longest intersection interval: 0.0950083
$\Longrightarrow$ Selective recursion: interval 1: $[0.848138, 0.855263]$,

1.14 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.848138, 0.855263]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 3.618 \cdot 10^{-07} X^3 + 5.30295 \cdot 10^{-05} X^2 + 0.00170017 X - 0.00132301$$
$$= -0.00132301 B_{0,3}(X) - 0.000756289 B_{1,3}(X) - 0.00017189 B_{2,3}(X) + 0.000430547 B_{3,3}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.754472, 0.761775\}$$

Intersection intervals with the x axis:

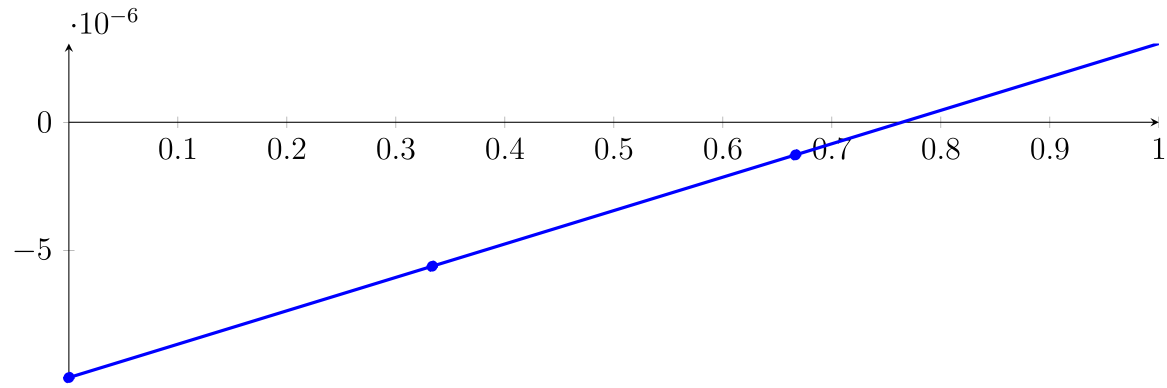
$$[0.754472, 0.761775]$$

Longest intersection interval: 0.00730249
$\Longrightarrow$ Selective recursion: interval 1: $[0.853514, 0.853566]$,

1.15 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.853514, 0.853566]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 1.4089 \cdot 10^{-13} X^3 + 2.87154 \cdot 10^{-09} X^2 + 1.30043 \cdot 10^{-05} X - 9.94098 \cdot 10^{-06}$$
$$= -9.94098 \cdot 10^{-06} B_{0,3}(X) - 5.60621 \cdot 10^{-06} B_{1,3}(X) - 1.27048 \cdot 10^{-06} B_{2,3}(X) + 3.0662 \cdot 10^{-06} B_{3,3}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.764269, 0.764321\}$$

Intersection intervals with the x axis:

$$[0.764269, 0.764321]$$

Longest intersection interval: $5.2035 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.853553, 0.853553]$,

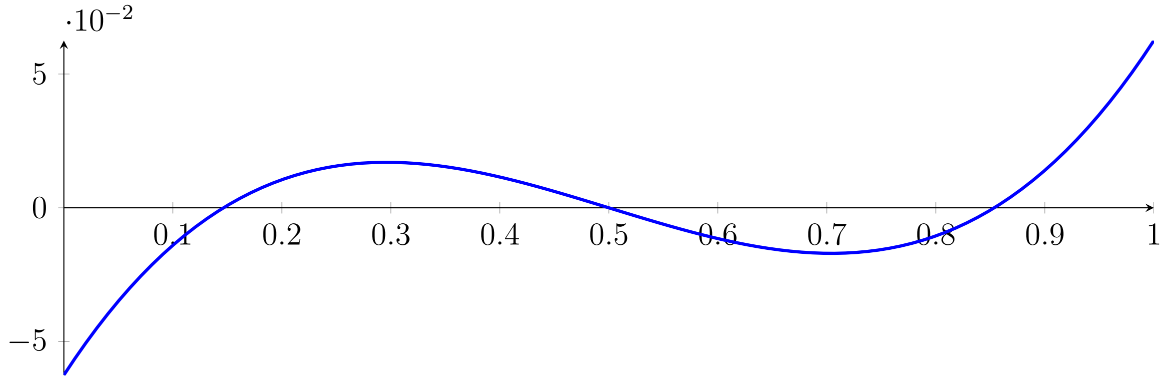
1.16 Recursion Branch 1 2 2 1 1 1 1 in Interval 1: $[0.853553, 0.853553]$

Found root in interval $[0.853553, 0.853553]$ at recursion depth 7!

1.17 Result: 3 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$



Result: Root Intervals

$$[0.146447, 0.146447],\ [0.5, 0.5],\ [0.853553, 0.853553]$$

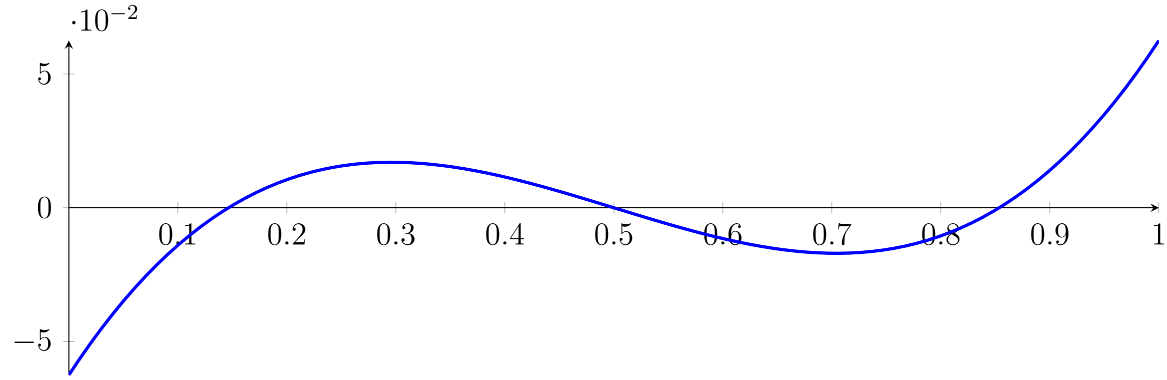
with precision $\varepsilon = 1 \cdot 10^{-06}$.

2 Running `QuadClip` on p3 with epsilon 6

$$1X^3 - 1.5X^2 + 0.625X - 0.0625$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

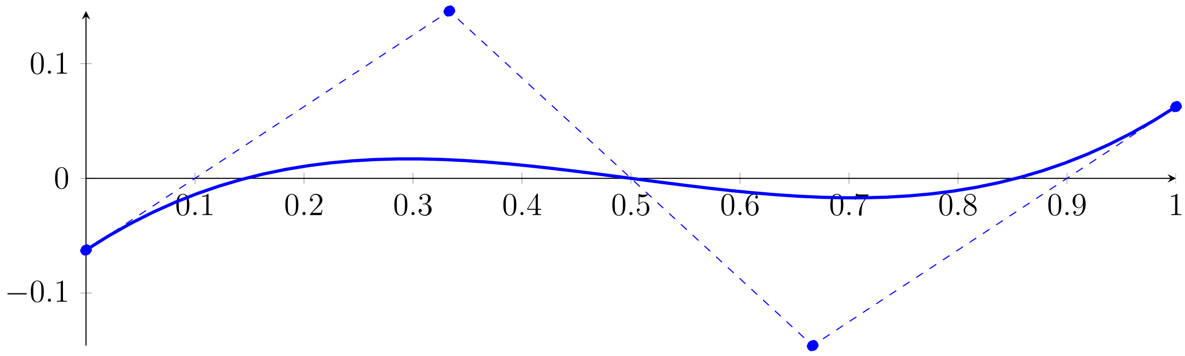
$$p = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$



2.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

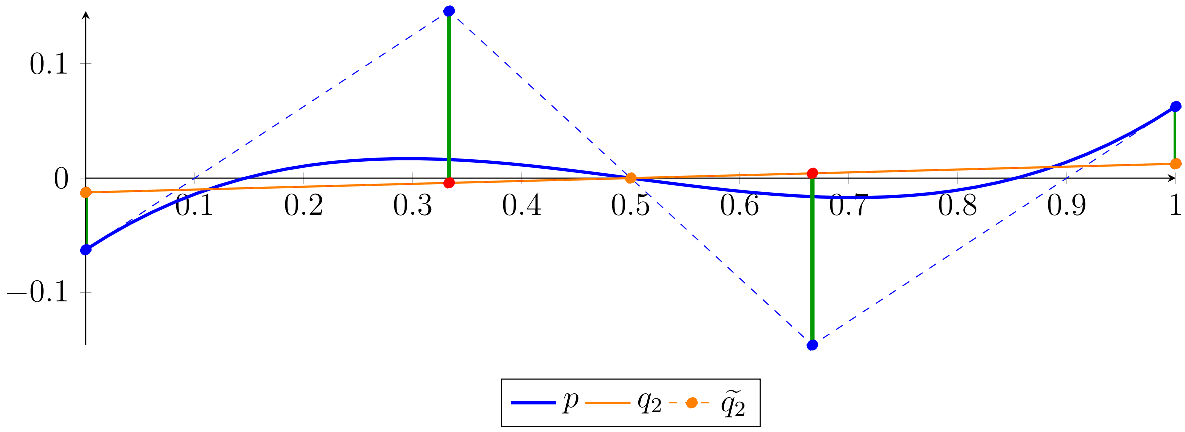
$$\begin{aligned} p &= 1X^3 - 1.5X^2 + 0.625X - 0.0625 \\ &= -0.0625B_{0,3}(X) + 0.145833B_{1,3}(X) - 0.145833B_{2,3}(X) + 0.0625B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.93124{\cdot}10^{-19}X^2 + 0.025X - 0.0125 \\ &= -0.0125B_{0,2} - 6.09864{\cdot}10^{-20}B_{1,2} + 0.0125B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.53313{\cdot}10^{-19}X^3 + 4.29446{\cdot}10^{-19}X^2 + 0.025X - 0.0125 \\ &= -0.0125B_{0,3} - 0.00416667B_{1,3} + 0.00416667B_{2,3} + 0.0125B_{3,3} \end{aligned}$$



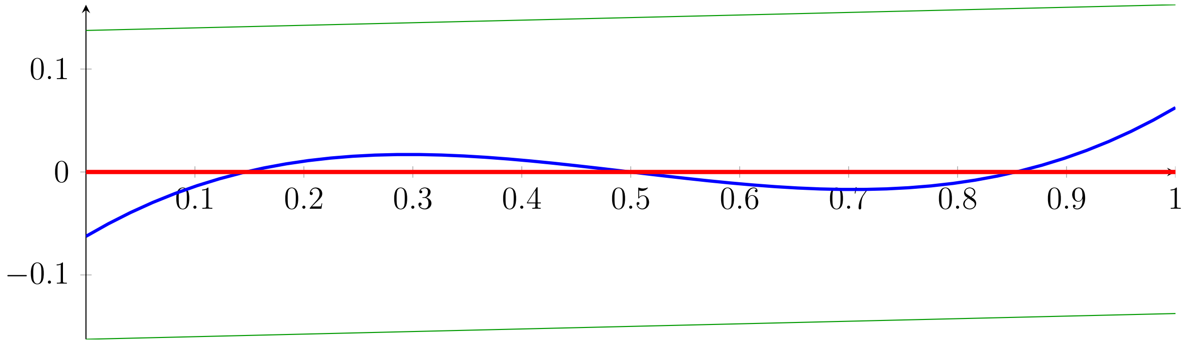
The maximum difference of the Bézier coefficients is $\delta = 0.15$.
Bounding polynomials M and m:

$$M = 1.89735 \cdot 10^{-19} X^2 + 0.025X + 0.1375$$
$$m = 1.89735 \cdot 10^{-19} X^2 + 0.025X - 0.1625$$

Root of M and m:

$$N(M) = \{-1.31762 \cdot 10^{17}, -5.83036\} \qquad N(m) = \{-1.31762 \cdot 10^{17}, 6.66964\}$$

Intersection intervals:



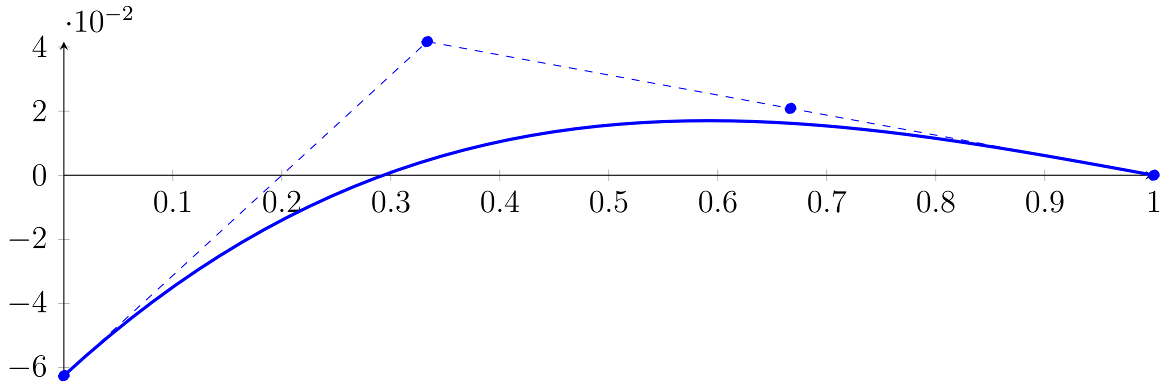
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$
Bisection point is very near to a root?!?

2.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

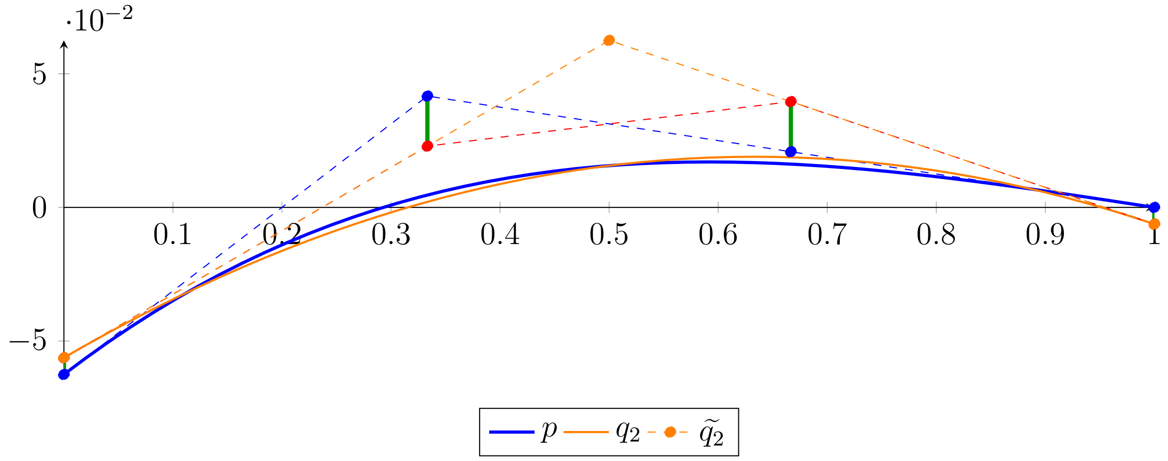
$$\begin{aligned} p &= 0.125X^3 - 0.375X^2 + 0.3125X - 0.0625 \\ &= -0.0625B_{0,3}(X) + 0.0416667B_{1,3}(X) + 0.0208333B_{2,3}(X) + 1.01644 \cdot 10^{-20} B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.1875X^2 + 0.2375X - 0.05625 \\ &= -0.05625B_{0,2} + 0.0625B_{1,2} - 0.00625B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -9.96111 \cdot 10^{-19} X^3 - 0.1875X^2 + 0.2375X - 0.05625 \\ &= -0.05625B_{0,3} + 0.0229167B_{1,3} + 0.0395833B_{2,3} - 0.00625B_{3,3} \end{aligned}$$



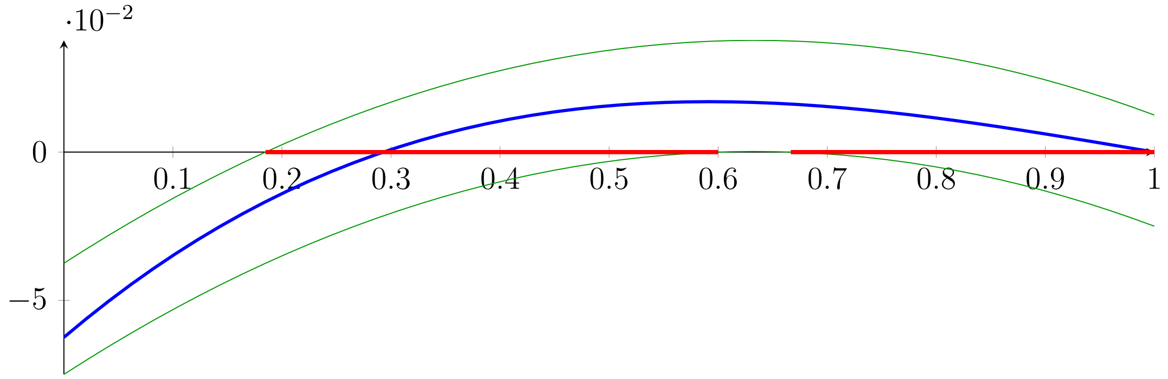
The maximum difference of the Bézier coefficients is $\delta = 0.01875$.
Bounding polynomials M and m:

$$M = -0.1875X^2 + 0.2375X - 0.0375$$
$$m = -0.1875X^2 + 0.2375X - 0.075$$

Root of M and m:

$$N(M) = \{0.184879, 1.08179\} \qquad N(m) = \{0.6, 0.666667\}$$

Intersection intervals:



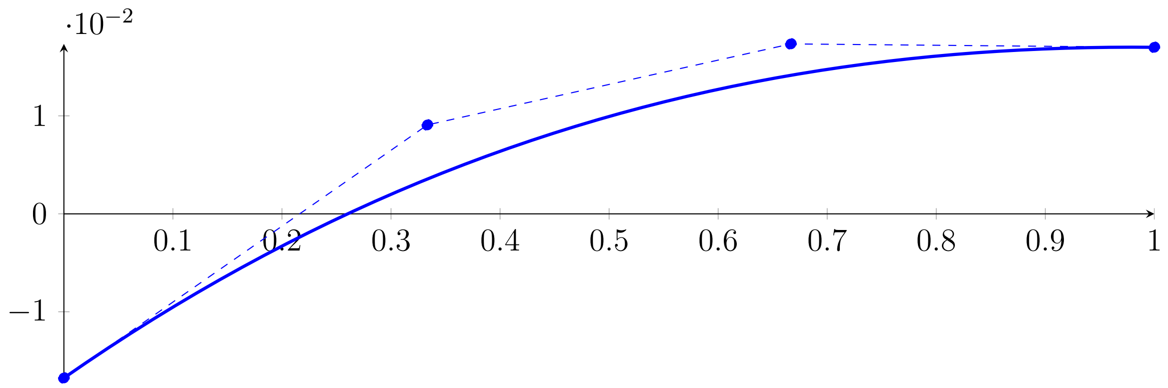
$$[0.184879, 0.6],\ [0.666667, 1]$$

Longest intersection interval: 0.415121
$\Longrightarrow$ Selective recursion: interval 1: $[0.0924396, 0.3]$, interval 2: $[0.333333, 0.5]$,

2.3 Recursion Branch 1 1 1 in Interval 1: $[0.0924396, 0.3]$

Normalized monomial und Bézier representations and the Bézier polygon:

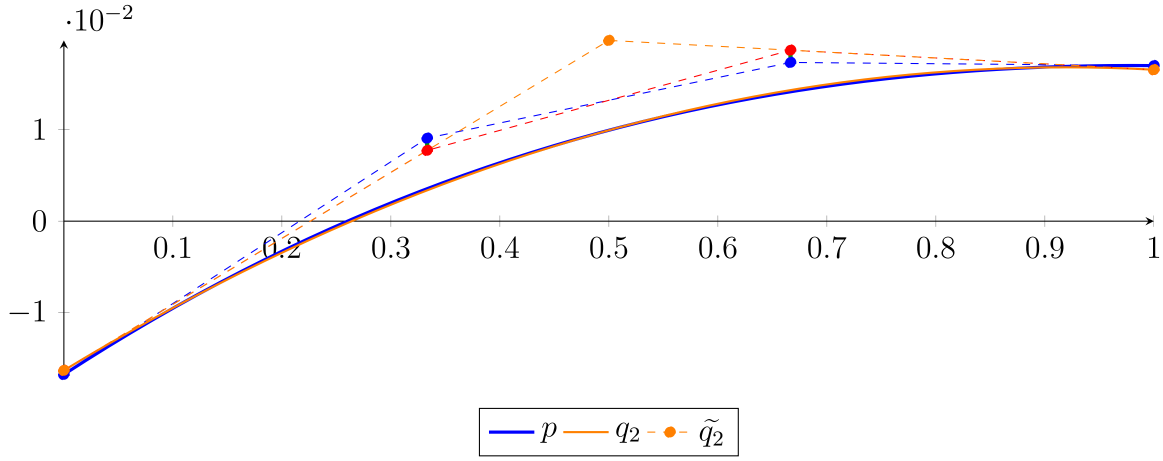
$$\begin{aligned} p &= 0.00894198X^3 - 0.0526747X^2 + 0.0774857X - 0.016753 \\ &= -0.016753B_{0,3}(X) + 0.0090756B_{1,3}(X) + 0.0173459B_{2,3}(X) + 0.017B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.0392618X^2 + 0.0721205X - 0.0163059 \\ &= -0.0163059B_{0,2} + 0.0197544B_{1,2} + 0.0165529B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -3.1679 \cdot 10^{-19}X^3 - 0.0392618X^2 + 0.0721205X - 0.0163059 \\ &= -0.0163059B_{0,3} + 0.00773431B_{1,3} + 0.0186872B_{2,3} + 0.0165529B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0013413$.
Bounding polynomials M and m:

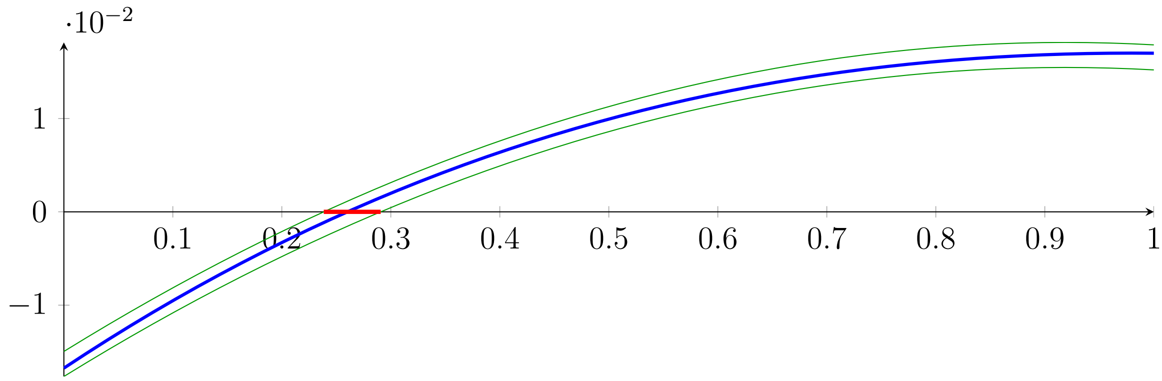
$$M = -0.0392618X^2 + 0.0721205X - 0.0149646$$

$$m = -0.0392618X^2 + 0.0721205X - 0.0176472$$

Root of M and m:

$$N(M) = \{0.238446, 1.59847\} \qquad N(m) = \{0.290692, 1.54622\}$$

Intersection intervals:



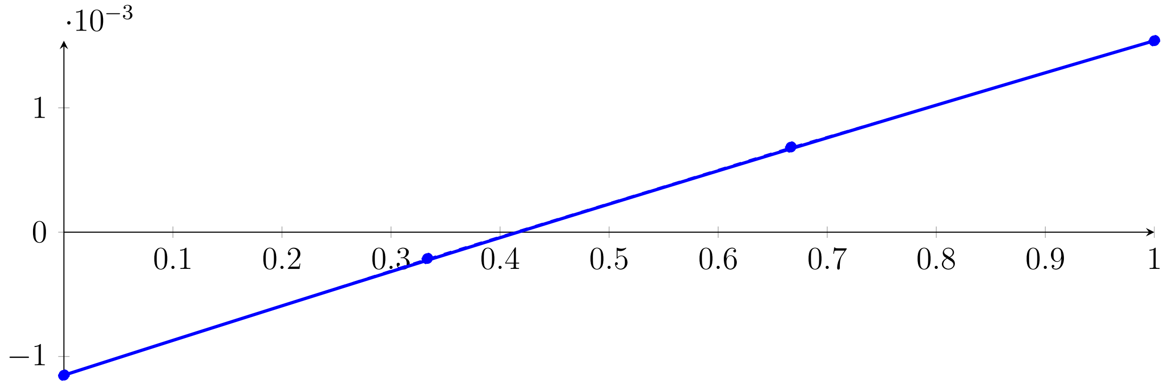
$$[0.238446, 0.290692]$$

Longest intersection interval: 0.0522458
$\Longrightarrow$ Selective recursion: interval 1: $[0.141932, 0.152776]$,

2.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.141932, 0.152776]$

Normalized monomial und Bézier representations and the Bézier polygon:

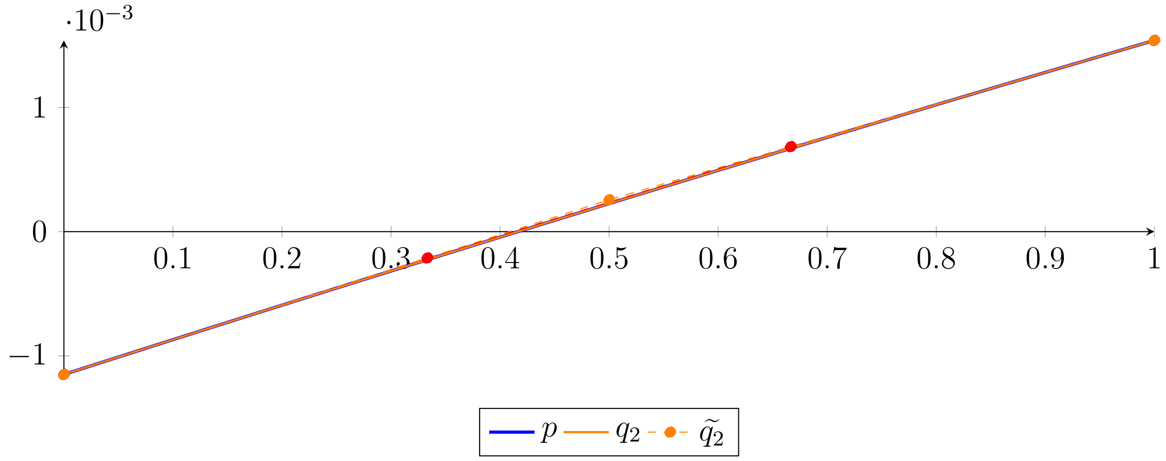
$$\begin{aligned} p &= 1.27523 \cdot 10^{-06} X^3 - 0.000126322 X^2 + 0.00281557 X - 0.00115047 \\ &= -0.00115047 B_{0,3}(X) - 0.000211952 B_{1,3}(X) + 0.000684462 B_{2,3}(X) + 0.00154004 B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.000124409 X^2 + 0.0028148 X - 0.00115041 \\ &= -0.00115041 B_{0,2} + 0.00025699 B_{1,2} + 0.00153998 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.60936 \cdot 10^{-20} X^3 - 0.000124409 X^2 + 0.0028148 X - 0.00115041 \\ &= -0.00115041 B_{0,3} - 0.000212143 B_{1,3} + 0.000684654 B_{2,3} + 0.00153998 B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.91284 \cdot 10^{-07}$.
Bounding polynomials M and m:

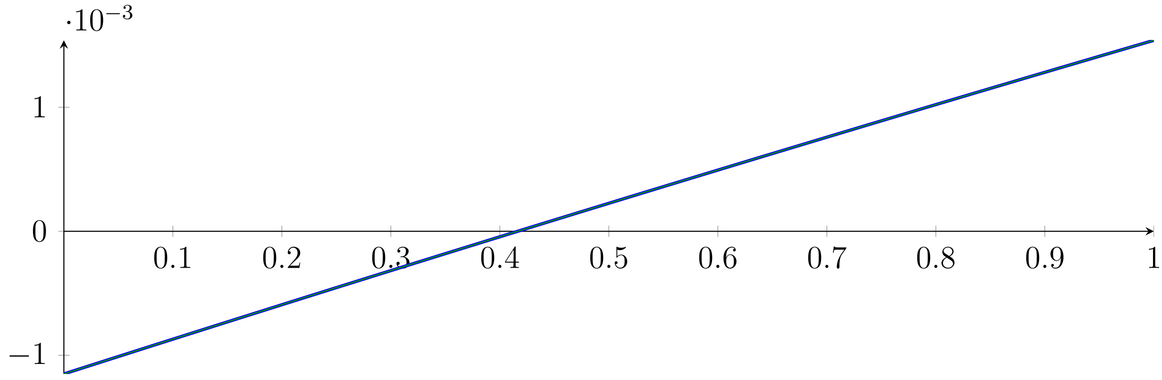
$$M = -0.000124409 X^2 + 0.0028148 X - 0.00115022$$

$$m = -0.000124409 X^2 + 0.0028148 X - 0.0011506$$

Root of M and m:

$$N(M) = \{0.416292, 22.2091\} \qquad N(m) = \{0.416433, 22.2089\}$$

Intersection intervals:



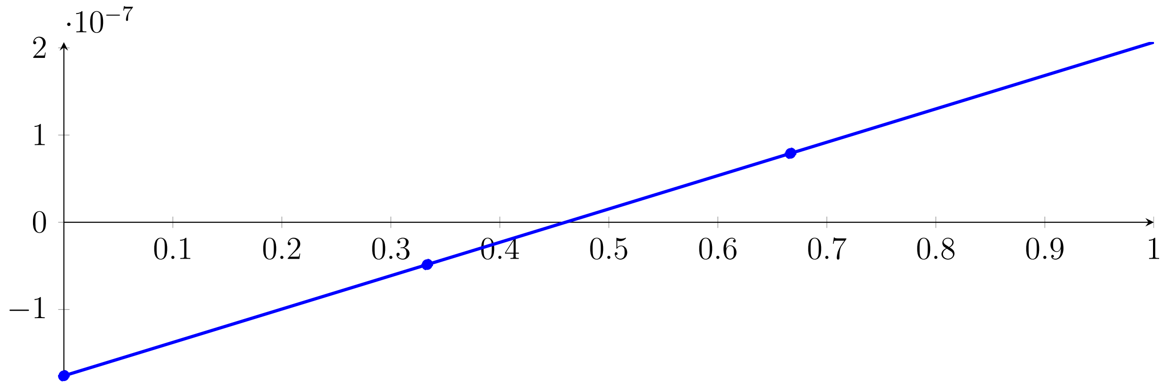
$[0.416292, 0.416433]$

Longest intersection interval: 0.000141106
$\Longrightarrow$ Selective recursion: interval 1: $[0.146446, 0.146447]$,

2.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.146446, 0.146447]$

Normalized monomial und Bézier representations and the Bézier polygon:

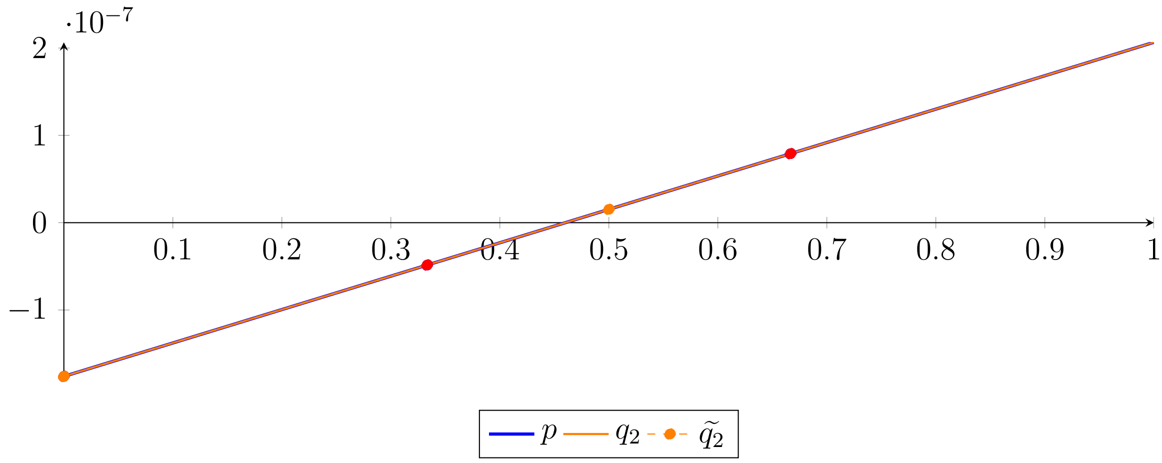
$$\begin{aligned} p &= 3.58283{\cdot}10^{-18}X^3 - 2.48349{\cdot}10^{-12}X^2 + 3.82547{\cdot}10^{-07}X - 1.7602{\cdot}10^{-07} \\ &= -1.7602{\cdot}10^{-07}B_{0,3}(X) - 4.85042{\cdot}10^{-08}B_{1,3}(X) + 7.90107{\cdot}10^{-08}B_{2,3}(X) + 2.06525{\cdot}10^{-07}B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.48348{\cdot}10^{-12}X^2 + 3.82547{\cdot}10^{-07}X - 1.7602{\cdot}10^{-07} \\ &= -1.7602{\cdot}10^{-07}B_{0,2} + 1.52536{\cdot}10^{-08}B_{1,2} + 2.06525{\cdot}10^{-07}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -2.3006{\cdot}10^{-24}X^3 - 2.48348{\cdot}10^{-12}X^2 + 3.82547{\cdot}10^{-07}X - 1.7602{\cdot}10^{-07} \\ &= -1.7602{\cdot}10^{-07}B_{0,3} - 4.85042{\cdot}10^{-08}B_{1,3} + 7.90107{\cdot}10^{-08}B_{2,3} + 2.06525{\cdot}10^{-07}B_{3,3} \end{aligned}$$



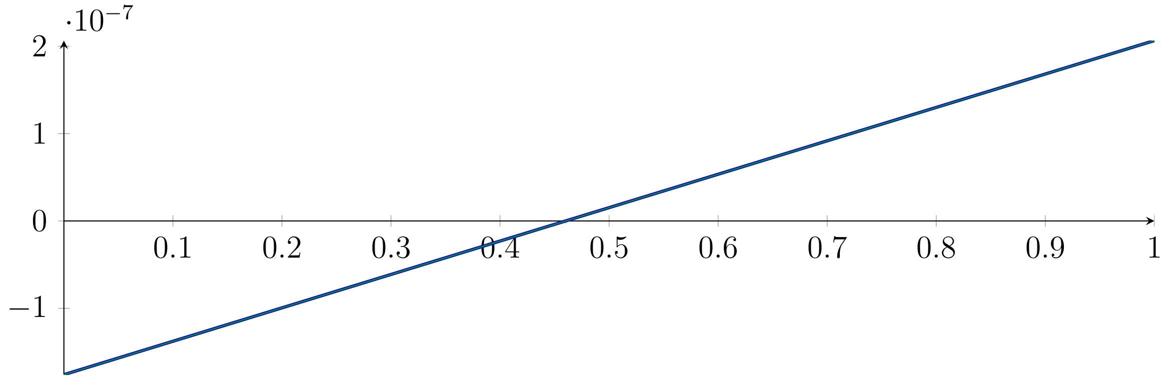
The maximum difference of the Bézier coefficients is $\delta = 5.37426 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -2.48348 \cdot 10^{-12} X^2 + 3.82547 \cdot 10^{-07} X - 1.7602 \cdot 10^{-07}$$
$$m = -2.48348 \cdot 10^{-12} X^2 + 3.82547 \cdot 10^{-07} X - 1.7602 \cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{0.460127, 154036\} \qquad N(m) = \{0.460127, 154036\}$$

Intersection intervals:



$$[0.460127, 0.460127]$$

Longest intersection interval: $2.8127 \cdot 10^{-12}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.146447, 0.146447]$,

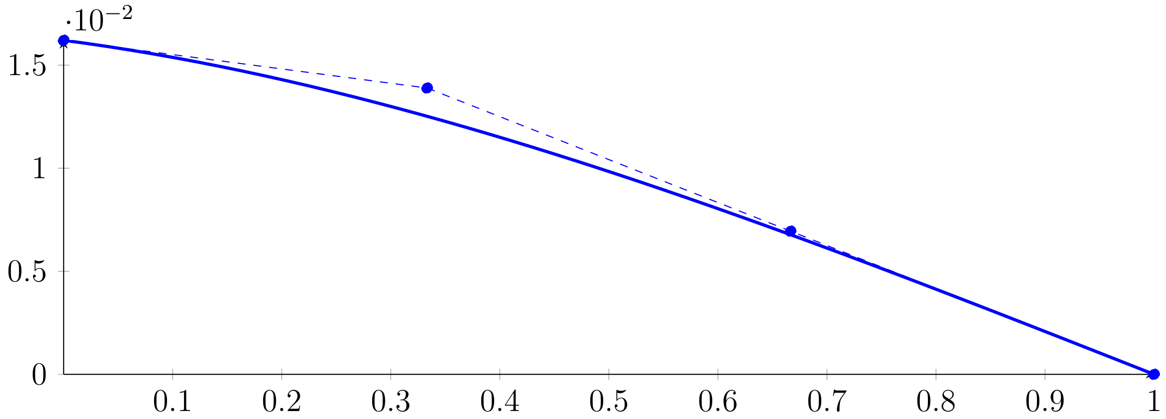
2.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.146447, 0.146447]$

Found root in interval $[0.146447, 0.146447]$ at recursion depth 6!

2.7 Recursion Branch 1 1 2 in Interval 2: $[0.333333, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

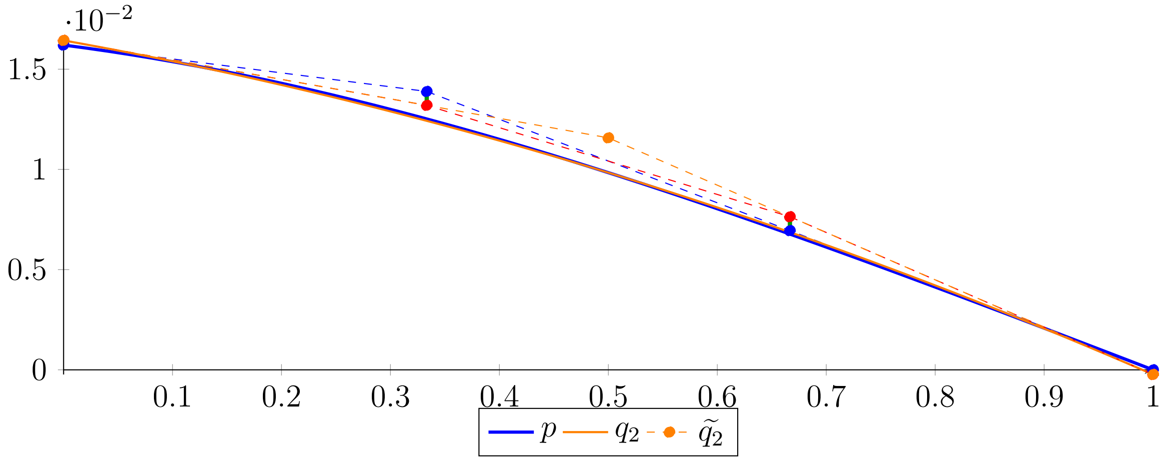
$$\begin{aligned} p &= 0.00462963 X^3 - 0.0138889 X^2 - 0.00694444 X + 0.0162037 \\ &= 0.0162037 B_{0,3}(X) + 0.0138889 B_{1,3}(X) + 0.00694444 B_{2,3}(X) + 1.01644 \cdot 10^{-20} B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00694444X^2 - 0.00972222X + 0.0164352 \\ &= 0.0164352B_{0,2} + 0.0115741B_{1,2} - 0.000231481B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.20279 \cdot 10^{-19}X^3 - 0.00694444X^2 - 0.00972222X + 0.0164352 \\ &= 0.0164352B_{0,3} + 0.0131944B_{1,3} + 0.00763889B_{2,3} - 0.000231481B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000694444$.
Bounding polynomials M and m:

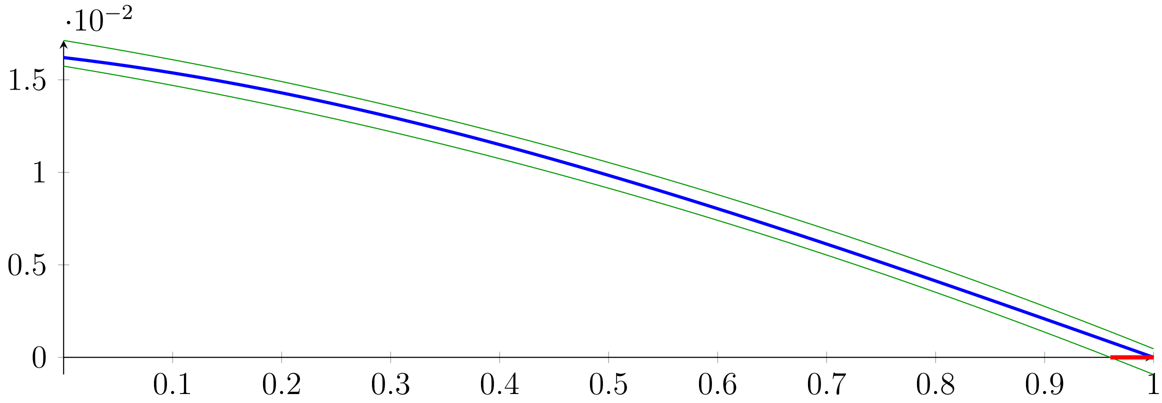
$$M = -0.00694444X^2 - 0.00972222X + 0.0171296$$

$$m = -0.00694444X^2 - 0.00972222X + 0.0157407$$

Root of M and m:

$$N(M) = \{-2.4195, 1.0195\} \qquad N(m) = \{-2.36032, 0.960321\}$$

Intersection intervals:



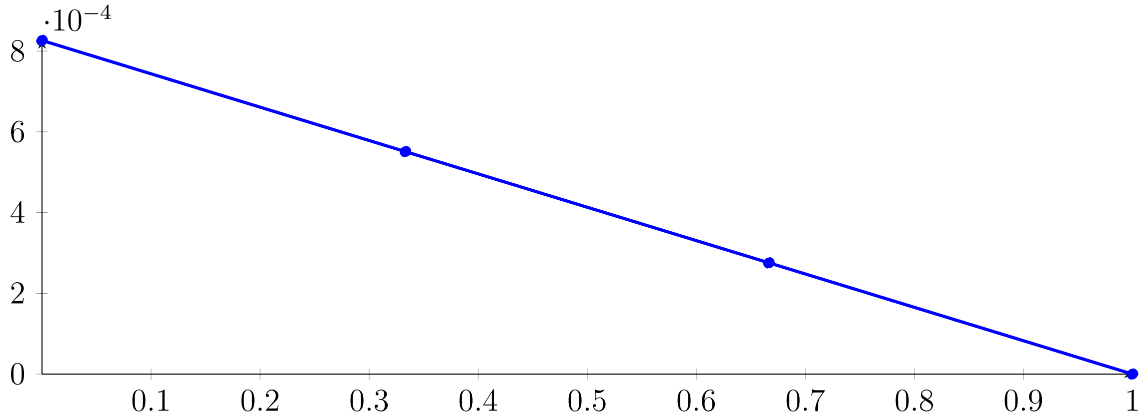
$$[0.960321, 1]$$

Longest intersection interval: 0.0396787
$\Longrightarrow$ Selective recursion: interval 1: $[0.493387, 0.5]$,

2.8 Recursion Branch 1 1 2 1 in Interval 1: $[0.493387, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

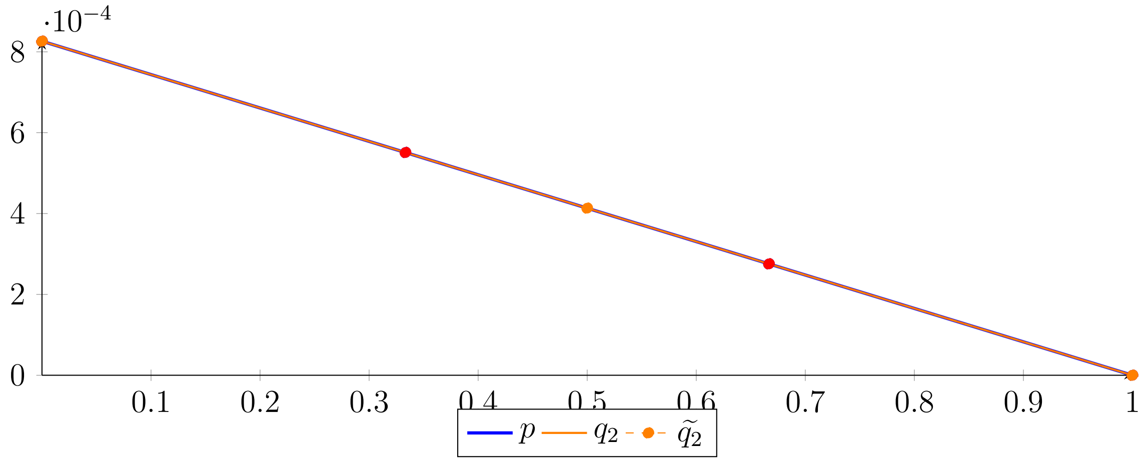
$$\begin{aligned} p &= 2.89214 \cdot 10^{-07} X^3 - 8.67643 \cdot 10^{-07} X^2 - 0.000825773 X + 0.000826351 \\ &= 0.000826351 B_{0,3}(X) + 0.000551094 B_{1,3}(X) + 0.000275547 B_{2,3}(X) + 1.01644 \cdot 10^{-20} B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -4.33822 \cdot 10^{-07} X^2 - 0.000825946 X + 0.000826366 \\ &= 0.000826366 B_{0,2} + 0.000413393 B_{1,2} - 1.44607 \cdot 10^{-08} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6.88214 \cdot 10^{-21} X^3 - 4.33822 \cdot 10^{-07} X^2 - 0.000825946 X + 0.000826366 \\ &= 0.000826366 B_{0,3} + 0.00055105 B_{1,3} + 0.00027559 B_{2,3} - 1.44607 \cdot 10^{-08} B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33822 \cdot 10^{-08}$.
Bounding polynomials M and m:

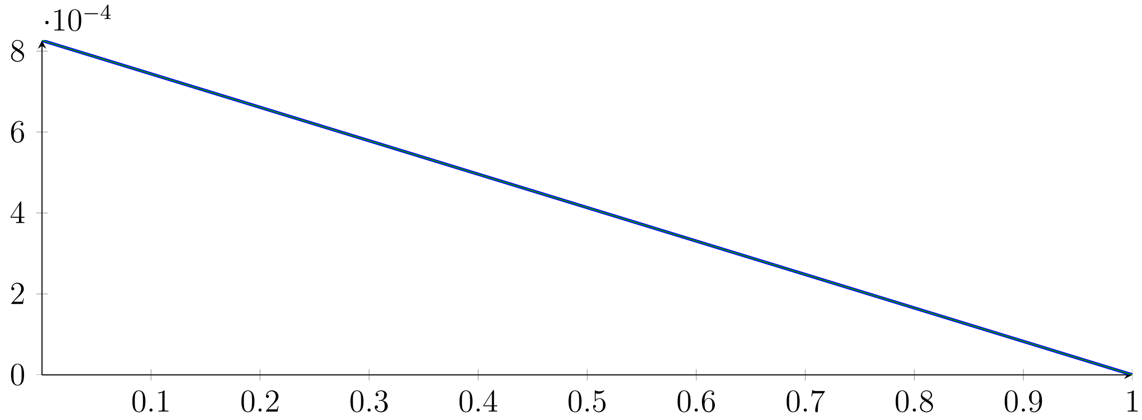
$$M = -4.33822 \cdot 10^{-07} X^2 - 0.000825946 X + 0.000826409$$

$$m = -4.33822 \cdot 10^{-07} X^2 - 0.000825946 X + 0.000826322$$

Root of M and m:

$$N(M) = \{-1904.88, 1.00003\} \qquad N(m) = \{-1904.88, 0.99993\}$$

Intersection intervals:



$[0.99993, 1]$

Longest intersection interval: $6.99588 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

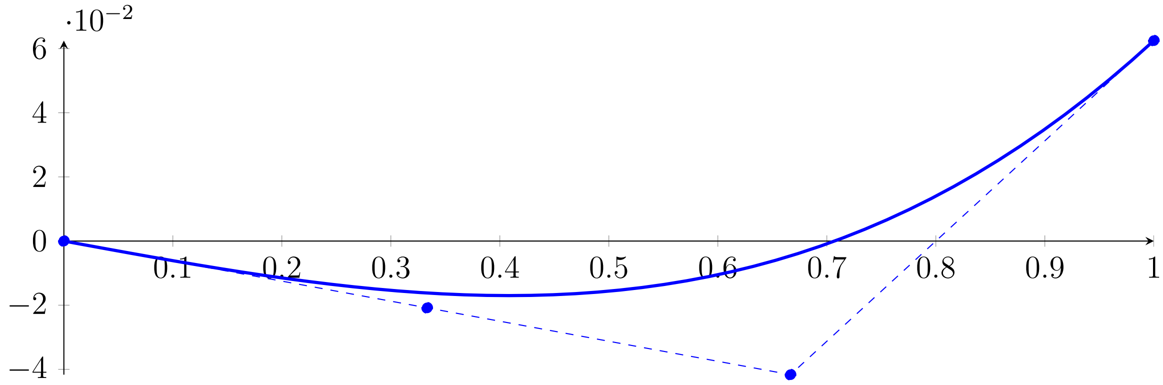
2.9 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

2.10 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

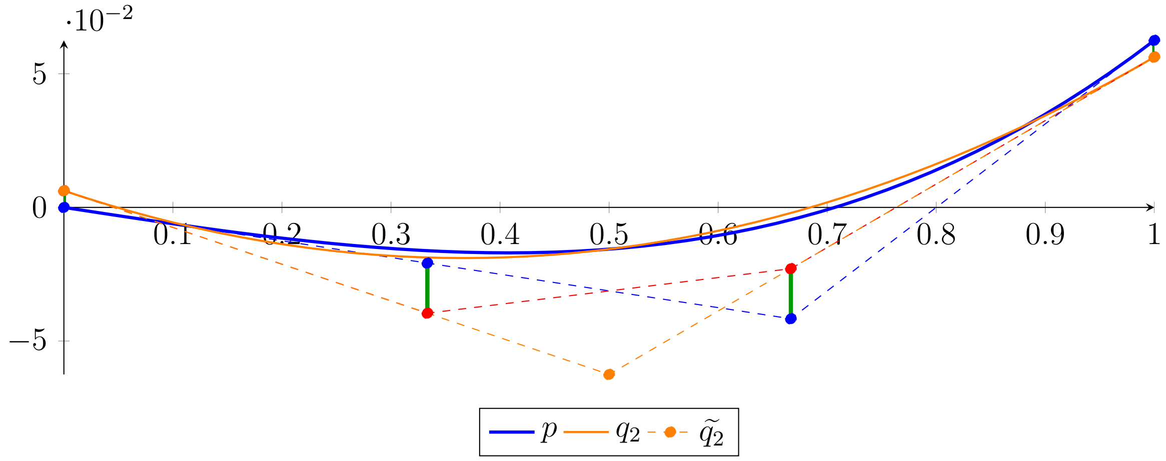
$$\begin{aligned} p &= 0.125X^3 - 1.01644 \cdot 10^{-20} X^2 - 0.0625X + 1.01644 \cdot 10^{-20} \\ &= 1.01644 \cdot 10^{-20} B_{0,3}(X) - 0.0208333 B_{1,3}(X) - 0.0416667 B_{2,3}(X) + 0.0625 B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.1875X^2 - 0.1375X + 0.00625 \\ &= 0.00625 B_{0,2} - 0.0625 B_{1,2} + 0.05625 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 3.6973 \cdot 10^{-19} X^3 + 0.1875X^2 - 0.1375X + 0.00625 \\ &= 0.00625 B_{0,3} - 0.0395833 B_{1,3} - 0.0229167 B_{2,3} + 0.05625 B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.01875$.

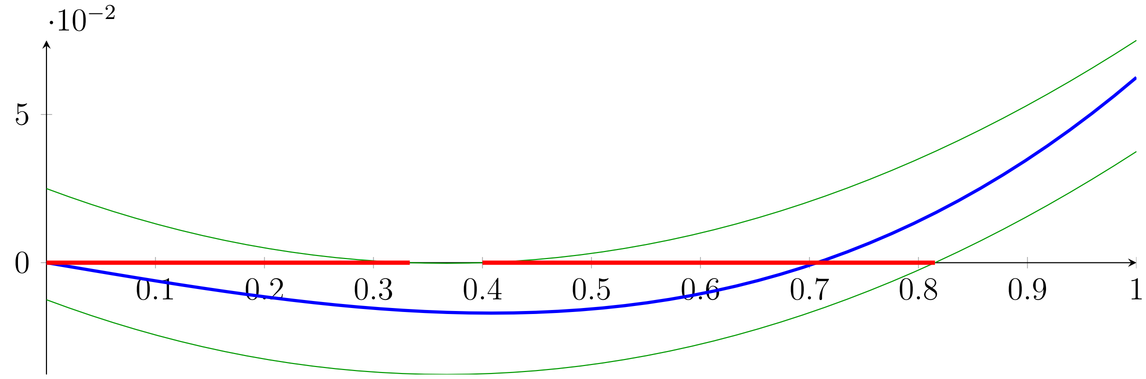
Bounding polynomials M and m:

$$M = 0.1875X^2 - 0.1375X + 0.025$$
$$m = 0.1875X^2 - 0.1375X - 0.0125$$

Root of M and m:

$$N(M) = \{0.333333, 0.4\} \qquad N(m) = \{-0.0817875, 0.815121\}$$

Intersection intervals:



$$[0, 0.333333],\ [0.4, 0.815121]$$

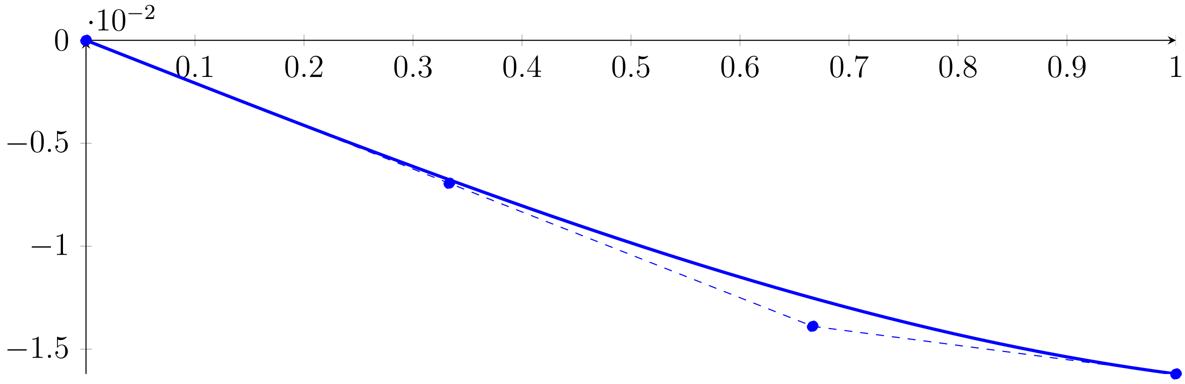
Longest intersection interval: 0.415121

$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.666667]$, interval 2: $[0.7, 0.90756]$,

2.11 Recursion Branch 1 2 1 in Interval 1: $[0.5, 0.666667]$

Normalized monomial und Bézier representations and the Bézier polygon:

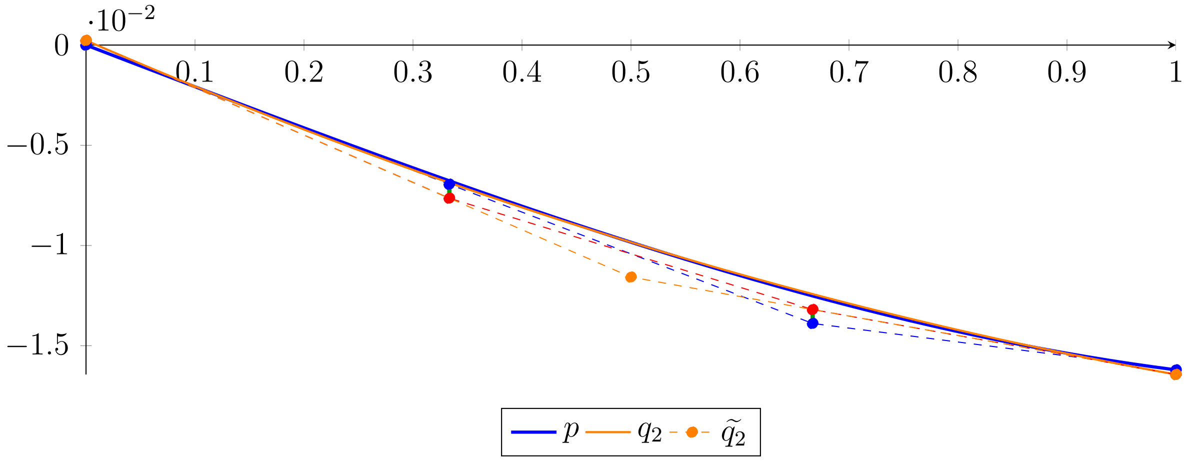
$$\begin{aligned} p &= 0.00462963X^3 - 0.0208333X + 1.01644 \cdot 10^{-20} \\ &= 1.01644 \cdot 10^{-20} B_{0,3}(X) - 0.00694444 B_{1,3}(X) - 0.0138889 B_{2,3}(X) - 0.0162037 B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00694444X^2 - 0.0236111X + 0.000231481 \\ &= 0.000231481B_{0,2} - 0.0115741B_{1,2} - 0.0164352B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 8.5868{\cdot}10^{-20}X^3 + 0.00694444X^2 - 0.0236111X + 0.000231481 \\ &= 0.000231481B_{0,3} - 0.00763889B_{1,3} - 0.0131944B_{2,3} - 0.0164352B_{3,3} \end{aligned}$$



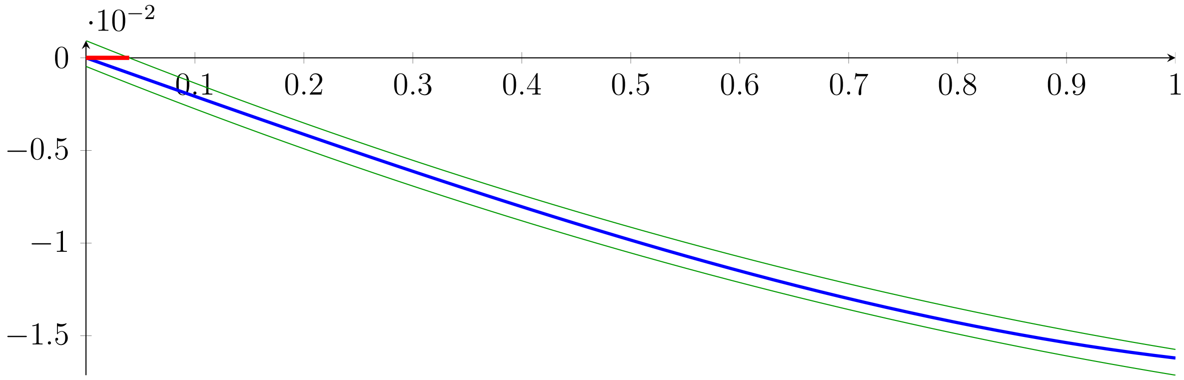
The maximum difference of the Bézier coefficients is $\delta = 0.000694444$.
Bounding polynomials M and m:

$$\begin{aligned} M &= 0.00694444X^2 - 0.0236111X + 0.000925926 \\ m &= 0.00694444X^2 - 0.0236111X - 0.000462963 \end{aligned}$$

Root of M and m:

$$N(M) = \{0.0396787, 3.36032\} \qquad N(m) = \{-0.0194961, 3.4195\}$$

Intersection intervals:



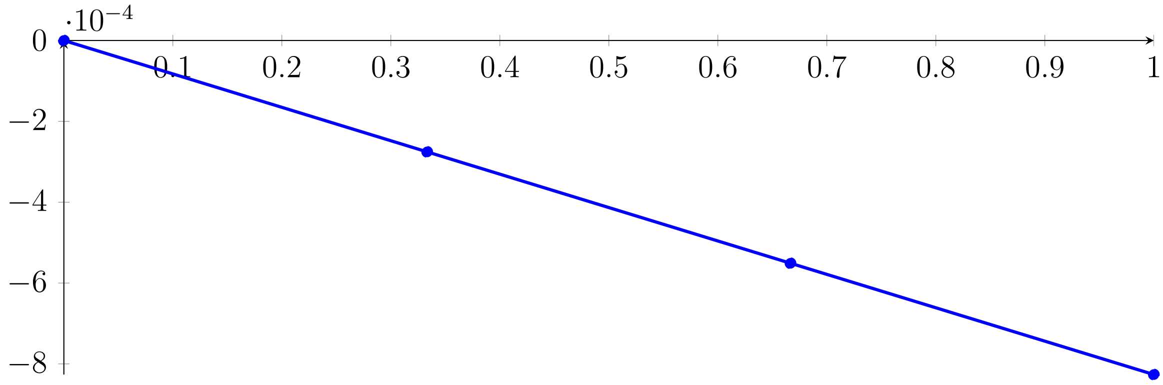
$$[0, 0.0396787]$$

Longest intersection interval: 0.0396787
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.506613]$,

2.12 Recursion Branch 1 2 1 1 in Interval 1: $[0.5, 0.506613]$

Normalized monomial und Bézier representations and the Bézier polygon:

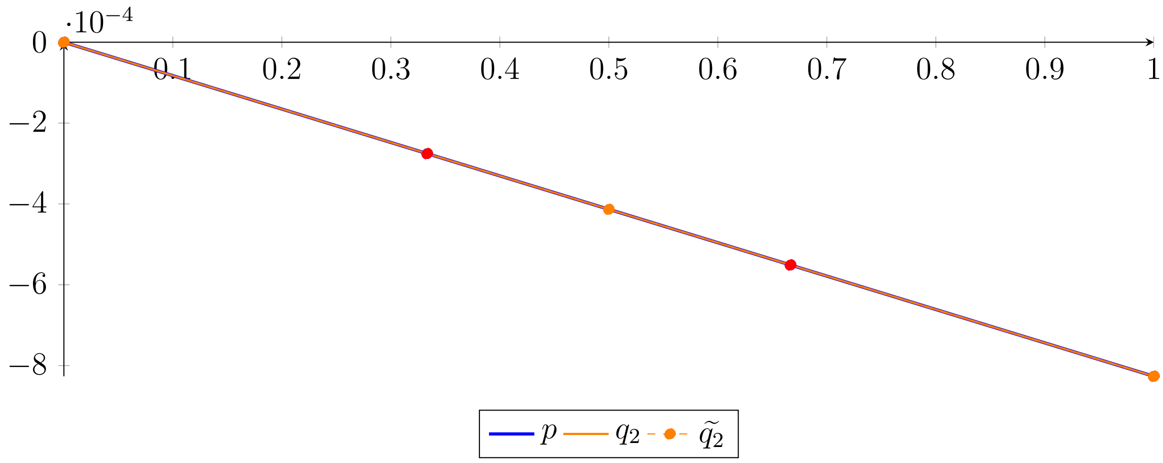
$$\begin{aligned} p &= 2.89214 \cdot 10^{-07} X^3 - 0.000826641 X + 1.01644 \cdot 10^{-20} \\ &= 1.01644 \cdot 10^{-20} B_{0,3}(X) - 0.000275547 B_{1,3}(X) - 0.000551094 B_{2,3}(X) - 0.000826351 B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 4.33822 \cdot 10^{-07} X^2 - 0.000826814 X + 1.44607 \cdot 10^{-08} \\ &= 1.44607 \cdot 10^{-08} B_{0,2} - 0.000413393 B_{1,2} - 0.000826366 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 3.27324 \cdot 10^{-21} X^3 + 4.33822 \cdot 10^{-07} X^2 - 0.000826814 X + 1.44607 \cdot 10^{-08} \\ &= 1.44607 \cdot 10^{-08} B_{0,3} - 0.00027559 B_{1,3} - 0.00055105 B_{2,3} - 0.000826366 B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33822 \cdot 10^{-08}$.

Bounding polynomials M and m:

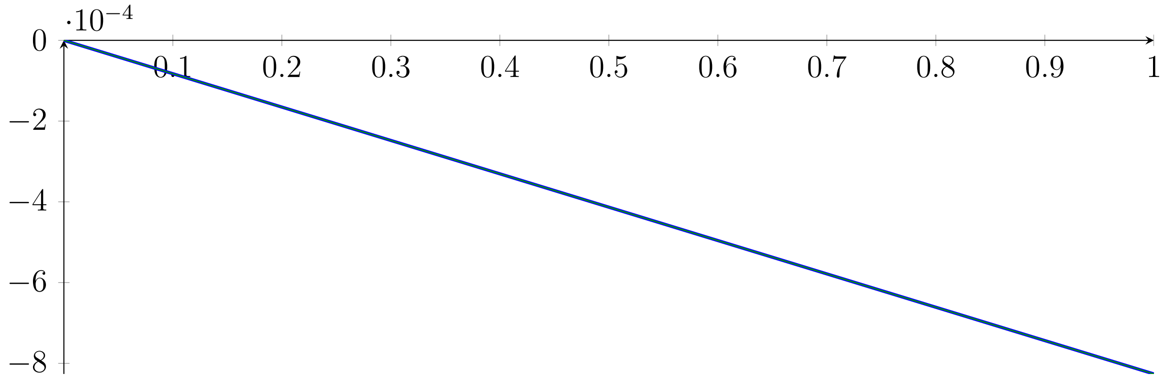
$$M = 4.33822 \cdot 10^{-07} X^2 - 0.000826814 X + 5.78429 \cdot 10^{-08}$$

$$m = 4.33822 \cdot 10^{-07} X^2 - 0.000826814 X - 2.89214 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{6.99588 \cdot 10^{-05}, 1905.88\} \qquad N(m) = \{-3.49794 \cdot 10^{-05}, 1905.88\}$$

Intersection intervals:



$[0, 6.99588e-05]$

Longest intersection interval: $6.99588 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

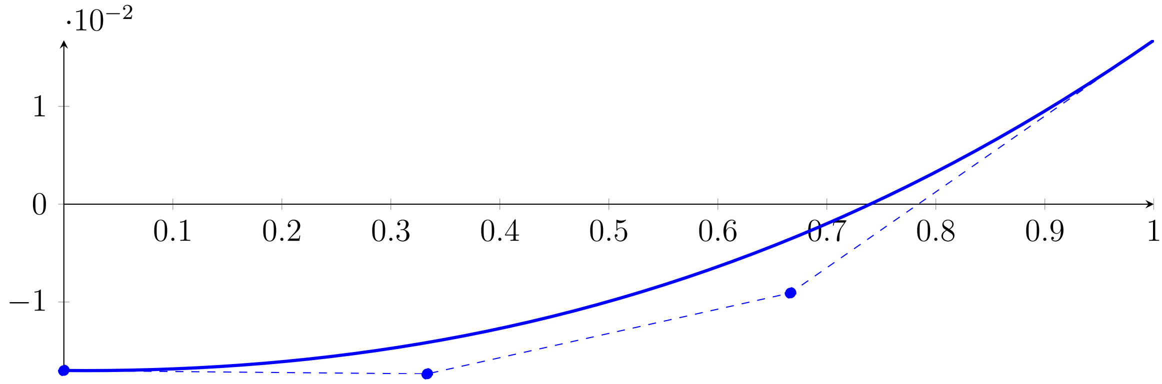
2.13 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

2.14 Recursion Branch 1 2 2 in Interval 2: $[0.7, 0.90756]$

Normalized monomial und Bézier representations and the Bézier polygon:

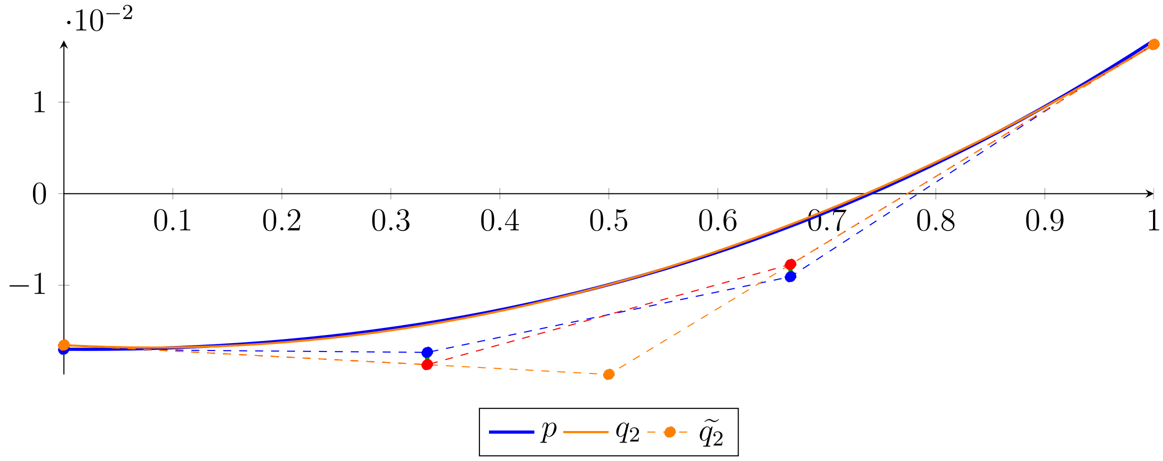
$$\begin{aligned} p &= 0.00894198X^3 + 0.0258488X^2 - 0.0010378X - 0.017 \\ &= -0.017B_{0,3}(X) - 0.0173459B_{1,3}(X) - 0.0090756B_{2,3}(X) + 0.016753B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0392618X^2 - 0.00640299X - 0.0165529 \\ &= -0.0165529B_{0,2} - 0.0197544B_{1,2} + 0.0163059B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -9.31736 \cdot 10^{-20}X^3 + 0.0392618X^2 - 0.00640299X - 0.0165529 \\ &= -0.0165529B_{0,3} - 0.0186872B_{1,3} - 0.00773431B_{2,3} + 0.0163059B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0013413$.

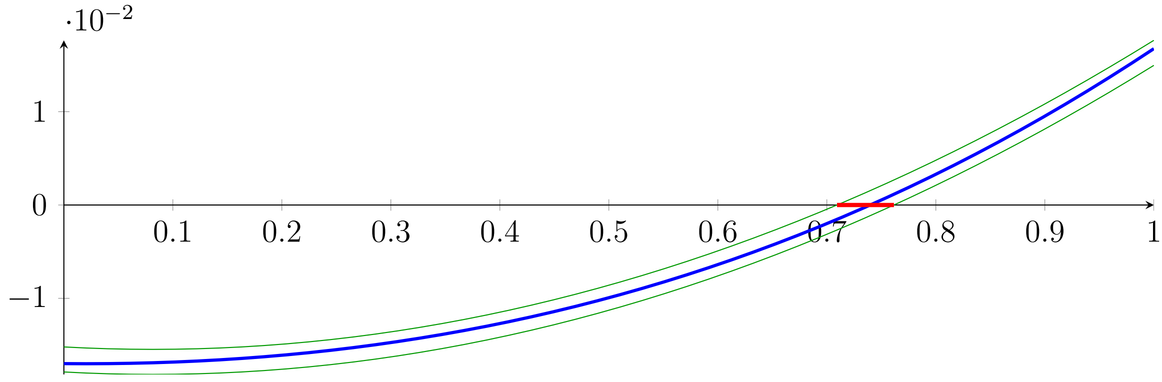
Bounding polynomials M and m:

$$M = 0.0392618X^2 - 0.00640299X - 0.0152116$$
$$m = 0.0392618X^2 - 0.00640299X - 0.0178942$$

Root of M and m:

$$N(M) = \{-0.546224, 0.709308\} \qquad N(m) = \{-0.598469, 0.761554\}$$

Intersection intervals:



$$[0.709308, 0.761554]$$

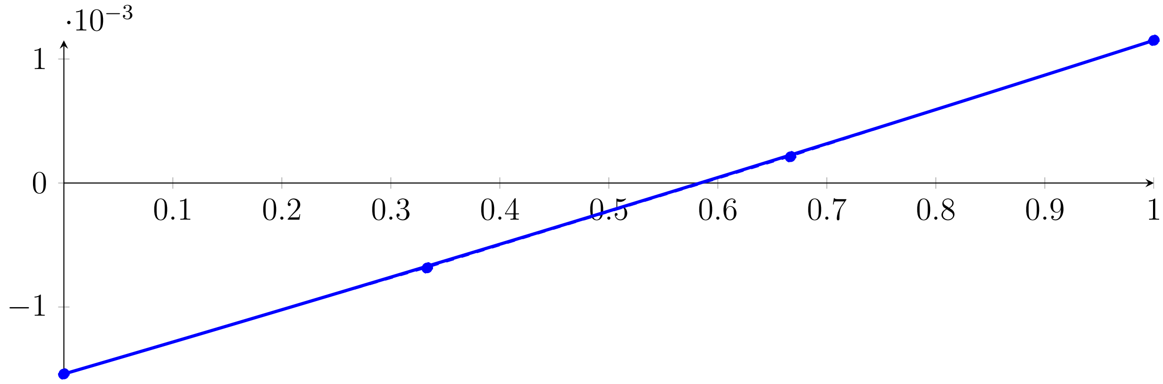
Longest intersection interval: 0.0522458

$\Longrightarrow$ Selective recursion: interval 1: $[0.847224, 0.858068]$,

2.15 Recursion Branch 1 2 2 1 in Interval 1: $[0.847224, 0.858068]$

Normalized monomial und Bézier representations and the Bézier polygon:

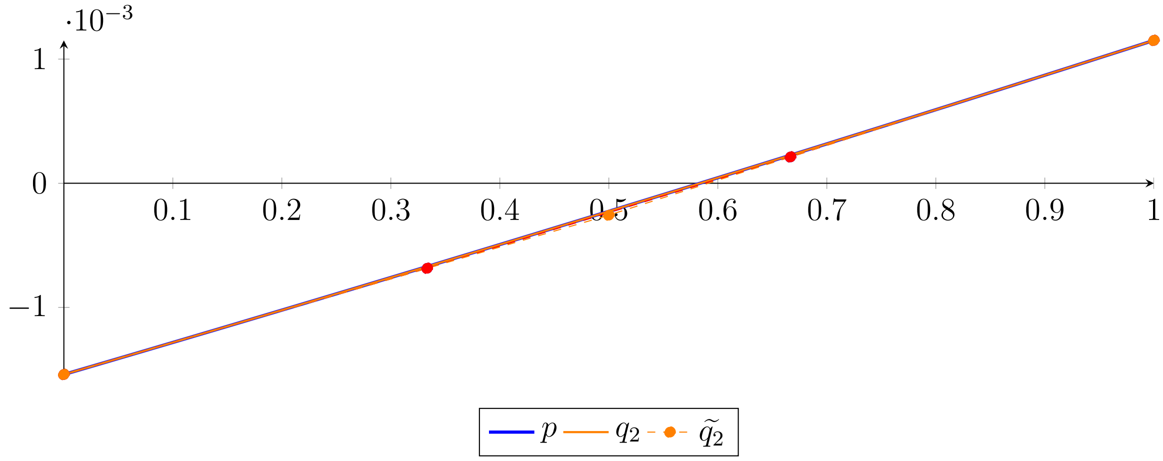
$$p = 1.27523 \cdot 10^{-06}X^3 + 0.000122496X^2 + 0.00256675X - 0.00154004$$
$$= -0.00154004B_{0,3}(X) - 0.000684462B_{1,3}(X) + 0.000211952B_{2,3}(X) + 0.00115047B_{3,3}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.000124409X^2 + 0.00256598X - 0.00153998 \\ &= -0.00153998B_{0,2} - 0.00025699B_{1,2} + 0.00115041B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.71524 \cdot 10^{-20}X^3 + 0.000124409X^2 + 0.00256598X - 0.00153998 \\ &= -0.00153998B_{0,3} - 0.000684654B_{1,3} + 0.000212143B_{2,3} + 0.00115041B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.91284 \cdot 10^{-07}$.
Bounding polynomials M and m:

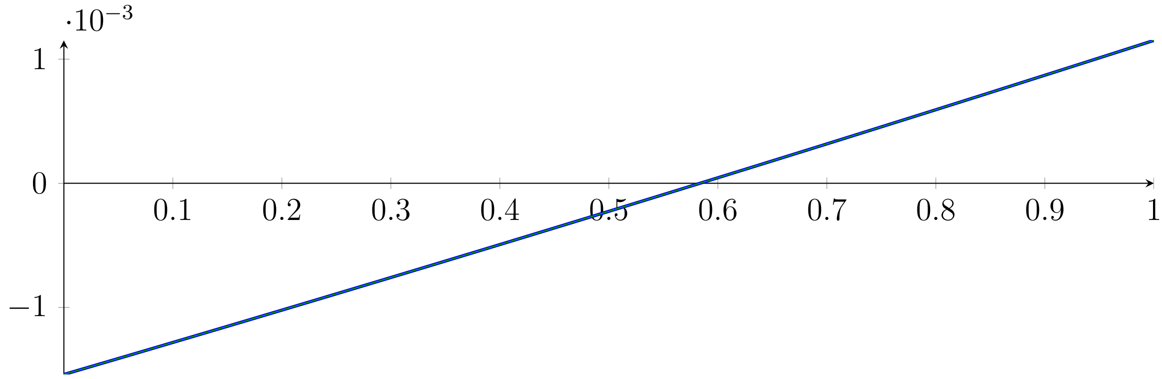
$$M = 0.000124409X^2 + 0.00256598X - 0.00153979$$

$$m = 0.000124409X^2 + 0.00256598X - 0.00154017$$

Root of M and m:

$$N(M) = \{-21.2089, 0.583567\} \qquad N(m) = \{-21.2091, 0.583708\}$$

Intersection intervals:



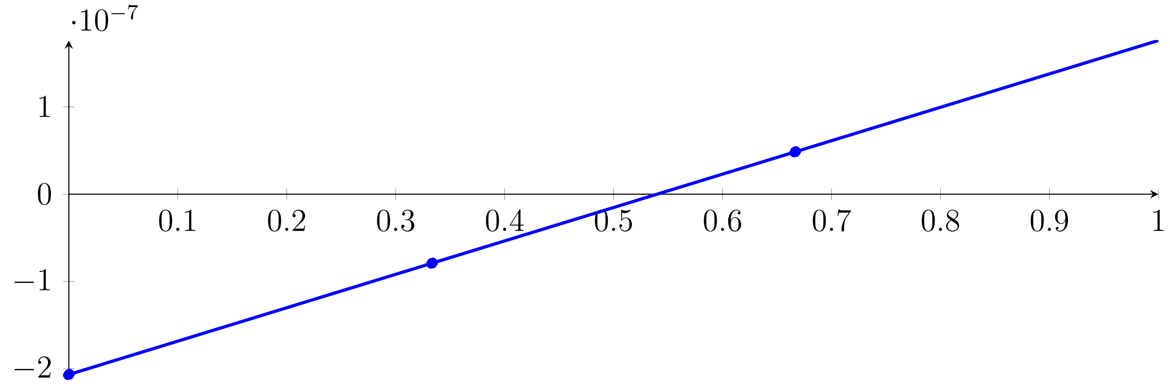
$$[0.583567, 0.583708]$$

Longest intersection interval: 0.000141106
$\Longrightarrow$ Selective recursion: interval 1: $[0.853553, 0.853554]$,

2.16 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.853553, 0.853554]$

Normalized monomial und Bézier representations and the Bézier polygon:

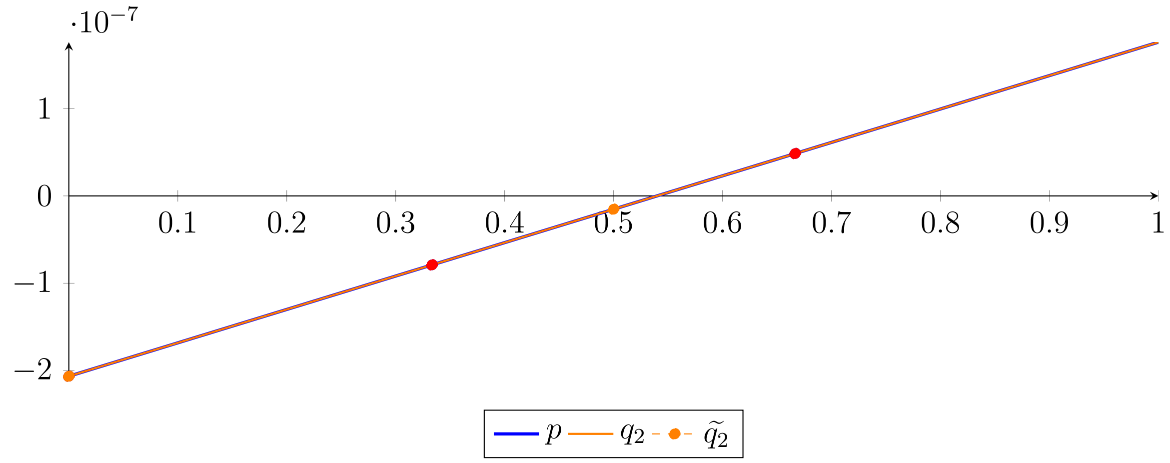
$$\begin{aligned} p &= 3.58283 \cdot 10^{-18} X^3 + 2.48347 \cdot 10^{-12} X^2 + 3.82542 \cdot 10^{-07} X - 2.06525 \cdot 10^{-07} \\ &= -2.06525 \cdot 10^{-07} B_{0,3}(X) - 7.90107 \cdot 10^{-08} B_{1,3}(X) + 4.85042 \cdot 10^{-08} B_{2,3}(X) + 1.7602 \cdot 10^{-07} B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 2.48348 \cdot 10^{-12} X^2 + 3.82542 \cdot 10^{-07} X - 2.06525 \cdot 10^{-07} \\ &= -2.06525 \cdot 10^{-07} B_{0,2} - 1.52536 \cdot 10^{-08} B_{1,2} + 1.7602 \cdot 10^{-07} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.42984 \cdot 10^{-24} X^3 + 2.48348 \cdot 10^{-12} X^2 + 3.82542 \cdot 10^{-07} X - 2.06525 \cdot 10^{-07} \\ &= -2.06525 \cdot 10^{-07} B_{0,3} - 7.90107 \cdot 10^{-08} B_{1,3} + 4.85042 \cdot 10^{-08} B_{2,3} + 1.7602 \cdot 10^{-07} B_{3,3} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.37426 \cdot 10^{-19}$.
Bounding polynomials M and m:

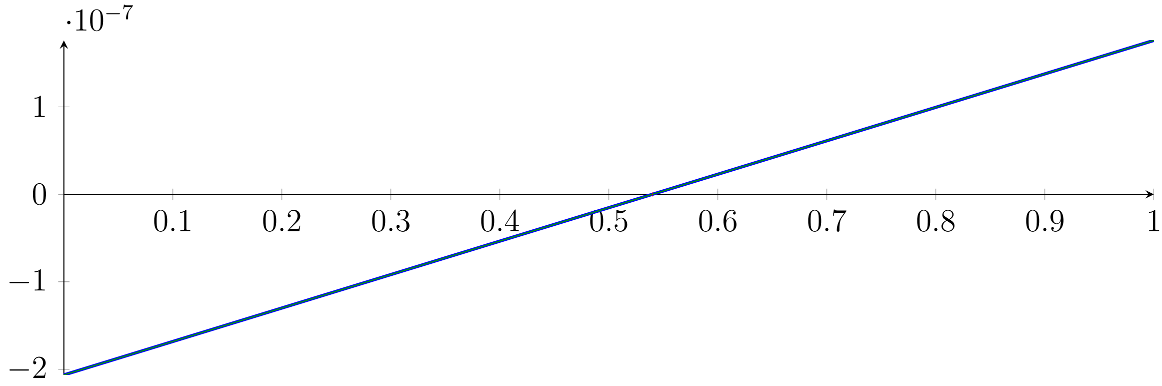
$$M = 2.48348 \cdot 10^{-12} X^2 + 3.82542 \cdot 10^{-07} X - 2.06525 \cdot 10^{-07}$$

$$m = 2.48348 \cdot 10^{-12} X^2 + 3.82542 \cdot 10^{-07} X - 2.06525 \cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{-154035, 0.539873\} \qquad N(m) = \{-154035, 0.539873\}$$

Intersection intervals:



$[0.539873, 0.539873]$

Longest intersection interval: $2.81551 \cdot 10^{-12}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.853553, 0.853553]$,

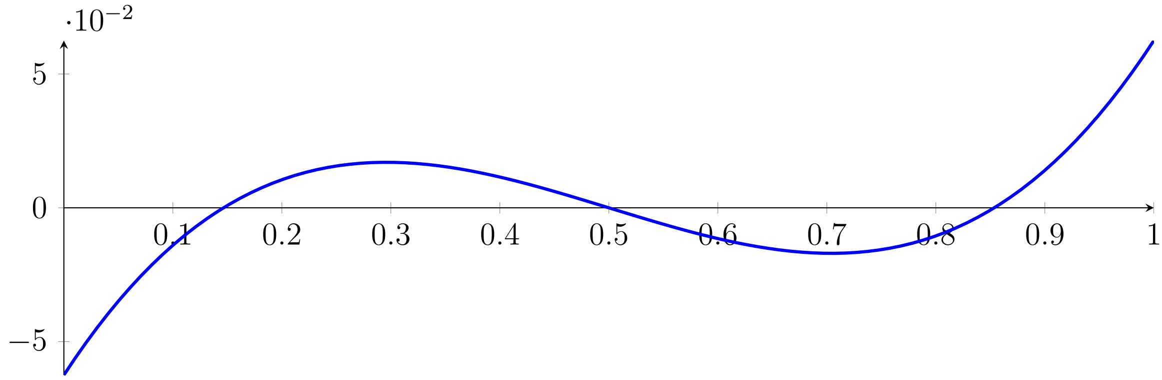
2.17 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.853553, 0.853553]$

Found root in interval $[0.853553, 0.853553]$ at recursion depth 6!

2.18 Result: 4 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$



Result: Root Intervals

$$[0.146447, 0.146447],\ [0.5, 0.5],\ [0.5, 0.5],\ [0.853553, 0.853553]$$

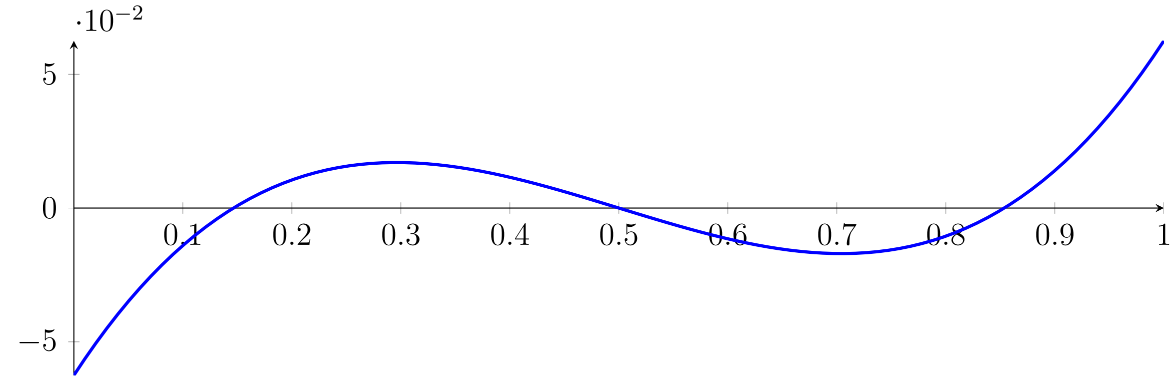
with precision $\varepsilon = 1{\cdot}10^{-06}$.

3 Running `CubeClip` on p3 with epsilon 6

$$1X^3 - 1.5X^2 + 0.625X - 0.0625$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

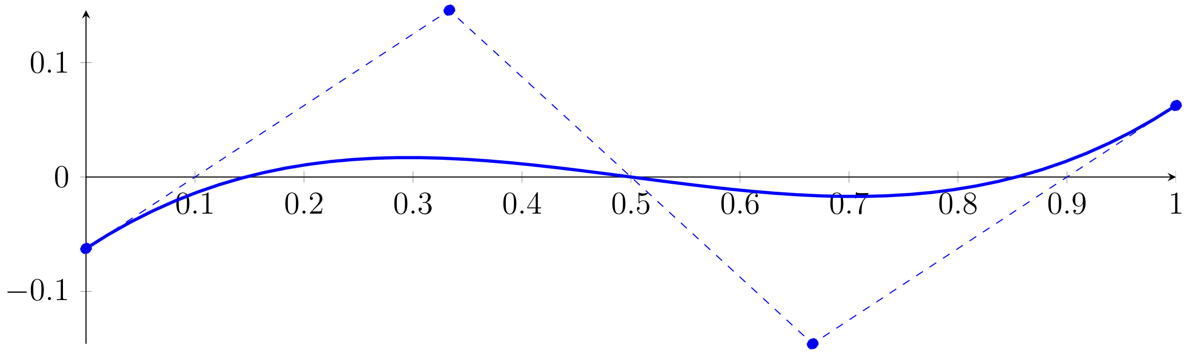
$$p = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$



3.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

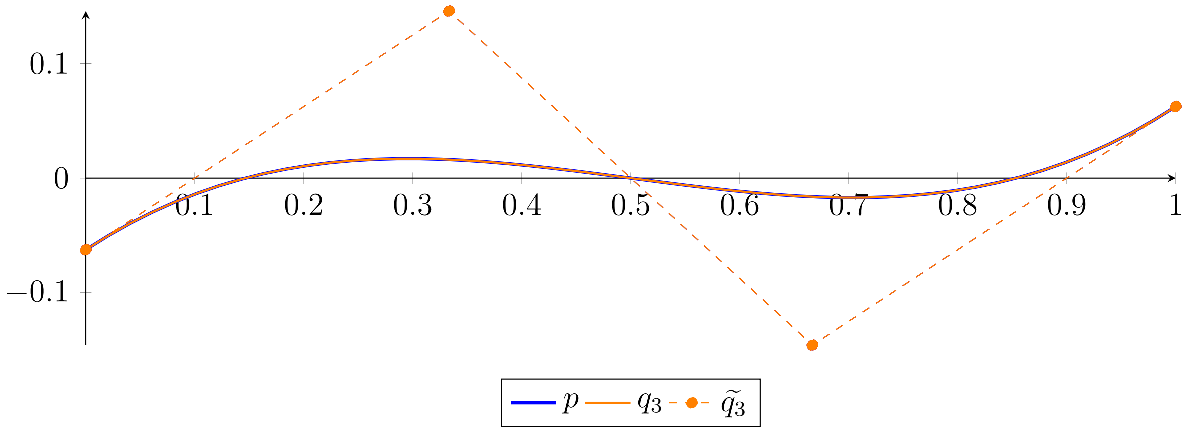
$$\begin{aligned} p &= 1X^3 - 1.5X^2 + 0.625X - 0.0625 \\ &= -0.0625B_{0,3}(X) + 0.145833B_{1,3}(X) - 0.145833B_{2,3}(X) + 0.0625B_{3,3}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1X^3 - 1.5X^2 + 0.625X - 0.0625 \\ &= -0.0625B_{0,3} + 0.145833B_{1,3} - 0.145833B_{2,3} + 0.0625B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 1X^3 - 1.5X^2 + 0.625X - 0.0625 \\ &= -0.0625B_{0,3} + 0.145833B_{1,3} - 0.145833B_{2,3} + 0.0625B_{3,3} \end{aligned}$$



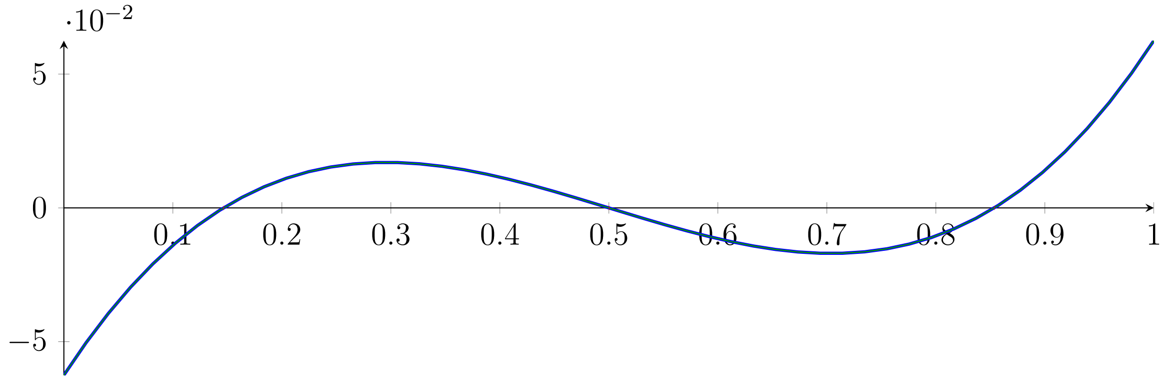
The maximum difference of the Bézier coefficients is $\delta = 1.87025 \cdot 10^{-18}$.
Bounding polynomials M and m:

$$M = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$
$$m = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$

Root of M and m:

$$N(M) = \{0.146447, 0.5, 0.853553\} \qquad N(m) = \{0.146447, 0.5, 0.853553\}$$

Intersection intervals:



$$[0.146447, 0.146447], \; [0.5, 0.5], \; [0.853553, 0.853553]$$

Longest intersection interval: $3.02221 \cdot 10^{-17}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.146447, 0.146447]$, interval 2: $[0.5, 0.5]$, interval 3: $[0.853553, 0.853553]$,

3.2 Recursion Branch 1 1 in Interval 1: $[0.146447, 0.146447]$

Found root in interval $[0.146447, 0.146447]$ at recursion depth 2!

3.3 Recursion Branch 1 2 in Interval 2: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 2!

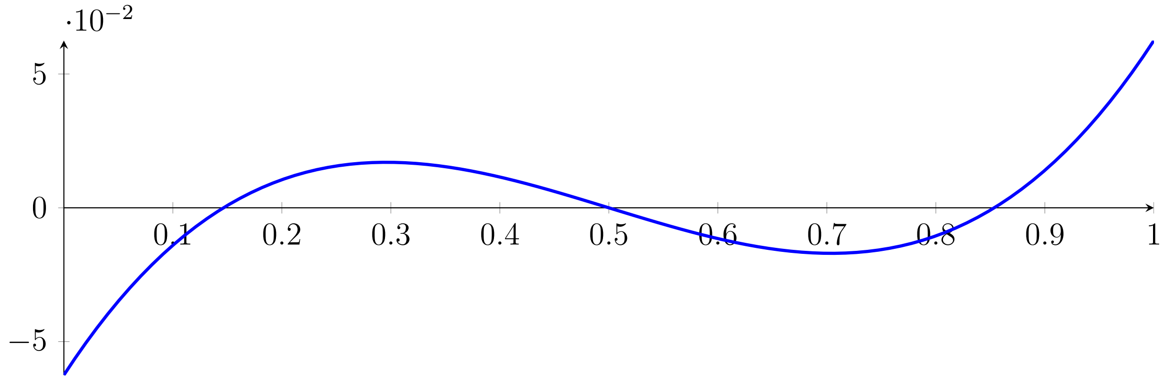
3.4 Recursion Branch 1 3 in Interval 3: $[0.853553, 0.853553]$

Found root in interval $[0.853553, 0.853553]$ at recursion depth 2!

3.5 Result: 3 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^3 - 1.5X^2 + 0.625X - 0.0625$$



Result: Root Intervals

$$[0.146447, 0.146447],\ [0.5, 0.5],\ [0.853553, 0.853553]$$

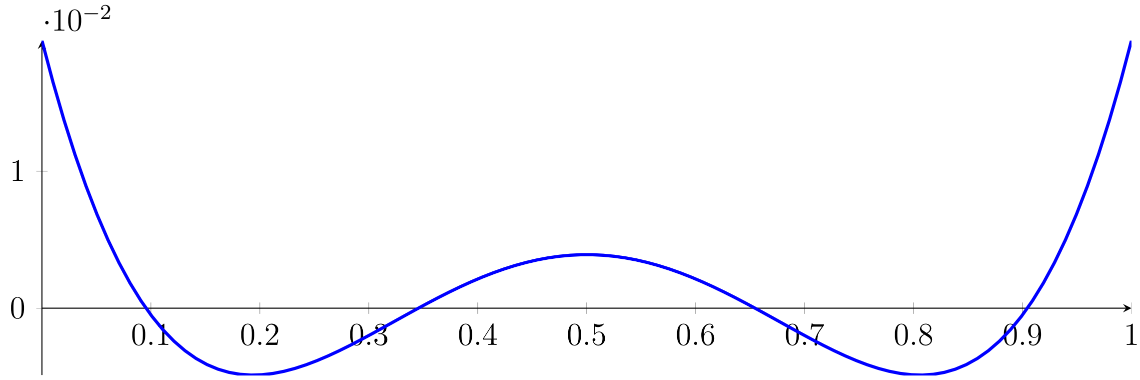
with precision $\varepsilon = 1 \cdot 10^{-06}$.

4 Running `BezClip` on p4 with epsilon 6

$$1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

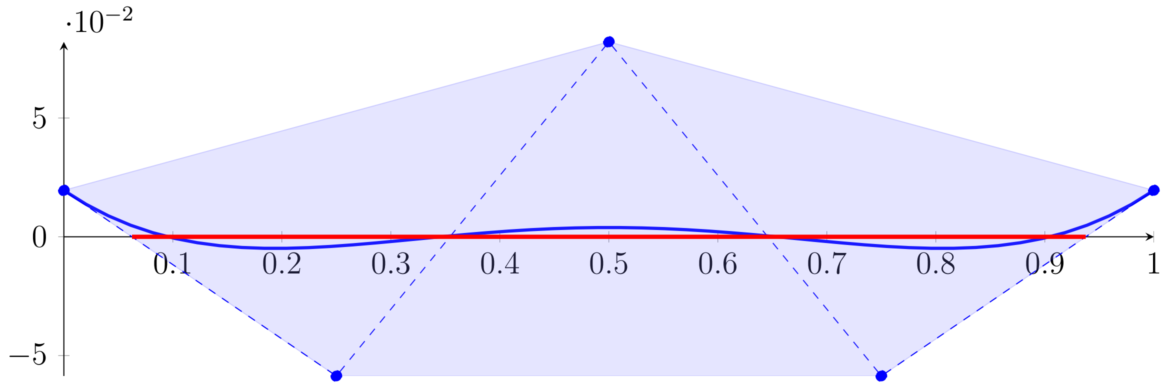
$$p = 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$



4.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) - 0.0585937B_{1,4}(X) + 0.0820312B_{2,4}(X) \\ &\quad - 0.0585937B_{3,4}(X) + 0.0195312B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0625, 0.9375\}$$

Intersection intervals with the x axis:

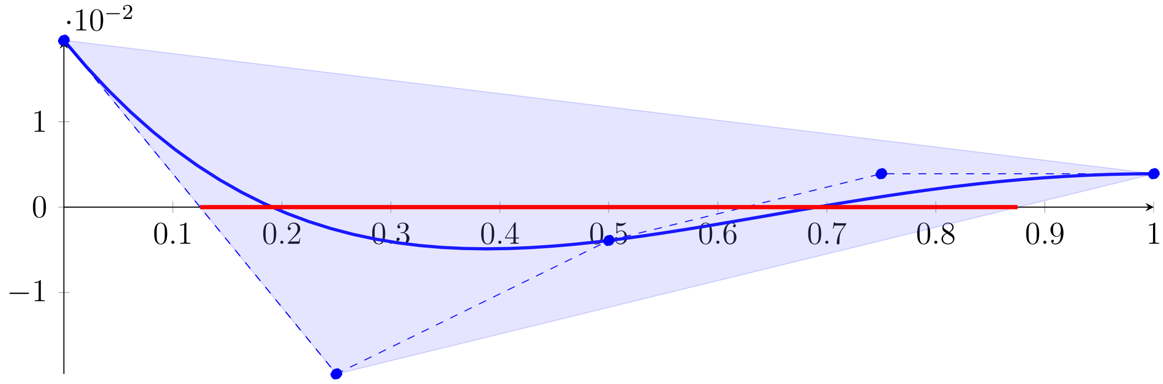
$$[0.0625, 0.9375]$$

Longest intersection interval: 0.875
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

4.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.0625X^4 - 0.25X^3 + 0.328125X^2 - 0.15625X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) - 0.0195312B_{1,4}(X) - 0.00390625B_{2,4}(X) \\ &\quad + 0.00390625B_{3,4}(X) + 0.00390625B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.125, 0.875\}$$

Intersection intervals with the x axis:

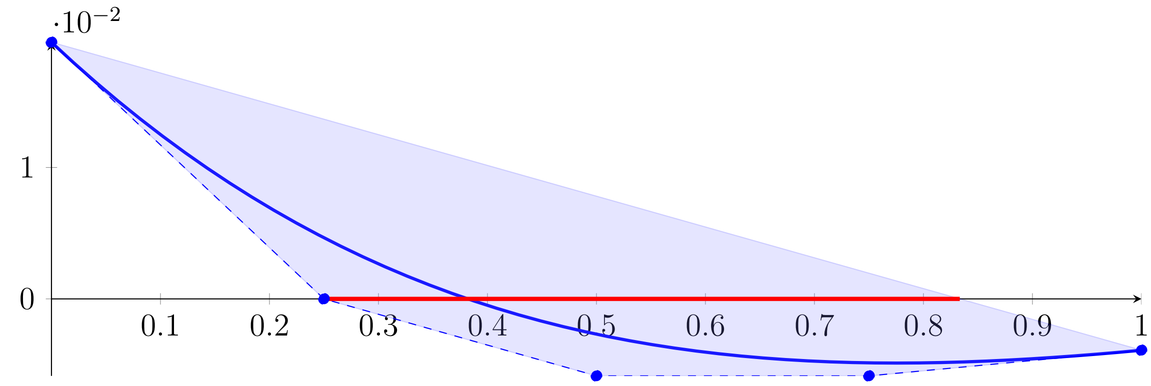
$$[0.125, 0.875]$$

Longest intersection interval: 0.75
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

4.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.00390625X^4 - 0.03125X^3 + 0.0820312X^2 - 0.078125X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) + 1.69407 \cdot 10^{-21}B_{1,4}(X) - 0.00585937B_{2,4}(X) \\ &\quad - 0.00585937B_{3,4}(X) - 0.00390625B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.833333\}$$

Intersection intervals with the x axis:

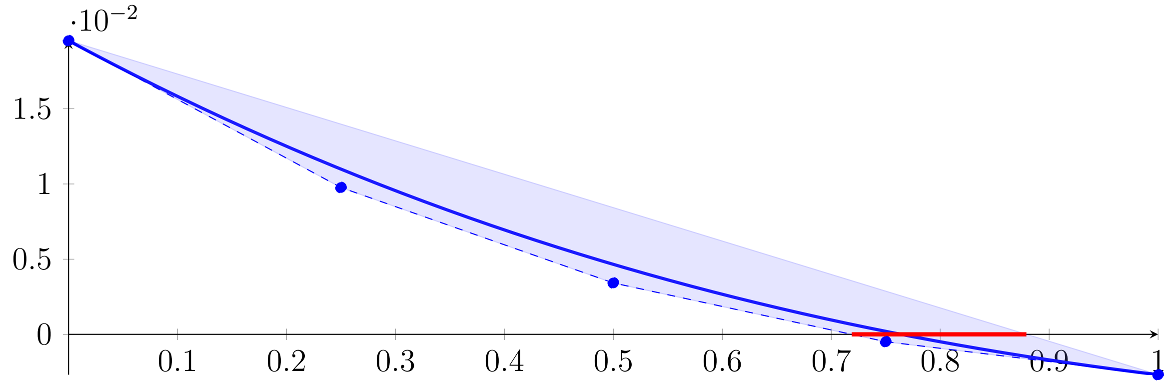
$$[0.25, 0.833333]$$

Longest intersection interval: 0.583333
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$

4.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000244141X^4 - 0.00390625X^3 + 0.0205078X^2 - 0.0390625X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) + 0.00976563B_{1,4}(X) + 0.00341797B_{2,4}(X) \\ &\quad - 0.000488281B_{3,4}(X) - 0.00268555B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.71875, 0.879121\}$$

Intersection intervals with the x axis:

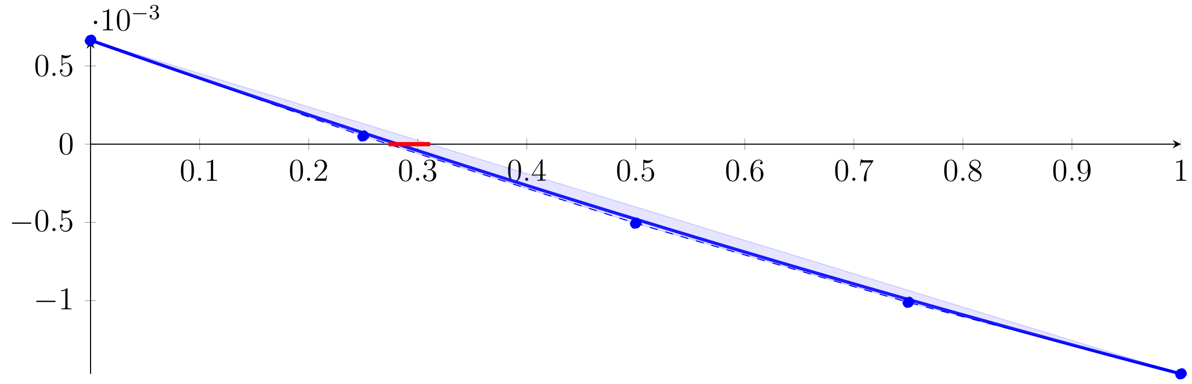
$$[0.71875, 0.879121]$$

Longest intersection interval: 0.160371
$\Longrightarrow$ Selective recursion: interval 1: $[0.0898438, 0.10989]$,

4.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0898438, 0.10989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.61489 \cdot 10^{-07}X^4 - 1.32165 \cdot 10^{-05}X^3 + 0.000330273X^2 - 0.00244948X + 0.000664182 \\ &= 0.000664182B_{0,4}(X) + 5.18125 \cdot 10^{-05}B_{1,4}(X) - 0.000505512B_{2,4}(X) \\ &\quad - 0.00101109B_{3,4}(X) - 0.00146808B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.273242, 0.311492\}$$

Intersection intervals with the x axis:

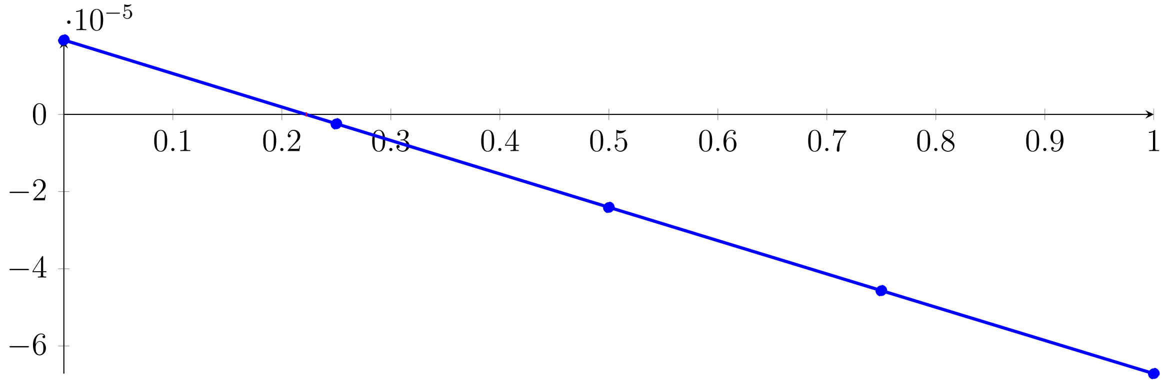
$$[0.273242, 0.311492]$$

Longest intersection interval: 0.0382504
$\Longrightarrow$ Selective recursion: interval 1: $[0.0953212, 0.096088]$,

4.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0953212, 0.096088]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 3.4569 \cdot 10^{-13} X^4 - 7.2977 \cdot 10^{-10} X^3 + 4.67476 \cdot 10^{-07} X^2 - 8.69026 \cdot 10^{-05} X + 1.92724 \cdot 10^{-05}$$
$$= 1.92724 \cdot 10^{-05} B_{0,4}(X) - 2.45322 \cdot 10^{-06} B_{1,4}(X) - 2.4101$$
$$\cdot 10^{-05} B_{2,4}(X) - 4.56709 \cdot 10^{-05} B_{3,4}(X) - 6.71634 \cdot 10^{-05} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.22177, 0.222968\}$$

Intersection intervals with the x axis:

$$[0.22177, 0.222968]$$

Longest intersection interval: 0.00119754
$\Longrightarrow$ Selective recursion: interval 1: $[0.0954913, 0.0954922]$,

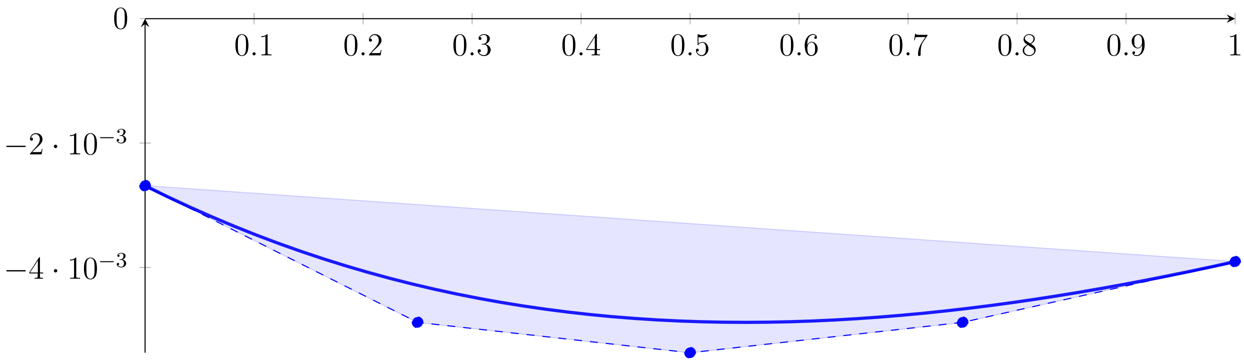
4.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.0954913, 0.0954922]$

Found root in interval $[0.0954913, 0.0954922]$ at recursion depth 7!

4.8 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.000244141 X^4 - 0.00292969 X^3 + 0.0102539 X^2 - 0.00878906 X - 0.00268555$$
$$= -0.00268555 B_{0,4}(X) - 0.00488281 B_{1,4}(X) - 0.00537109 B_{2,4}(X)$$
$$- 0.00488281 B_{3,4}(X) - 0.00390625 B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

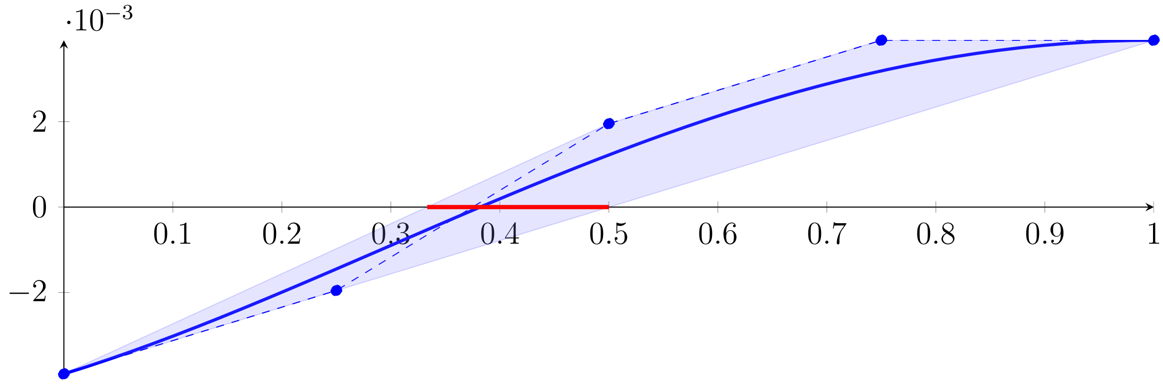
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

4.9 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.00390625X^4 - 0.015625X^3 + 0.0117188X^2 + 0.0078125X - 0.00390625 \\ &= -0.00390625B_{0,4}(X) - 0.00195312B_{1,4}(X) + 0.00195313B_{2,4}(X) \\ &\quad + 0.00390625B_{3,4}(X) + 0.00390625B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.333333, 0.5\}$$

Intersection intervals with the x axis:

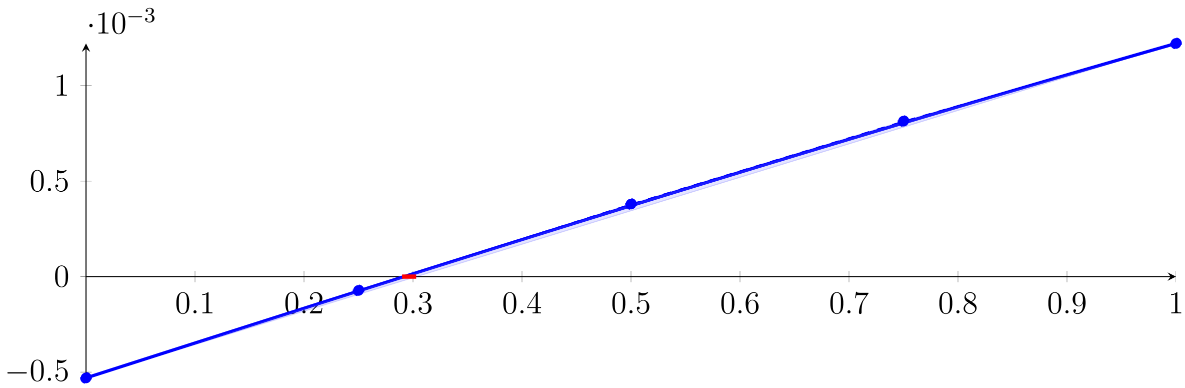
$$[0.333333, 0.5]$$

Longest intersection interval: 0.166667
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.375]$,

4.10 Recursion Branch 1 1 2 1 in Interval 1: $[0.333333, 0.375]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.01408\cdot 10^{-06}X^4 - 4.82253\cdot 10^{-05}X^3 - 3.6169\cdot 10^{-05}X^2 + 0.00183256X - 0.000530478 \\ &= -0.000530478B_{0,4}(X) - 7.2338\cdot 10^{-05}B_{1,4}(X) + 0.000379774B_{2,4}(X) \\ &\quad + 0.000813802B_{3,4}(X) + 0.0012207B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.29, 0.302926\}$$

Intersection intervals with the x axis:

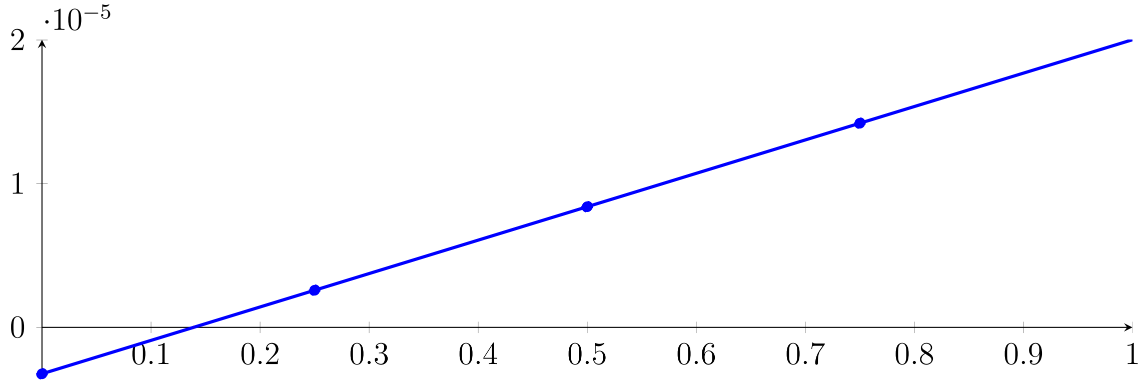
$$[0.29, 0.302926]$$

Longest intersection interval: 0.012926
$\Longrightarrow$ Selective recursion: interval 1: $[0.345417, 0.345955]$,

4.11 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.345417, 0.345955]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 8.41415 \cdot 10^{-14} X^4 - 9.66007 \cdot 10^{-11} X^3 - 1.27991 \cdot 10^{-08} X^2 + 2.3263 \cdot 10^{-05} X - 3.23215 \cdot 10^{-06}$$
$$= -3.23215 \cdot 10^{-06} B_{0,4}(X) + 2.58361 \cdot 10^{-06} B_{1,4}(X) + 8.39723 \cdot 10^{-06} B_{2,4}(X) + 1.42087 \cdot 10^{-05} B_{3,4}(X) + 2.0018 \cdot 10^{-05} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.138939, 0.139017\}$$

Intersection intervals with the x axis:

$$[0.138939, 0.139017]$$

Longest intersection interval: $7.70623 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.345491, 0.345492]$,

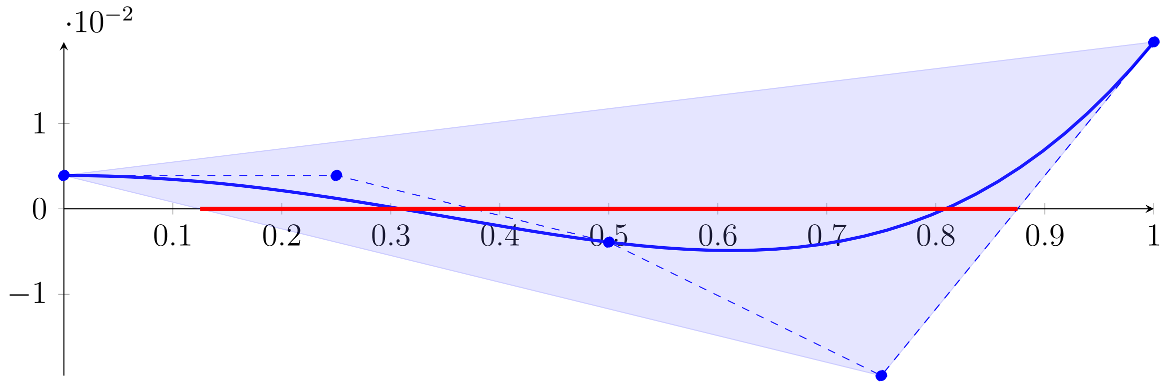
4.12 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.345491, 0.345492]$

Found root in interval $[0.345491, 0.345492]$ at recursion depth 6!

4.13 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.0625 X^4 - 3.38813 \cdot 10^{-21} X^3 - 0.046875 X^2 + 3.38813 \cdot 10^{-21} X + 0.00390625$$
$$= 0.00390625 B_{0,4}(X) + 0.00390625 B_{1,4}(X) - 0.00390625 B_{2,4}(X) - 0.0195312 B_{3,4}(X) + 0.0195312 B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.125, 0.875\}$$

Intersection intervals with the x axis:

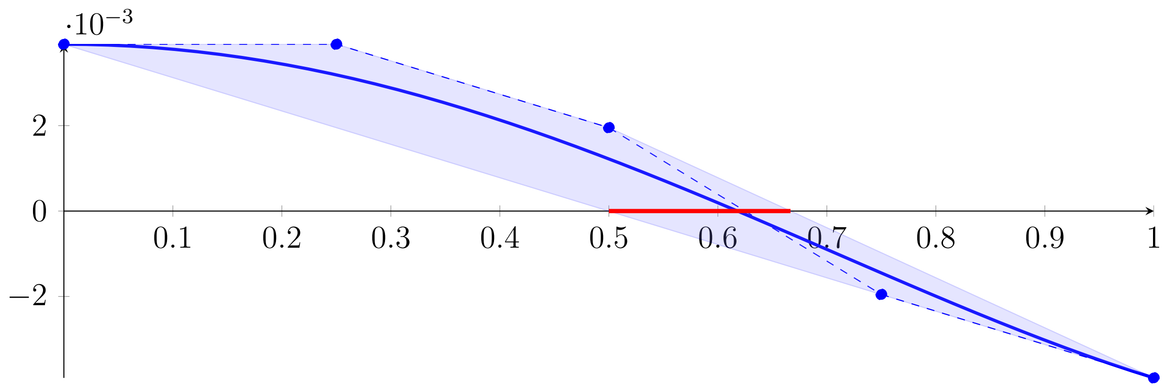
$$[0.125, 0.875]$$

Longest intersection interval: 0.75
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

4.14 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.00390625X^4 - 0.0117188X^2 + 1.69407\cdot 10^{-21}X + 0.00390625 \\ &= 0.00390625B_{0,4}(X) + 0.00390625B_{1,4}(X) + 0.00195313B_{2,4}(X) \\ &\quad - 0.00195312B_{3,4}(X) - 0.00390625B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.5, 0.666667\}$$

Intersection intervals with the x axis:

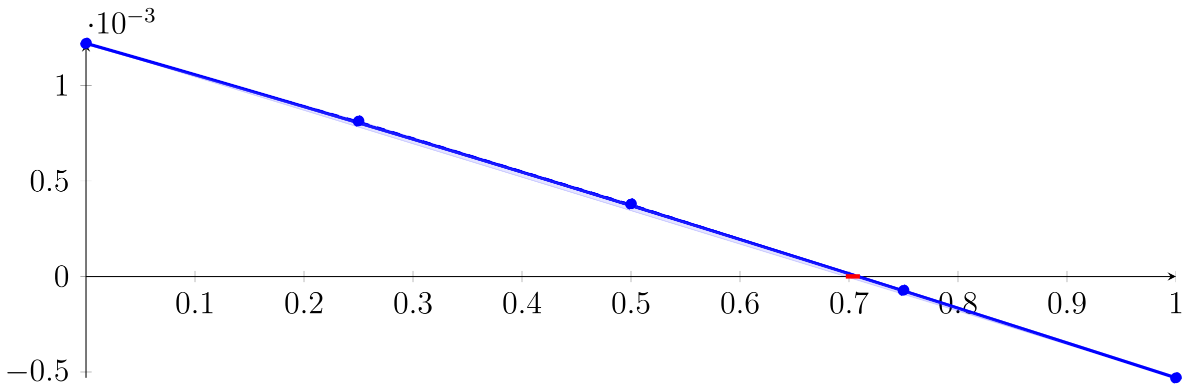
$$[0.5, 0.666667]$$

Longest intersection interval: 0.166667
$\Longrightarrow$ Selective recursion: interval 1: $[0.625, 0.666667]$,

4.15 Recursion Branch 1 2 1 1 in Interval 1: $[0.625, 0.666667]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.01408\cdot 10^{-06}X^4 + 3.6169\cdot 10^{-05}X^3 - 0.00016276X^2 - 0.0016276X + 0.0012207 \\ &= 0.0012207B_{0,4}(X) + 0.000813802B_{1,4}(X) + 0.000379774B_{2,4}(X) \\ &\quad - 7.2338\cdot 10^{-05}B_{3,4}(X) - 0.000530478B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.697074, 0.71\}$$

Intersection intervals with the x axis:

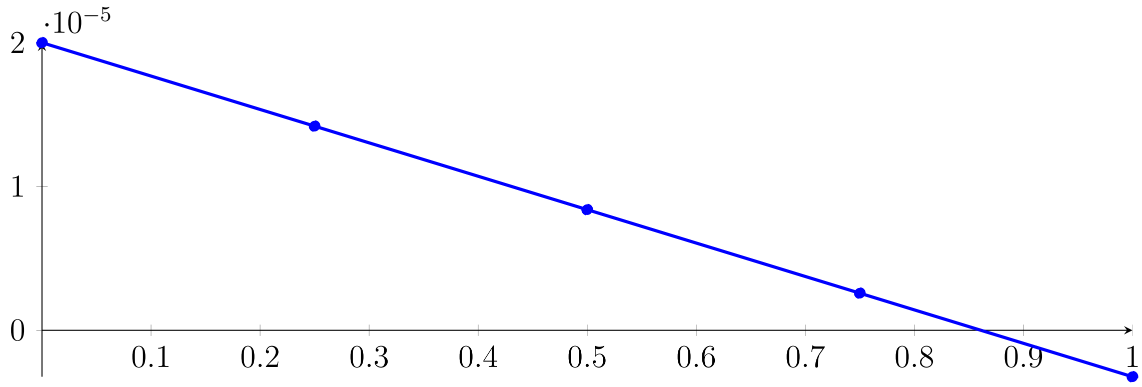
$$[0.697074, 0.71]$$

Longest intersection interval: 0.012926
$\Longrightarrow$ Selective recursion: interval 1: $[0.654045, 0.654583]$,

4.16 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.654045, 0.654583]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 8.41415 \cdot 10^{-14} X^4 + 9.62642 \cdot 10^{-11} X^3 - 1.30884 \cdot 10^{-08} X^2 - 2.32372 \cdot 10^{-05} X + 2.0018 \cdot 10^{-05} \\ &= 2.0018 \cdot 10^{-05} B_{0,4}(X) + 1.42087 \cdot 10^{-05} B_{1,4}(X) + 8.39723 \\ &\quad \cdot 10^{-06} B_{2,4}(X) + 2.58361 \cdot 10^{-06} B_{3,4}(X) - 3.23215 \cdot 10^{-06} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.860983, 0.861061\}$$

Intersection intervals with the x axis:

$$[0.860983, 0.861061]$$

Longest intersection interval: $7.70623 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.654508, 0.654509]$,

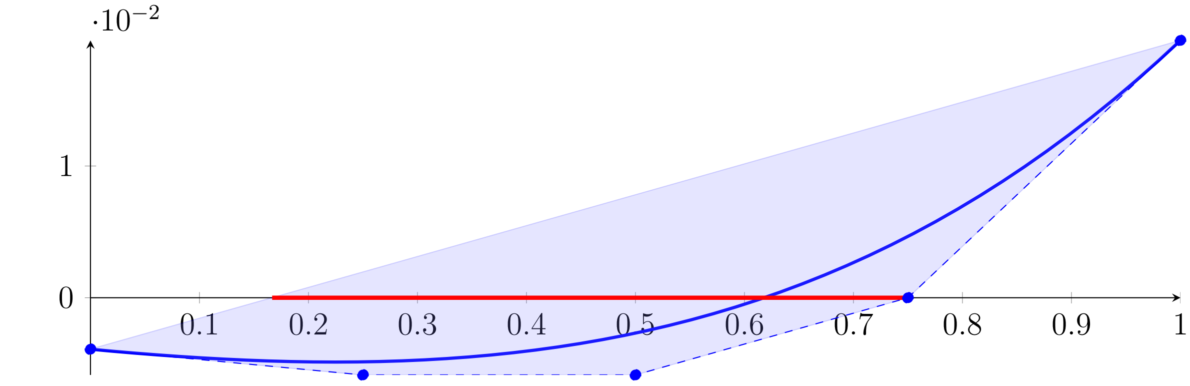
4.17 Recursion Branch 1 2 1 1 1 1 in Interval 1: $[0.654508, 0.654509]$

Found root in interval $[0.654508, 0.654509]$ at recursion depth 6!

4.18 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.00390625 X^4 + 0.015625 X^3 + 0.0117187 X^2 - 0.0078125 X - 0.00390625 \\ &= -0.00390625 B_{0,4}(X) - 0.00585937 B_{1,4}(X) - 0.00585937 B_{2,4}(X) \\ &\quad + 3.38813 \cdot 10^{-21} B_{3,4}(X) + 0.0195312 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.166667, 0.75\}$$

Intersection intervals with the x axis:

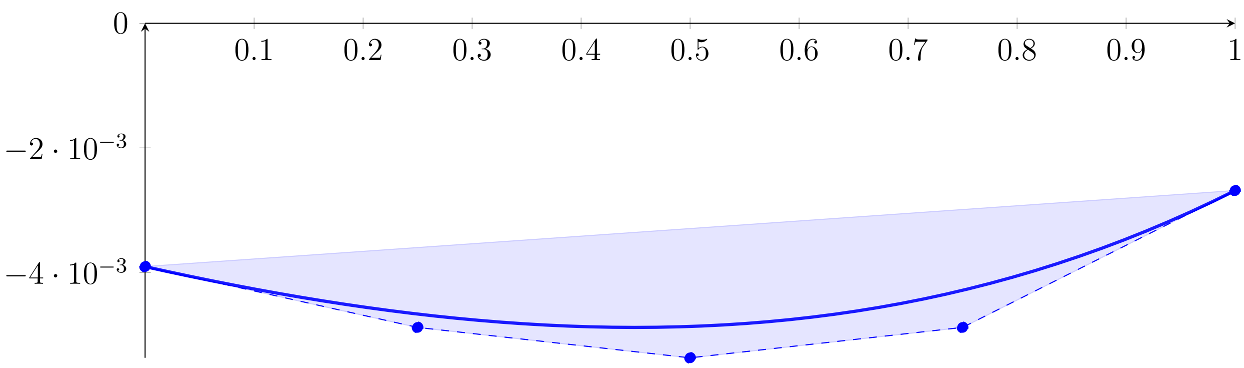
$$[0.166667, 0.75]$$

Longest intersection interval: 0.583333
$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$

4.19 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000244141X^4 + 0.00195312X^3 + 0.00292969X^2 - 0.00390625X - 0.00390625 \\ &= -0.00390625B_{0,4}(X) - 0.00488281B_{1,4}(X) - 0.00537109B_{2,4}(X) \\ &\quad - 0.00488281B_{3,4}(X) - 0.00268555B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

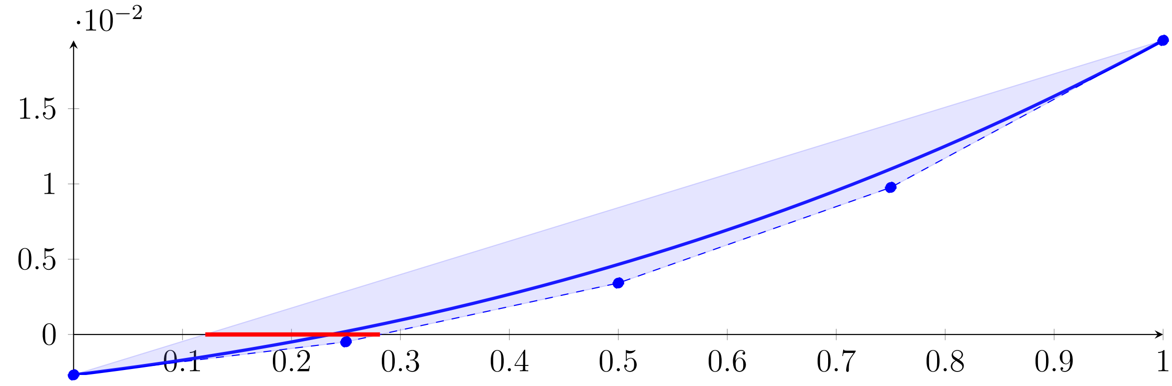
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

4.20 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000244141X^4 + 0.00292969X^3 + 0.0102539X^2 + 0.00878906X - 0.00268555 \\ &= -0.00268555B_{0,4}(X) - 0.000488281B_{1,4}(X) \\ &\quad + 0.00341797B_{2,4}(X) + 0.00976563B_{3,4}(X) + 0.0195312B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.120879, 0.28125\}$$

Intersection intervals with the x axis:

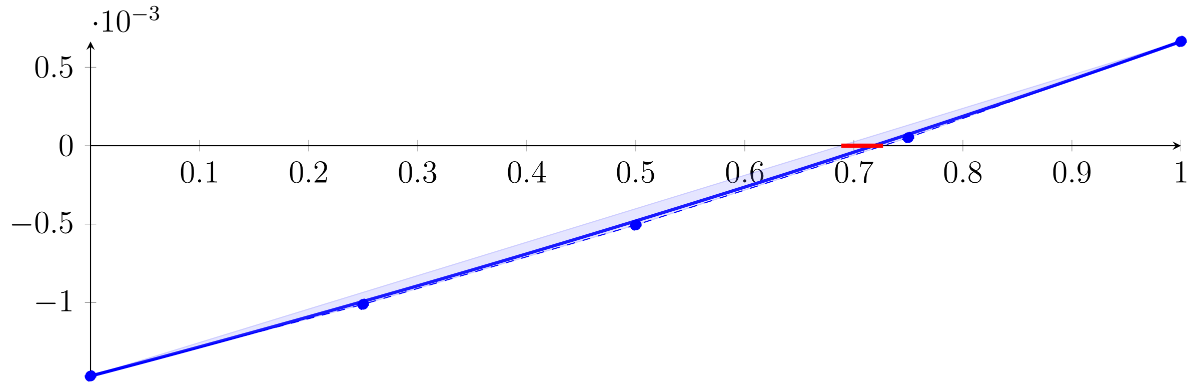
$$[0.120879, 0.28125]$$

Longest intersection interval: 0.160371
$\Longrightarrow$ Selective recursion: interval 1: $[0.89011, 0.910156]$,

4.21 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.89011, 0.910156]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.61489\cdot 10^{-07}X^4 + 1.25705\cdot 10^{-05}X^3 + 0.000291593X^2 + 0.00182794X - 0.00146808 \\ &= -0.00146808B_{0,4}(X) - 0.00101109B_{1,4}(X) - 0.000505512B_{2,4}(X) \\ &\quad + 5.18125\cdot 10^{-05}B_{3,4}(X) + 0.000664182B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.688508, 0.726758\}$$

Intersection intervals with the x axis:

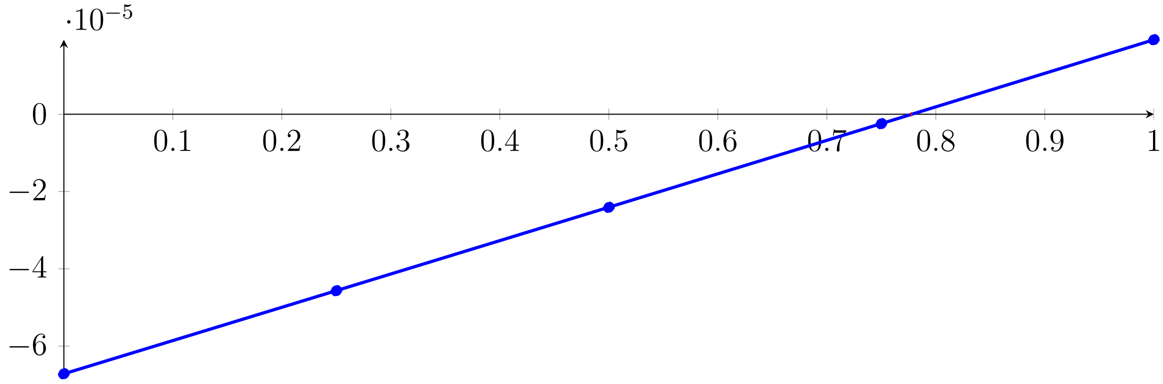
$$[0.688508, 0.726758]$$

Longest intersection interval: 0.0382504
$\Longrightarrow$ Selective recursion: interval 1: $[0.903912, 0.904679]$,

4.22 Recursion Branch 1 2 2 2 1 1 in Interval 1: $[0.903912, 0.904679]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.4569{\cdot}10^{-13}X^4 + 7.28387{\cdot}10^{-10}X^3 + 4.65289{\cdot}10^{-07}X^2 + 8.59698{\cdot}10^{-05}X - 6.71634{\cdot}10^{-05}\\ &= -6.71634{\cdot}10^{-05}B_{0,4}(X) - 4.56709{\cdot}10^{-05}B_{1,4}(X) - 2.4101\\ &\quad {\cdot}10^{-05}B_{2,4}(X) - 2.45322{\cdot}10^{-06}B_{3,4}(X) + 1.92724{\cdot}10^{-05}B_{4,4}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.777032, 0.77823\}$$

Intersection intervals with the x axis:

$$[0.777032, 0.77823]$$

Longest intersection interval: 0.00119754
$\Longrightarrow$ Selective recursion: interval 1: $[0.904508, 0.904509]$,

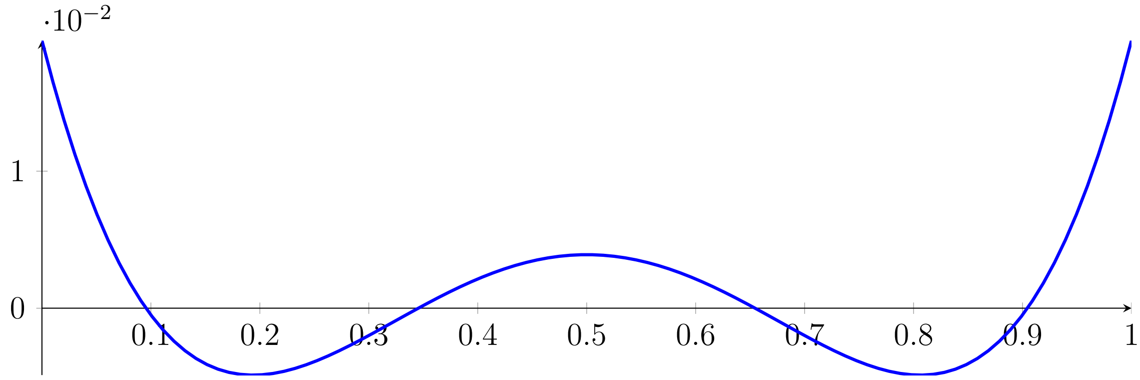
4.23 Recursion Branch 1 2 2 2 1 1 1 in Interval 1: $[0.904508, 0.904509]$

Found root in interval $[0.904508, 0.904509]$ at recursion depth 7!

4.24 Result: 4 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$



Result: Root Intervals

$$[0.0954913, 0.0954922],\ [0.345491, 0.345492],\ [0.654508, 0.654509],\ [0.904508, 0.904509]$$

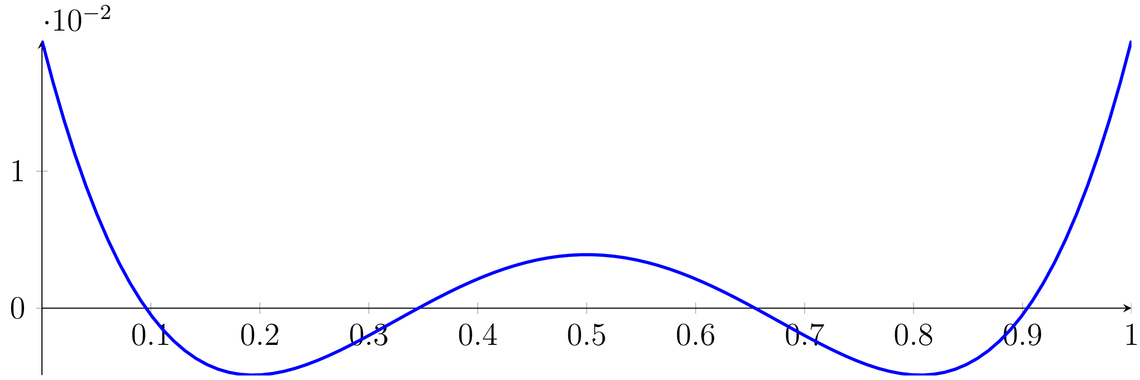
with precision $\varepsilon = 1 \cdot 10^{-06}$.

5 Running `QuadClip` on p4 with epsilon 6

$$1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

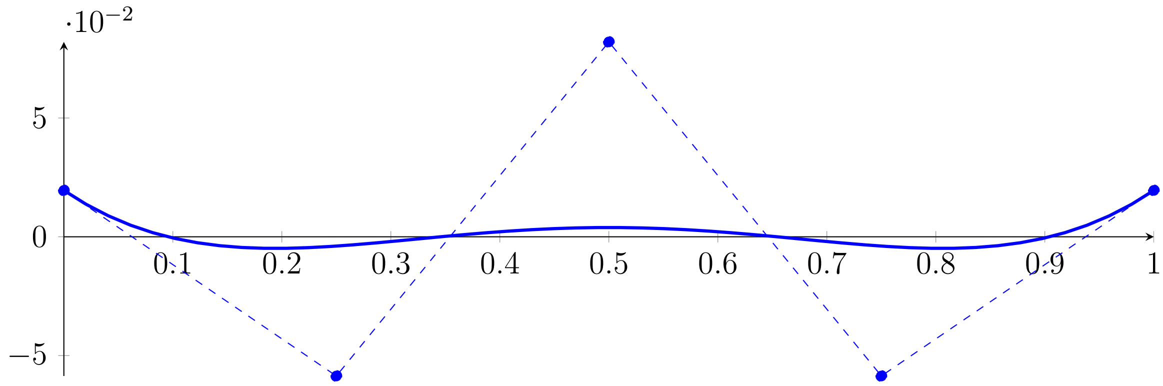
$$p = 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$



5.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

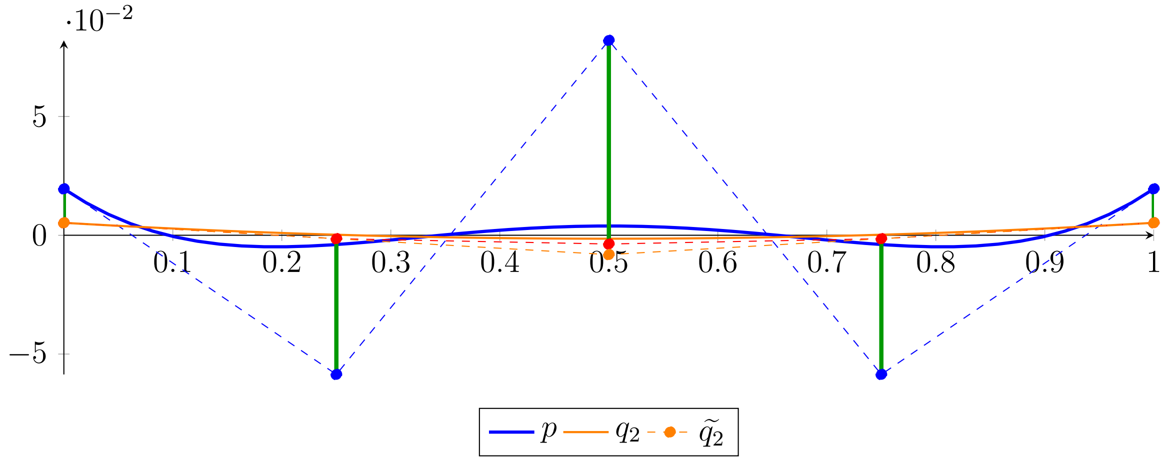
$$\begin{aligned} p &= 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) - 0.0585937B_{1,4}(X) + 0.0820312B_{2,4}(X) \\ &\quad - 0.0585937B_{3,4}(X) + 0.0195312B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0267857X^2 - 0.0267857X + 0.00524554 \\ &= 0.00524554B_{0,2} - 0.00814732B_{1,2} + 0.00524554B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43141 \cdot 10^{-18}X^4 - 4.95684 \cdot 10^{-18}X^3 + 0.0267857X^2 - 0.0267857X + 0.00524554 \\ &= 0.00524554B_{0,4} - 0.00145089B_{1,4} - 0.00368304B_{2,4} - 0.00145089B_{3,4} + 0.00524554B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

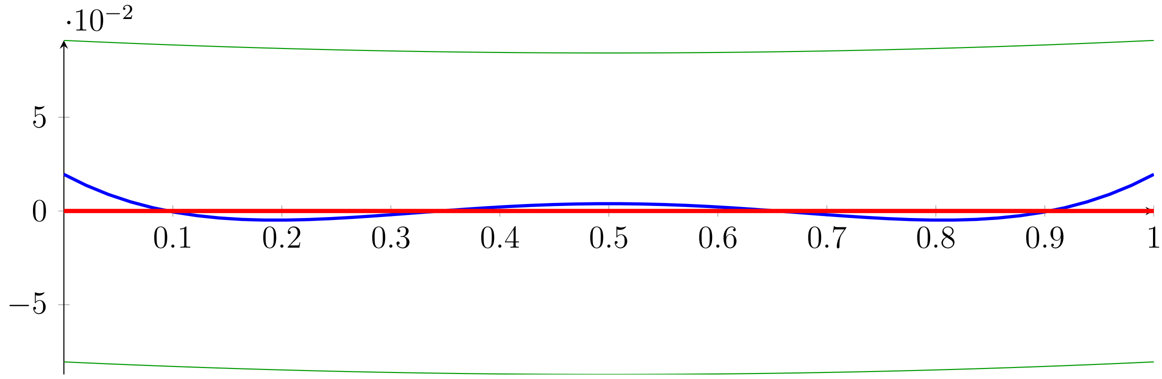
$$M = 0.0267857X^2 - 0.0267857X + 0.0909598$$

$$m = 0.0267857X^2 - 0.0267857X - 0.0804687$$

Root of M and m:

$$N(M) = \{\} \qquad\qquad N(m) = \{-1.30393, 2.30393\}$$

Intersection intervals:



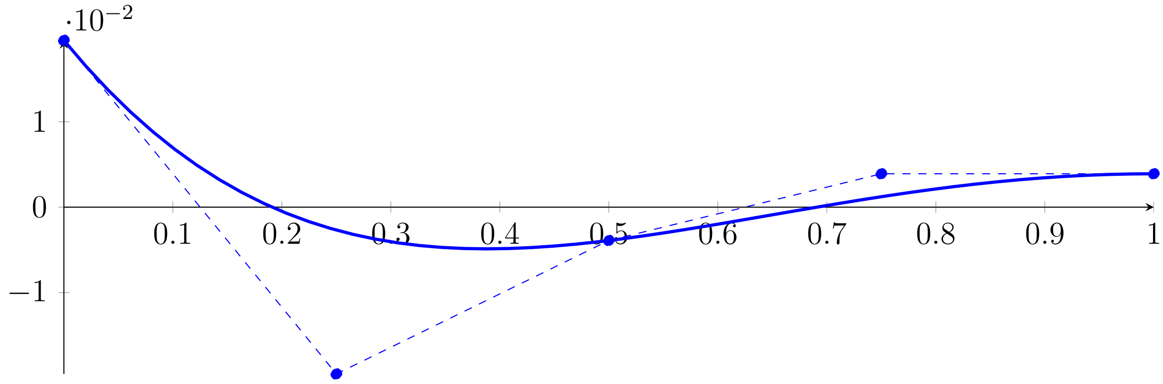
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

5.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

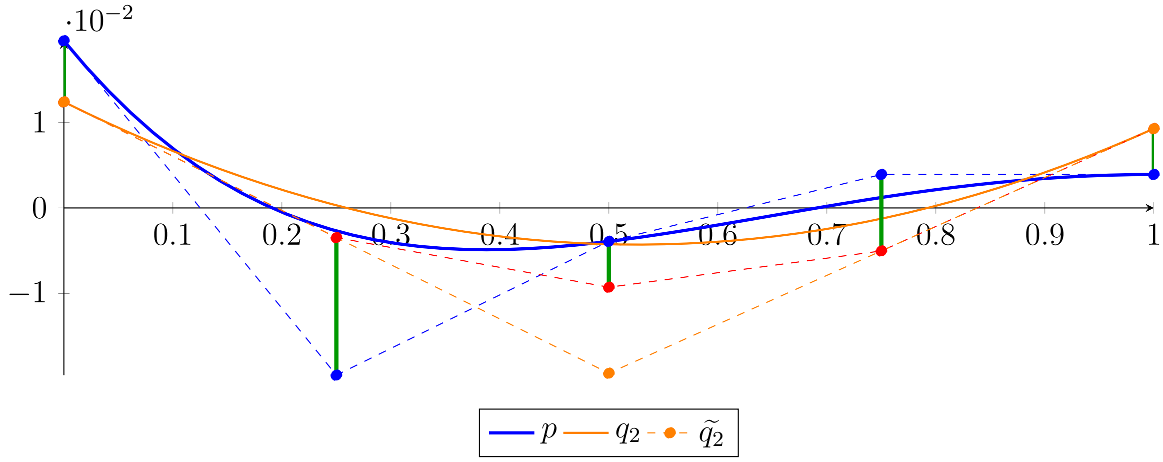
$$\begin{aligned} p &= 0.0625X^4 - 0.25X^3 + 0.328125X^2 - 0.15625X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) - 0.0195312B_{1,4}(X) - 0.00390625B_{2,4}(X) \\ &\quad + 0.00390625B_{3,4}(X) + 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0602679X^2 - 0.0633929X + 0.0123884 \\ &= 0.0123884B_{0,2} - 0.019308B_{1,2} + 0.00926339B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.46167\cdot 10^{-18}X^4 - 1.1252\cdot 10^{-17}X^3 + 0.0602679X^2 - 0.0633929X + 0.0123884 \\ &= 0.0123884B_{0,4} - 0.00345982B_{1,4} - 0.00926339B_{2,4} - 0.00502232B_{3,4} + 0.00926339B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0160714$.

Bounding polynomials M and m:

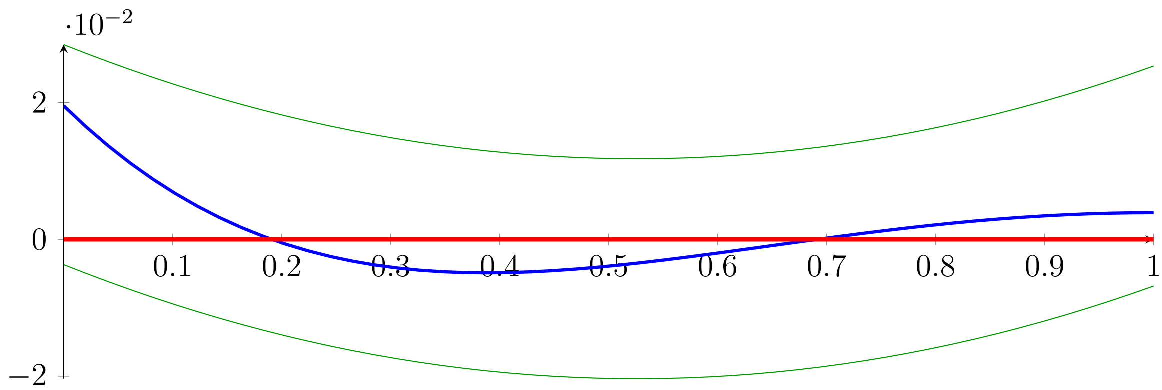
$$M = 0.0602679X^2 - 0.0633929X + 0.0284598$$

$$m = 0.0602679X^2 - 0.0633929X - 0.00368304$$

Root of M and m:

$$N(M) = \{\} \qquad\qquad N(m) = \{-0.0552016, 1.10705\}$$

Intersection intervals:



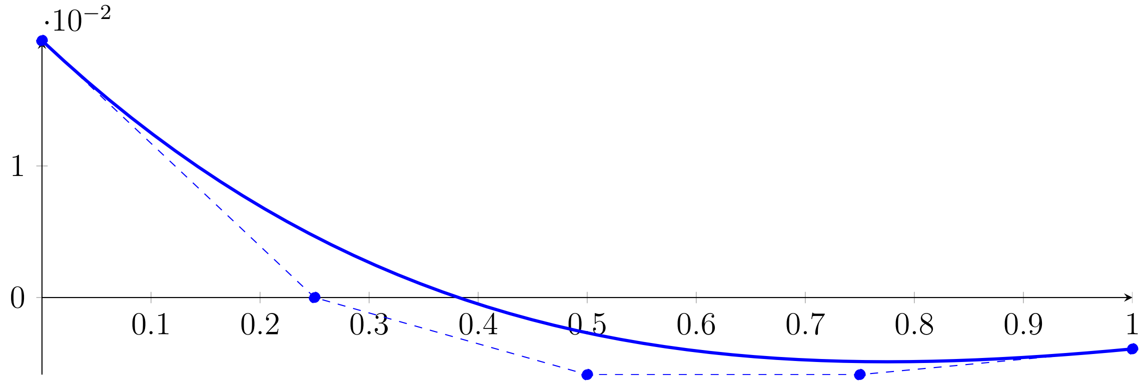
$$[0, 1]$$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$
Bisection point is very near to a root?!?

5.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

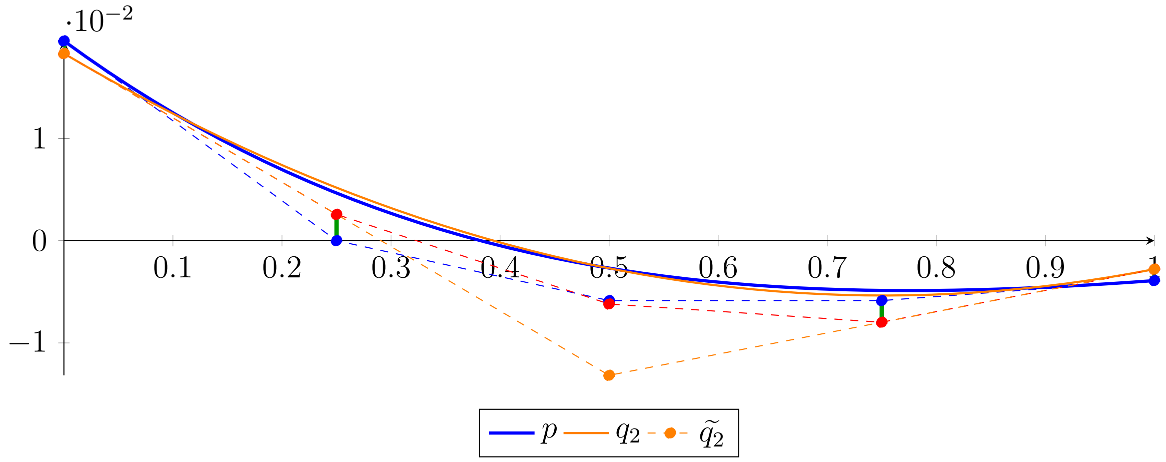
$$\begin{aligned} p &= 0.00390625X^4 - 0.03125X^3 + 0.0820312X^2 - 0.078125X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) + 1.69407\cdot 10^{-21}B_{1,4}(X) - 0.00585937B_{2,4}(X) \\ &\quad - 0.00585937B_{3,4}(X) - 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0418527X^2 - 0.0629464X + 0.0183036 \\ &= 0.0183036B_{0,2} - 0.0131696B_{1,2} - 0.00279018B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 3.87941\cdot 10^{-18}X^4 - 8.1654\cdot 10^{-18}X^3 + 0.0418527X^2 - 0.0629464X + 0.0183036 \\ &= 0.0183036B_{0,4} + 0.00256696B_{1,4} - 0.0061942B_{2,4} - 0.00797991B_{3,4} - 0.00279018B_{4,4} \end{aligned}$$



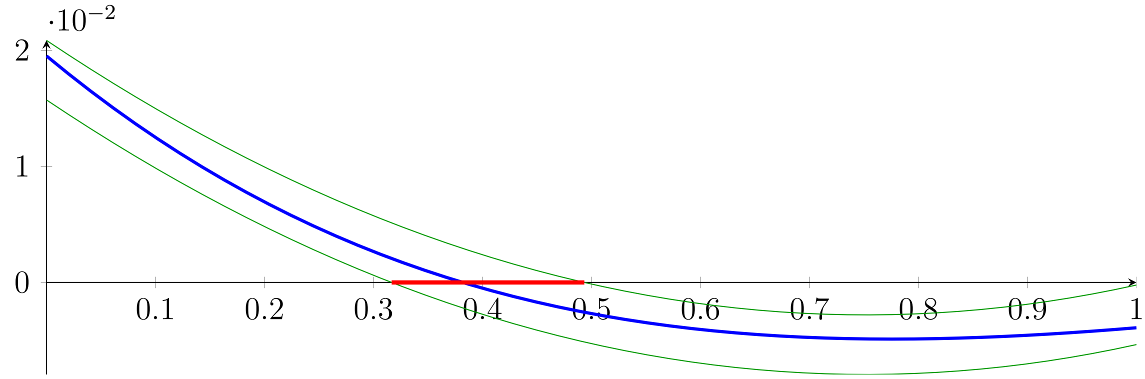
The maximum difference of the Bézier coefficients is $\delta = 0.00256696$.
Bounding polynomials M and m:

$$M = 0.0418527X^2 - 0.0629464X + 0.0208705$$
$$m = 0.0418527X^2 - 0.0629464X + 0.0157366$$

Root of M and m:

$$N(M) = \{0.493471, 1.01053\} \qquad N(m) = \{0.316679, 1.18732\}$$

Intersection intervals:



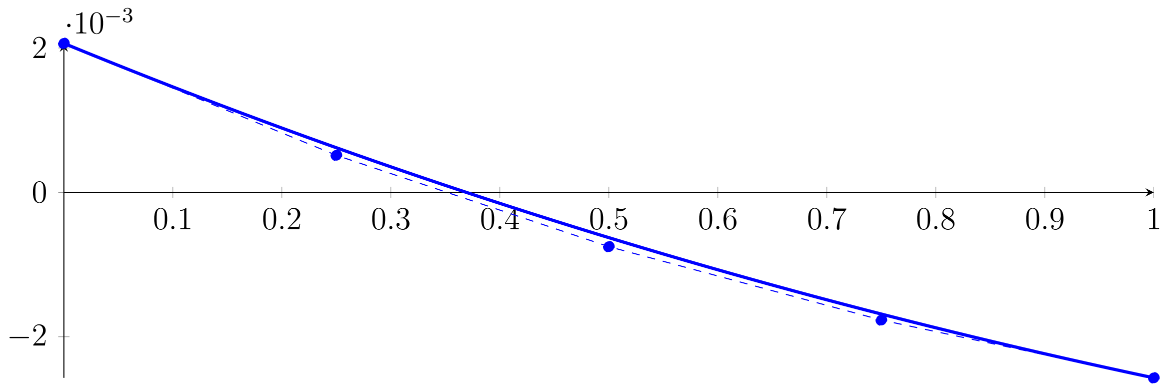
$[0.316679, 0.493471]$

Longest intersection interval: 0.176791
$\Longrightarrow$ Selective recursion: interval 1: $[0.0791699, 0.123368]$,

5.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.0791699, 0.123368]$

Normalized monomial und Bézier representations and the Bézier polygon:

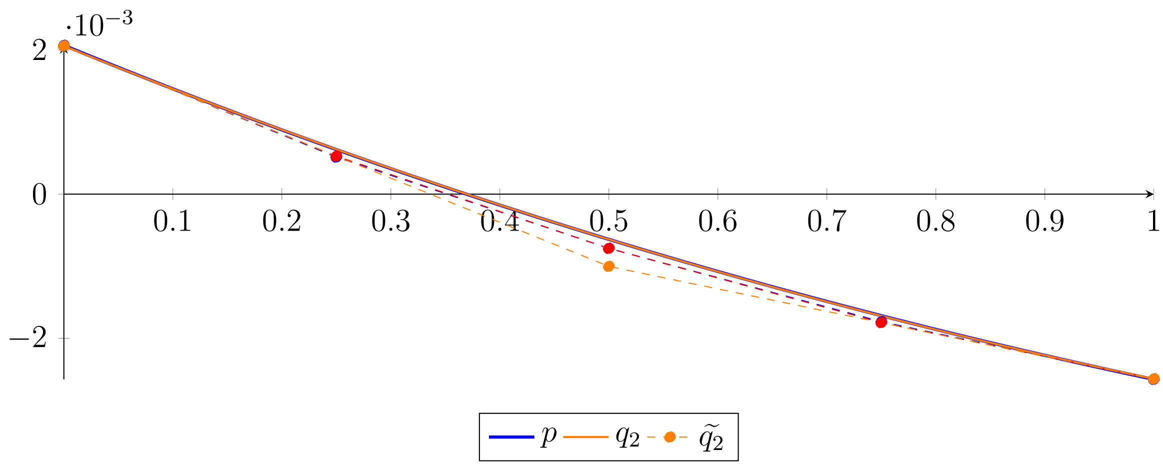
$$
\begin{aligned}
p &= 3.81597{\cdot}10^{-06}X^4 - 0.000145335X^3 + 0.00170944X^2 - 0.00620102X + 0.00206408\\
&= 0.00206408B_{0,4}(X) + 0.000513823B_{1,4}(X) - 0.000751526B_{2,4}(X)\\
&\quad - 0.0017683B_{3,4}(X) - 0.00256902B_{4,4}(X)
\end{aligned}
$$



Degree reduction and raising:

$$
\begin{aligned}
q_2 &= 0.00149798X^2 - 0.00611731X + 0.00205714\\
&= 0.00205714B_{0,2} - 0.00100152B_{1,2} - 0.00256219B_{2,2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q_2} &= 1.45266{\cdot}10^{-19}X^4 - 3.9387{\cdot}10^{-19}X^3 + 0.00149798X^2 - 0.00611731X + 0.00205714\\
&= 0.00205714B_{0,4} + 0.000527811B_{1,4} - 0.000751853B_{2,4} - 0.00178185B_{3,4} - 0.00256219B_{4,4}
\end{aligned}
$$



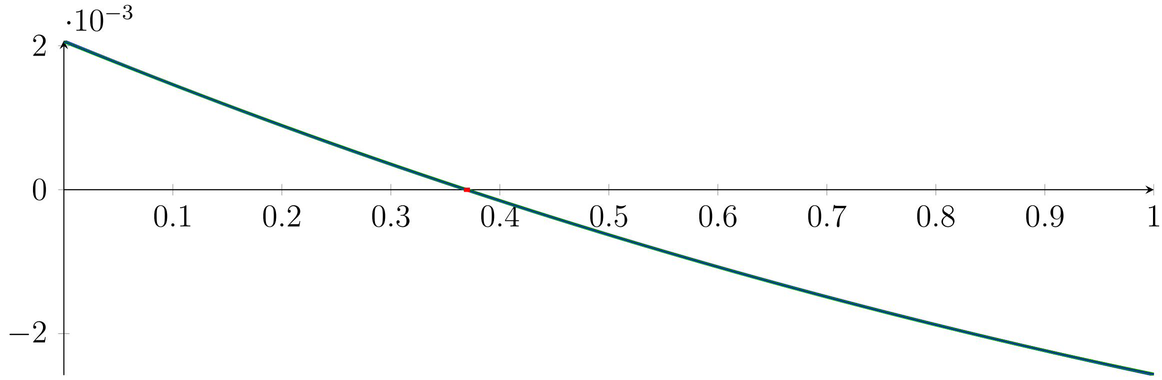
The maximum difference of the Bézier coefficients is $\delta = 1.39884 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = 0.00149798X^2 - 0.00611731X + 0.00207113$$
$$m = 0.00149798X^2 - 0.00611731X + 0.00204315$$

Root of M and m:

$$N(M) = \{0.372557, 3.71115\} \qquad N(m) = \{0.366972, 3.71673\}$$

Intersection intervals:



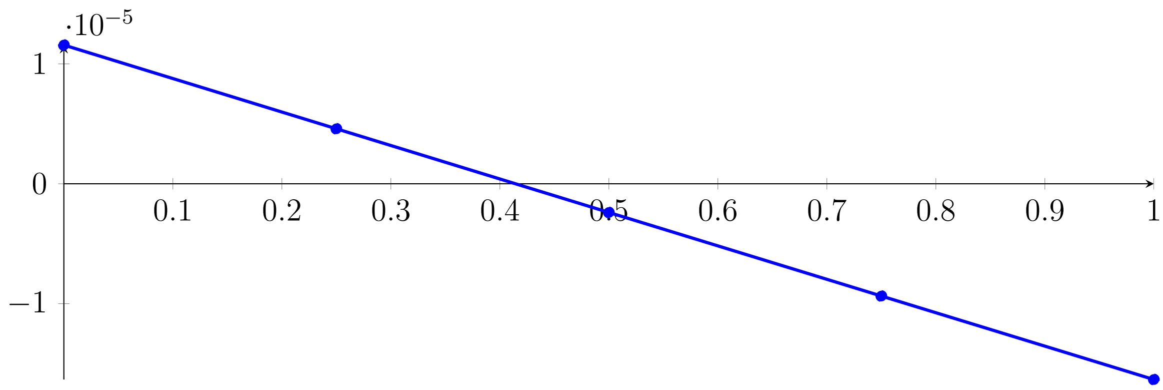
$[0.366972, 0.372557]$

Longest intersection interval: 0.00558472
$\Longrightarrow$ Selective recursion: interval 1: $[0.0953892, 0.0956361]$,

5.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0953892, 0.0956361]$

Normalized monomial und Bézier representations and the Bézier polygon:

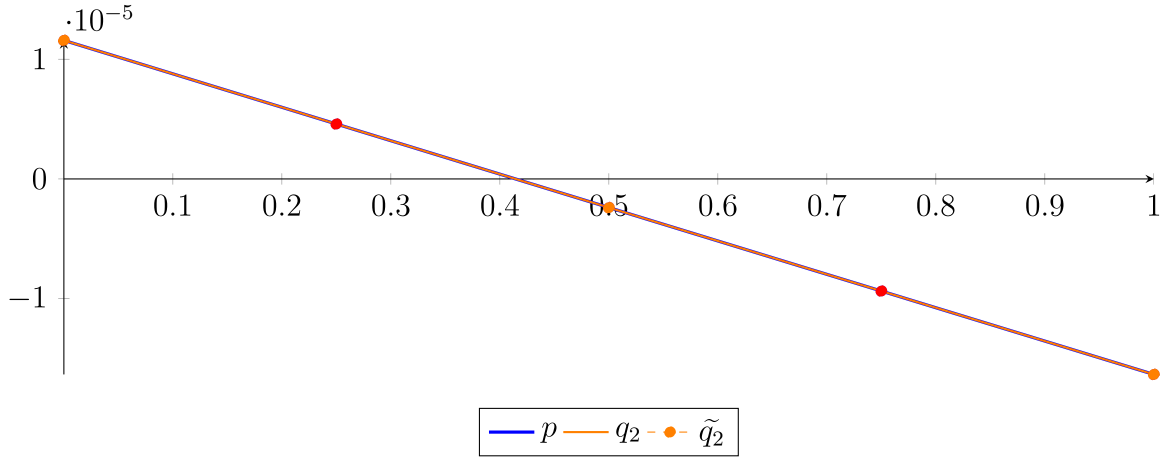
$$\begin{aligned} p &= 3.71203 \cdot 10^{-15} X^4 - 2.43392 \cdot 10^{-11} X^3 + 4.84218 \cdot 10^{-08} X^2 - 2.79479 \cdot 10^{-05} X + 1.15719 \cdot 10^{-05} \\ &= 1.15719 \cdot 10^{-05} B_{0,4}(X) + 4.58497 \cdot 10^{-06} B_{1,4}(X) - 2.39394 \\ &\quad \cdot 10^{-06} B_{2,4}(X) - 9.36478 \cdot 10^{-06} B_{3,4}(X) - 1.63276 \cdot 10^{-05} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 4.83853 \cdot 10^{-08} X^2 - 2.79479 \cdot 10^{-05} X + 1.15719 \cdot 10^{-05} \\ &= 1.15719 \cdot 10^{-05} B_{0,2} - 2.402 \cdot 10^{-06} B_{1,2} - 1.63276 \cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 7.36191 \cdot 10^{-23} X^4 - 6.55127 \cdot 10^{-22} X^3 + 4.83853 \cdot 10^{-08} X^2 - 2.79479 \cdot 10^{-05} X + 1.15719 \cdot 10^{-05} \\ &= 1.15719 \cdot 10^{-05} B_{0,4} + 4.58497 \cdot 10^{-06} B_{1,4} - 2.39394 \cdot 10^{-06} B_{2,4} - 9.36478 \cdot 10^{-06} B_{3,4} - 1.63276 \cdot 10^{-05} B_{4,4} \end{aligned}$$



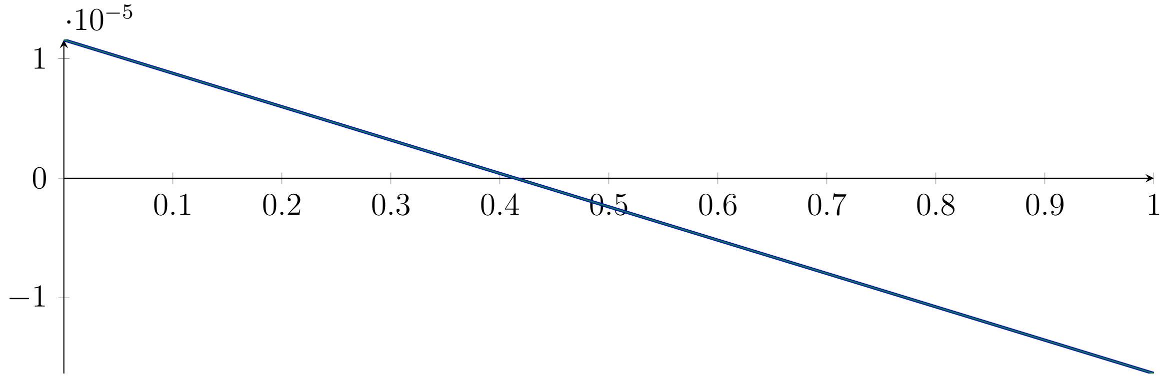
The maximum difference of the Bézier coefficients is $\delta = 2.43339 \cdot 10^{-12}$.
Bounding polynomials M and m:

$$M = 4.83853 \cdot 10^{-08} X^2 - 2.79479 \cdot 10^{-05} X + 1.15719 \cdot 10^{-05}$$
$$m = 4.83853 \cdot 10^{-08} X^2 - 2.79479 \cdot 10^{-05} X + 1.15719 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.414352, 577.197\} \qquad N(m) = \{0.414351, 577.197\}$$

Intersection intervals:



$$[0.414351, 0.414352]$$

Longest intersection interval: $1.74388 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0954915, 0.0954915]$,

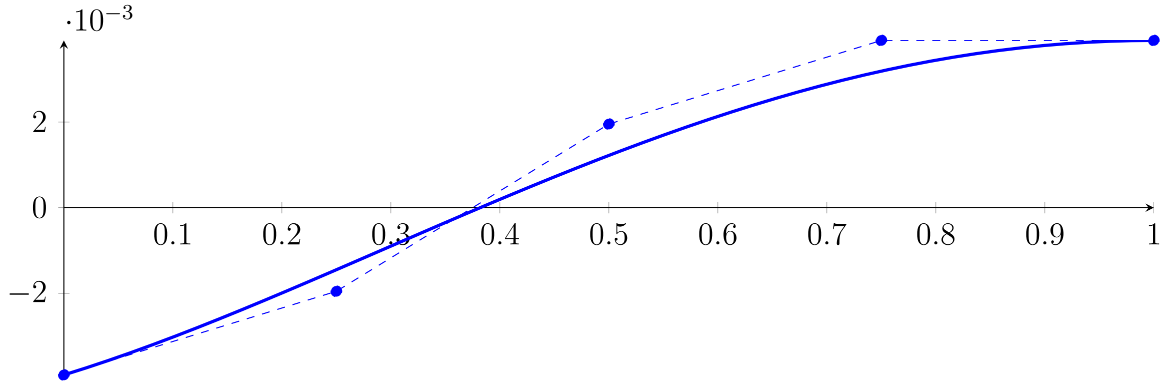
5.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0954915, 0.0954915]$

Found root in interval $[0.0954915, 0.0954915]$ at recursion depth 6!

5.7 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

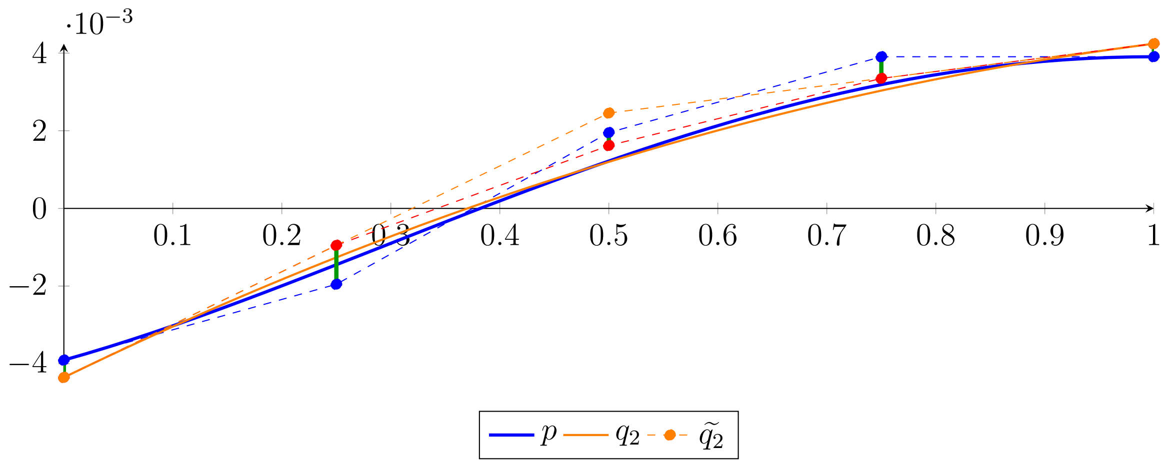
$$\begin{aligned} p &= 0.00390625X^4 - 0.015625X^3 + 0.0117188X^2 + 0.0078125X - 0.00390625 \\ &= -0.00390625B_{0,4}(X) - 0.00195312B_{1,4}(X) + 0.00195313B_{2,4}(X) \\ &\quad + 0.00390625B_{3,4}(X) + 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00502232X^2 + 0.0136161X - 0.00435268 \\ &= -0.00435268B_{0,2} + 0.00245536B_{1,2} + 0.00424107B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.74338 \cdot 10^{-19}X^4 + 1.14011 \cdot 10^{-18}X^3 - 0.00502232X^2 + 0.0136161X - 0.00435268 \\ &= -0.00435268B_{0,4} - 0.000948661B_{1,4} + 0.0016183B_{2,4} + 0.00334821B_{3,4} + 0.00424107B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00100446$.

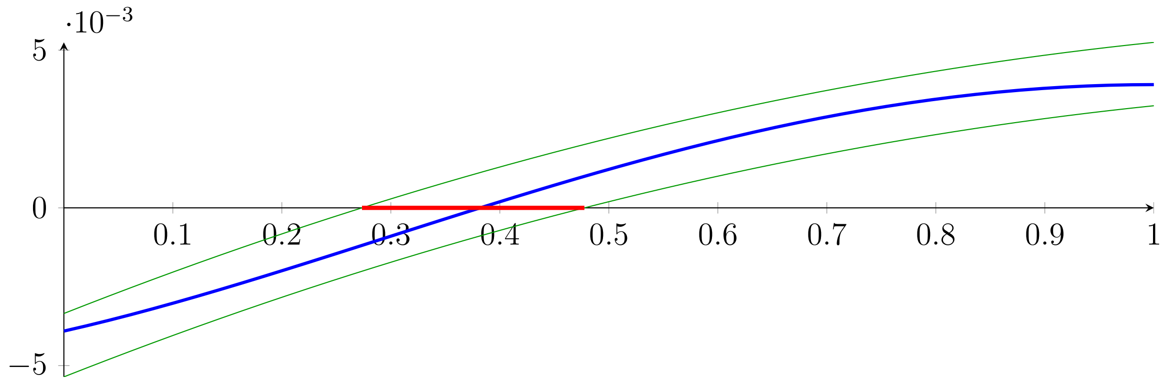
Bounding polynomials M and m:

$$\begin{aligned} M &= -0.00502232X^2 + 0.0136161X - 0.00334821 \\ m &= -0.00502232X^2 + 0.0136161X - 0.00535714 \end{aligned}$$

Root of M and m:

$$N(M) = \{0.273491, 2.43762\} \qquad N(m) = \{0.477567, 2.23354\}$$

Intersection intervals:



$$[0.273491, 0.477567]$$

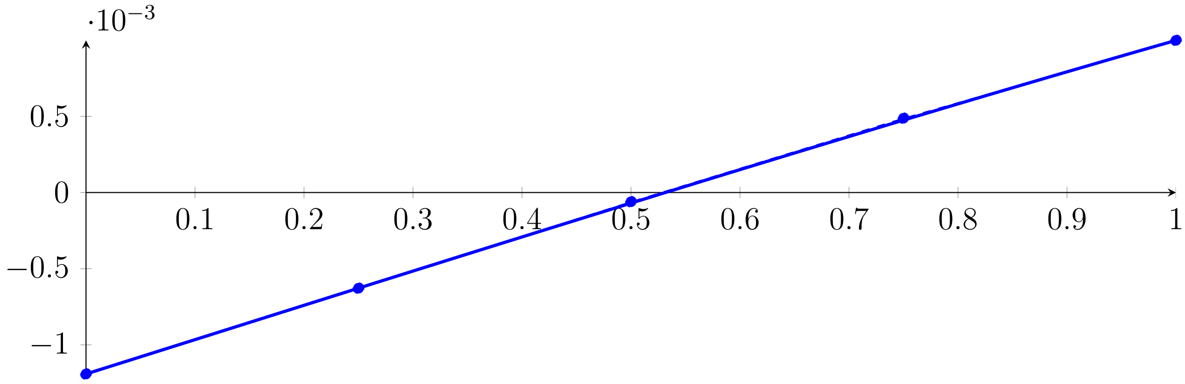
Longest intersection interval: 0.204076

$\Longrightarrow$ Selective recursion: interval 1: $[0.318373, 0.369392]$,

5.8 Recursion Branch 1 1 2 1 in Interval 1: $[0.318373, 0.369392]$

Normalized monomial und Bézier representations and the Bézier polygon:

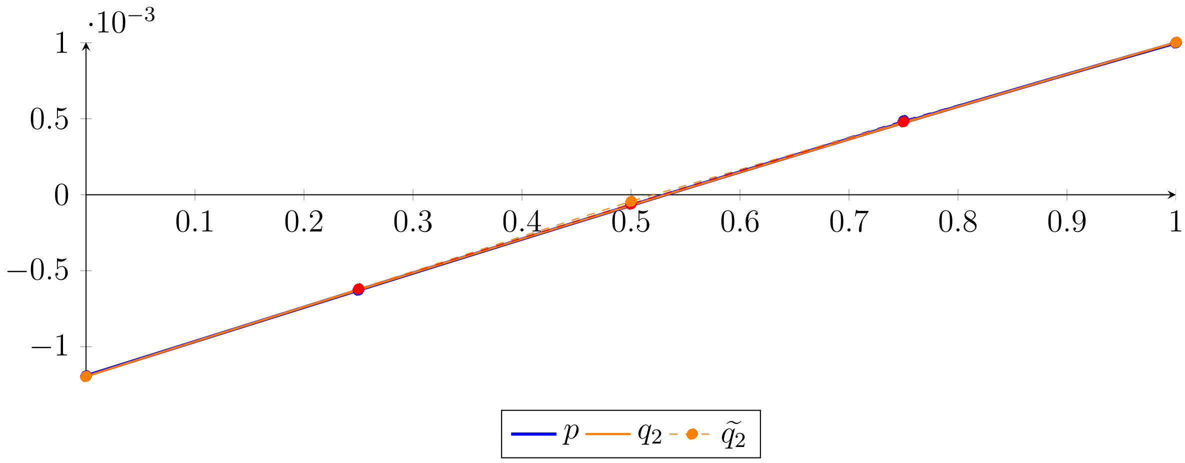
$$\begin{aligned}p &= 6.7753\cdot10^{-06}X^4 - 9.648\cdot10^{-05}X^3 + 2.71509\cdot10^{-05}X^2 + 0.00225217X - 0.00119085\\ &= -0.00119085B_{0,4}(X) - 0.000627807B_{1,4}(X) - 6.02381\\ &\quad\cdot10^{-05}B_{2,4}(X) + 0.000487736B_{3,4}(X) + 0.00099877B_{4,4}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -0.000105954X^2 + 0.00230387X - 0.00119509\\ &= -0.00119509B_{0,2} - 4.31598\cdot10^{-05}B_{1,2} + 0.00100282B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -2.06464\cdot10^{-20}X^4 + 5.71747\cdot10^{-20}X^3 - 0.000105954X^2 + 0.00230387X - 0.00119509\\ &= -0.00119509B_{0,4} - 0.000619127B_{1,4} - 6.08189\cdot10^{-05}B_{2,4} + 0.00047983B_{3,4} + 0.00100282B_{4,4}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.6801\cdot10^{-06}$.
Bounding polynomials M and m:

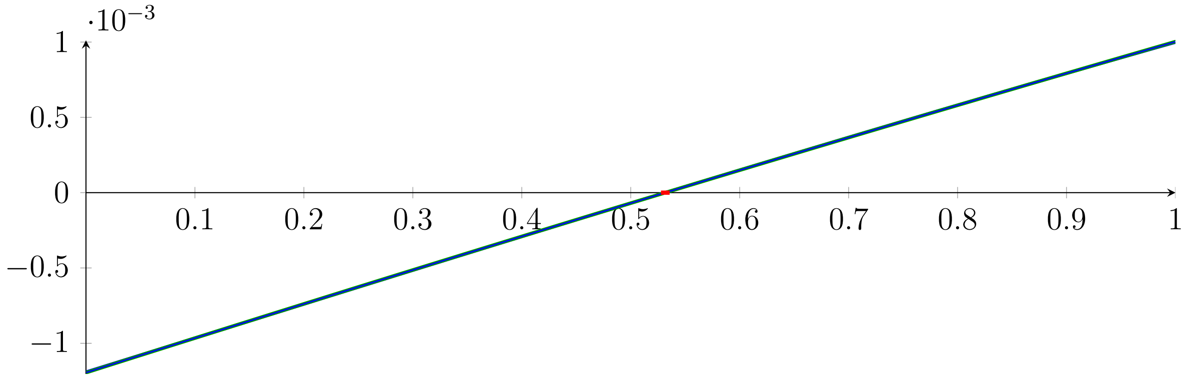
$$M = -0.000105954X^2 + 0.00230387X - 0.00118641$$

$$m = -0.000105954X^2 + 0.00230387X - 0.00120377$$

Root of M and m:

$$N(M) = \{0.527776, 21.2162\} \qquad N(m) = \{0.535699, 21.2083\}$$

Intersection intervals:



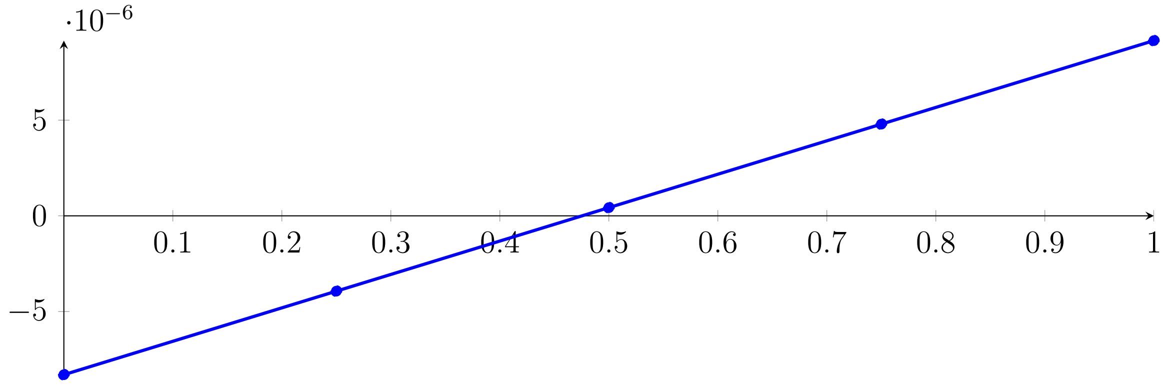
$[0.527776, 0.535699]$

Longest intersection interval: 0.00792274
$\Longrightarrow$ Selective recursion: interval 1: $[0.345299, 0.345704]$,

5.9 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.345299, 0.345704]$

Normalized monomial und Bézier representations and the Bézier polygon:

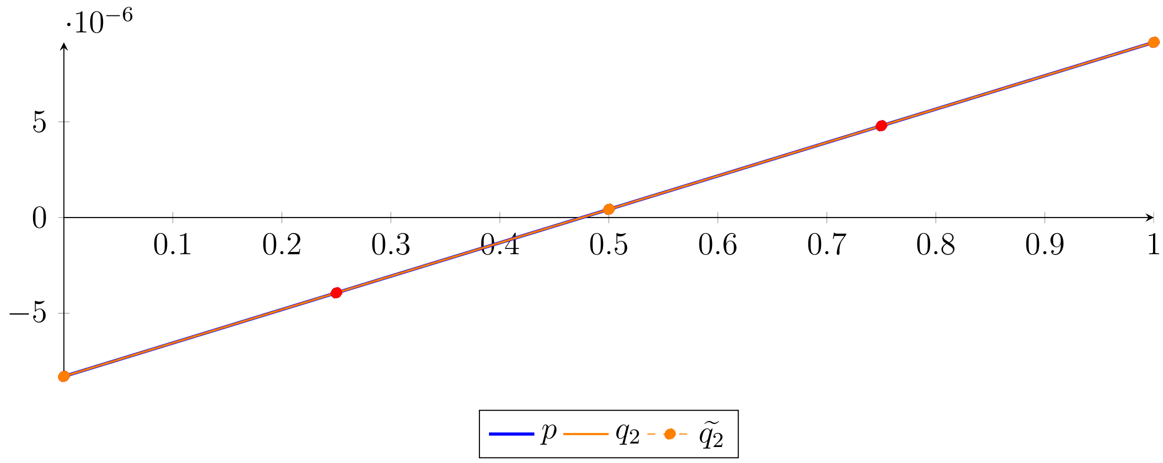
$$p = 2.6695 \cdot 10^{-14} X^4 - 4.08672 \cdot 10^{-11} X^3 - 7.17366 \cdot 10^{-09} X^2 + 1.74633 \cdot 10^{-05} X - 8.30118 \cdot 10^{-06}$$
$$= -8.30118 \cdot 10^{-06} B_{0,4}(X) - 3.93536 \cdot 10^{-06} B_{1,4}(X) + 4.29254 \cdot 10^{-07} B_{2,4}(X) + 4.79267 \cdot 10^{-06} B_{3,4}(X) + 9.15486 \cdot 10^{-06} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -7.23491 \cdot 10^{-09} X^2 + 1.74633 \cdot 10^{-05} X - 8.30118 \cdot 10^{-06}$$
$$= -8.30118 \cdot 10^{-06} B_{0,2} + 4.3046 \cdot 10^{-07} B_{1,2} + 9.15486 \cdot 10^{-06} B_{2,2}$$

$$\widetilde{q}_2 = -6.61744 \cdot 10^{-23} X^4 + 3.54033 \cdot 10^{-22} X^3 - 7.23491 \cdot 10^{-09} X^2 + 1.74633 \cdot 10^{-05} X - 8.30118 \cdot 10^{-06}$$
$$= -8.30118 \cdot 10^{-06} B_{0,4} - 3.93536 \cdot 10^{-06} B_{1,4} + 4.29254 \cdot 10^{-07} B_{2,4} + 4.79266 \cdot 10^{-06} B_{3,4} + 9.15486 \cdot 10^{-06} B_{4,4}$$



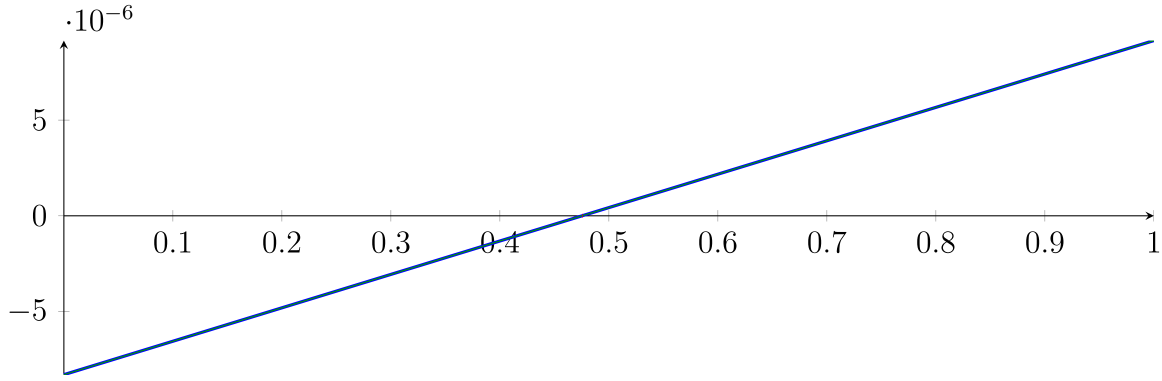
The maximum difference of the Bézier coefficients is $\delta = 4.0829 \cdot 10^{-12}$.
Bounding polynomials M and m:

$$M = -7.23491 \cdot 10^{-09} X^2 + 1.74633 \cdot 10^{-05} X - 8.30118 \cdot 10^{-06}$$
$$m = -7.23491 \cdot 10^{-09} X^2 + 1.74633 \cdot 10^{-05} X - 8.30118 \cdot 10^{-06}$$

Root of M and m:

$$N(M) = \{0.475444, 2413.28\} \qquad N(m) = \{0.475444, 2413.28\}$$

Intersection intervals:



$$[0.475444, 0.475444]$$

Longest intersection interval: $4.67783 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.345492, 0.345492]$,

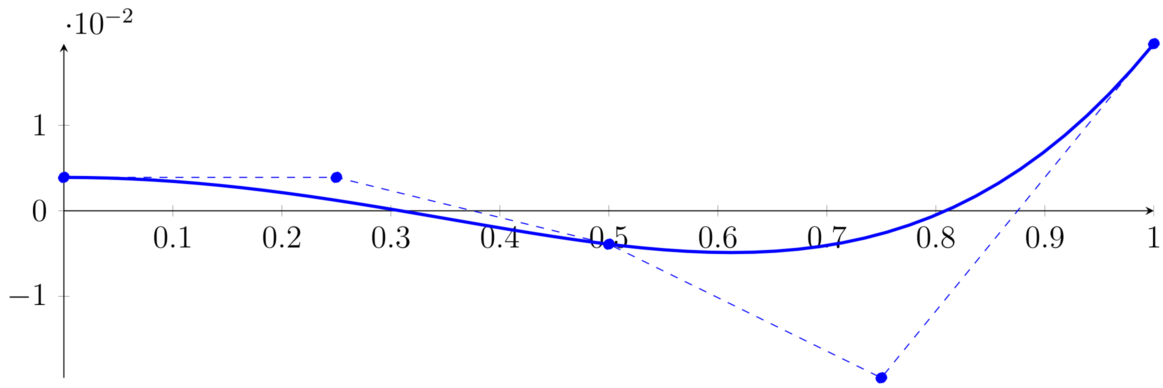
5.10 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.345492, 0.345492]$

Found root in interval $[0.345492, 0.345492]$ at recursion depth 6!

5.11 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

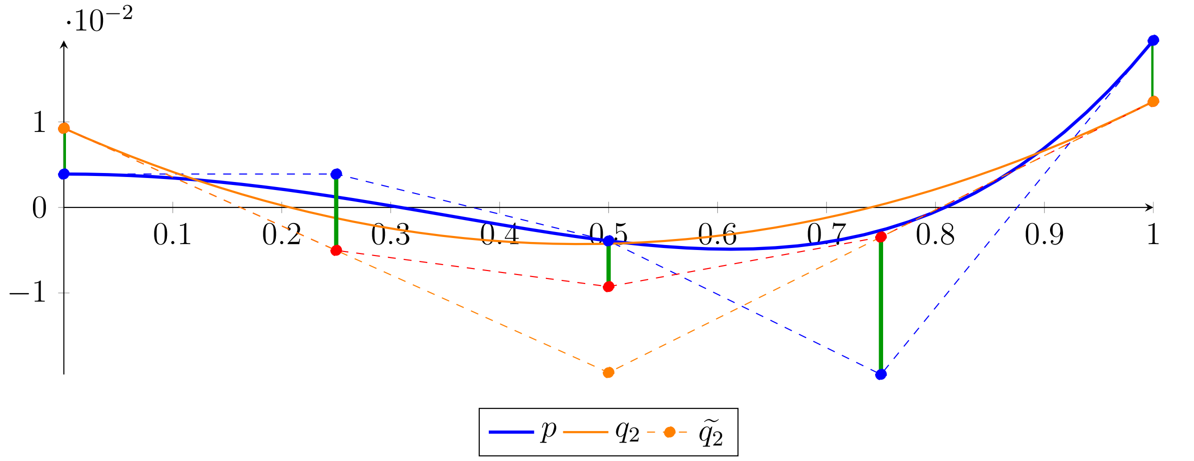
$$\begin{aligned} p &= 0.0625X^4 - 3.38813 \cdot 10^{-21} X^3 - 0.046875X^2 + 3.38813 \cdot 10^{-21} X + 0.00390625 \\ &= 0.00390625 B_{0,4}(X) + 0.00390625 B_{1,4}(X) - 0.00390625 B_{2,4}(X) \\ &\quad - 0.0195312 B_{3,4}(X) + 0.0195312 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0602679X^2 - 0.0571429X + 0.00926339 \\ &= 0.00926339B_{0,2} - 0.019308B_{1,2} + 0.0123884B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.4371\cdot 10^{-18}X^4 - 1.11368\cdot 10^{-17}X^3 + 0.0602679X^2 - 0.0571429X + 0.00926339 \\ &= 0.00926339B_{0,4} - 0.00502232B_{1,4} - 0.00926339B_{2,4} - 0.00345982B_{3,4} + 0.0123884B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0160714$.

Bounding polynomials M and m:

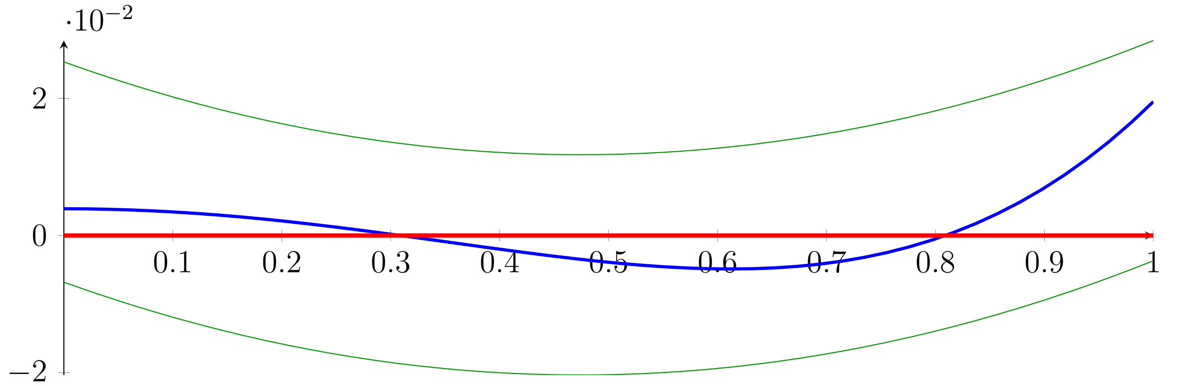
$$M = 0.0602679X^2 - 0.0571429X + 0.0253348$$

$$m = 0.0602679X^2 - 0.0571429X - 0.00680804$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-0.107053, 1.0552\}$$

Intersection intervals:



$[0, 1]$

Longest intersection interval: 1

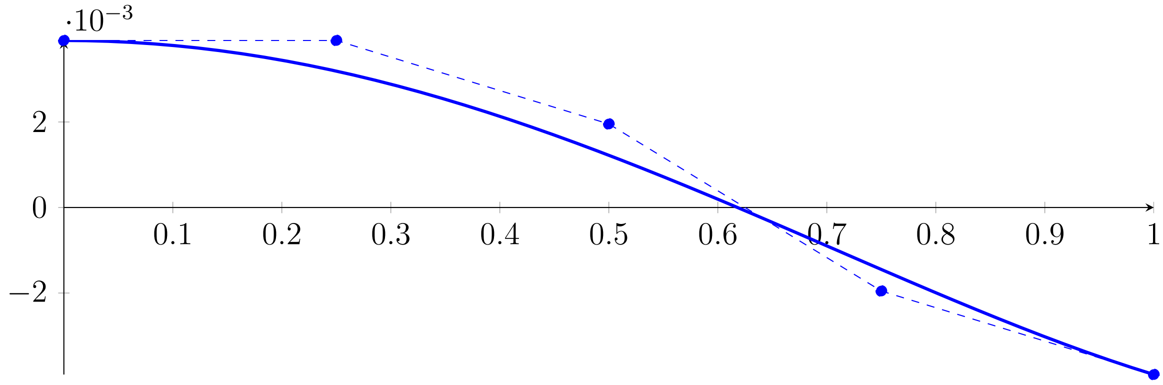
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

Bisection point is very near to a root?!?

5.12 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

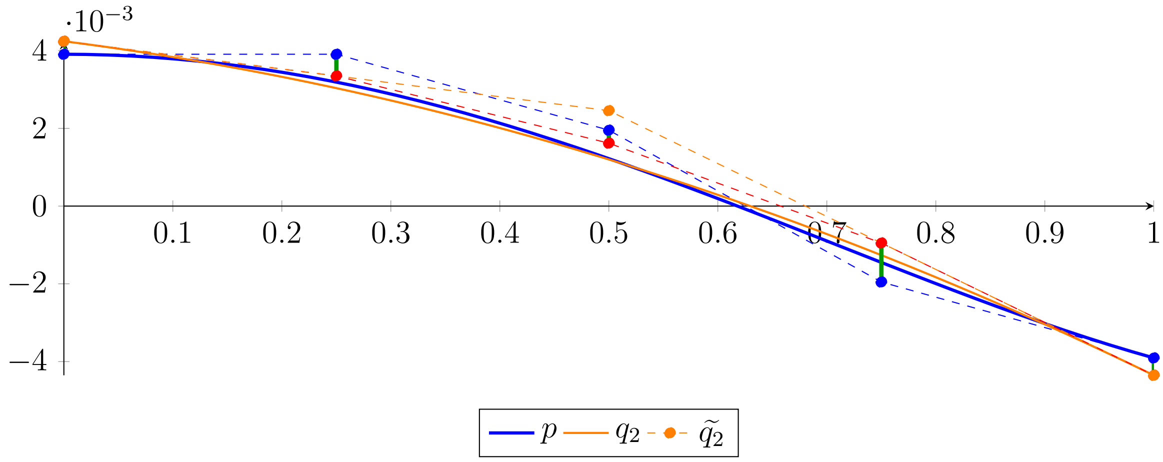
$$\begin{aligned} p &= 0.00390625X^4 - 0.0117188X^2 + 1.69407\cdot 10^{-21}X + 0.00390625 \\ &= 0.00390625B_{0,4}(X) + 0.00390625B_{1,4}(X) + 0.00195313B_{2,4}(X) \\ &\quad - 0.00195312B_{3,4}(X) - 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00502232X^2 - 0.00357143X + 0.00424107 \\ &= 0.00424107B_{0,2} + 0.00245536B_{1,2} - 0.00435268B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.00647\cdot 10^{-19}X^4 + 8.13152\cdot 10^{-19}X^3 - 0.00502232X^2 - 0.00357143X + 0.00424107 \\ &= 0.00424107B_{0,4} + 0.00334821B_{1,4} + 0.0016183B_{2,4} - 0.000948661B_{3,4} - 0.00435268B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00100446$.

Bounding polynomials M and m:

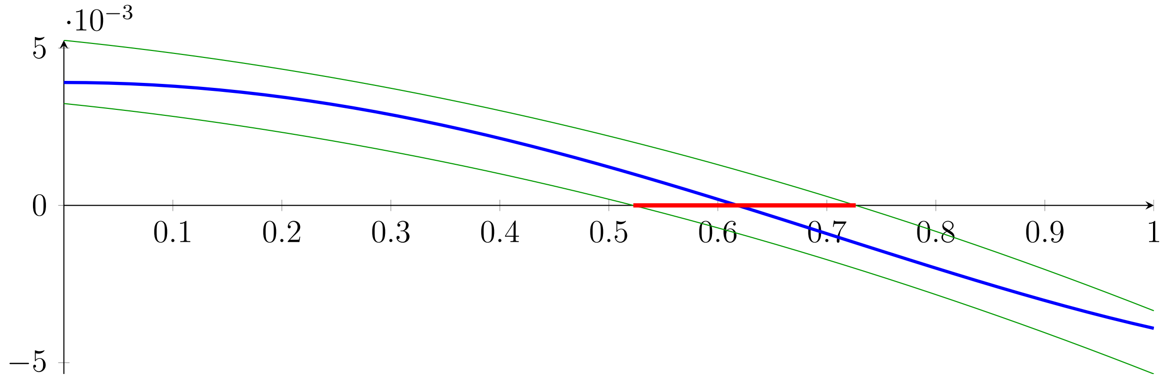
$$M = -0.00502232X^2 - 0.00357143X + 0.00524554$$

$$m = -0.00502232X^2 - 0.00357143X + 0.00323661$$

Root of M and m:

$$N(M) = \{-1.43762, 0.726509\} \qquad N(m) = \{-1.23354, 0.522433\}$$

Intersection intervals:



$[0.522433, 0.726509]$

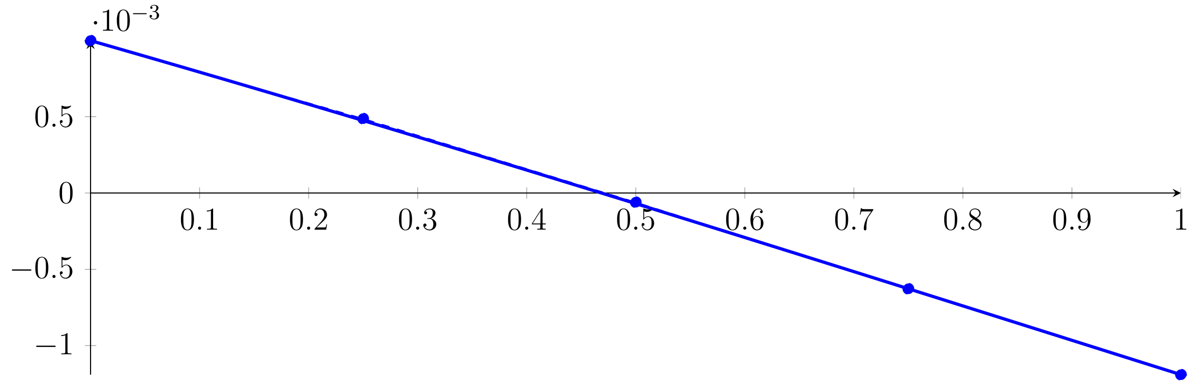
Longest intersection interval: 0.204076

$\Longrightarrow$ Selective recursion: interval 1: $[0.630608, 0.681627]$,

5.13 Recursion Branch 1 2 1 1 in Interval 1: $[0.630608, 0.681627]$

Normalized monomial und Bézier representations and the Bézier polygon:

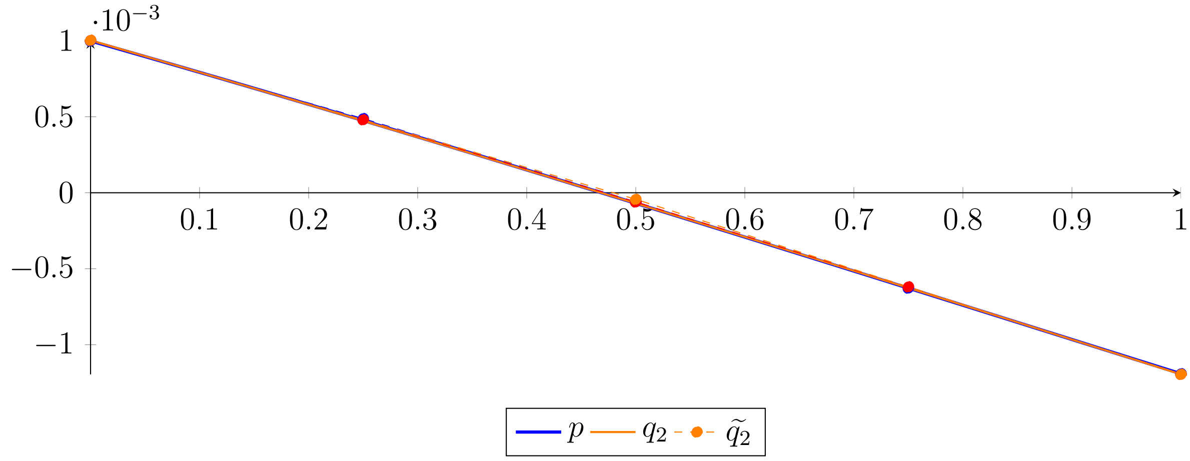
$$\begin{aligned} p &= 6.7753 \cdot 10^{-06} X^4 + 6.93788 \cdot 10^{-05} X^3 - 0.000221637 X^2 - 0.00204414 X + 0.00099877 \\ &= 0.00099877 B_{0,4}(X) + 0.000487736 B_{1,4}(X) - 6.02381 \\ &\quad \cdot 10^{-05} B_{2,4}(X) - 0.000627807 B_{3,4}(X) - 0.00119085 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.000105954 X^2 - 0.00209196 X + 0.00100282 \\ &= 0.00100282 B_{0,2} - 4.31598 \cdot 10^{-05} B_{1,2} - 0.00119509 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.69407 \cdot 10^{-21} X^4 - 2.6258 \cdot 10^{-20} X^3 - 0.000105954 X^2 - 0.00209196 X + 0.00100282 \\ &= 0.00100282 B_{0,4} + 0.00047983 B_{1,4} - 6.08189 \cdot 10^{-05} B_{2,4} - 0.000619127 B_{3,4} - 0.00119509 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.6801 \cdot 10^{-06}$.

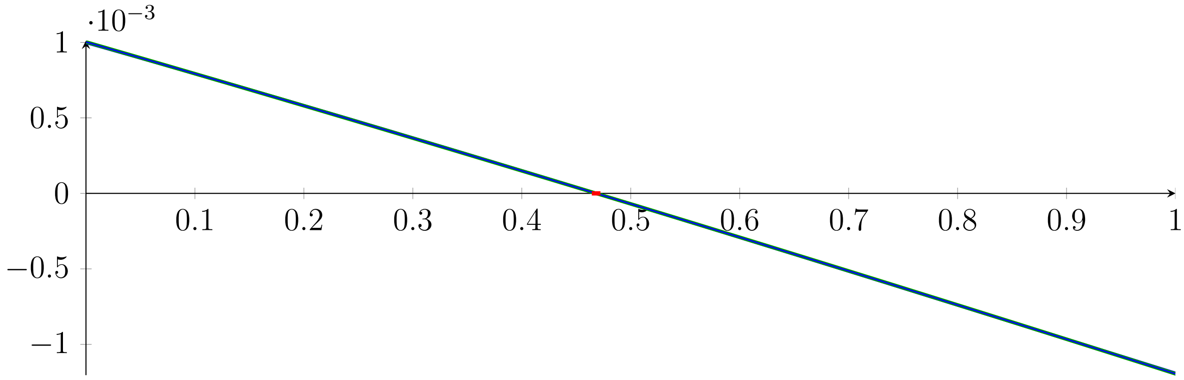
Bounding polynomials M and m:

$$M = -0.000105954X^2 - 0.00209196X + 0.0010115$$
$$m = -0.000105954X^2 - 0.00209196X + 0.000994139$$

Root of M and m:

$$N(M) = \{-20.2162, 0.472224\} \qquad N(m) = \{-20.2083, 0.464301\}$$

Intersection intervals:



$$[0.464301, 0.472224]$$

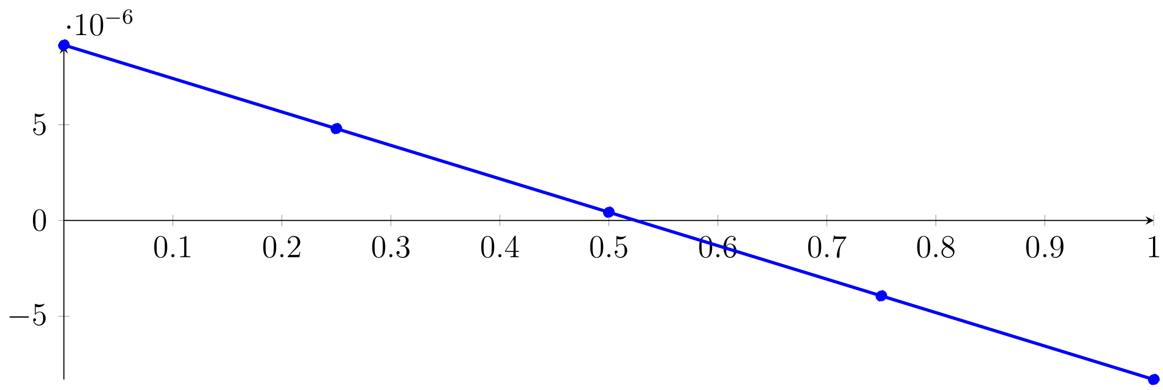
Longest intersection interval: 0.00792274

$\Longrightarrow$ Selective recursion: interval 1: $[0.654296, 0.654701]$,

5.14 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.654296, 0.654701]$

Normalized monomial und Bézier representations and the Bézier polygon:

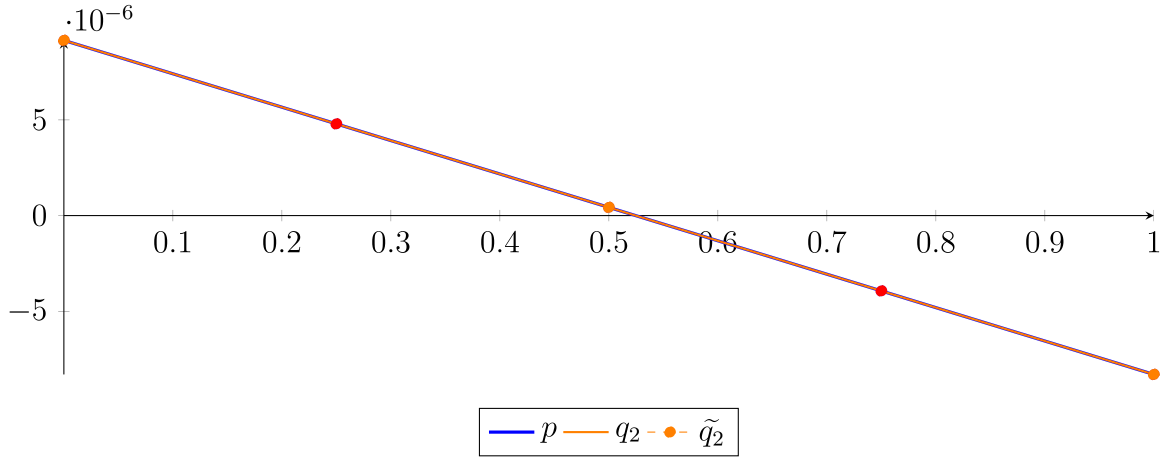
$$\begin{aligned} p &= 2.6695 \cdot 10^{-14} X^4 + 4.07604 \cdot 10^{-11} X^3 - 7.2961 \cdot 10^{-09} X^2 - 1.74488 \cdot 10^{-05} X + 9.15486 \cdot 10^{-06} \\ &= 9.15486 \cdot 10^{-06} B_{0,4}(X) + 4.79267 \cdot 10^{-06} B_{1,4}(X) + 4.29254 \\ &\quad \cdot 10^{-07} B_{2,4}(X) - 3.93536 \cdot 10^{-06} B_{3,4}(X) - 8.30118 \cdot 10^{-06} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -7.23491 \cdot 10^{-09} X^2 - 1.74488 \cdot 10^{-05} X + 9.15486 \cdot 10^{-06} \\ &= 9.15486 \cdot 10^{-06} B_{0,2} + 4.3046 \cdot 10^{-07} B_{1,2} - 8.30118 \cdot 10^{-06} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 8.27181 \cdot 10^{-23} X^4 - 3.07711 \cdot 10^{-22} X^3 - 7.23491 \cdot 10^{-09} X^2 - 1.74488 \cdot 10^{-05} X + 9.15486 \cdot 10^{-06} \\ &= 9.15486 \cdot 10^{-06} B_{0,4} + 4.79266 \cdot 10^{-06} B_{1,4} + 4.29254 \cdot 10^{-07} B_{2,4} - 3.93536 \cdot 10^{-06} B_{3,4} - 8.30118 \cdot 10^{-06} B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.0829 \cdot 10^{-12}$.
Bounding polynomials M and m:

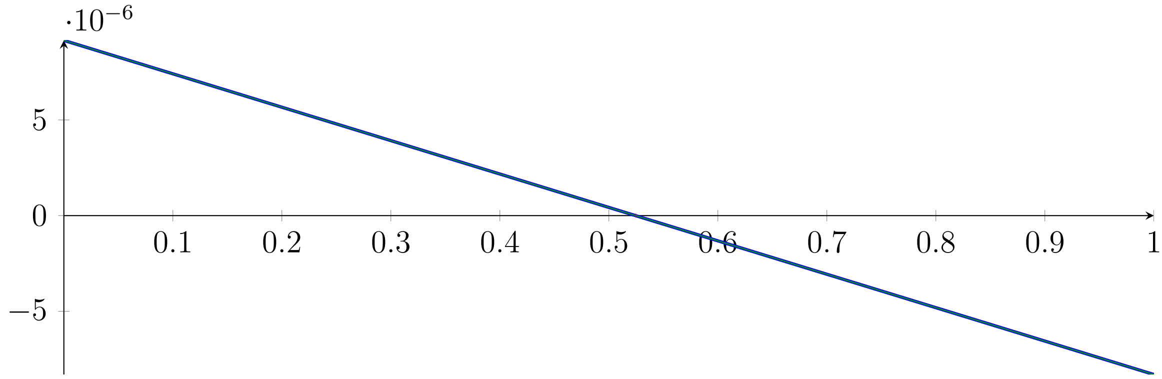
$$M = -7.23491 \cdot 10^{-09} X^2 - 1.74488 \cdot 10^{-05} X + 9.15487 \cdot 10^{-06}$$

$$m = -7.23491 \cdot 10^{-09} X^2 - 1.74488 \cdot 10^{-05} X + 9.15486 \cdot 10^{-06}$$

Root of M and m:

$$N(M) = \{-2412.28, 0.524556\} \qquad N(m) = \{-2412.28, 0.524556\}$$

Intersection intervals:



$$[0.524556, 0.524556]$$

Longest intersection interval: $4.67783 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.654508, 0.654508]$,

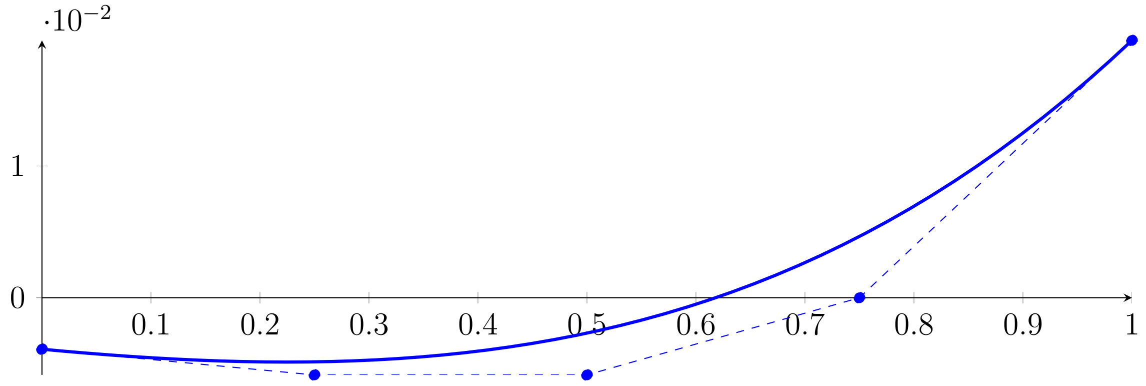
5.15 Recursion Branch 1 2 1 1 1 1 in Interval 1: $[0.654508, 0.654508]$

Found root in interval $[0.654508, 0.654508]$ at recursion depth 6!

5.16 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

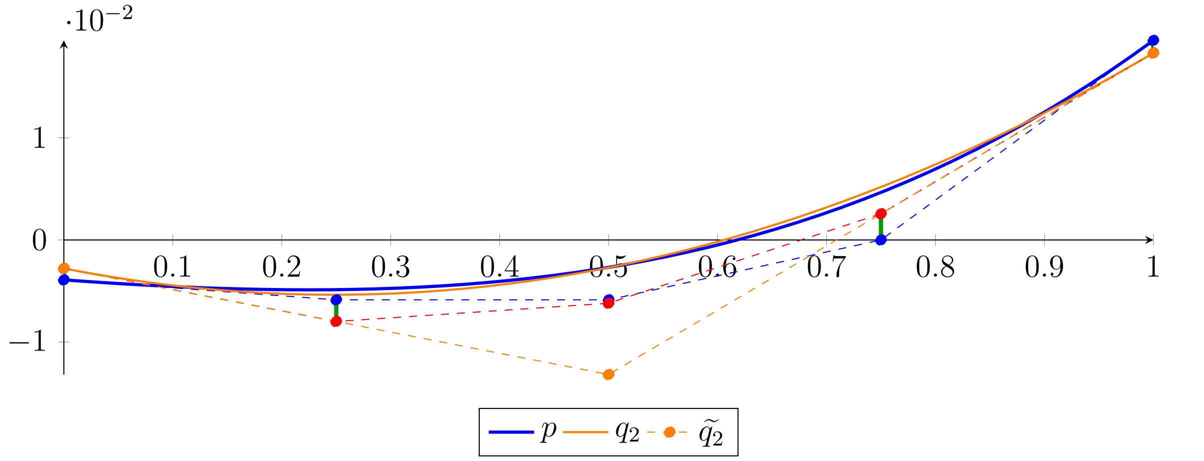
$$\begin{aligned} p &= 0.00390625 X^4 + 0.015625 X^3 + 0.0117187 X^2 - 0.0078125 X - 0.00390625 \\ &= -0.00390625 B_{0,4}(X) - 0.00585937 B_{1,4}(X) - 0.00585937 B_{2,4}(X) \\ &\quad + 3.38813 \cdot 10^{-21} B_{3,4}(X) + 0.0195312 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0418527X^2 - 0.0207589X - 0.00279018 \\ &= -0.00279018B_{0,2} - 0.0131696B_{1,2} + 0.0183036B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 3.70831\cdot 10^{-18}X^4 - 7.38613\cdot 10^{-18}X^3 + 0.0418527X^2 - 0.0207589X - 0.00279018 \\ &= -0.00279018B_{0,4} - 0.00797991B_{1,4} - 0.0061942B_{2,4} + 0.00256696B_{3,4} + 0.0183036B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00256696$.

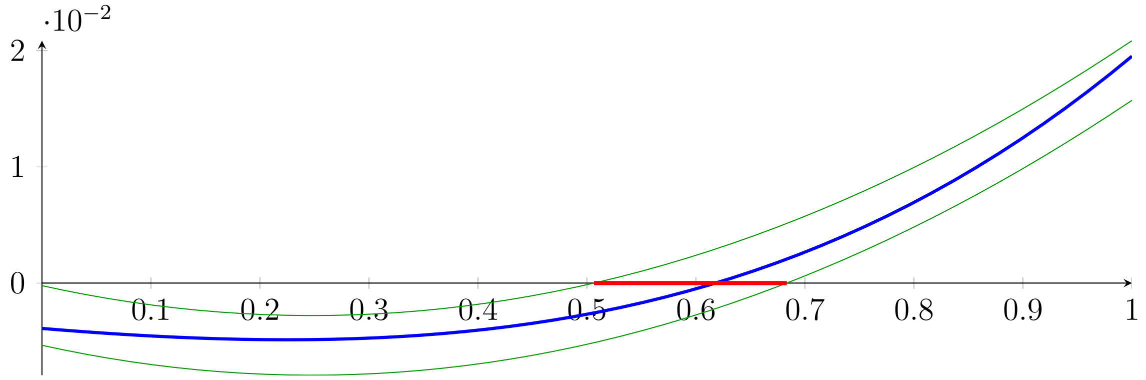
Bounding polynomials M and m:

$$\begin{aligned} M &= 0.0418527X^2 - 0.0207589X - 0.000223214 \\ m &= 0.0418527X^2 - 0.0207589X - 0.00535714 \end{aligned}$$

Root of M and m:

$$N(M) = \{-0.0105292, 0.506529\} \qquad N(m) = \{-0.187321, 0.683321\}$$

Intersection intervals:



$$[0.506529, 0.683321]$$

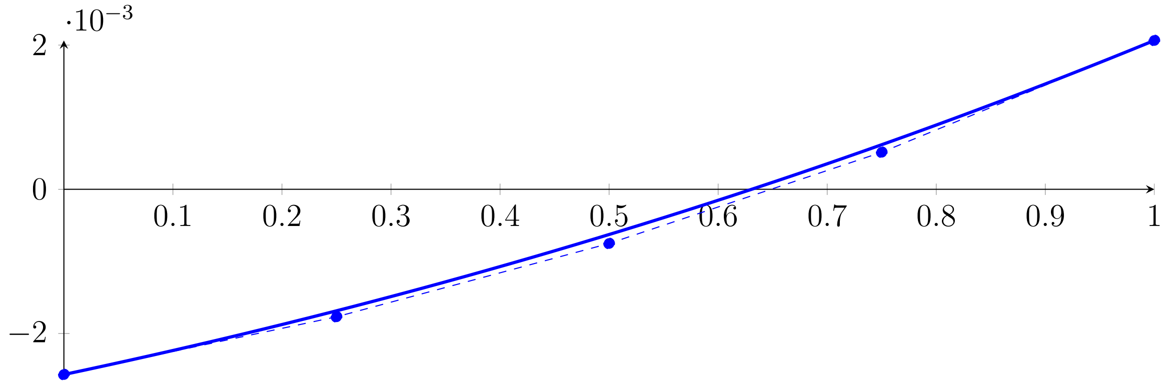
Longest intersection interval: 0.176791

$\Longrightarrow$ Selective recursion: interval 1: $[0.876632, 0.92083]$,

5.17 Recursion Branch 1 2 2 1 in Interval 1: $[0.876632, 0.92083]$

Normalized monomial und Bézier representations and the Bézier polygon:

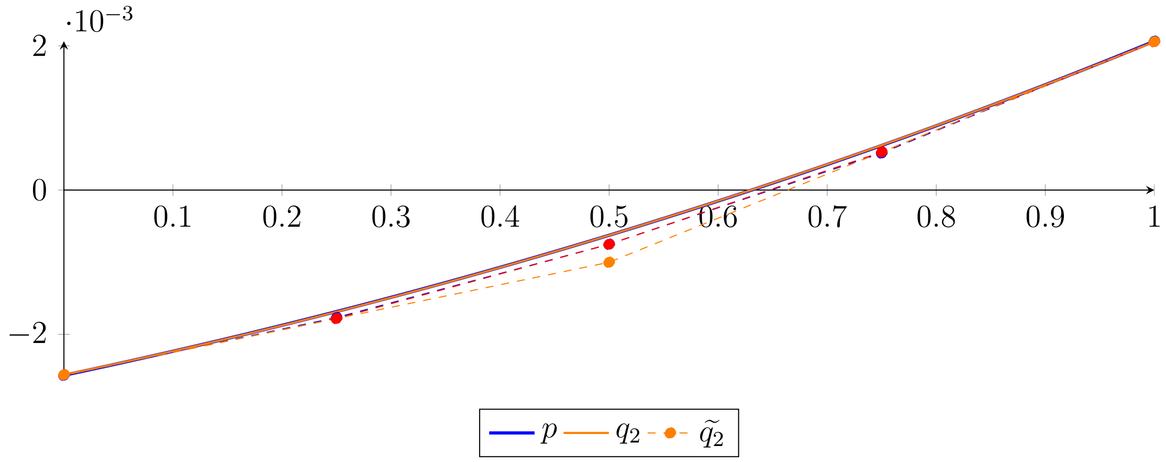
$$\begin{aligned} p &= 3.81597 \cdot 10^{-06} X^4 + 0.000130071 X^3 + 0.00129633 X^2 + 0.00320288 X - 0.00256902 \\ &= -0.00256902 B_{0,4}(X) - 0.0017683 B_{1,4}(X) - 0.000751526 B_{2,4}(X) \\ &\quad + 0.000513823 B_{3,4}(X) + 0.00206408 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00149798 X^2 + 0.00312135 X - 0.00256219 \\ &= -0.00256219 B_{0,2} - 0.00100152 B_{1,2} + 0.00205714 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.05244 \cdot 10^{-19} X^4 - 2.1684 \cdot 10^{-19} X^3 + 0.00149798 X^2 + 0.00312135 X - 0.00256219 \\ &= -0.00256219 B_{0,4} - 0.00178185 B_{1,4} - 0.000751853 B_{2,4} + 0.000527811 B_{3,4} + 0.00205714 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.39884 \cdot 10^{-05}$.

Bounding polynomials M and m:

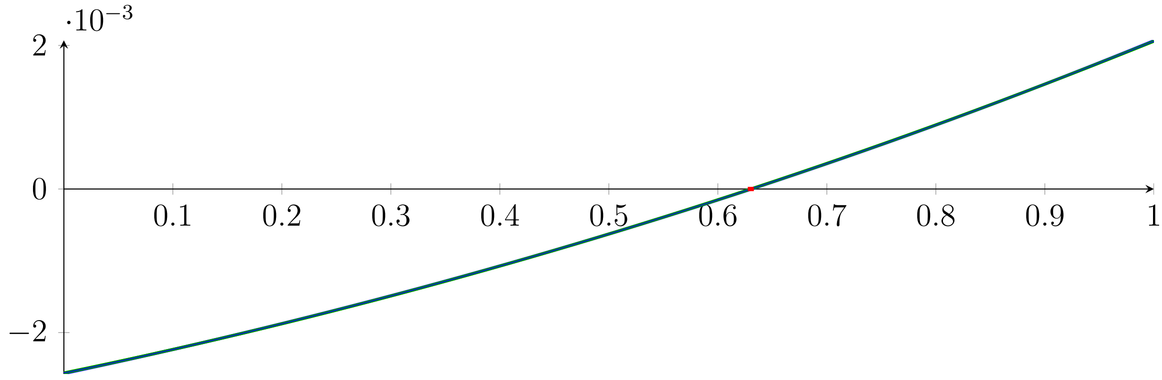
$$M = 0.00149798 X^2 + 0.00312135 X - 0.0025482$$

$$m = 0.00149798 X^2 + 0.00312135 X - 0.00257618$$

Root of M and m:

$$N(M) = \{-2.71115, 0.627443\} \qquad N(m) = \{-2.71673, 0.633028\}$$

Intersection intervals:



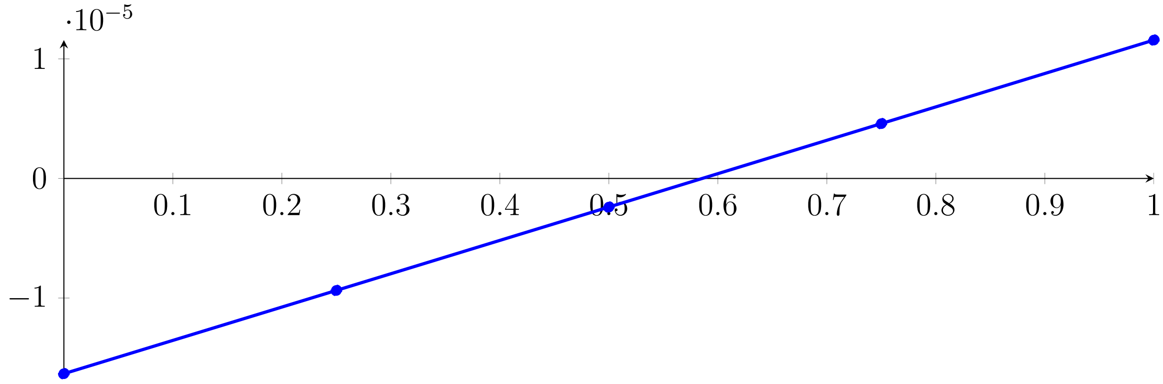
$[0.627443, 0.633028]$

Longest intersection interval: 0.00558472
$\Longrightarrow$ Selective recursion: interval 1: $[0.904364, 0.904611]$,

5.18 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.904364, 0.904611]$

Normalized monomial und Bézier representations and the Bézier polygon:

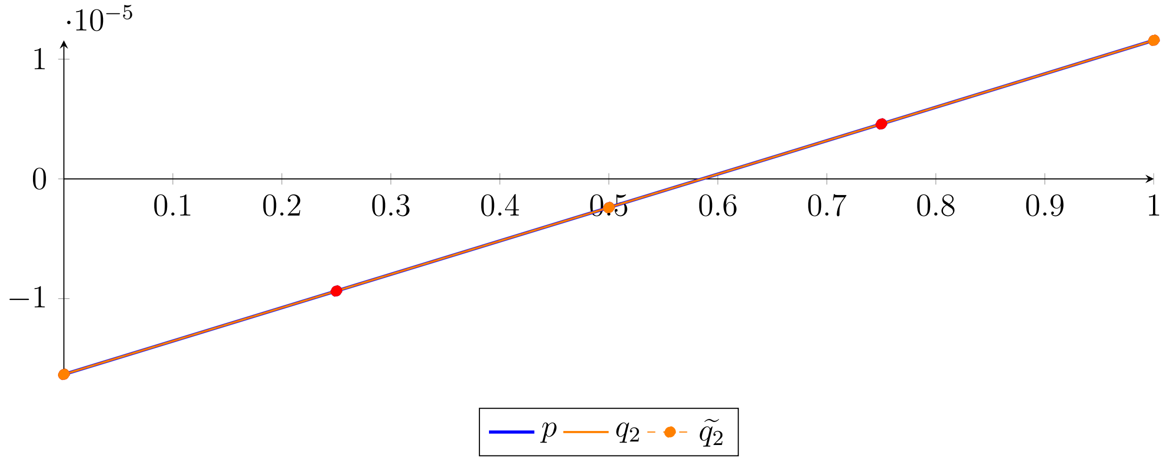
$$p = 3.71203{\cdot}10^{-15}X^4 + 2.43243{\cdot}10^{-11}X^3 + 4.83488{\cdot}10^{-08}X^2 + 2.78511{\cdot}10^{-05}X - 1.63276{\cdot}10^{-05}$$
$$= -1.63276{\cdot}10^{-05}B_{0,4}(X) - 9.36478{\cdot}10^{-06}B_{1,4}(X) - 2.39394 \cdot 10^{-06}B_{2,4}(X) + 4.58497{\cdot}10^{-06}B_{3,4}(X) + 1.15719{\cdot}10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 4.83853{\cdot}10^{-08}X^2 + 2.78511{\cdot}10^{-05}X - 1.63276{\cdot}10^{-05}$$
$$= -1.63276{\cdot}10^{-05}B_{0,2} - 2.402{\cdot}10^{-06}B_{1,2} + 1.15719{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -1.63782{\cdot}10^{-22}X^4 + 4.10282{\cdot}10^{-22}X^3 + 4.83853{\cdot}10^{-08}X^2 + 2.78511{\cdot}10^{-05}X - 1.63276{\cdot}10^{-05}$$
$$= -1.63276{\cdot}10^{-05}B_{0,4} - 9.36478{\cdot}10^{-06}B_{1,4} - 2.39394{\cdot}10^{-06}B_{2,4} + 4.58497{\cdot}10^{-06}B_{3,4} + 1.15719{\cdot}10^{-05}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.43339{\cdot}10^{-12}$.
Bounding polynomials M and m:

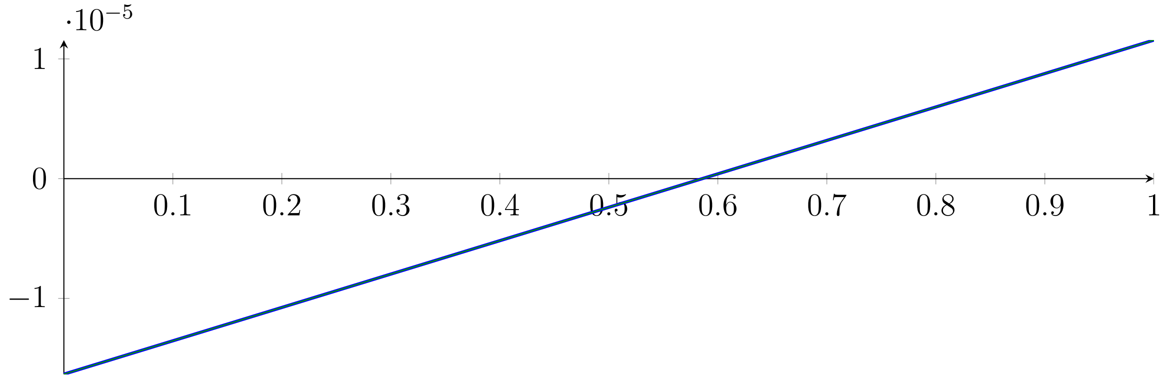
$$M = 4.83853{\cdot}10^{-08}X^2 + 2.78511{\cdot}10^{-05}X - 1.63276{\cdot}10^{-05}$$

$$m = 4.83853{\cdot}10^{-08}X^2 + 2.78511{\cdot}10^{-05}X - 1.63276{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-576.197, 0.585648\} \qquad N(m) = \{-576.197, 0.585649\}$$

Intersection intervals:



$$[0.585648, 0.585649]$$

Longest intersection interval: $1.74388{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.904508, 0.904508]$,

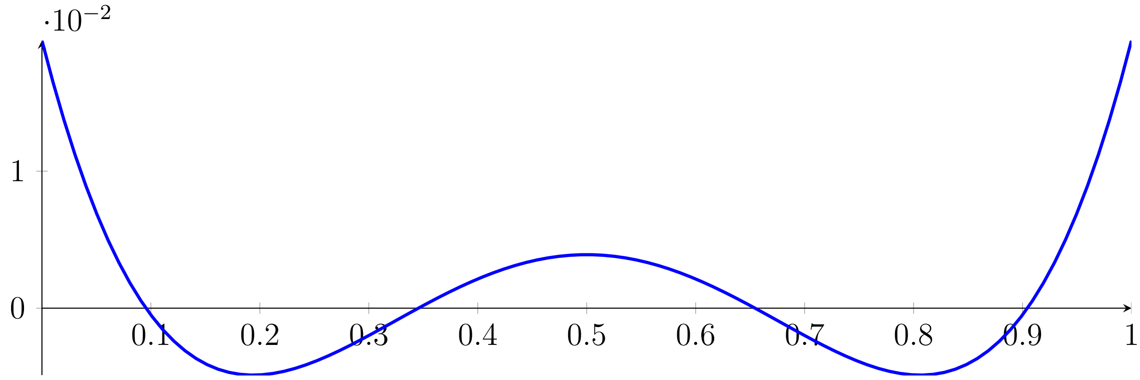
5.19 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.904508, 0.904508]$

Found root in interval $[0.904508, 0.904508]$ at recursion depth 6!

5.20 Result: 4 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$



Result: Root Intervals

$$[0.0954915, 0.0954915], [0.345492, 0.345492], [0.654508, 0.654508], [0.904508, 0.904508]$$

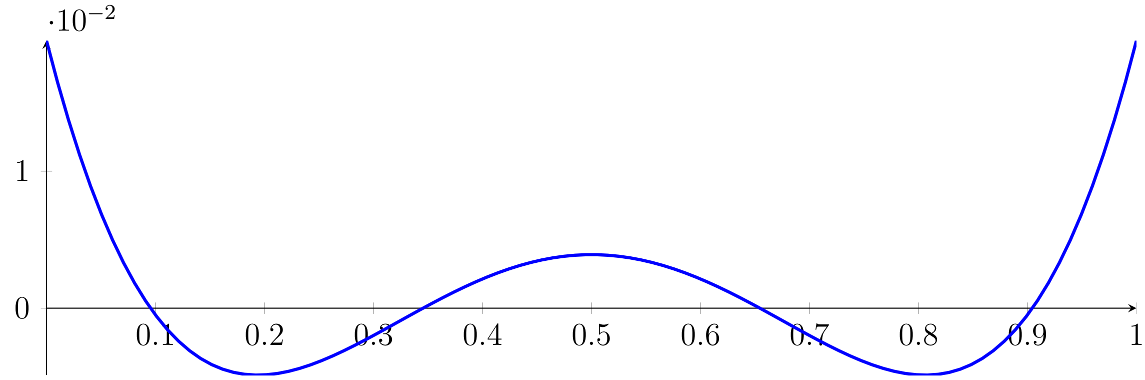
with precision $\varepsilon = 1 \cdot 10^{-06}$.

6 Running `CubeClip` on p4 with epsilon 6

$$1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

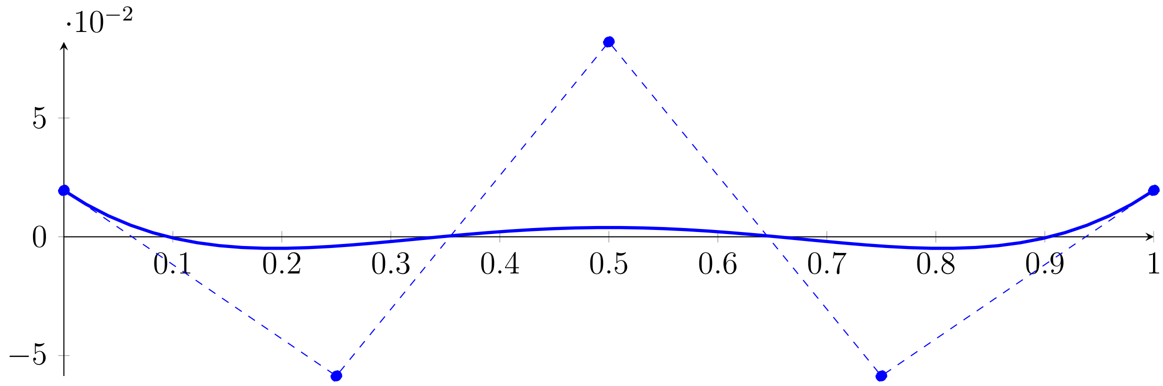
$$p = 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$



6.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

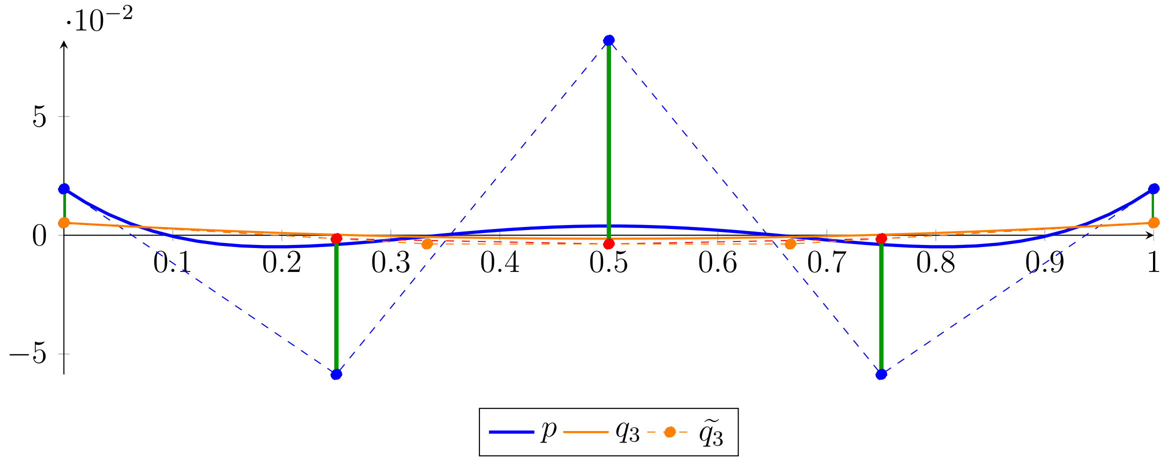
$$\begin{aligned} p &= 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) - 0.0585937B_{1,4}(X) + 0.0820312B_{2,4}(X) \\ &\quad - 0.0585937B_{3,4}(X) + 0.0195312B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.06218 \cdot 10^{-18} X^3 + 0.0267857X^2 - 0.0267857X + 0.00524554 \\ &= 0.00524554B_{0,3} - 0.00368304B_{1,3} - 0.00368304B_{2,3} + 0.00524554B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -4.10811 \cdot 10^{-20} X^4 - 1.03338 \cdot 10^{-18} X^3 + 0.0267857X^2 - 0.0267857X + 0.00524554 \\ &= 0.00524554B_{0,4} - 0.00145089B_{1,4} - 0.00368304B_{2,4} - 0.00145089B_{3,4} + 0.00524554B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

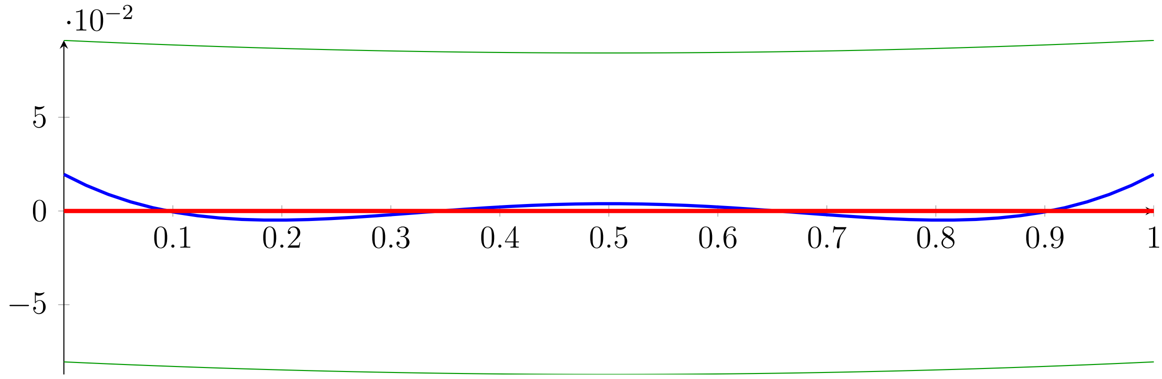
$$M = -1.05032 \cdot 10^{-18} X^3 + 0.0267857 X^2 - 0.0267857 X + 0.0909598$$

$$m = -1.03677 \cdot 10^{-18} X^3 + 0.0267857 X^2 - 0.0267857 X - 0.0804687$$

Root of M and m:

$$N(M) = \{2.55024 \cdot 10^{16}\} \qquad N(m) = \{-1.93311, 1.7168, 2.58358 \cdot 10^{16}\}$$

Intersection intervals:



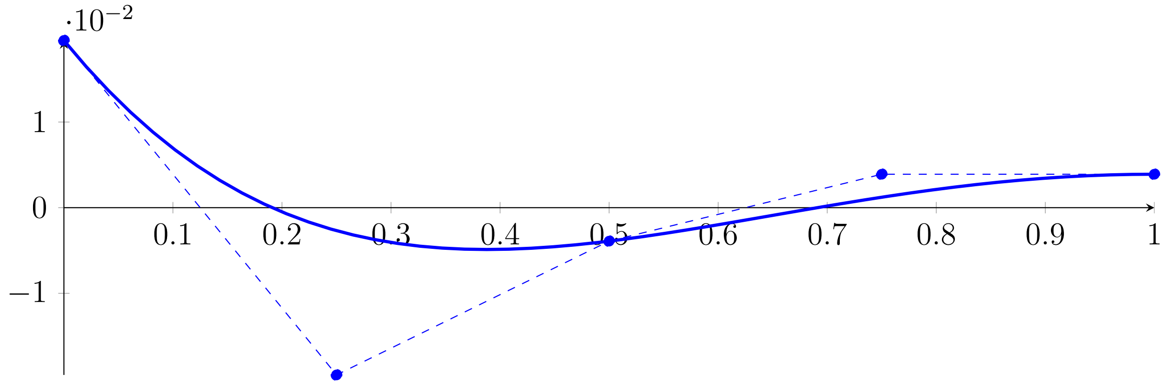
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

6.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

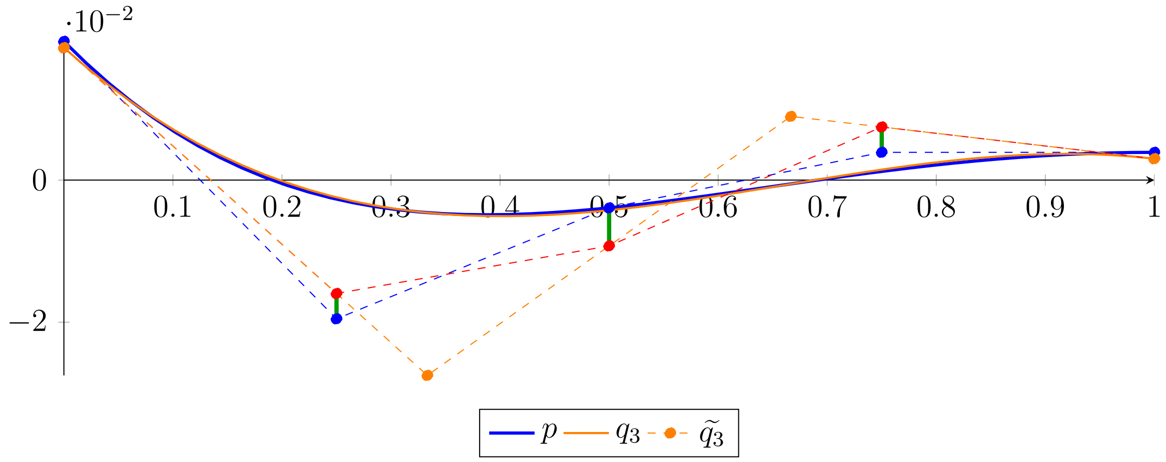
$$\begin{aligned} p &= 0.0625X^4 - 0.25X^3 + 0.328125X^2 - 0.15625X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) - 0.0195312B_{1,4}(X) - 0.00390625B_{2,4}(X) \\ &\quad + 0.00390625B_{3,4}(X) + 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.125X^3 + 0.247768X^2 - 0.138393X + 0.0186384 \\ &= 0.0186384B_{0,3} - 0.0274926B_{1,3} + 0.00896577B_{2,3} + 0.00301339B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 2.19551\cdot 10^{-18}X^4 - 0.125X^3 + 0.247768X^2 - 0.138393X + 0.0186384 \\ &= 0.0186384B_{0,4} - 0.0159598B_{1,4} - 0.00926339B_{2,4} + 0.00747768B_{3,4} + 0.00301339B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00535714$.
Bounding polynomials M and m:

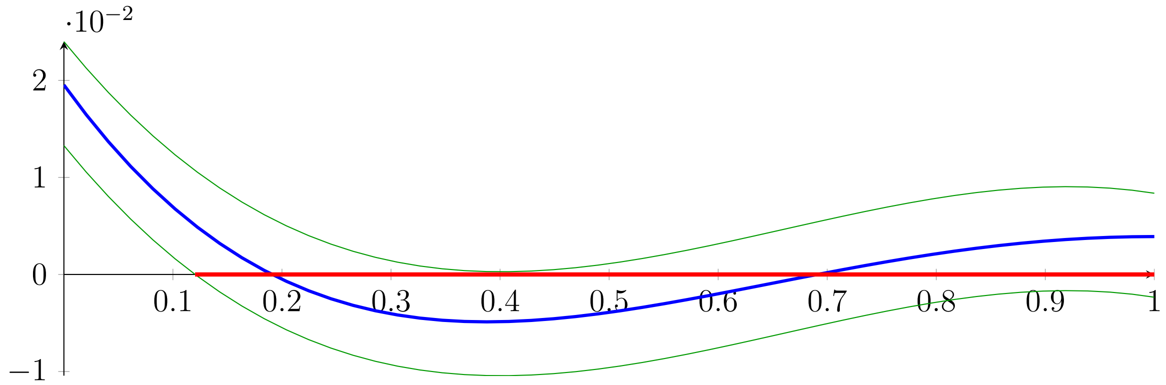
$$M = -0.125X^3 + 0.247768X^2 - 0.138393X + 0.0239955$$

$$m = -0.125X^3 + 0.247768X^2 - 0.138393X + 0.0132813$$

Root of M and m:

$$N(M) = \{1.18398\} \qquad N(m) = \{0.120308\}$$

Intersection intervals:



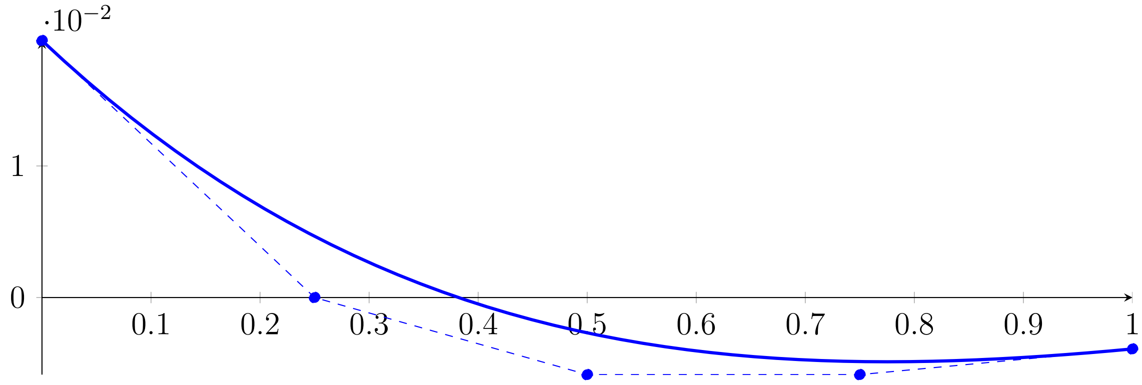
$$[0.120308, 1]$$

Longest intersection interval: 0.879692
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$
Bisection point is very near to a root?!?

6.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

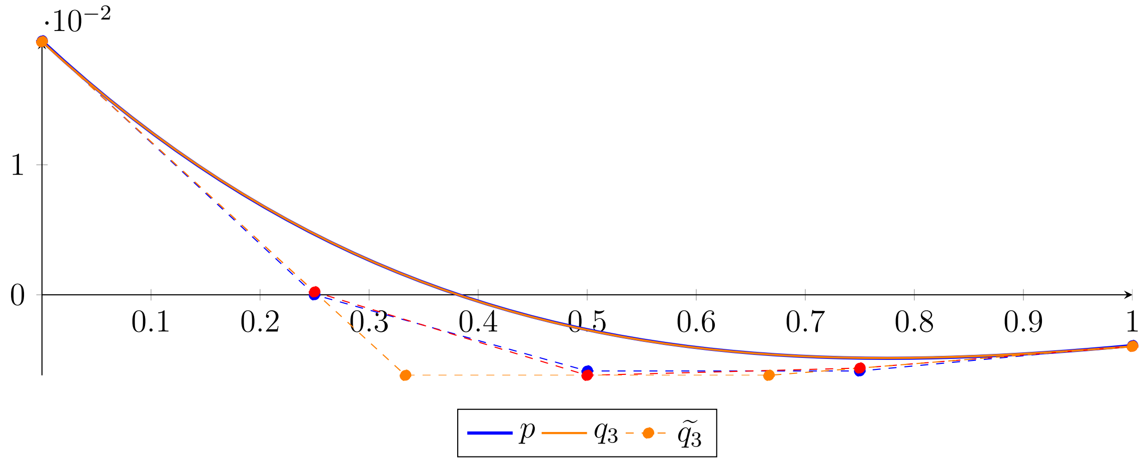
$$\begin{aligned} p &= 0.00390625X^4 - 0.03125X^3 + 0.0820312X^2 - 0.078125X + 0.0195312 \\ &= 0.0195312B_{0,4}(X) + 1.69407{\cdot}10^{-21}B_{1,4}(X) - 0.00585937B_{2,4}(X) \\ &\quad - 0.00585937B_{3,4}(X) - 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.0234375X^3 + 0.0770089X^2 - 0.0770089X + 0.0194754 \\ &= 0.0194754B_{0,3} - 0.0061942B_{1,3} - 0.0061942B_{2,3} - 0.00396205B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.38913{\cdot}10^{-19}X^4 - 0.0234375X^3 + 0.0770089X^2 - 0.0770089X + 0.0194754 \\ &= 0.0194754B_{0,4} + 0.000223214B_{1,4} - 0.0061942B_{2,4} - 0.00563616B_{3,4} - 0.00396205B_{4,4} \end{aligned}$$



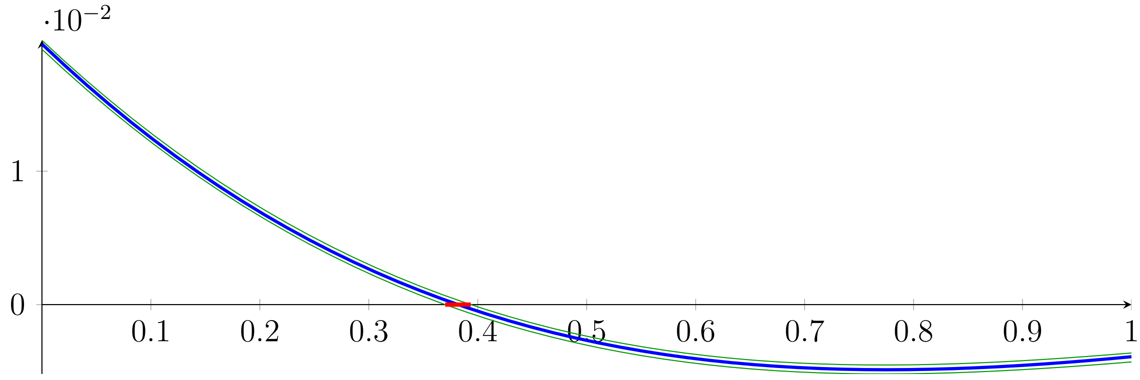
The maximum difference of the Bézier coefficients is $\delta = 0.000334821$.
Bounding polynomials M and m:

$$M = -0.0234375X^3 + 0.0770089X^2 - 0.0770089X + 0.0198103$$
$$m = -0.0234375X^3 + 0.0770089X^2 - 0.0770089X + 0.0191406$$

Root of M and m:

$$N(M) = \{0.393626\} \qquad\qquad N(m) = \{0.37009\}$$

Intersection intervals:



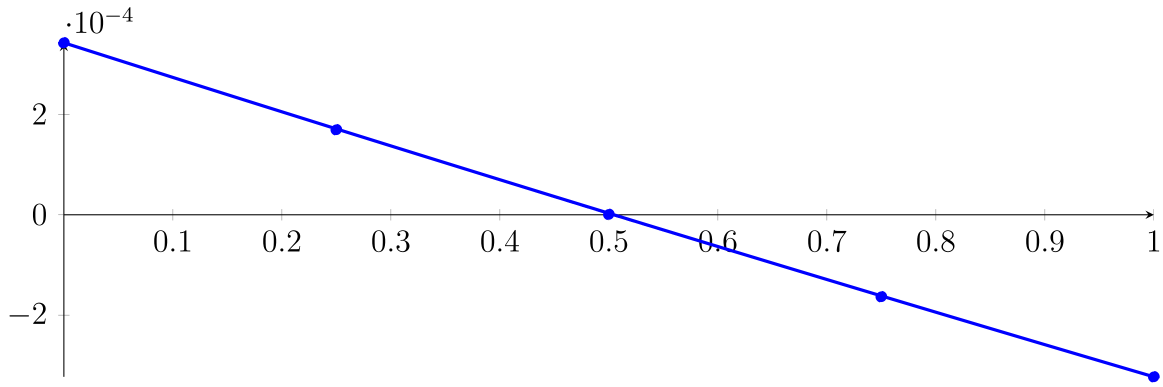
$[0.37009, 0.393626]$

Longest intersection interval: 0.023536
$\Longrightarrow$ Selective recursion: interval 1: $[0.0925225, 0.0984065]$,

6.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.0925225, 0.0984065]$

Normalized monomial und Bézier representations and the Bézier polygon:

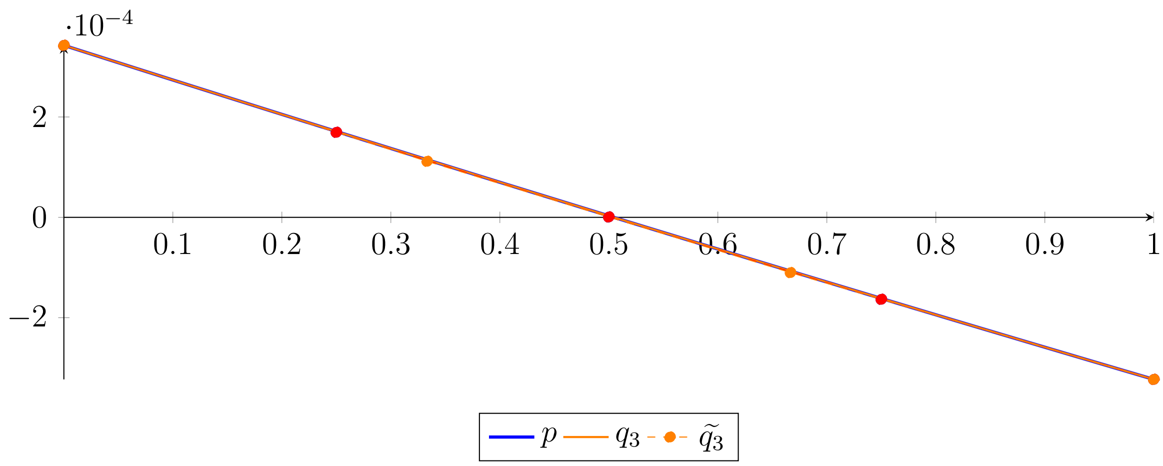
$$\begin{aligned} p &= 1.19864 \cdot 10^{-09} X^4 - 3.32032 \cdot 10^{-07} X^3 + 2.79993 \cdot 10^{-05} X^2 - 0.000693269X + 0.000342734 \\ &= 0.000342734 B_{0,4}(X) + 0.000169417 B_{1,4}(X) + 7.65889 \\ &\quad \cdot 10^{-07} B_{2,4}(X) - 0.000163301 B_{3,4}(X) - 0.000322867 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -3.29635 \cdot 10^{-07} X^3 + 2.79977 \cdot 10^{-05} X^2 - 0.000693268X + 0.000342734 \\ &= 0.000342734 B_{0,3} + 0.000111644 B_{1,3} - 0.000110113 B_{2,3} - 0.000322867 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -7.94093 \cdot 10^{-21} X^4 - 3.29635 \cdot 10^{-07} X^3 + 2.79977 \cdot 10^{-05} X^2 - 0.000693268X + 0.000342734 \\ &= 0.000342734 B_{0,4} + 0.000169417 B_{1,4} + 7.65786 \cdot 10^{-07} B_{2,4} - 0.000163301 B_{3,4} - 0.000322867 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.0274 \cdot 10^{-10}$.
Bounding polynomials M and m:

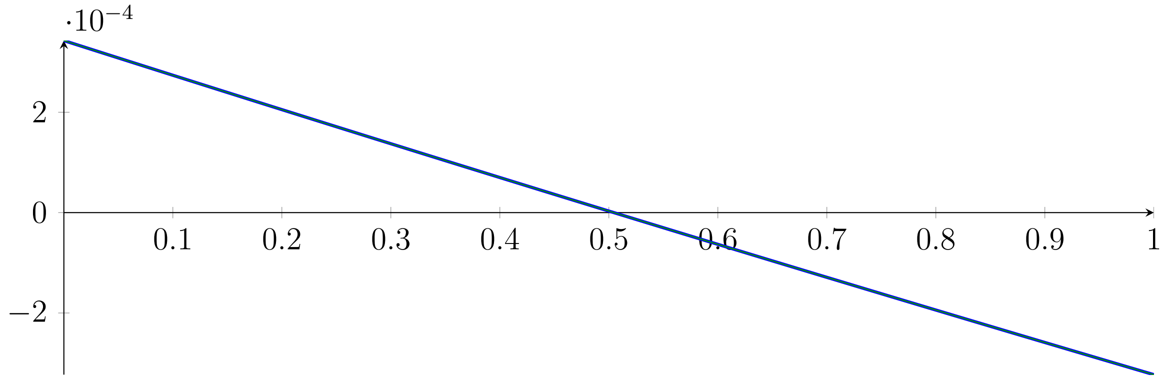
$$M = -3.29635 \cdot 10^{-07} X^3 + 2.79977 \cdot 10^{-05} X^2 - 0.000693268 X + 0.000342734$$

$$m = -3.29635 \cdot 10^{-07} X^3 + 2.79977 \cdot 10^{-05} X^2 - 0.000693268 X + 0.000342734$$

Root of M and m:

$$N(M) = \{0.504596\} \qquad N(m) = \{0.504595\}$$

Intersection intervals:



$$[0.504595, 0.504596]$$

Longest intersection interval: $3.08871 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0954915, 0.0954915]$,

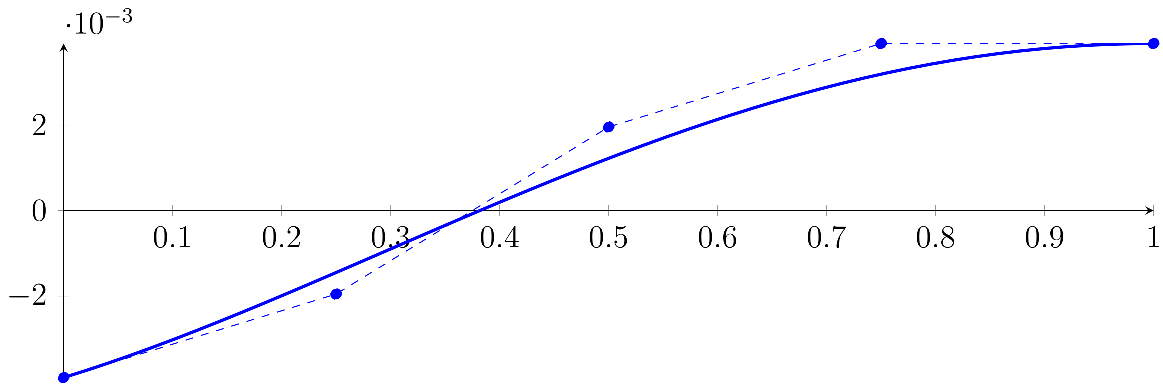
6.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0954915, 0.0954915]$

Found root in interval $[0.0954915, 0.0954915]$ at recursion depth 5!

6.6 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

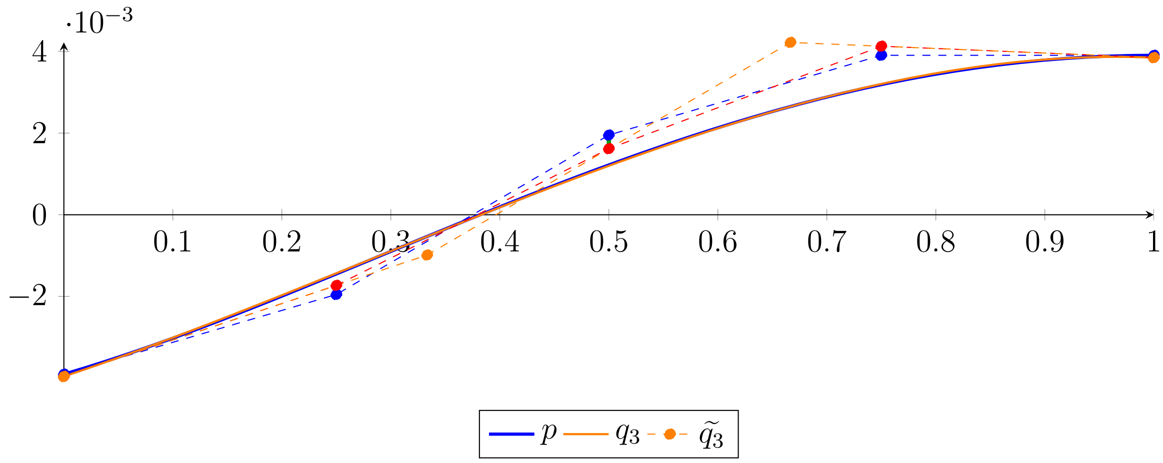
$$\begin{aligned} p &= 0.00390625 X^4 - 0.015625 X^3 + 0.0117188 X^2 + 0.0078125 X - 0.00390625 \\ &= -0.00390625 B_{0,4}(X) - 0.00195312 B_{1,4}(X) + 0.00195313 B_{2,4}(X) \\ &\quad + 0.00390625 B_{3,4}(X) + 0.00390625 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_3 &= -0.0078125X^3 + 0.00669643X^2 + 0.00892857X - 0.00396205 \\
&= -0.00396205B_{0,3} - 0.000985863B_{1,3} + 0.00422247B_{2,3} + 0.00385045B_{3,3}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_3 &= 2.0117\cdot 10^{-19}X^4 - 0.0078125X^3 + 0.00669643X^2 + 0.00892857X - 0.00396205 \\
&= -0.00396205B_{0,4} - 0.00172991B_{1,4} + 0.0016183B_{2,4} + 0.00412946B_{3,4} + 0.00385045B_{4,4}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 0.000334821$.
Bounding polynomials M and m:

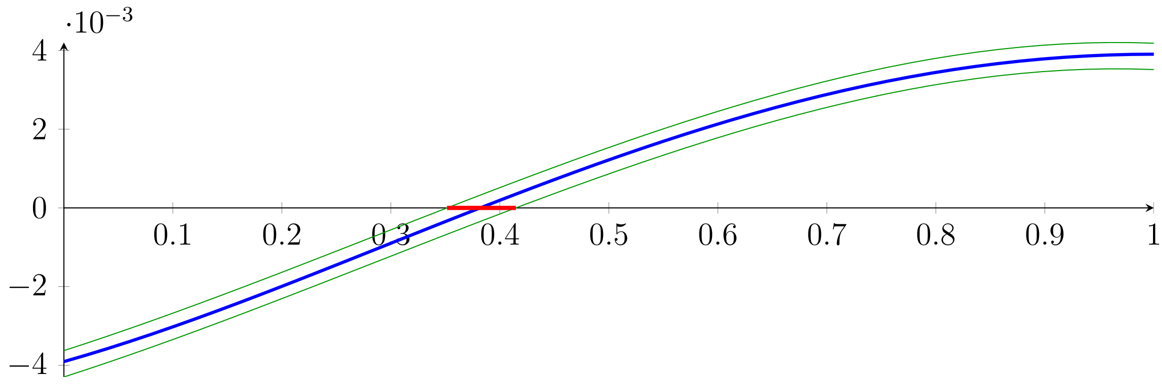
$$M = -0.0078125X^3 + 0.00669643X^2 + 0.00892857X - 0.00362723$$

$$m = -0.0078125X^3 + 0.00669643X^2 + 0.00892857X - 0.00429688$$

Root of M and m:

$$N(M) = \{-0.923863, 0.351571, 1.42943\} \qquad N(m) = \{-0.951491, 0.414676, 1.39396\}$$

Intersection intervals:



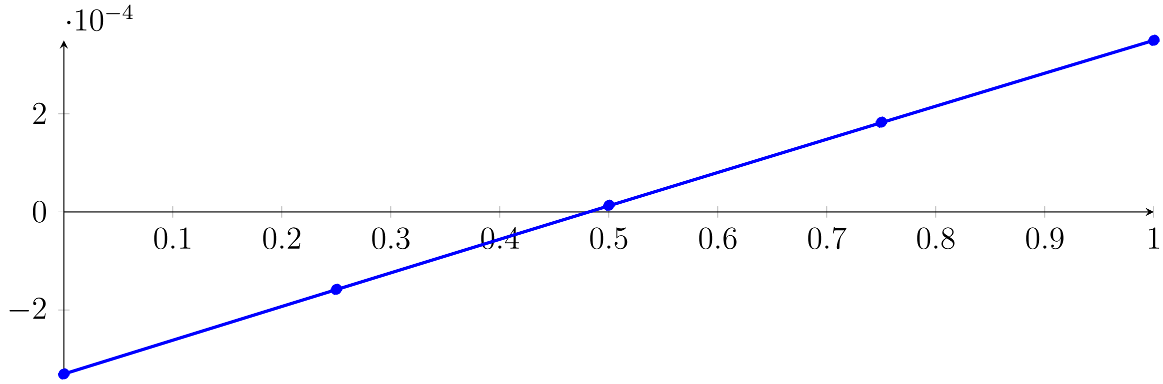
$$[0.351571, 0.414676]$$

Longest intersection interval: 0.0631044
$\Longrightarrow$ Selective recursion: interval 1: $[0.337893, 0.353669]$,

6.7 Recursion Branch 1 1 2 1 in Interval 1: $[0.337893, 0.353669]$

Normalized monomial und Bézier representations and the Bézier polygon:

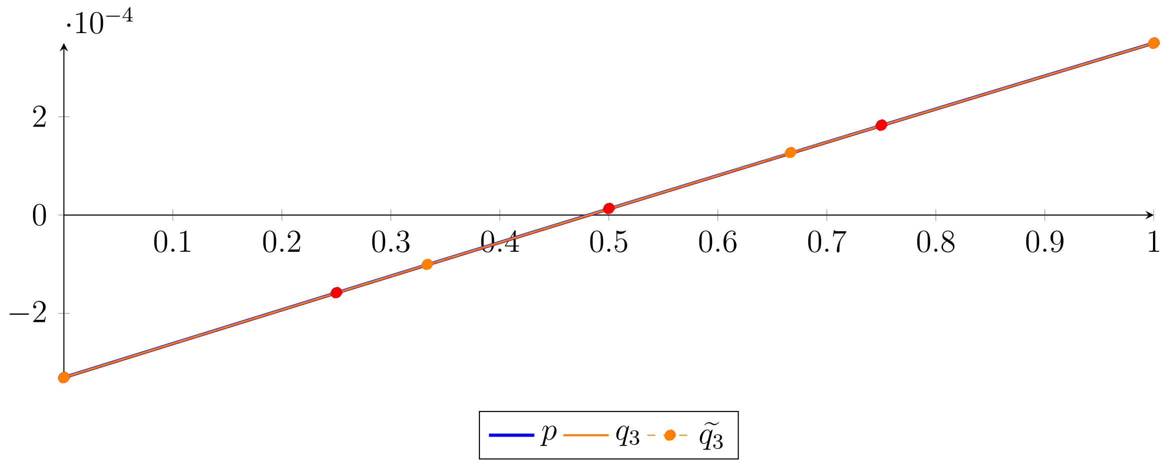
$$\begin{aligned} p &= 6.19438 \cdot 10^{-08} X^4 - 2.54601 \cdot 10^{-06} X^3 - 7.42365 \cdot 10^{-06} X^2 + 0.000690209 X - 0.00033044 \\ &= -0.00033044 B_{0,4}(X) - 0.000157888 B_{1,4}(X) + 1.34269 \\ &\quad \cdot 10^{-05} B_{2,4}(X) + 0.000182868 B_{3,4}(X) + 0.000349861 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.42213 \cdot 10^{-06} X^3 - 7.50329 \cdot 10^{-06} X^2 + 0.000690227 X - 0.000330441 \\ &= -0.000330441 B_{0,3} - 0.000100366 B_{1,3} + 0.000127209 B_{2,3} + 0.00034986 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 7.25272 \cdot 10^{-21} X^4 - 2.42213 \cdot 10^{-06} X^3 - 7.50329 \cdot 10^{-06} X^2 + 0.000690227 X - 0.000330441 \\ &= -0.000330441 B_{0,4} - 0.000157885 B_{1,4} + 1.34216 \cdot 10^{-05} B_{2,4} + 0.000182872 B_{3,4} + 0.00034986 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.30947 \cdot 10^{-09}$.
Bounding polynomials M and m:

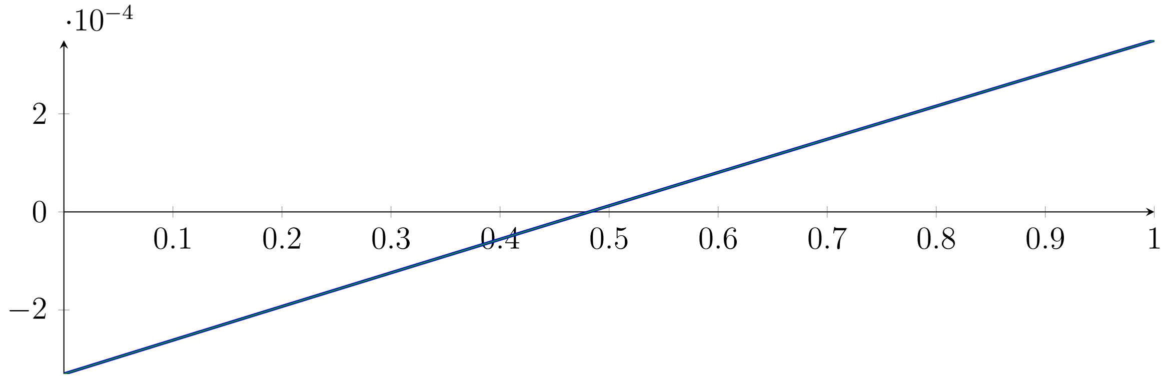
$$M = -2.42213 \cdot 10^{-06} X^3 - 7.50329 \cdot 10^{-06} X^2 + 0.000690227 X - 0.000330436$$

$$m = -2.42213 \cdot 10^{-06} X^3 - 7.50329 \cdot 10^{-06} X^2 + 0.000690227 X - 0.000330446$$

Root of M and m:

$$N(M) = \{-18.7145, 0.481649, 15.135\} \qquad N(m) = \{-18.7145, 0.481665, 15.135\}$$

Intersection intervals:



$[0.481649, 0.481665]$

Longest intersection interval: $1.5586 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.345491, 0.345492]$,

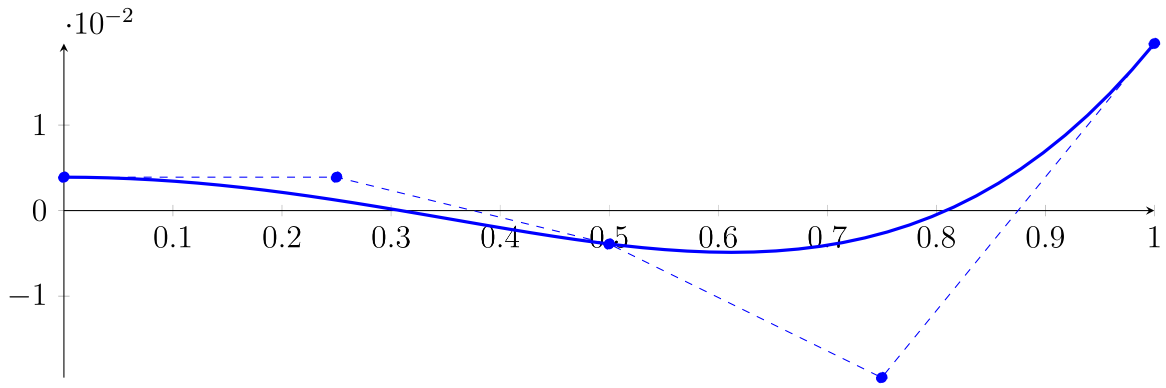
6.8 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.345491, 0.345492]$

Found root in interval $[0.345491, 0.345492]$ at recursion depth 5!

6.9 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

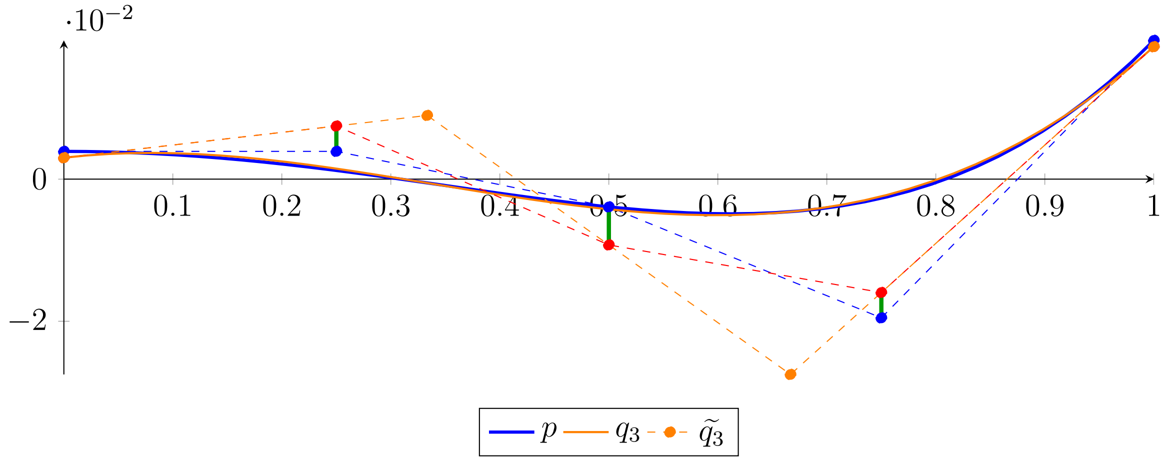
$$\begin{aligned} p &= 0.0625X^4 - 3.38813 \cdot 10^{-21} X^3 - 0.046875X^2 + 3.38813 \cdot 10^{-21} X + 0.00390625 \\ &= 0.00390625 B_{0,4}(X) + 0.00390625 B_{1,4}(X) - 0.00390625 B_{2,4}(X) \\ &\quad - 0.0195312 B_{3,4}(X) + 0.0195312 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.125X^3 - 0.127232X^2 + 0.0178571X + 0.00301339 \\ &= 0.00301339 B_{0,3} + 0.00896577 B_{1,3} - 0.0274926 B_{2,3} + 0.0186384 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -2.26899 \cdot 10^{-18} X^4 + 0.125X^3 - 0.127232X^2 + 0.0178571X + 0.00301339 \\ &= 0.00301339 B_{0,4} + 0.00747768 B_{1,4} - 0.00926339 B_{2,4} - 0.0159598 B_{3,4} + 0.0186384 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00535714$.
Bounding polynomials M and m:

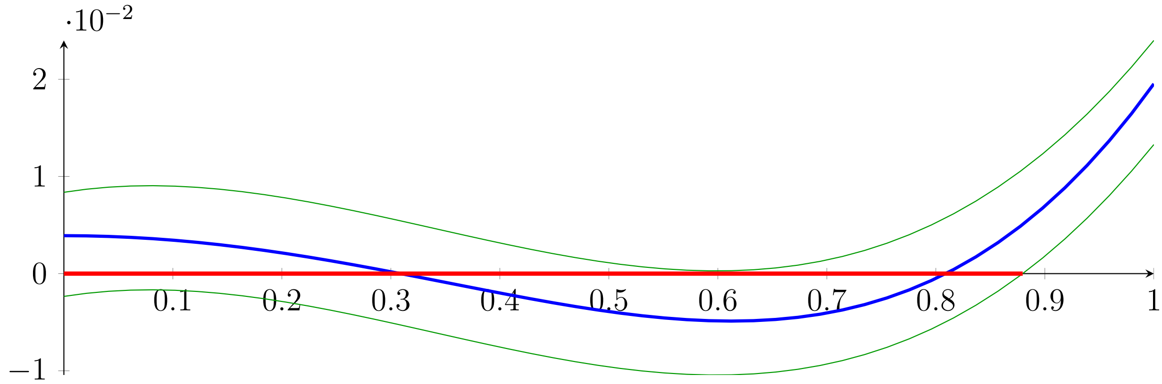
$$M = 0.125X^3 - 0.127232X^2 + 0.0178571X + 0.00837054$$

$$m = 0.125X^3 - 0.127232X^2 + 0.0178571X - 0.00234375$$

Root of M and m:

$$N(M) = \{-0.183981\} \qquad\qquad N(m) = \{0.879692\}$$

Intersection intervals:



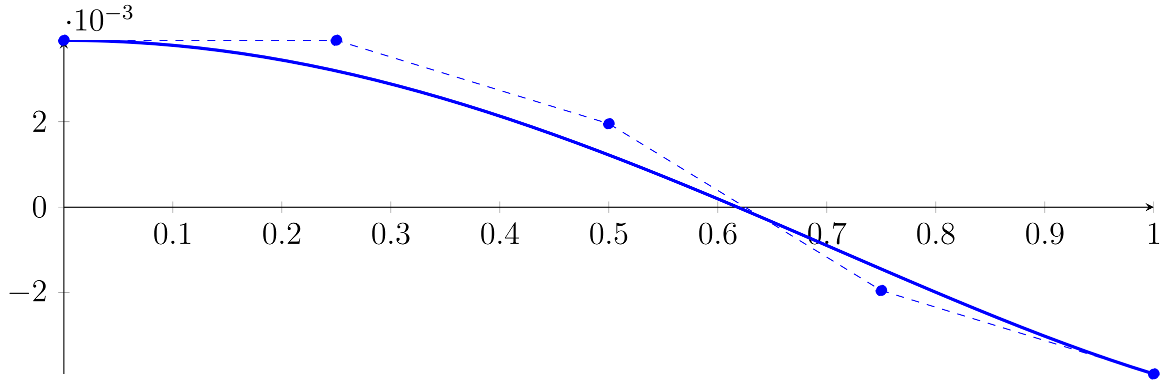
$$[0, 0.879692]$$

Longest intersection interval: 0.879692
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$
Bisection point is very near to a root?!?

6.10 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

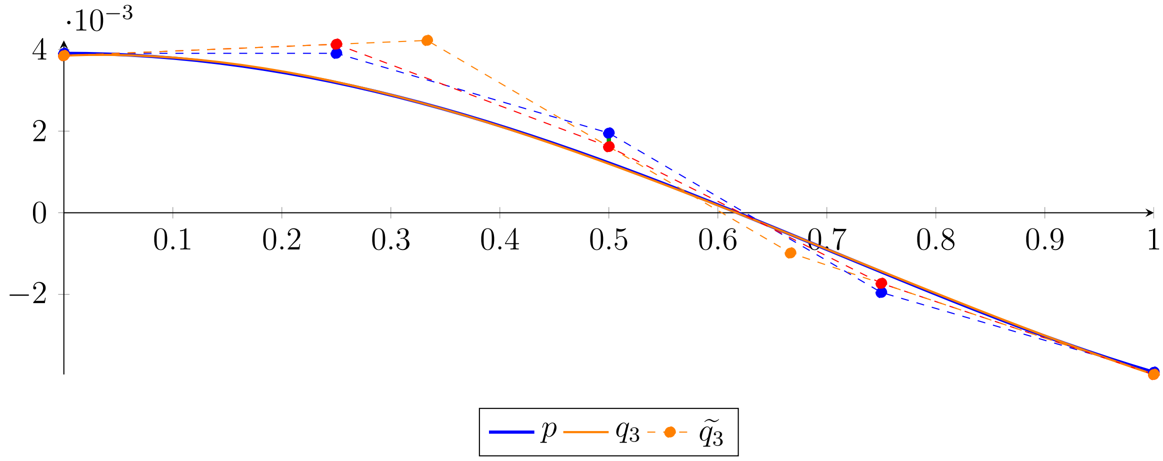
$$\begin{aligned} p &= 0.00390625X^4 - 0.0117188X^2 + 1.69407{\cdot}10^{-21}X + 0.00390625 \\ &= 0.00390625B_{0,4}(X) + 0.00390625B_{1,4}(X) + 0.00195313B_{2,4}(X) \\ &\quad - 0.00195312B_{3,4}(X) - 0.00390625B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.0078125X^3 - 0.0167411X^2 + 0.00111607X + 0.00385045 \\ &= 0.00385045B_{0,3} + 0.00422247B_{1,3} - 0.000985863B_{2,3} - 0.00396205B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -2.80156\cdot 10^{-19}X^4 + 0.0078125X^3 - 0.0167411X^2 + 0.00111607X + 0.00385045 \\ &= 0.00385045B_{0,4} + 0.00412946B_{1,4} + 0.0016183B_{2,4} - 0.00172991B_{3,4} - 0.00396205B_{4,4} \end{aligned}$$



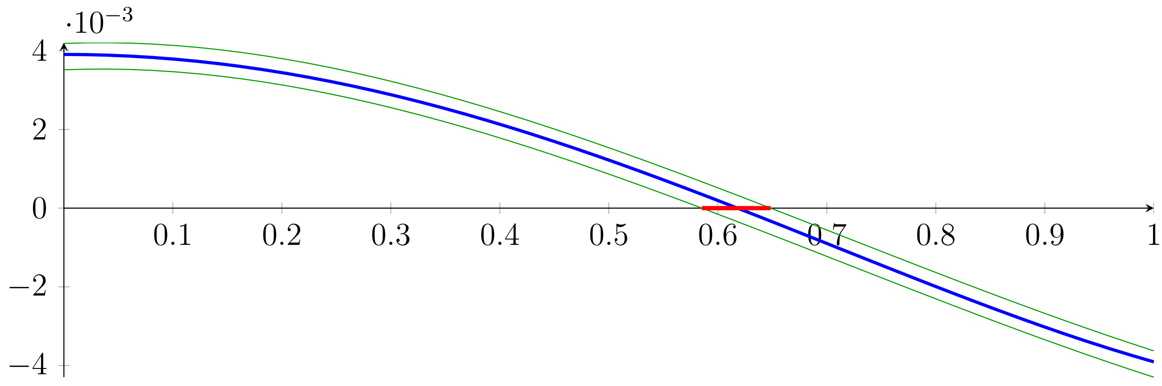
The maximum difference of the Bézier coefficients is $\delta = 0.000334821$.
Bounding polynomials M and m:

$$M = 0.0078125X^3 - 0.0167411X^2 + 0.00111607X + 0.00418527$$
$$m = 0.0078125X^3 - 0.0167411X^2 + 0.00111607X + 0.00351562$$

Root of M and m:

$$N(M) = \{-0.429434, 0.648429, 1.92386\} \qquad N(m) = \{-0.393958, 0.585324, 1.95149\}$$

Intersection intervals:



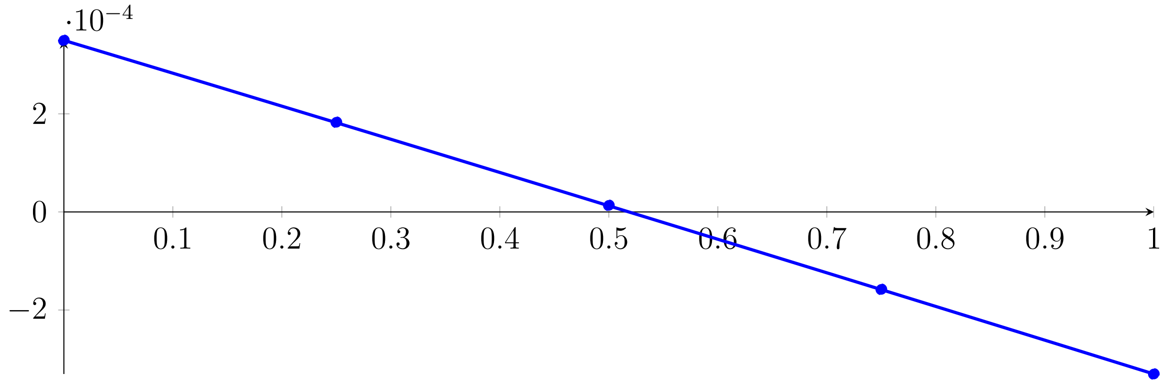
$$[0.585324, 0.648429]$$

Longest intersection interval: 0.0631044
$\implies$ Selective recursion: interval 1: $[0.646331, 0.662107]$,

6.11 Recursion Branch 1 2 1 1 in Interval 1: $[0.646331, 0.662107]$

Normalized monomial und Bézier representations and the Bézier polygon:

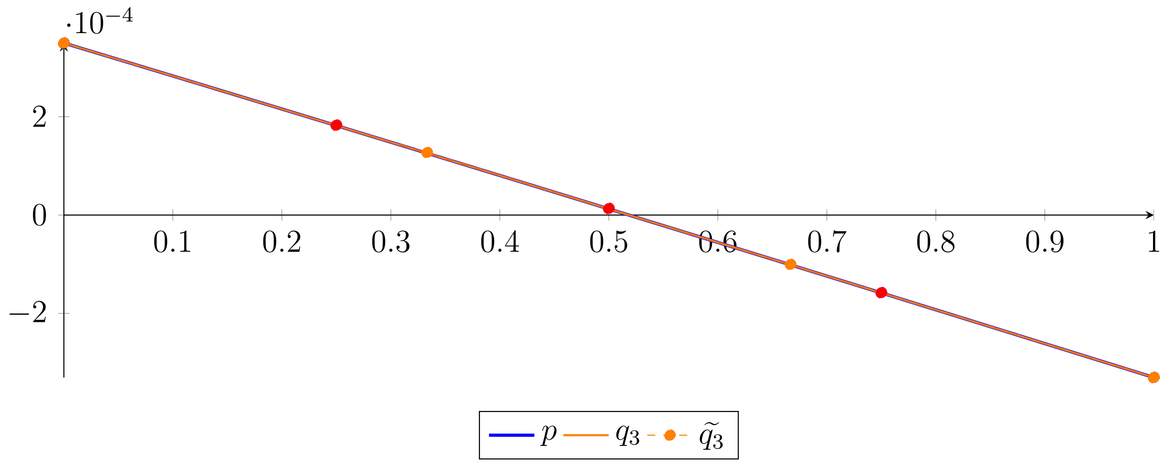
$$p = 6.19438 \cdot 10^{-08} X^4 + 2.29824 \cdot 10^{-06} X^3 - 1.469 \cdot 10^{-05} X^2 - 0.000667971 X + 0.000349861$$
$$= 0.000349861 B_{0,4}(X) + 0.000182868 B_{1,4}(X) + 1.34269$$
$$\cdot 10^{-05} B_{2,4}(X) - 0.000157888 B_{3,4}(X) - 0.00033044 B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = 2.42213 \cdot 10^{-06} X^3 - 1.47697 \cdot 10^{-05} X^2 - 0.000667954 X + 0.00034986$$
$$= 0.00034986 B_{0,3} + 0.000127209 B_{1,3} - 0.000100366 B_{2,3} - 0.000330441 B_{3,3}$$

$$\widetilde{q}_3 = -8.54974 \cdot 10^{-21} X^4 + 2.42213 \cdot 10^{-06} X^3 - 1.47697 \cdot 10^{-05} X^2 - 0.000667954 X + 0.00034986$$
$$= 0.00034986 B_{0,4} + 0.000182872 B_{1,4} + 1.34216 \cdot 10^{-05} B_{2,4} - 0.000157885 B_{3,4} - 0.000330441 B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 5.30947 \cdot 10^{-09}$.
Bounding polynomials M and m:

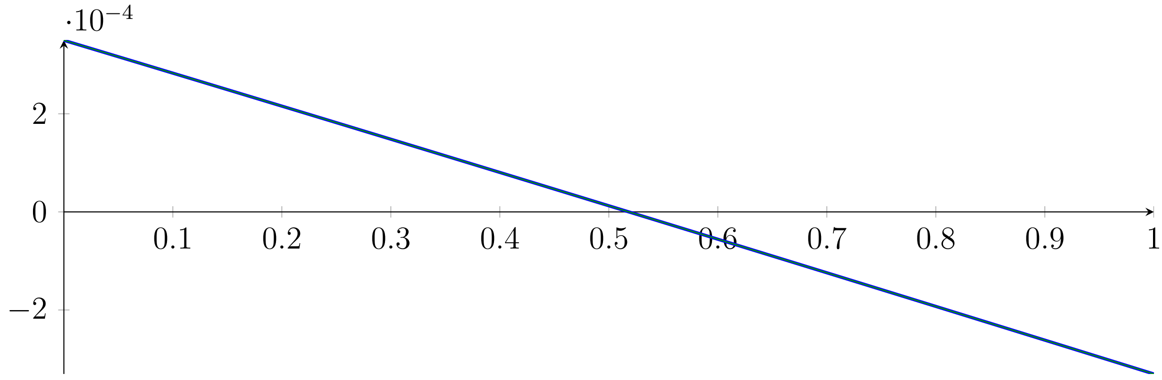
$$M = 2.42213 \cdot 10^{-06} X^3 - 1.47697 \cdot 10^{-05} X^2 - 0.000667954 X + 0.000349865$$

$$m = 2.42213 \cdot 10^{-06} X^3 - 1.47697 \cdot 10^{-05} X^2 - 0.000667954 X + 0.000349855$$

Root of M and m:

$$N(M) = \{-14.135, 0.518351, 19.7145\} \qquad N(m) = \{-14.135, 0.518335, 19.7145\}$$

Intersection intervals:



$[0.518335, 0.518351]$

Longest intersection interval: $1.5586 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.654508, 0.654509]$,

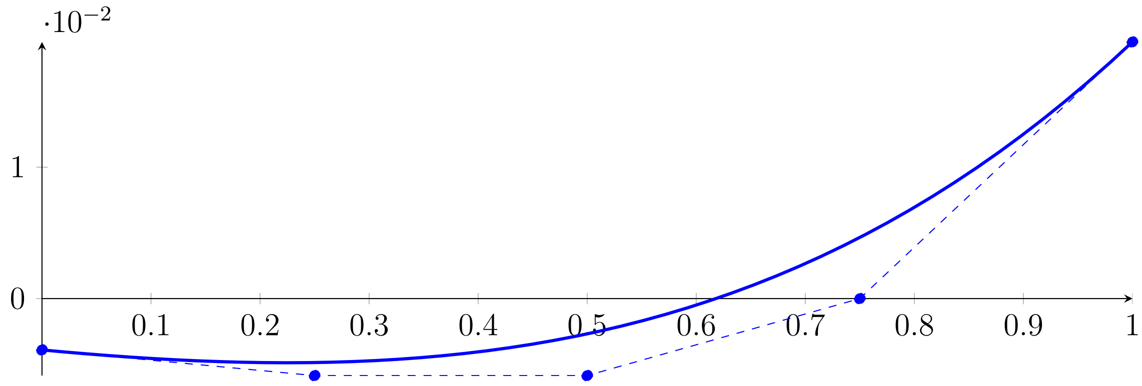
6.12 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.654508, 0.654509]$

Found root in interval $[0.654508, 0.654509]$ at recursion depth 5!

6.13 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

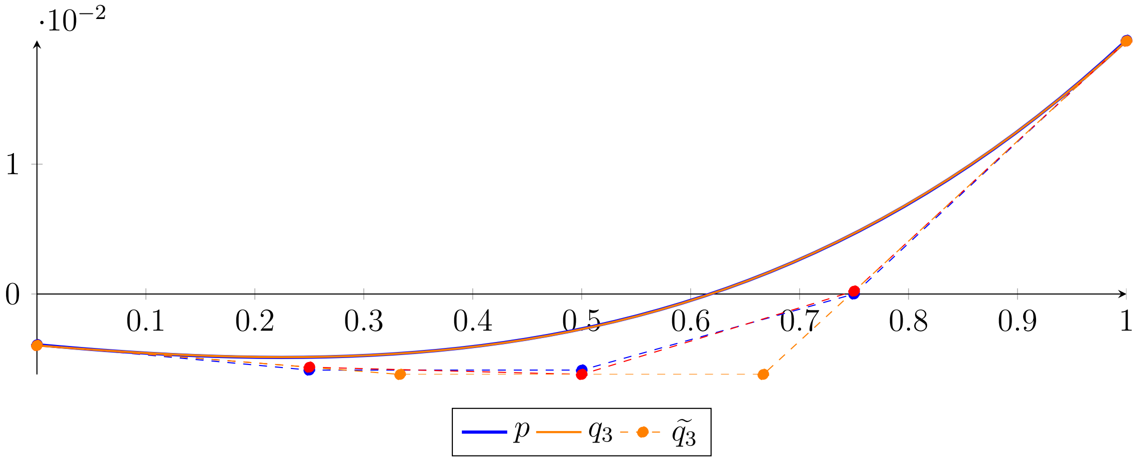
$$\begin{aligned} p &= 0.00390625X^4 + 0.015625X^3 + 0.0117187X^2 - 0.0078125X - 0.00390625 \\ &= -0.00390625B_{0,4}(X) - 0.00585937B_{1,4}(X) - 0.00585937B_{2,4}(X) \\ &\quad + 3.38813 \cdot 10^{-21}B_{3,4}(X) + 0.0195312B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.0234375X^3 + 0.00669643X^2 - 0.00669643X - 0.00396205 \\ &= -0.00396205B_{0,3} - 0.0061942B_{1,3} - 0.0061942B_{2,3} + 0.0194754B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -2.21923 \cdot 10^{-19}X^4 + 0.0234375X^3 + 0.00669643X^2 - 0.00669643X - 0.00396205 \\ &= -0.00396205B_{0,4} - 0.00563616B_{1,4} - 0.0061942B_{2,4} + 0.000223214B_{3,4} + 0.0194754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000334821$.
Bounding polynomials M and m:

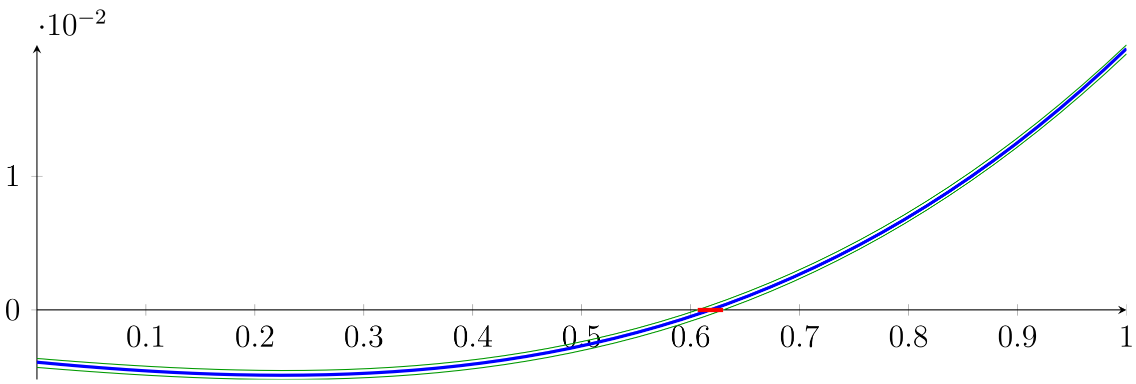
$$M = 0.0234375X^3 + 0.00669643X^2 - 0.00669643X - 0.00362723$$

$$m = 0.0234375X^3 + 0.00669643X^2 - 0.00669643X - 0.00429688$$

Root of M and m:

$$N(M) = \{0.606374\} \qquad N(m) = \{0.62991\}$$

Intersection intervals:



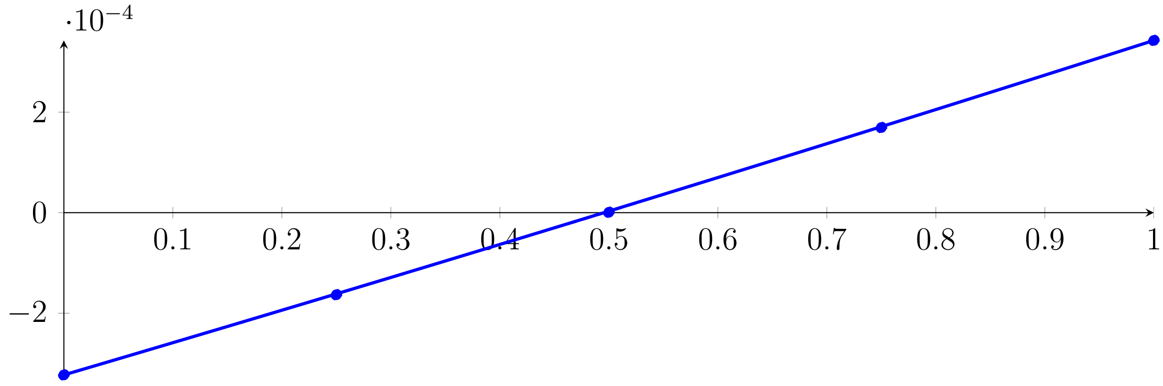
$$[0.606374, 0.62991]$$

Longest intersection interval: 0.023536
$\Longrightarrow$ Selective recursion: interval 1: $[0.901594, 0.907478]$,

6.14 Recursion Branch 1 2 2 1 in Interval 1: $[0.901594, 0.907478]$

Normalized monomial und Bézier representations and the Bézier polygon:

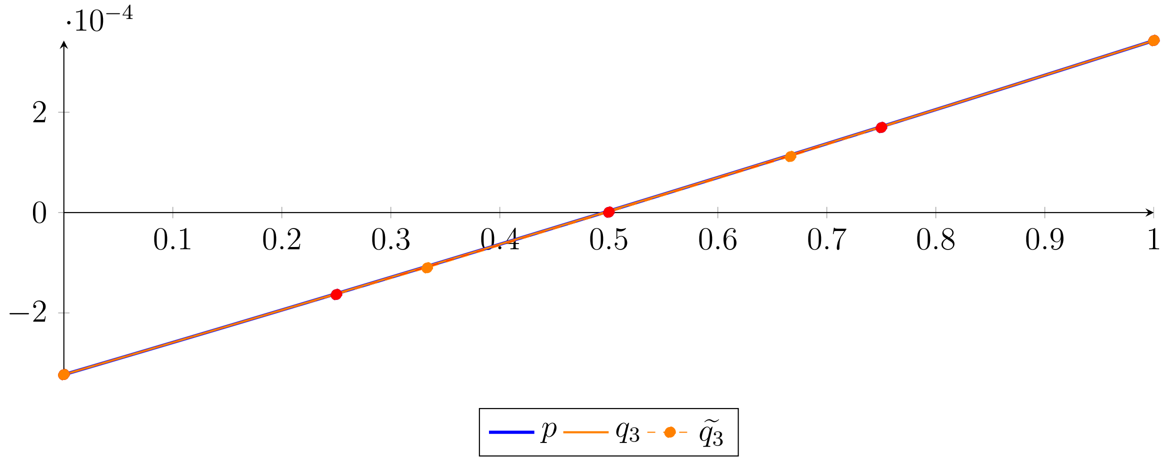
$$\begin{aligned} p &= 1.19864 \cdot 10^{-09}X^4 + 3.27237 \cdot 10^{-07}X^3 + 2.70103 \cdot 10^{-05}X^2 + 0.000638261X - 0.000322867 \\ &= -0.000322867B_{0,4}(X) - 0.000163301B_{1,4}(X) + 7.65889 \\ &\quad \cdot 10^{-07}B_{2,4}(X) + 0.000169417B_{3,4}(X) + 0.000342734B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 3.29635 \cdot 10^{-07} X^3 + 2.70088 \cdot 10^{-05} X^2 + 0.000638262X - 0.000322867$$
$$= -0.000322867B_{0,3} - 0.000110113B_{1,3} + 0.000111644B_{2,3} + 0.000342734B_{3,3}$$

$$\widetilde{q_3} = 7.41154 \cdot 10^{-21} X^4 + 3.29635 \cdot 10^{-07} X^3 + 2.70088 \cdot 10^{-05} X^2 + 0.000638262X - 0.000322867$$
$$= -0.000322867B_{0,4} - 0.000163301B_{1,4} + 7.65786 \cdot 10^{-07} B_{2,4} + 0.000169417B_{3,4} + 0.000342734B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.0274 \cdot 10^{-10}$.

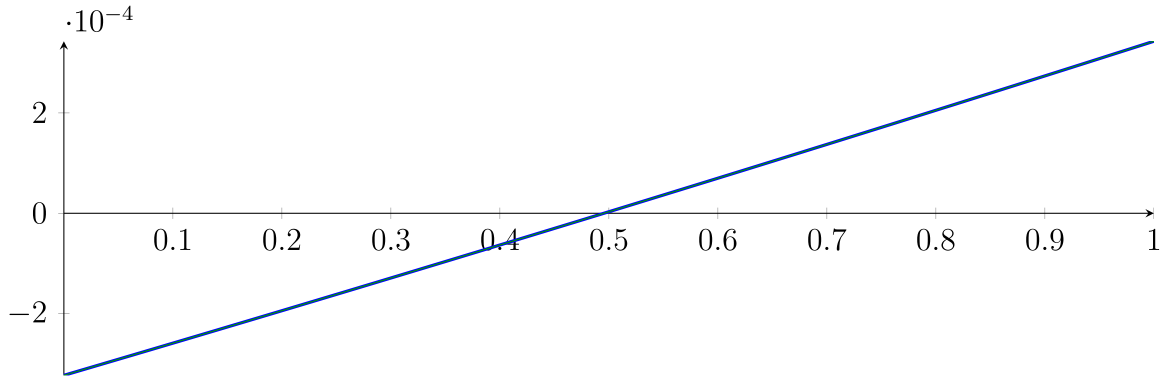
Bounding polynomials M and m:

$$M = 3.29635 \cdot 10^{-07} X^3 + 2.70088 \cdot 10^{-05} X^2 + 0.000638262X - 0.000322866$$
$$m = 3.29635 \cdot 10^{-07} X^3 + 2.70088 \cdot 10^{-05} X^2 + 0.000638262X - 0.000322867$$

Root of M and m:

$$N(M) = \{0.495404\} \qquad N(m) = \{0.495405\}$$

Intersection intervals:



$$[0.495404, 0.495405]$$

Longest intersection interval: $3.08871 \cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.904508, 0.904508]$,

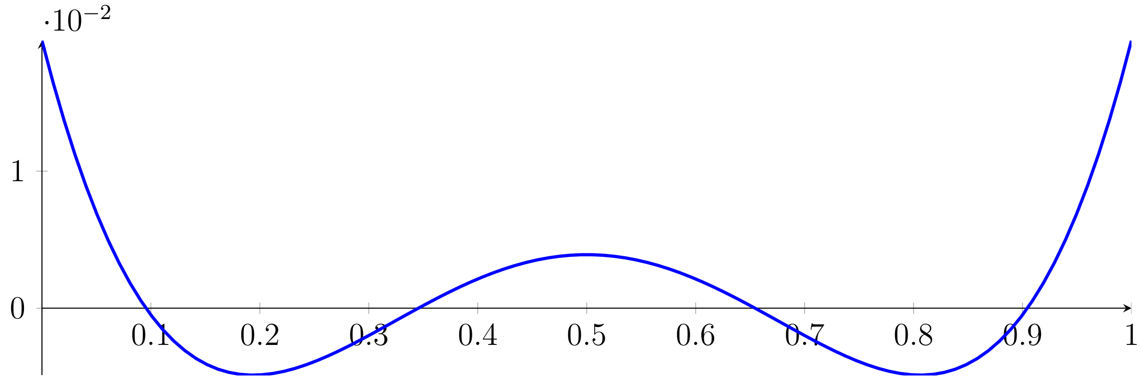
6.15 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.904508, 0.904508]$

Found root in interval $[0.904508, 0.904508]$ at recursion depth 5!

6.16 Result: 4 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^4 - 2X^3 + 1.3125X^2 - 0.3125X + 0.0195312$$



Result: Root Intervals

$$[0.0954915, 0.0954915], [0.345491, 0.345492], [0.654508, 0.654509], [0.904508, 0.904508]$$

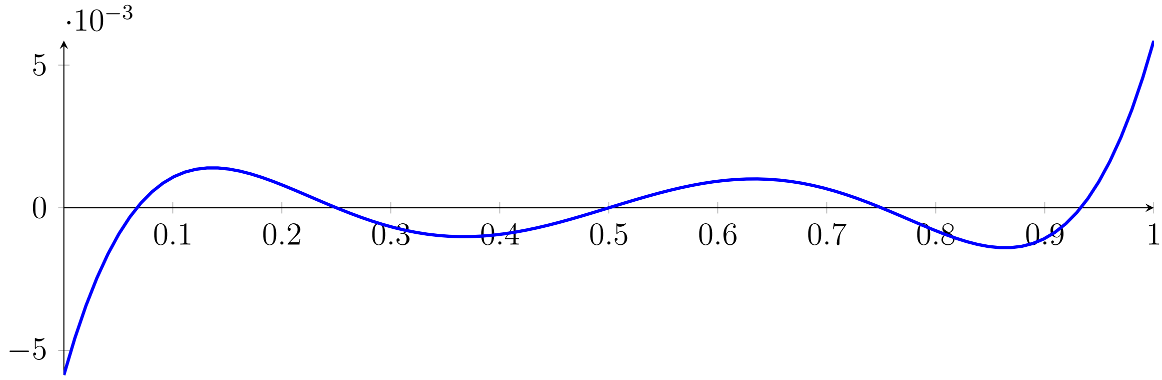
with precision $\varepsilon = 1 \cdot 10^{-06}$.

7 Running `BezClip` on p5 with epsilon 6

$$1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

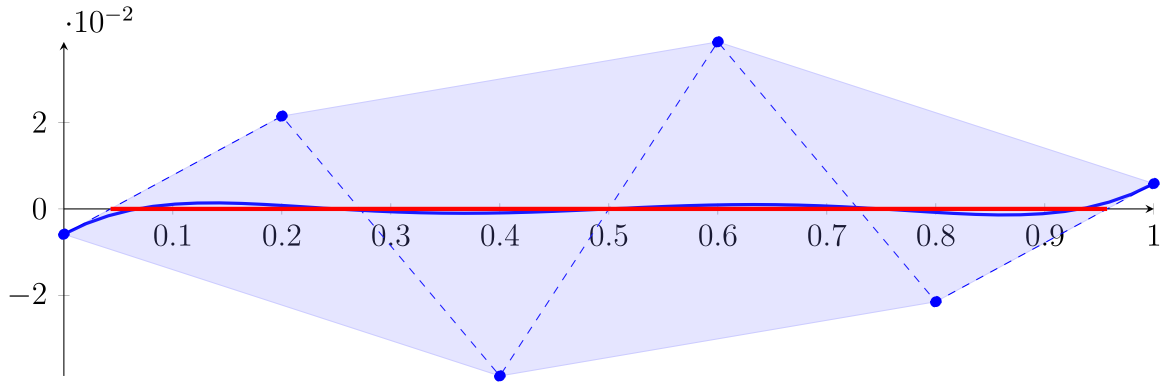
$$p = 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$



7.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.0214844B_{1,5}(X) - 0.0386719B_{2,5}(X) \\ &\quad + 0.0386719B_{3,5}(X) - 0.0214844B_{4,5}(X) + 0.00585938B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0428571, 0.957143\}$$

Intersection intervals with the x axis:

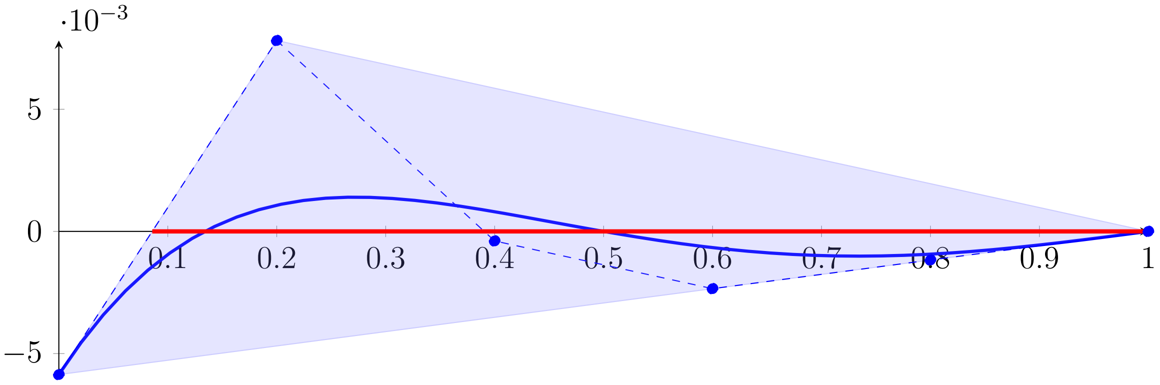
$$[0.0428571, 0.957143]$$

Longest intersection interval: 0.914286
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

7.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.03125X^5 - 0.15625X^4 + 0.28125X^3 - 0.21875X^2 + 0.0683594X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.0078125B_{1,5}(X) - 0.000390625B_{2,5}(X) \\ &\quad - 0.00234375B_{3,5}(X) - 0.00117187B_{4,5}(X) + 6.89273 \cdot 10^{-20}B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0857143, 1\}$$

Intersection intervals with the x axis:

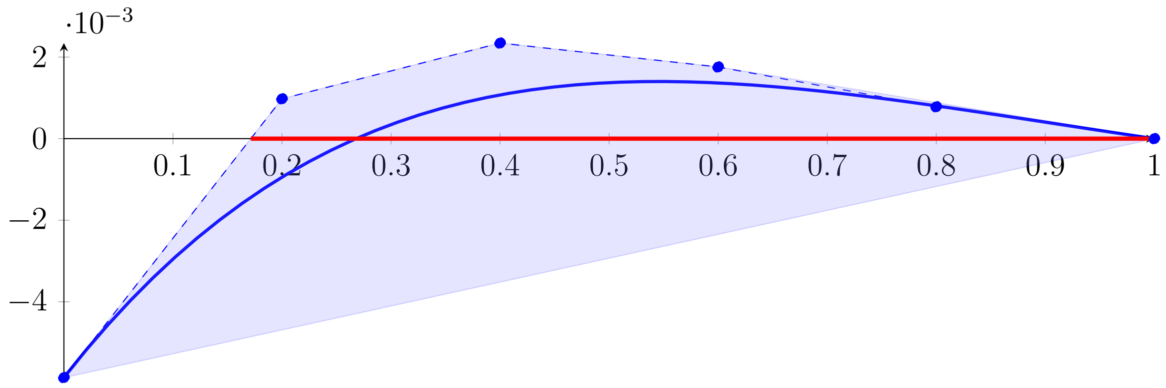
$$[0.0857143, 1]$$

Longest intersection interval: 0.914286
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

7.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000976563X^5 - 0.00976562X^4 + 0.0351563X^3 - 0.0546875X^2 + 0.0341797X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.000976563B_{1,5}(X) + 0.00234375B_{2,5}(X) \\ &\quad + 0.00175781B_{3,5}(X) + 0.00078125B_{4,5}(X) + 1.55642 \cdot 10^{-20}B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.171429, 1\}$$

Intersection intervals with the x axis:

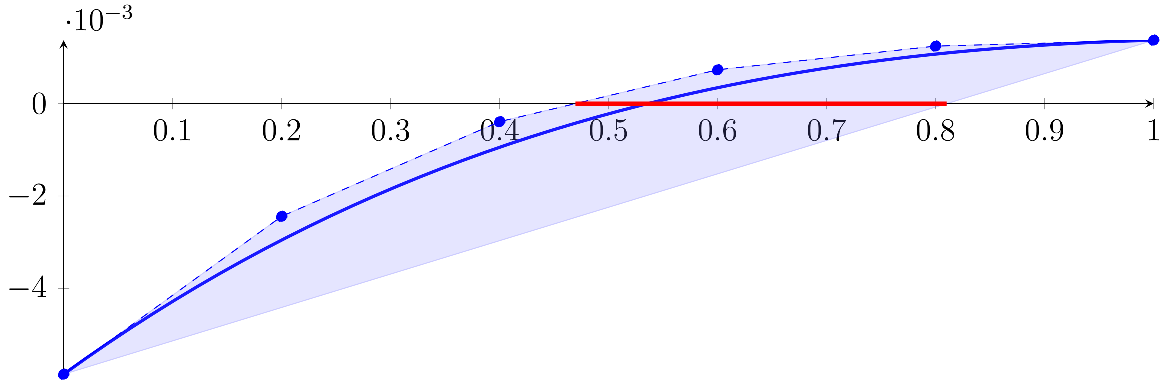
$$[0.171429, 1]$$

Longest intersection interval: 0.828571
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$

7.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.05176 \cdot 10^{-05} X^5 - 0.000610352 X^4 + 0.00439453 X^3 - 0.0136719 X^2 + 0.0170898 X - 0.00585938 \\ &= -0.00585938 B_{0,5}(X) - 0.00244141 B_{1,5}(X) - 0.000390625 B_{2,5}(X) \\ &\quad + 0.000732422 B_{3,5}(X) + 0.00124512 B_{4,5}(X) + 0.00137329 B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.469565, 0.810127\}$$

Intersection intervals with the x axis:

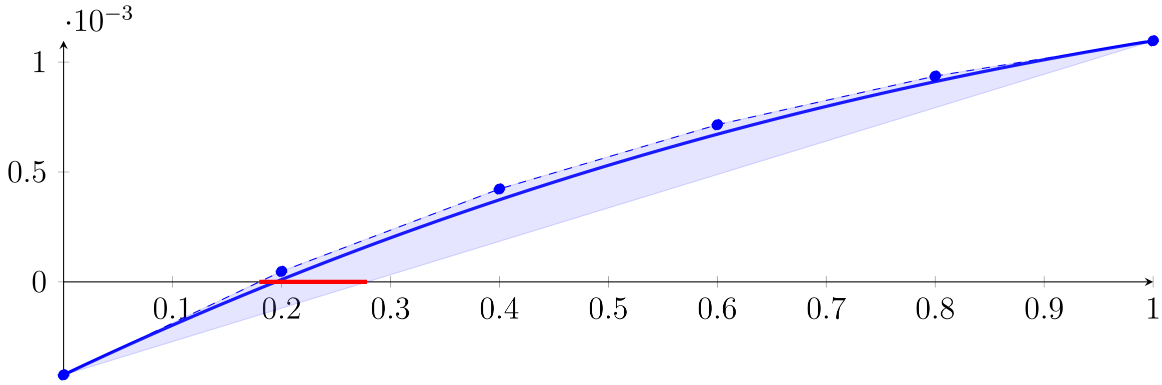
$$[0.469565, 0.810127]$$

Longest intersection interval: 0.340561
$\Longrightarrow$ Selective recursion: interval 1: $[0.0586957, 0.101266]$,

7.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0586957, 0.101266]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 1.39806 \cdot 10^{-07} X^5 - 7.24652 \cdot 10^{-06} X^4 + 0.000130956 X^3 \\ & - 0.000957685 X^2 + 0.00235385 X - 0.000423099 \\ = {} & -0.000423099 B_{0,5}(X) + 4.76717 \cdot 10^{-05} B_{1,5}(X) + 0.000422674 B_{2,5}(X) \\ & + 0.000715003 B_{3,5}(X) + 0.000936306 B_{4,5}(X) + 0.00109692 B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.179747, 0.278351\}$$

Intersection intervals with the x axis:

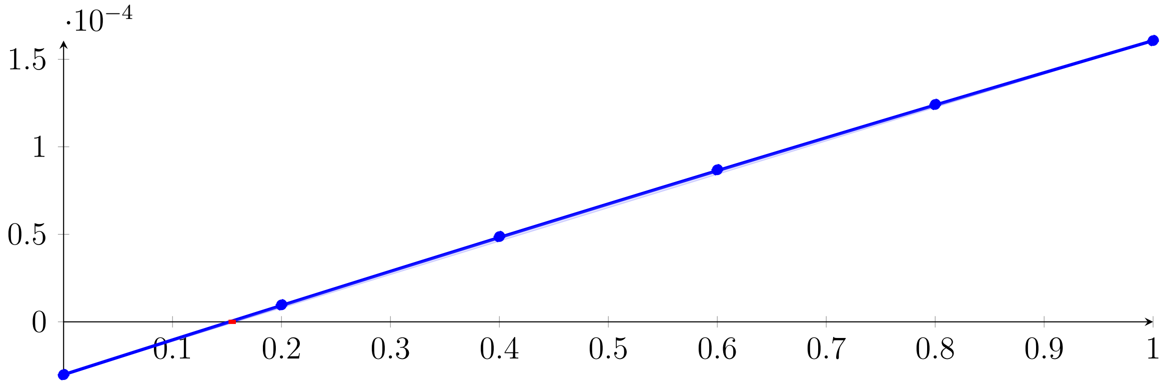
$$[0.179747, 0.278351]$$

Longest intersection interval: 0.098604
$\Longrightarrow$ Selective recursion: interval 1: $[0.0663475, 0.0705451]$,

7.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0663475, 0.0705451]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.30317\cdot 10^{-12}X^5 - 6.73151\cdot 10^{-10}X^4 + 1.20596\cdot 10^{-07}X^3 \\ &\quad - 8.63833\cdot 10^{-06}X^2 + 0.000199387X - 3.0189\cdot 10^{-05} \\ &= -3.0189\cdot 10^{-05}B_{0,5}(X) + 9.68839\cdot 10^{-06}B_{1,5}(X) + 4.87019\cdot 10^{-05}B_{2,5}(X) \\ &\quad + 8.68637\cdot 10^{-05}B_{3,5}(X) + 0.000124186B_{4,5}(X) + 0.000160679B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.151409, 0.158166\}$$

Intersection intervals with the x axis:

$$[0.151409, 0.158166]$$

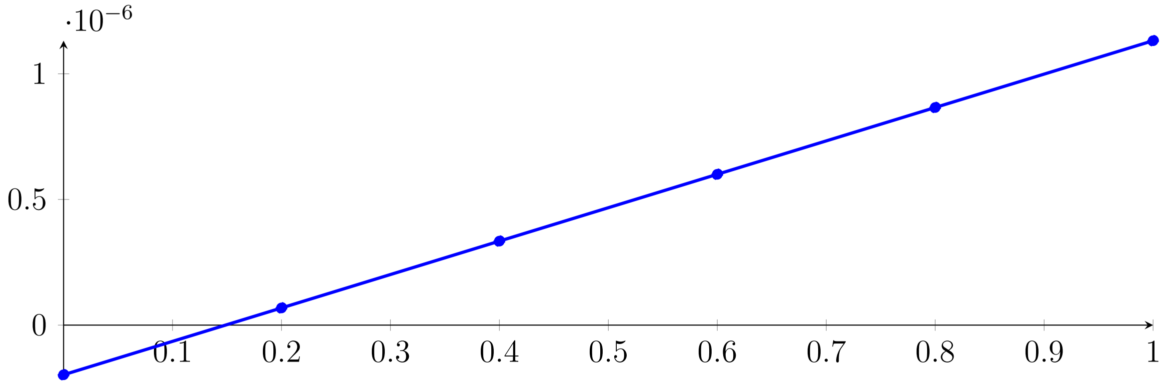
Longest intersection interval: 0.00675734

$\Longrightarrow$ Selective recursion: interval 1: $[0.0669831, 0.0670114]$,

7.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.0669831, 0.0670114]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2.0085\cdot 10^{-23}X^5 - 1.40146\cdot 10^{-18}X^4 + 3.70844\cdot 10^{-14}X^3 \\ &\quad - 3.91944\cdot 10^{-10}X^2 + 1.32971\cdot 10^{-06}X - 1.97613\cdot 10^{-07} \\ &= -1.97613\cdot 10^{-07}B_{0,5}(X) + 6.83282\cdot 10^{-08}B_{1,5}(X) + 3.3423\cdot 10^{-07}B_{2,5}(X) \\ &\quad + 6.00093\cdot 10^{-07}B_{3,5}(X) + 8.65916\cdot 10^{-07}B_{4,5}(X) + 1.1317\cdot 10^{-06}B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.148614, 0.148658\}$$

Intersection intervals with the x axis:

$$[0.148614, 0.148658]$$

Longest intersection interval: $4.38142\cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.0669873, 0.0669873]$,

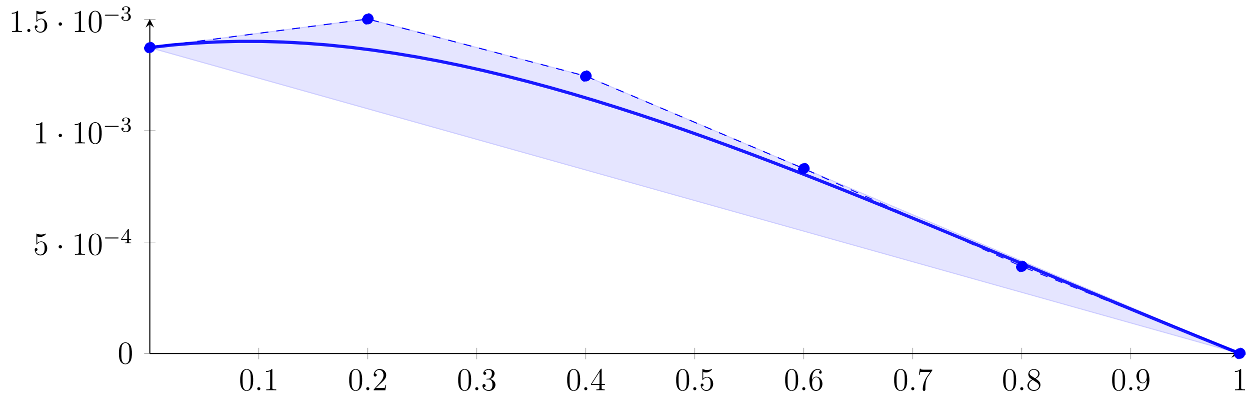
7.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.0669873, 0.0669873]$

Found root in interval $[0.0669873, 0.0669873]$ at recursion depth 8!

7.9 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.05176 \cdot 10^{-05} X^5 - 0.000457764 X^4 + 0.0022583 X^3 - 0.00384521 X^2 + 0.000640869 X + 0.00137329 \\ &= 0.00137329 B_{0,5}(X) + 0.00150146 B_{1,5}(X) + 0.00124512 B_{2,5}(X) \\ &\quad + 0.000830078 B_{3,5}(X) + 0.000390625 B_{4,5}(X) + 1.55642 \cdot 10^{-20} B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{\}$$

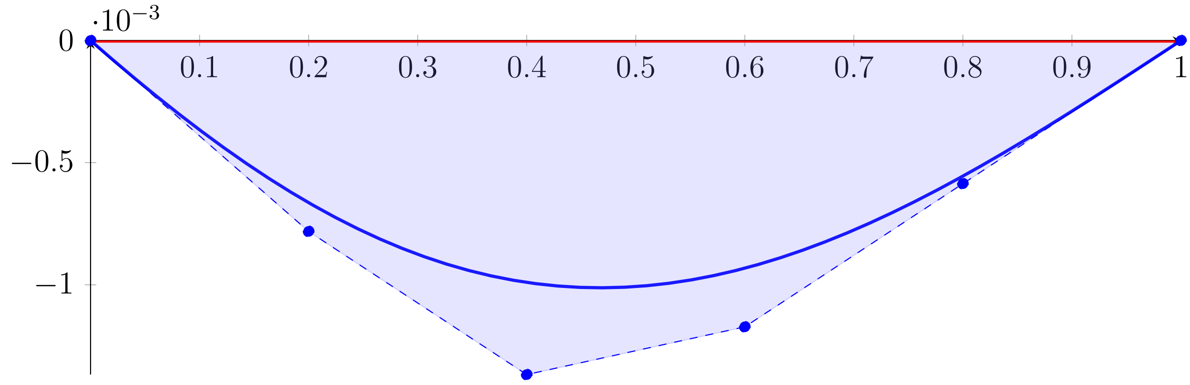
Intersection intervals with the x axis:

No intersection with the x axis. Done.

7.10 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000976562 X^5 - 0.00488281 X^4 + 0.00585938 X^3 + 0.00195313 X^2 - 0.00390625 X + 1.55642 \cdot 10^{-20} \\ &= 1.55642 \cdot 10^{-20} B_{0,5}(X) - 0.00078125 B_{1,5}(X) - 0.00136719 B_{2,5}(X) \\ &\quad - 0.00117187 B_{3,5}(X) - 0.000585937 B_{4,5}(X) + 6.89273 \cdot 10^{-20} B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{1.5084e - 17, 1\}$$

Intersection intervals with the x axis:

$$[1.5084e - 17, 1]$$

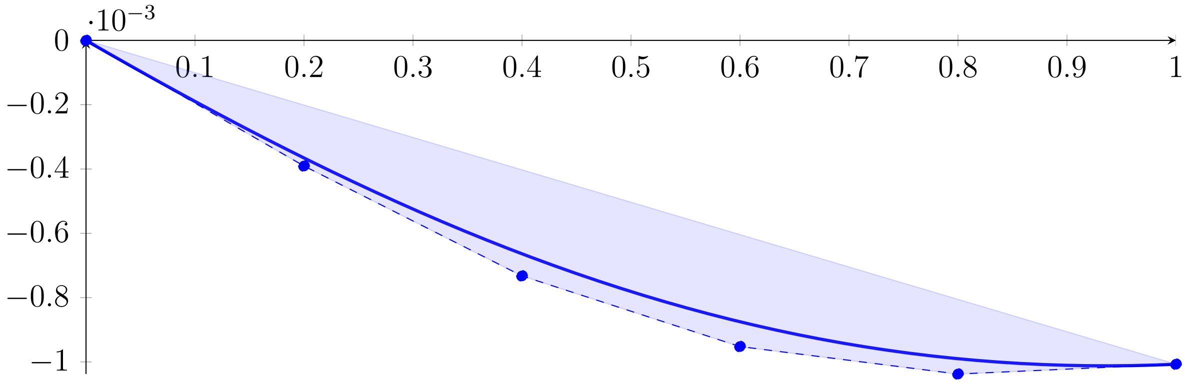
Longest intersection interval: 1

$\Longrightarrow$ Bisection: first half $[0.25, 0.375]$ und second half $[0.375, 0.5]$

7.11 Recursion Branch 1 1 2 1 on the First Half $[0.25, 0.375]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p &= 3.05176\cdot 10^{-05}X^5 - 0.000305176X^4 + 0.000732422X^3 \\
&\quad + 0.000488281X^2 - 0.00195312X + 1.55642\cdot 10^{-20} \\
&= 1.55642\cdot 10^{-20}B_{0,5}(X) - 0.000390625B_{1,5}(X) - 0.000732422B_{2,5}(X) \\
&\quad - 0.000952148B_{3,5}(X) - 0.0010376B_{4,5}(X) - 0.00100708B_{5,5}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{1.54548e - 17, 1.90684e - 17\}$$

Intersection intervals with the x axis:

$$[1.54548e - 17, 1.90684e - 17]$$

Longest intersection interval: $3.6136\cdot 10^{-18}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.25]$,

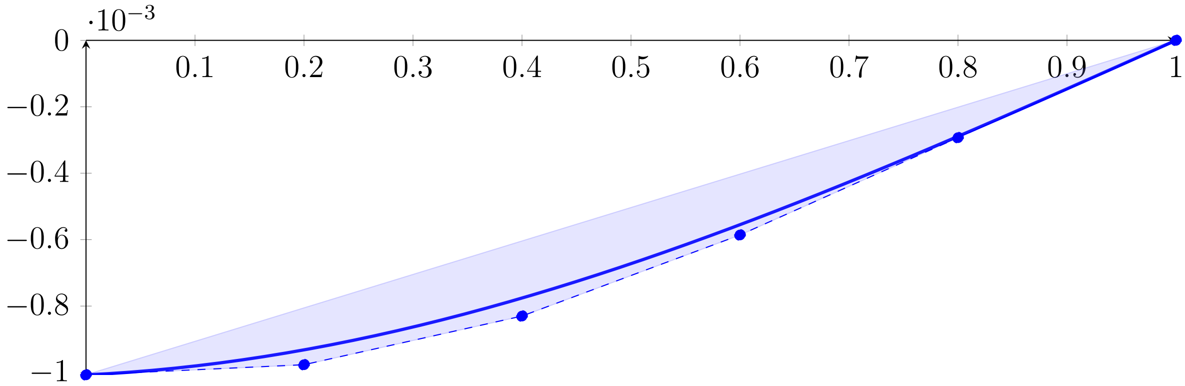
7.12 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.25, 0.25]$

Found root in interval $[0.25, 0.25]$ at recursion depth 5!

7.13 Recursion Branch 1 1 2 2 on the Second Half $[0.375, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p &= 3.05176\cdot 10^{-05}X^5 - 0.000152588X^4 - 0.000183105X^3 + 0.00115967X^2 + 0.000152588X - 0.00100708 \\
&= -0.00100708B_{0,5}(X) - 0.000976562B_{1,5}(X) - 0.000830078B_{2,5}(X) \\
&\quad - 0.000585937B_{3,5}(X) - 0.000292969B_{4,5}(X) + 6.89273\cdot 10^{-20}B_{5,5}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{1, 1\}$$

Intersection intervals with the x axis:

$$[1, 1]$$

Longest intersection interval: $2.1413 \cdot 10^{-17}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

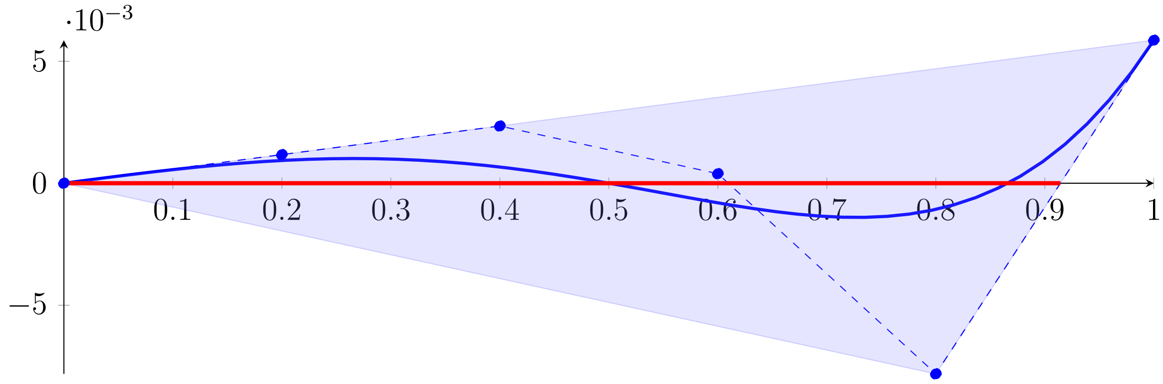
7.14 Recursion Branch 1 1 2 2 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

7.15 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.03125X^5 + 5.45277 \cdot 10^{-20}X^4 - 0.03125X^3 + 1.76818 \cdot 10^{-19}X^2 + 0.00585938X + 6.89273 \cdot 10^{-20} \\ &= 6.89273 \cdot 10^{-20}B_{0,5}(X) + 0.00117188B_{1,5}(X) + 0.00234375B_{2,5}(X) \\ &\quad + 0.000390625B_{3,5}(X) - 0.0078125B_{4,5}(X) + 0.00585938B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{5.14996e - 17, 0.914286\}$$

Intersection intervals with the x axis:

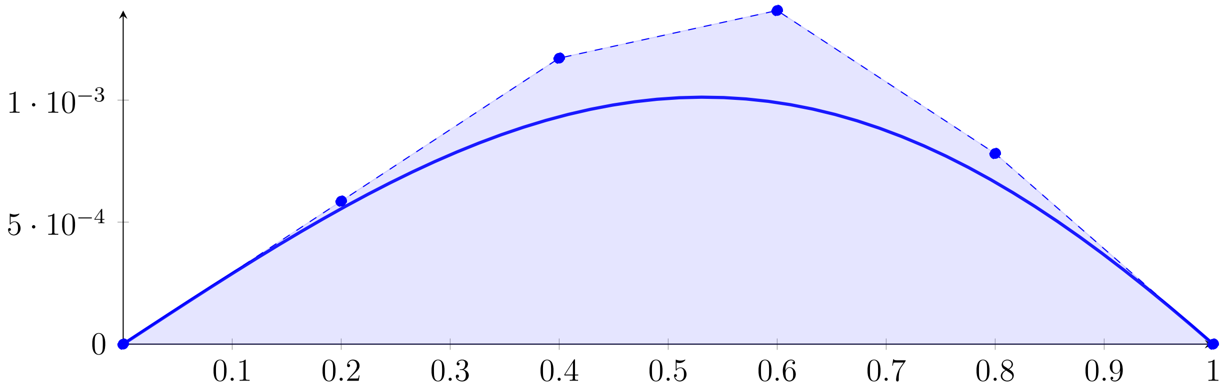
$$[5.14996e - 17, 0.914286]$$

Longest intersection interval: 0.914286
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

7.16 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000976563X^5 + 3.70577 \cdot 10^{-21}X^4 - 0.00390625X^3 \\ &\quad + 4.44692 \cdot 10^{-20}X^2 + 0.00292969X + 6.89273 \cdot 10^{-20} \\ &= 6.89273 \cdot 10^{-20}B_{0,5}(X) + 0.000585938B_{1,5}(X) + 0.00117188B_{2,5}(X) \\ &\quad + 0.00136719B_{3,5}(X) + 0.00078125B_{4,5}(X) + 2.17237 \cdot 10^{-19}B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

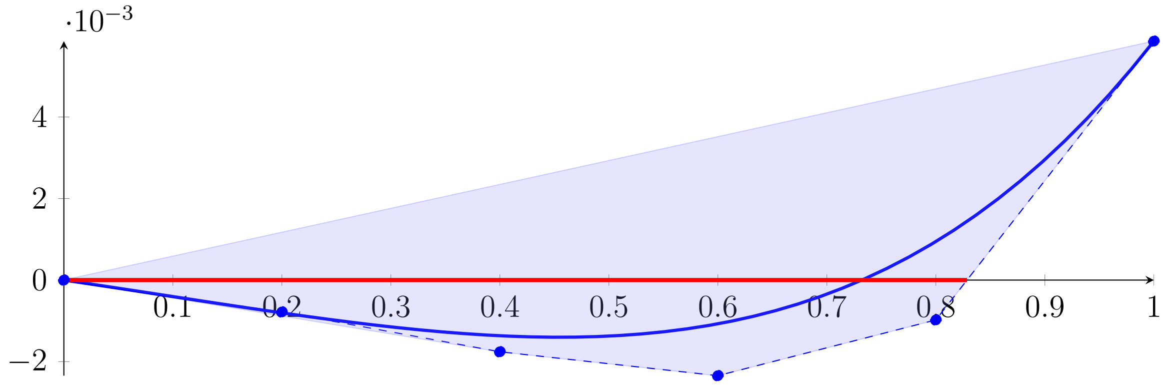
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

7.17 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000976562X^5 + 0.00488281X^4 + 0.00585938X^3 - 0.00195312X^2 - 0.00390625X + 2.17237{\cdot}10^{-19} \\ &= 2.17237{\cdot}10^{-19}B_{0,5}(X) - 0.00078125B_{1,5}(X) - 0.00175781B_{2,5}(X) \\ &\quad - 0.00234375B_{3,5}(X) - 0.000976562B_{4,5}(X) + 0.00585938B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{7.16387e - 17, 0.828571\}$$

Intersection intervals with the x axis:

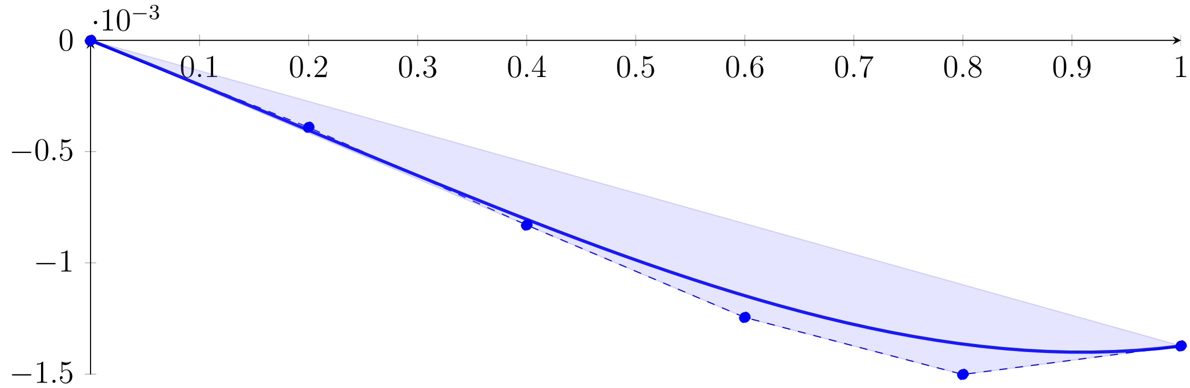
$$[7.16387e - 17, 0.828571]$$

Longest intersection interval: 0.828571
$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$

7.18 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.05176{\cdot}10^{-05}X^5 + 0.000305176X^4 + 0.000732422X^3 \\ &\quad - 0.000488281X^2 - 0.00195312X + 2.17237{\cdot}10^{-19} \\ &= 2.17237{\cdot}10^{-19}B_{0,5}(X) - 0.000390625B_{1,5}(X) - 0.000830078B_{2,5}(X) \\ &\quad - 0.00124512B_{3,5}(X) - 0.00150146B_{4,5}(X) - 0.00137329B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{1.26906e-16, 1.58188e-16\}$$

Intersection intervals with the x axis:

$$[1.26906e-16, 1.58188e-16]$$

Longest intersection interval: $3.12816 \cdot 10^{-17}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.75]$,

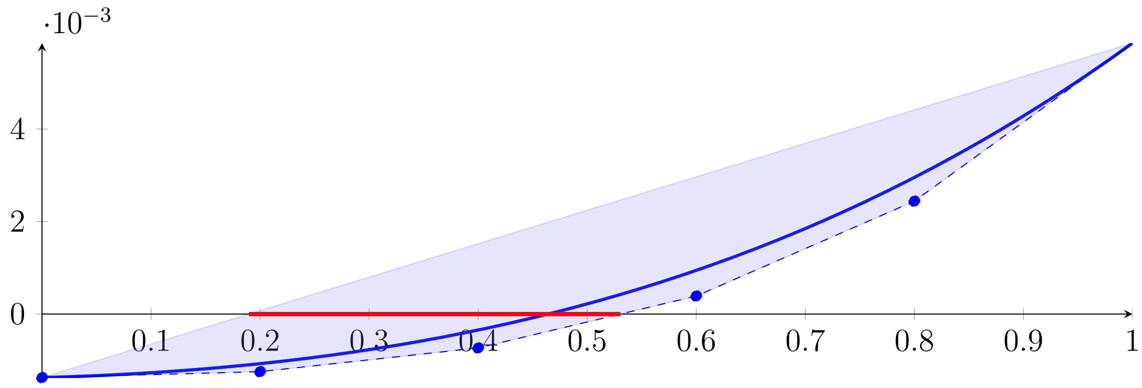
7.19 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.75, 0.75]$

Found root in interval $[0.75, 0.75]$ at recursion depth 5!

7.20 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.05176 \cdot 10^{-05} X^5 + 0.000457764 X^4 + 0.0022583 X^3 + 0.00384521 X^2 + 0.000640869 X - 0.00137329 \\ &= -0.00137329 B_{0,5}(X) - 0.00124512 B_{1,5}(X) - 0.000732422 B_{2,5}(X) \\ &\quad + 0.000390625 B_{3,5}(X) + 0.00244141 B_{4,5}(X) + 0.00585938 B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.189873, 0.530435\}$$

Intersection intervals with the x axis:

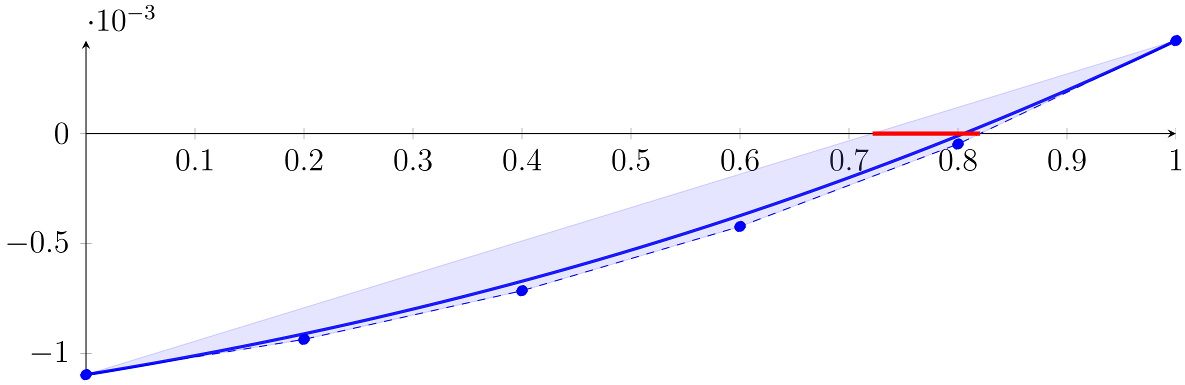
$$[0.189873, 0.530435]$$

Longest intersection interval: 0.340561
$\Longrightarrow$ Selective recursion: interval 1: $[0.898734, 0.941304]$,

7.21 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.898734, 0.941304]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.39806 \cdot 10^{-07} X^5 + 6.54749 \cdot 10^{-06} X^4 + 0.000103368 X^3 \\ &\quad + 0.000606899 X^2 + 0.000803063 X - 0.00109692 \\ &= -0.00109692 B_{0,5}(X) - 0.000936306 B_{1,5}(X) - 0.000715003 B_{2,5}(X) \\ &\quad - 0.000422674 B_{3,5}(X) - 4.76717 \cdot 10^{-05} B_{4,5}(X) + 0.000423099 B_{5,5}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.721649, 0.820253\}$$

Intersection intervals with the x axis:

$$[0.721649, 0.820253]$$

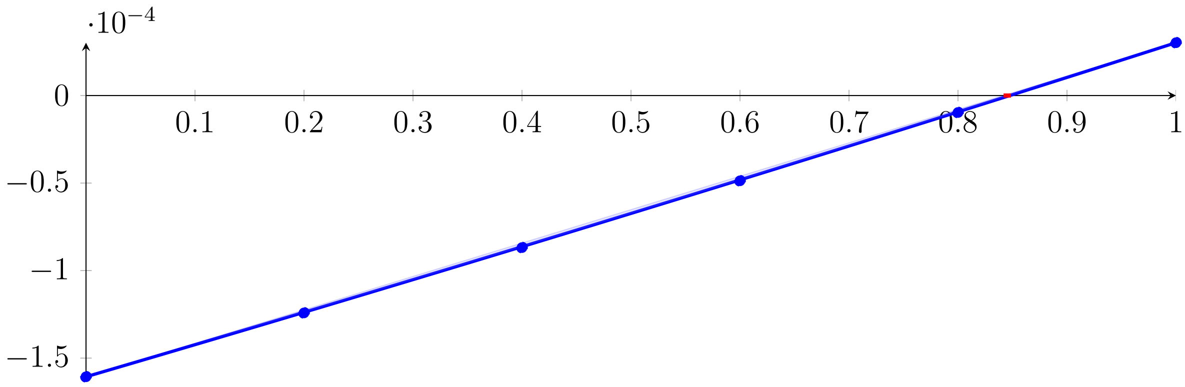
Longest intersection interval: 0.098604

$\Longrightarrow$ Selective recursion: interval 1: $[0.929455, 0.933652]$,

7.22 Recursion Branch 1 2 2 2 1 1 in Interval 1: $[0.929455, 0.933652]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.30317 \cdot 10^{-12} X^5 + 6.66635 \cdot 10^{-10} X^4 + 1.17916 \cdot 10^{-07} X^3 \\ &\quad + 8.28057 \cdot 10^{-06} X^2 + 0.000182469 X - 0.000160679 \\ &= -0.000160679 B_{0,5}(X) - 0.000124186 B_{1,5}(X) - 8.68637 \cdot 10^{-05} B_{2,5}(X) \\ &\quad - 4.87019 \cdot 10^{-05} B_{3,5}(X) - 9.68839 \cdot 10^{-06} B_{4,5}(X) + 3.0189 \cdot 10^{-05} B_{5,5}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.841834, 0.848591\}$$

Intersection intervals with the x axis:

$$[0.841834, 0.848591]$$

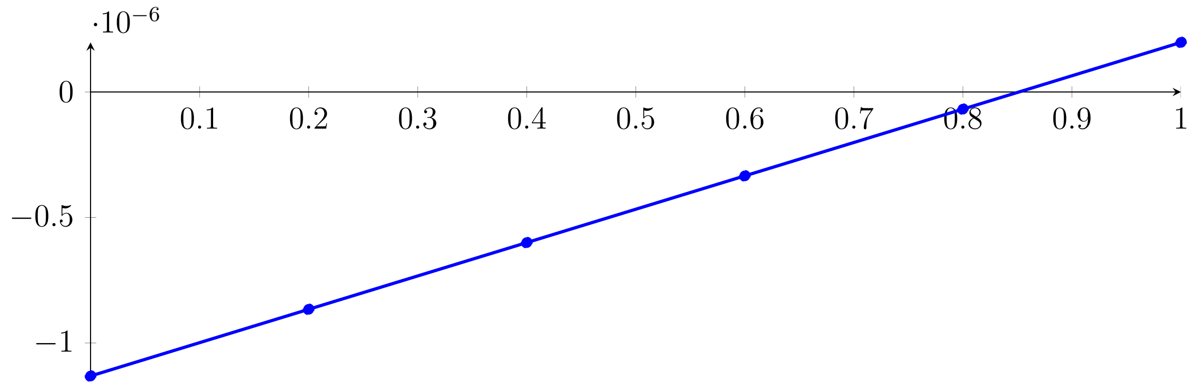
Longest intersection interval: 0.00675734

$\Longrightarrow$ Selective recursion: interval 1: $[0.932989, 0.933017]$,

7.23 Recursion Branch 1 2 2 2 1 1 1 in Interval 1: $[0.932989, 0.933017]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.86116 \cdot 10^{-23} X^5 + 1.40137 \cdot 10^{-18} X^4 + 3.70788 \cdot 10^{-14} X^3 \\ &\quad + 3.91833 \cdot 10^{-10} X^2 + 1.32892 \cdot 10^{-06} X - 1.1317 \cdot 10^{-06} \\ &= -1.1317 \cdot 10^{-06} B_{0,5}(X) - 8.65916 \cdot 10^{-07} B_{1,5}(X) - 6.00093 \cdot 10^{-07} B_{2,5}(X) \\ &\quad - 3.3423 \cdot 10^{-07} B_{3,5}(X) - 6.83282 \cdot 10^{-08} B_{4,5}(X) + 1.97613 \cdot 10^{-07} B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.851342, 0.851386\}$$

Intersection intervals with the x axis:

$$[0.851342, 0.851386]$$

Longest intersection interval: $4.38142 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.933013, 0.933013]$,

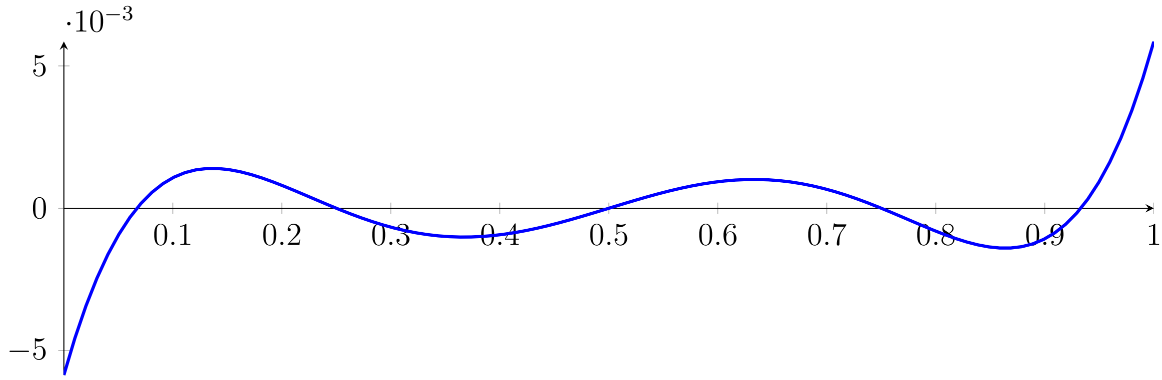
7.24 Recursion Branch 1 2 2 2 1 1 1 1 in Interval 1: $[0.933013, 0.933013]$

Found root in interval $[0.933013, 0.933013]$ at recursion depth 8!

7.25 Result: 5 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$



Result: Root Intervals

$$[0.0669873, 0.0669873],\ [0.25, 0.25],\ [0.5, 0.5],\ [0.75, 0.75],\ [0.933013, 0.933013]$$

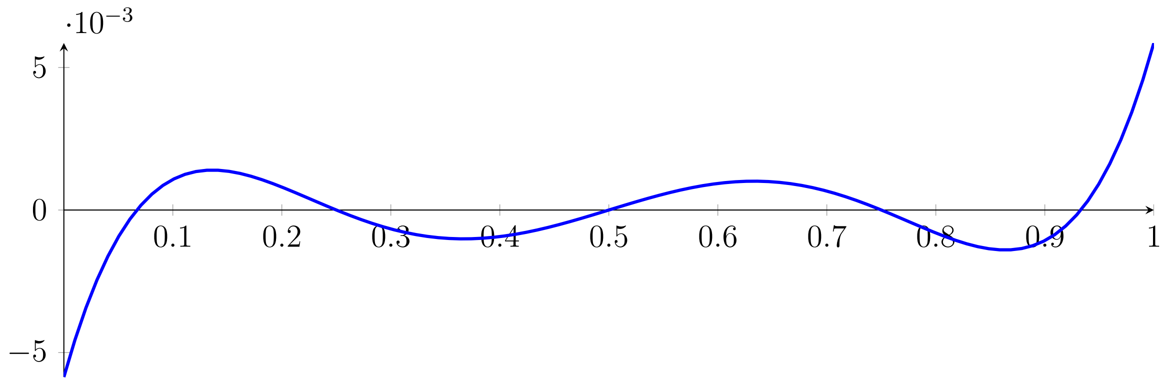
with precision $\varepsilon = 1 \cdot 10^{-06}$.

8 Running `QuadClip` on p5 with epsilon 6

$$1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

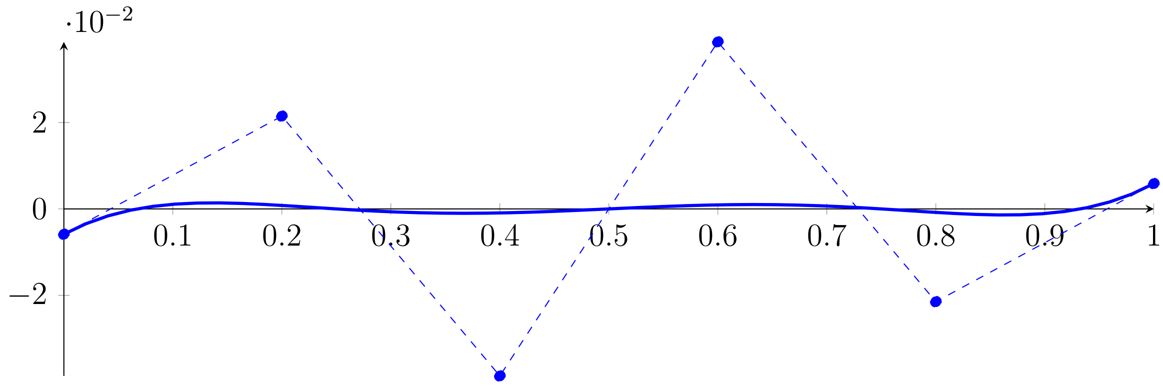
$$p = 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$



8.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

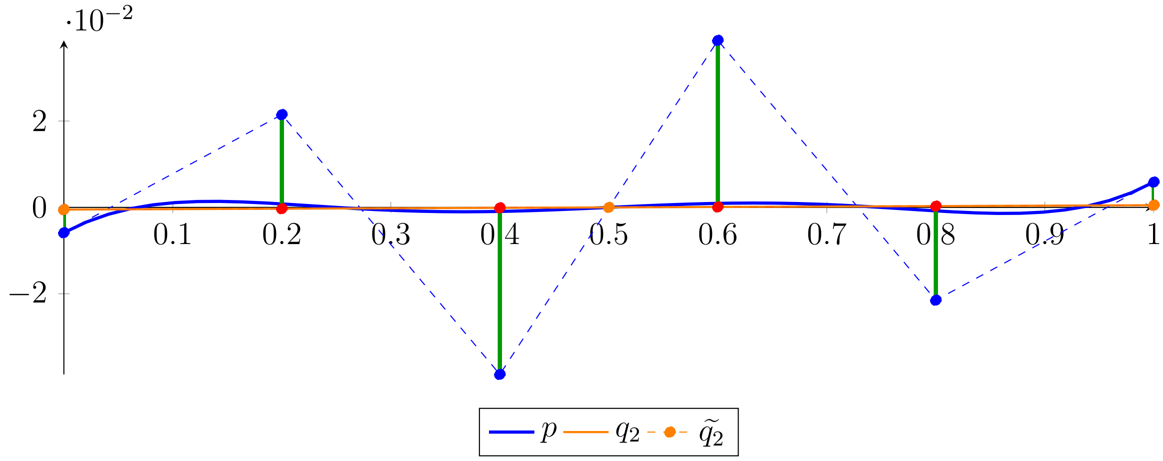
$$\begin{aligned} p &= 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.0214844B_{1,5}(X) - 0.0386719B_{2,5}(X) \\ &\quad + 0.0386719B_{3,5}(X) - 0.0214844B_{4,5}(X) + 0.00585938B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.25592 \cdot 10^{-19}X^2 + 0.00100446X - 0.000502232 \\ &= -0.000502232B_{0,2} - 1.37431 \cdot 10^{-19}B_{1,2} + 0.000502232B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.84971 \cdot 10^{-19}X^5 + 3.3246 \cdot 10^{-19}X^4 - 1.54054 \cdot 10^{-19}X^3 \\ &\quad + 8.20563 \cdot 10^{-19}X^2 + 0.00100446X - 0.000502232 \\ &= -0.000502232B_{0,5} - 0.000301339B_{1,5} - 0.000100446B_{2,5} \\ &\quad + 0.000100446B_{3,5} + 0.000301339B_{4,5} + 0.000502232B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0385714$.
Bounding polynomials M and m:

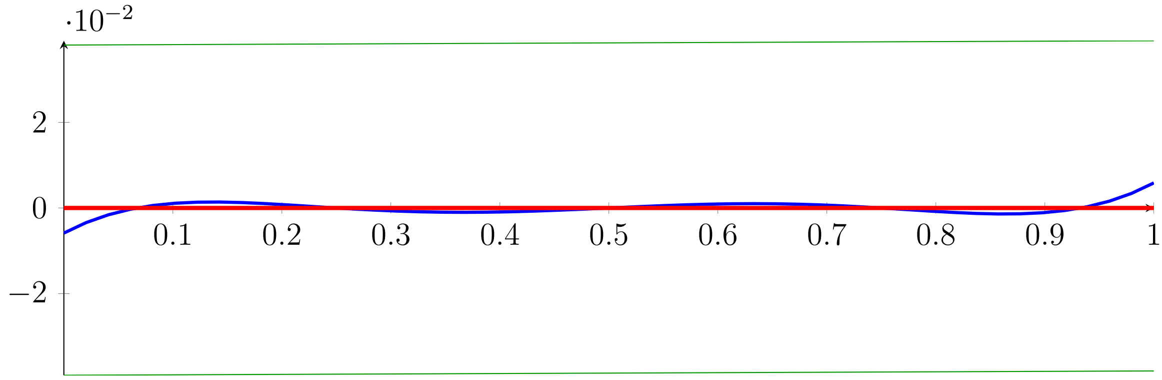
$$M = 8.30092{\cdot}10^{-19}X^2 + 0.00100446X + 0.0380692$$

$$m = 8.30092{\cdot}10^{-19}X^2 + 0.00100446X - 0.0390737$$

Root of M and m:

$$N(M) = \{-1.21006{\cdot}10^{15}, -37.9479\} \qquad N(m) = \{-1.21006{\cdot}10^{15}, 38.8775\}$$

Intersection intervals:



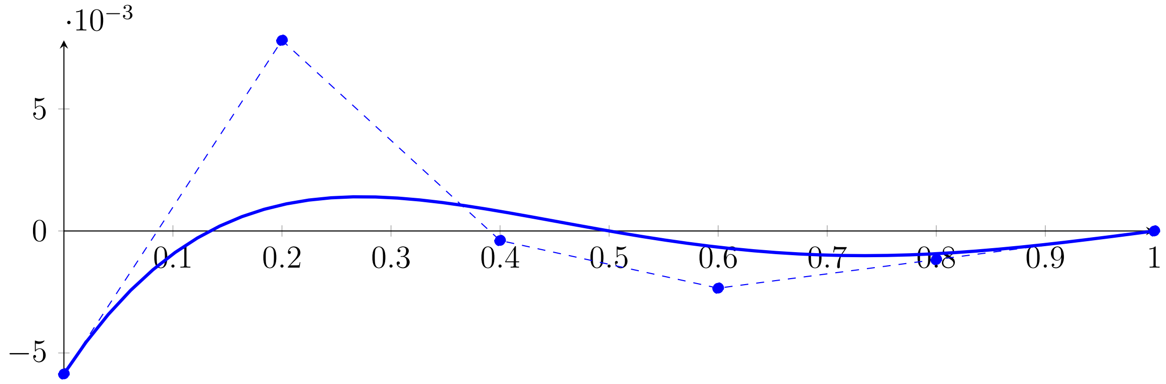
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$
Bisection point is very near to a root?!?

8.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

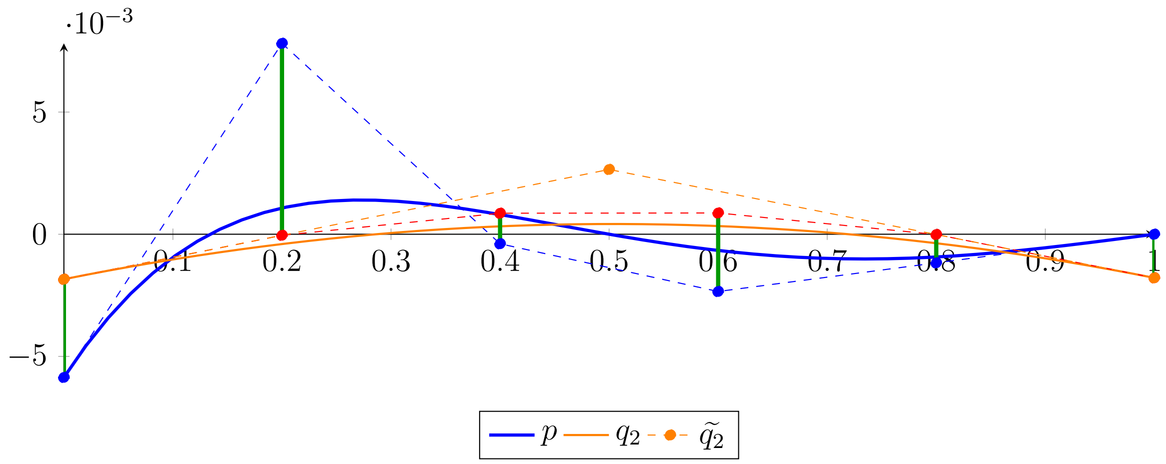
$$\begin{aligned} p &= 0.03125X^5 - 0.15625X^4 + 0.28125X^3 - 0.21875X^2 + 0.0683594X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.0078125B_{1,5}(X) - 0.000390625B_{2,5}(X) \\ &\quad - 0.00234375B_{3,5}(X) - 0.00117187B_{4,5}(X) + 6.89273{\cdot}10^{-20}B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00892857X^2 + 0.00898437X - 0.00184152 \\ &= -0.00184152B_{0,2} + 0.00265067B_{1,2} - 0.00178571B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 =&\, 2.0639{\cdot}10^{-18}X^5 - 5.42895{\cdot}10^{-18}X^4 + 5.21031{\cdot}10^{-18}X^3 - 0.00892857X^2 + 0.00898438X - 0.00184152 \\ =&\, -0.00184152B_{0,5} - 4.46429\cdot 10^{-05}B_{1,5} + 0.000859375B_{2,5} \\ &+ 0.000870536B_{3,5} - 1.11607\cdot 10^{-05}B_{4,5} - 0.00178571B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00785714$.

Bounding polynomials M and m:

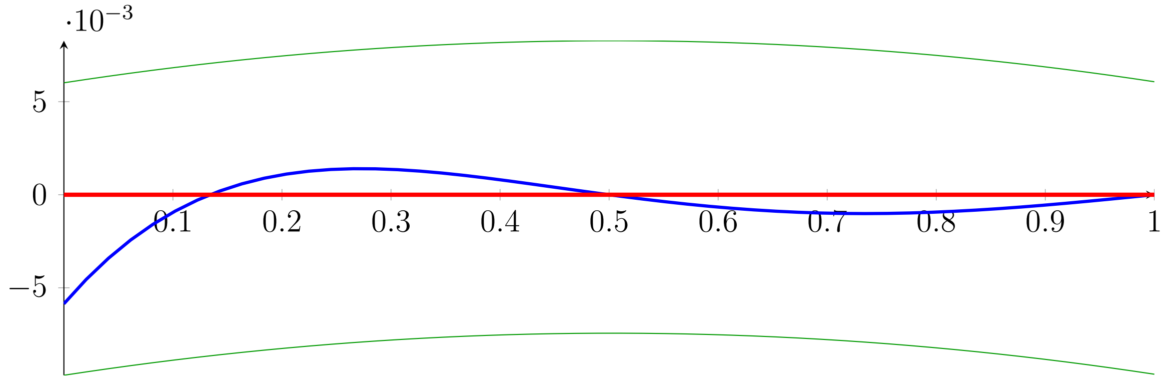
$$M = -0.00892857X^2 + 0.00898437X + 0.00601562$$

$$m = -0.00892857X^2 + 0.00898437X - 0.00969866$$

Root of M and m:

$$N(M) = \{-0.459624, 1.46587\} \qquad N(m) = \{\}$$

Intersection intervals:



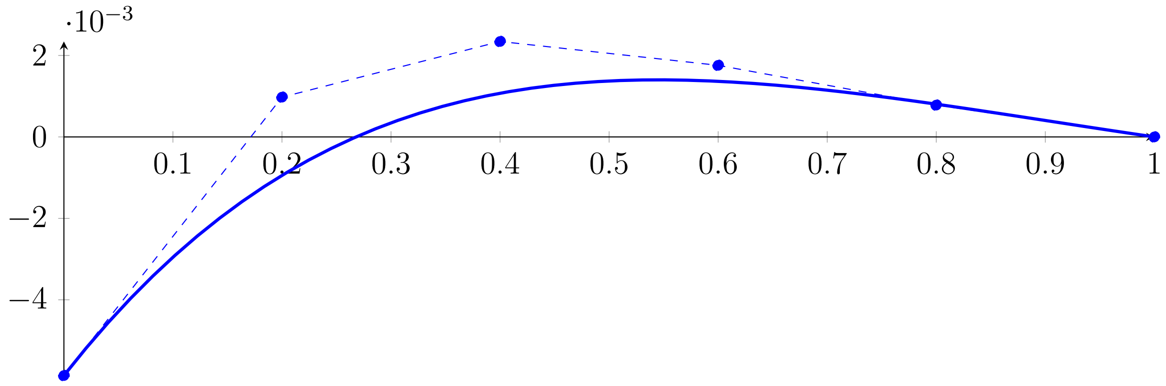
$$[0, 1]$$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$
Bisection point is very near to a root?!?

8.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

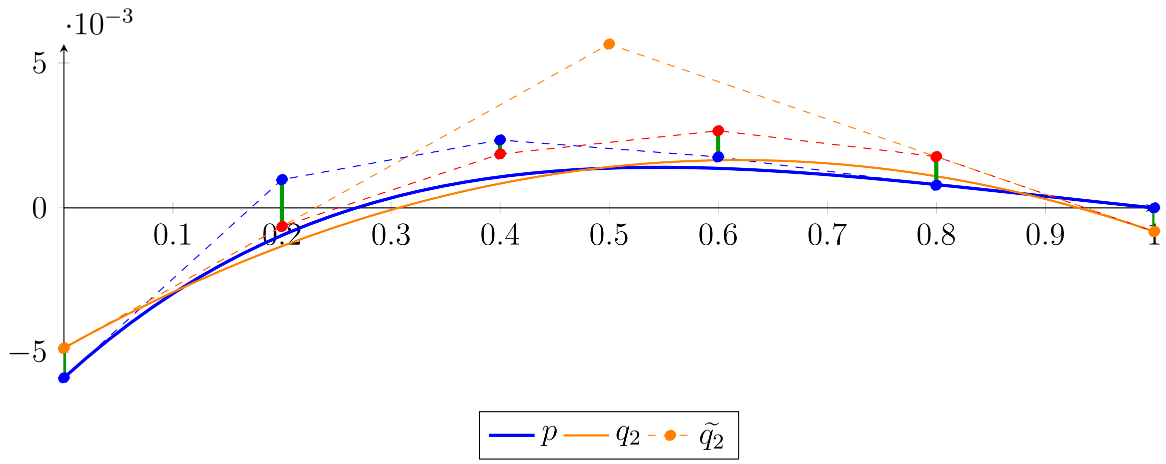
$$\begin{aligned}p &= 0.000976563X^5 - 0.00976562X^4 + 0.0351563X^3 - 0.0546875X^2 + 0.0341797X - 0.00585938\\ &= -0.00585938B_{0,5}(X) + 0.000976563B_{1,5}(X) + 0.00234375B_{2,5}(X)\\ &\quad + 0.00175781B_{3,5}(X) + 0.00078125B_{4,5}(X) + 1.55642{\cdot}10^{-20}B_{5,5}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -0.0169503X^2 + 0.0209682X - 0.00483398\\ &= -0.00483398B_{0,2} + 0.00565011B_{1,2} - 0.000816127B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= 2.23363{\cdot}10^{-18}X^5 - 6.64921{\cdot}10^{-18}X^4 + 7.1659{\cdot}10^{-18}X^3 - 0.0169503X^2 + 0.0209682X - 0.00483398\\ &= -0.00483398B_{0,5} - 0.000640346B_{1,5} + 0.00185826B_{2,5}\\ &\quad + 0.00266183B_{3,5} + 0.00177037B_{4,5} - 0.000816127B_{5,5}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00161691$.
Bounding polynomials M and m:

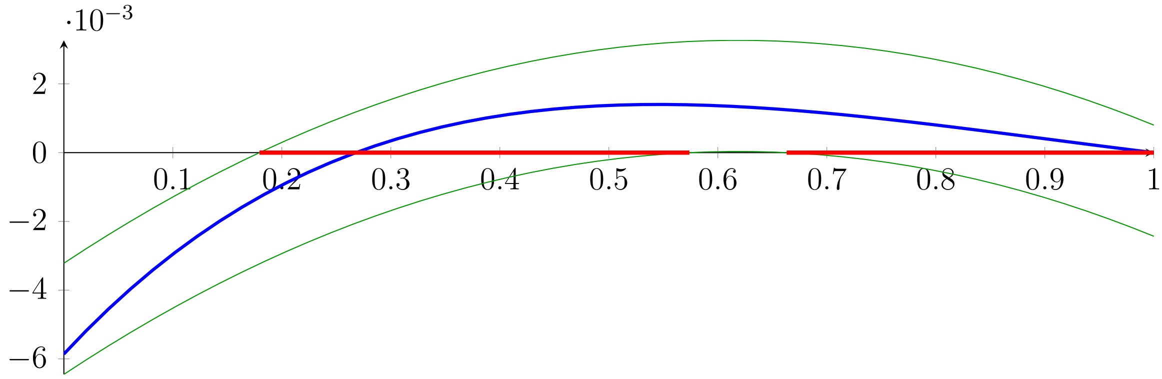
$$M = -0.0169503X^2 + 0.0209682X - 0.00321708$$

$$m = -0.0169503X^2 + 0.0209682X - 0.00645089$$

Root of M and m:

$$N(M) = \{0.179462, 1.05758\} \qquad\qquad N(m) = \{0.57392, 0.663117\}$$

Intersection intervals:



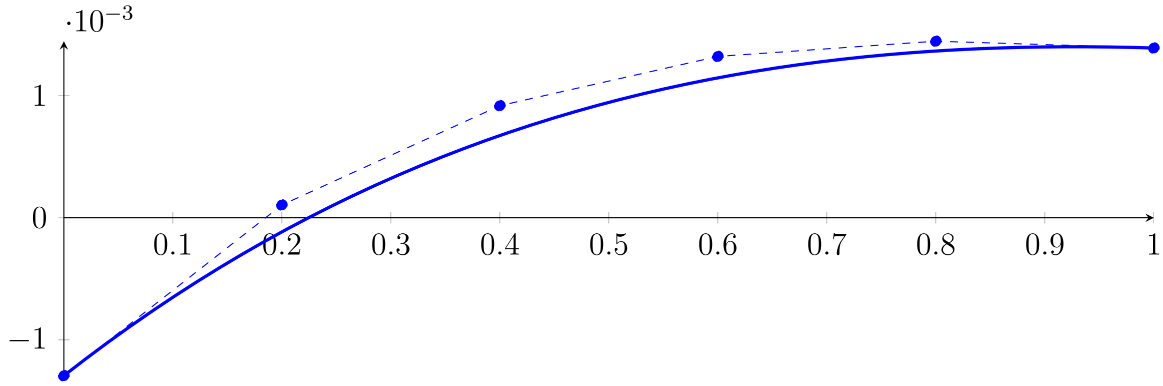
$$[0.179462, 0.57392],\ [0.663117, 1]$$

Longest intersection interval: 0.394458
$\Longrightarrow$ Selective recursion: interval 1: $[0.0448654, 0.14348]$, interval 2: $[0.165779, 0.25]$,

8.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.0448654, 0.14348]$

Normalized monomial und Bézier representations and the Bézier polygon:

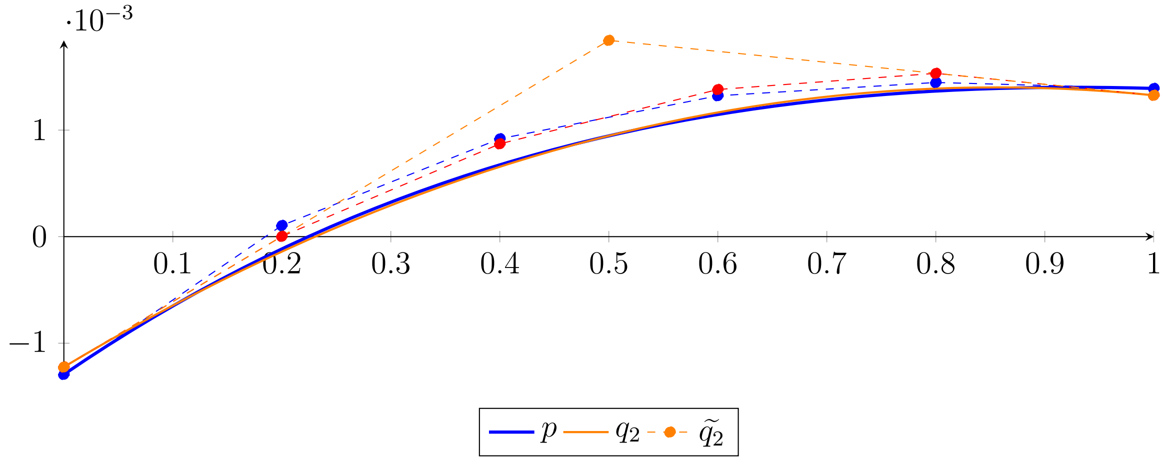
$$\begin{aligned}p &= 9.32622\cdot 10^{-06}X^5 - 0.000215216X^4 + 0.00174681X^3 - 0.00584899X^2 + 0.00699262X - 0.00129347\\ &= -0.00129347B_{0,5}(X) + 0.000105049B_{1,5}(X) + 0.000918673B_{2,5}(X)\\ &\quad + 0.00132208B_{3,5}(X) + 0.00144691B_{4,5}(X) + 0.00139107B_{5,5}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -0.00358106X^2 + 0.0061313X - 0.00122358\\ &= -0.00122358B_{0,2} + 0.00184207B_{1,2} + 0.00132666B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -8.49786\cdot 10^{-19}X^5 + 1.62736\cdot 10^{-18}X^4 - 9.67735\\ &\quad \cdot 10^{-19}X^3 - 0.00358106X^2 + 0.0061313X - 0.00122358\\ &= -0.00122358B_{0,5} + 2.6796\cdot 10^{-06}B_{1,5} + 0.000870835B_{2,5}\\ &\quad + 0.00138088B_{3,5} + 0.00153283B_{4,5} + 0.00132666B_{5,5}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00010237$.

Bounding polynomials M and m:

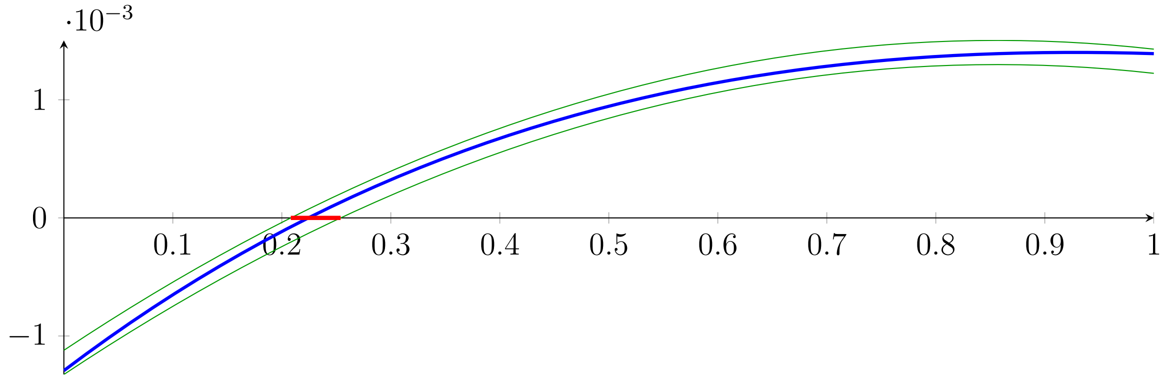
$$M = -0.00358106X^2 + 0.0061313X - 0.00112121$$

$$m = -0.00358106X^2 + 0.0061313X - 0.00132595$$

Root of M and m:

$$N(M) = \{0.208179, 1.50397\} \qquad N(m) = \{0.253915, 1.45823\}$$

Intersection intervals:



$$[0.208179, 0.253915]$$

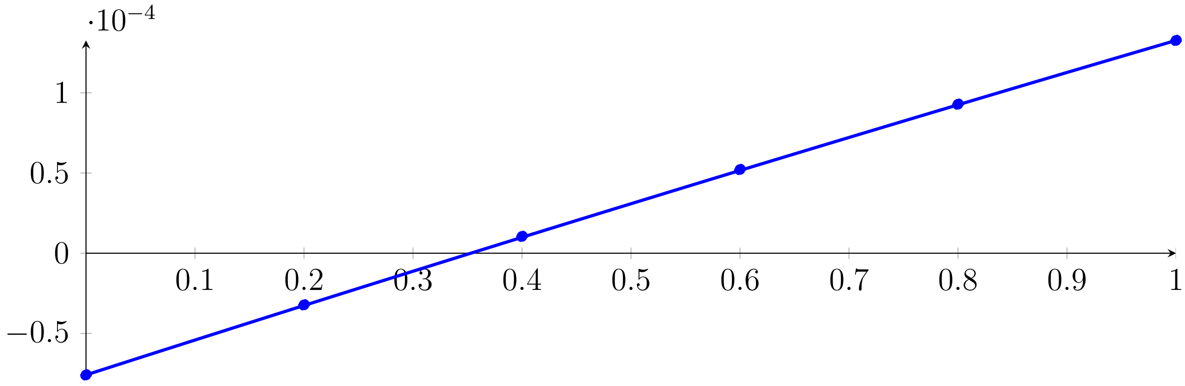
Longest intersection interval: 0.0457362

$\Longrightarrow$ Selective recursion: interval 1: $[0.0653949, 0.0699052]$,

8.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0653949, 0.0699052]$

Normalized monomial und Bézier representations and the Bézier polygon:

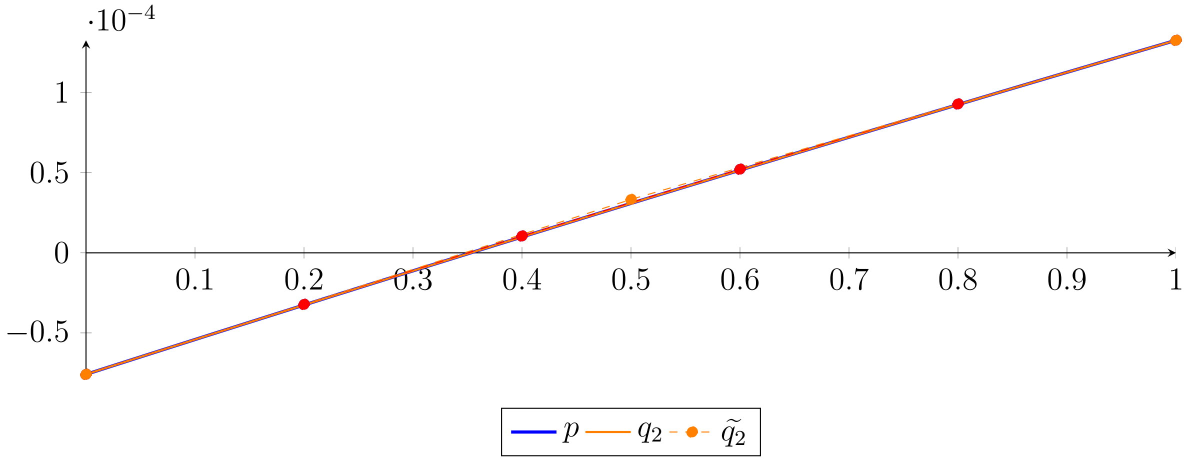
$$\begin{aligned} p &= 1.86641 \cdot 10^{-12} X^5 - 8.99231 \cdot 10^{-10} X^4 + 1.50361 \cdot 10^{-07} X^3 \\ &\quad - 1.00682 \cdot 10^{-05} X^2 + 0.000218472X - 7.58847 \cdot 10^{-05} \\ &= -7.58847 \cdot 10^{-05} B_{0,5}(X) - 3.21903 \cdot 10^{-05} B_{1,5}(X) + 1.04972 \cdot 10^{-05} B_{2,5}(X) \\ &\quad + 5.21929 \cdot 10^{-05} B_{3,5}(X) + 9.29117 \cdot 10^{-05} B_{4,5}(X) + 0.000132668 B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -9.84419\cdot 10^{-06}X^2 + 0.000218382X - 7.58772\cdot 10^{-05} \\ &= -7.58772\cdot 10^{-05}B_{0,2} + 3.33139\cdot 10^{-05}B_{1,2} + 0.000132661B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -8.33467\cdot 10^{-20}X^5 + 1.78704\cdot 10^{-19}X^4 - 1.30297\cdot 10^{-19}X^3 \\ &\quad - 9.84419\cdot 10^{-06}X^2 + 0.000218382X - 7.58772\cdot 10^{-05} \\ &= -7.58772\cdot 10^{-05}B_{0,5} - 3.22008\cdot 10^{-05}B_{1,5} + 1.04913\cdot 10^{-05}B_{2,5} \\ &\quad + 5.21989\cdot 10^{-05}B_{3,5} + 9.29221\cdot 10^{-05}B_{4,5} + 0.000132661B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.04381\cdot 10^{-08}$.

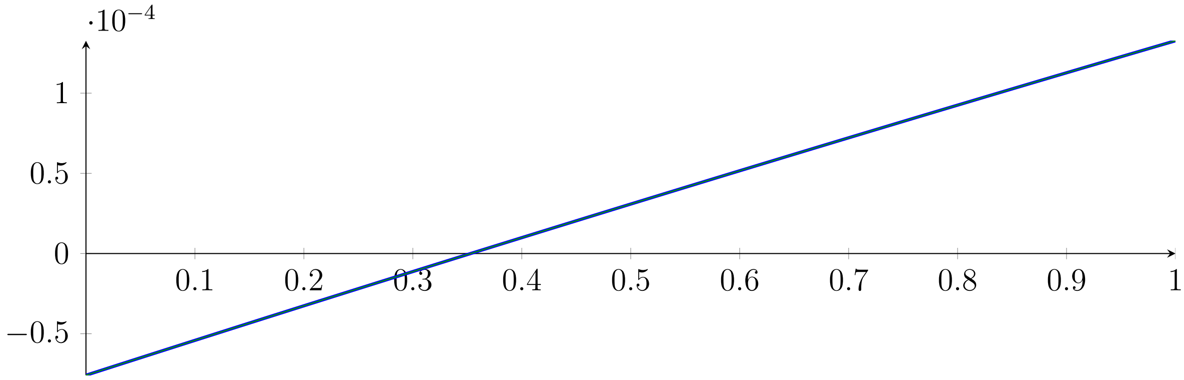
Bounding polynomials M and m:

$$M = -9.84419\cdot 10^{-06}X^2 + 0.000218382X - 7.58668\cdot 10^{-05}$$
$$m = -9.84419\cdot 10^{-06}X^2 + 0.000218382X - 7.58877\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.353021, 21.8309\} \qquad N(m) = \{0.35312, 21.8308\}$$

Intersection intervals:



$$[0.353021, 0.35312]$$

Longest intersection interval: $9.87378 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0669871, 0.0669876]$,

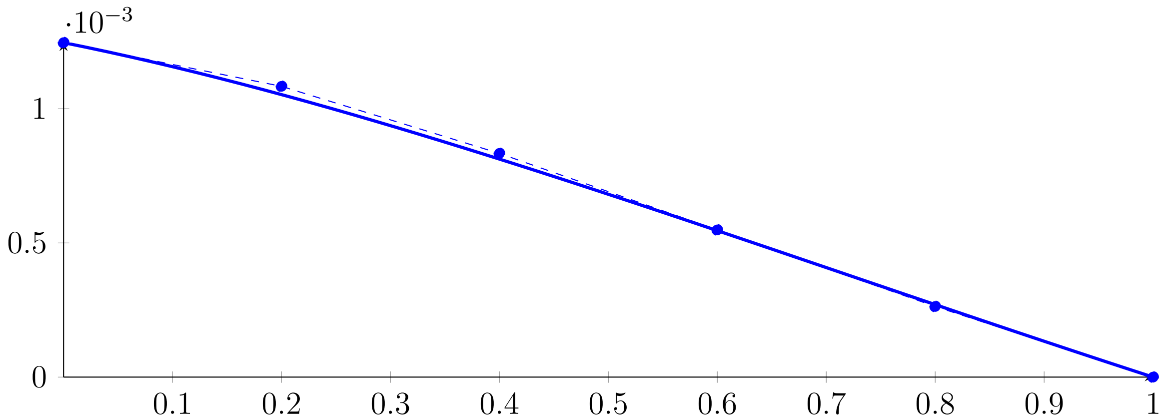
8.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0669871, 0.0669876]$

Found root in interval $[0.0669871, 0.0669876]$ at recursion depth 6!

8.7 Recursion Branch 1 1 1 2 in Interval 2: $[0.165779, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

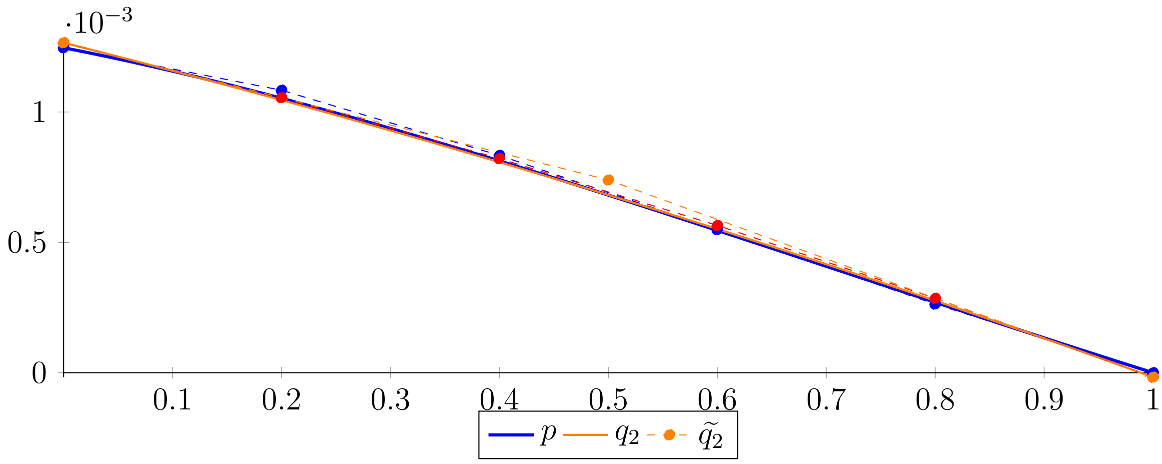
$$\begin{aligned}
p &= 4.23736 \cdot 10^{-06} X^5 - 8.40775 \cdot 10^{-05} X^4 + 0.000517957 X^3 \\
&\quad - 0.00087012 X^2 - 0.000814458 X + 0.00124646 \\
&= 0.00124646 B_{0,5}(X) + 0.00108357 B_{1,5}(X) + 0.000833666 B_{2,5}(X) \\
&\quad + 0.000548546 B_{3,5}(X) + 0.00026319 B_{4,5}(X) + 1.55642 \cdot 10^{-20} B_{5,5}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= -0.00022975 X^2 - 0.0010529 X + 0.00126561 \\
&= 0.00126561 B_{0,2} + 0.000739155 B_{1,2} - 1.70457 \cdot 10^{-05} B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_2 &= -7.546 \cdot 10^{-19} X^5 + 1.99106 \cdot 10^{-18} X^4 - 1.96723 \cdot 10^{-18} X^3 - 0.00022975 X^2 - 0.0010529 X + 0.00126561 \\
&= 0.00126561 B_{0,5} + 0.00105503 B_{1,5} + 0.00082147 B_{2,5} \\
&\quad + 0.00056494 B_{3,5} + 0.000285435 B_{4,5} - 1.70457 \cdot 10^{-05} B_{5,5}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.85435{\cdot}10^{-05}$.

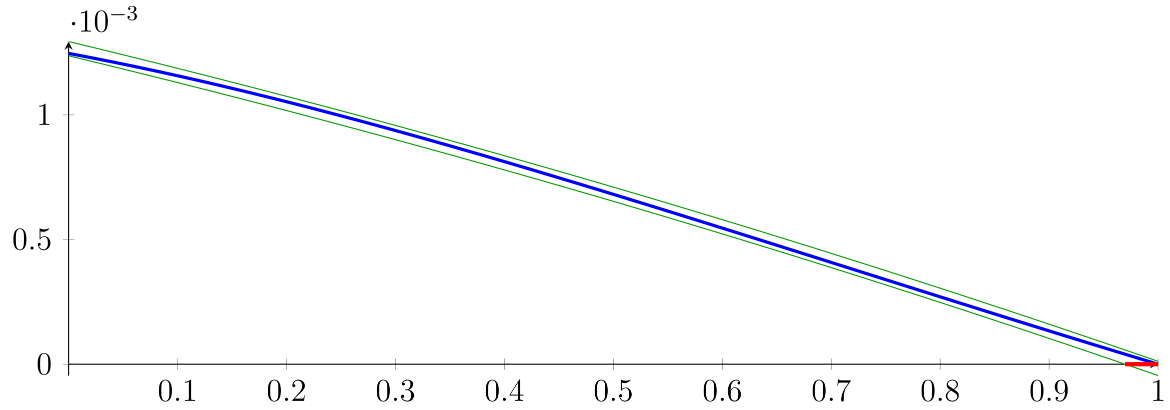
Bounding polynomials M and m:

$$M = -0.00022975X^2 - 0.0010529X + 0.00129415$$
$$m = -0.00022975X^2 - 0.0010529X + 0.00123706$$

Root of M and m:

$$N(M) = \{-5.5904, 1.00759\} \qquad N(m) = \{-5.55253, 0.969717\}$$

Intersection intervals:



$$[0.969717, 1]$$

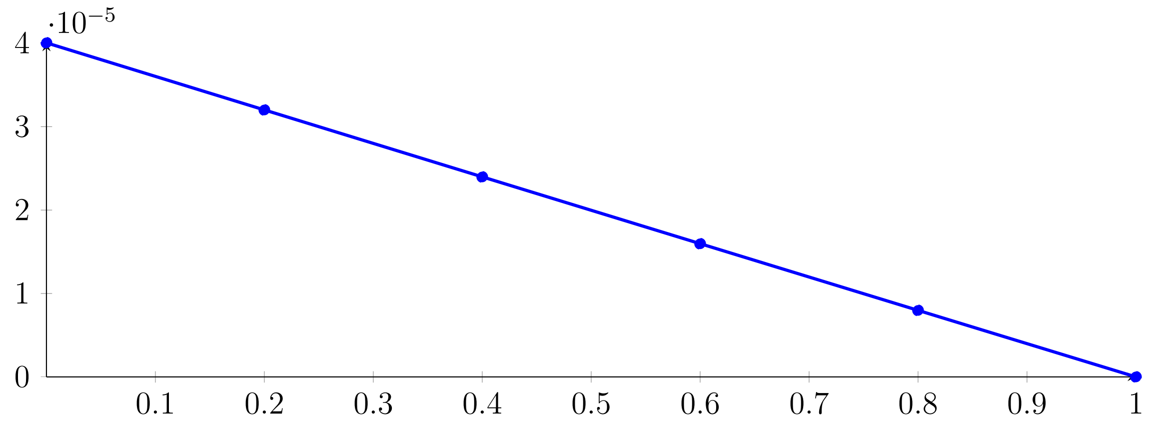
Longest intersection interval: 0.0302829

$\Longrightarrow$ Selective recursion: interval 1: $[0.24745, 0.25]$,

8.8 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.24745, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

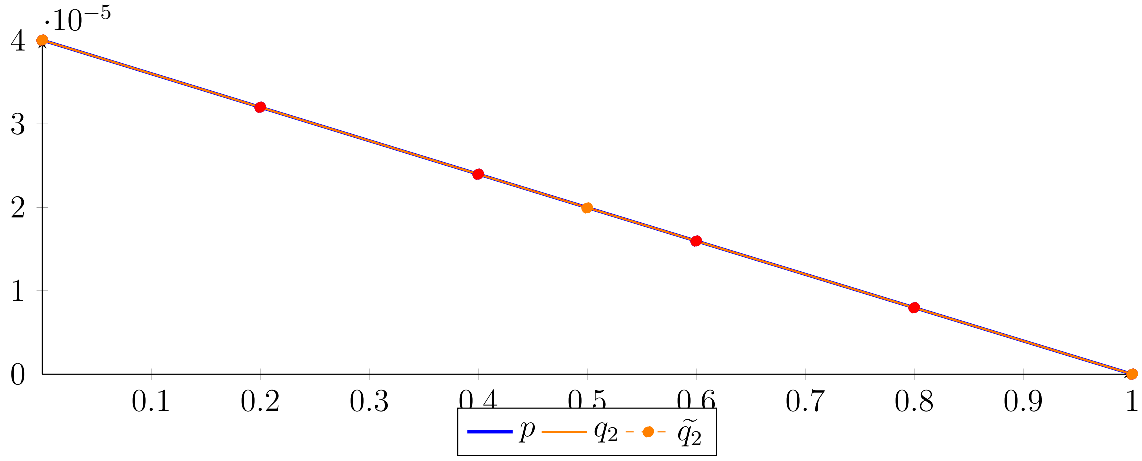
$$\begin{aligned} p &= 1.07915{\cdot}10^{-13}X^5 - 5.34296{\cdot}10^{-11}X^4 + 6.43391{\cdot}10^{-09}X^3 \\ &\quad + 1.84292{\cdot}10^{-07}X^2 - 4.02384{\cdot}10^{-05}X + 4.00477{\cdot}10^{-05} \\ &= 4.00477{\cdot}10^{-05}B_{0,5}(X) + 3.2{\cdot}10^{-05}B_{1,5}(X) + 2.39708{\cdot}10^{-05}B_{2,5}(X) \\ &\quad + 1.59606{\cdot}10^{-05}B_{3,5}(X) + 7.97014{\cdot}10^{-06}B_{4,5}(X) + 1.55642{\cdot}10^{-20}B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.93851{\cdot}10^{-07}X^2 - 4.02422{\cdot}10^{-05}X + 4.0048{\cdot}10^{-05} \\ &= 4.0048{\cdot}10^{-05}B_{0,2} + 1.99269{\cdot}10^{-05}B_{1,2} - 3.15609{\cdot}10^{-10}B_{2,2} \end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -2.37864\cdot 10^{-20}X^5 + 6.29815\cdot 10^{-20}X^4 - 6.24356\cdot 10^{-20}X^3\\ &\quad + 1.93851\cdot 10^{-07}X^2 - 4.02422\cdot 10^{-05}X + 4.0048\cdot 10^{-05}\\ &= 4.0048\cdot 10^{-05}B_{0,5} + 3.19996\cdot 10^{-05}B_{1,5} + 2.39705\cdot 10^{-05}B_{2,5}\\ &\quad + 1.59609\cdot 10^{-05}B_{3,5} + 7.97058\cdot 10^{-06}B_{4,5} - 3.15609\cdot 10^{-10}B_{5,5}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.45195\cdot 10^{-10}$.
Bounding polynomials M and m:

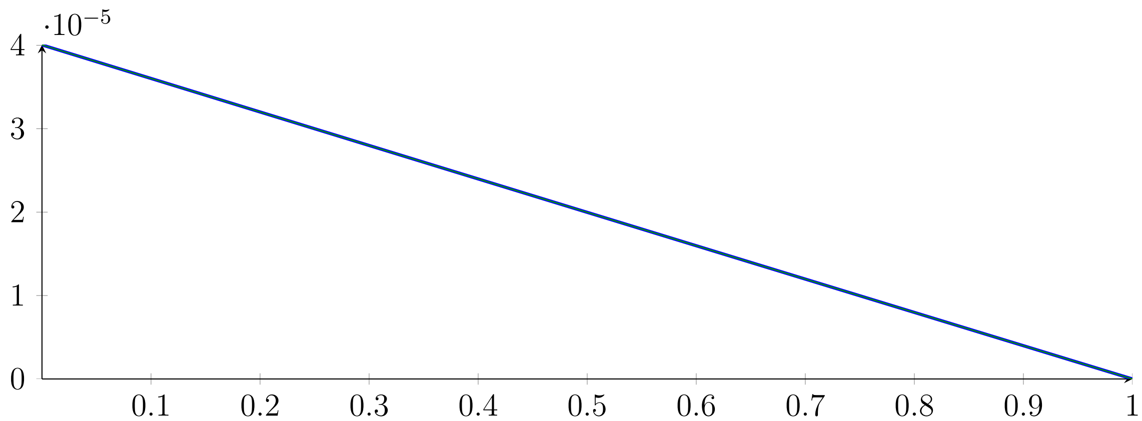
$$M = 1.93851\cdot 10^{-07}X^2 - 4.02422\cdot 10^{-05}X + 4.00485\cdot 10^{-05}$$

$$m = 1.93851\cdot 10^{-07}X^2 - 4.02422\cdot 10^{-05}X + 4.00476\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{1, 206.593\} \qquad N(m) = \{0.999981, 206.593\}$$

Intersection intervals:



$$[0.999981, 1]$$

Longest intersection interval: $1.90895\cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.25]$,

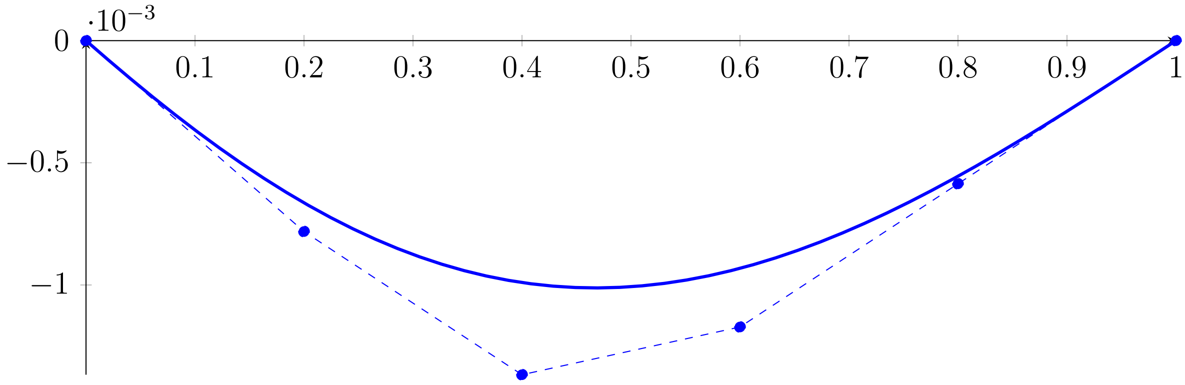
8.9 Recursion Branch 1 1 1 2 1 1 in Interval 1: $[0.25, 0.25]$

Found root in interval $[0.25, 0.25]$ at recursion depth 6!

8.10 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

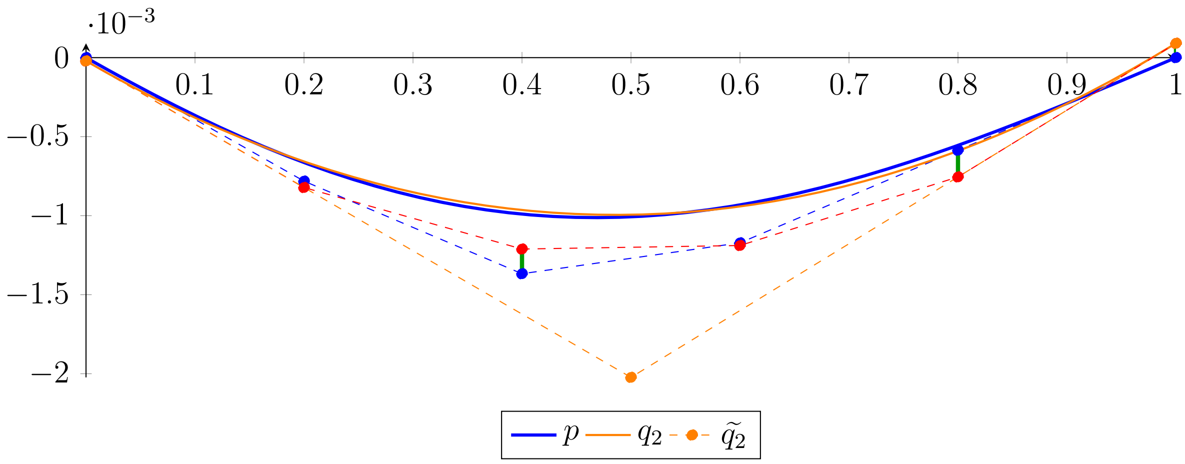
$$p = 0.000976562X^5 - 0.00488281X^4 + 0.00585938X^3 + 0.00195313X^2 - 0.00390625X + 1.55642{\cdot}10^{-20}$$
$$= 1.55642{\cdot}10^{-20}B_{0,5}(X) - 0.00078125B_{1,5}(X) - 0.00136719B_{2,5}(X)$$
$$- 0.00117187B_{3,5}(X) - 0.000585937B_{4,5}(X) + 6.89273{\cdot}10^{-20}B_{5,5}(X)$$



Degree reduction and raising:

$$q_2 = 0.00411551X^2 - 0.00400391X - 2.09263{\cdot}10^{-05}$$
$$= -2.09263{\cdot}10^{-05}B_{0,2} - 0.00202288B_{1,2} + 9.06808{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 2.71186{\cdot}10^{-19}X^5 - 5.03604{\cdot}10^{-19}X^4 + 3.23229{\cdot}10^{-19}X^3$$
$$+ 0.00411551X^2 - 0.00400391X - 2.09263{\cdot}10^{-05}$$
$$= -2.09263{\cdot}10^{-05}B_{0,5} - 0.000821708B_{1,5} - 0.00121094B_{2,5}$$
$$- 0.00118862B_{3,5} - 0.000754743B_{4,5} + 9.06808{\cdot}10^{-05}B_{5,5}$$



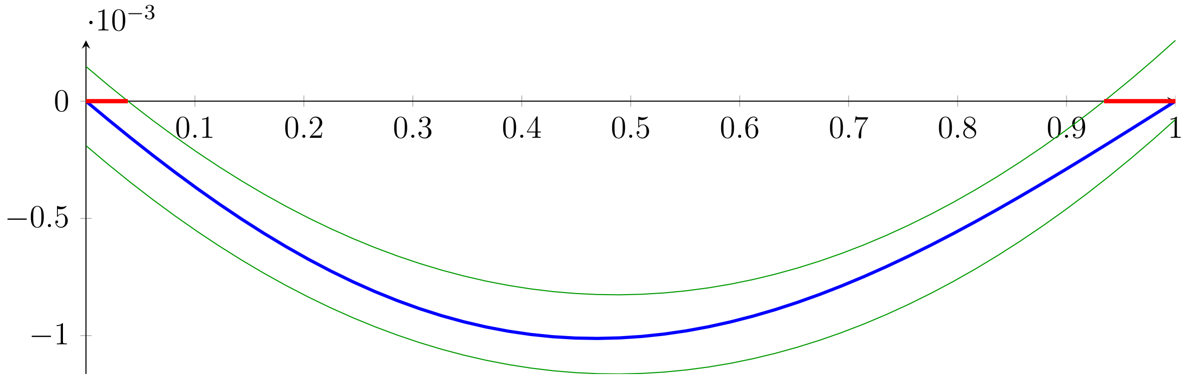
The maximum difference of the Bézier coefficients is $\delta = 0.000168806$.
Bounding polynomials M and m:

$$M = 0.00411551X^2 - 0.00400391X + 0.000147879$$
$$m = 0.00411551X^2 - 0.00400391X - 0.000189732$$

Root of M and m:

$$N(M) = \{0.0384537, 0.934428\} \qquad N(m) = \{-0.0452794, 1.01816\}$$

Intersection intervals:



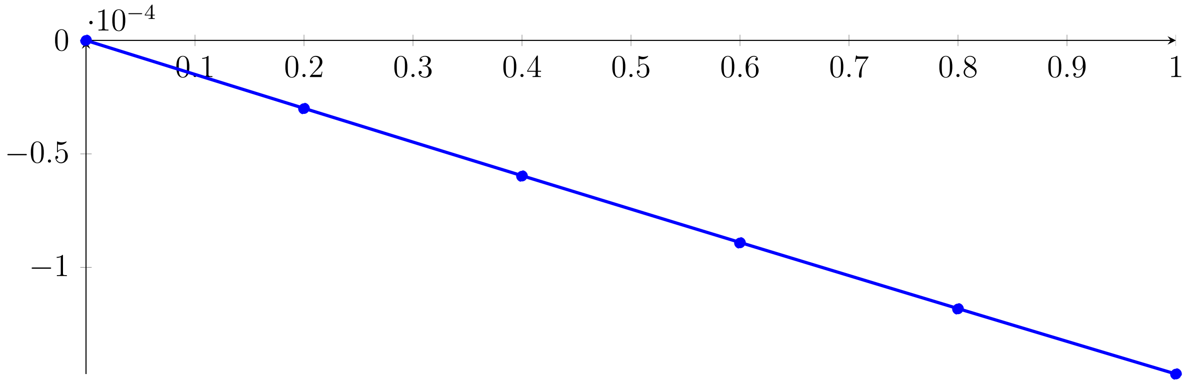
$$[0, 0.0384537], [0.934428, 1]$$

Longest intersection interval: 0.0655723
$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.259613]$, interval 2: $[0.483607, 0.5]$,

8.11 Recursion Branch 1 1 2 1 in Interval 1: $[0.25, 0.259613]$

Normalized monomial und Bézier representations and the Bézier polygon:

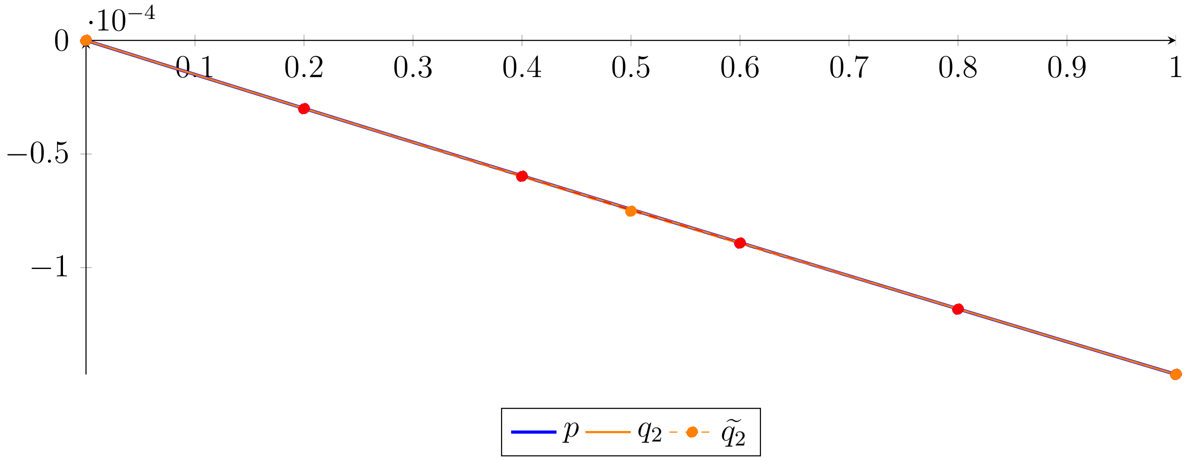
$$\begin{aligned} p &= 8.2109{\cdot}10^{-11}X^5 - 1.06763{\cdot}10^{-08}X^4 + 3.3317{\cdot}10^{-07}X^3 \\ &\quad + 2.88806{\cdot}10^{-06}X^2 - 0.00015021X + 1.55642{\cdot}10^{-20} \\ &= 1.55642{\cdot}10^{-20}B_{0,5}(X) - 3.0042{\cdot}10^{-05}B_{1,5}(X) - 5.97951{\cdot}10^{-05}B_{2,5}(X) \\ &\quad - 8.92261{\cdot}10^{-05}B_{3,5}(X) - 0.000118304B_{4,5}(X) - 0.000146999B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 3.36966{\cdot}10^{-06}X^2 - 0.0001504X + 1.57522{\cdot}10^{-08} \\ &= 1.57522{\cdot}10^{-08}B_{0,2} - 7.51843{\cdot}10^{-05}B_{1,2} - 0.000147015B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.41788{\cdot}10^{-19}X^5 - 3.29266{\cdot}10^{-19}X^4 + 2.7472{\cdot}10^{-19}X^3 \\ &\quad + 3.36966{\cdot}10^{-06}X^2 - 0.0001504X + 1.57522{\cdot}10^{-08} \\ &= 1.57522{\cdot}10^{-08}B_{0,5} - 3.00642{\cdot}10^{-05}B_{1,5} - 5.98073{\cdot}10^{-05}B_{2,5} \\ &\quad - 8.92134{\cdot}10^{-05}B_{3,5} - 0.000118282B_{4,5} - 0.000147015B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22936 \cdot 10^{-08}$.

Bounding polynomials M and m:

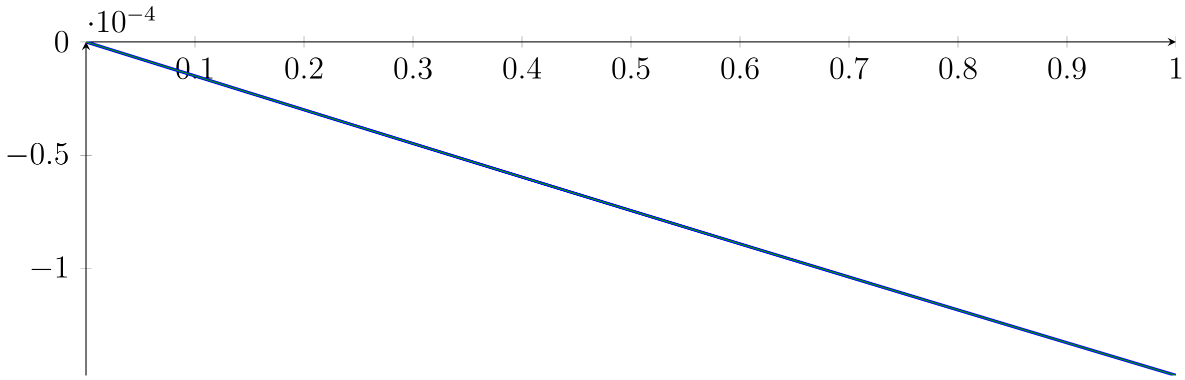
$$M = 3.36966 \cdot 10^{-06} X^2 - 0.0001504X + 3.80457 \cdot 10^{-08}$$

$$m = 3.36966 \cdot 10^{-06} X^2 - 0.0001504X - 6.54138 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{0.000252965, 44.6333\} \qquad N(m) = \{-4.34932 \cdot 10^{-05}, 44.6336\}$$

Intersection intervals:



$$[0, 0.000252965]$$

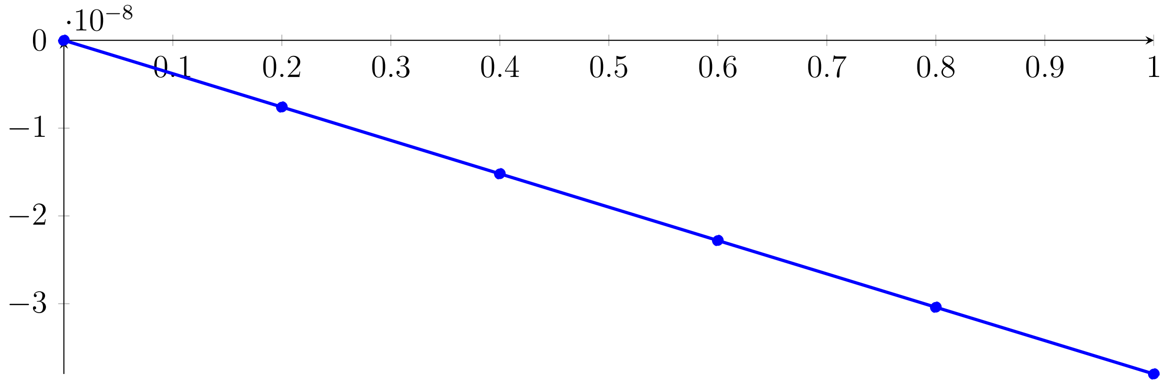
Longest intersection interval: 0.000252965

$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.250002]$,

8.12 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.25, 0.250002]$

Normalized monomial und Bézier representations and the Bézier polygon:

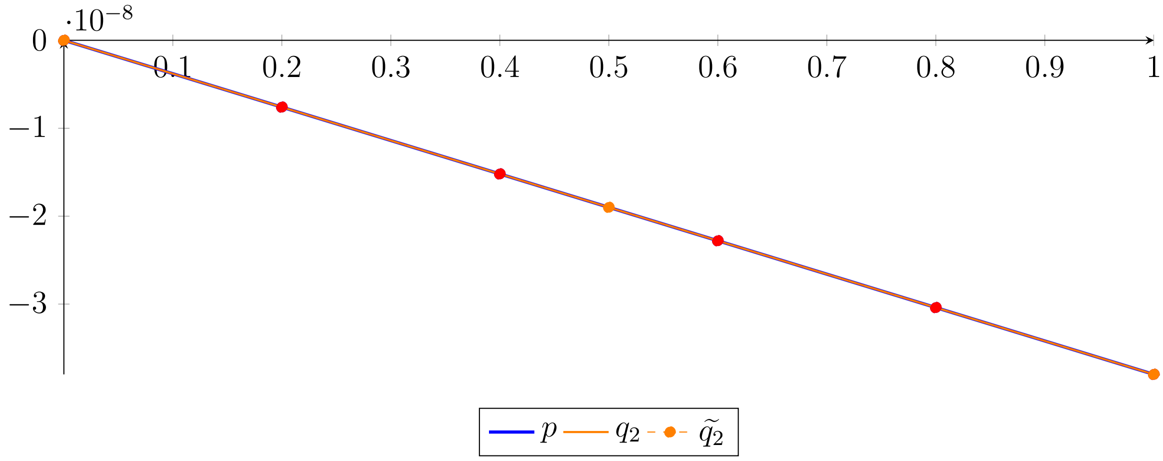
$$\begin{aligned} p = &-5.493 \cdot 10^{-26} X^5 - 4.36693 \cdot 10^{-23} X^4 + 5.39321 \cdot 10^{-18} X^3 \\ &+ 1.84811 \cdot 10^{-13} X^2 - 3.79978 \cdot 10^{-08} X + 1.55642 \cdot 10^{-20} \\ = &\ 1.55642 \cdot 10^{-20} B_{0,5}(X) - 7.59957 \cdot 10^{-09} B_{1,5}(X) - 1.51991 \cdot 10^{-08} B_{2,5}(X) \\ &- 2.27986 \cdot 10^{-08} B_{3,5}(X) - 3.03982 \cdot 10^{-08} B_{4,5}(X) - 3.79976 \cdot 10^{-08} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.84819{\cdot}10^{-13}X^2 - 3.79978{\cdot}10^{-08}X + 2.85221{\cdot}10^{-19} \\ &= 2.85221{\cdot}10^{-19}B_{0,2} - 1.89989{\cdot}10^{-08}B_{1,2} - 3.79976{\cdot}10^{-08}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 3.65833{\cdot}10^{-23}X^5 - 8.49314{\cdot}10^{-23}X^4 + 7.08919{\cdot}10^{-23}X^3 \\ &\quad + 1.84819{\cdot}10^{-13}X^2 - 3.79978{\cdot}10^{-08}X + 2.85221{\cdot}10^{-19} \\ &= 2.85221{\cdot}10^{-19}B_{0,5} - 7.59957{\cdot}10^{-09}B_{1,5} - 1.51991{\cdot}10^{-08}B_{2,5} \\ &\quad - 2.27986{\cdot}10^{-08}B_{3,5} - 3.03982{\cdot}10^{-08}B_{4,5} - 3.79976{\cdot}10^{-08}B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.7752{\cdot}10^{-19}$.

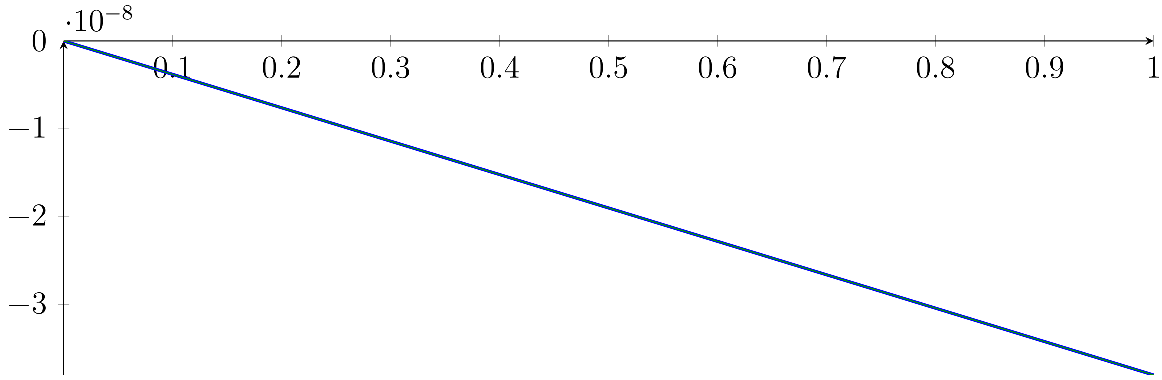
Bounding polynomials M and m:

$$\begin{aligned} M &= 1.84819{\cdot}10^{-13}X^2 - 3.79978{\cdot}10^{-08}X + 6.62741{\cdot}10^{-19} \\ m &= 1.84819{\cdot}10^{-13}X^2 - 3.79978{\cdot}10^{-08}X - 9.22988{\cdot}10^{-20} \end{aligned}$$

Root of M and m:

$$N(M) = \{1.74479{\cdot}10^{-11}, 205595\} \qquad N(m) = \{-2.43012{\cdot}10^{-12}, 205595\}$$

Intersection intervals:



$$[0, 1.74479e - 11]$$

Longest intersection interval: $1.74479 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.25]$,

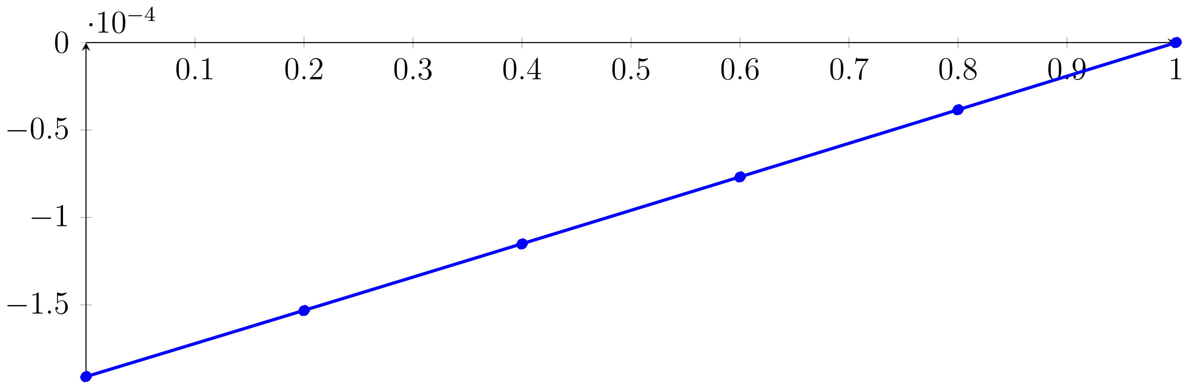
8.13 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.25, 0.25]$

Found root in interval $[0.25, 0.25]$ at recursion depth 6!

8.14 Recursion Branch 1 1 2 2 in Interval 2: $[0.483607, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

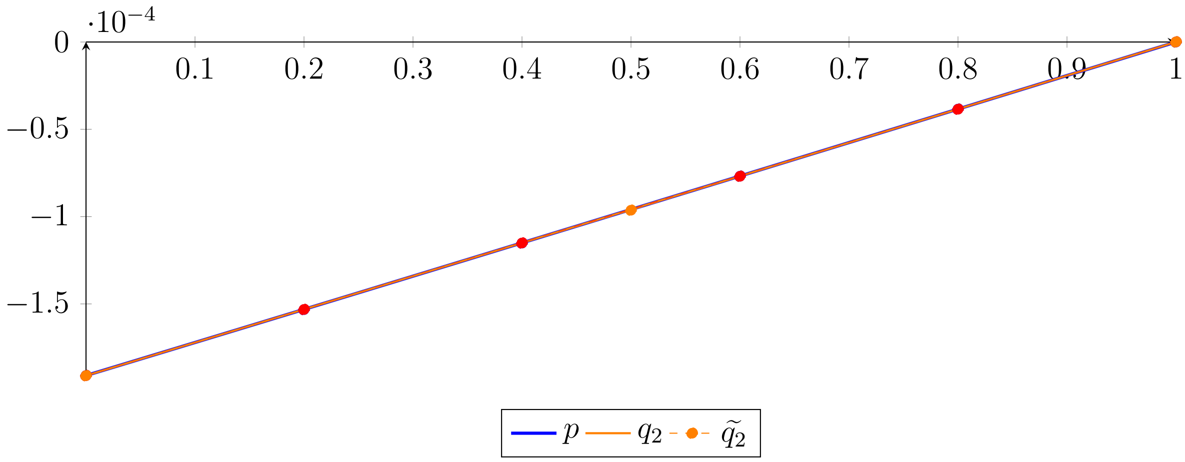
$$\begin{aligned} p &= 1.18387 \cdot 10^{-09} X^5 - 5.91935 \cdot 10^{-09} X^4 - 1.0895 \cdot 10^{-06} X^3 \\ &\quad + 3.29219 \cdot 10^{-06} X^2 + 0.000188808 X - 0.000191006 \\ &= -0.000191006 B_{0,5}(X) - 0.000153245 B_{1,5}(X) - 0.000115154 B_{2,5}(X) \\ &\quad - 7.68426 \cdot 10^{-05} B_{3,5}(X) - 3.84213 \cdot 10^{-05} B_{4,5}(X) + 6.89273 \cdot 10^{-20} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.6499 \cdot 10^{-06} X^2 + 0.000189466 X - 0.000191061 \\ &= -0.000191061 B_{0,2} - 9.63281 \cdot 10^{-05} B_{1,2} + 5.49403 \cdot 10^{-08} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 1.13648 \cdot 10^{-19} X^5 - 3.0116 \cdot 10^{-19} X^4 + 2.98447 \cdot 10^{-19} X^3 \\ &\quad + 1.6499 \cdot 10^{-06} X^2 + 0.000189466 X - 0.000191061 \\ &= -0.000191061 B_{0,5} - 0.000153168 B_{1,5} - 0.00011511 B_{2,5} \\ &\quad - 7.68865 \cdot 10^{-05} B_{3,5} - 3.84983 \cdot 10^{-05} B_{4,5} + 5.49403 \cdot 10^{-08} B_{5,5} \end{aligned}$$



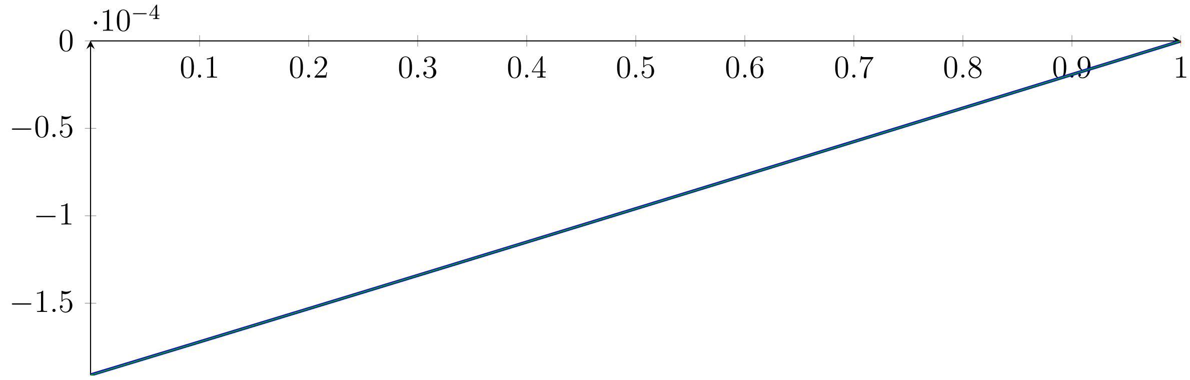
The maximum difference of the Bézier coefficients is $\delta = 7.69671{\cdot}10^{-08}$.
Bounding polynomials M and m:

$$M = 1.6499{\cdot}10^{-06}X^2 + 0.000189466X - 0.000190984$$
$$m = 1.6499{\cdot}10^{-06}X^2 + 0.000189466X - 0.000191138$$

Root of M and m:

$$N(M) = \{-115.834, 0.999316\} \qquad N(m) = \{-115.835, 1.00011\}$$

Intersection intervals:



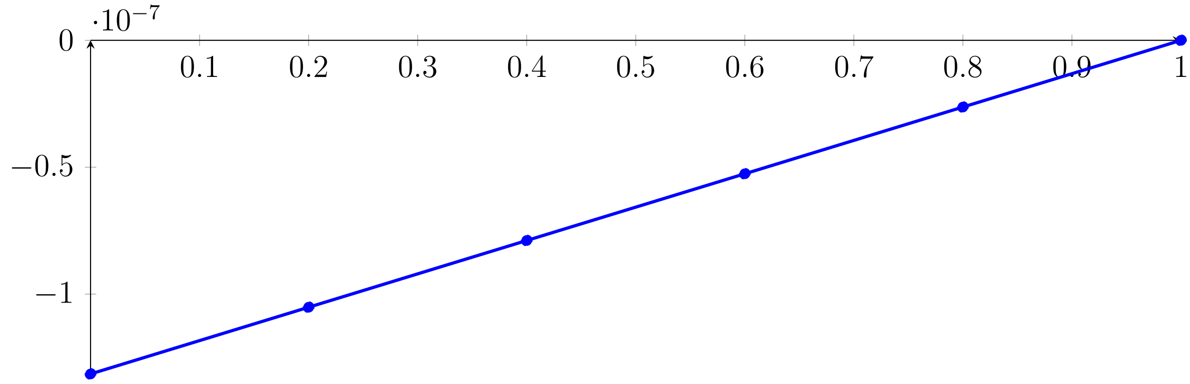
$[0.999316, 1]$

Longest intersection interval: 0.000684291
$\Longrightarrow$ Selective recursion: interval 1: $[0.499989, 0.5]$,

8.15 Recursion Branch 1 1 2 2 1 in Interval 1: $[0.499989, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

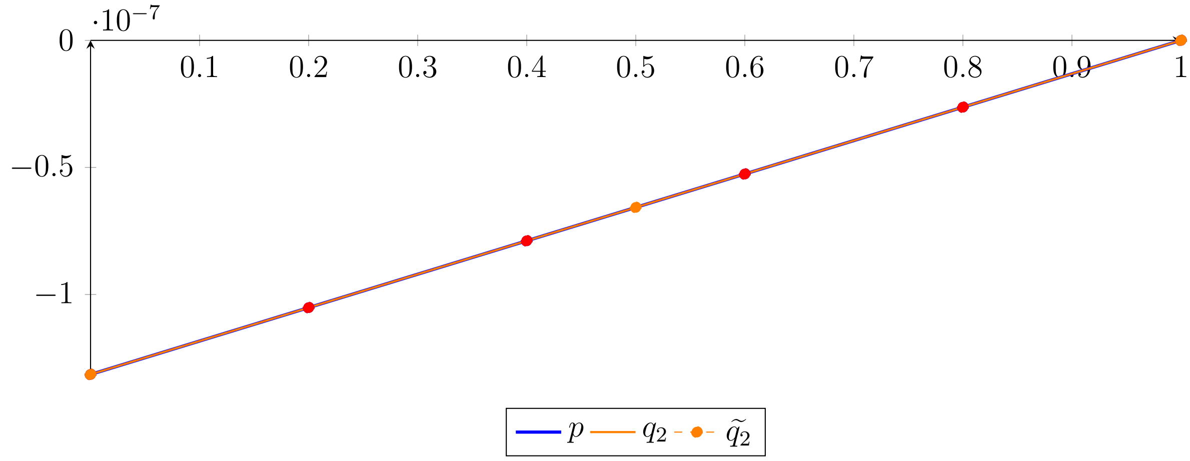
$$\begin{aligned}
p &= 1.68021{\cdot}10^{-25}X^5 - 7.10858{\cdot}10^{-25}X^4 - 3.52895{\cdot}10^{-16}X^3 \\
&\quad + 1.05868{\cdot}10^{-15}X^2 + 1.31457{\cdot}10^{-07}X - 1.31457{\cdot}10^{-07} \\
&= -1.31457{\cdot}10^{-07}B_{0,5}(X) - 1.05165{\cdot}10^{-07}B_{1,5}(X) - 7.88741{\cdot}10^{-08}B_{2,5}(X) \\
&\quad - 5.25827{\cdot}10^{-08}B_{3,5}(X) - 2.62914{\cdot}10^{-08}B_{4,5}(X) + 6.89273{\cdot}10^{-20}B_{5,5}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= 5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 1.31457{\cdot}10^{-07} \\
&= -1.31457{\cdot}10^{-07}B_{0,2} - 6.57284{\cdot}10^{-08}B_{1,2} + 1.77137{\cdot}10^{-17}B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_2 &= 7.81427{\cdot}10^{-23}X^5 - 2.06795{\cdot}10^{-22}X^4 + 2.05115{\cdot}10^{-22}X^3 \\
&\quad + 5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 1.31457{\cdot}10^{-07} \\
&= -1.31457{\cdot}10^{-07}B_{0,5} - 1.05165{\cdot}10^{-07}B_{1,5} - 7.88741{\cdot}10^{-08}B_{2,5} \\
&\quad - 5.25827{\cdot}10^{-08}B_{3,5} - 2.62914{\cdot}10^{-08}B_{4,5} + 1.77137{\cdot}10^{-17}B_{5,5}
\end{aligned}$$



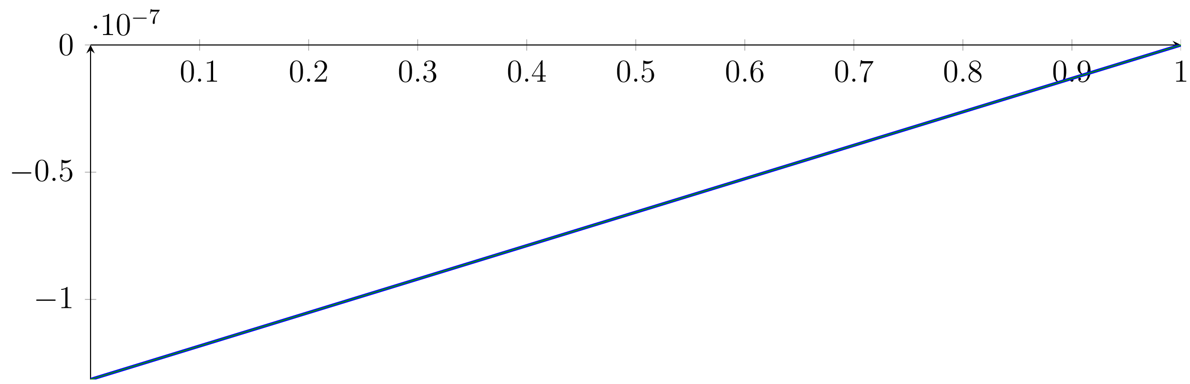
The maximum difference of the Bézier coefficients is $\delta = 2.47027{\cdot}10^{-17}$.
Bounding polynomials M and m:

$$M = 5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 1.31457{\cdot}10^{-07}$$
$$m = 5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 1.31457{\cdot}10^{-07}$$

Root of M and m:

$$N(M) = \{-2.4834{\cdot}10^{8}, 1\} \qquad N(m) = \{-2.4834{\cdot}10^{8}, 1\}$$

Intersection intervals:



$[1, 1]$

Longest intersection interval: $3.29623{\cdot}10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

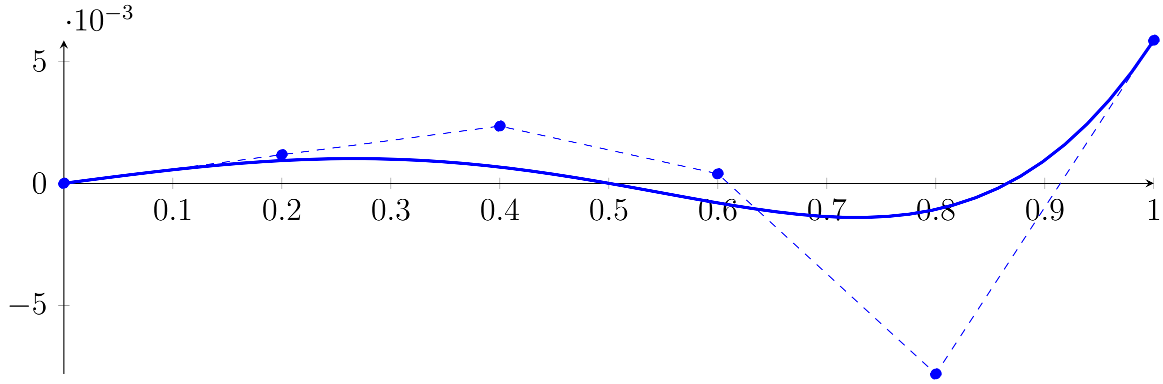
8.16 Recursion Branch 1 1 2 2 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 6!

8.17 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

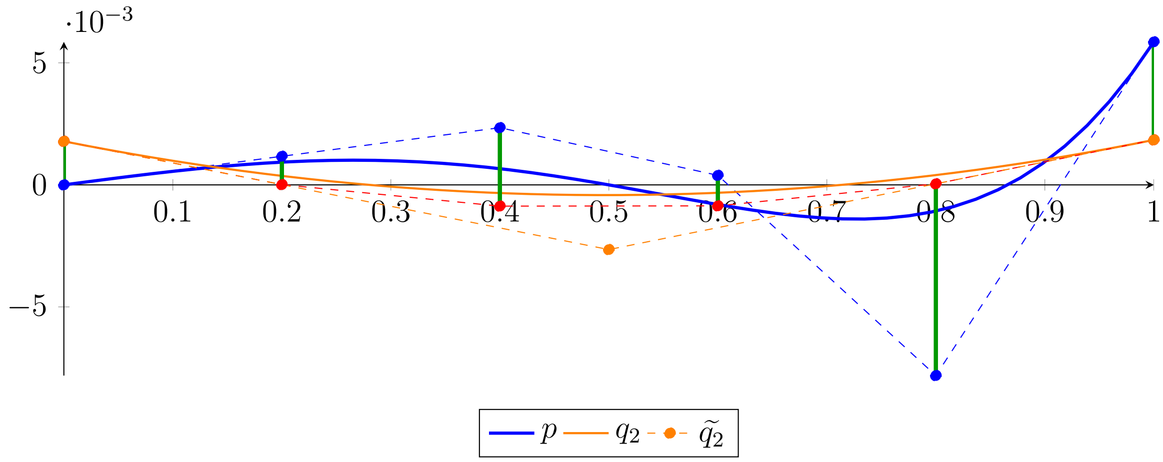
$$\begin{aligned} p &= 0.03125X^5 + 5.45277{\cdot}10^{-20}X^4 - 0.03125X^3 + 1.76818{\cdot}10^{-19}X^2 + 0.00585938X + 6.89273{\cdot}10^{-20} \\ &= 6.89273{\cdot}10^{-20}B_{0,5}(X) + 0.00117188B_{1,5}(X) + 0.00234375B_{2,5}(X) \\ &\quad + 0.000390625B_{3,5}(X) - 0.0078125B_{4,5}(X) + 0.00585938B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = 0.00892857X^2 - 0.00887277X + 0.00178571$$
$$= 0.00178571B_{0,2} - 0.00265067B_{1,2} + 0.00184152B_{2,2}$$

$$\widetilde{q_2} = -2.08169\cdot 10^{-18}X^5 + 5.46124\cdot 10^{-18}X^4 - 5.21878$$
$$\cdot 10^{-18}X^3 + 0.00892857X^2 - 0.00887277X + 0.00178571$$
$$= 0.00178571B_{0,5} + 1.11607\cdot 10^{-05}B_{1,5} - 0.000870536B_{2,5}$$
$$- 0.000859375B_{3,5} + 4.46429\cdot 10^{-05}B_{4,5} + 0.00184152B_{5,5}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00785714$.

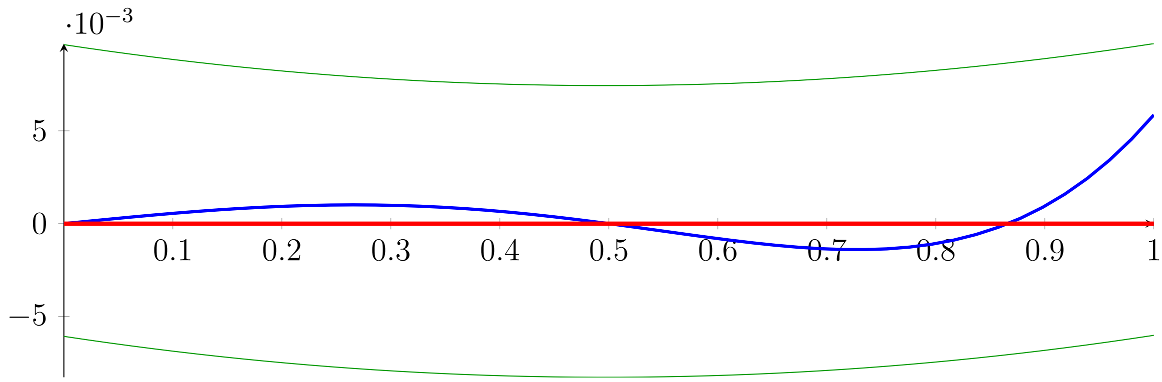
Bounding polynomials M and m:

$$M = 0.00892857X^2 - 0.00887277X + 0.00964286$$
$$m = 0.00892857X^2 - 0.00887277X - 0.00607143$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-0.465874, 1.45962\}$$

Intersection intervals:



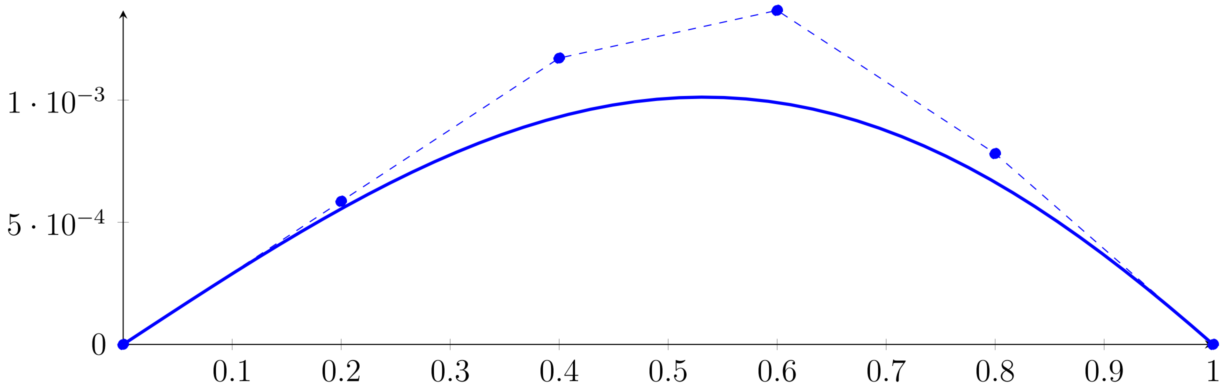
$$[0, 1]$$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$
Bisection point is very near to a root?!?

8.18 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

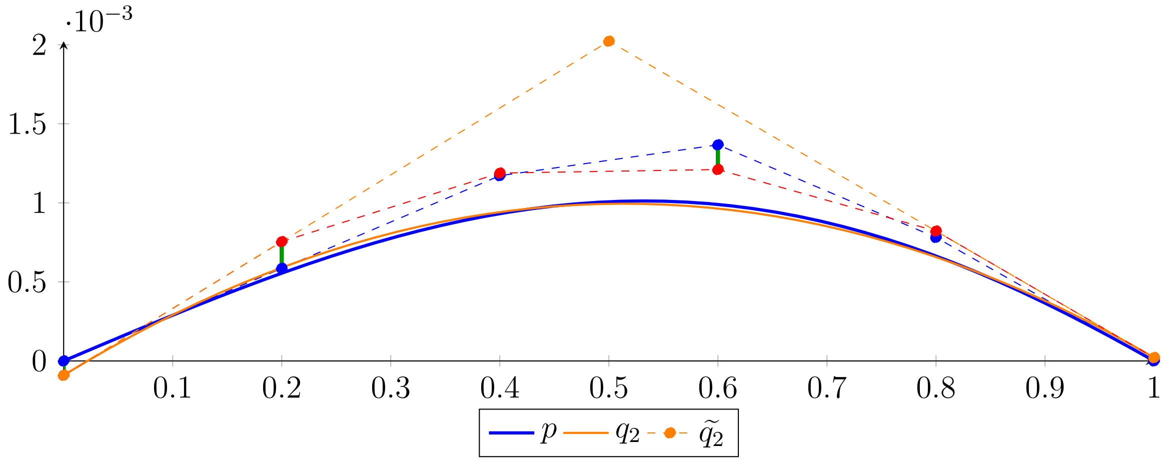
$$\begin{aligned} p &= 0.000976563X^5 + 3.70577{\cdot}10^{-21}X^4 - 0.00390625X^3 \\ &\quad + 4.44692{\cdot}10^{-20}X^2 + 0.00292969X + 6.89273{\cdot}10^{-20} \\ &= 6.89273{\cdot}10^{-20}B_{0,5}(X) + 0.000585938B_{1,5}(X) + 0.00117188B_{2,5}(X) \\ &\quad + 0.00136719B_{3,5}(X) + 0.00078125B_{4,5}(X) + 2.17237{\cdot}10^{-19}B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00411551X^2 + 0.00422712X - 9.06808{\cdot}10^{-05} \\ &= -9.06808{\cdot}10^{-05}B_{0,2} + 0.00202288B_{1,2} + 2.09263{\cdot}10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -3.10941{\cdot}10^{-19}X^5 + 5.76379{\cdot}10^{-19}X^4 - 3.58666{\cdot}10^{-19}X^3 \\ &\quad - 0.00411551X^2 + 0.00422712X - 9.06808{\cdot}10^{-05} \\ &= -9.06808{\cdot}10^{-05}B_{0,5} + 0.000754743B_{1,5} + 0.00118862B_{2,5} \\ &\quad + 0.00121094B_{3,5} + 0.000821708B_{4,5} + 2.09263{\cdot}10^{-05}B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000168806$.
Bounding polynomials M and m:

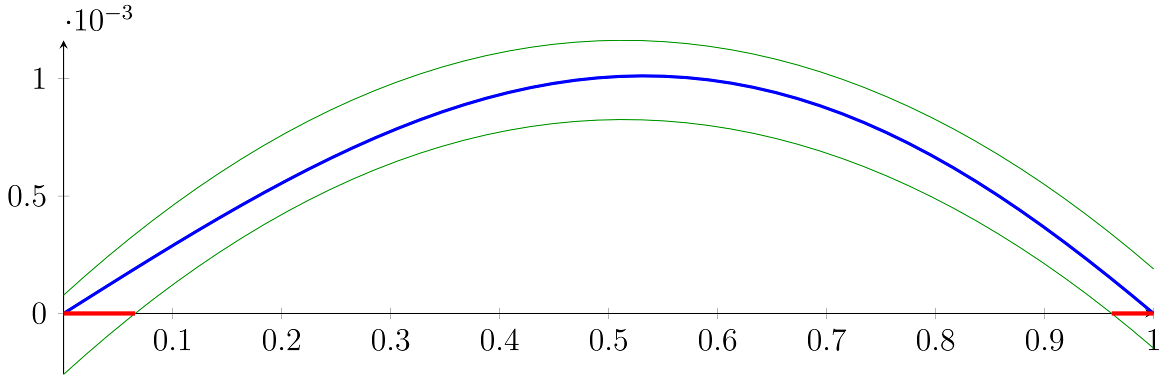
$$M = -0.00411551X^2 + 0.00422712X + 7.8125{\cdot}10^{-05}$$

$$m = -0.00411551X^2 + 0.00422712X - 0.000259487$$

Root of M and m:

$$N(M) = \{-0.0181607, 1.04528\} \qquad N(m) = \{0.0655723, 0.961546\}$$

Intersection intervals:



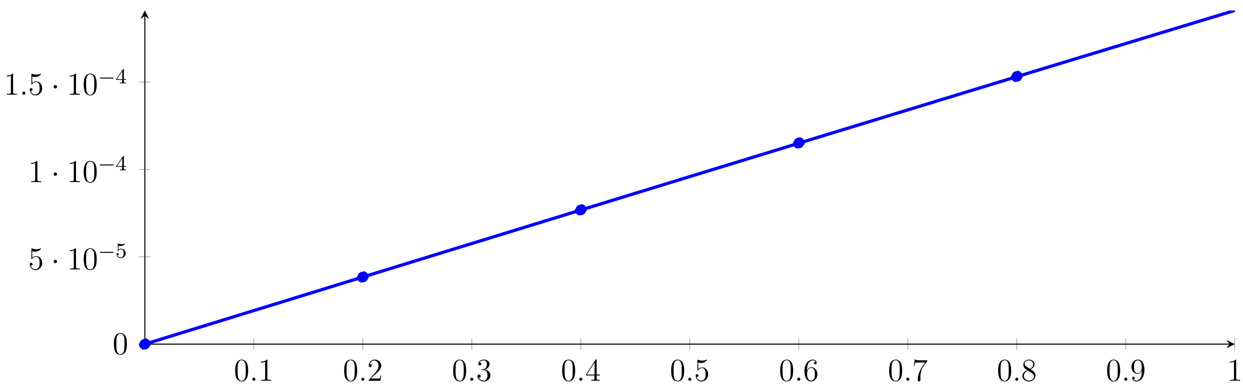
$$[0, 0.0655723], [0.961546, 1]$$

Longest intersection interval: 0.0655723
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.516393]$, interval 2: $[0.740387, 0.75]$,

8.19 Recursion Branch 1 2 1 1 in Interval 1: $[0.5, 0.516393]$

Normalized monomial und Bézier representations and the Bézier polygon:

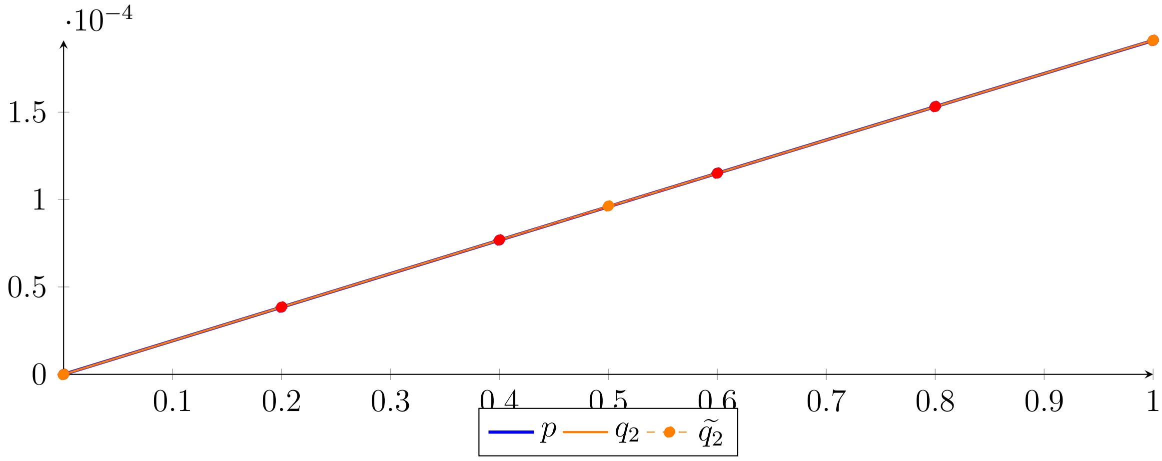
$$\begin{aligned} p &= 1.18387{\cdot}10^{-09}X^5 - 2.64698{\cdot}10^{-22}X^4 - 1.10134{\cdot}10^{-06}X^3 \\ &\quad + 1.98523{\cdot}10^{-22}X^2 + 0.000192106X + 6.89273{\cdot}10^{-20} \\ &= 6.89273{\cdot}10^{-20}B_{0,5}(X) + 3.84213{\cdot}10^{-05}B_{1,5}(X) + 7.68426{\cdot}10^{-05}B_{2,5}(X) \\ &\quad + 0.000115154B_{3,5}(X) + 0.000153245B_{4,5}(X) + 0.000191006B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.6499{\cdot}10^{-06}X^2 + 0.000192766X - 5.49403{\cdot}10^{-08} \\ &= -5.49403{\cdot}10^{-08}B_{0,2} + 9.63281{\cdot}10^{-05}B_{1,2} + 0.000191061B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.83929{\cdot}10^{-19}X^5 + 4.27243{\cdot}10^{-19}X^4 - 3.56523{\cdot}10^{-19}X^3 \\ &\quad - 1.6499{\cdot}10^{-06}X^2 + 0.000192766X - 5.49403{\cdot}10^{-08} \\ &= -5.49403{\cdot}10^{-08}B_{0,5} + 3.84983{\cdot}10^{-05}B_{1,5} + 7.68865{\cdot}10^{-05}B_{2,5} \\ &\quad + 0.00011511B_{3,5} + 0.000153168B_{4,5} + 0.000191061B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.69671 \cdot 10^{-08}$.

Bounding polynomials M and m:

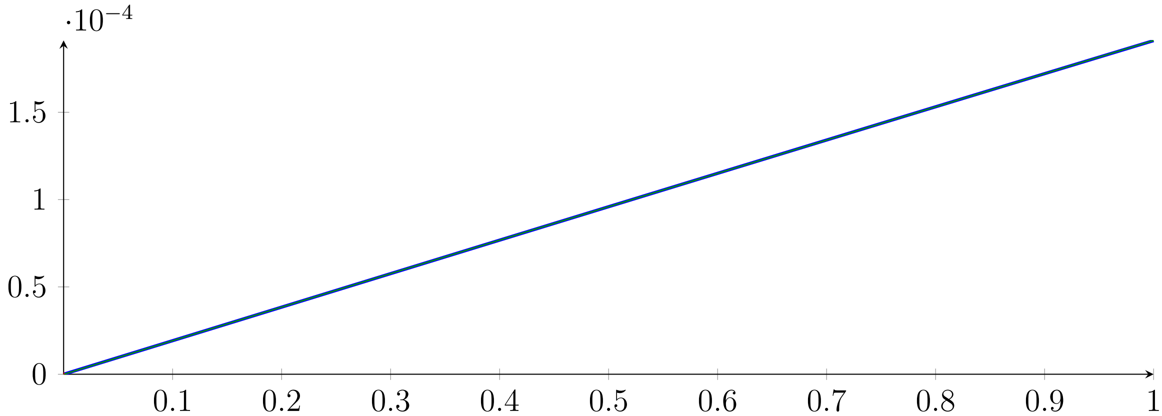
$$M = -1.6499 \cdot 10^{-06} X^2 + 0.000192766X + 2.20268 \cdot 10^{-08}$$

$$m = -1.6499 \cdot 10^{-06} X^2 + 0.000192766X - 1.31907 \cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{-0.000114267, 116.835\} \qquad N(m) = \{0.000684291, 116.834\}$$

Intersection intervals:



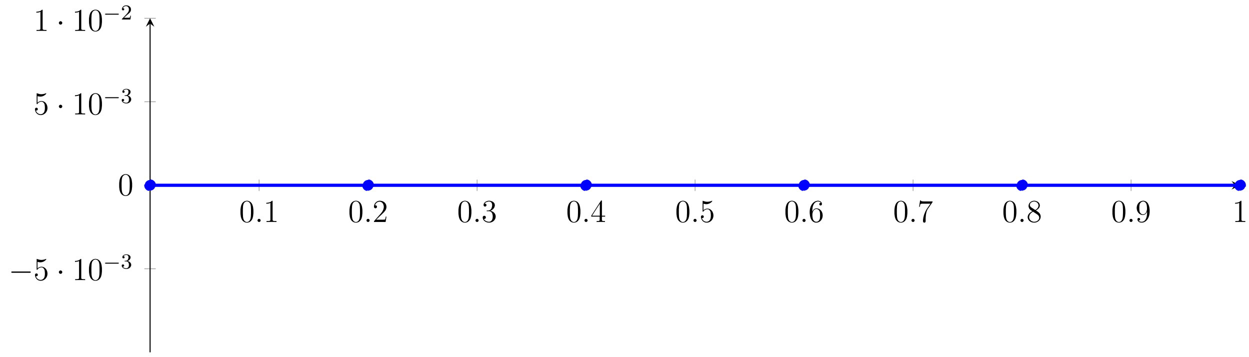
$[0, 0.000684291]$

Longest intersection interval: 0.000684291
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.500011]$,

8.20 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.5, 0.500011]$

Normalized monomial und Bézier representations and the Bézier polygon:

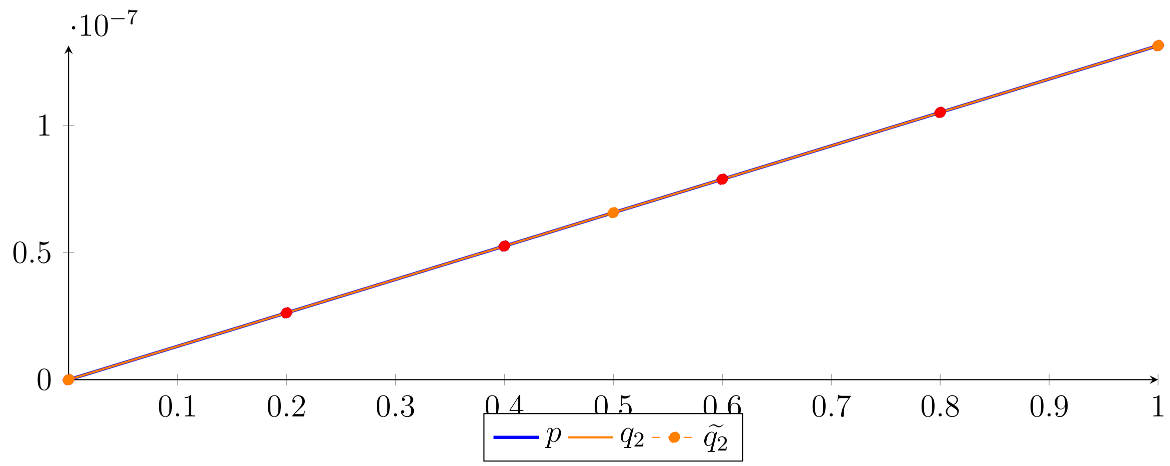
$$\begin{aligned} p &= 4.3944 \cdot 10^{-25} X^5 - 1.9387 \cdot 10^{-25} X^4 - 3.52895 \cdot 10^{-16} X^3 + 1.31457 \cdot 10^{-07} X + 6.89273 \cdot 10^{-20} \\ &= 6.89273 \cdot 10^{-20} B_{0,5}(X) + 2.62914 \cdot 10^{-08} B_{1,5}(X) + 5.25827 \cdot 10^{-08} B_{2,5}(X) \\ &\quad + 7.88741 \cdot 10^{-08} B_{3,5}(X) + 1.05165 \cdot 10^{-07} B_{4,5}(X) + 1.31457 \cdot 10^{-07} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 1.75758{\cdot}10^{-17} \\ &= -1.75758{\cdot}10^{-17}B_{0,2} + 6.57284{\cdot}10^{-08}B_{1,2} + 1.31457{\cdot}10^{-07}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.26544{\cdot}10^{-22}X^5 + 2.93998{\cdot}10^{-22}X^4 - 2.45361{\cdot}10^{-22}X^3 \\ &\quad - 5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 1.75758{\cdot}10^{-17} \\ &= -1.75758{\cdot}10^{-17}B_{0,5} + 2.62914{\cdot}10^{-08}B_{1,5} + 5.25827{\cdot}10^{-08}B_{2,5} \\ &\quad + 7.88741{\cdot}10^{-08}B_{3,5} + 1.05165{\cdot}10^{-07}B_{4,5} + 1.31457{\cdot}10^{-07}B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.47026{\cdot}10^{-17}$.

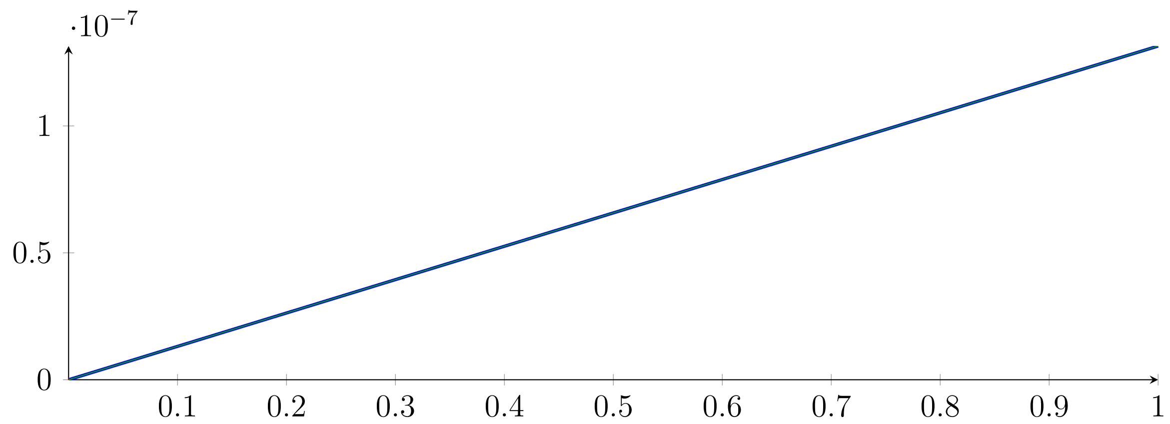
Bounding polynomials M and m:

$$\begin{aligned} M &= -5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X + 7.12682{\cdot}10^{-18} \\ m &= -5.29342{\cdot}10^{-16}X^2 + 1.31457{\cdot}10^{-07}X - 4.22785{\cdot}10^{-17} \end{aligned}$$

Root of M and m:

$$N(M) = \{-6.10413{\cdot}10^{-11}, 2.4834{\cdot}10^{8}\} \qquad N(m) = \{3.29623{\cdot}10^{-10}, 2.4834{\cdot}10^{8}\}$$

Intersection intervals:



$$[0, 3.29623e - 10]$$

Longest intersection interval: $3.29623 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

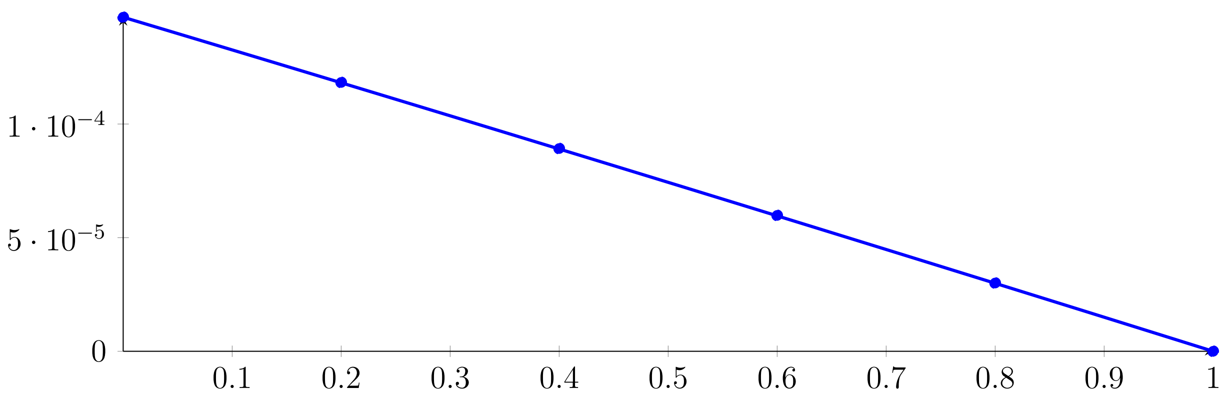
8.21 Recursion Branch 1 2 1 1 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 6!

8.22 Recursion Branch 1 2 1 2 in Interval 2: $[0.740387, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

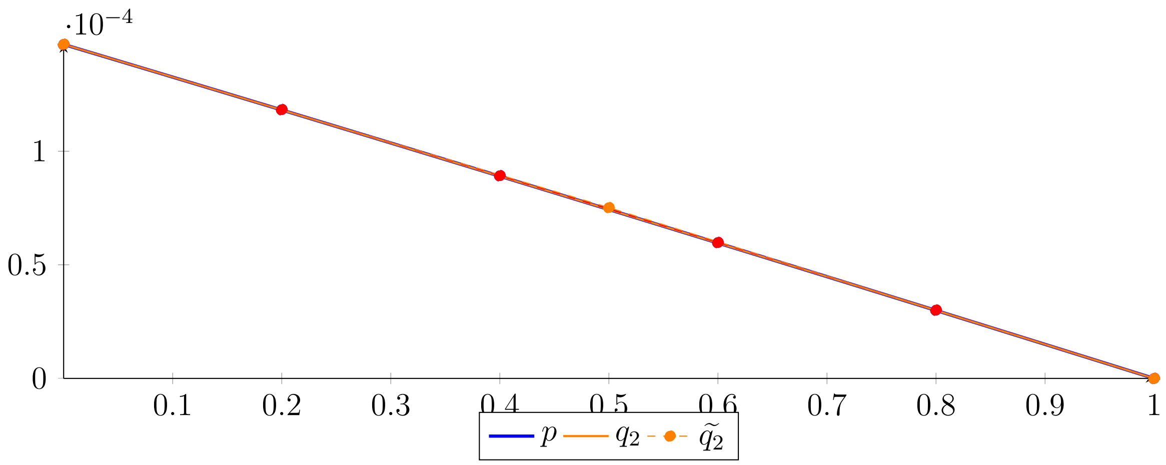
$$\begin{aligned} p &= 8.2109 \cdot 10^{-11} X^5 + 1.02658 \cdot 10^{-08} X^4 + 2.91286 \cdot 10^{-07} X^3 \\ &\quad - 3.82433 \cdot 10^{-06} X^2 - 0.000143476 X + 0.000146999 \\ &= 0.000146999 B_{0,5}(X) + 0.000118304 B_{1,5}(X) + 8.92261 \cdot 10^{-05} B_{2,5}(X) \\ &\quad + 5.97951 \cdot 10^{-05} B_{3,5}(X) + 3.0042 \cdot 10^{-05} B_{4,5}(X) + 2.17237 \cdot 10^{-19} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.36966 \cdot 10^{-06} X^2 - 0.000143661 X + 0.000147015 \\ &= 0.000147015 B_{0,2} + 7.51843 \cdot 10^{-05} B_{1,2} - 1.57522 \cdot 10^{-08} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -8.76017 \cdot 10^{-20} X^5 + 2.31809 \cdot 10^{-19} X^4 - 2.29625 \cdot 10^{-19} X^3 \\ &\quad - 3.36966 \cdot 10^{-06} X^2 - 0.000143661 X + 0.000147015 \\ &= 0.000147015 B_{0,5} + 0.000118282 B_{1,5} + 8.92134 \cdot 10^{-05} B_{2,5} \\ &\quad + 5.98073 \cdot 10^{-05} B_{3,5} + 3.00642 \cdot 10^{-05} B_{4,5} - 1.57522 \cdot 10^{-08} B_{5,5} \end{aligned}$$



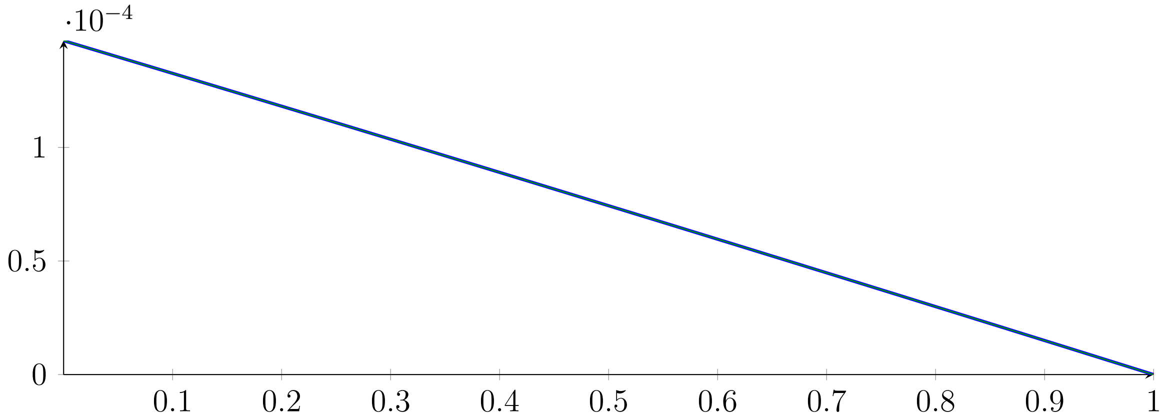
The maximum difference of the Bézier coefficients is $\delta = 2.22936 \cdot 10^{-08}$.
Bounding polynomials M and m:

$$M = -3.36966 \cdot 10^{-06} X^2 - 0.000143661X + 0.000147037$$
$$m = -3.36966 \cdot 10^{-06} X^2 - 0.000143661X + 0.000146992$$

Root of M and m:

$$N(M) = \{-43.6336, 1.00004\} \qquad N(m) = \{-43.6333, 0.999747\}$$

Intersection intervals:



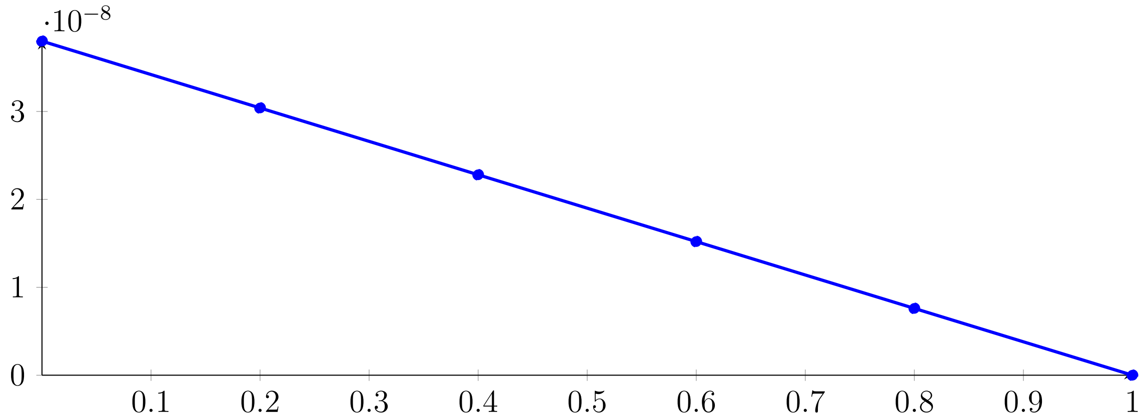
$[0.999747, 1]$

Longest intersection interval: 0.000252965
$\Longrightarrow$ Selective recursion: interval 1: $[0.749998, 0.75]$,

8.23 Recursion Branch 1 2 1 2 1 in Interval 1: $[0.749998, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

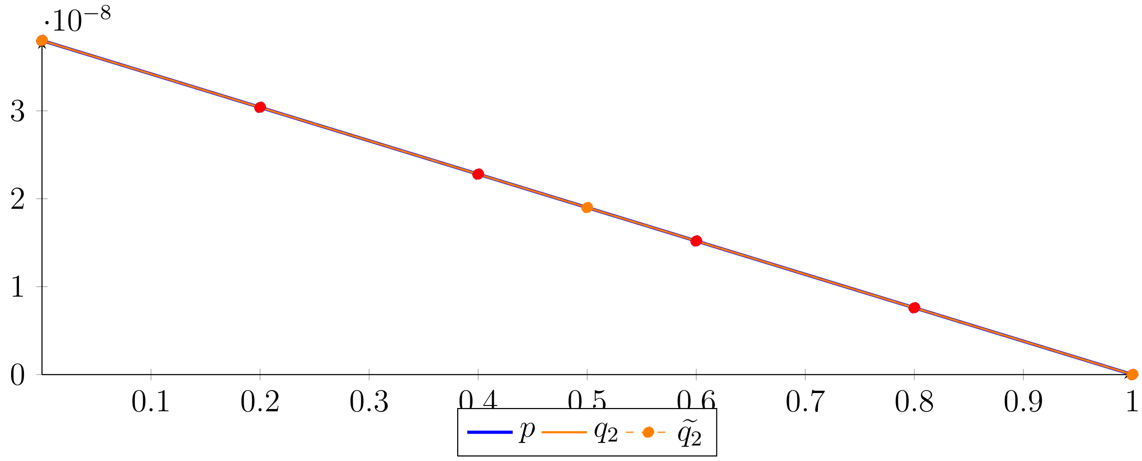
$$\begin{aligned} p = &-6.46235 \cdot 10^{-27} X^5 + 4.37178 \cdot 10^{-23} X^4 + 5.39304 \cdot 10^{-18} X^3 \\ &- 1.84827 \cdot 10^{-13} X^2 - 3.79975 \cdot 10^{-08} X + 3.79976 \cdot 10^{-08} \\ = &\, 3.79976 \cdot 10^{-08} B_{0,5}(X) + 3.03982 \cdot 10^{-08} B_{1,5}(X) + 2.27986 \cdot 10^{-08} B_{2,5}(X) \\ &+ 1.51991 \cdot 10^{-08} B_{3,5}(X) + 7.59957 \cdot 10^{-09} B_{4,5}(X) + 2.17237 \cdot 10^{-19} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.84819{\cdot}10^{-13}X^2 - 3.79975{\cdot}10^{-08}X + 3.79976{\cdot}10^{-08} \\ &= 3.79976{\cdot}10^{-08}B_{0,2} + 1.89989{\cdot}10^{-08}B_{1,2} - 5.24195{\cdot}10^{-20}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.25924{\cdot}10^{-23}X^5 + 5.98413{\cdot}10^{-23}X^4 - 5.93244{\cdot}10^{-23}X^3 \\ &\quad - 1.84819{\cdot}10^{-13}X^2 - 3.79975{\cdot}10^{-08}X + 3.79976{\cdot}10^{-08} \\ &= 3.79976{\cdot}10^{-08}B_{0,5} + 3.03982{\cdot}10^{-08}B_{1,5} + 2.27986{\cdot}10^{-08}B_{2,5} \\ &\quad + 1.51991{\cdot}10^{-08}B_{3,5} + 7.59957{\cdot}10^{-09}B_{4,5} - 5.24197{\cdot}10^{-20}B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.77521{\cdot}10^{-19}$.

Bounding polynomials M and m:

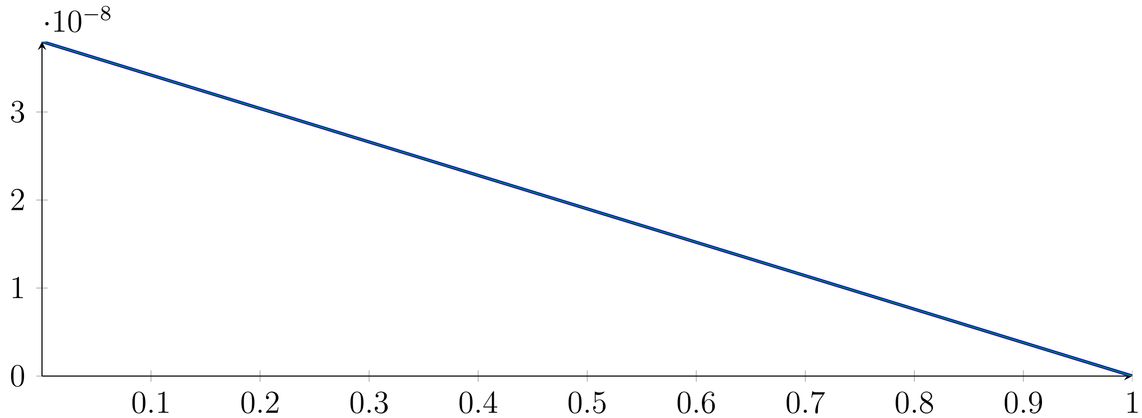
$$M = -1.84819{\cdot}10^{-13}X^2 - 3.79975{\cdot}10^{-08}X + 3.79976{\cdot}10^{-08}$$

$$m = -1.84819{\cdot}10^{-13}X^2 - 3.79975{\cdot}10^{-08}X + 3.79976{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{-205594, 1\} \qquad N(m) = \{-205594, 1\}$$

Intersection intervals:

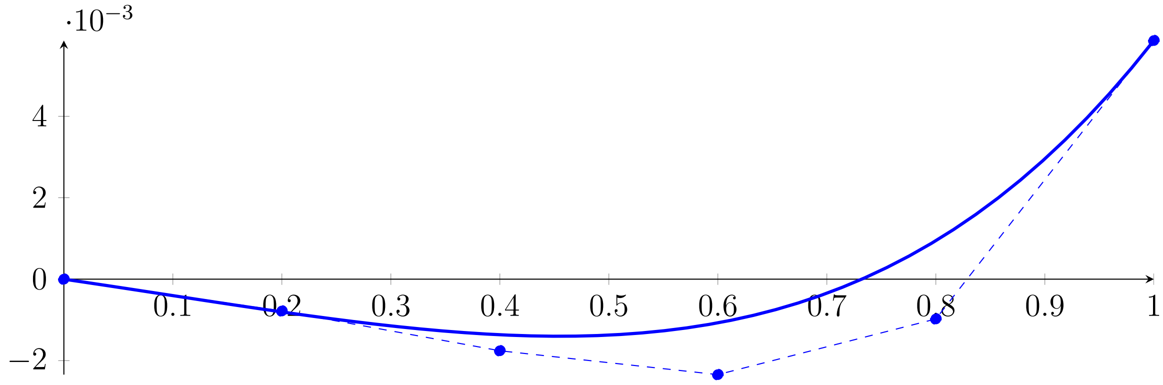


No intersection intervals with the x axis.

8.24 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

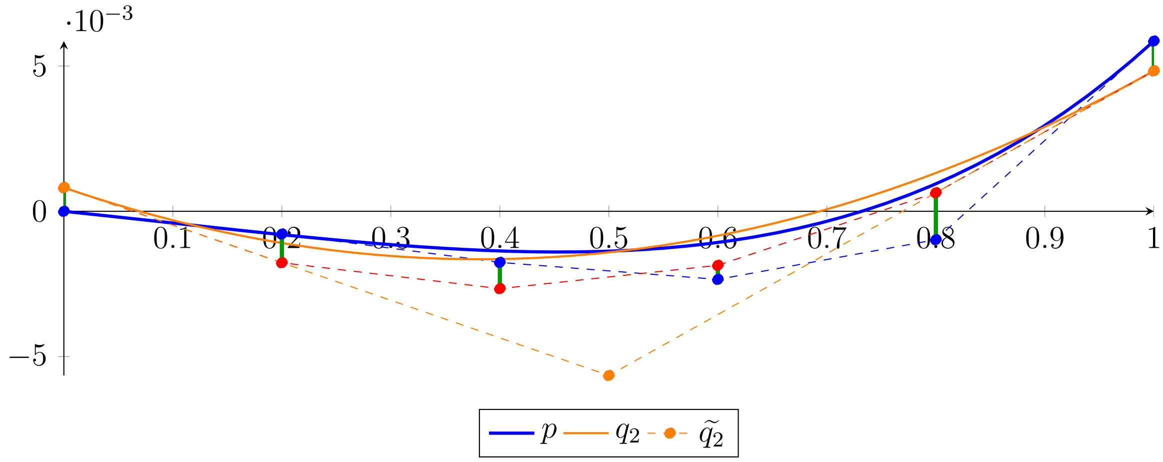
$$\begin{aligned}p &= 0.000976562X^5 + 0.00488281X^4 + 0.00585938X^3 - 0.00195312X^2 - 0.00390625X + 2.17237 \cdot 10^{-19}\\ &= 2.17237 \cdot 10^{-19} B_{0,5}(X) - 0.00078125 B_{1,5}(X) - 0.00175781 B_{2,5}(X)\\ &\quad - 0.00234375 B_{3,5}(X) - 0.000976562 B_{4,5}(X) + 0.00585938 B_{5,5}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= 0.0169503X^2 - 0.0129325X + 0.000816127\\ &= 0.000816127B_{0,2} - 0.00565011B_{1,2} + 0.00483398B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -3.71509 \cdot 10^{-18} X^5 + 9.31524 \cdot 10^{-18} X^4 - 8.38986\\ &\quad \cdot 10^{-18} X^3 + 0.0169503X^2 - 0.0129325X + 0.000816127\\ &= 0.000816127B_{0,5} - 0.00177037B_{1,5} - 0.00266183B_{2,5}\\ &\quad - 0.00185826B_{3,5} + 0.000640346B_{4,5} + 0.00483398B_{5,5}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00161691$.
Bounding polynomials M and m:

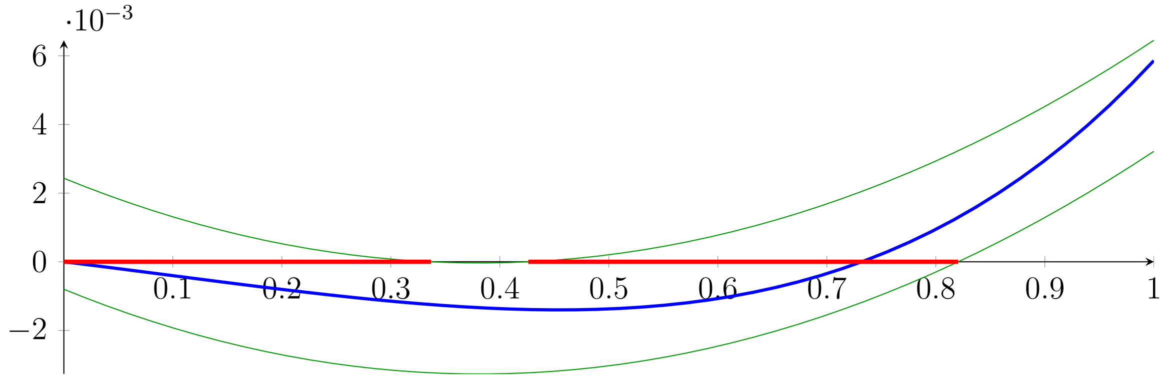
$$M = 0.0169503X^2 - 0.0129325X + 0.00243304$$

$$m = 0.0169503X^2 - 0.0129325X - 0.000800781$$

Root of M and m:

$$N(M) = \{0.336883, 0.42608\} \qquad N(m) = \{-0.0575754, 0.820538\}$$

Intersection intervals:



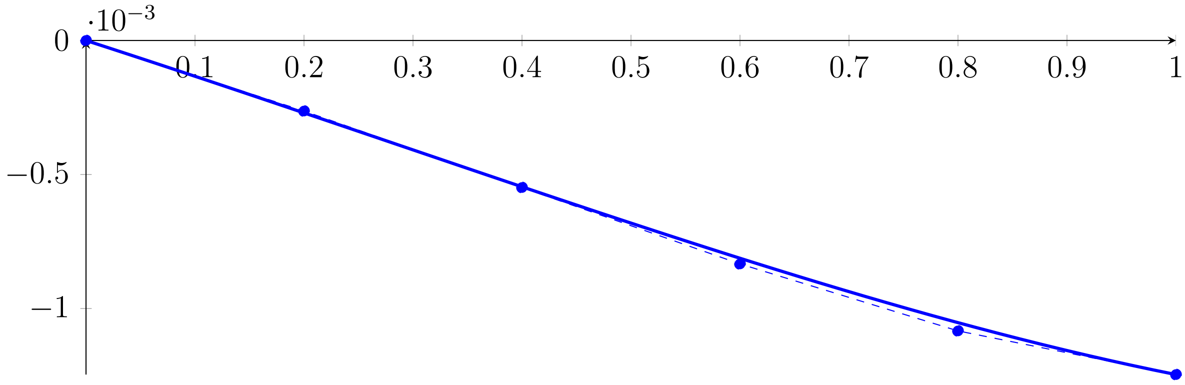
$[0, 0.336883]$, $[0.42608, 0.820538]$

Longest intersection interval: 0.394458
$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.834221]$, interval 2: $[0.85652, 0.955135]$,

8.25 Recursion Branch 1 2 2 1 in Interval 1: $[0.75, 0.834221]$

Normalized monomial und Bézier representations and the Bézier polygon:

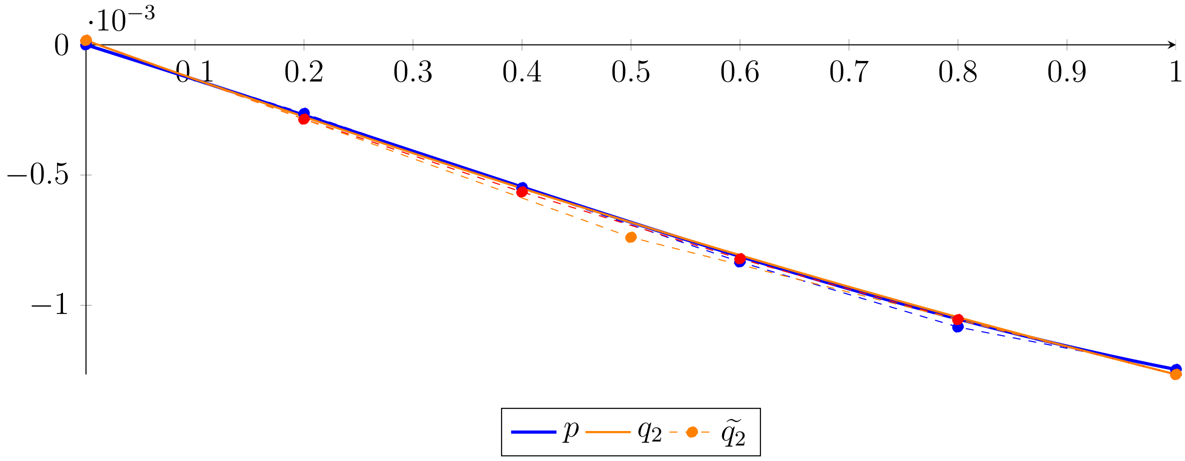
$$\begin{aligned} p &= 4.23736\cdot 10^{-06}X^5 + 6.28907\cdot 10^{-05}X^4 + 0.000224021X^3 \\ &\quad - 0.00022166X^2 - 0.00131595X + 2.17237\cdot 10^{-19} \\ &= 2.17237\cdot 10^{-19}B_{0,5}(X) - 0.00026319B_{1,5}(X) - 0.000548546B_{2,5}(X) \\ &\quad - 0.000833666B_{3,5}(X) - 0.00108357B_{4,5}(X) - 0.00124646B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00022975X^2 - 0.0015124X + 1.70457\cdot 10^{-05} \\ &= 1.70457\cdot 10^{-05}B_{0,2} - 0.000739155B_{1,2} - 0.00126561B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.2276\cdot 10^{-18}X^5 - 2.8409\cdot 10^{-18}X^4 + 2.36249\cdot 10^{-18}X^3 + 0.00022975X^2 - 0.0015124X + 1.70457\cdot 10^{-05} \\ &= 1.70457\cdot 10^{-05}B_{0,5} - 0.000285435B_{1,5} - 0.00056494B_{2,5} \\ &\quad - 0.00082147B_{3,5} - 0.00105503B_{4,5} - 0.00126561B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.85435 \cdot 10^{-05}$.

Bounding polynomials M and m:

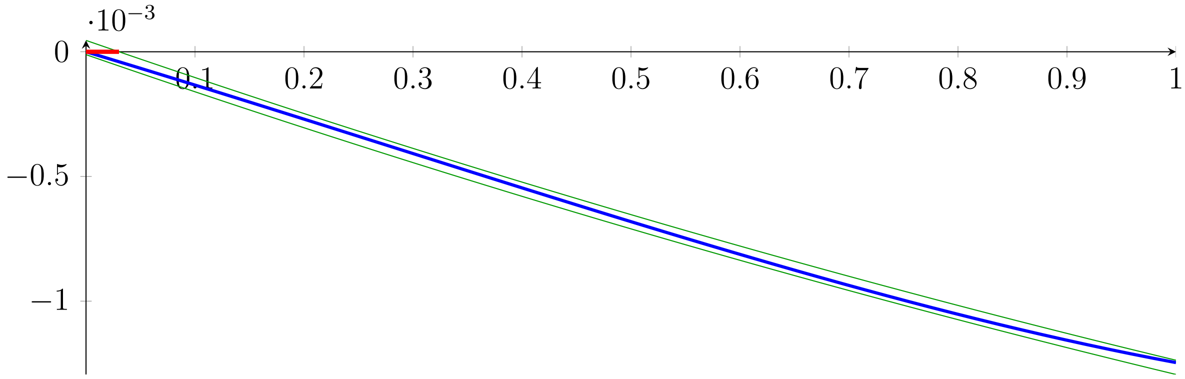
$$M = 0.00022975X^2 - 0.0015124X + 4.55892 \cdot 10^{-05}$$

$$m = 0.00022975X^2 - 0.0015124X - 1.14978 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.0302829, 6.55253\} \qquad N(m) = \{-0.00759359, 6.5904\}$$

Intersection intervals:



$$[0, 0.0302829]$$

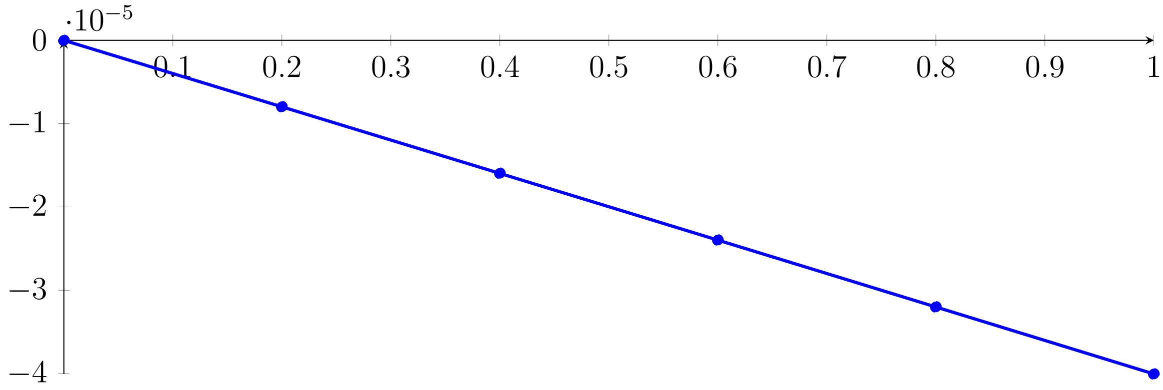
Longest intersection interval: 0.0302829

$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.75255]$,

8.26 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.75, 0.75255]$

Normalized monomial und Bézier representations and the Bézier polygon:

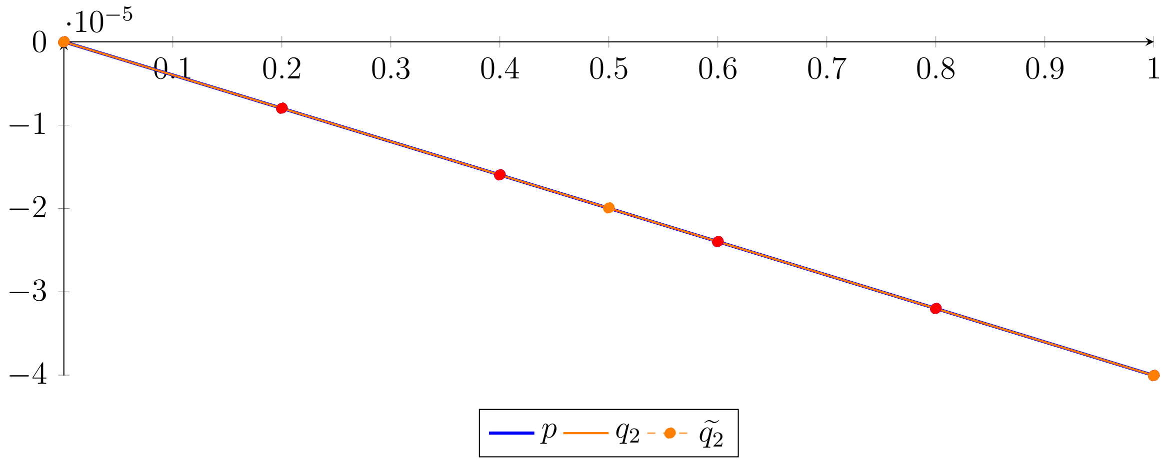
$$\begin{aligned} p &= 1.07915 \cdot 10^{-13} X^5 + 5.289 \cdot 10^{-11} X^4 + 6.22127 \cdot 10^{-09} X^3 \\ &\quad - 2.03274 \cdot 10^{-07} X^2 - 3.98507 \cdot 10^{-05} X + 2.17237 \cdot 10^{-19} \\ &= 2.17237 \cdot 10^{-19} B_{0,5}(X) - 7.97014 \cdot 10^{-06} B_{1,5}(X) - 1.59606 \cdot 10^{-05} B_{2,5}(X) \\ &\quad - 2.39708 \cdot 10^{-05} B_{3,5}(X) - 3.2 \cdot 10^{-05} B_{4,5}(X) - 4.00477 \cdot 10^{-05} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.93851\cdot 10^{-07}X^2 - 3.98545\cdot 10^{-05}X + 3.15609\cdot 10^{-10} \\ &= 3.15609\cdot 10^{-10}B_{0,2} - 1.99269\cdot 10^{-05}B_{1,2} - 4.0048\cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 3.85207\cdot 10^{-20}X^5 - 8.94541\cdot 10^{-20}X^4 + 7.47166\cdot 10^{-20}X^3 \\ &\quad - 1.93851\cdot 10^{-07}X^2 - 3.98545\cdot 10^{-05}X + 3.15609\cdot 10^{-10} \\ &= 3.15609\cdot 10^{-10}B_{0,5} - 7.97058\cdot 10^{-06}B_{1,5} - 1.59609\cdot 10^{-05}B_{2,5} \\ &\quad - 2.39705\cdot 10^{-05}B_{3,5} - 3.19996\cdot 10^{-05}B_{4,5} - 4.0048\cdot 10^{-05}B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.45195\cdot 10^{-10}$.

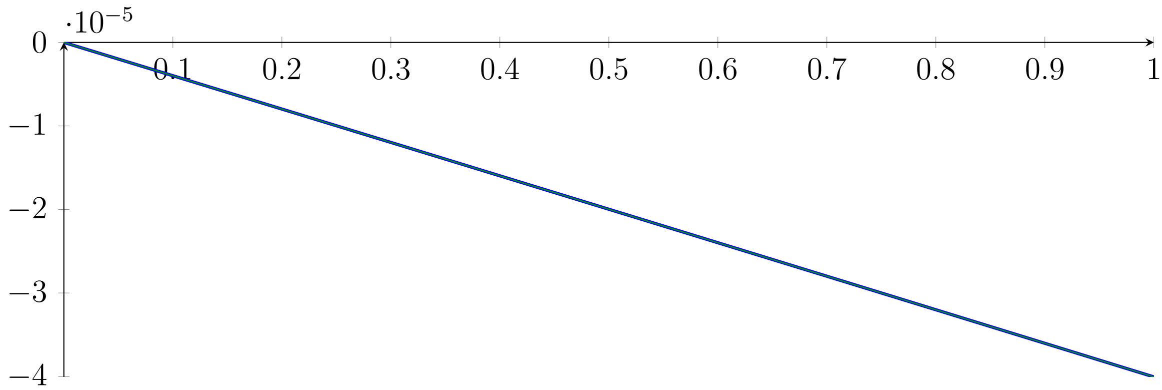
Bounding polynomials M and m:

$$\begin{aligned} M &= -1.93851\cdot 10^{-07}X^2 - 3.98545\cdot 10^{-05}X + 7.60804\cdot 10^{-10} \\ m &= -1.93851\cdot 10^{-07}X^2 - 3.98545\cdot 10^{-05}X - 1.29586\cdot 10^{-10} \end{aligned}$$

Root of M and m:

$$N(M) = \{-205.593, 1.90895\cdot 10^{-05}\} \qquad N(m) = \{-205.593, -3.25149\cdot 10^{-06}\}$$

Intersection intervals:



$$[0, 1.90895e - 05]$$

Longest intersection interval: $1.90895 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.75]$,

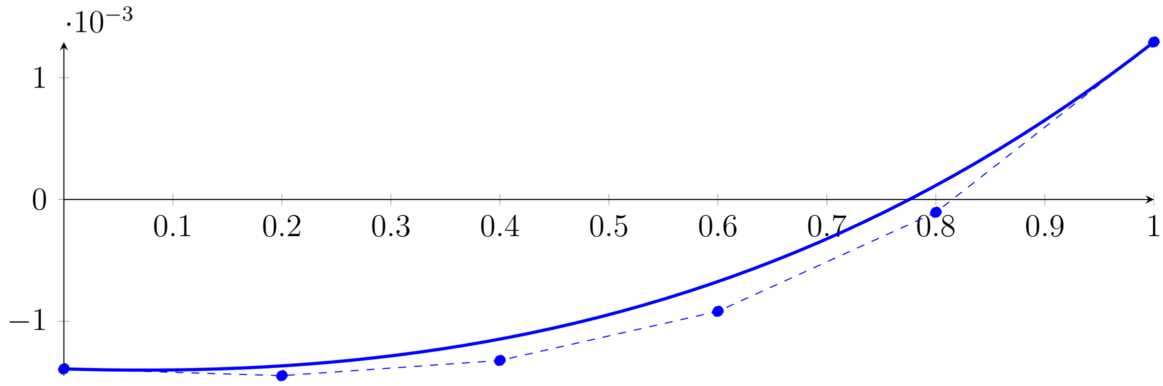
8.27 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.75, 0.75]$

Found root in interval $[0.75, 0.75]$ at recursion depth 6!

8.28 Recursion Branch 1 2 2 2 in Interval 2: $[0.85652, 0.955135]$

Normalized monomial und Bézier representations and the Bézier polygon:

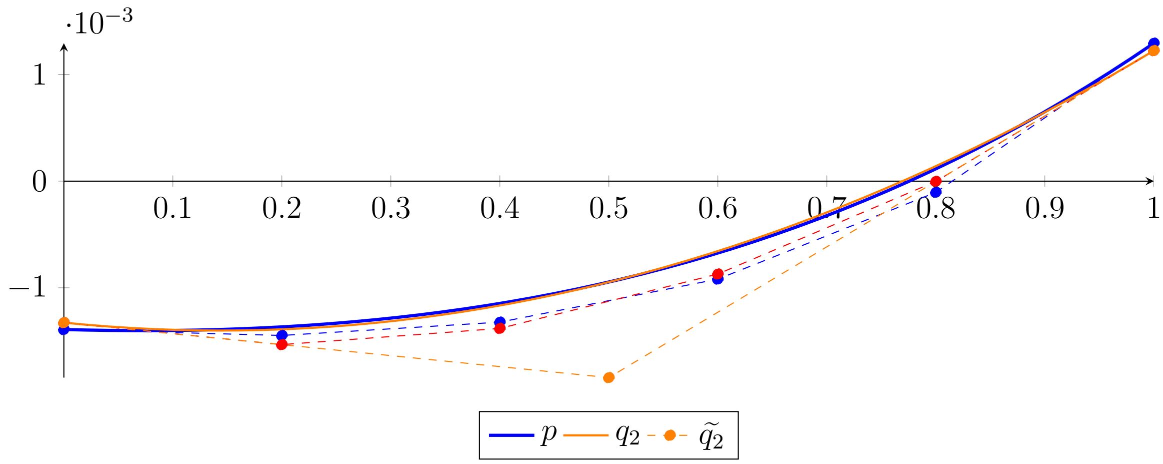
$$\begin{aligned} p &= 9.32622 \cdot 10^{-06} X^5 + 0.000168585 X^4 + 0.000979212 X^3 + 0.00180658 X^2 - 0.000279159 X - 0.00139107 \\ &= -0.00139107 B_{0,5}(X) - 0.00144691 B_{1,5}(X) - 0.00132208 B_{2,5}(X) \\ &\quad - 0.000918673 B_{3,5}(X) - 0.000105049 B_{4,5}(X) + 0.00129347 B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00358106 X^2 - 0.00103081 X - 0.00132666 \\ &= -0.00132666 B_{0,2} - 0.00184207 B_{1,2} + 0.00122358 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -8.84091 \cdot 10^{-20} X^5 + 6.51157 \cdot 10^{-20} X^4 + 1.89524 \\ &\quad \cdot 10^{-19} X^3 + 0.00358106 X^2 - 0.00103081 X - 0.00132666 \\ &= -0.00132666 B_{0,5} - 0.00153283 B_{1,5} - 0.00138088 B_{2,5} \\ &\quad - 0.000870835 B_{3,5} - 2.6796 \cdot 10^{-06} B_{4,5} + 0.00122358 B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00010237$.

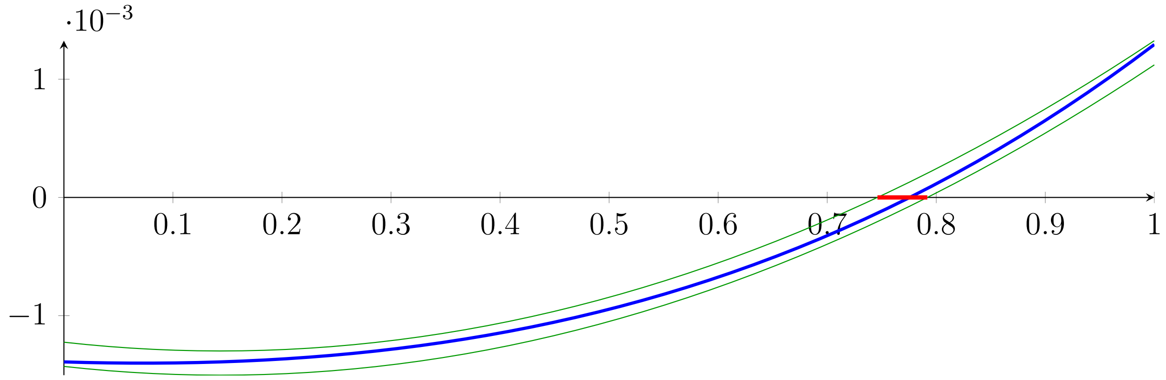
Bounding polynomials M and m:

$$M = 0.00358106X^2 - 0.00103081X - 0.00122429$$
$$m = 0.00358106X^2 - 0.00103081X - 0.00142903$$

Root of M and m:

$$N(M) = \{-0.458233, 0.746085\} \qquad N(m) = \{-0.503969, 0.791821\}$$

Intersection intervals:



$[0.746085, 0.791821]$

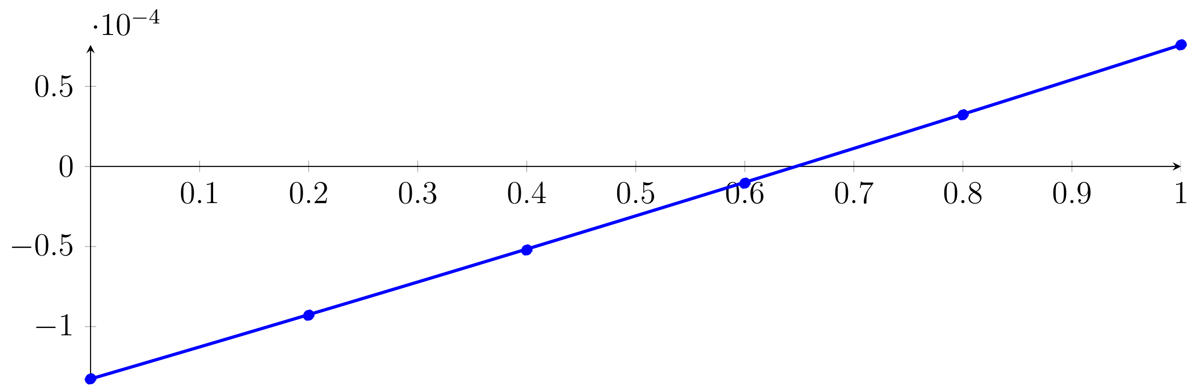
Longest intersection interval: 0.0457362

$\Longrightarrow$ Selective recursion: interval 1: $[0.930095, 0.934605]$,

8.29 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.930095, 0.934605]$

Normalized monomial und Bézier representations and the Bézier polygon:

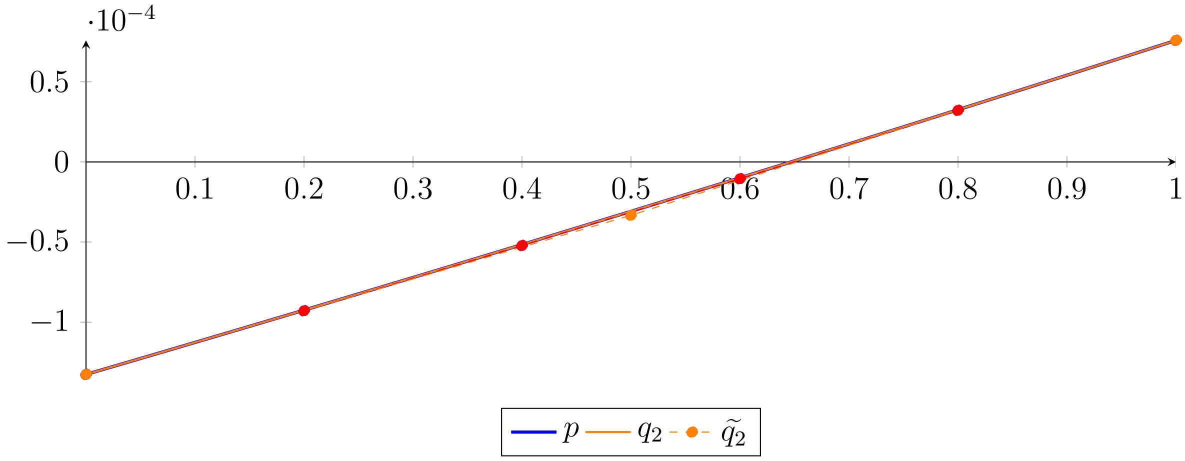
$$\begin{aligned} p &= 1.86641 \cdot 10^{-12} X^5 + 8.89899 \cdot 10^{-10} X^4 + 1.46783 \cdot 10^{-07} X^3 \\ &\quad + 9.62249 \cdot 10^{-06} X^2 + 0.000198783X - 0.000132668 \\ &= -0.000132668 B_{0,5}(X) - 9.29117 \cdot 10^{-05} B_{1,5}(X) - 5.21929 \cdot 10^{-05} B_{2,5}(X) \\ &\quad - 1.04972 \cdot 10^{-05} B_{3,5}(X) + 3.21903 \cdot 10^{-05} B_{4,5}(X) + 7.58847 \cdot 10^{-05} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 9.84419 \cdot 10^{-06} X^2 + 0.000198694X - 0.000132661 \\ &= -0.000132661 B_{0,2} - 3.33139 \cdot 10^{-05} B_{1,2} + 7.58772 \cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6.73656 \cdot 10^{-21} X^5 - 4.08296 \cdot 10^{-20} X^4 + 6.65715 \cdot 10^{-20} X^3 \\ &\quad + 9.84419 \cdot 10^{-06} X^2 + 0.000198694X - 0.000132661 \\ &= -0.000132661 B_{0,5} - 9.29221 \cdot 10^{-05} B_{1,5} - 5.21989 \cdot 10^{-05} B_{2,5} \\ &\quad - 1.04913 \cdot 10^{-05} B_{3,5} + 3.22008 \cdot 10^{-05} B_{4,5} + 7.58772 \cdot 10^{-05} B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.04381 \cdot 10^{-08}$.
Bounding polynomials M and m:

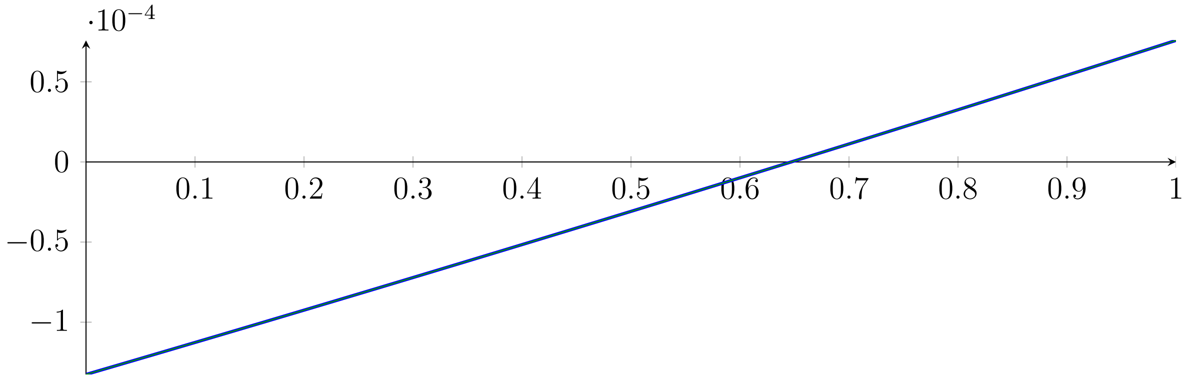
$$M = 9.84419 \cdot 10^{-06} X^2 + 0.000198694X - 0.00013265$$

$$m = 9.84419 \cdot 10^{-06} X^2 + 0.000198694X - 0.000132671$$

Root of M and m:

$$N(M) = \{-20.8308, 0.64688\} \qquad N(m) = \{-20.8309, 0.646979\}$$

Intersection intervals:



$$[0.64688, 0.646979]$$

Longest intersection interval: $9.87378 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.933012, 0.933013]$,

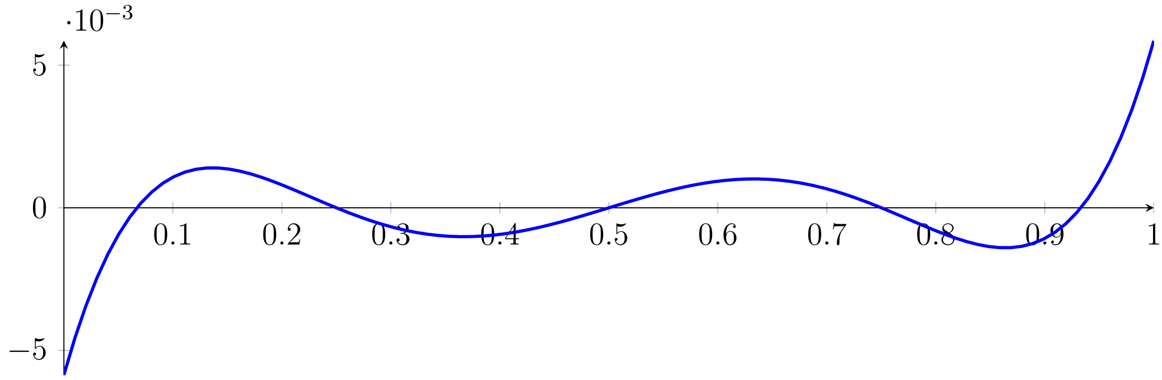
8.30 Recursion Branch 1 2 2 2 1 1 in Interval 1: $[0.933012, 0.933013]$

Found root in interval $[0.933012, 0.933013]$ at recursion depth 6!

8.31 Result: 7 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$



Result: Root Intervals

$[0.0669871, 0.0669876]$, $[0.25, 0.25]$, $[0.25, 0.25]$, $[0.5, 0.5]$, $[0.5, 0.5]$, $[0.75, 0.75]$, $[0.933012, 0.933013]$

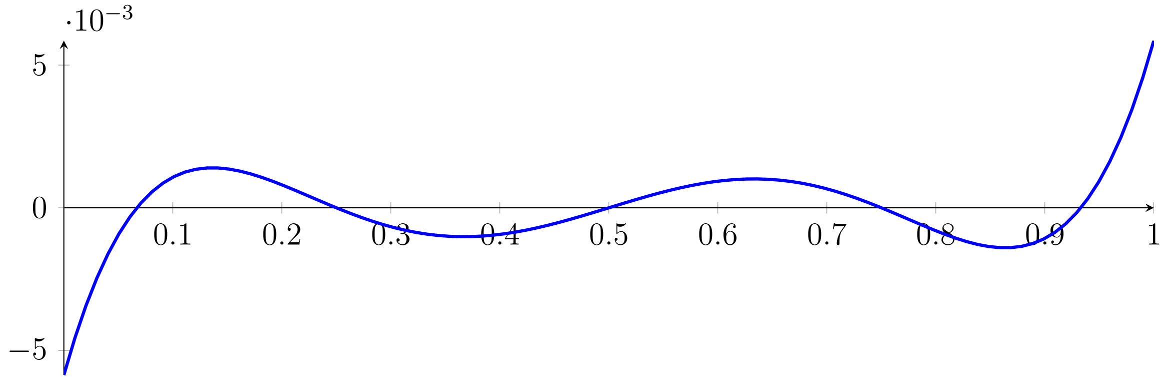
with precision $\varepsilon = 1{\cdot}10^{-06}$.

9 Running CubeClip on p5 with epsilon 6

$$1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$

Called CubeClip with input polynomial on interval $[0, 1]$:

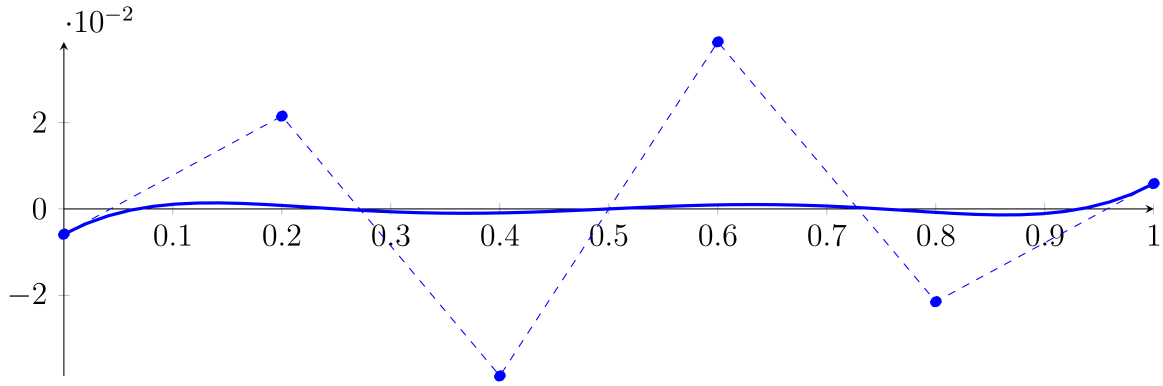
$$p = 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$



9.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

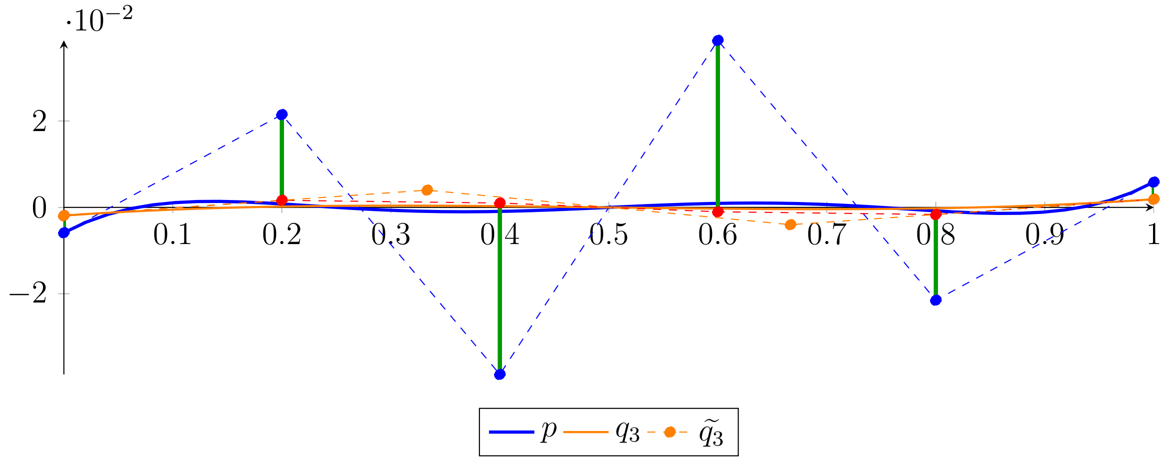
$$\begin{aligned}p &= 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938\\ &= -0.00585938B_{0,5}(X) + 0.0214844B_{1,5}(X) - 0.0386719B_{2,5}(X)\\ &\quad + 0.0386719B_{3,5}(X) - 0.0214844B_{4,5}(X) + 0.00585938B_{5,5}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 0.0277778X^3 - 0.0416667X^2 + 0.0176711X - 0.00189112\\ &= -0.00189112B_{0,3} + 0.00399926B_{1,3} - 0.00399926B_{2,3} + 0.00189112B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 9.41647{\cdot}10^{-18}X^5 - 2.35708{\cdot}10^{-17}X^4 + 0.0277778X^3 - 0.0416667X^2 + 0.0176711X - 0.00189112\\ &= -0.00189112B_{0,5} + 0.00164311B_{1,5} + 0.00101066B_{2,5}\\ &\quad - 0.00101066B_{3,5} - 0.00164311B_{4,5} + 0.00189112B_{5,5}\end{aligned}$$



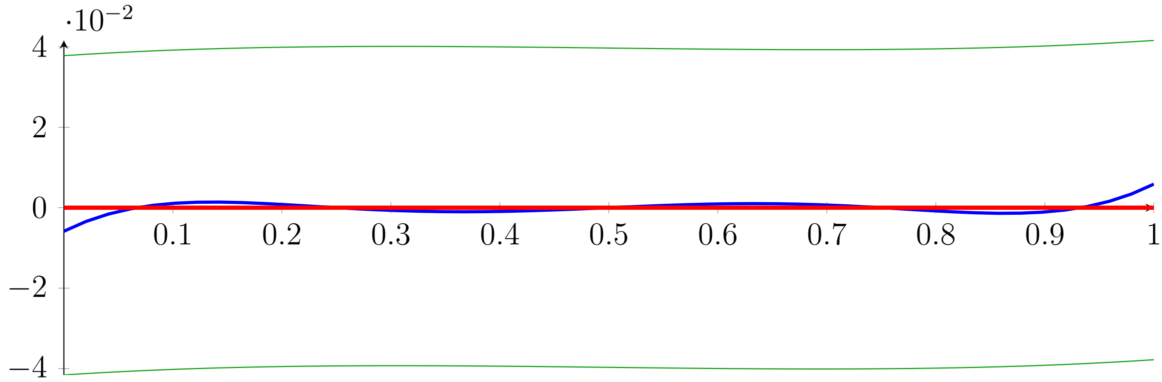
The maximum difference of the Bézier coefficients is $\delta = 0.0396825$.
Bounding polynomials M and m:

$$M = 0.0277778X^3 - 0.0416667X^2 + 0.0176711X + 0.0377914$$
$$m = 0.0277778X^3 - 0.0416667X^2 + 0.0176711X - 0.0415737$$

Root of M and m:

$$N(M) = \{-0.659931\} \qquad N(m) = \{1.65993\}$$

Intersection intervals:



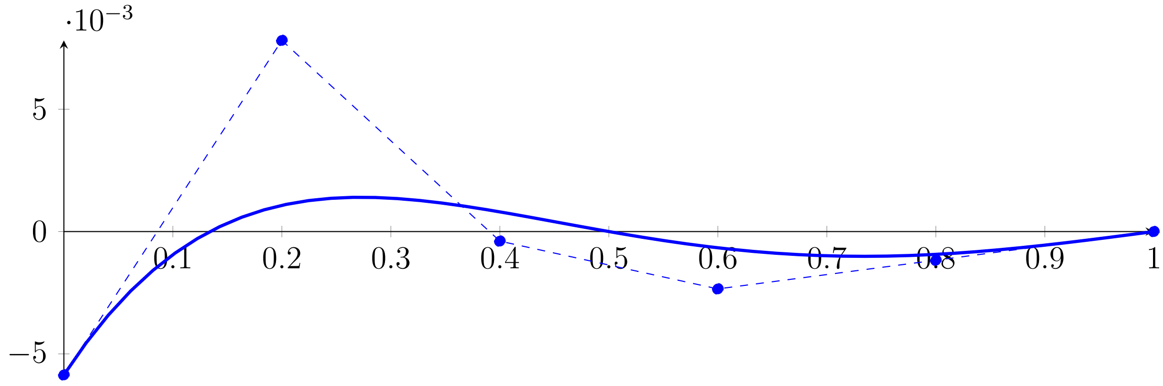
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$
Bisection point is very near to a root?!?

9.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

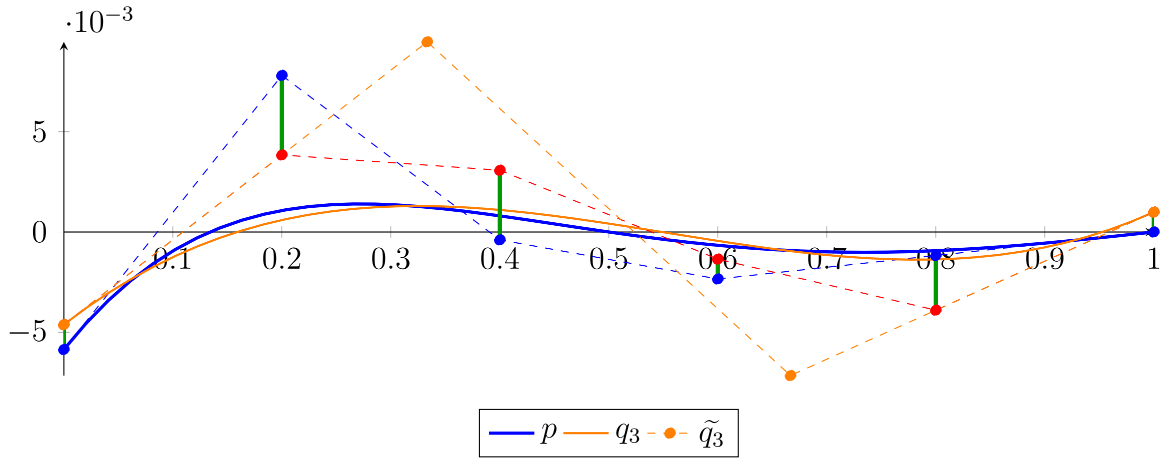
$$\begin{aligned} p &= 0.03125X^5 - 0.15625X^4 + 0.28125X^3 - 0.21875X^2 + 0.0683594X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.0078125B_{1,5}(X) - 0.000390625B_{2,5}(X) \\ &\quad - 0.00234375B_{3,5}(X) - 0.00117187B_{4,5}(X) + 6.89273 \cdot 10^{-20}B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_3 &= 0.0555556X^3 - 0.0922619X^2 + 0.0423177X - 0.0046193 \\
&= -0.0046193B_{0,3} + 0.00948661B_{1,3} - 0.00716146B_{2,3} + 0.000992063B_{3,3}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_3 &= 2.06845\cdot 10^{-17}X^5 - 5.0894\cdot 10^{-17}X^4 + 0.0555556X^3 - 0.0922619X^2 + 0.0423177X - 0.0046193 \\
&= -0.0046193B_{0,5} + 0.00384425B_{1,5} + 0.0030816B_{2,5} \\
&\quad - 0.00135169B_{3,5} - 0.00390005B_{4,5} + 0.000992063B_{5,5}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 0.00396825$.

Bounding polynomials M and m:

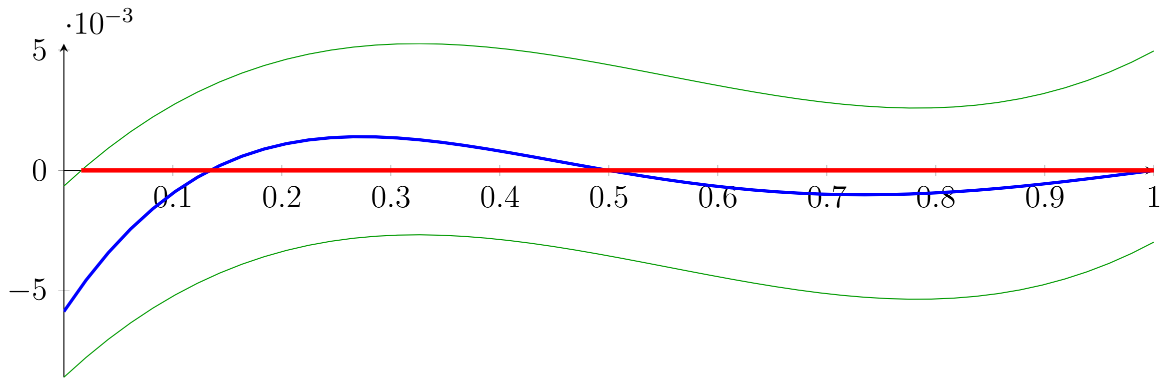
$$M = 0.0555556X^3 - 0.0922619X^2 + 0.0423177X - 0.000651042$$

$$m = 0.0555556X^3 - 0.0922619X^2 + 0.0423177X - 0.00858755$$

Root of M and m:

$$N(M) = \{0.0159328\} \qquad N(m) = \{1.09333\}$$

Intersection intervals:



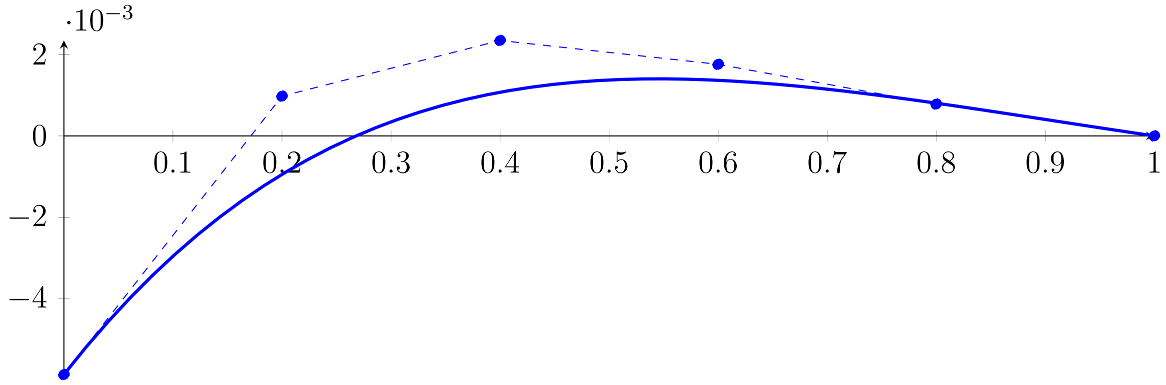
$$[0.0159328, 1]$$

Longest intersection interval: 0.984067
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$
Bisection point is very near to a root?!?

9.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

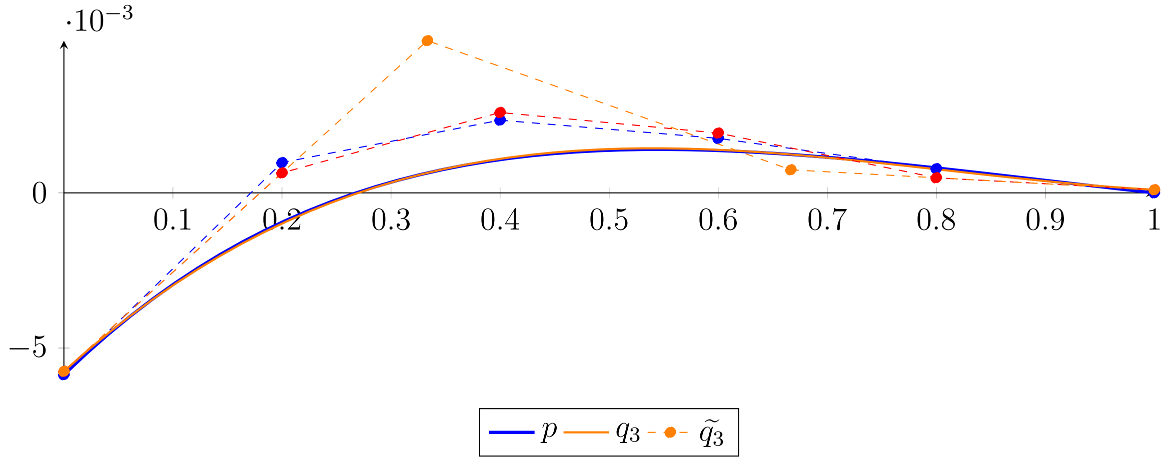
$$\begin{aligned} p &= 0.000976563X^5 - 0.00976562X^4 + 0.0351563X^3 - 0.0546875X^2 + 0.0341797X - 0.00585938 \\ &= -0.00585938B_{0,5}(X) + 0.000976563B_{1,5}(X) + 0.00234375B_{2,5}(X) \\ &\quad + 0.00175781B_{3,5}(X) + 0.00078125B_{4,5}(X) + 1.55642\cdot 10^{-20}B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.0183377X^3 - 0.0444568X^2 + 0.0319708X - 0.00575087 \\ &= -0.00575087B_{0,3} + 0.00490606B_{1,3} + 0.000744048B_{2,3} + 0.000100756B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.43445\cdot 10^{-18}X^5 - 3.12343\cdot 10^{-18}X^4 + 0.0183377X^3 - 0.0444568X^2 + 0.0319708X - 0.00575087 \\ &= -0.00575087B_{0,5} + 0.000643291B_{1,5} + 0.00259177B_{2,5} \\ &\quad + 0.00192832B_{3,5} + 0.000486731B_{4,5} + 0.000100756B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000333271$.
Bounding polynomials M and m:

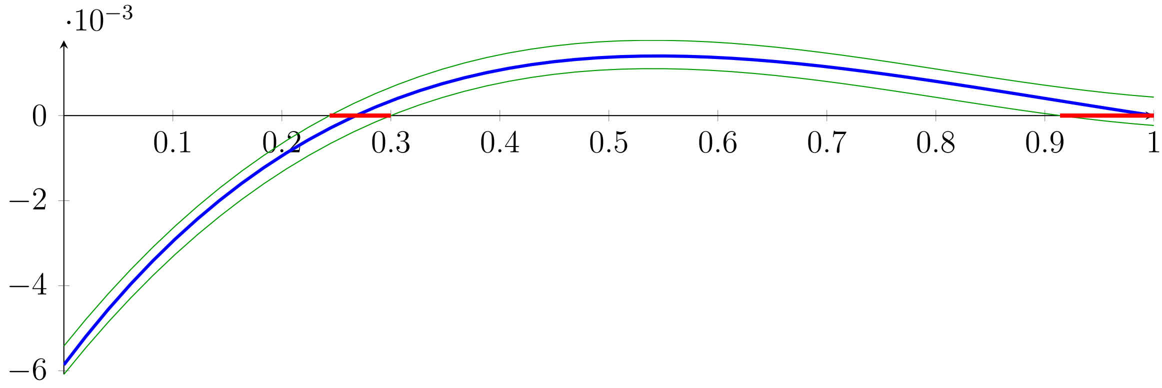
$$M = 0.0183377X^3 - 0.0444568X^2 + 0.0319708X - 0.0054176$$

$$m = 0.0183377X^3 - 0.0444568X^2 + 0.0319708X - 0.00608414$$

Root of M and m:

$$N(M) = \{0.243787\} \qquad N(m) = \{0.299893, 0.913959, 1.21049\}$$

Intersection intervals:



$$[0.243787, 0.299893],\ [0.913959, 1]$$

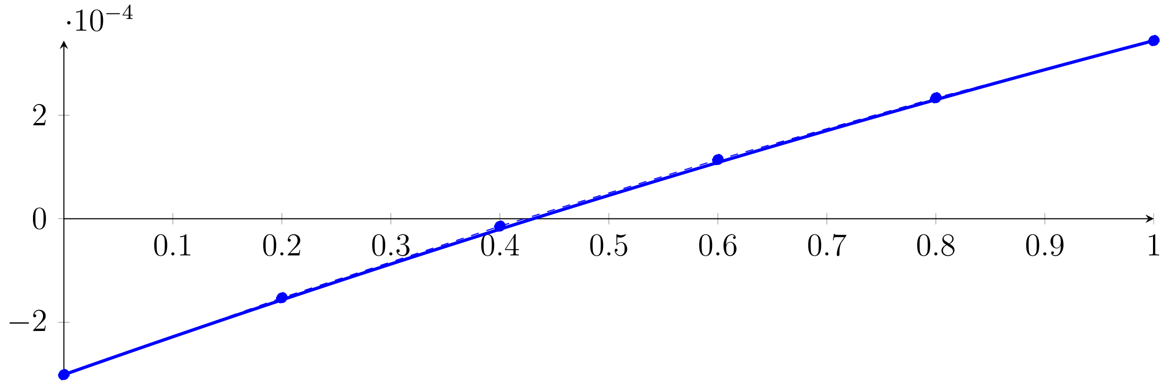
Longest intersection interval: 0.0860408

$\Longrightarrow$ Selective recursion: interval 1: $[0.0609469, 0.0749732]$, interval 2: $[0.22849, 0.25]$,

9.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.0609469, 0.0749732]$

Normalized monomial und Bézier representations and the Bézier polygon:

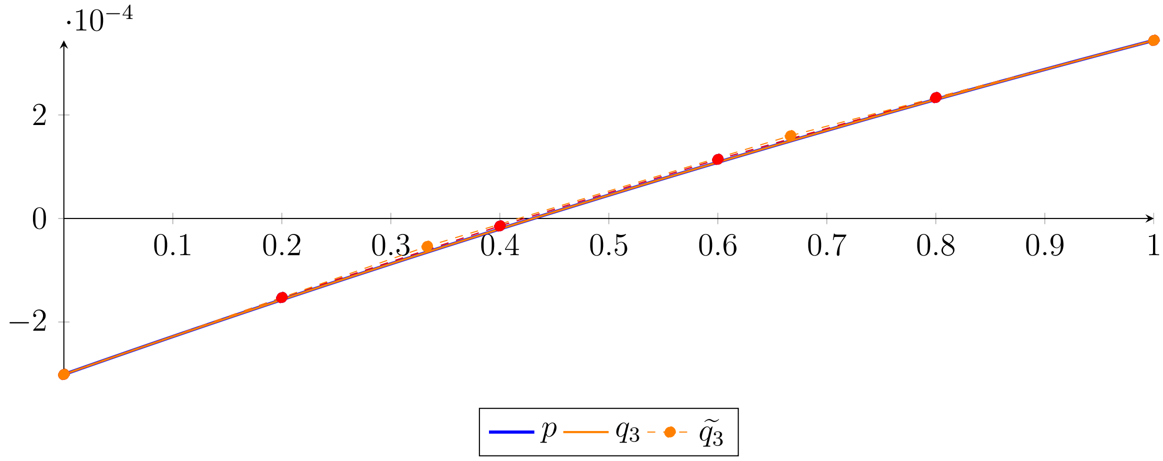
$$\begin{aligned} p &= 5.42896{\cdot}10^{-10}X^5 - 8.4969{\cdot}10^{-08}X^4 + 4.62954{\cdot}10^{-06}X^3 \\ &\quad - 0.000101725X^2 + 0.000742551X - 0.000301281 \\ &= -0.000301281B_{0,5}(X) - 0.000152771B_{1,5}(X) - 1.44334{\cdot}10^{-05}B_{2,5}(X) \\ &\quad + 0.000114195B_{3,5}(X) + 0.000233559B_{4,5}(X) + 0.000344089B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 4.46111{\cdot}10^{-06}X^3 - 0.000101618X^2 + 0.000742527X - 0.00030128 \\ &= -0.00030128B_{0,3} - 5.3771{\cdot}10^{-05}B_{1,3} + 0.000159865B_{2,3} + 0.00034409B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -7.31572{\cdot}10^{-19}X^5 + 1.73218{\cdot}10^{-18}X^4 + 4.46111{\cdot}10^{-06}X^3 \\ &\quad - 0.000101618X^2 + 0.000742527X - 0.00030128 \\ &= -0.00030128B_{0,5} - 0.000152775B_{1,5} - 1.4431{\cdot}10^{-05}B_{2,5} \\ &\quad + 0.000114197B_{3,5} + 0.000233555B_{4,5} + 0.00034409B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.59413 \cdot 10^{-09}$.

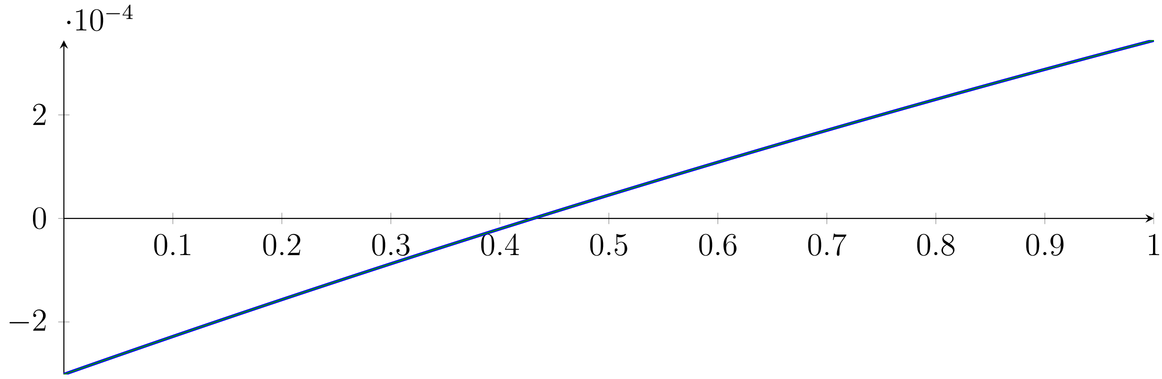
Bounding polynomials M and m:

$$M = 4.46111 \cdot 10^{-06} X^3 - 0.000101618 X^2 + 0.000742527 X - 0.000301276$$
$$m = 4.46111 \cdot 10^{-06} X^3 - 0.000101618 X^2 + 0.000742527 X - 0.000301284$$

Root of M and m:

$$N(M) = \{0.430645\} \qquad N(m) = \{0.430656\}$$

Intersection intervals:



$$[0.430645, 0.430656]$$

Longest intersection interval: $1.0933 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0669872, 0.0669874]$,

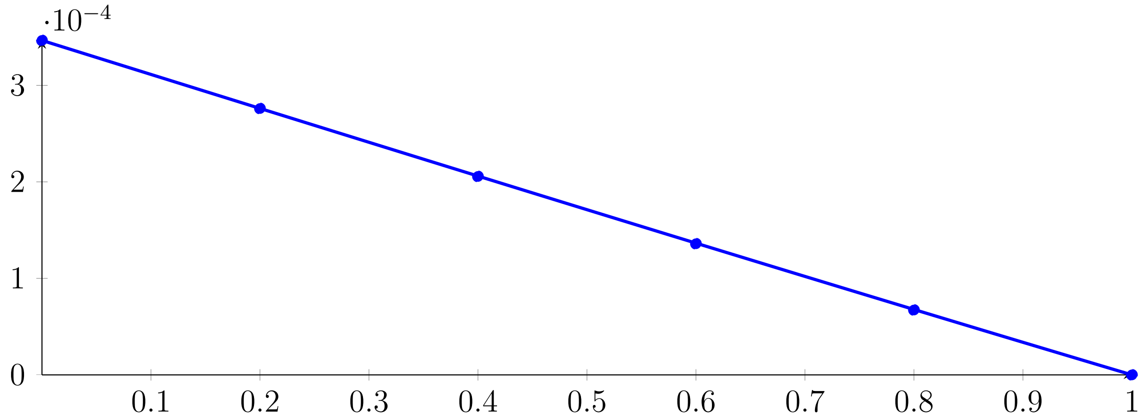
9.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0669872, 0.0669874]$

Found root in interval $[0.0669872, 0.0669874]$ at recursion depth 5!

9.6 Recursion Branch 1 1 1 2 in Interval 2: $[0.22849, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

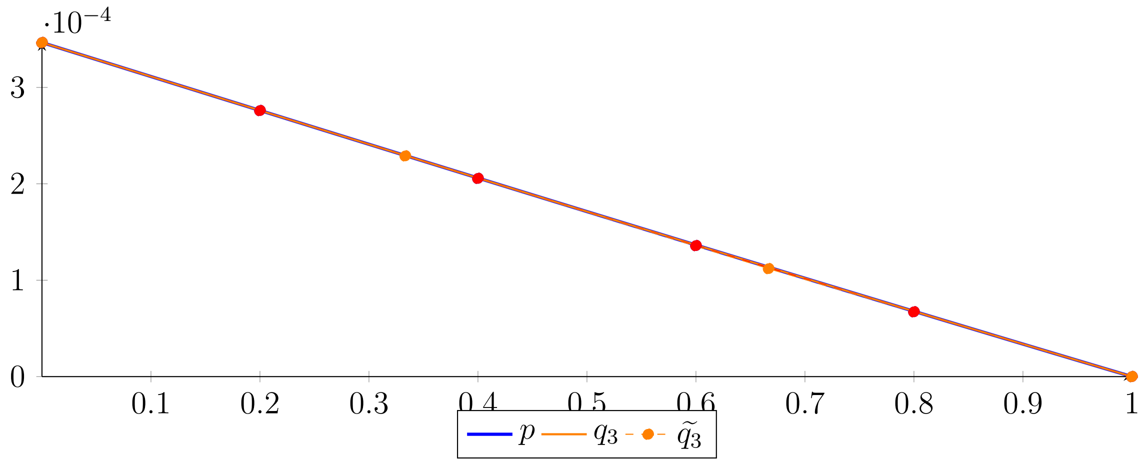
$$\begin{aligned} p &= 4.60491 \cdot 10^{-09} X^5 - 2.90625 \cdot 10^{-07} X^4 + 4.84864 \cdot 10^{-06} X^3 \\ &\quad + 1.61078 \cdot 10^{-06} X^2 - 0.000352725 X + 0.000346551 \\ &= 0.000346551 B_{0,5}(X) + 0.000276006 B_{1,5}(X) + 0.000205622 B_{2,5}(X) \\ &\quad + 0.000135885 B_{3,5}(X) + 6.72193 \cdot 10^{-05} B_{4,5}(X) + 1.55642 \cdot 10^{-20} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 4.28018\cdot 10^{-06}X^3 + 1.97348\cdot 10^{-06}X^2 - 0.000352805X + 0.000346555\\&= 0.000346555B_{0,3} + 0.000228954B_{1,3} + 0.00011201B_{2,3} + 3.96905\cdot 10^{-09}B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 6.88479\cdot 10^{-20}X^5 - 1.96538\cdot 10^{-19}X^4 + 4.28018\cdot 10^{-06}X^3\\&\quad + 1.97348\cdot 10^{-06}X^2 - 0.000352805X + 0.000346555\\&= 0.000346555B_{0,5} + 0.000275994B_{1,5} + 0.000205631B_{2,5}\\&\quad + 0.000135892B_{3,5} + 6.72075\cdot 10^{-05}B_{4,5} + 3.96905\cdot 10^{-09}B_{5,5}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.20533\cdot 10^{-08}$.

Bounding polynomials M and m:

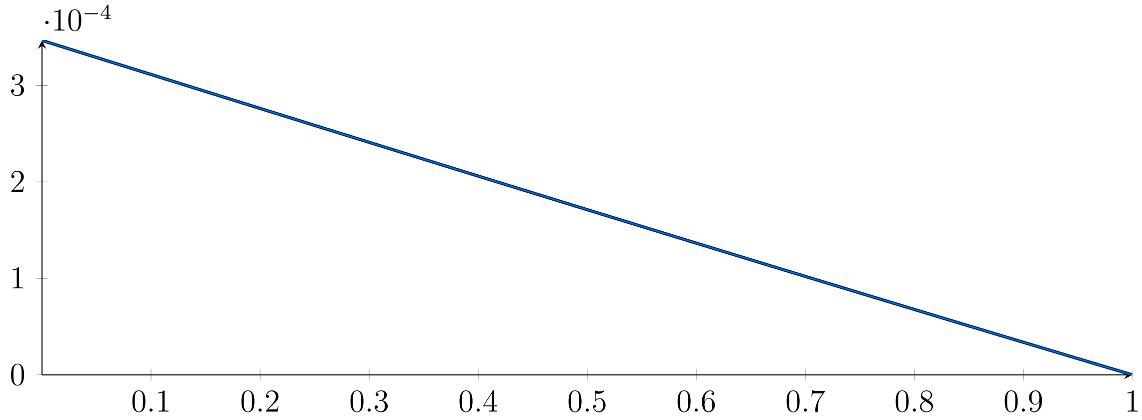
$$M = 4.28018\cdot 10^{-06}X^3 + 1.97348\cdot 10^{-06}X^2 - 0.000352805X + 0.000346567$$

$$m = 4.28018\cdot 10^{-06}X^3 + 1.97348\cdot 10^{-06}X^2 - 0.000352805X + 0.000346543$$

Root of M and m:

$$N(M) = \{-9.7583, 1.00005, 8.29718\} \qquad N(m) = \{-9.75827, 0.999976, 8.29722\}$$

Intersection intervals:



$[0.999976, 1]$

Longest intersection interval: $2.40591 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.249999, 0.25]$,

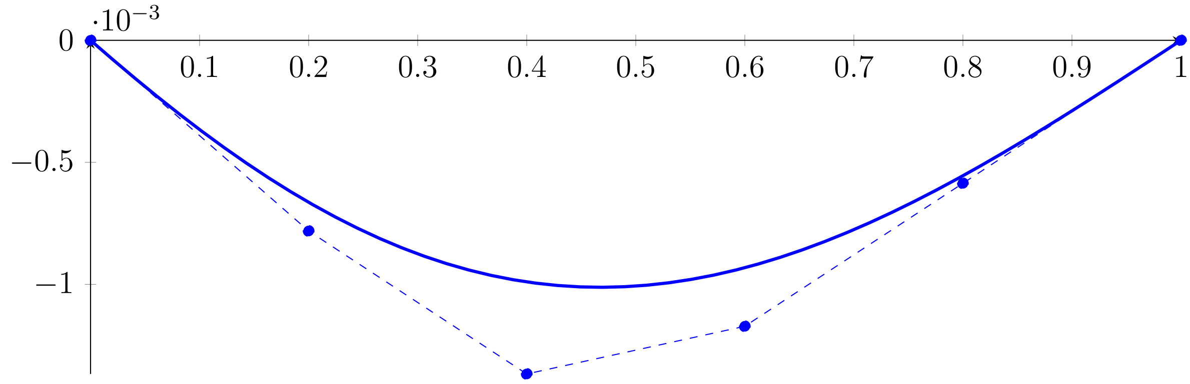
9.7 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.249999, 0.25]$

Found root in interval $[0.249999, 0.25]$ at recursion depth 5!

9.8 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

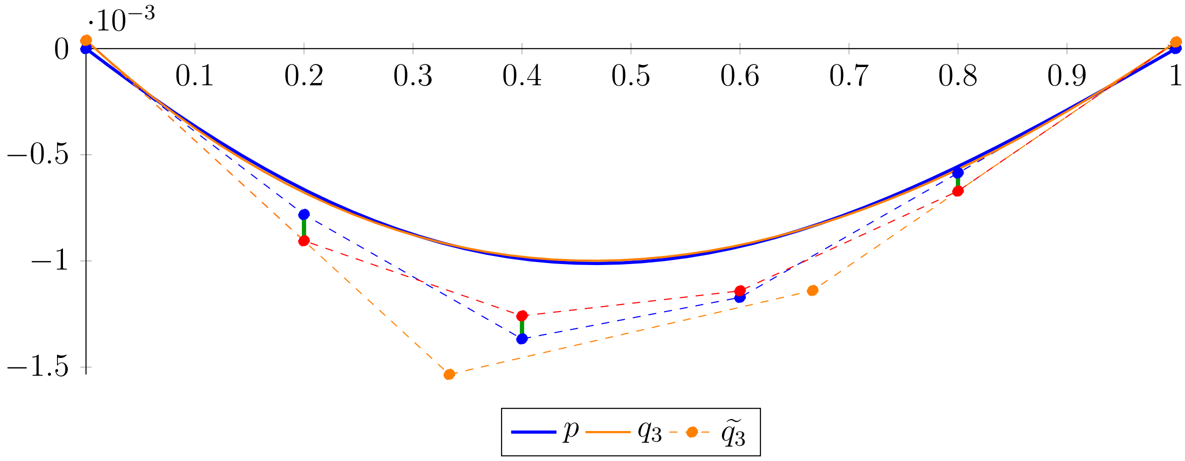
$$\begin{aligned}
p &= 0.000976562X^5 - 0.00488281X^4 + 0.00585938X^3 + 0.00195313X^2 - 0.00390625X + 1.55642 \cdot 10^{-20} \\
&= 1.55642 \cdot 10^{-20} B_{0,5}(X) - 0.00078125 B_{1,5}(X) - 0.00136719 B_{2,5}(X) \\
&\quad - 0.00117187 B_{3,5}(X) - 0.000585937 B_{4,5}(X) + 6.89273 \cdot 10^{-20} B_{5,5}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= -0.00119358X^3 + 0.00590588X^2 - 0.00472005X + 3.87525 \cdot 10^{-05} \\
&= 3.87525 \cdot 10^{-05} B_{0,3} - 0.0015346 B_{1,3} - 0.00113932 B_{2,3} + 3.1002 \cdot 10^{-05} B_{3,3}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_3 &= 8.56231 \cdot 10^{-19} X^5 - 2.12016 \cdot 10^{-18} X^4 - 0.00119358X^3 \\
&\quad + 0.00590588X^2 - 0.00472005X + 3.87525 \cdot 10^{-05} \\
&= 3.87525 \cdot 10^{-05} B_{0,5} - 0.000905258 B_{1,5} - 0.00125868 B_{2,5} \\
&\quad - 0.00114087 B_{3,5} - 0.000671193 B_{4,5} + 3.1002 \cdot 10^{-05} B_{5,5}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000124008$.
Bounding polynomials M and m:

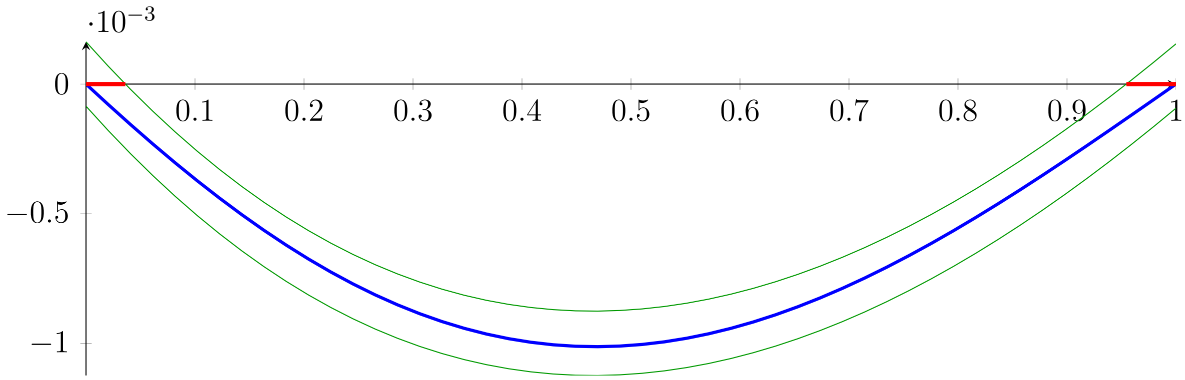
$$M = -0.00119358X^3 + 0.00590588X^2 - 0.00472005X + 0.00016276$$

$$m = -0.00119358X^3 + 0.00590588X^2 - 0.00472005X - 8.52555{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.0361016, 0.954443, 3.95751\} \qquad N(m) = \{-0.0176703, 1.02605, 3.93968\}$$

Intersection intervals:



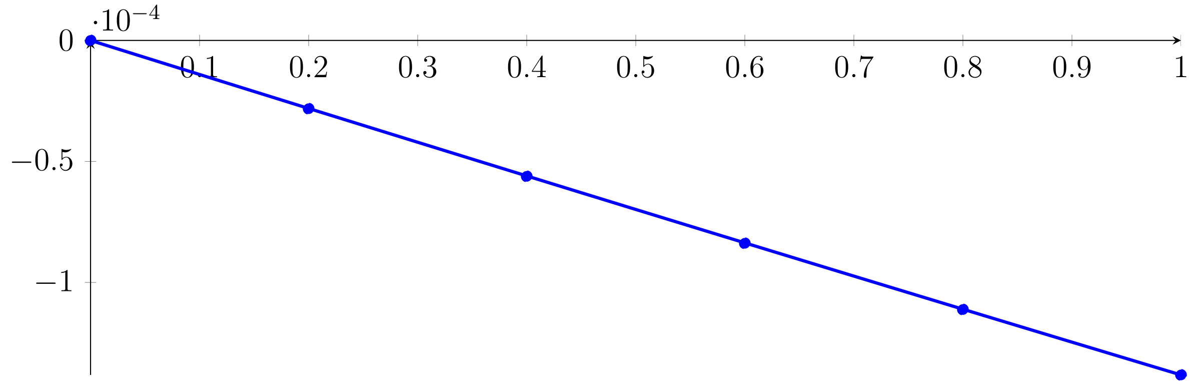
$$[0, 0.0361016],\ [0.954443, 1]$$

Longest intersection interval: 0.0455567
$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.259025]$, interval 2: $[0.488611, 0.5]$,

9.9 Recursion Branch 1 1 2 1 in Interval 1: $[0.25, 0.259025]$

Normalized monomial und Bézier representations and the Bézier polygon:

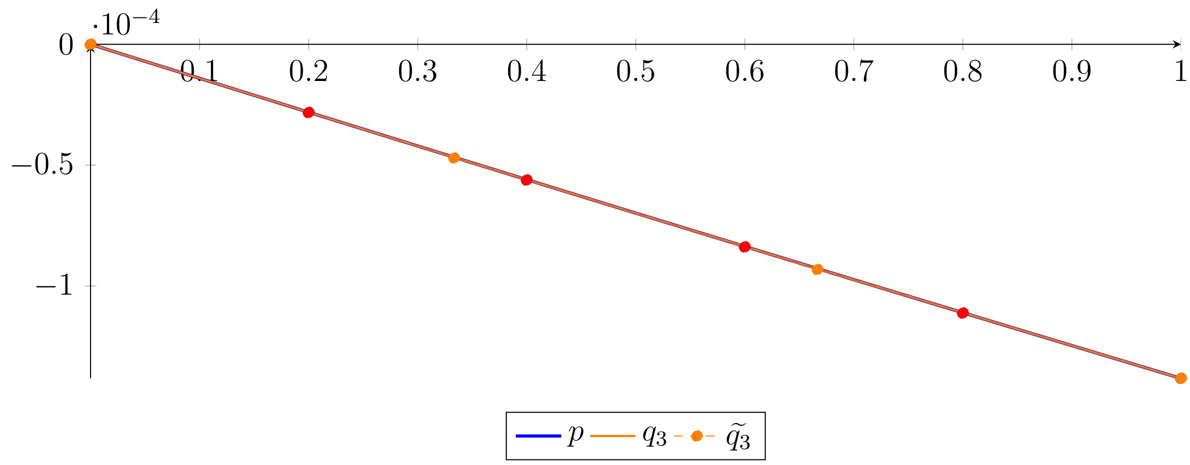
$$\begin{aligned} p &= 5.98872{\cdot}10^{-11}X^5 - 8.29425{\cdot}10^{-09}X^4 + 2.75697{\cdot}10^{-07}X^3 \\ &\quad + 2.54556{\cdot}10^{-06}X^2 - 0.000141022X + 1.55642{\cdot}10^{-20} \\ &= 1.55642{\cdot}10^{-20}B_{0,5}(X) - 2.82044{\cdot}10^{-05}B_{1,5}(X) - 5.61542{\cdot}10^{-05}B_{2,5}(X) \\ &\quad - 8.38219{\cdot}10^{-05}B_{3,5}(X) - 0.000111182B_{4,5}(X) - 0.000138209B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.59275\cdot 10^{-07}X^3 + 2.55608\cdot 10^{-06}X^2 - 0.000141024X + 1.16588\cdot 10^{-10} \\ &= 1.16588\cdot 10^{-10}B_{0,3} - 4.7008\cdot 10^{-05}B_{1,3} - 9.31641\cdot 10^{-05}B_{2,3} - 0.000138209B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.59785\cdot 10^{-19}X^5 - 6.02371\cdot 10^{-19}X^4 + 2.59275\cdot 10^{-07}X^3 \\ &\quad + 2.55608\cdot 10^{-06}X^2 - 0.000141024X + 1.16588\cdot 10^{-10} \\ &= 1.16588\cdot 10^{-10}B_{0,5} - 2.82047\cdot 10^{-05}B_{1,5} - 5.6154\cdot 10^{-05}B_{2,5} \\ &\quad - 8.38217\cdot 10^{-05}B_{3,5} - 0.000111182B_{4,5} - 0.000138209B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.5024\cdot 10^{-10}$.

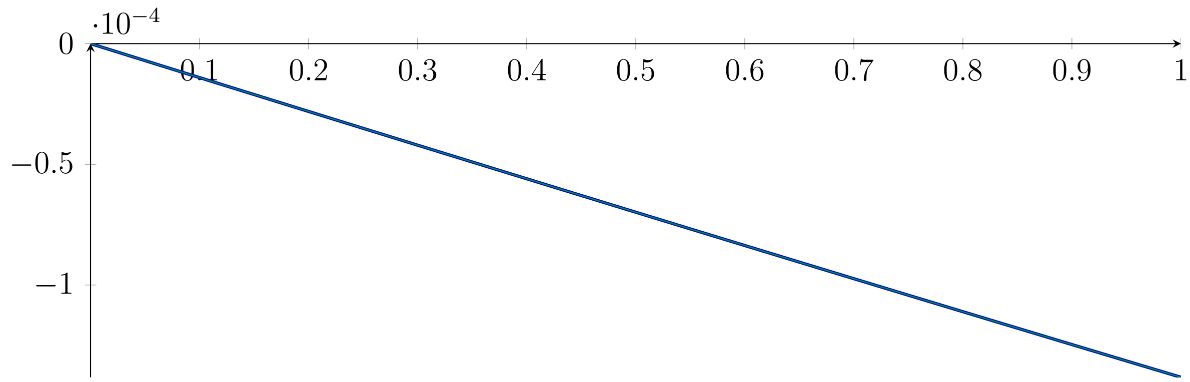
Bounding polynomials M and m:

$$\begin{aligned} M &= 2.59275\cdot 10^{-07}X^3 + 2.55608\cdot 10^{-06}X^2 - 0.000141024X + 4.66828\cdot 10^{-10} \\ m &= 2.59275\cdot 10^{-07}X^3 + 2.55608\cdot 10^{-06}X^2 - 0.000141024X - 2.33651\cdot 10^{-10} \end{aligned}$$

Root of M and m:

$$N(M) = \{-28.7666, 3.31026\cdot 10^{-06}, 18.908\} \qquad N(m) = \{-28.7666, -1.65682\cdot 10^{-06}, 18.908\}$$

Intersection intervals:



$$[0, 3.31026e - 06]$$

Longest intersection interval: $3.31026 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.25, 0.25]$,

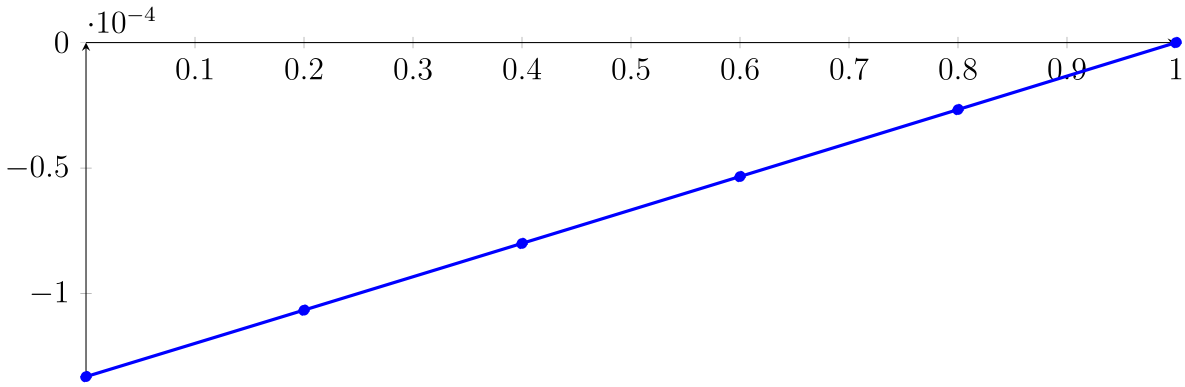
9.10 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.25, 0.25]$

Found root in interval $[0.25, 0.25]$ at recursion depth 5!

9.11 Recursion Branch 1 1 2 2 in Interval 2: $[0.488611, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

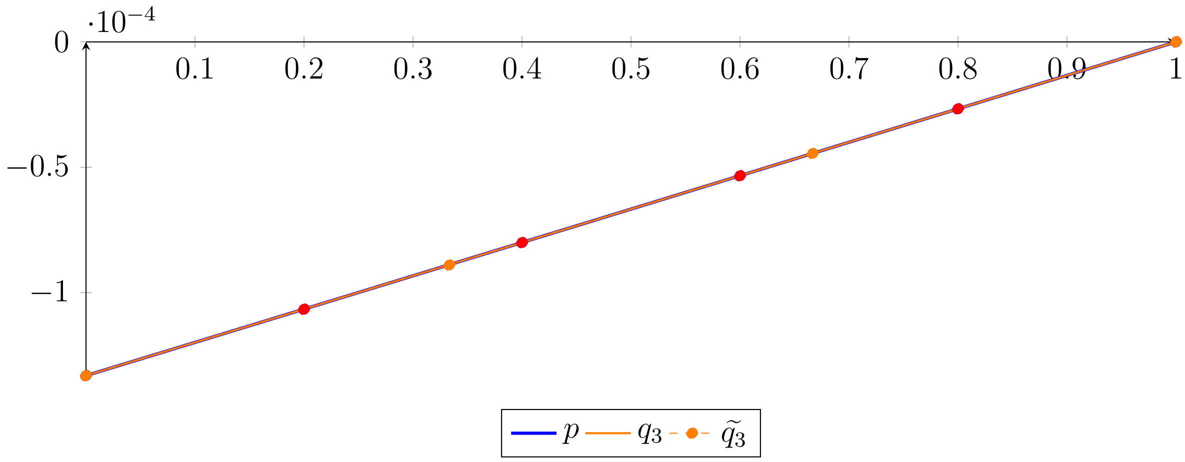
$$\begin{aligned} p &= 1.91629 \cdot 10^{-10} X^5 - 9.58145 \cdot 10^{-10} X^4 - 3.67416 \cdot 10^{-07} X^3 \\ &\quad + 1.10608 \cdot 10^{-06} X^2 + 0.00013236 X - 0.000133098 \\ &= -0.000133098 B_{0,5}(X) - 0.000106626 B_{1,5}(X) - 8.00432 \cdot 10^{-05} B_{2,5}(X) \\ &\quad - 5.33868 \cdot 10^{-05} B_{3,5}(X) - 2.66934 \cdot 10^{-05} B_{4,5}(X) + 6.89273 \cdot 10^{-20} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -3.688 \cdot 10^{-07} X^3 + 1.10685 \cdot 10^{-06} X^2 + 0.00013236 X - 0.000133098 \\ &= -0.000133098 B_{0,3} - 8.89778 \cdot 10^{-05} B_{1,3} - 4.4489 \cdot 10^{-05} B_{2,3} + 6.08346 \cdot 10^{-12} B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.46301 \cdot 10^{-20} X^5 + 7.11375 \cdot 10^{-20} X^4 - 3.688 \cdot 10^{-07} X^3 \\ &\quad + 1.10685 \cdot 10^{-06} X^2 + 0.00013236 X - 0.000133098 \\ &= -0.000133098 B_{0,5} - 0.000106626 B_{1,5} - 8.00432 \cdot 10^{-05} B_{2,5} \\ &\quad - 5.33867 \cdot 10^{-05} B_{3,5} - 2.66934 \cdot 10^{-05} B_{4,5} + 6.08346 \cdot 10^{-12} B_{5,5} \end{aligned}$$



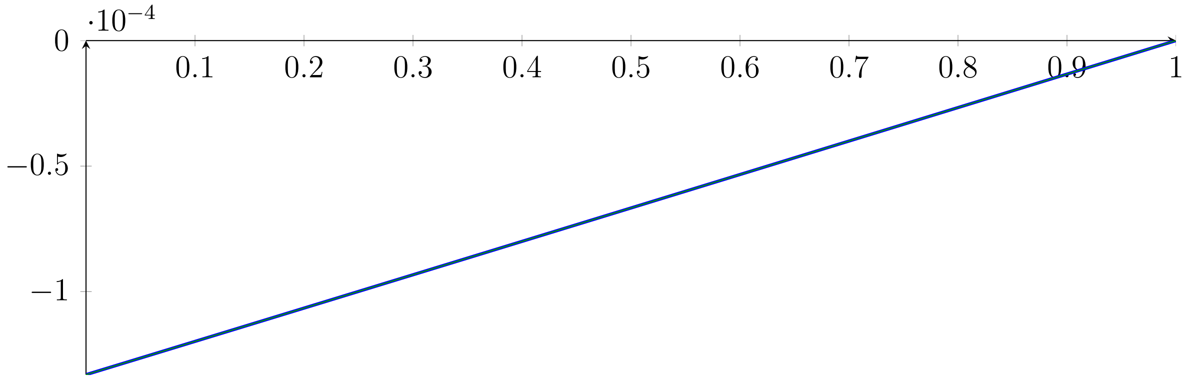
The maximum difference of the Bézier coefficients is $\delta = 2.43338 \cdot 10^{-11}$.
Bounding polynomials M and m:

$$M = -3.688 \cdot 10^{-07} X^3 + 1.10685 \cdot 10^{-06} X^2 + 0.00013236 X - 0.000133098$$
$$m = -3.688 \cdot 10^{-07} X^3 + 1.10685 \cdot 10^{-06} X^2 + 0.00013236 X - 0.000133098$$

Root of M and m:

$$N(M) = \{-18.0229, 1, 20.0242\} \qquad N(m) = \{-18.0229, 1, 20.0242\}$$

Intersection intervals:



$[1, 1]$

Longest intersection interval: $2.27901 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

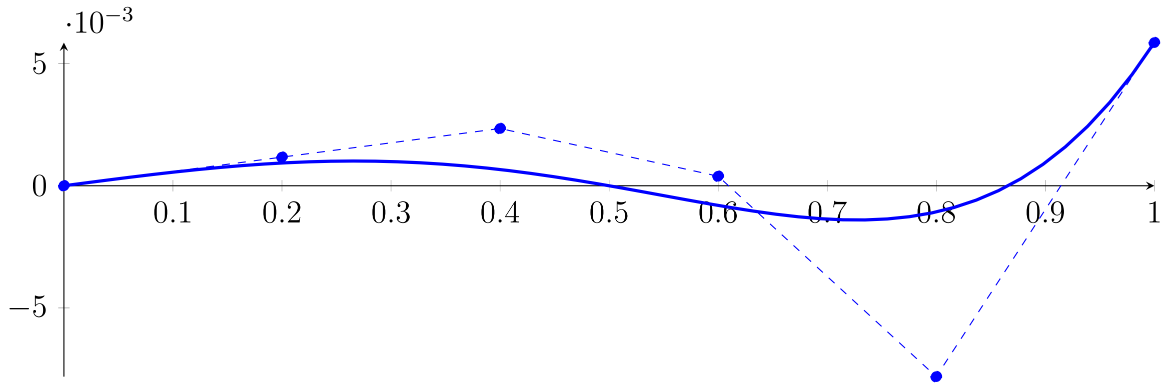
9.12 Recursion Branch 1 1 2 2 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

9.13 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

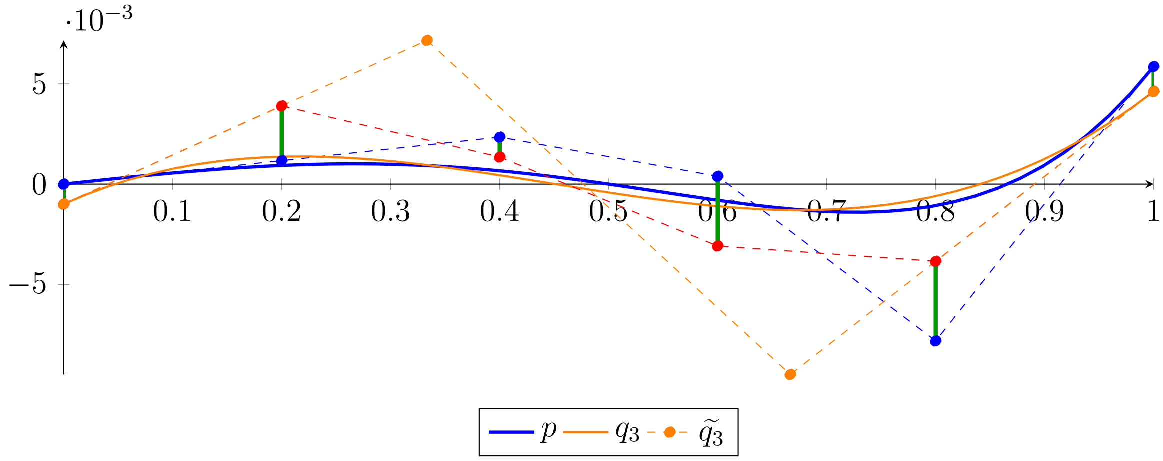
$$\begin{aligned} p &= 0.03125 X^5 + 5.45277 \cdot 10^{-20} X^4 - 0.03125 X^3 + 1.76818 \cdot 10^{-19} X^2 + 0.00585938 X + 6.89273 \cdot 10^{-20} \\ &= 6.89273 \cdot 10^{-20} B_{0,5}(X) + 0.00117188 B_{1,5}(X) + 0.00234375 B_{2,5}(X) \\ &\quad + 0.000390625 B_{3,5}(X) - 0.0078125 B_{4,5}(X) + 0.00585938 B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 0.0555556X^3 - 0.0744048X^2 + 0.0244606X - 0.000992063$$
$$= -0.000992063B_{0,3} + 0.00716146B_{1,3} - 0.00948661B_{2,3} + 0.0046193B_{3,3}$$

$$\widetilde{q}_3 = 2.09885{\cdot}10^{-17}X^5 - 5.29051{\cdot}10^{-17}X^4 + 0.0555556X^3 - 0.0744048X^2 + 0.0244606X - 0.000992063$$
$$= -0.000992063B_{0,5} + 0.00390005B_{1,5} + 0.00135169B_{2,5}$$
$$- 0.0030816B_{3,5} - 0.00384425B_{4,5} + 0.0046193B_{5,5}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00396825$.

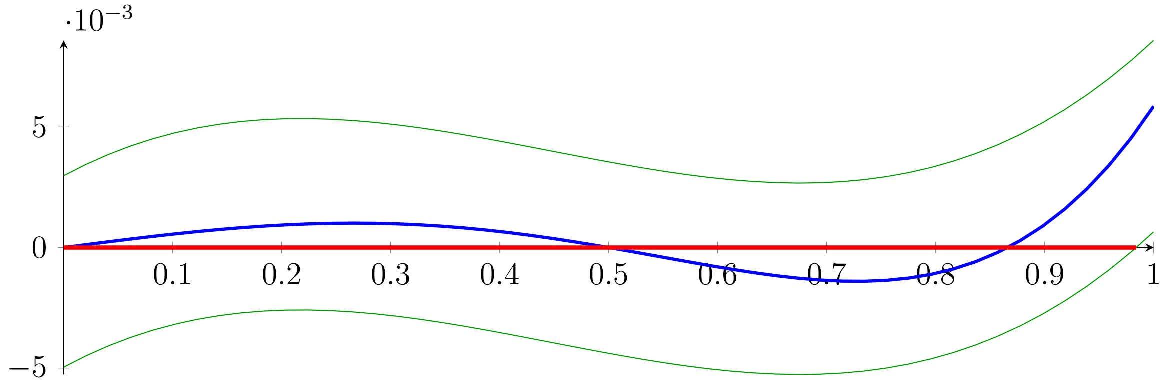
Bounding polynomials M and m:

$$M = 0.0555556X^3 - 0.0744048X^2 + 0.0244606X + 0.00297619$$
$$m = 0.0555556X^3 - 0.0744048X^2 + 0.0244606X - 0.00496032$$

Root of M and m:

$$N(M) = \{-0.0933305\} \qquad\qquad N(m) = \{0.984067\}$$

Intersection intervals:



$$[0, 0.984067]$$

Longest intersection interval: 0.984067

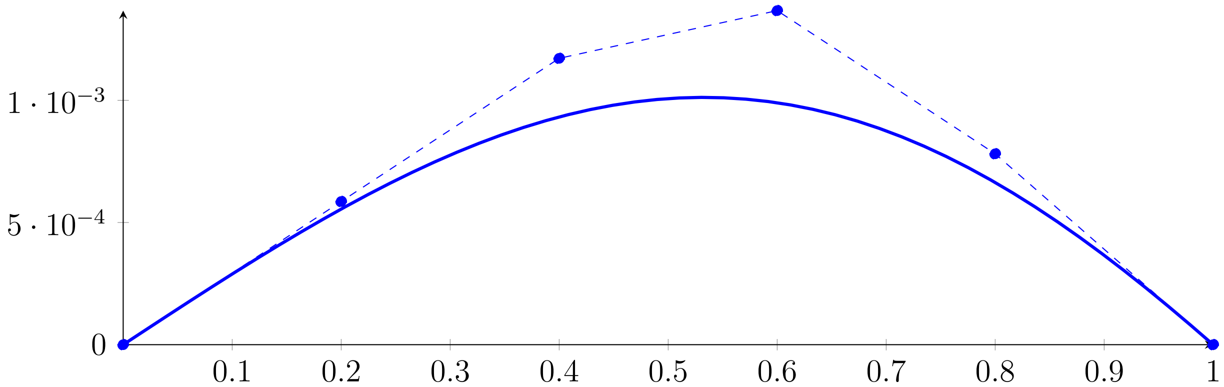
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

Bisection point is very near to a root?!?

9.14 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

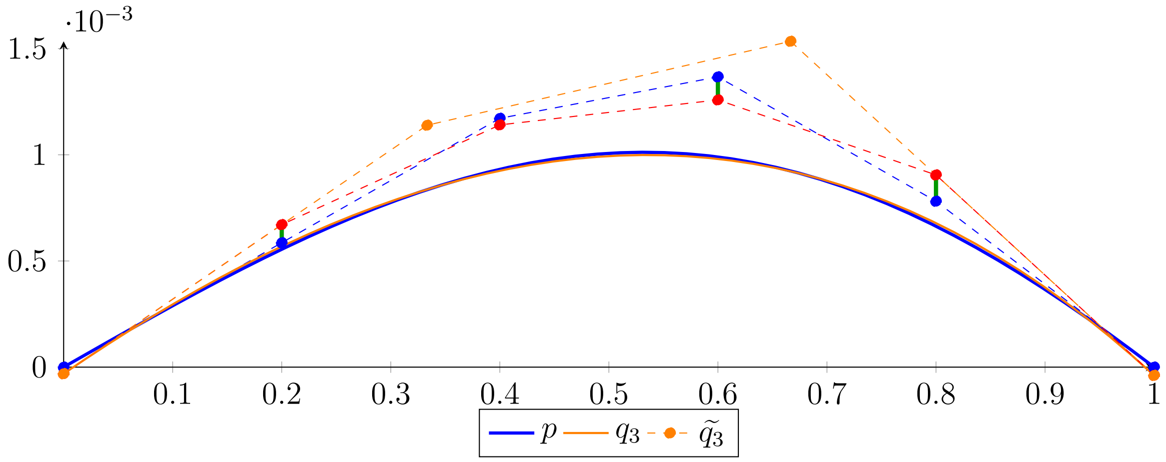
$$\begin{aligned} p &= 0.000976563X^5 + 3.70577{\cdot}10^{-21}X^4 - 0.00390625X^3 \\ &\quad + 4.44692{\cdot}10^{-20}X^2 + 0.00292969X + 6.89273{\cdot}10^{-20} \\ &= 6.89273{\cdot}10^{-20}B_{0,5}(X) + 0.000585938B_{1,5}(X) + 0.00117188B_{2,5}(X) \\ &\quad + 0.00136719B_{3,5}(X) + 0.00078125B_{4,5}(X) + 2.17237{\cdot}10^{-19}B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.00119358X^3 - 0.00232515X^2 + 0.00351097X - 3.1002{\cdot}10^{-05} \\ &= -3.1002{\cdot}10^{-05}B_{0,3} + 0.00113932B_{1,3} + 0.0015346B_{2,3} - 3.87525{\cdot}10^{-05}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.98441{\cdot}10^{-18}X^5 + 4.90232{\cdot}10^{-18}X^4 - 0.00119358X^3 \\ &\quad - 0.00232515X^2 + 0.00351097X - 3.1002{\cdot}10^{-05} \\ &= -3.1002{\cdot}10^{-05}B_{0,5} + 0.000671193B_{1,5} + 0.00114087B_{2,5} \\ &\quad + 0.00125868B_{3,5} + 0.000905258B_{4,5} - 3.87525{\cdot}10^{-05}B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000124008$.
Bounding polynomials M and m:

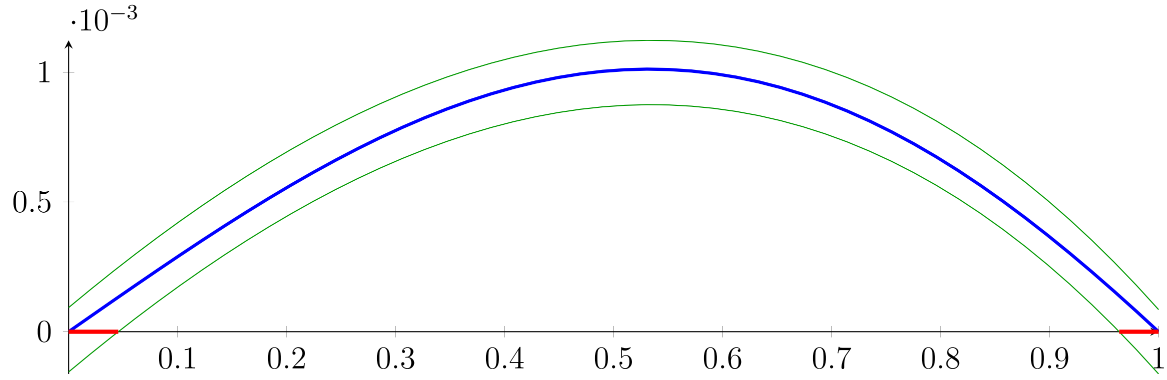
$$M = -0.00119358X^3 - 0.00232515X^2 + 0.00351097X + 9.3006{\cdot}10^{-05}$$

$$m = -0.00119358X^3 - 0.00232515X^2 + 0.00351097X - 0.00015501$$

Root of M and m:

$$N(M) = \{-2.93968, -0.0260468, 1.01767\} \qquad N(m) = \{-2.95751, 0.0455567, 0.963898\}$$

Intersection intervals:



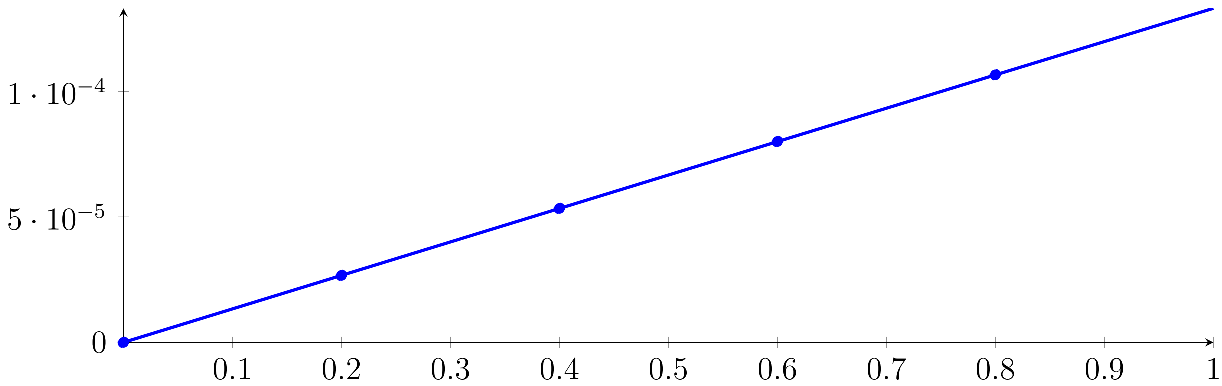
$$[0, 0.0455567],\ [0.963898, 1]$$

Longest intersection interval: 0.0455567
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.511389]$, interval 2: $[0.740975, 0.75]$,

9.15 Recursion Branch 1 2 1 1 in Interval 1: $[0.5, 0.511389]$

Normalized monomial und Bézier representations and the Bézier polygon:

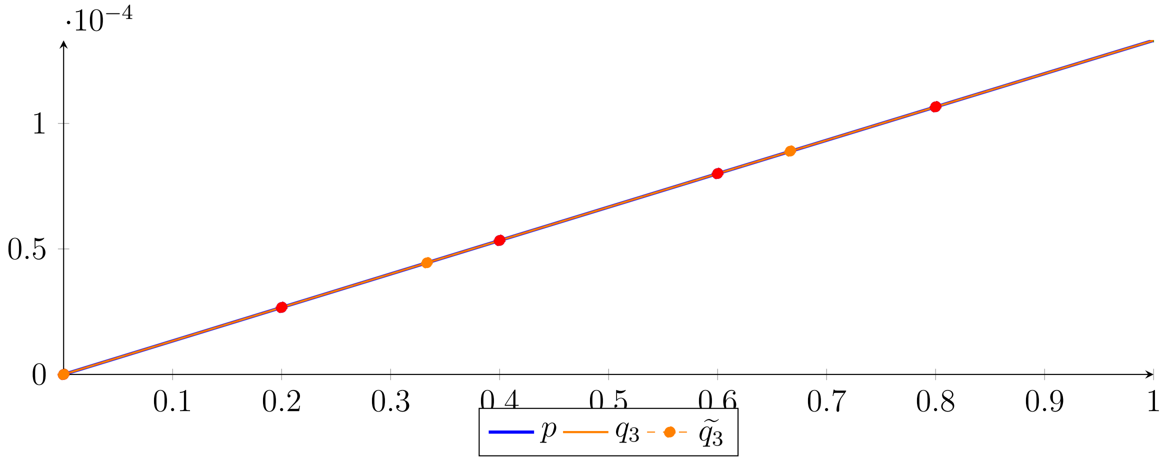
$$\begin{aligned}
p &= 1.91629{\cdot}10^{-10}X^5 - 1.32349{\cdot}10^{-22}X^4 - 3.69332{\cdot}10^{-07}X^3 \\
&\quad + 1.32349{\cdot}10^{-22}X^2 + 0.000133467X + 6.89273{\cdot}10^{-20} \\
&= 6.89273{\cdot}10^{-20}B_{0,5}(X) + 2.66934{\cdot}10^{-05}B_{1,5}(X) + 5.33868{\cdot}10^{-05}B_{2,5}(X) \\
&\quad + 8.00432{\cdot}10^{-05}B_{3,5}(X) + 0.000106626B_{4,5}(X) + 0.000133098B_{5,5}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= -3.688{\cdot}10^{-07}X^3 - 4.5626{\cdot}10^{-10}X^2 + 0.000133467X - 6.08346{\cdot}10^{-12} \\
&= -6.08346{\cdot}10^{-12}B_{0,3} + 4.4489{\cdot}10^{-05}B_{1,3} + 8.89778{\cdot}10^{-05}B_{2,3} + 0.000133098B_{3,3}
\end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= -2.49389{\cdot}10^{-19}X^5 + 5.78121{\cdot}10^{-19}X^4 - 3.688{\cdot}10^{-07}X^3 \\
&\quad - 4.5626{\cdot}10^{-10}X^2 + 0.000133467X - 6.08346{\cdot}10^{-12} \\
&= -6.08346{\cdot}10^{-12}B_{0,5} + 2.66934{\cdot}10^{-05}B_{1,5} + 5.33867{\cdot}10^{-05}B_{2,5} \\
&\quad + 8.00432{\cdot}10^{-05}B_{3,5} + 0.000106626B_{4,5} + 0.000133098B_{5,5}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.43338 \cdot 10^{-11}$.

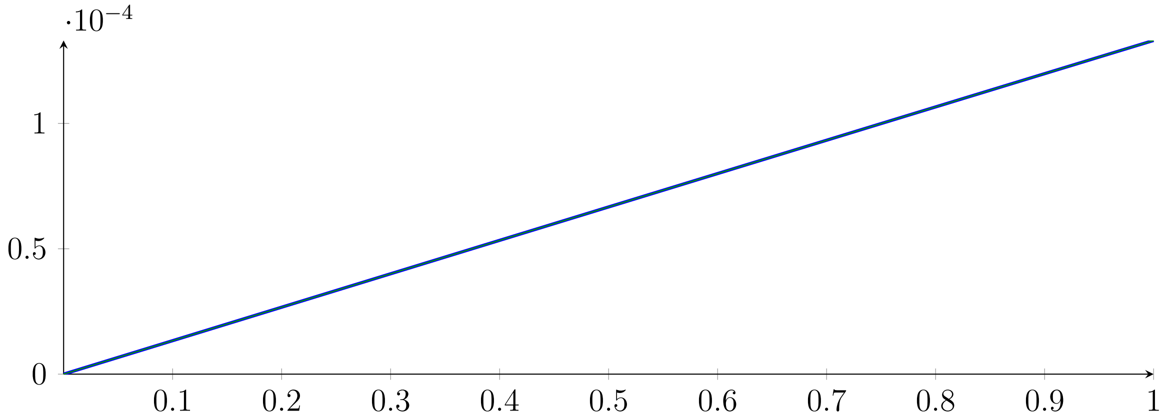
Bounding polynomials M and m:

$$M = -3.688 \cdot 10^{-07} X^3 - 4.5626 \cdot 10^{-10} X^2 + 0.000133467X + 1.82504 \cdot 10^{-11}$$
$$m = -3.688 \cdot 10^{-07} X^3 - 4.5626 \cdot 10^{-10} X^2 + 0.000133467X - 3.04173 \cdot 10^{-11}$$

Root of M and m:

$$N(M) = \{-19.0242, -1.36741 \cdot 10^{-07}, 19.0229\} \qquad N(m) = \{-19.0242, 2.27901 \cdot 10^{-07}, 19.0229\}$$

Intersection intervals:



$[0, 2.27901e - 07]$

Longest intersection interval: $2.27901 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

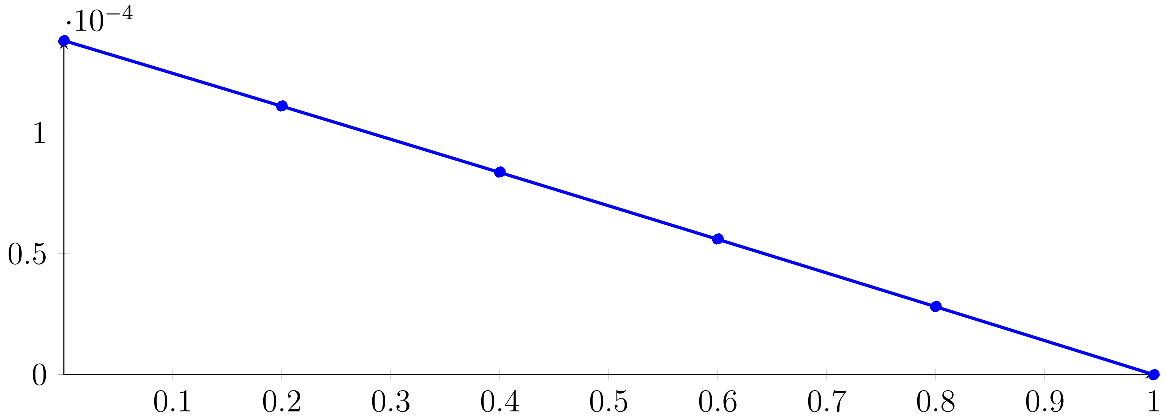
9.16 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

9.17 Recursion Branch 1 2 1 2 in Interval 2: $[0.740975, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

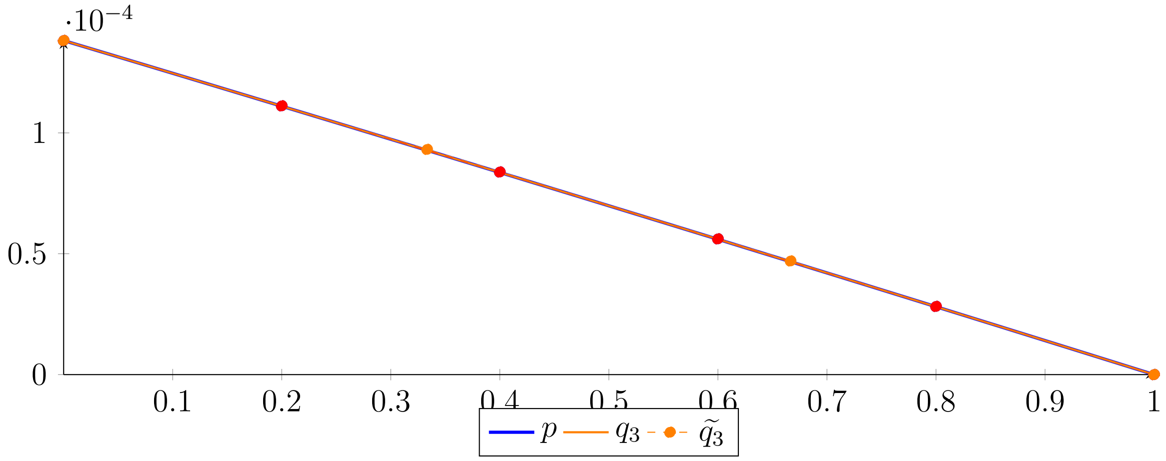
$$\begin{aligned} p = {} & 5.98872 \cdot 10^{-11} X^5 + 7.99481 \cdot 10^{-09} X^4 + 2.43119 \cdot 10^{-07} X^3 \\ & - 3.32348 \cdot 10^{-06} X^2 - 0.000135137X + 0.000138209 \\ = {} & 0.000138209 B_{0,5}(X) + 0.000111182 B_{1,5}(X) + 8.38219 \cdot 10^{-05} B_{2,5}(X) \\ & + 5.61542 \cdot 10^{-05} B_{3,5}(X) + 2.82044 \cdot 10^{-05} B_{4,5}(X) + 2.17237 \cdot 10^{-19} B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.59275 \cdot 10^{-07} X^3 - 3.33391 \cdot 10^{-06} X^2 - 0.000135134X + 0.000138209 \\ &= 0.000138209 B_{0,3} + 9.31641 \cdot 10^{-05} B_{1,3} + 4.7008 \cdot 10^{-05} B_{2,3} - 1.16588 \cdot 10^{-10} B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.47228 \cdot 10^{-20} X^5 - 7.16008 \cdot 10^{-20} X^4 + 2.59275 \cdot 10^{-07} X^3 \\ &\quad - 3.33391 \cdot 10^{-06} X^2 - 0.000135134X + 0.000138209 \\ &= 0.000138209 B_{0,5} + 0.000111182 B_{1,5} + 8.38217 \cdot 10^{-05} B_{2,5} \\ &\quad + 5.6154 \cdot 10^{-05} B_{3,5} + 2.82047 \cdot 10^{-05} B_{4,5} - 1.16588 \cdot 10^{-10} B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.5024 \cdot 10^{-10}$.

Bounding polynomials M and m:

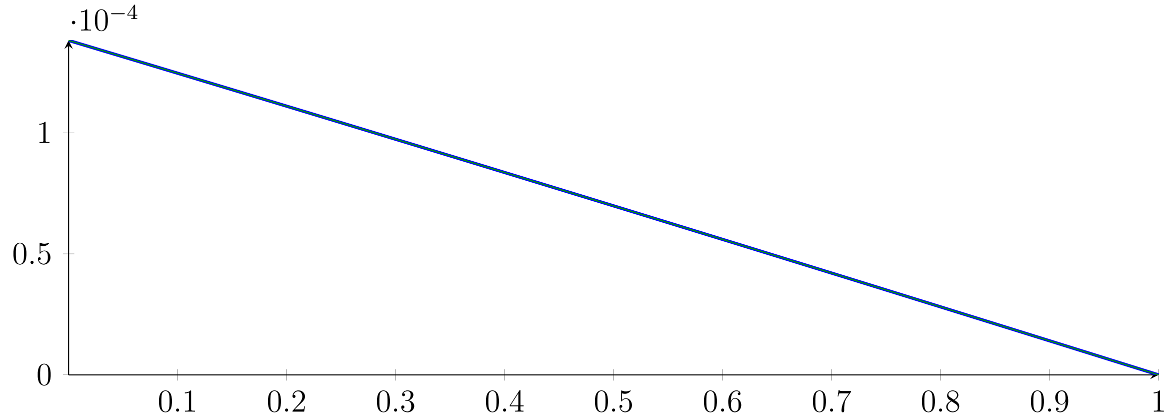
$$M = 2.59275 \cdot 10^{-07} X^3 - 3.33391 \cdot 10^{-06} X^2 - 0.000135134X + 0.000138209$$

$$m = 2.59275 \cdot 10^{-07} X^3 - 3.33391 \cdot 10^{-06} X^2 - 0.000135134X + 0.000138208$$

Root of M and m:

$$N(M) = \{-17.908, 1, 29.7666\} \qquad N(m) = \{-17.908, 0.999997, 29.7666\}$$

Intersection intervals:



$[0.999997, 1]$

Longest intersection interval: $3.31026 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.75]$,

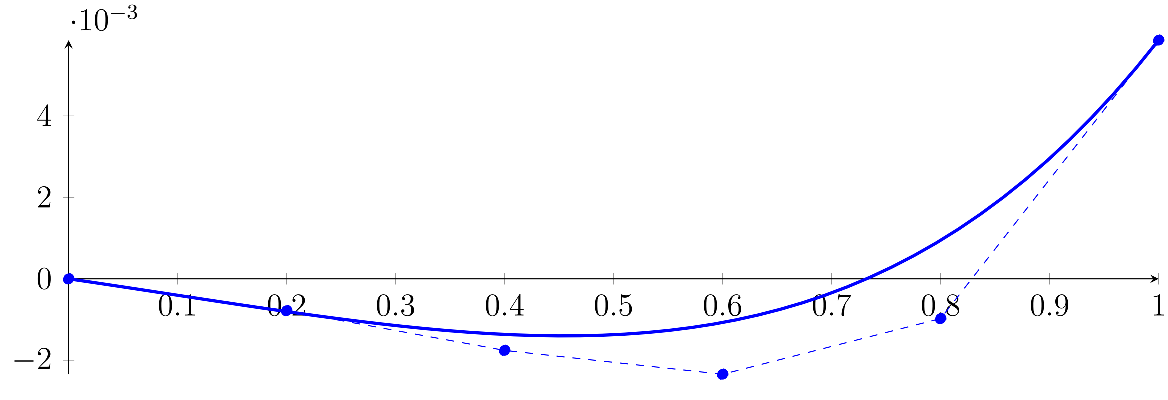
9.18 Recursion Branch 1 2 1 2 1 in Interval 1: $[0.75, 0.75]$

Found root in interval $[0.75, 0.75]$ at recursion depth 5!

9.19 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

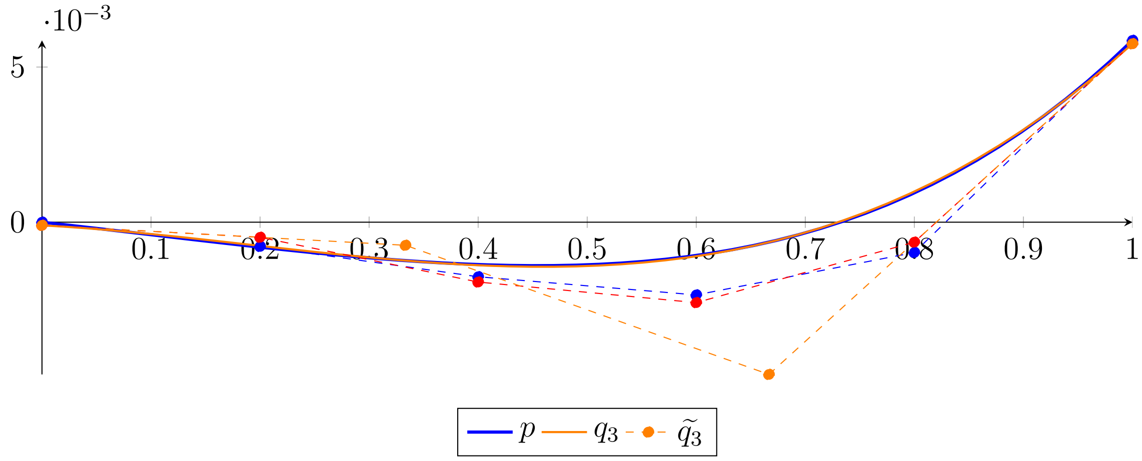
$$\begin{aligned}p &= 0.000976562X^5 + 0.00488281X^4 + 0.00585938X^3 - 0.00195312X^2 - 0.00390625X + 2.17237 \cdot 10^{-19}\\ &= 2.17237 \cdot 10^{-19}B_{0,5}(X) - 0.00078125B_{1,5}(X) - 0.00175781B_{2,5}(X)\\ &\quad - 0.00234375B_{3,5}(X) - 0.000976562B_{4,5}(X) + 0.00585938B_{5,5}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 0.0183377X^3 - 0.0105562X^2 - 0.00192987X - 0.000100756\\ &= -0.000100756B_{0,3} - 0.000744048B_{1,3} - 0.00490606B_{2,3} + 0.00575087B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 4.09923 \cdot 10^{-18}X^5 - 1.16605 \cdot 10^{-17}X^4 + 0.0183377X^3 - 0.0105562X^2 - 0.00192987X - 0.000100756\\ &= -0.000100756B_{0,5} - 0.000486731B_{1,5} - 0.00192832B_{2,5}\\ &\quad - 0.00259177B_{3,5} - 0.000643291B_{4,5} + 0.00575087B_{5,5}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000333271$.
Bounding polynomials M and m:

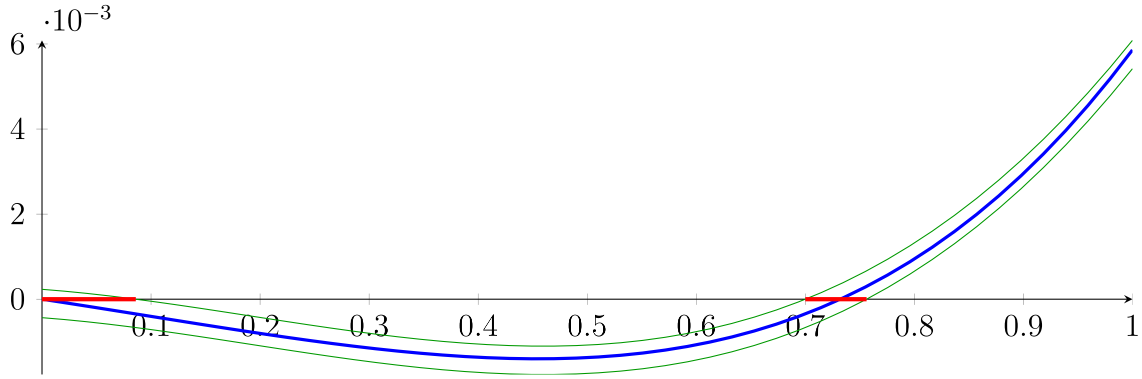
$$M = 0.0183377X^3 - 0.0105562X^2 - 0.00192987X + 0.000232515$$

$$m = 0.0183377X^3 - 0.0105562X^2 - 0.00192987X - 0.000434028$$

Root of M and m:

$$N(M) = \{-0.210493, 0.0860408, 0.700107\} \qquad N(m) = \{0.756213\}$$

Intersection intervals:



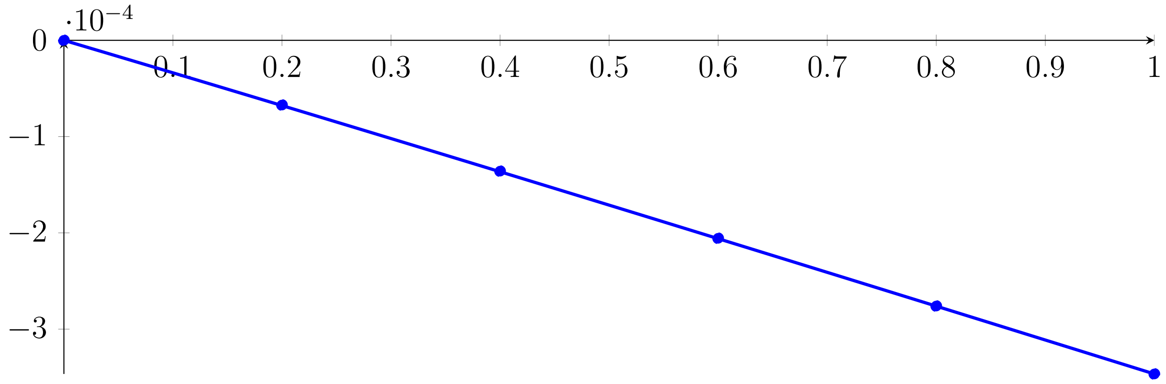
$$[0, 0.0860408],\ [0.700107, 0.756213]$$

Longest intersection interval: 0.0860408
$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.77151]$, interval 2: $[0.925027, 0.939053]$,

9.20 Recursion Branch 1 2 2 1 in Interval 1: $[0.75, 0.77151]$

Normalized monomial und Bézier representations and the Bézier polygon:

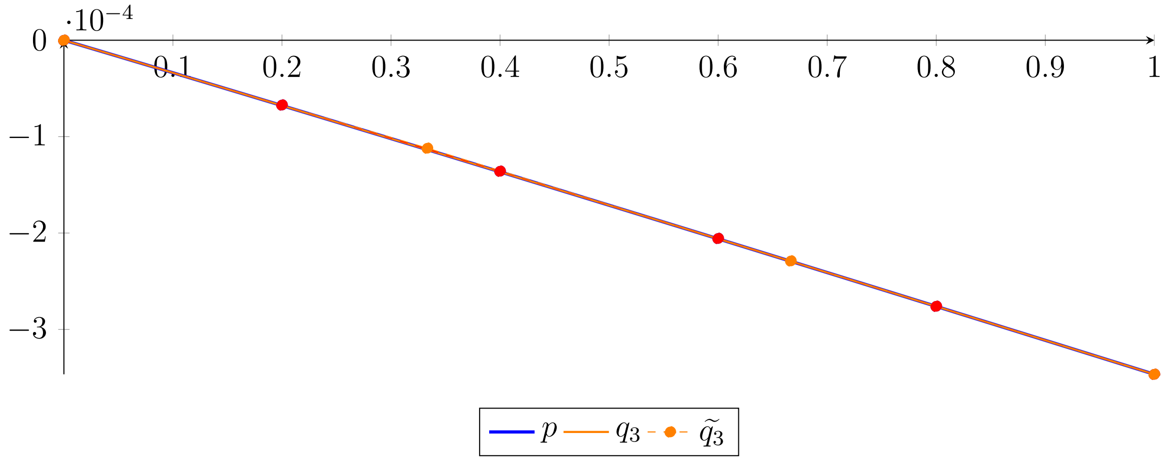
$$\begin{aligned} p &= 4.60491 \cdot 10^{-09}X^5 + 2.676 \cdot 10^{-07}X^4 + 3.73219 \cdot 10^{-06}X^3 \\ &\quad - 1.4459 \cdot 10^{-05}X^2 - 0.000336097X + 2.17237 \cdot 10^{-19} \\ &= 2.17237 \cdot 10^{-19}B_{0,5}(X) - 6.72193 \cdot 10^{-05}B_{1,5}(X) - 0.000135885B_{2,5}(X) \\ &\quad - 0.000205622B_{3,5}(X) - 0.000276006B_{4,5}(X) - 0.000346551B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 4.28018 \cdot 10^{-06} X^3 - 1.4814 \cdot 10^{-05} X^2 - 0.000336017 X - 3.96905 \cdot 10^{-09} \\ &= -3.96905 \cdot 10^{-09} B_{0,3} - 0.00011201 B_{1,3} - 0.000228954 B_{2,3} - 0.000346555 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 6.47547 \cdot 10^{-19} X^5 - 1.50066 \cdot 10^{-18} X^4 + 4.28018 \cdot 10^{-06} X^3 \\ &\quad - 1.4814 \cdot 10^{-05} X^2 - 0.000336017 X - 3.96905 \cdot 10^{-09} \\ &= -3.96905 \cdot 10^{-09} B_{0,5} - 6.72075 \cdot 10^{-05} B_{1,5} - 0.000135892 B_{2,5} \\ &\quad - 0.000205631 B_{3,5} - 0.000275994 B_{4,5} - 0.000346555 B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.20533 \cdot 10^{-08}$.

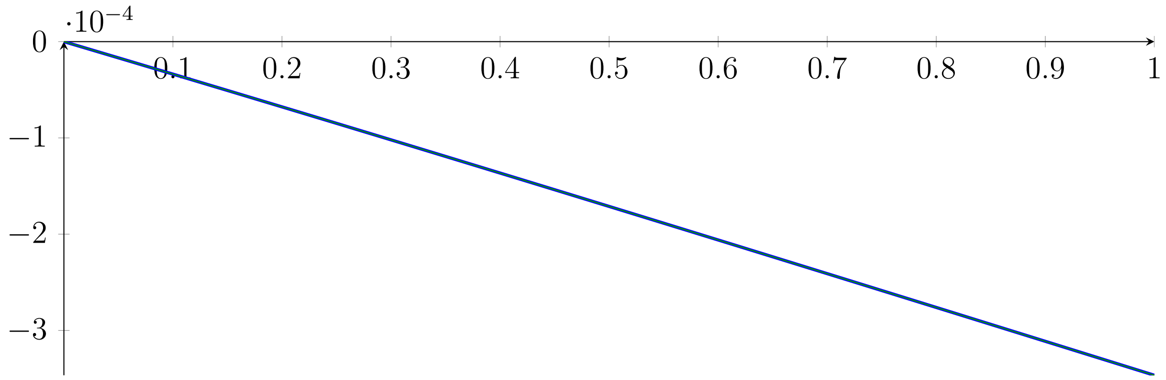
Bounding polynomials M and m:

$$M = 4.28018 \cdot 10^{-06} X^3 - 1.4814 \cdot 10^{-05} X^2 - 0.000336017 X + 8.08429 \cdot 10^{-09}$$
$$m = 4.28018 \cdot 10^{-06} X^3 - 1.4814 \cdot 10^{-05} X^2 - 0.000336017 X - 1.60224 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{-7.29722, 2.40591 \cdot 10^{-05}, 10.7583\} \qquad N(m) = \{-7.29718, -4.76833 \cdot 10^{-05}, 10.7583\}$$

Intersection intervals:



$$[0, 2.40591e - 05]$$

Longest intersection interval: $2.40591 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.75, 0.750001]$,

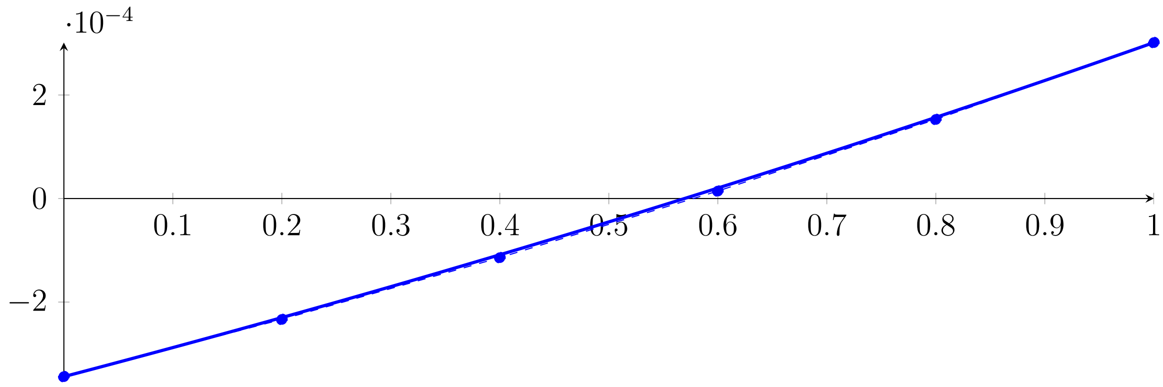
9.21 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.75, 0.750001]$

Found root in interval $[0.75, 0.750001]$ at recursion depth 5!

9.22 Recursion Branch 1 2 2 2 in Interval 2: $[0.925027, 0.939053]$

Normalized monomial und Bézier representations and the Bézier polygon:

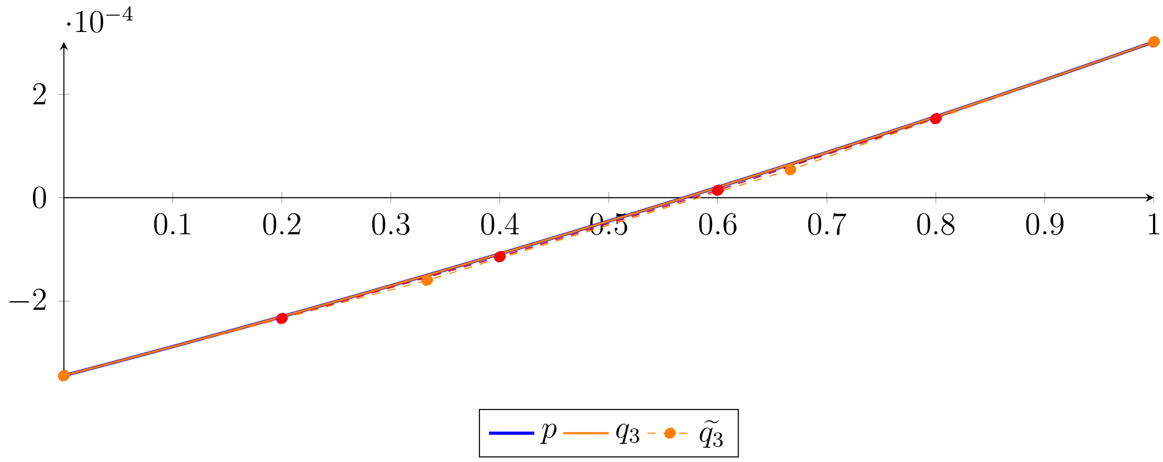
$$\begin{aligned} p &= 5.42896 \cdot 10^{-10} X^5 + 8.22545 \cdot 10^{-08} X^4 + 4.2951 \cdot 10^{-06} X^3 \\ &\quad + 8.83412 \cdot 10^{-05} X^2 + 0.000552651 X - 0.000344089 \\ &= -0.000344089 B_{0,5}(X) - 0.000233559 B_{1,5}(X) - 0.000114195 B_{2,5}(X) \\ &\quad + 1.44334 \cdot 10^{-05} B_{3,5}(X) + 0.000152771 B_{4,5}(X) + 0.000301281 B_{5,5}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 4.46111 \cdot 10^{-06} X^3 + 8.82342 \cdot 10^{-05} X^2 + 0.000552675 X - 0.00034409 \\ &= -0.00034409 B_{0,3} - 0.000159865 B_{1,3} + 5.3771 \cdot 10^{-05} B_{2,3} + 0.00030128 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -5.91044 \cdot 10^{-19} X^5 + 1.40118 \cdot 10^{-18} X^4 + 4.46111 \cdot 10^{-06} X^3 \\ &\quad + 8.82342 \cdot 10^{-05} X^2 + 0.000552675 X - 0.00034409 \\ &= -0.00034409 B_{0,5} - 0.000233555 B_{1,5} - 0.000114197 B_{2,5} \\ &\quad + 1.4431 \cdot 10^{-05} B_{3,5} + 0.000152775 B_{4,5} + 0.00030128 B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.59413 \cdot 10^{-09}$.
Bounding polynomials M and m:

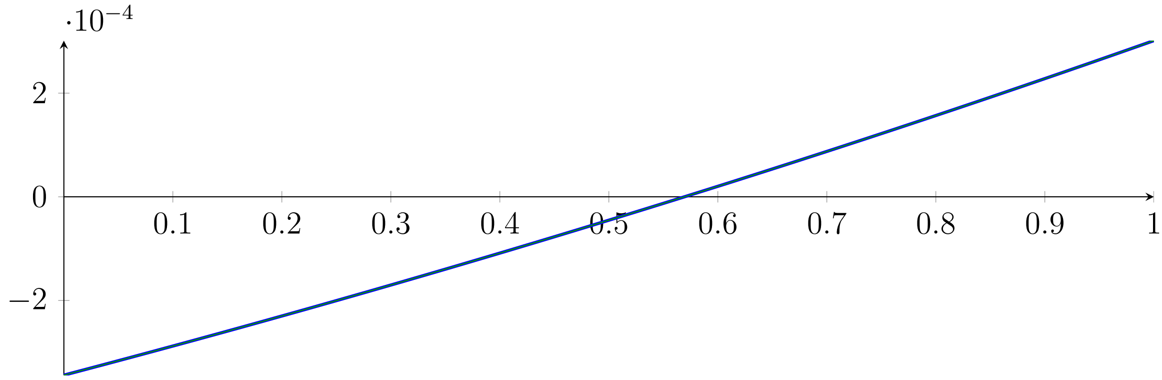
$$M = 4.46111 \cdot 10^{-06} X^3 + 8.82342 \cdot 10^{-05} X^2 + 0.000552675 X - 0.000344087$$

$$m = 4.46111 \cdot 10^{-06} X^3 + 8.82342 \cdot 10^{-05} X^2 + 0.000552675 X - 0.000344094$$

Root of M and m:

$$N(M) = \{0.569344\} \qquad N(m) = \{0.569355\}$$

Intersection intervals:



$$[0.569344, 0.569355]$$

Longest intersection interval: $1.0933 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.933013, 0.933013]$,

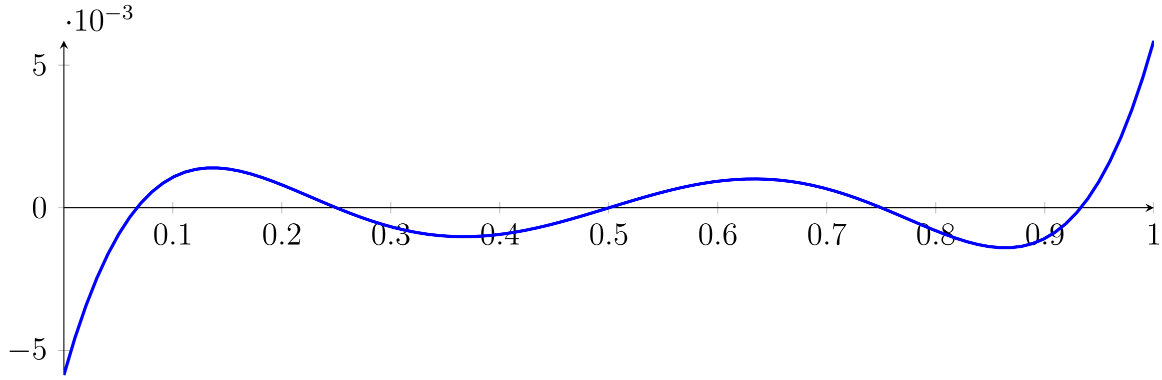
9.23 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.933013, 0.933013]$

Found root in interval $[0.933013, 0.933013]$ at recursion depth 5!

9.24 Result: 8 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^5 - 2.5X^4 + 2.25X^3 - 0.875X^2 + 0.136719X - 0.00585938$$



Result: Root Intervals

$$[0.0669872, 0.0669874],\ [0.249999, 0.25],\ [0.25, 0.25],\ [0.5, 0.5],\ [0.5, 0.5],\ [0.75, 0.75],\ [0.75, 0.750001],\ [0.933013, 0.933013]$$

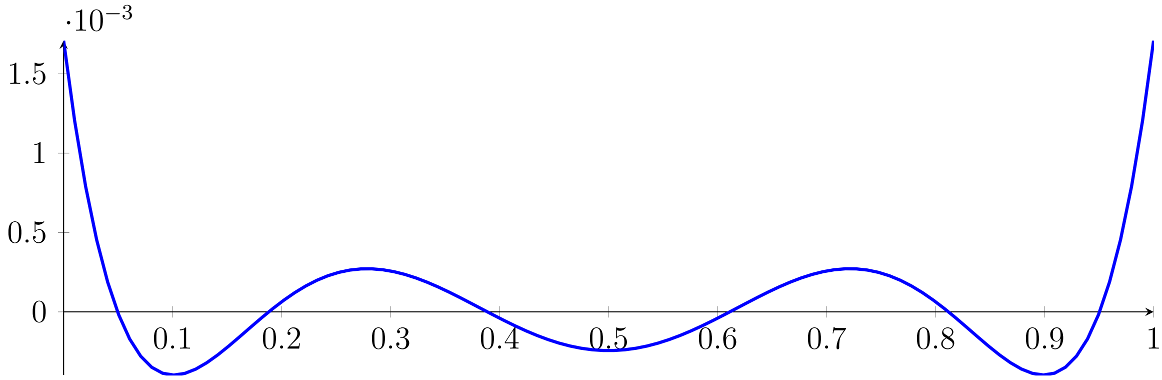
with precision $\varepsilon = 1 \cdot 10^{-06}$.

10 Running `BezClip` on p6 with epsilon 6

$$1X^6-3X^5+3.4375X^4-1.875X^3+0.492188X^2-0.0546875X+0.00170898$$

Called `BezClip` with input polynomial on interval $[0,1]$:

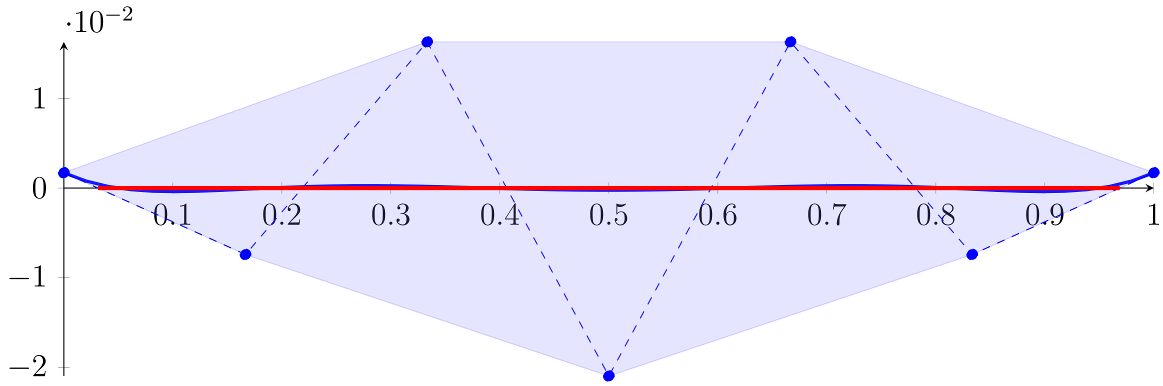
$$p = 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$



10.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898 \\ &= 0.00170898B_{0,6}(X) - 0.0074056B_{1,6}(X) + 0.0162923B_{2,6}(X) - 0.0209473B_{3,6}(X) \\ &\quad + 0.0162923B_{4,6}(X) - 0.0074056B_{5,6}(X) + 0.00170898B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.03125, 0.96875\}$$

Intersection intervals with the x axis:

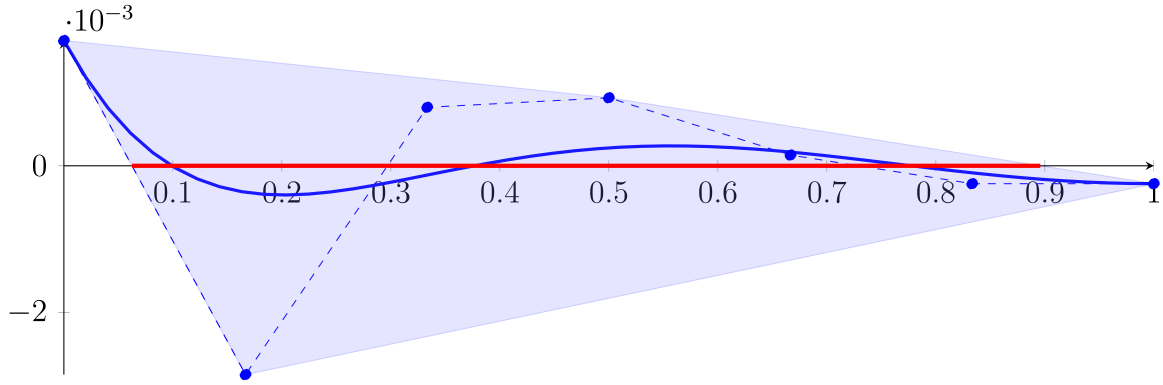
$$[0.03125, 0.96875]$$

Longest intersection interval: 0.9375
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

10.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 0.015625X^6 - 0.09375X^5 + 0.214844X^4 - 0.234375X^3 + 0.123047X^2 - 0.0273438X + 0.00170898\\ &= 0.00170898B_{0,6}(X) - 0.00284831B_{1,6}(X) + 0.000797526B_{2,6}(X) + 0.000927734B_{3,6}(X)\\ &\quad + 0.000146484B_{4,6}(X) - 0.000244141B_{5,6}(X) - 0.000244141B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0625, 0.895833\}$$

Intersection intervals with the x axis:

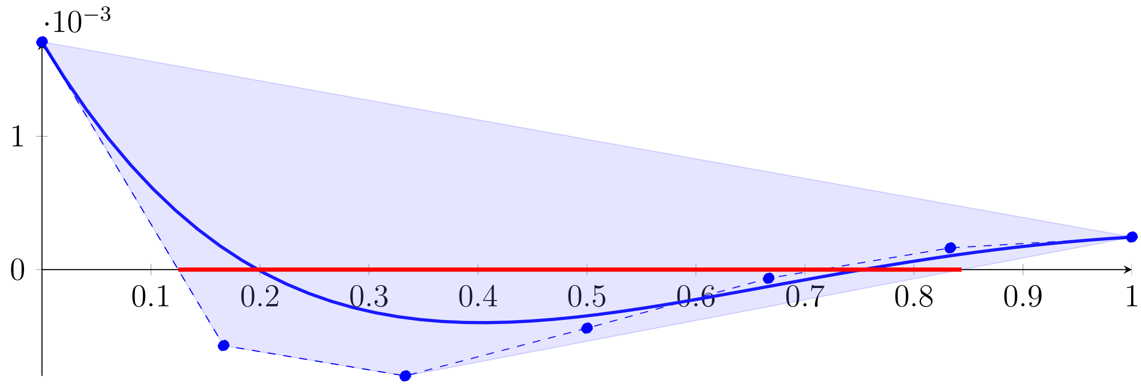
$$[0.0625, 0.895833]$$

Longest intersection interval: 0.833333
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

10.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 0.000244141X^6 - 0.00292969X^5 + 0.0134277X^4\\ &\quad - 0.0292969X^3 + 0.0307617X^2 - 0.0136719X + 0.00170898\\ &= 0.00170898B_{0,6}(X) - 0.000569661B_{1,6}(X) - 0.000797526B_{2,6}(X) - 0.000439453B_{3,6}(X)\\ &\quad - 6.51042\cdot 10^{-05}B_{4,6}(X) + 0.00016276B_{5,6}(X) + 0.000244141B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.125, 0.84375\}$$

Intersection intervals with the x axis:

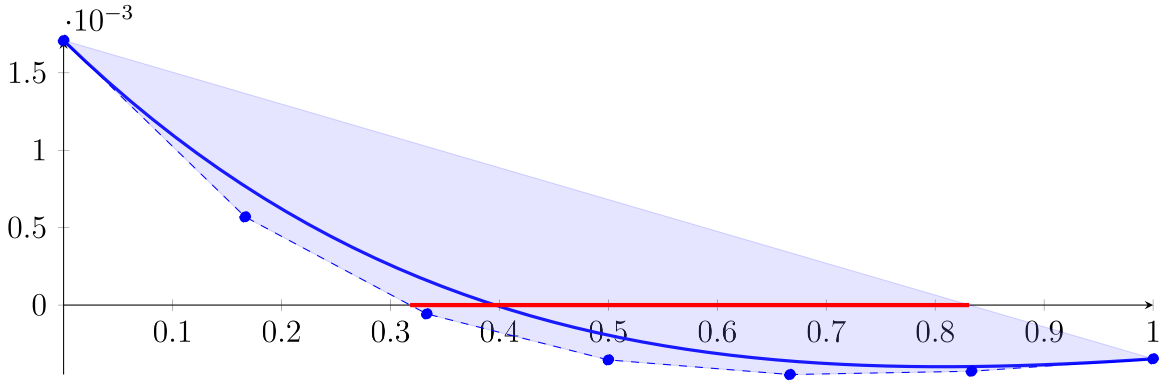
$$[0.125, 0.84375]$$

Longest intersection interval: 0.71875
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$

10.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.8147 \cdot 10^{-06} X^6 - 9.15527 \cdot 10^{-05} X^5 + 0.000839233 X^4 \\ &\quad - 0.00366211 X^3 + 0.00769043 X^2 - 0.00683594 X + 0.00170898 \\ &= 0.00170898 B_{0,6}(X) + 0.000569661 B_{1,6}(X) - 5.69661 \cdot 10^{-05} B_{2,6}(X) - 0.000354004 B_{3,6}(X) \\ &\quad - 0.000448608 B_{4,6}(X) - 0.000427246 B_{5,6}(X) - 0.000347137 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.318182, 0.831169\}$$

Intersection intervals with the x axis:

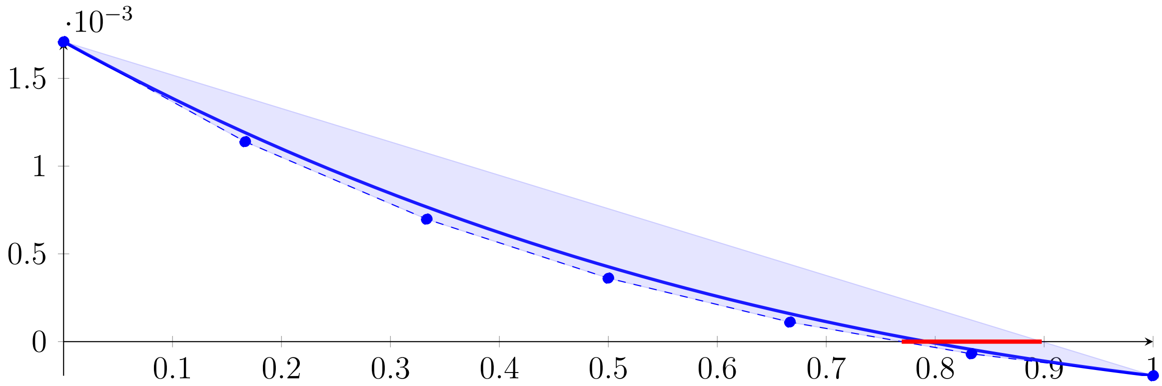
$$[0.318182, 0.831169]$$

Longest intersection interval: 0.512987
$\Longrightarrow$ Bisection: first half $[0, 0.0625]$ und second half $[0.0625, 0.125]$

10.5 Recursion Branch 1 1 1 1 1 on the First Half $[0, 0.0625]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 5.96046 \cdot 10^{-08} X^6 - 2.86102 \cdot 10^{-06} X^5 + 5.24521 \cdot 10^{-05} X^4 \\ &\quad - 0.000457764 X^3 + 0.00192261 X^2 - 0.00341797 X + 0.00170898 \\ &= 0.00170898 B_{0,6}(X) + 0.00113932 B_{1,6}(X) + 0.000697835 B_{2,6}(X) + 0.000361633 B_{3,6}(X) \\ &\quad + 0.000111326 B_{4,6}(X) - 6.94593 \cdot 10^{-05} B_{5,6}(X) - 0.00019449 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.769298, 0.897824\}$$

Intersection intervals with the x axis:

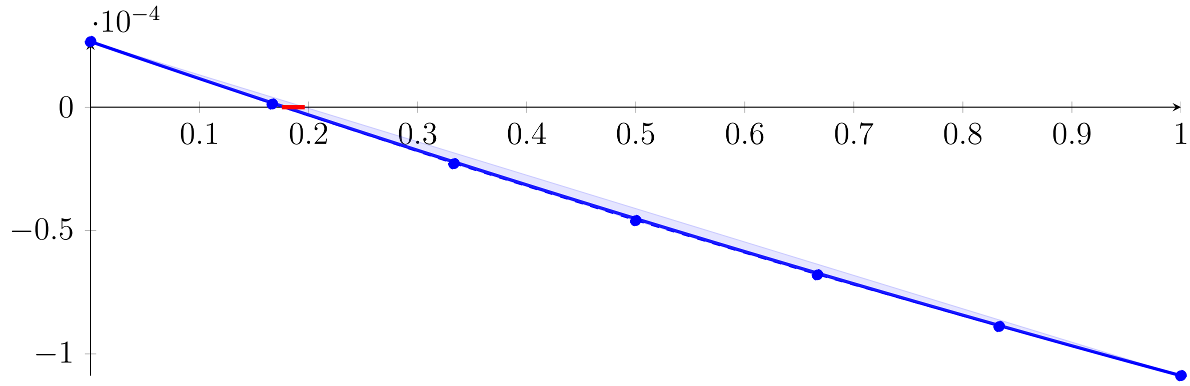
$$[0.769298, 0.897824]$$

Longest intersection interval: 0.128525
$\Longrightarrow$ Selective recursion: interval 1: $[0.0480811, 0.056114]$,

10.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0480811, 0.056114]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 2.68666 \cdot 10^{-13} X^6 - 9.06891 \cdot 10^{-11} X^5 + 1.1454 \cdot 10^{-08} X^4 - 6.63988 \\ &\quad \cdot 10^{-07} X^3 + 1.71742 \cdot 10^{-05} X^2 - 0.000151915 X + 2.65834 \cdot 10^{-05} \\ &= 2.65834 \cdot 10^{-05} B_{0,6}(X) + 1.2643 \cdot 10^{-06} B_{1,6}(X) - 2.29099 \cdot 10^{-05} B_{2,6}(X) - 4.59723 \\ &\quad \cdot 10^{-05} B_{3,6}(X) - 6.79555 \cdot 10^{-05} B_{4,6}(X) - 8.8891 \cdot 10^{-05} B_{5,6}(X) - 0.00010881 B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.175383, 0.196342\}$$

Intersection intervals with the x axis:

$$[0.175383, 0.196342]$$

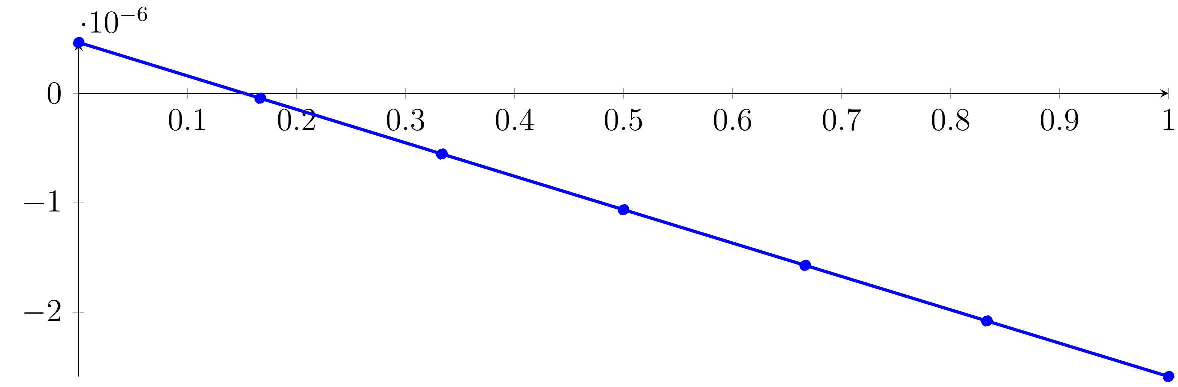
Longest intersection interval: 0.0209591

$\Longrightarrow$ Selective recursion: interval 1: $[0.04949, 0.0496583]$,

10.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.04949, 0.0496583]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.90584 \cdot 10^{-23} X^6 - 3.65673 \cdot 10^{-19} X^5 + 2.19498 \cdot 10^{-15} X^4 - 6.03963 \\ &\quad \cdot 10^{-12} X^3 + 7.39181 \cdot 10^{-09} X^2 - 3.05902 \cdot 10^{-06} X + 4.64814 \cdot 10^{-07} \\ &= 4.64814 \cdot 10^{-07} B_{0,6}(X) - 4.50225 \cdot 10^{-08} B_{1,6}(X) - 5.54366 \cdot 10^{-07} B_{2,6}(X) - 1.06322 \\ &\quad \cdot 10^{-06} B_{3,6}(X) - 1.57158 \cdot 10^{-06} B_{4,6}(X) - 2.07944 \cdot 10^{-06} B_{5,6}(X) - 2.58682 \cdot 10^{-06} B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.151949, 0.152316\}$$

Intersection intervals with the x axis:

$$[0.151949, 0.152316]$$

Longest intersection interval: 0.000367757

$\Longrightarrow$ Selective recursion: interval 1: $[0.0495156, 0.0495156]$,

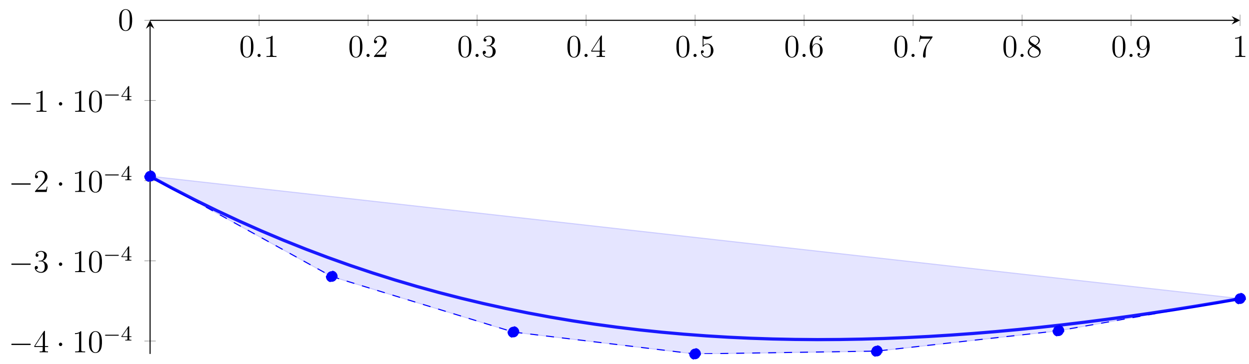
10.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.0495156, 0.0495156]$

Found root in interval $[0.0495156, 0.0495156]$ at recursion depth 8!

10.9 Recursion Branch 1 1 1 1 2 on the Second Half $[0.0625, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 5.96046 \cdot 10^{-08} X^6 - 2.5034 \cdot 10^{-06} X^5 + 3.9041 \cdot 10^{-05} X^4 \\ & - 0.000275373 X^3 + 0.000836313 X^2 - 0.000750184 X - 0.00019449 \\ = {} & -0.00019449 B_{0,6}(X) - 0.000319521 B_{1,6}(X) - 0.000388797 B_{2,6}(X) - 0.000416088 B_{3,6}(X) \\ & - 0.00041256 B_{4,6}(X) - 0.000387192 B_{5,6}(X) - 0.000347137 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

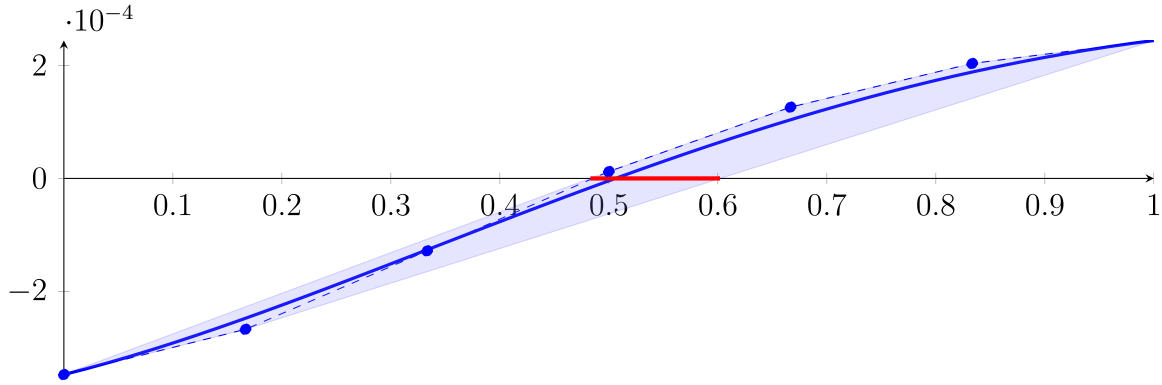
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

10.10 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 3.8147 \cdot 10^{-06} X^6 - 6.86646 \cdot 10^{-05} X^5 + 0.00043869 X^4 \\ & - 0.00114441 X^3 + 0.000881195 X^2 + 0.000480652 X - 0.000347137 \\ = {} & -0.000347137 B_{0,6}(X) - 0.000267029 B_{1,6}(X) - 0.000128174 B_{2,6}(X) + 1.2207 \\ & \cdot 10^{-05} B_{3,6}(X) + 0.000126139 B_{4,6}(X) + 0.000203451 B_{5,6}(X) + 0.000244141 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.483015, 0.60199\}$$

Intersection intervals with the x axis:

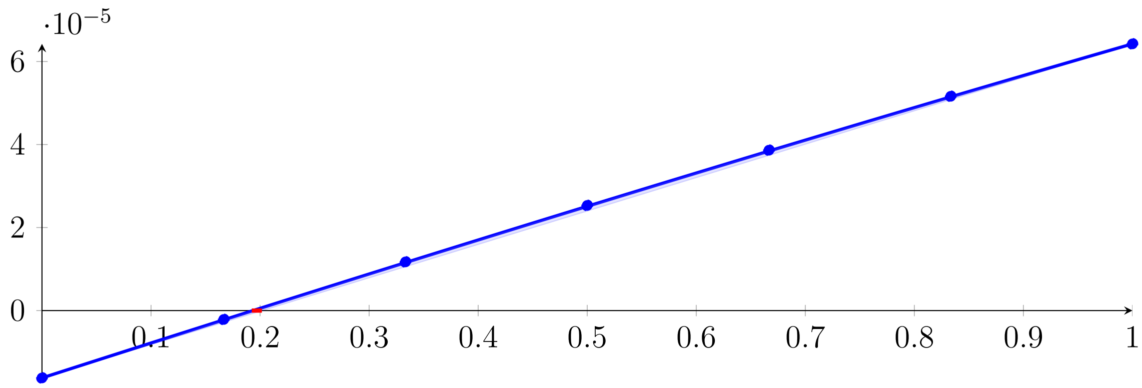
$$[0.483015, 0.60199]$$

Longest intersection interval: 0.118975
$\Longrightarrow$ Selective recursion: interval 1: $[0.185377, 0.200249]$,

10.11 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.185377, 0.200249]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.08193 \cdot 10^{-11} X^6 - 1.37333 \cdot 10^{-09} X^5 + 5.73469 \cdot 10^{-08} X^4 - 7.55208 \\ &\quad \cdot 10^{-07} X^3 - 3.35873 \cdot 10^{-06} X^2 + 8.45421 \cdot 10^{-05} X - 1.62307 \cdot 10^{-05} \\ &= -1.62307 \cdot 10^{-05} B_{0,6}(X) - 2.14035 \cdot 10^{-06} B_{1,6}(X) + 1.17261 \cdot 10^{-05} B_{2,6}(X) + 2.53308 \\ &\quad \cdot 10^{-05} B_{3,6}(X) + 3.864 \cdot 10^{-05} B_{4,6}(X) + 5.16232 \cdot 10^{-05} B_{5,6}(X) + 6.42534 \cdot 10^{-05} B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.192393, 0.201663\}$$

Intersection intervals with the x axis:

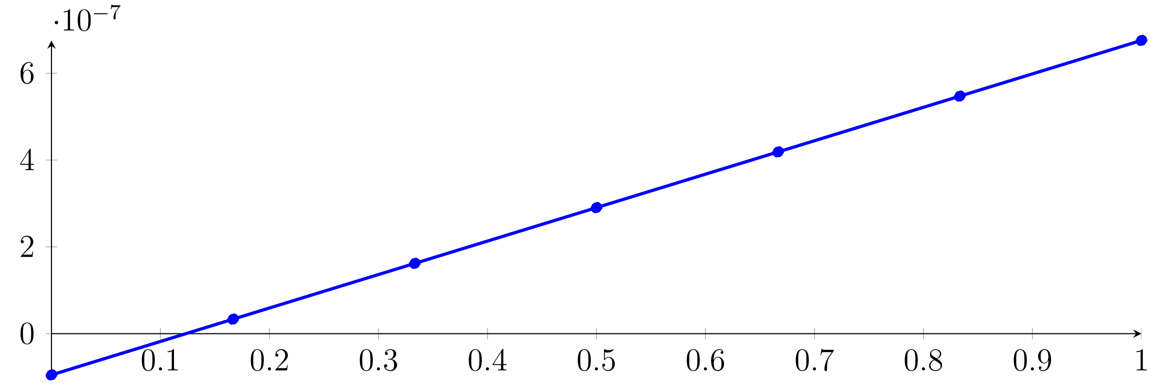
$$[0.192393, 0.201663]$$

Longest intersection interval: 0.00927086
$\Longrightarrow$ Selective recursion: interval 1: $[0.188238, 0.188376]$,

10.12 Recursion Branch 1 1 1 2 1 1 in Interval 1: $[0.188238, 0.188376]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.02438 \cdot 10^{-24} X^6 - 9.31939 \cdot 10^{-20} X^5 + 4.13918 \cdot 10^{-16} X^4 - 5.67002 \\ &\quad \cdot 10^{-13} X^3 - 3.25056 \cdot 10^{-10} X^2 + 7.71033 \cdot 10^{-07} X - 9.50603 \cdot 10^{-08} \\ &= -9.50603 \cdot 10^{-08} B_{0,6}(X) + 3.34452 \cdot 10^{-08} B_{1,6}(X) + 1.61929 \cdot 10^{-07} B_{2,6}(X) + 2.90391 \\ &\quad \cdot 10^{-07} B_{3,6}(X) + 4.18832 \cdot 10^{-07} B_{4,6}(X) + 5.47251 \cdot 10^{-07} B_{5,6}(X) + 6.75647 \cdot 10^{-07} B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.12329, 0.123342\}$$

Intersection intervals with the x axis:

$$[0.12329, 0.123342]$$

Longest intersection interval: $5.20896 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.188255, 0.188255]$,

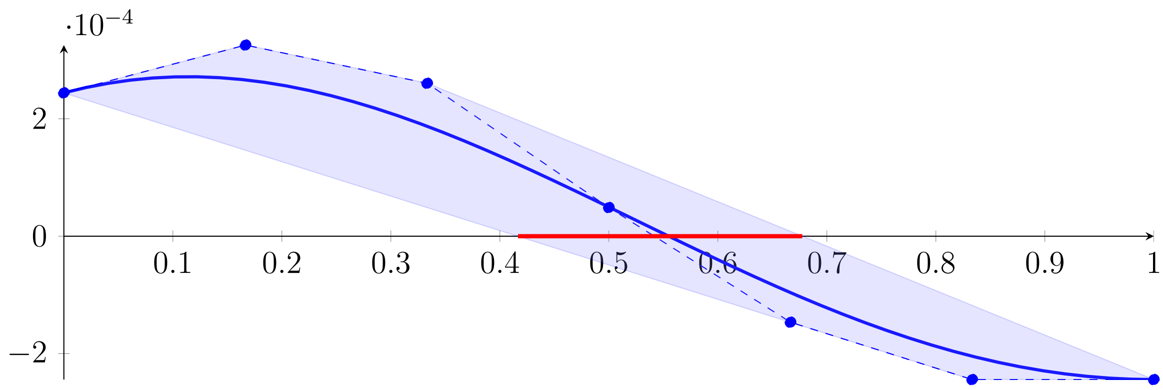
10.13 Recursion Branch 1 1 1 2 1 1 1 in Interval 1: $[0.188255, 0.188255]$

Found root in interval $[0.188255, 0.188255]$ at recursion depth 7!

10.14 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000244141X^6 - 0.00146484X^5 + 0.00244141X^4 - 3.17637 \\ &\quad \cdot 10^{-21}X^3 - 0.00219727X^2 + 0.000488281X + 0.000244141 \\ &= 0.000244141B_{0,6}(X) + 0.000325521B_{1,6}(X) + 0.000260417B_{2,6}(X) + 4.88281 \\ &\quad \cdot 10^{-05}B_{3,6}(X) - 0.000146484B_{4,6}(X) - 0.000244141B_{5,6}(X) - 0.000244141B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.416667, 0.677419\}$$

Intersection intervals with the x axis:

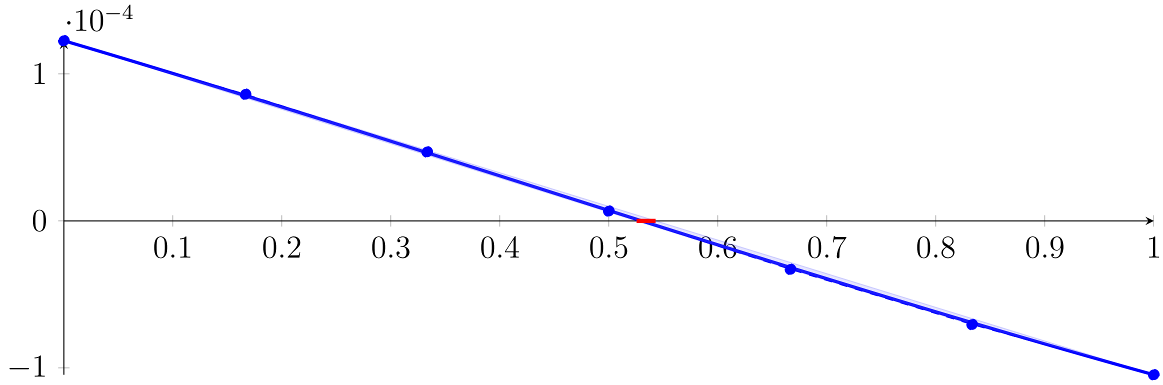
$$[0.416667, 0.677419]$$

Longest intersection interval: 0.260753
$\Longrightarrow$ Selective recursion: interval 1: $[0.354167, 0.419355]$,

10.15 Recursion Branch 1 1 2 1 in Interval 1: $[0.354167, 0.419355]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 7.67384{\cdot}10^{-08}X^6 - 1.03004{\cdot}10^{-06}X^5 + 1.17567{\cdot}10^{-07}X^4 + 3.33146 {\cdot}10^{-05}X^3 - 4.10259{\cdot}10^{-05}X^2 - 0.000218696X + 0.000122588$$
$$= 0.000122588B_{0,6}(X) + 8.61391{\cdot}10^{-05}B_{1,6}(X) + 4.69548{\cdot}10^{-05}B_{2,6}(X) + 6.70112 {\cdot}10^{-06}B_{3,6}(X) - 3.29483{\cdot}10^{-05}B_{4,6}(X) - 7.04838{\cdot}10^{-05}B_{5,6}(X) - 0.000104654B_{6,6}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.525442, 0.542899\}$$

Intersection intervals with the x axis:

$$[0.525442, 0.542899]$$

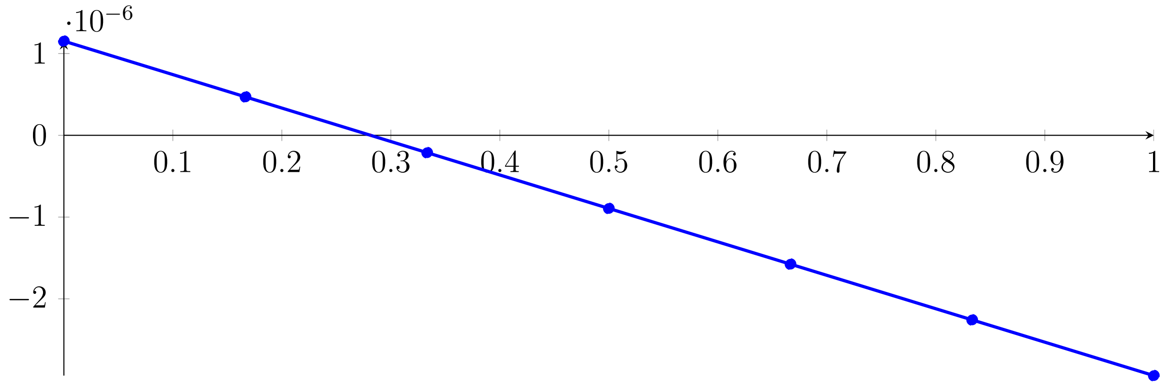
Longest intersection interval: 0.0174562

$\Longrightarrow$ Selective recursion: interval 1: $[0.388419, 0.389557]$,

10.16 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.388419, 0.389557]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 2.17127{\cdot}10^{-18}X^6 - 1.27742{\cdot}10^{-15}X^5 - 2.10848{\cdot}10^{-13}X^4 + 1.6458 {\cdot}10^{-10}X^3 + 3.1316{\cdot}10^{-09}X^2 - 4.09386{\cdot}10^{-06}X + 1.15182{\cdot}10^{-06}$$
$$= 1.15182{\cdot}10^{-06}B_{0,6}(X) + 4.69506{\cdot}10^{-07}B_{1,6}(X) - 2.12594{\cdot}10^{-07}B_{2,6}(X) - 8.94478 {\cdot}10^{-07}B_{3,6}(X) - 1.57614{\cdot}10^{-06}B_{4,6}(X) - 2.25756{\cdot}10^{-06}B_{5,6}(X) - 2.93875{\cdot}10^{-06}B_{6,6}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.281387, 0.281579\}$$

Intersection intervals with the x axis:

$$[0.281387, 0.281579]$$

Longest intersection interval: 0.000191597

$\Longrightarrow$ Selective recursion: interval 1: $[0.38874, 0.38874]$,

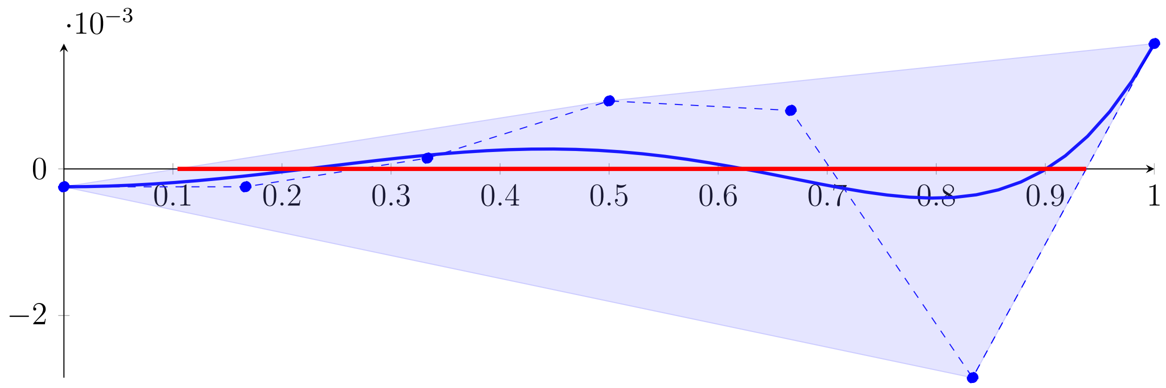
10.17 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.38874, 0.38874]$

Found root in interval $[0.38874, 0.38874]$ at recursion depth 6!

10.18 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 0.015625X^6 - 3.97047\cdot 10^{-20}X^5 - 0.0195313X^4 - 3.02814 \\ &\quad \cdot 10^{-19}X^3 + 0.00585937X^2 + 8.76679\cdot 10^{-20}X - 0.000244141 \\ &= -0.000244141B_{0,6}(X) - 0.000244141B_{1,6}(X) + 0.000146484B_{2,6}(X) \\ &\quad + 0.000927734B_{3,6}(X) + 0.000797526B_{4,6}(X) - 0.00284831B_{5,6}(X) + 0.00170898B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.104167, 0.9375\}$$

Intersection intervals with the x axis:

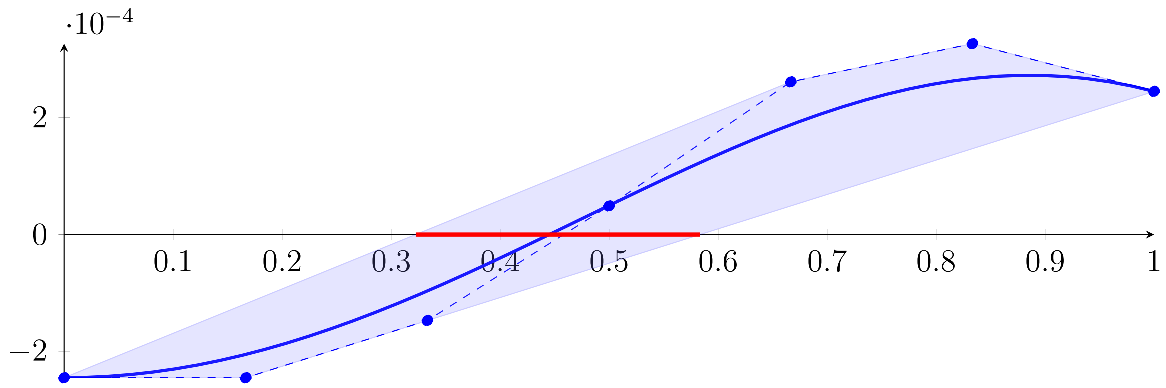
$$[0.104167, 0.9375]$$

Longest intersection interval: 0.833333
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

10.19 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 0.000244141X^6 + 6.35275\cdot 10^{-22}X^5 - 0.0012207X^4 - 3.75871 \\ &\quad \cdot 10^{-20}X^3 + 0.00146484X^2 + 4.3834\cdot 10^{-20}X - 0.000244141 \\ &= -0.000244141B_{0,6}(X) - 0.000244141B_{1,6}(X) - 0.000146484B_{2,6}(X) + 4.88281 \\ &\quad \cdot 10^{-05}B_{3,6}(X) + 0.000260417B_{4,6}(X) + 0.000325521B_{5,6}(X) + 0.000244141B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.322581, 0.583333\}$$

Intersection intervals with the x axis:

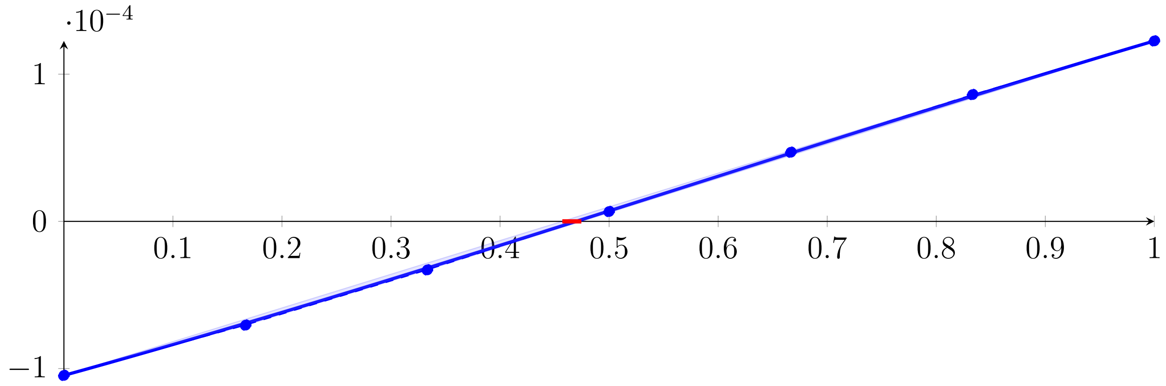
$$[0.322581, 0.583333]$$

Longest intersection interval: 0.260753
$\Longrightarrow$ Selective recursion: interval 1: $[0.580645, 0.645833]$,

10.20 Recursion Branch 1 2 1 1 in Interval 1: $[0.580645, 0.645833]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 7.67384 \cdot 10^{-08} X^6 + 5.69605 \cdot 10^{-07} X^5 - 3.88153 \cdot 10^{-06} X^4 - 2.50193 \\ &\quad \cdot 10^{-05} X^3 + 5.0474 \cdot 10^{-05} X^2 + 0.000205023 X - 0.000104654 \\ &= -0.000104654 B_{0,6}(X) - 7.04838 \cdot 10^{-05} B_{1,6}(X) - 3.29483 \cdot 10^{-05} B_{2,6}(X) + 6.70112 \\ &\quad \cdot 10^{-06} B_{3,6}(X) + 4.69548 \cdot 10^{-05} B_{4,6}(X) + 8.61391 \cdot 10^{-05} B_{5,6}(X) + 0.000122588 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.457101, 0.474558\}$$

Intersection intervals with the x axis:

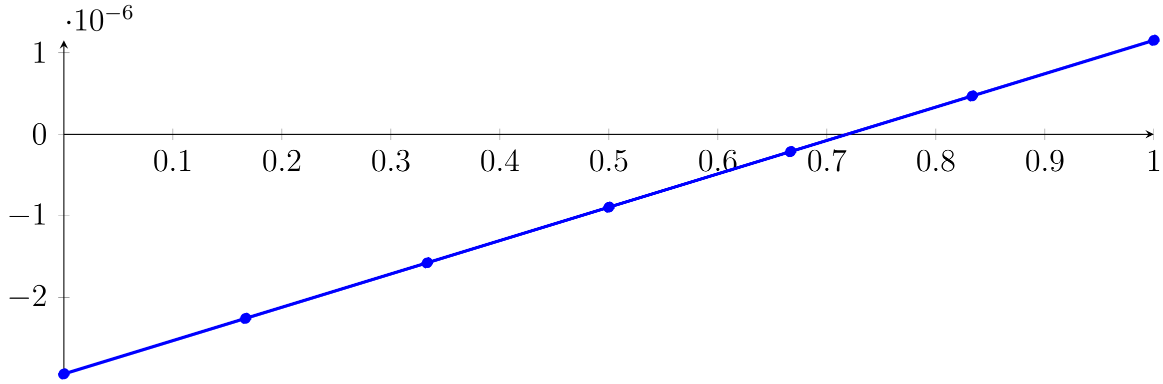
$$[0.457101, 0.474558]$$

Longest intersection interval: 0.0174562
$\Longrightarrow$ Selective recursion: interval 1: $[0.610443, 0.611581]$,

10.21 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.610443, 0.611581]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2.17127 \cdot 10^{-18} X^6 + 1.26439 \cdot 10^{-15} X^5 - 2.17202 \cdot 10^{-13} X^4 - 1.63724 \\ &\quad \cdot 10^{-10} X^3 + 3.62406 \cdot 10^{-09} X^2 + 4.0871 \cdot 10^{-06} X - 2.93875 \cdot 10^{-06} \\ &= -2.93875 \cdot 10^{-06} B_{0,6}(X) - 2.25756 \cdot 10^{-06} B_{1,6}(X) - 1.57614 \cdot 10^{-06} B_{2,6}(X) - 8.94478 \\ &\quad \cdot 10^{-07} B_{3,6}(X) - 2.12594 \cdot 10^{-07} B_{4,6}(X) + 4.69506 \cdot 10^{-07} B_{5,6}(X) + 1.15182 \cdot 10^{-06} B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.718421, 0.718613\}$$

Intersection intervals with the x axis:

$$[0.718421, 0.718613]$$

Longest intersection interval: 0.000191597
$\Longrightarrow$ Selective recursion: interval 1: $[0.61126, 0.61126]$,

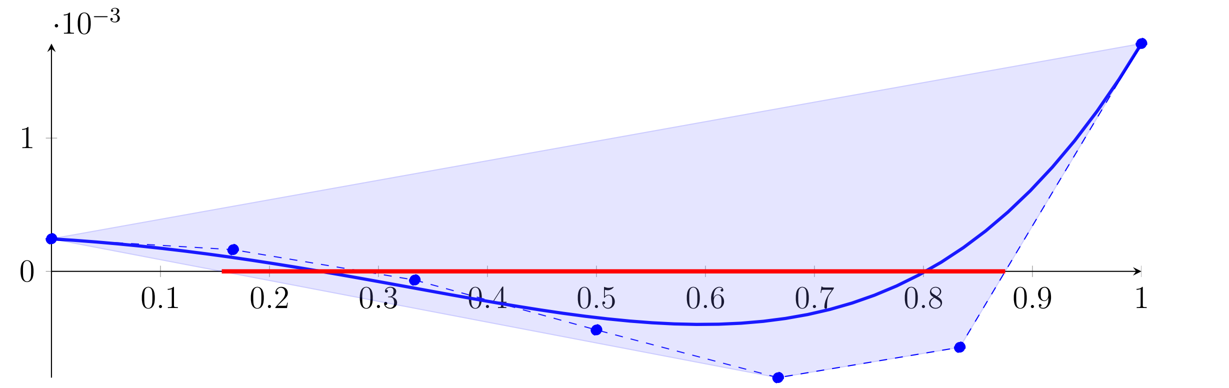
10.22 Recursion Branch 1 2 1 1 1 1 in Interval 1: $[0.61126, 0.61126]$

Found root in interval $[0.61126, 0.61126]$ at recursion depth 6!

10.23 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 0.000244141X^6 + 0.00146484X^5 + 0.00244141X^4 - 1.07997\\ &\quad \cdot 10^{-19}X^3 - 0.00219727X^2 - 0.000488281X + 0.000244141\\ &= 0.000244141B_{0,6}(X) + 0.00016276B_{1,6}(X) - 6.51042 \cdot 10^{-05}B_{2,6}(X)\\ &\quad - 0.000439453B_{3,6}(X) - 0.000797526B_{4,6}(X) - 0.000569661B_{5,6}(X) + 0.00170898B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.15625, 0.875\}$$

Intersection intervals with the x axis:

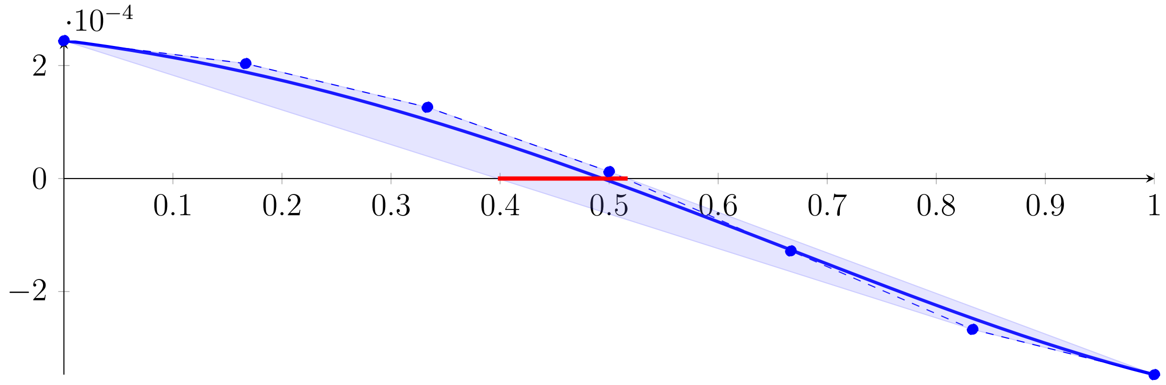
$$[0.15625, 0.875]$$

Longest intersection interval: 0.71875
$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$

10.24 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.8147{\cdot}10^{-06}X^6 + 4.57764{\cdot}10^{-05}X^5 + 0.000152588X^4 - 1.37643\\ &\quad {\cdot}10^{-20}X^3 - 0.000549316X^2 - 0.000244141X + 0.000244141\\ &= 0.000244141B_{0,6}(X) + 0.000203451B_{1,6}(X) + 0.000126139B_{2,6}(X) + 1.2207\\ &\quad {\cdot}10^{-05}B_{3,6}(X) - 0.000128174B_{4,6}(X) - 0.000267029B_{5,6}(X) - 0.000347137B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.39801, 0.516985\}$$

Intersection intervals with the x axis:

$$[0.39801, 0.516985]$$

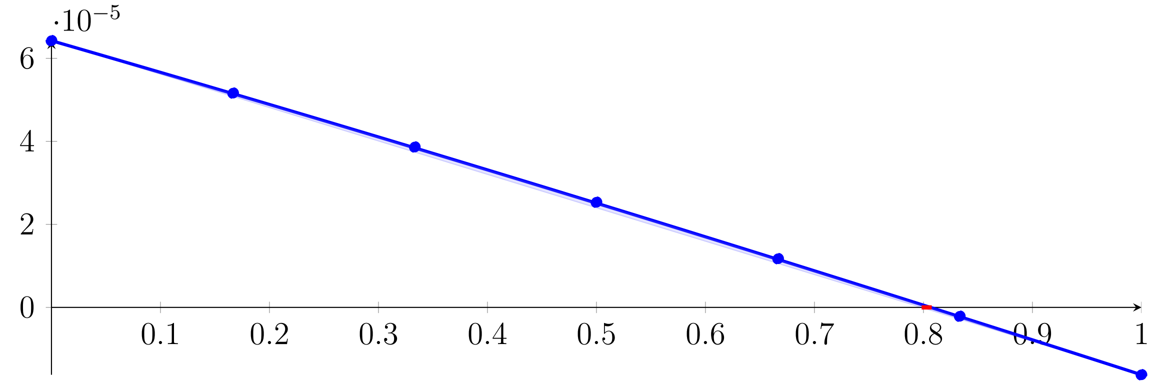
Longest intersection interval: 0.118975

$\Longrightarrow$ Selective recursion: interval 1: $[0.799751, 0.814623]$,

10.25 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.799751, 0.814623]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.08193{\cdot}10^{-11}X^6 + 1.30841{\cdot}10^{-09}X^5 + 5.06426{\cdot}10^{-08}X^4 + 5.39337\\ &\quad {\cdot}10^{-07}X^3 - 5.29384{\cdot}10^{-06}X^2 - 7.57816{\cdot}10^{-05}X + 6.42534{\cdot}10^{-05}\\ &= 6.42534{\cdot}10^{-05}B_{0,6}(X) + 5.16232{\cdot}10^{-05}B_{1,6}(X) + 3.864{\cdot}10^{-05}B_{2,6}(X) + 2.53308\\ &\quad {\cdot}10^{-05}B_{3,6}(X) + 1.17261{\cdot}10^{-05}B_{4,6}(X) - 2.14035{\cdot}10^{-06}B_{5,6}(X) - 1.62307{\cdot}10^{-05}B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.798337, 0.807607\}$$

Intersection intervals with the x axis:

$$[0.798337, 0.807607]$$

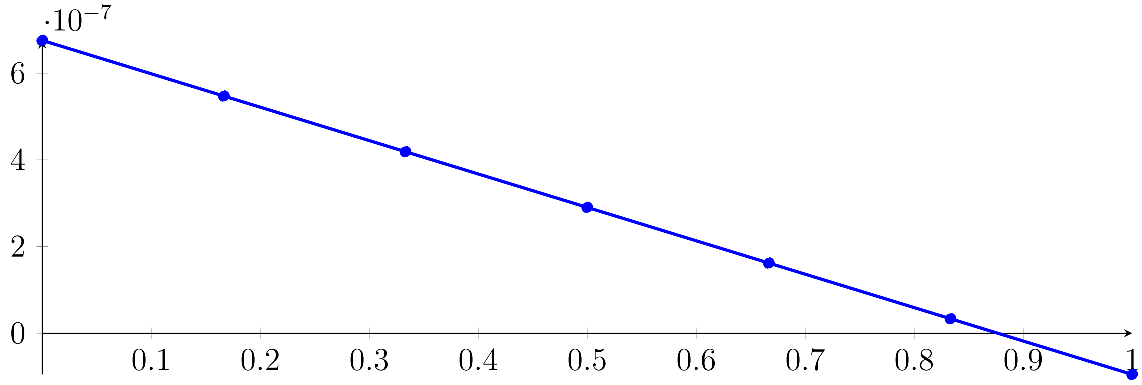
Longest intersection interval: 0.00927086

$\Longrightarrow$ Selective recursion: interval 1: $[0.811624, 0.811762]$,

10.26 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.811624, 0.811762]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.41212 \cdot 10^{-24} X^6 + 9.31577 \cdot 10^{-20} X^5 + 4.13452 \cdot 10^{-16} X^4 + 5.65347 \\ &\quad \cdot 10^{-13} X^3 - 3.26755 \cdot 10^{-10} X^2 - 7.70382 \cdot 10^{-07} X + 6.75647 \cdot 10^{-07} \\ &= 6.75647 \cdot 10^{-07} B_{0,6}(X) + 5.47251 \cdot 10^{-07} B_{1,6}(X) + 4.18832 \cdot 10^{-07} B_{2,6}(X) + 2.90391 \\ &\quad \cdot 10^{-07} B_{3,6}(X) + 1.61929 \cdot 10^{-07} B_{4,6}(X) + 3.34452 \cdot 10^{-08} B_{5,6}(X) - 9.50603 \cdot 10^{-08} B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.876658, 0.87671\}$$

Intersection intervals with the x axis:

$$[0.876658, 0.87671]$$

Longest intersection interval: $5.20896 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.811745, 0.811745]$,

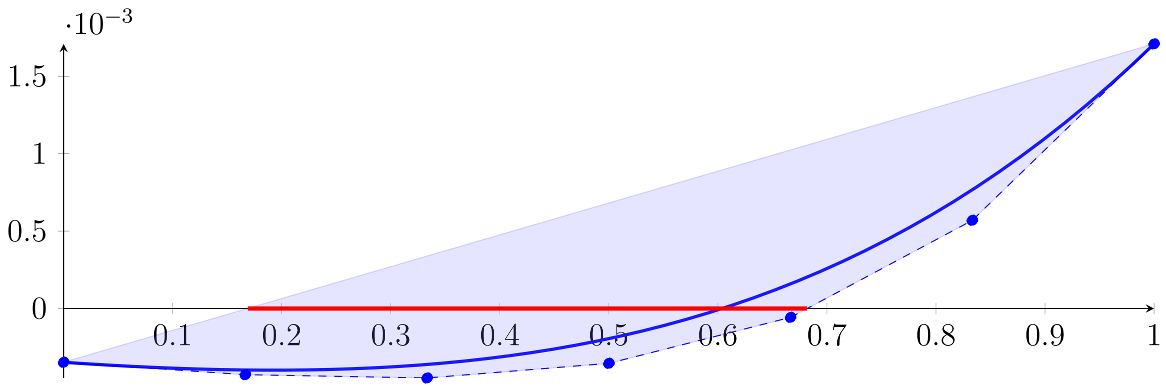
10.27 Recursion Branch 1 2 2 1 1 1 1 in Interval 1: $[0.811745, 0.811745]$

Found root in interval $[0.811745, 0.811745]$ at recursion depth 7!

10.28 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.8147 \cdot 10^{-06} X^6 + 6.86646 \cdot 10^{-05} X^5 + 0.00043869 X^4 + 0.00114441 X^3 \\ &\quad + 0.000881195 X^2 - 0.000480652 X - 0.000347137 \\ &= -0.000347137 B_{0,6}(X) - 0.000427246 B_{1,6}(X) - 0.000448608 B_{2,6}(X) \\ &\quad - 0.000354004 B_{3,6}(X) - 5.69661 \cdot 10^{-05} B_{4,6}(X) + 0.000569661 B_{5,6}(X) + 0.00170898 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.168831, 0.681818\}$$

Intersection intervals with the x axis:

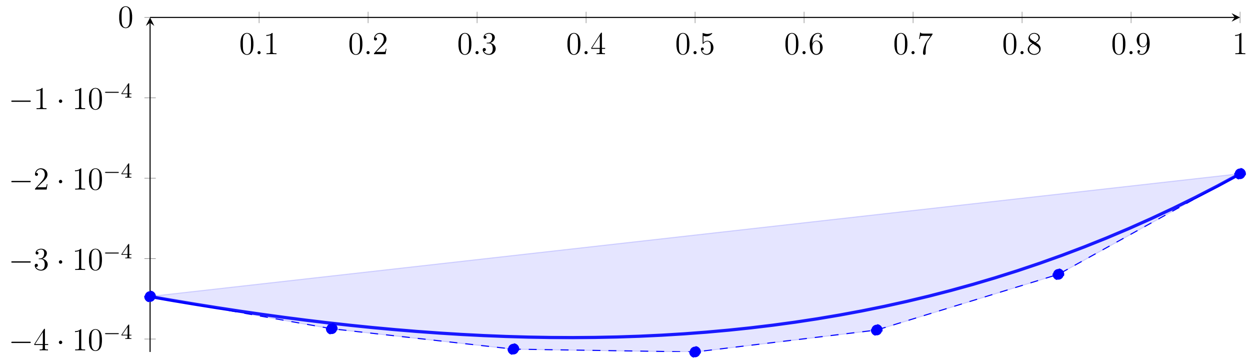
$$[0.168831, 0.681818]$$

Longest intersection interval: 0.512987
$\implies$ Bisection: first half $[0.875, 0.9375]$ und second half $[0.9375, 1]$

10.29 Recursion Branch 1 2 2 2 1 on the First Half $[0.875, 0.9375]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 5.96046 \cdot 10^{-08} X^6 + 2.14577 \cdot 10^{-06} X^5 + 2.74181 \cdot 10^{-05} X^4 \\ &\quad + 0.000143051 X^3 + 0.000220299 X^2 - 0.000240326 X - 0.000347137 \\ &= -0.000347137 B_{0,6}(X) - 0.000387192 B_{1,6}(X) - 0.00041256 B_{2,6}(X) - 0.000416088 B_{3,6}(X) \\ &\quad - 0.000388797 B_{4,6}(X) - 0.000319521 B_{5,6}(X) - 0.00019449 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

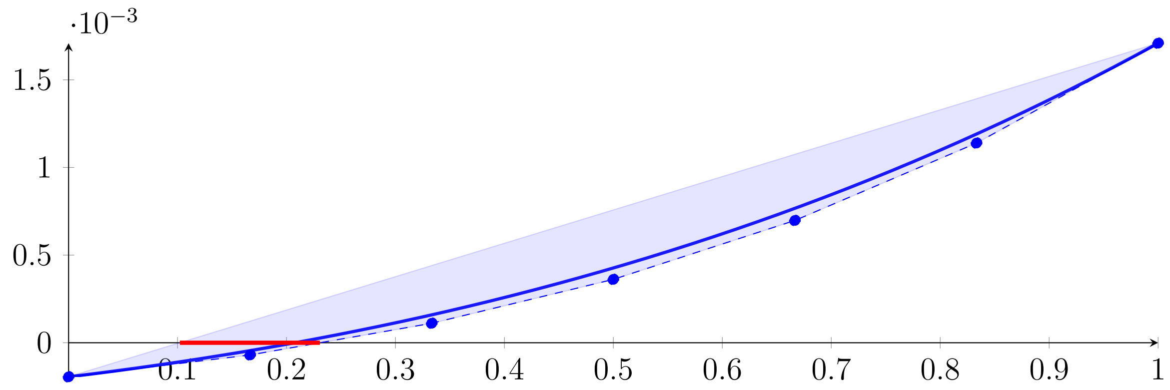
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

10.30 Recursion Branch 1 2 2 2 2 on the Second Half $[0.9375, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 5.96046 \cdot 10^{-08} X^6 + 2.5034 \cdot 10^{-06} X^5 + 3.9041 \cdot 10^{-05} X^4 \\ &\quad + 0.000275373 X^3 + 0.000836313 X^2 + 0.000750184 X - 0.00019449 \\ &= -0.00019449 B_{0,6}(X) - 6.94593 \cdot 10^{-05} B_{1,6}(X) + 0.000111326 B_{2,6}(X) \\ &\quad + 0.000361633 B_{3,6}(X) + 0.000697835 B_{4,6}(X) + 0.00113932 B_{5,6}(X) + 0.00170898 B_{6,6}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.102176, 0.230702\}$$

Intersection intervals with the x axis:

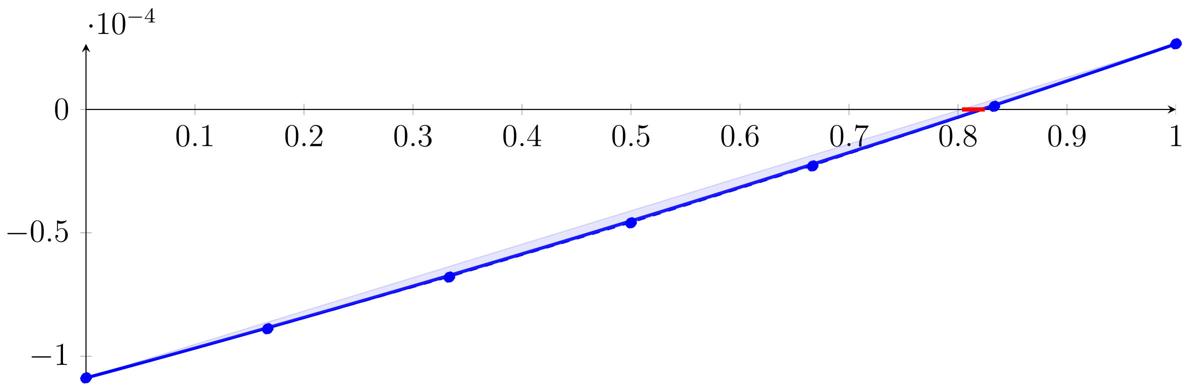
$$[0.102176, 0.230702]$$

Longest intersection interval: 0.128525
$\Longrightarrow$ Selective recursion: interval 1: $[0.943886, 0.951919]$,

10.31 Recursion Branch 1 2 2 2 2 1 in Interval 1: $[0.943886, 0.951919]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 2.68666\cdot 10^{-13}X^6 + 8.90771\cdot 10^{-11}X^5 + 1.10046\cdot 10^{-08}X^4 + 6.19073\\&\quad\cdot 10^{-07}X^3 + 1.525\cdot 10^{-05}X^2 + 0.000119513X - 0.00010881\\&= -0.00010881B_{0,6}(X) - 8.8891\cdot 10^{-05}B_{1,6}(X) - 6.79555\cdot 10^{-05}B_{2,6}(X) - 4.59723\\&\quad\cdot 10^{-05}B_{3,6}(X) - 2.29099\cdot 10^{-05}B_{4,6}(X) + 1.2643\cdot 10^{-06}B_{5,6}(X) + 2.65834\cdot 10^{-05}B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.803658, 0.824617\}$$

Intersection intervals with the x axis:

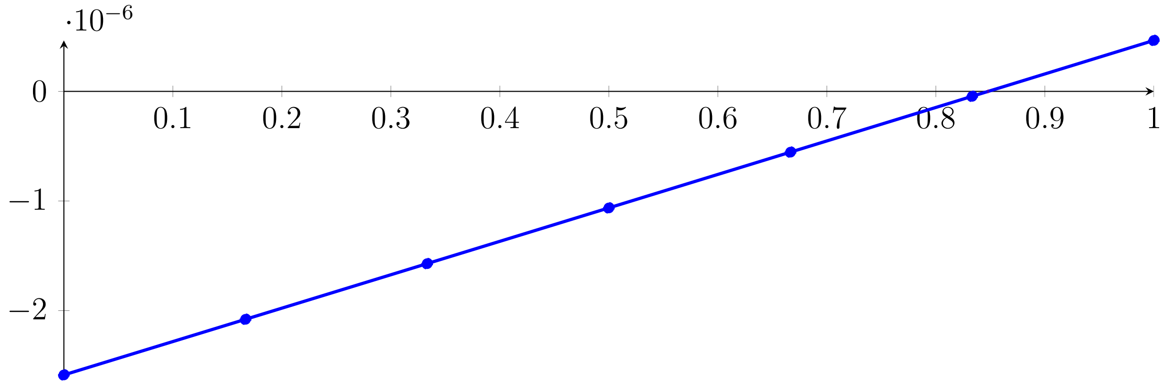
$$[0.803658, 0.824617]$$

Longest intersection interval: 0.0209591
$\Longrightarrow$ Selective recursion: interval 1: $[0.950342, 0.95051]$,

10.32 Recursion Branch 1 2 2 2 2 1 1 in Interval 1: $[0.950342, 0.95051]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.51552\cdot 10^{-23}X^6 + 3.65509\cdot 10^{-19}X^5 + 2.19315\cdot 10^{-15}X^4 + 6.03085\\&\quad\cdot 10^{-12}X^3 + 7.37371\cdot 10^{-09}X^2 + 3.04425\cdot 10^{-06}X - 2.58682\cdot 10^{-06}\\&= -2.58682\cdot 10^{-06}B_{0,6}(X) - 2.07944\cdot 10^{-06}B_{1,6}(X) - 1.57158\cdot 10^{-06}B_{2,6}(X) - 1.06322\\&\quad\cdot 10^{-06}B_{3,6}(X) - 5.54366\cdot 10^{-07}B_{4,6}(X) - 4.50225\cdot 10^{-08}B_{5,6}(X) + 4.64814\cdot 10^{-07}B_{6,6}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.847684, 0.848051\}$$

Intersection intervals with the x axis:

$$[0.847684, 0.848051]$$

Longest intersection interval: 0.000367757
$\Longrightarrow$ Selective recursion: interval 1: $[0.950484, 0.950484]$,

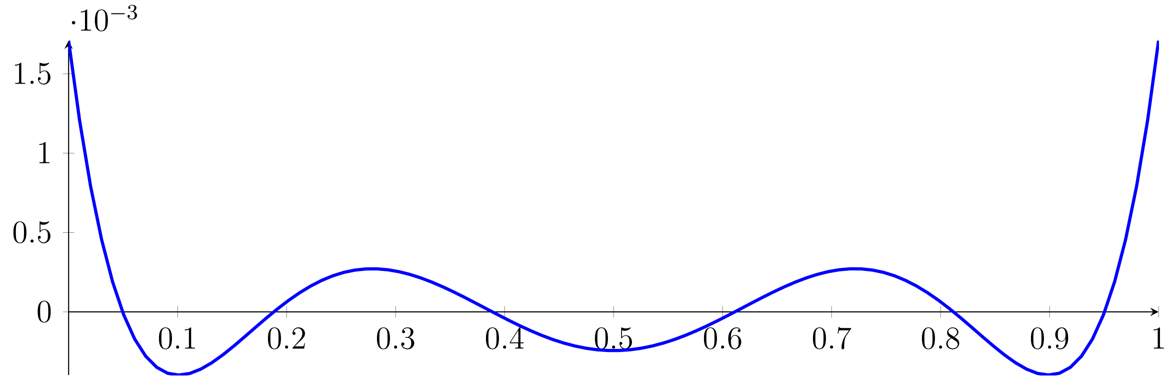
10.33 Recursion Branch 1 2 2 2 2 1 1 1 in Interval 1: $[0.950484, 0.950484]$

Found root in interval $[0.950484, 0.950484]$ at recursion depth 8!

10.34 Result: 6 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$



Result: Root Intervals

$$[0.0495156, 0.0495156], [0.188255, 0.188255], [0.38874, 0.38874], [0.61126, 0.61126], [0.811745, 0.811745], [0.950484, 0.950484]$$

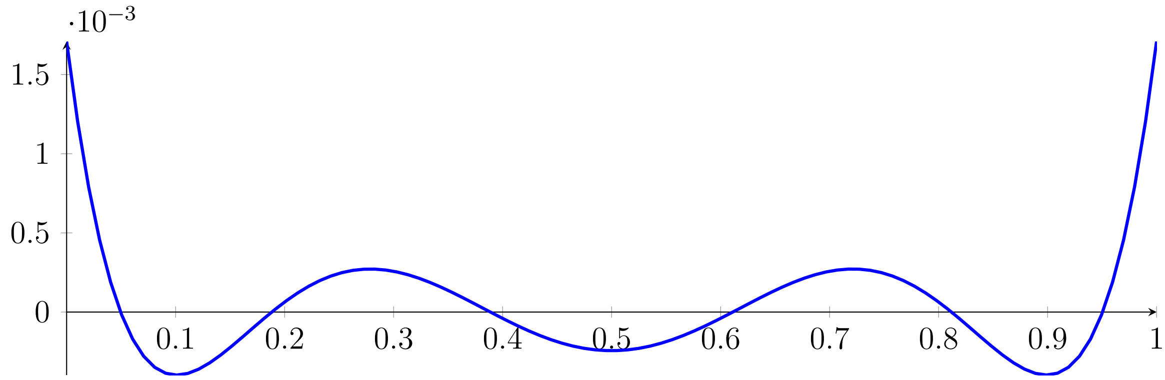
with precision $\varepsilon = 1 \cdot 10^{-06}$.

11 Running QuadClip on p6 with epsilon 6

$$1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$

Called QuadClip with input polynomial on interval $[0, 1]$:

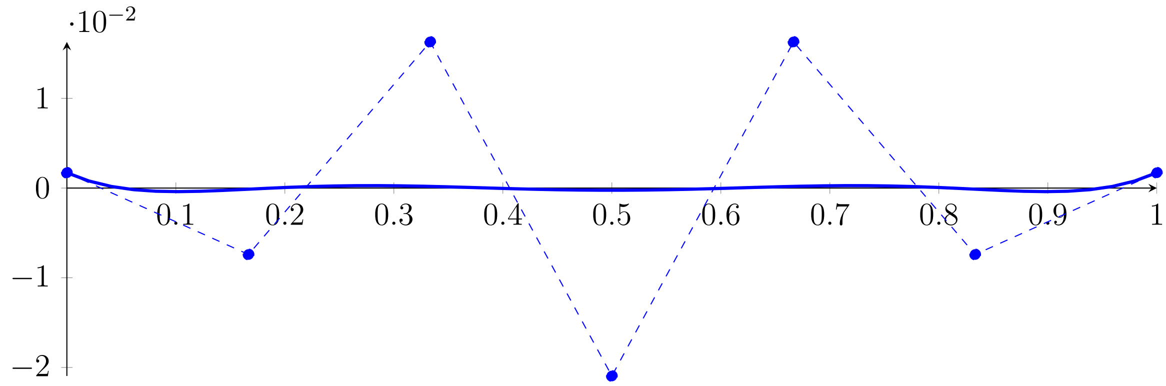
$$p = 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$



11.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

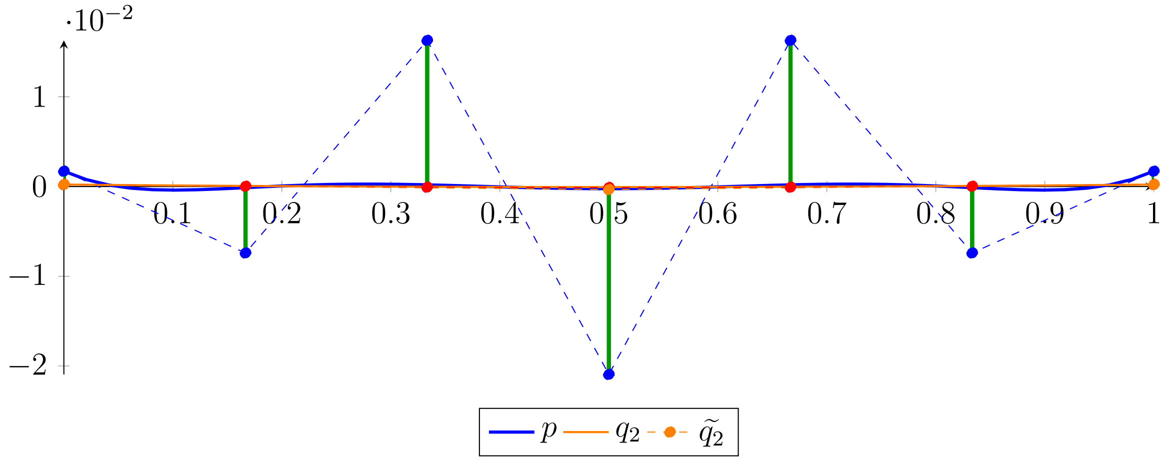
$$\begin{aligned} p &= 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898 \\ &= 0.00170898B_{0,6}(X) - 0.0074056B_{1,6}(X) + 0.0162923B_{2,6}(X) - 0.0209473B_{3,6}(X) \\ &\quad + 0.0162923B_{4,6}(X) - 0.0074056B_{5,6}(X) + 0.00170898B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00111607X^2 - 0.00111607X + 0.000220889 \\ &= 0.000220889B_{0,2} - 0.000337147B_{1,2} + 0.000220889B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 6.89948 \cdot 10^{-18}X^6 - 2.09161 \cdot 10^{-17}X^5 + 2.39427 \cdot 10^{-17}X^4 - 1.27595 \\ &\quad \cdot 10^{-17}X^3 + 0.00111607X^2 - 0.00111607X + 0.000220889 \\ &= 0.000220889B_{0,6} + 3.48772 \cdot 10^{-05}B_{1,6} - 7.67299 \cdot 10^{-05}B_{2,6} - 0.000113932B_{3,6} \\ &\quad - 7.67299 \cdot 10^{-05}B_{4,6} + 3.48772 \cdot 10^{-05}B_{5,6} + 0.000220889B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0208333$.
Bounding polynomials M and m:

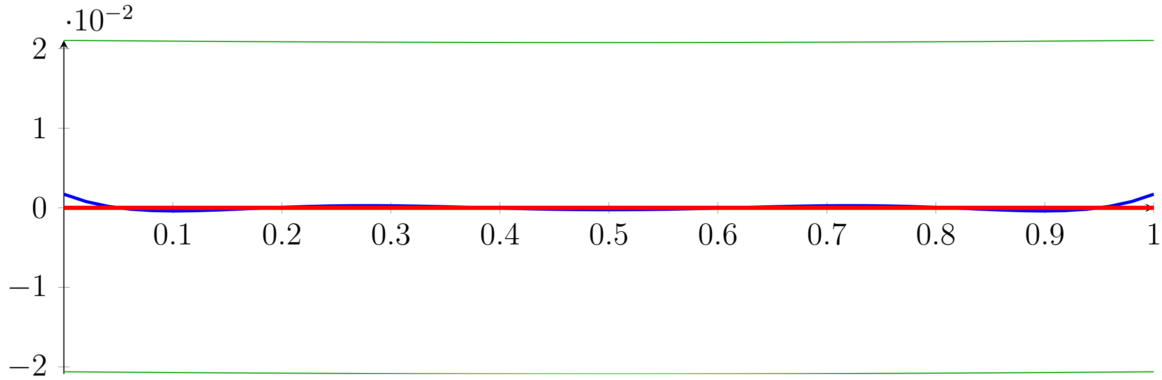
$$M = 0.00111607X^2 - 0.00111607X + 0.0210542$$

$$m = 0.00111607X^2 - 0.00111607X - 0.0206124$$

Root of M and m:

$$N(M) = \{\} \qquad\qquad N(m) = \{-3.82652, 4.82652\}$$

Intersection intervals:



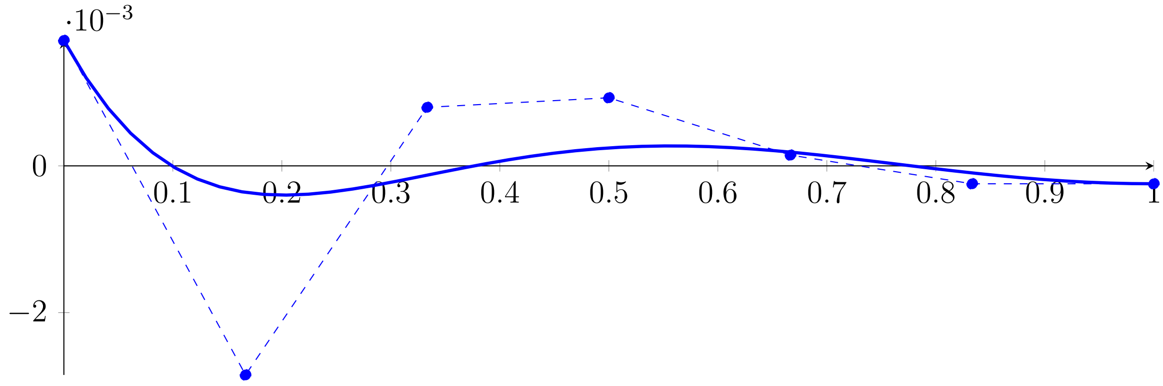
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$
Bisection point is very near to a root?!?

11.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

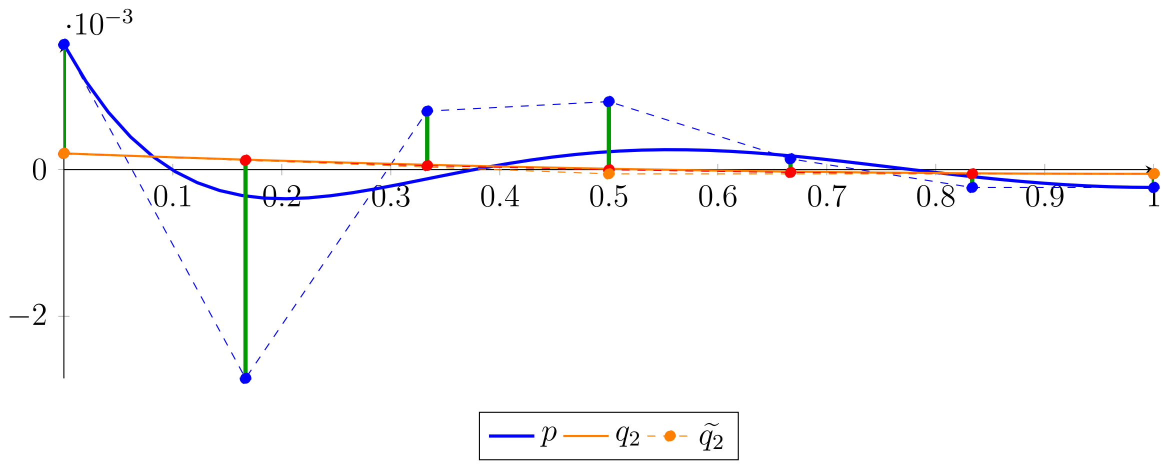
$$\begin{aligned}p &= 0.015625X^6 - 0.09375X^5 + 0.214844X^4 - 0.234375X^3 + 0.123047X^2 - 0.0273438X + 0.00170898\\ &= 0.00170898B_{0,6}(X) - 0.00284831B_{1,6}(X) + 0.000797526B_{2,6}(X) + 0.000927734B_{3,6}(X)\\ &\quad + 0.000146484B_{4,6}(X) - 0.000244141B_{5,6}(X) - 0.000244141B_{6,6}(X)\end{aligned}$$



Degree reduction and raising:

$$q_2 = 0.000279018X^2 - 0.000558036X + 0.000220889$$
$$= 0.000220889B_{0,2} - 5.81287 \cdot 10^{-05}B_{1,2} - 5.81287 \cdot 10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 7.83479 \cdot 10^{-19}X^6 - 1.9333 \cdot 10^{-18}X^5 + 1.59454 \cdot 10^{-18}X^4 - 4.18487 \cdot 10^{-19}X^3 + 0.000279018X^2 - 0.000558036X + 0.000220889$$
$$= 0.000220889B_{0,6} + 0.000127883B_{1,6} + 5.34784 \cdot 10^{-05}B_{2,6} - 2.32515 \cdot 10^{-06}B_{3,6} - 3.95275 \cdot 10^{-05}B_{4,6} - 5.81287 \cdot 10^{-05}B_{5,6} - 5.81287 \cdot 10^{-05}B_{6,6}$$



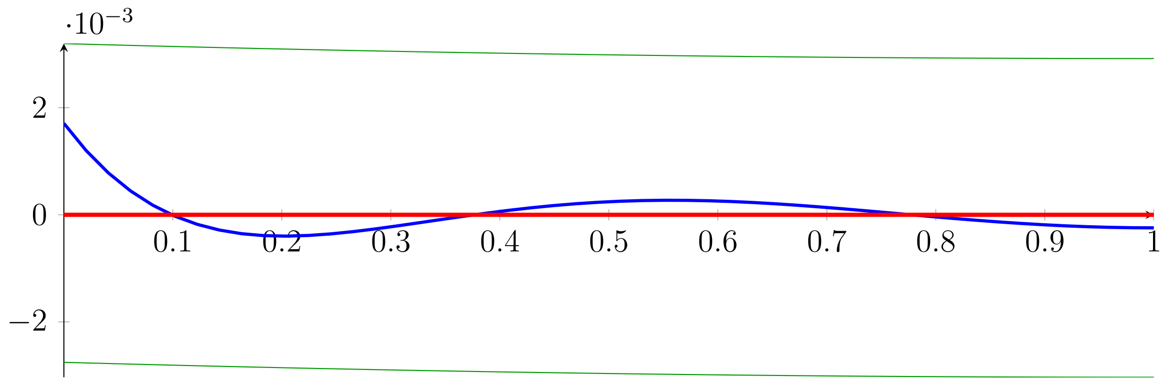
The maximum difference of the Bézier coefficients is $\delta = 0.00297619$.
Bounding polynomials M and m:

$$M = 0.000279018X^2 - 0.000558036X + 0.00319708$$
$$m = 0.000279018X^2 - 0.000558036X - 0.0027553$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-2.29773, 4.29773\}$$

Intersection intervals:



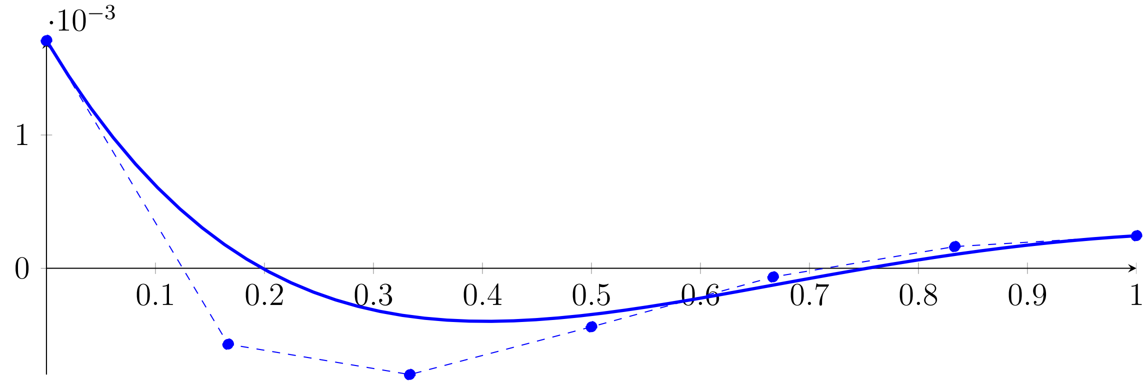
$$[0,1]$$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

11.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

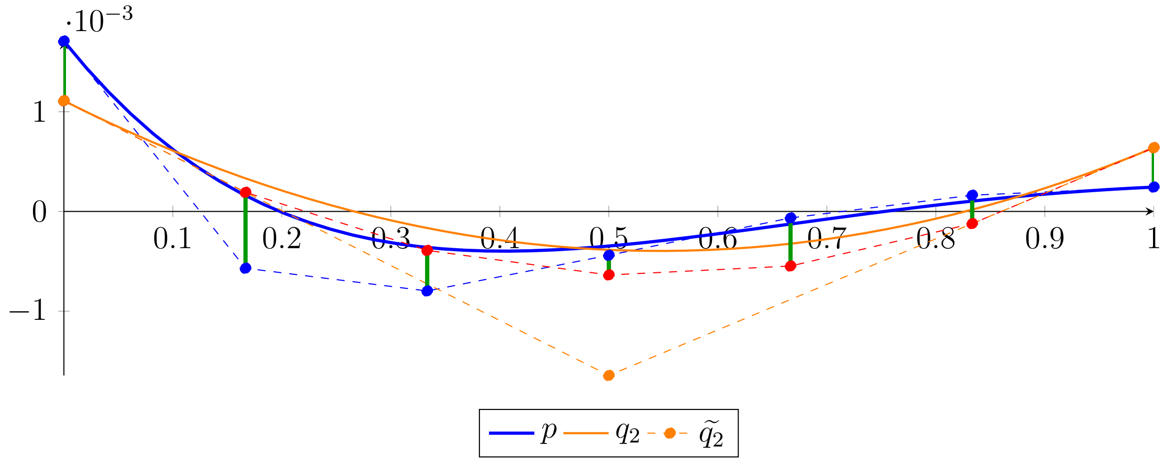
$$\begin{aligned} p &= 0.000244141X^6 - 0.00292969X^5 + 0.0134277X^4 \\ &\quad - 0.0292969X^3 + 0.0307617X^2 - 0.0136719X + 0.00170898 \\ &= 0.00170898B_{0,6}(X) - 0.000569661B_{1,6}(X) - 0.000797526B_{2,6}(X) - 0.000439453B_{3,6}(X) \\ &\quad - 6.51042\cdot 10^{-05}B_{4,6}(X) + 0.00016276B_{5,6}(X) + 0.000244141B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00503976X^2 - 0.0055106X + 0.00111026 \\ &= 0.00111026B_{0,2} - 0.00164504B_{1,2} + 0.000639416B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 3.62485\cdot 10^{-17}X^6 - 1.09647\cdot 10^{-16}X^5 + 1.24933\cdot 10^{-16}X^4 \\ &\quad - 6.60008\cdot 10^{-17}X^3 + 0.00503976X^2 - 0.0055106X + 0.00111026 \\ &= 0.00111026B_{0,6} + 0.000191825B_{1,6} - 0.000390625B_{2,6} - 0.000637091B_{3,6} \\ &\quad - 0.000547573B_{4,6} - 0.00012207B_{5,6} + 0.000639416B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000761486$.
Bounding polynomials M and m:

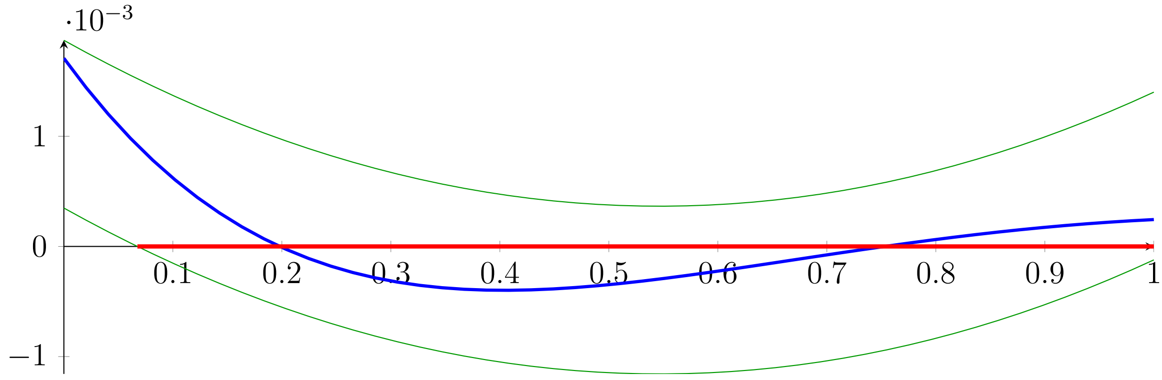
$$M = 0.00503976X^2 - 0.0055106X + 0.00187174$$

$$m = 0.00503976X^2 - 0.0055106X + 0.000348772$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{0.0674522, 1.02597\}$$

Intersection intervals:



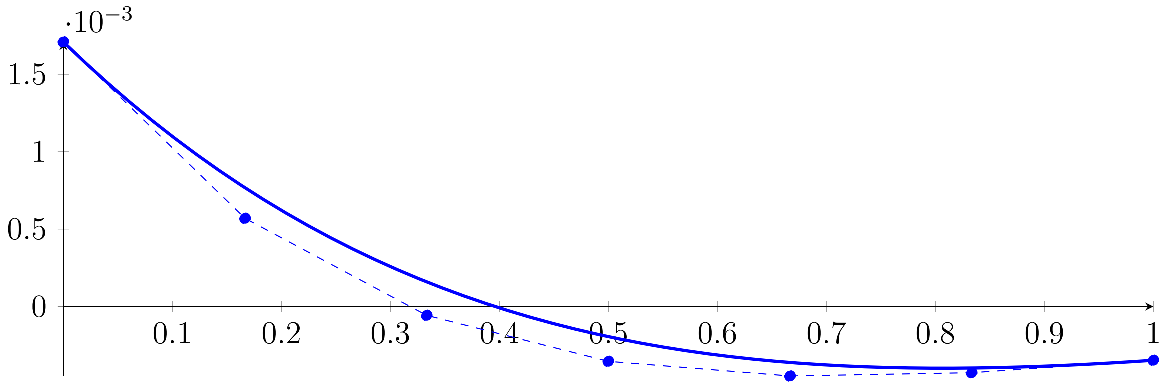
$$[0.0674522, 1]$$

Longest intersection interval: 0.932548
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$
Bisection point is very near to a root?!?

11.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

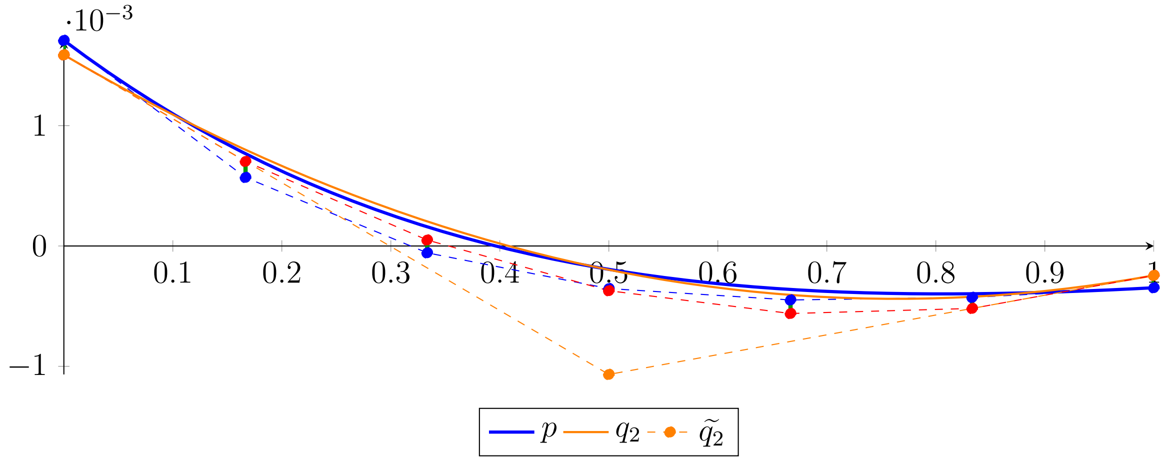
$$\begin{aligned} p &= 3.8147 \cdot 10^{-06} X^6 - 9.15527 \cdot 10^{-05} X^5 + 0.000839233 X^4 \\ &\quad - 0.00366211 X^3 + 0.00769043 X^2 - 0.00683594 X + 0.00170898 \\ &= 0.00170898 B_{0,6}(X) + 0.000569661 B_{1,6}(X) - 5.69661 \cdot 10^{-05} B_{2,6}(X) - 0.000354004 B_{3,6}(X) \\ &\quad - 0.000448608 B_{4,6}(X) - 0.000427246 B_{5,6}(X) - 0.000347137 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00347928 X^2 - 0.00531224 X + 0.00158846 \\ &= 0.00158846 B_{0,2} - 0.00106766 B_{1,2} - 0.000244504 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.28758 \cdot 10^{-17} X^6 - 6.70145 \cdot 10^{-17} X^5 + 7.31773 \cdot 10^{-17} X^4 \\ &\quad - 3.63314 \cdot 10^{-17} X^3 + 0.00347928 X^2 - 0.00531224 X + 0.00158846 \\ &= 0.00158846 B_{0,6} + 0.000703085 B_{1,6} + 4.96637 \cdot 10^{-05} B_{2,6} - 0.000371806 B_{3,6} \\ &\quad - 0.000561324 B_{4,6} - 0.00051889 B_{5,6} - 0.000244504 B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000133424$.
Bounding polynomials M and m:

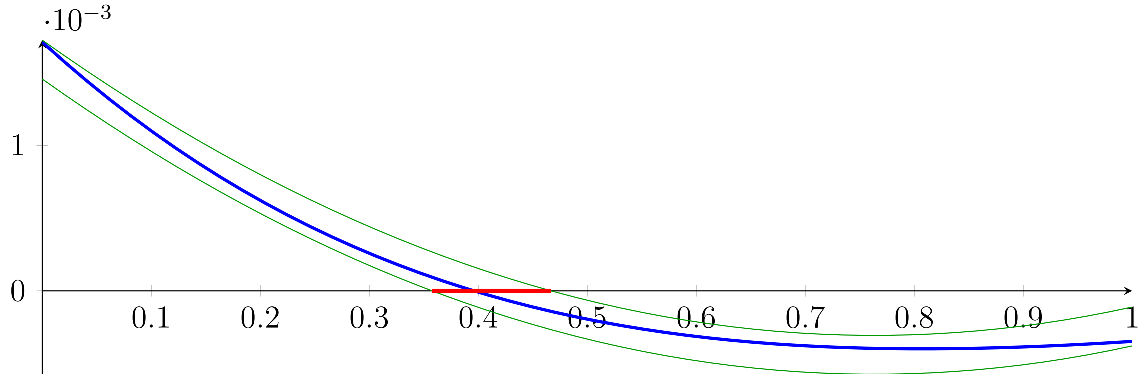
$$M = 0.00347928X^2 - 0.00531224X + 0.00172188$$

$$m = 0.00347928X^2 - 0.00531224X + 0.00145503$$

Root of M and m:

$$N(M) = \{0.466931, 1.05989\} \qquad N(m) = \{0.357706, 1.16912\}$$

Intersection intervals:



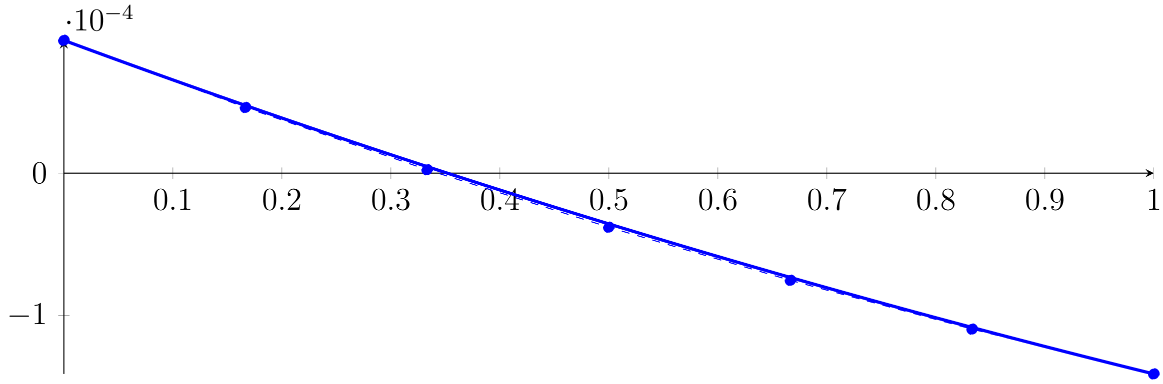
$$[0.357706, 0.466931]$$

Longest intersection interval: 0.109225
$\Longrightarrow$ Selective recursion: interval 1: $[0.0447133, 0.0583664]$,

11.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0447133, 0.0583664]$

Normalized monomial und Bézier representations and the Bézier polygon:

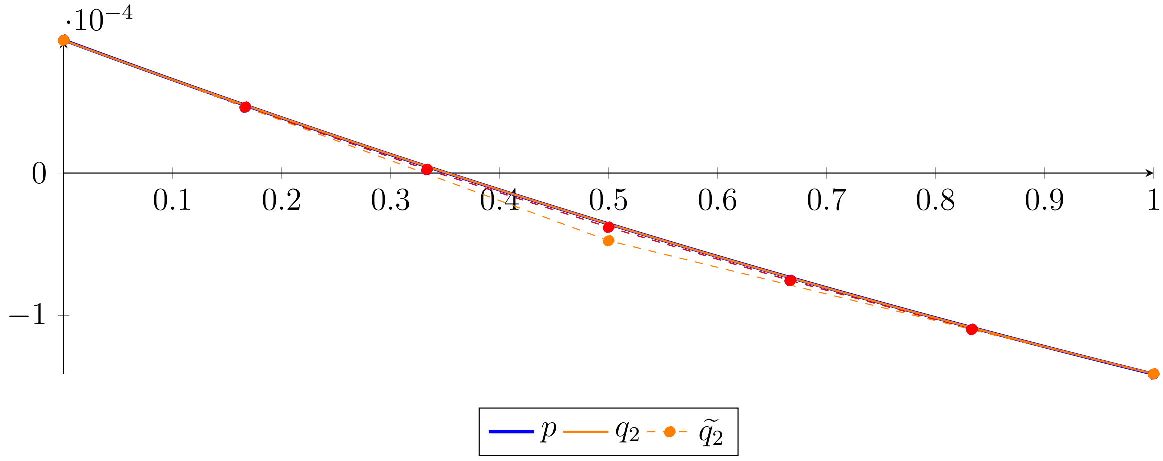
$$\begin{aligned} p &= 6.47727 \cdot 10^{-12} X^6 - 1.29597 \cdot 10^{-09} X^5 + 9.71824 \cdot 10^{-08} X^4 - 3.35535 \\ &\quad \cdot 10^{-06} X^3 + 5.20615 \cdot 10^{-05} X^2 - 0.000283282X + 9.33437 \cdot 10^{-05} \\ &= 9.33437 \cdot 10^{-05} B_{0,6}(X) + 4.61301 \cdot 10^{-05} B_{1,6}(X) + 2.38719 \cdot 10^{-06} B_{2,6}(X) - 3.80527 \\ &\quad \cdot 10^{-05} B_{3,6}(X) - 7.53509 \cdot 10^{-05} B_{4,6}(X) - 0.000109662 B_{5,6}(X) - 0.000141136 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 4.71928{\cdot}10^{-05}X^2 - 0.000281356X + 9.31842{\cdot}10^{-05} \\ &= 9.31842{\cdot}10^{-05}B_{0,2} - 4.74939{\cdot}10^{-05}B_{1,2} - 0.000140979B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.71289{\cdot}10^{-18}X^6 - 4.98889{\cdot}10^{-18}X^5 + 5.34673{\cdot}10^{-18}X^4 - 2.53673 \\ &\quad {\cdot}10^{-18}X^3 + 4.71928{\cdot}10^{-05}X^2 - 0.000281356X + 9.31842{\cdot}10^{-05} \\ &= 9.31842{\cdot}10^{-05}B_{0,6} + 4.62915{\cdot}10^{-05}B_{1,6} + 2.54494{\cdot}10^{-06}B_{2,6} - 3.80554 \\ &\quad {\cdot}10^{-05}B_{3,6} - 7.55095{\cdot}10^{-05}B_{4,6} - 0.000109817B_{5,6} - 0.000140979B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.61381{\cdot}10^{-07}$.

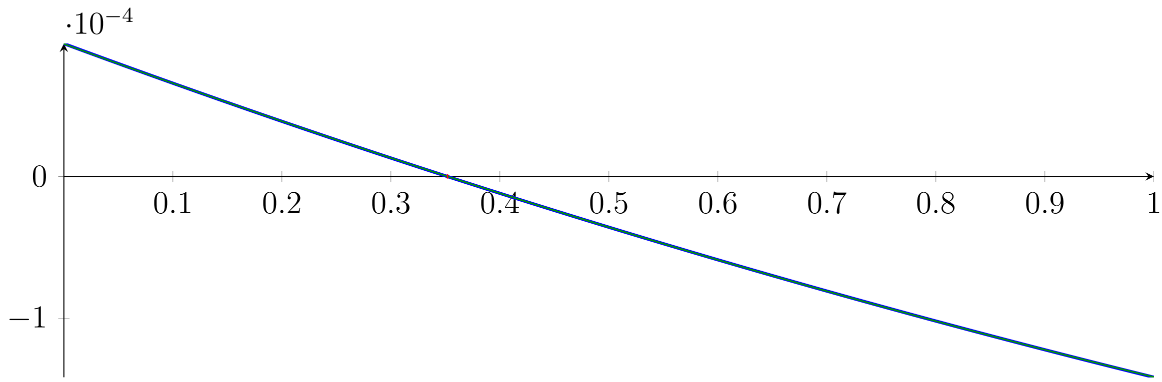
Bounding polynomials M and m:

$$M = 4.71928{\cdot}10^{-05}X^2 - 0.000281356X + 9.33455{\cdot}10^{-05}$$
$$m = 4.71928{\cdot}10^{-05}X^2 - 0.000281356X + 9.30228{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.352627, 5.60922\} \qquad N(m) = \{0.351326, 5.61052\}$$

Intersection intervals:



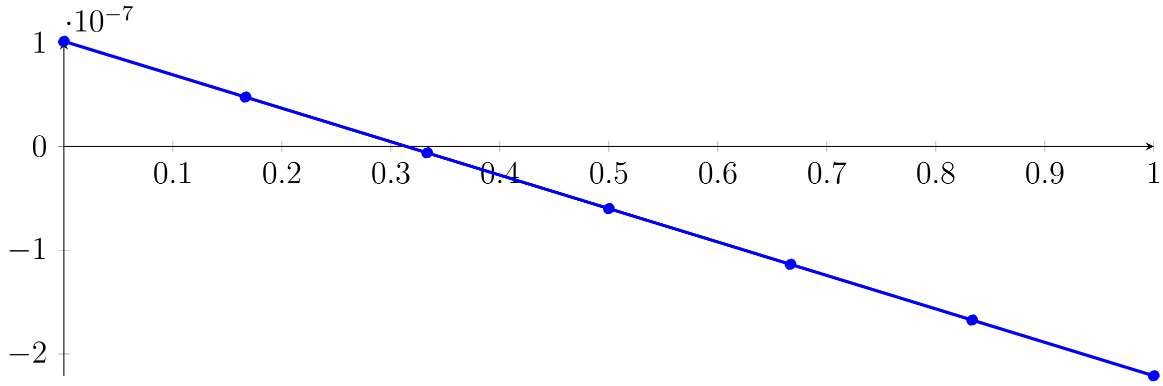
$$[0.351326, 0.352627]$$

Longest intersection interval: 0.00130075
$\Longrightarrow$ Selective recursion: interval 1: $[0.04951, 0.0495277]$,

11.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.04951, 0.0495277]$

Normalized monomial und Bézier representations and the Bézier polygon:

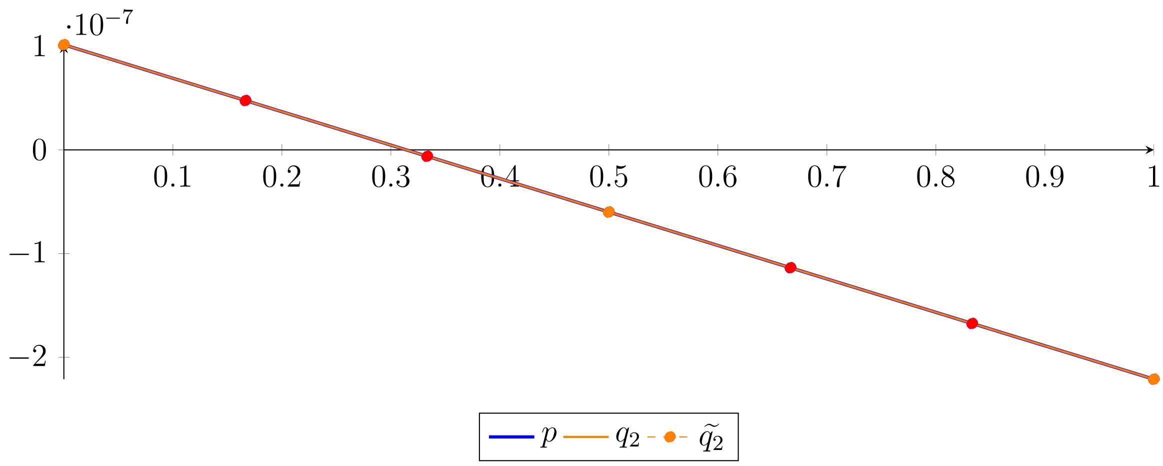
$$\begin{aligned} p = & 1.00813 \cdot 10^{-24} X^6 - 6.2814 \cdot 10^{-24} X^5 + 2.71721 \cdot 10^{-19} X^4 - 7.08741 \\ & \cdot 10^{-15} X^3 + 8.22229 \cdot 10^{-11} X^2 - 3.2249 \cdot 10^{-07} X + 1.01313 \cdot 10^{-07} \\ = & 1.01313 \cdot 10^{-07} B_{0,6}(X) + 4.75647 \cdot 10^{-08} B_{1,6}(X) - 6.17822 \cdot 10^{-09} B_{2,6}(X) - 5.99157 \\ & \cdot 10^{-08} B_{3,6}(X) - 1.13648 \cdot 10^{-07} B_{4,6}(X) - 1.67374 \cdot 10^{-07} B_{5,6}(X) - 2.21095 \cdot 10^{-07} B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = & 8.22123 \cdot 10^{-11} X^2 - 3.2249 \cdot 10^{-07} X + 1.01313 \cdot 10^{-07} \\ = & 1.01313 \cdot 10^{-07} B_{0,2} - 5.99321 \cdot 10^{-08} B_{1,2} - 2.21095 \cdot 10^{-07} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} = & 2.52407 \cdot 10^{-21} X^6 - 7.41333 \cdot 10^{-21} X^5 + 8.0243 \cdot 10^{-21} X^4 - 3.85867 \\ & \cdot 10^{-21} X^3 + 8.22123 \cdot 10^{-11} X^2 - 3.2249 \cdot 10^{-07} X + 1.01313 \cdot 10^{-07} \\ = & 1.01313 \cdot 10^{-07} B_{0,6} + 4.75647 \cdot 10^{-08} B_{1,6} - 6.17822 \cdot 10^{-09} B_{2,6} - 5.99157 \\ & \cdot 10^{-08} B_{3,6} - 1.13648 \cdot 10^{-07} B_{4,6} - 1.67374 \cdot 10^{-07} B_{5,6} - 2.21095 \cdot 10^{-07} B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.54353 \cdot 10^{-16}$.
Bounding polynomials M and m:

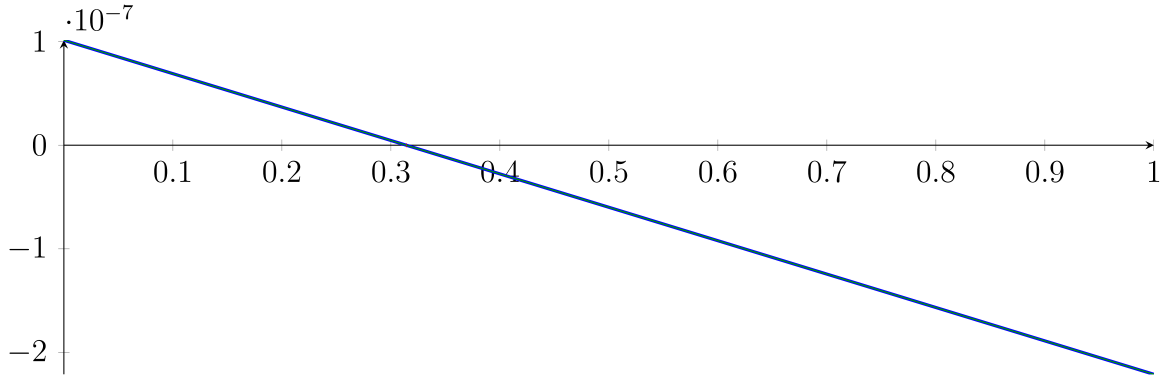
$$M = 8.22123 \cdot 10^{-11} X^2 - 3.2249 \cdot 10^{-07} X + 1.01313 \cdot 10^{-07}$$

$$m = 8.22123 \cdot 10^{-11} X^2 - 3.2249 \cdot 10^{-07} X + 1.01313 \cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{0.314184, 3922.34\} \qquad N(m) = \{0.314184, 3922.34\}$$

Intersection intervals:



$[0.314184, 0.314184]$

Longest intersection interval: $2.19795 \cdot 10^{-09}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.0495156, 0.0495156]$,

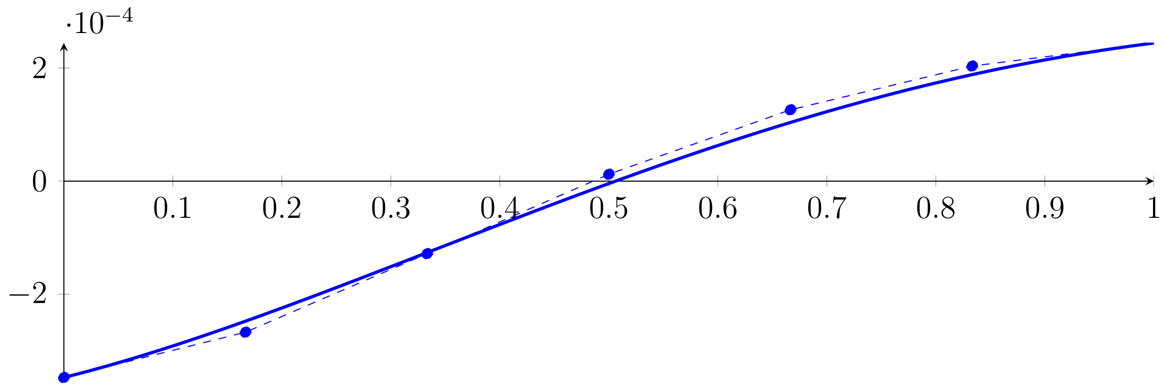
11.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.0495156, 0.0495156]$

Found root in interval $[0.0495156, 0.0495156]$ at recursion depth 7!

11.8 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

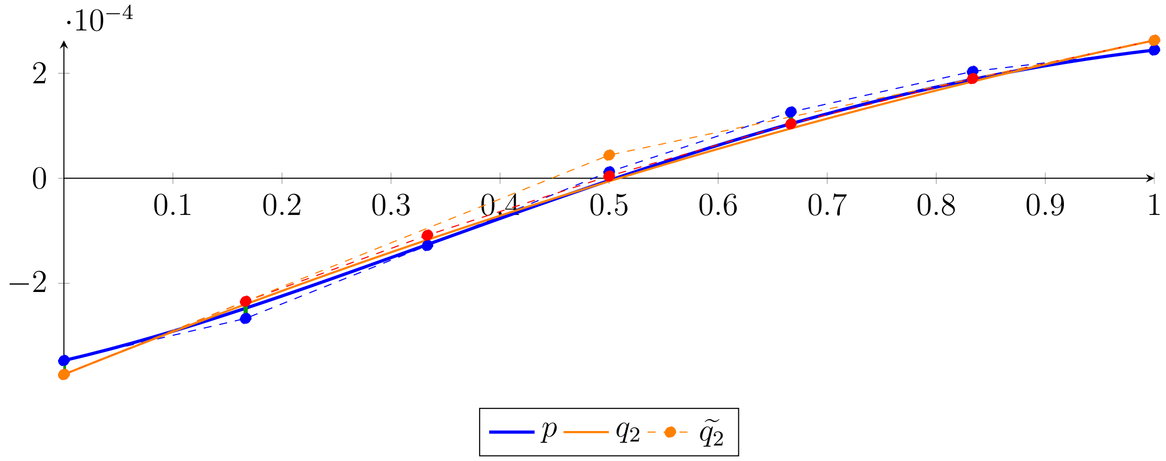
$$\begin{aligned} p = {} & 3.8147 \cdot 10^{-06} X^6 - 6.86646 \cdot 10^{-05} X^5 + 0.00043869 X^4 \\ & - 0.00114441 X^3 + 0.000881195 X^2 + 0.000480652 X - 0.000347137 \\ = {} & -0.000347137 B_{0,6}(X) - 0.000267029 B_{1,6}(X) - 0.000128174 B_{2,6}(X) + 1.2207 \\ & \cdot 10^{-05} B_{3,6}(X) + 0.000126139 B_{4,6}(X) + 0.000203451 B_{5,6}(X) + 0.000244141 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.000199182 X^2 + 0.000835419 X - 0.000373659 \\ &= -0.000373659 B_{0,2} + 4.40507 \cdot 10^{-05} B_{1,2} + 0.000262578 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -8.28107 \cdot 10^{-19} X^6 + 1.63472 \cdot 10^{-18} X^5 - 5.83262 \cdot 10^{-19} X^4 - 6.34216 \\ & \quad \cdot 10^{-19} X^3 - 0.000199182 X^2 + 0.000835419 X - 0.000373659 \\ &= -0.000373659 B_{0,6} - 0.000234422 B_{1,6} - 0.000108465 B_{2,6} + 4.21433 \\ & \quad \cdot 10^{-06} B_{3,6} + 0.000103614 B_{4,6} + 0.000189736 B_{5,6} + 0.000262578 B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.26066{\cdot}10^{-05}$.
Bounding polynomials M and m:

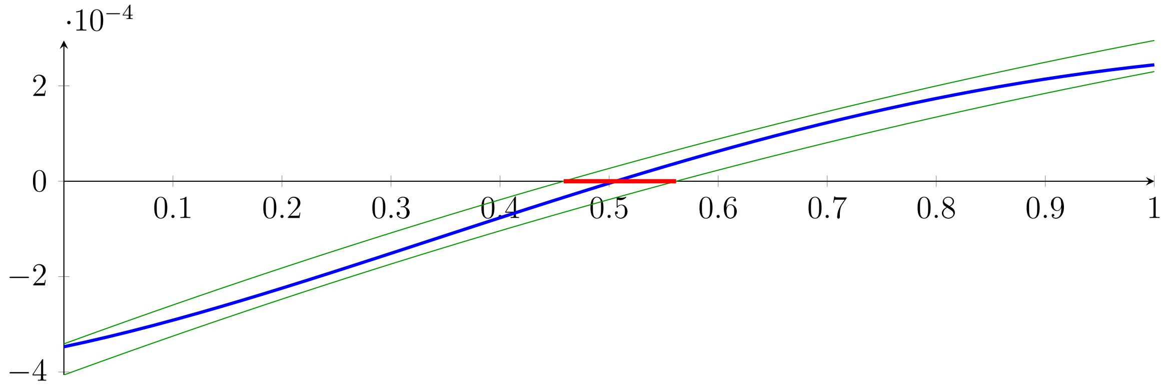
$$M = -0.000199182X^2 + 0.000835419X - 0.000341052$$

$$m = -0.000199182X^2 + 0.000835419X - 0.000406265$$

Root of M and m:

$$N(M) = \{0.458324, 3.73593\} \qquad N(m) = \{0.561461, 3.63279\}$$

Intersection intervals:



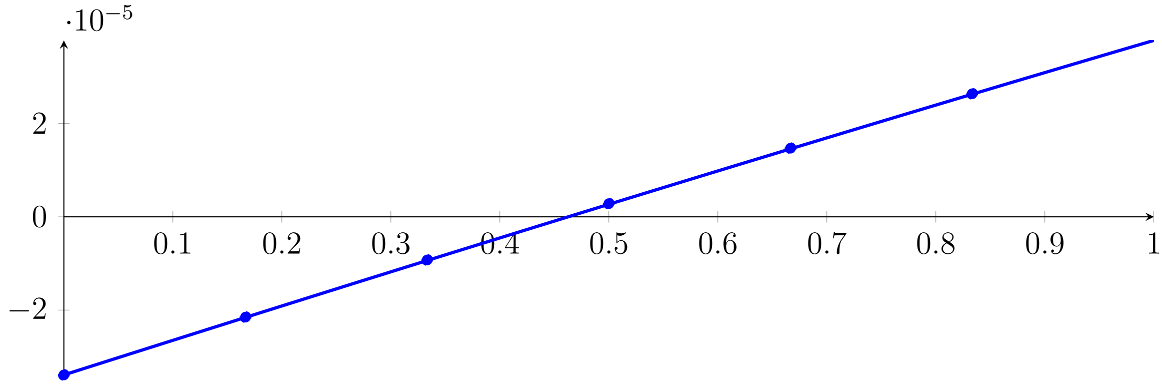
$$[0.458324, 0.561461]$$

Longest intersection interval: 0.103137
$\Longrightarrow$ Selective recursion: interval 1: $[0.18229, 0.195183]$,

11.9 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.18229, 0.195183]$

Normalized monomial und Bézier representations and the Bézier polygon:

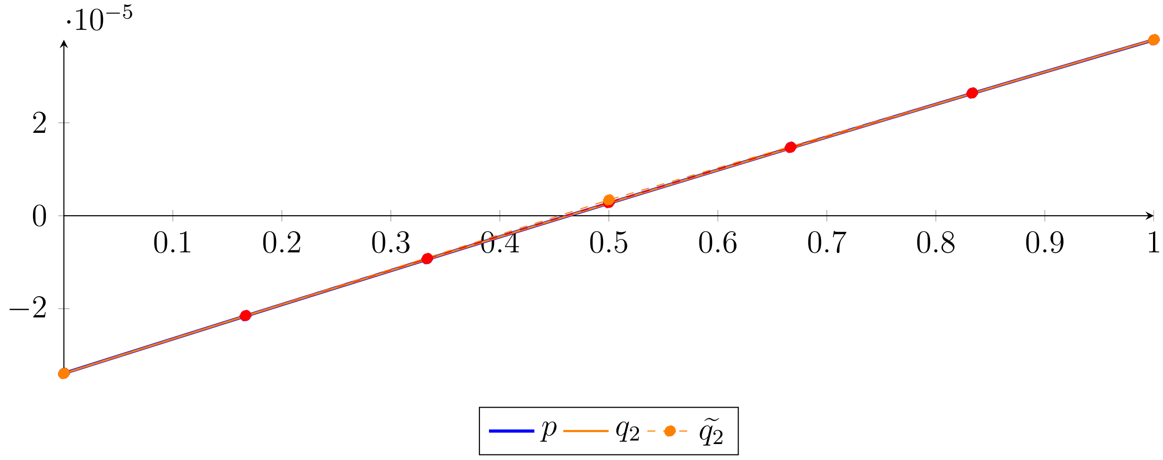
$$\begin{aligned} p &= 4.59143{\cdot}10^{-12}X^6 - 6.78898{\cdot}10^{-10}X^5 + 3.31936{\cdot}10^{-08}X^4 - 5.23371 \\ &\quad {\cdot}10^{-07}X^3 - 2.15945{\cdot}10^{-06}X^2 + 7.44098{\cdot}10^{-05}X - 3.39135{\cdot}10^{-05} \\ &= -3.39135{\cdot}10^{-05}B_{0,6}(X) - 2.15119{\cdot}10^{-05}B_{1,6}(X) - 9.25422{\cdot}10^{-06}B_{2,6}(X) + 2.83332 \\ &\quad {\cdot}10^{-06}B_{3,6}(X) + 1.47268{\cdot}10^{-05}B_{4,6}(X) + 2.64043{\cdot}10^{-05}B_{5,6}(X) + 3.7846{\cdot}10^{-05}B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.8888{\cdot}10^{-06}X^2 + 7.46942{\cdot}10^{-05}X - 3.39369{\cdot}10^{-05} \\ &= -3.39369{\cdot}10^{-05}B_{0,2} + 3.41018{\cdot}10^{-06}B_{1,2} + 3.78685{\cdot}10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.46947{\cdot}10^{-19}X^6 + 3.59804{\cdot}10^{-19}X^5 - 2.78876{\cdot}10^{-19}X^4 + 4.58589 \\ &\quad {\cdot}10^{-20}X^3 - 2.8888{\cdot}10^{-06}X^2 + 7.46942{\cdot}10^{-05}X - 3.39369{\cdot}10^{-05} \\ &= -3.39369{\cdot}10^{-05}B_{0,6} - 2.14879{\cdot}10^{-05}B_{1,6} - 9.23144{\cdot}10^{-06}B_{2,6} + 2.83242 \\ &\quad {\cdot}10^{-06}B_{3,6} + 1.47037{\cdot}10^{-05}B_{4,6} + 2.63824{\cdot}10^{-05}B_{5,6} + 3.78685{\cdot}10^{-05}B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.40038{\cdot}10^{-08}$.

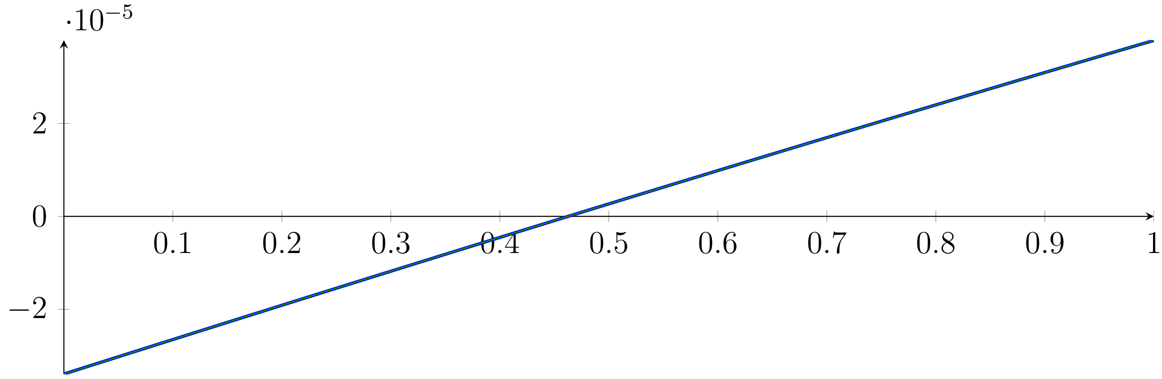
Bounding polynomials M and m:

$$M = -2.8888{\cdot}10^{-06}X^2 + 7.46942{\cdot}10^{-05}X - 3.39129{\cdot}10^{-05}$$
$$m = -2.8888{\cdot}10^{-06}X^2 + 7.46942{\cdot}10^{-05}X - 3.39609{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.462289, 25.3942\} \qquad N(m) = \{0.462955, 25.3935\}$$

Intersection intervals:



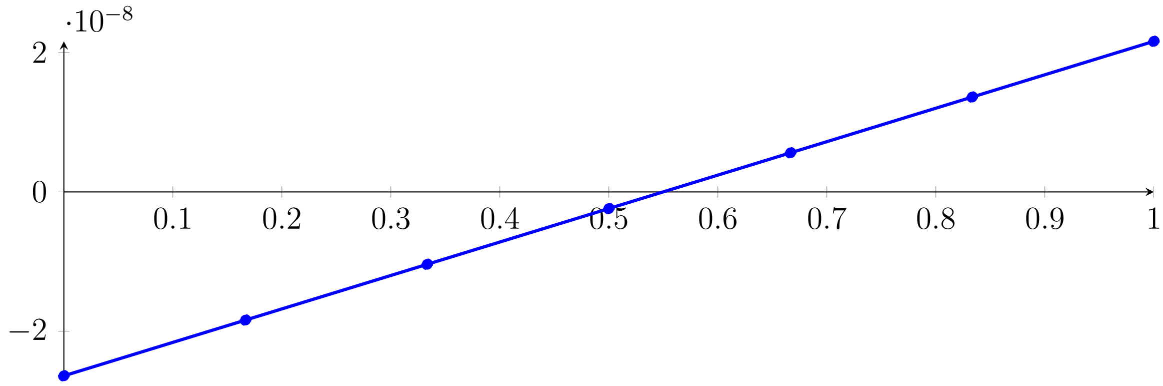
$$[0.462289, 0.462955]$$

Longest intersection interval: 0.000666575
$\Longrightarrow$ Selective recursion: interval 1: $[0.18825, 0.188259]$,

11.10 Recursion Branch 1 1 1 2 1 1 in Interval 1: $[0.18825, 0.188259]$

Normalized monomial und Bézier representations and the Bézier polygon:

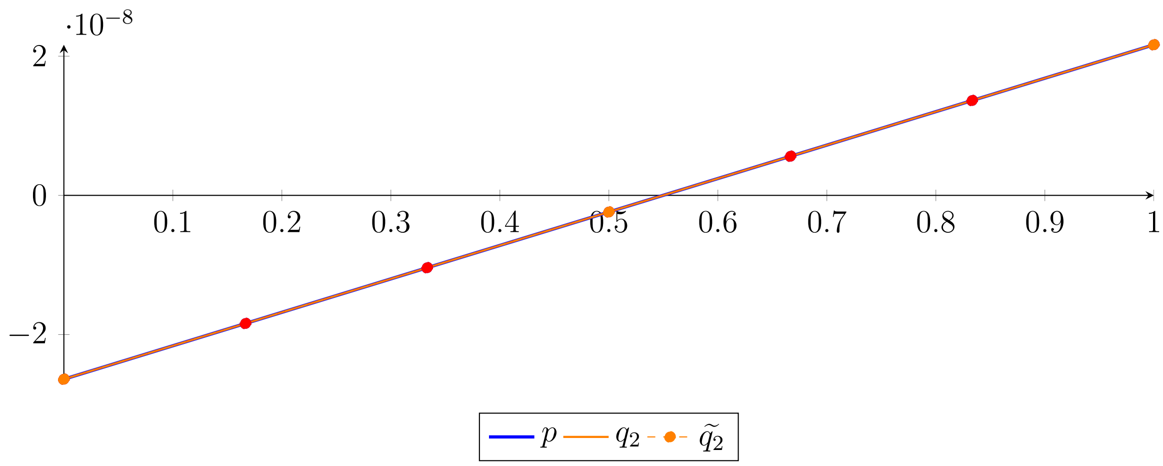
$$\begin{aligned} p = & 1.29247 \cdot 10^{-26} X^6 + 6.24631 \cdot 10^{-21} X^4 - 1.37257 \cdot 10^{-16} X^3 \\ & - 1.26339 \cdot 10^{-12} X^2 + 4.80538 \cdot 10^{-08} X - 2.64152 \cdot 10^{-08} \\ = & -2.64152 \cdot 10^{-08} B_{0,6}(X) - 1.84062 \cdot 10^{-08} B_{1,6}(X) - 1.03973 \cdot 10^{-08} B_{2,6}(X) - 2.3885 \\ & \cdot 10^{-09} B_{3,6}(X) + 5.62022 \cdot 10^{-09} B_{4,6}(X) + 1.36289 \cdot 10^{-08} B_{5,6}(X) + 2.16374 \cdot 10^{-08} B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.26359 \cdot 10^{-12} X^2 + 4.80538 \cdot 10^{-08} X - 2.64152 \cdot 10^{-08} \\ &= -2.64152 \cdot 10^{-08} B_{0,2} - 2.38824 \cdot 10^{-09} B_{1,2} + 2.16374 \cdot 10^{-08} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & 7.73382 \cdot 10^{-23} X^6 - 3.04202 \cdot 10^{-22} X^5 + 4.48713 \cdot 10^{-22} X^4 - 3.11518 \\ & \cdot 10^{-22} X^3 - 1.26359 \cdot 10^{-12} X^2 + 4.80538 \cdot 10^{-08} X - 2.64152 \cdot 10^{-08} \\ = & -2.64152 \cdot 10^{-08} B_{0,6} - 1.84062 \cdot 10^{-08} B_{1,6} - 1.03973 \cdot 10^{-08} B_{2,6} - 2.3885 \\ & \cdot 10^{-09} B_{3,6} + 5.62022 \cdot 10^{-09} B_{4,6} + 1.36289 \cdot 10^{-08} B_{5,6} + 2.16374 \cdot 10^{-08} B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.86243 \cdot 10^{-18}$.
Bounding polynomials M and m:

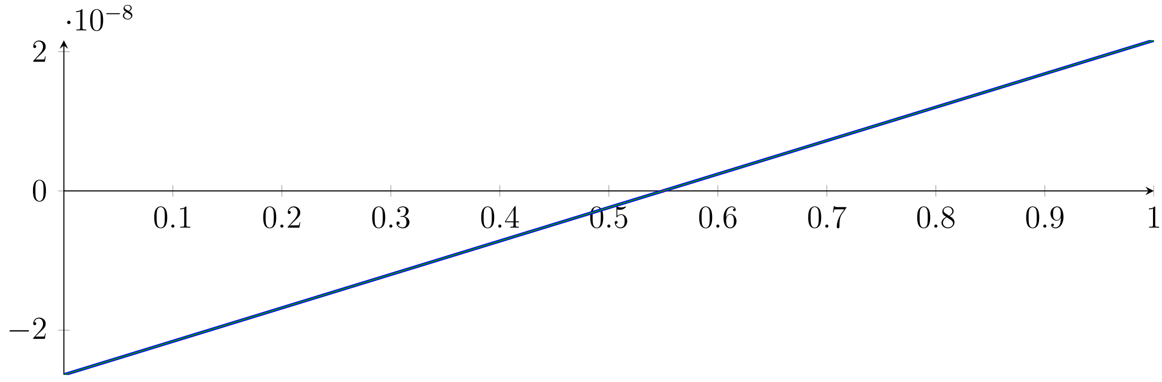
$$M = -1.26359 \cdot 10^{-12} X^2 + 4.80538 \cdot 10^{-08} X - 2.64152 \cdot 10^{-08}$$

$$m = -1.26359\cdot 10^{-12}X^2 + 4.80538\cdot 10^{-08}X - 2.64152\cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{0.549707, 38029\} \qquad N(m) = \{0.549707, 38029\}$$

Intersection intervals:



$$[0.549707, 0.549707]$$

Longest intersection interval: $2.85622\cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.188255, 0.188255]$,

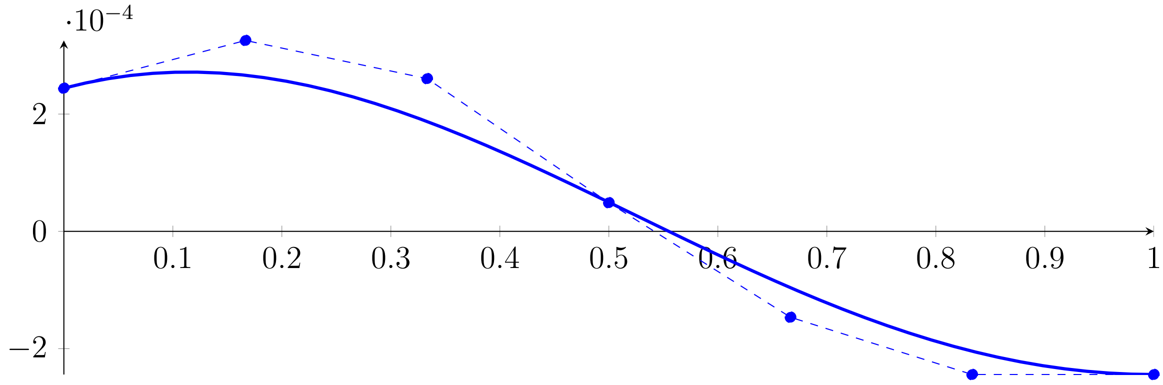
11.11 Recursion Branch 1 1 1 2 1 1 1 in Interval 1: $[0.188255, 0.188255]$

Found root in interval $[0.188255, 0.188255]$ at recursion depth 7!

11.12 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

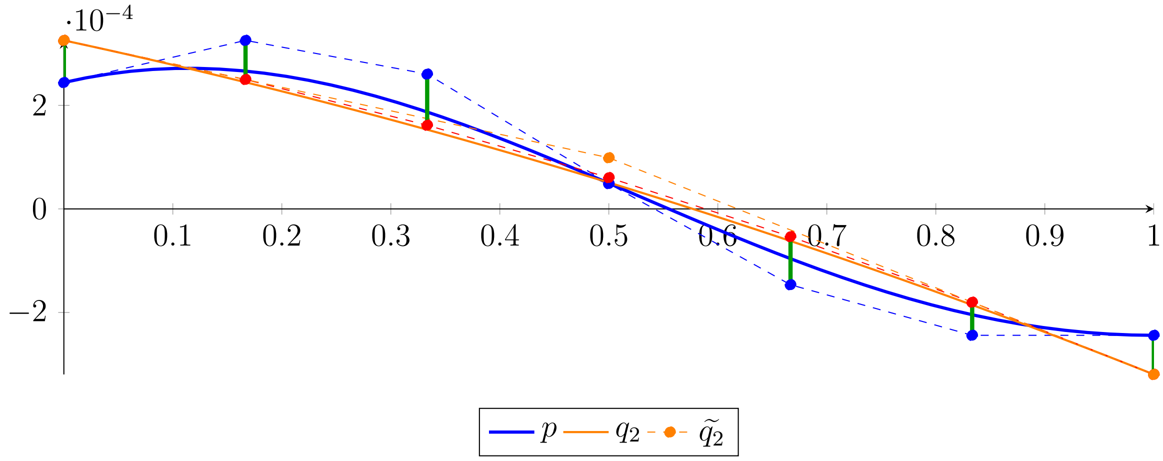
$$\begin{aligned} p &= 0.000244141X^6 - 0.00146484X^5 + 0.00244141X^4 - 3.17637 \\ &\quad \cdot 10^{-21}X^3 - 0.00219727X^2 + 0.000488281X + 0.000244141 \\ &= 0.000244141B_{0,6}(X) + 0.000325521B_{1,6}(X) + 0.000260417B_{2,6}(X) + 4.88281 \\ &\quad \cdot 10^{-05}B_{3,6}(X) - 0.000146484B_{4,6}(X) - 0.000244141B_{5,6}(X) - 0.000244141B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.000191825X^2 - 0.000453404X + 0.000325521 \\ &= 0.000325521B_{0,2} + 9.88188\cdot 10^{-05}B_{1,2} - 0.000319708B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.57704\cdot 10^{-18}X^6 + 8.77473\cdot 10^{-18}X^5 - 1.14199\cdot 10^{-17}X^4 + 7.06425 \\ &\quad \cdot 10^{-18}X^3 - 0.000191825X^2 - 0.000453404X + 0.000325521 \\ &= 0.000325521B_{0,6} + 0.000249953B_{1,6} + 0.000161598B_{2,6} + 6.04539 \\ &\quad \cdot 10^{-05}B_{3,6} - 5.34784\cdot 10^{-05}B_{4,6} - 0.000180199B_{5,6} - 0.000319708B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 9.88188 \cdot 10^{-05}$.
Bounding polynomials M and m:

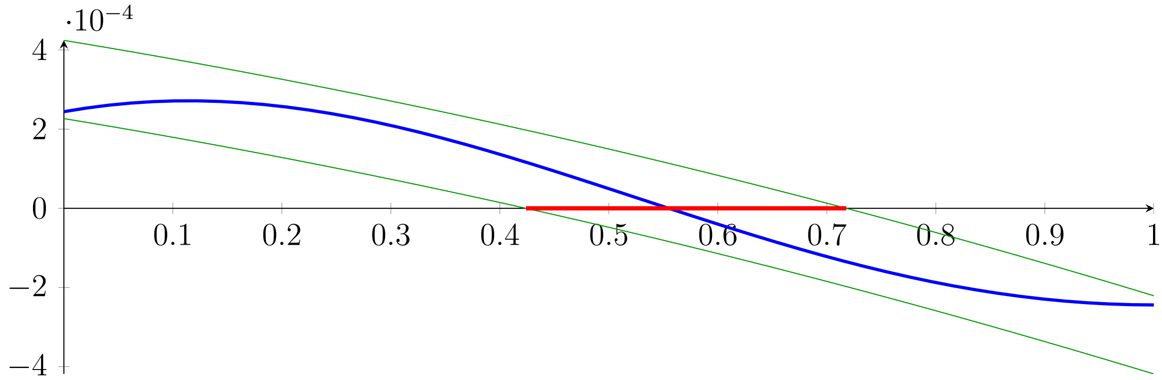
$$M = -0.000191825X^2 - 0.000453404X + 0.00042434$$

$$m = -0.000191825X^2 - 0.000453404X + 0.000226702$$

Root of M and m:

$$N(M) = \{-3.08151, 0.71787\} \qquad N(m) = \{-2.78759, 0.423957\}$$

Intersection intervals:



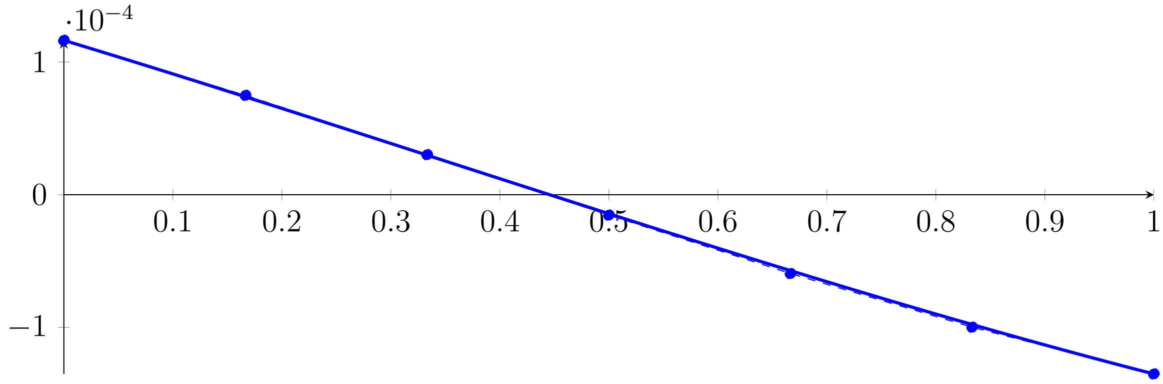
$$[0.423957, 0.71787]$$

Longest intersection interval: 0.293914
$\Longrightarrow$ Selective recursion: interval 1: $[0.355989, 0.429468]$,

11.13 Recursion Branch 1 1 2 1 in Interval 1: $[0.355989, 0.429468]$

Normalized monomial und Bézier representations and the Bézier polygon:

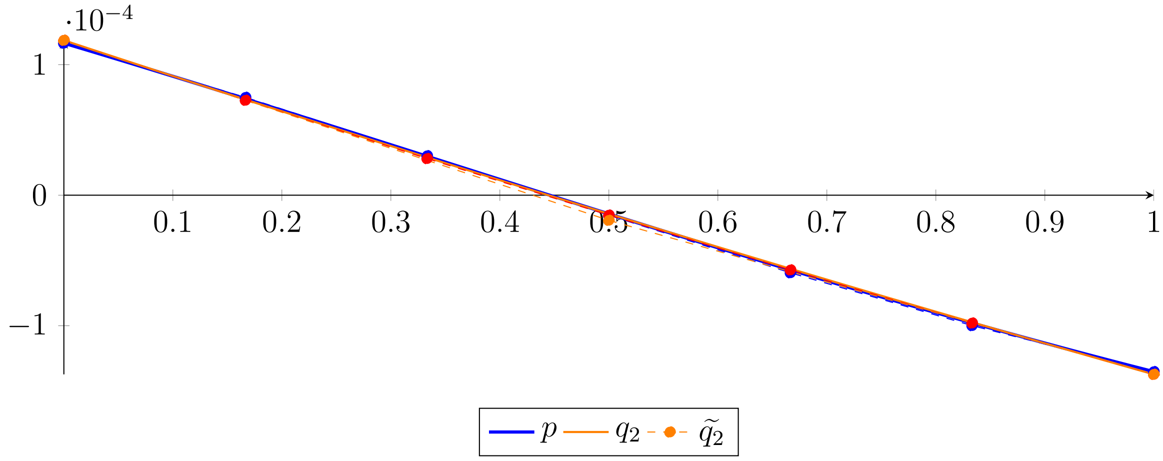
$$\begin{aligned} p = {} & 1.57383 \cdot 10^{-07} X^6 - 1.85074 \cdot 10^{-06} X^5 - 4.11901 \cdot 10^{-08} X^4 + 4.77171 \\ & \cdot 10^{-05} X^3 - 4.85738 \cdot 10^{-05} X^2 - 0.000249006X + 0.000116443 \\ = {} & 0.000116443 B_{0,6}(X) + 7.49421 \cdot 10^{-05} B_{1,6}(X) + 3.02029 \cdot 10^{-05} B_{2,6}(X) - 1.53887 \\ & \cdot 10^{-05} B_{3,6}(X) - 5.94496 \cdot 10^{-05} B_{4,6}(X) - 9.99079 \cdot 10^{-05} B_{5,6}(X) - 0.000135154 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.99073{\cdot}10^{-05}X^2 - 0.000275795X + 0.000118646 \\ &= 0.000118646B_{0,2} - 1.92518{\cdot}10^{-05}B_{1,2} - 0.000137242B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 7.51199{\cdot}10^{-19}X^6 - 1.98484{\cdot}10^{-18}X^5 + 1.80934{\cdot}10^{-18}X^4 - 6.01923 \\ &\quad {\cdot}10^{-19}X^3 + 1.99073{\cdot}10^{-05}X^2 - 0.000275795X + 0.000118646 \\ &= 0.000118646B_{0,6} + 7.26799{\cdot}10^{-05}B_{1,6} + 2.80412{\cdot}10^{-05}B_{2,6} - 1.52703 \\ &\quad {\cdot}10^{-05}B_{3,6} - 5.72547{\cdot}10^{-05}B_{4,6} - 9.7912{\cdot}10^{-05}B_{5,6} - 0.000137242B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.26215{\cdot}10^{-06}$.

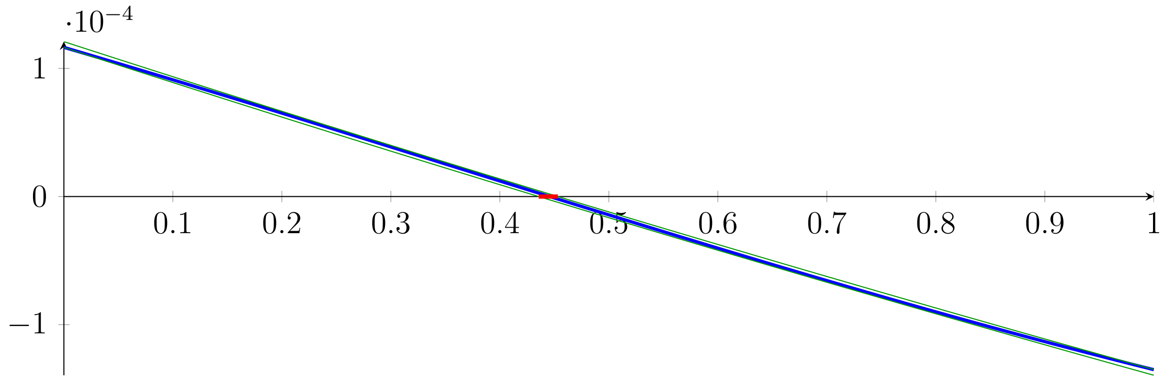
Bounding polynomials M and m:

$$M = 1.99073{\cdot}10^{-05}X^2 - 0.000275795X + 0.000120908$$
$$m = 1.99073{\cdot}10^{-05}X^2 - 0.000275795X + 0.000116384$$

Root of M and m:

$$N(M) = \{0.453225, 13.4007\} \qquad N(m) = \{0.435695, 13.4183\}$$

Intersection intervals:



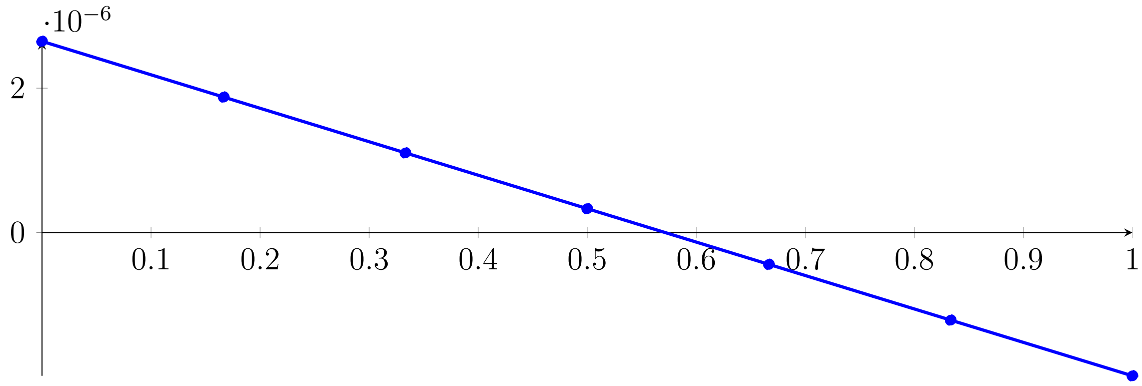
$$[0.435695, 0.453225]$$

Longest intersection interval: 0.0175293
$\Longrightarrow$ Selective recursion: interval 1: $[0.388003, 0.389291]$,

11.14 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.388003, 0.389291]$

Normalized monomial und Bézier representations and the Bézier polygon:

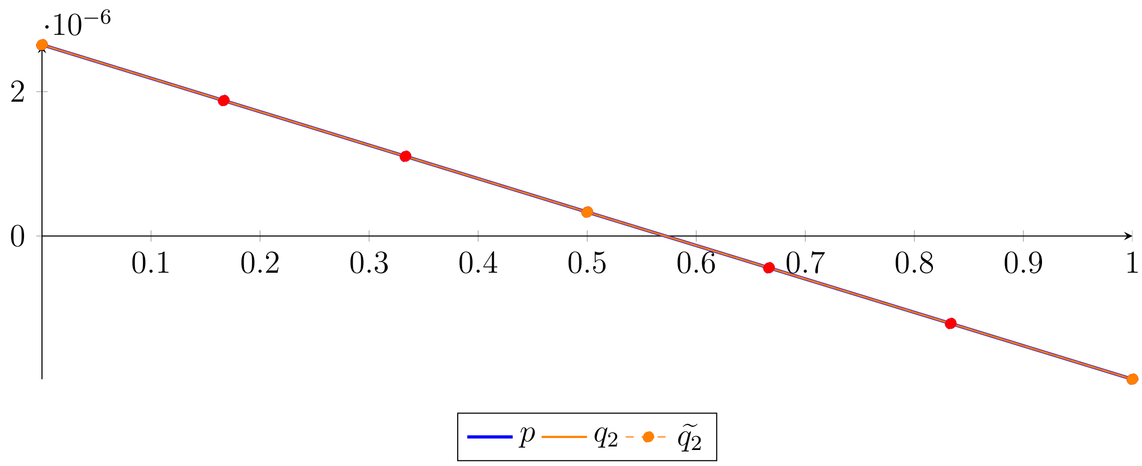
$$\begin{aligned} p = & 4.56614 \cdot 10^{-18} X^6 - 2.38222 \cdot 10^{-15} X^5 - 3.42255 \cdot 10^{-13} X^4 + 2.39113 \\ & \cdot 10^{-10} X^3 + 3.78071 \cdot 10^{-09} X^2 - 4.63634 \cdot 10^{-06} X + 2.64878 \cdot 10^{-06} \\ = & 2.64878 \cdot 10^{-06} B_{0,6}(X) + 1.87606 \cdot 10^{-06} B_{1,6}(X) + 1.10358 \cdot 10^{-06} B_{2,6}(X) + 3.31378 \\ & \cdot 10^{-07} B_{3,6}(X) - 4.40553 \cdot 10^{-07} B_{4,6}(X) - 1.2122 \cdot 10^{-06} B_{5,6}(X) - 1.98354 \cdot 10^{-06} B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = & 4.13879 \cdot 10^{-09} X^2 - 4.63648 \cdot 10^{-06} X + 2.64879 \cdot 10^{-06} \\ = & 2.64879 \cdot 10^{-06} B_{0,2} + 3.3055 \cdot 10^{-07} B_{1,2} - 1.98355 \cdot 10^{-06} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} = & -1.15009 \cdot 10^{-20} X^6 + 4.18872 \cdot 10^{-20} X^5 - 5.79285 \cdot 10^{-20} X^4 + 3.79428 \\ & \cdot 10^{-20} X^3 + 4.13879 \cdot 10^{-09} X^2 - 4.63648 \cdot 10^{-06} X + 2.64879 \cdot 10^{-06} \\ = & 2.64879 \cdot 10^{-06} B_{0,6} + 1.87604 \cdot 10^{-06} B_{1,6} + 1.10357 \cdot 10^{-06} B_{2,6} + 3.31378 \\ & \cdot 10^{-07} B_{3,6} - 4.40541 \cdot 10^{-07} B_{4,6} - 1.21218 \cdot 10^{-06} B_{5,6} - 1.98355 \cdot 10^{-06} B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.19327 \cdot 10^{-11}$.
Bounding polynomials M and m:

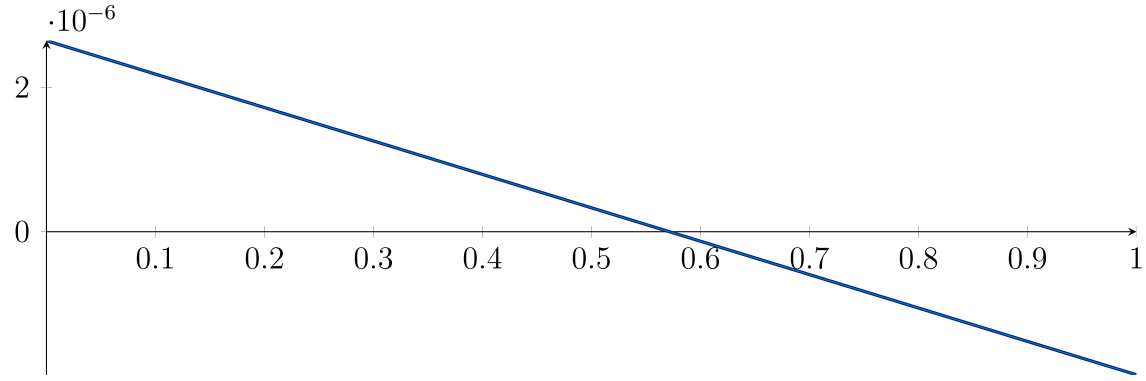
$$M = 4.13879 \cdot 10^{-09} X^2 - 4.63648 \cdot 10^{-06} X + 2.6488 \cdot 10^{-06}$$

$$m = 4.13879{\cdot}10^{-09}X^2 - 4.63648{\cdot}10^{-06}X + 2.64878{\cdot}10^{-06}$$

Root of M and m:

$$N(M) = \{0.571588, 1119.68\} \qquad N(m) = \{0.571582, 1119.68\}$$

Intersection intervals:



$$[0.571582, 0.571588]$$

Longest intersection interval: $5.15256{\cdot}10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.38874, 0.38874]$,

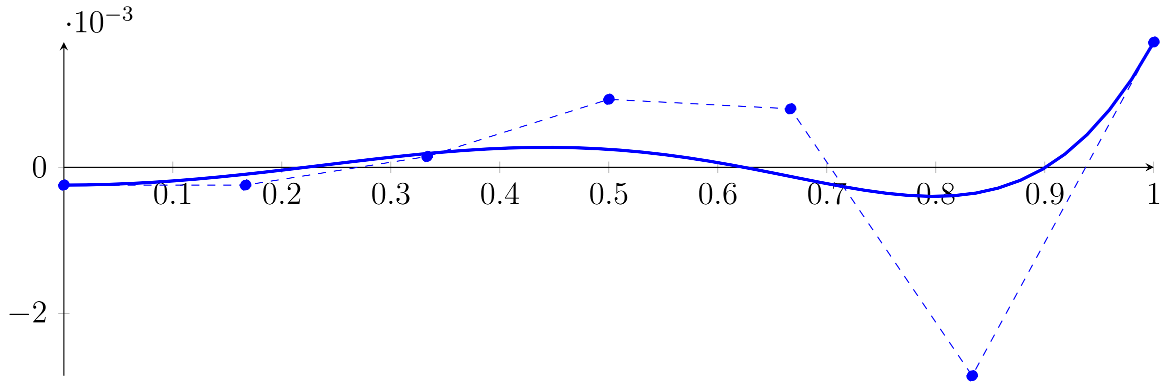
11.15 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.38874, 0.38874]$

Found root in interval $[0.38874, 0.38874]$ at recursion depth 6!

11.16 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

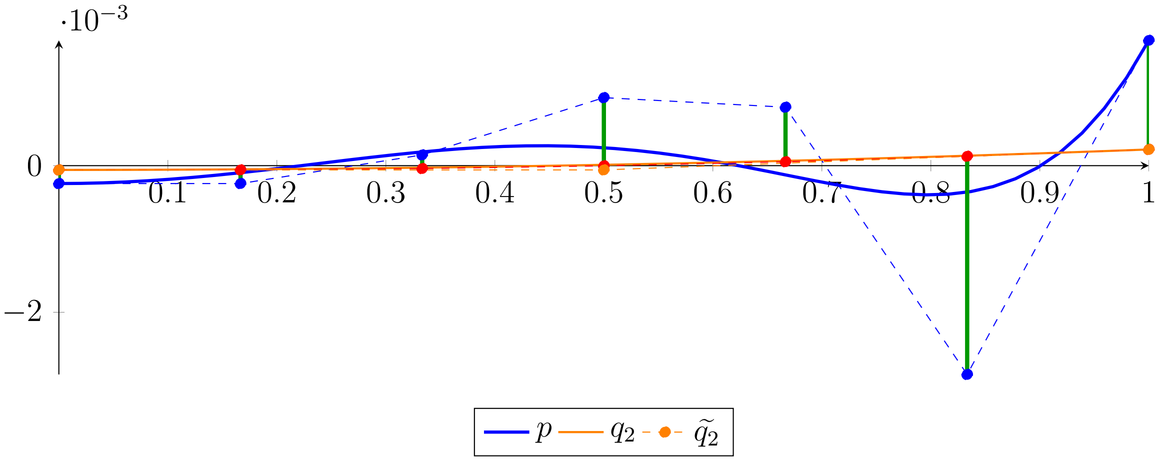
$$\begin{aligned} p &= 0.015625X^6 - 3.97047{\cdot}10^{-20}X^5 - 0.0195313X^4 - 3.02814 \\ &\quad {\cdot}10^{-19}X^3 + 0.00585937X^2 + 8.76679{\cdot}10^{-20}X - 0.000244141 \\ &= -0.000244141B_{0,6}(X) - 0.000244141B_{1,6}(X) + 0.000146484B_{2,6}(X) \\ &\quad + 0.000927734B_{3,6}(X) + 0.000797526B_{4,6}(X) - 0.00284831B_{5,6}(X) + 0.00170898B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.000279018X^2 + 5.30031{\cdot}10^{-19}X - 5.81287{\cdot}10^{-05} \\ &= -5.81287{\cdot}10^{-05}B_{0,2} - 5.81287{\cdot}10^{-05}B_{1,2} + 0.000220889B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.70229{\cdot}10^{-19}X^6 - 2.01249{\cdot}10^{-18}X^5 + 2.77501{\cdot}10^{-18}X^4 - 1.86301 \\ &\quad {\cdot}10^{-18}X^3 + 0.000279018X^2 + 4.53368{\cdot}10^{-19}X - 5.81287{\cdot}10^{-05} \\ &= -5.81287{\cdot}10^{-05}B_{0,6} - 5.81287{\cdot}10^{-05}B_{1,6} - 3.95275{\cdot}10^{-05}B_{2,6} - 2.32515 \\ &\quad {\cdot}10^{-06}B_{3,6} + 5.34784{\cdot}10^{-05}B_{4,6} + 0.000127883B_{5,6} + 0.000220889B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00297619$.
Bounding polynomials M and m:

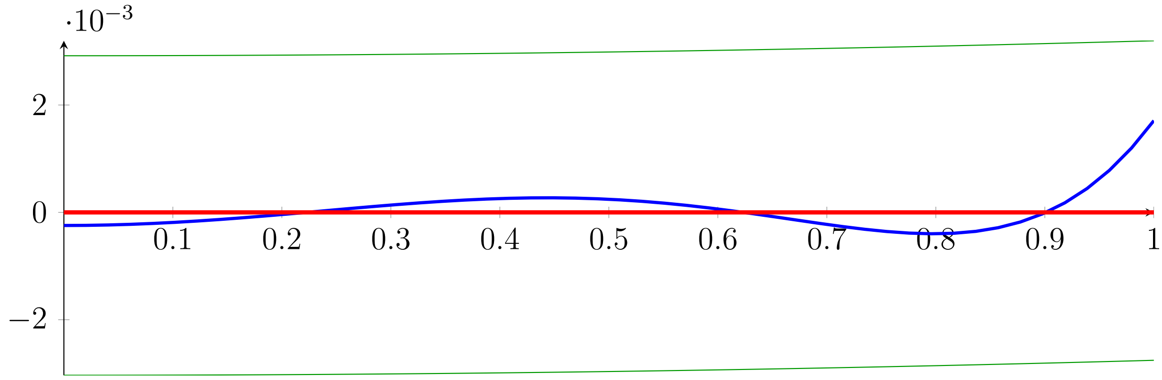
$$M = 0.000279018X^2 + 5.29819 \cdot 10^{-19}X + 0.00291806$$

$$m = 0.000279018X^2 + 5.29819 \cdot 10^{-19}X - 0.00303432$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-3.29773, 3.29773\}$$

Intersection intervals:



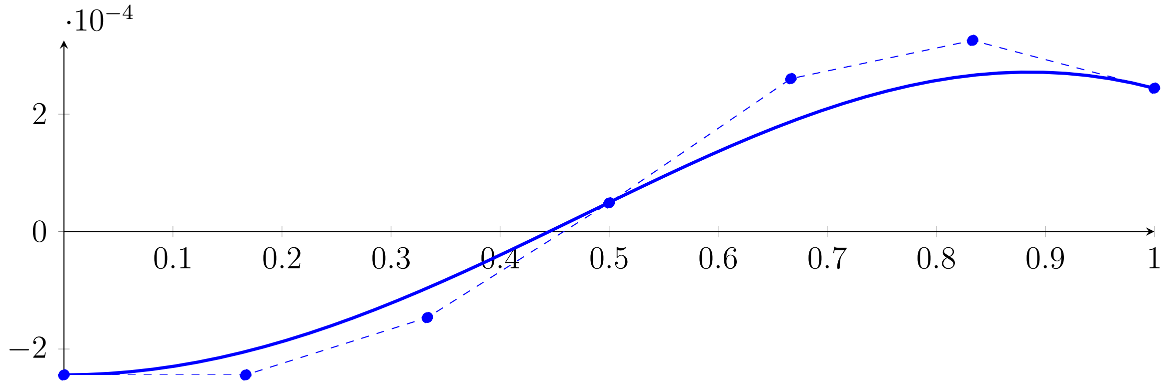
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

11.17 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

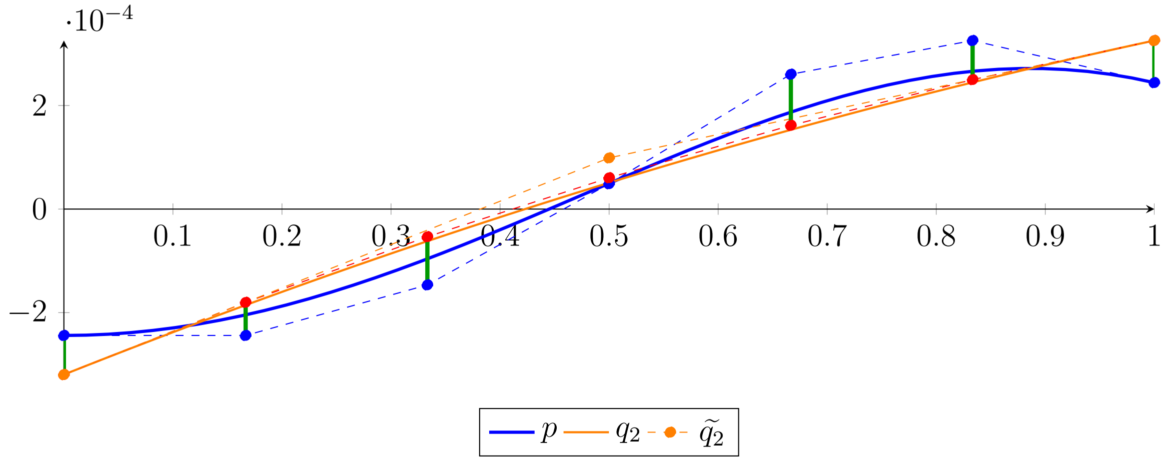
$$\begin{aligned} p &= 0.000244141X^6 + 6.35275 \cdot 10^{-22}X^5 - 0.0012207X^4 - 3.75871 \\ &\quad \cdot 10^{-20}X^3 + 0.00146484X^2 + 4.3834 \cdot 10^{-20}X - 0.000244141 \\ &= -0.000244141B_{0,6}(X) - 0.000244141B_{1,6}(X) - 0.000146484B_{2,6}(X) + 4.88281 \\ &\quad \cdot 10^{-05}B_{3,6}(X) + 0.000260417B_{4,6}(X) + 0.000325521B_{5,6}(X) + 0.000244141B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_2 &= -0.000191825X^2 + 0.000837054X - 0.000319708 \\
&= -0.000319708B_{0,2} + 9.88188\cdot 10^{-05}B_{1,2} + 0.000325521B_{2,2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_2 &= -3.06981\cdot 10^{-18}X^6 + 8.5905\cdot 10^{-18}X^5 - 8.68818\cdot 10^{-18}X^4 + 3.72271 \\
&\quad \cdot 10^{-18}X^3 - 0.000191825X^2 + 0.000837054X - 0.000319708 \\
&= -0.000319708B_{0,6} - 0.000180199B_{1,6} - 5.34784\cdot 10^{-05}B_{2,6} + 6.04539 \\
&\quad \cdot 10^{-05}B_{3,6} + 0.000161598B_{4,6} + 0.000249953B_{5,6} + 0.000325521B_{6,6}
\end{aligned}
$$



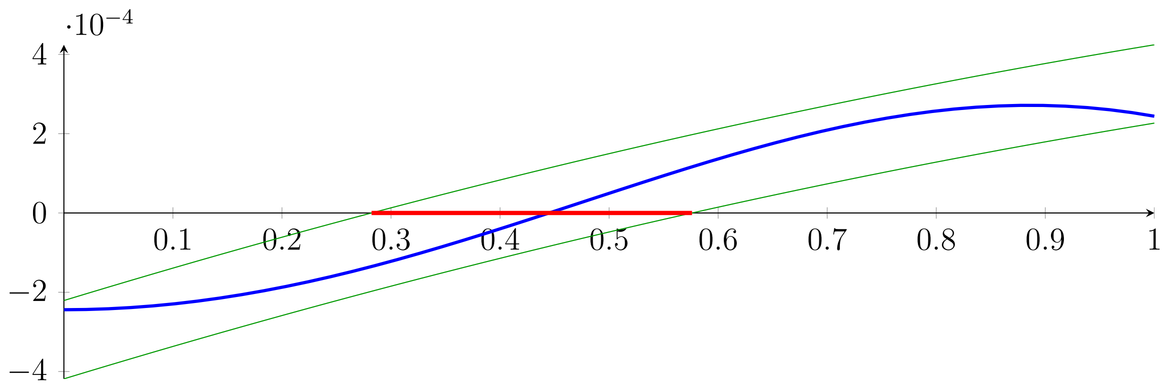
The maximum difference of the Bézier coefficients is $\delta = 9.88188\cdot 10^{-05}$.
Bounding polynomials M and m:

$$
\begin{aligned}
M &= -0.000191825X^2 + 0.000837054X - 0.000220889 \\
m &= -0.000191825X^2 + 0.000837054X - 0.000418527
\end{aligned}
$$

Root of M and m:

$$N(M) = \{0.28213, 4.08151\} \qquad N(m) = \{0.576043, 3.78759\}$$

Intersection intervals:



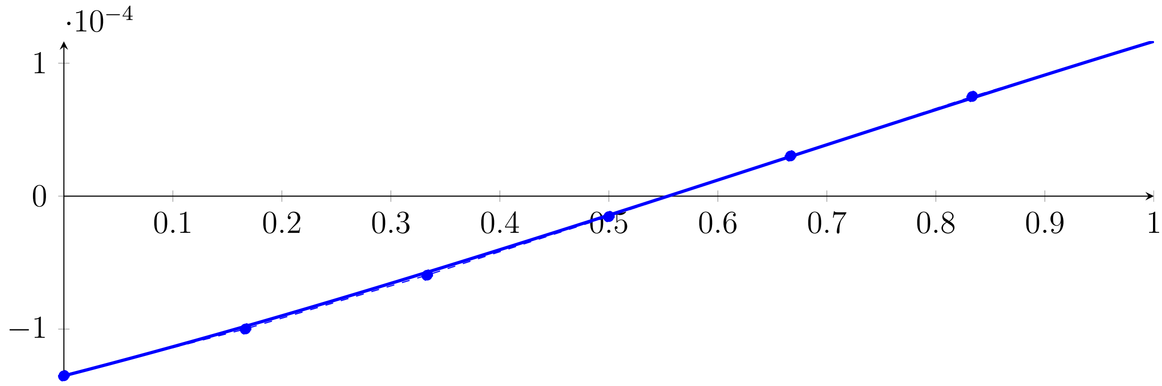
$$[0.28213, 0.576043]$$

Longest intersection interval: 0.293914
$\Longrightarrow$ Selective recursion: interval 1: $[0.570532, 0.644011]$,

11.18 Recursion Branch 1 2 1 1 in Interval 1: $[0.570532, 0.644011]$

Normalized monomial und Bézier representations and the Bézier polygon:

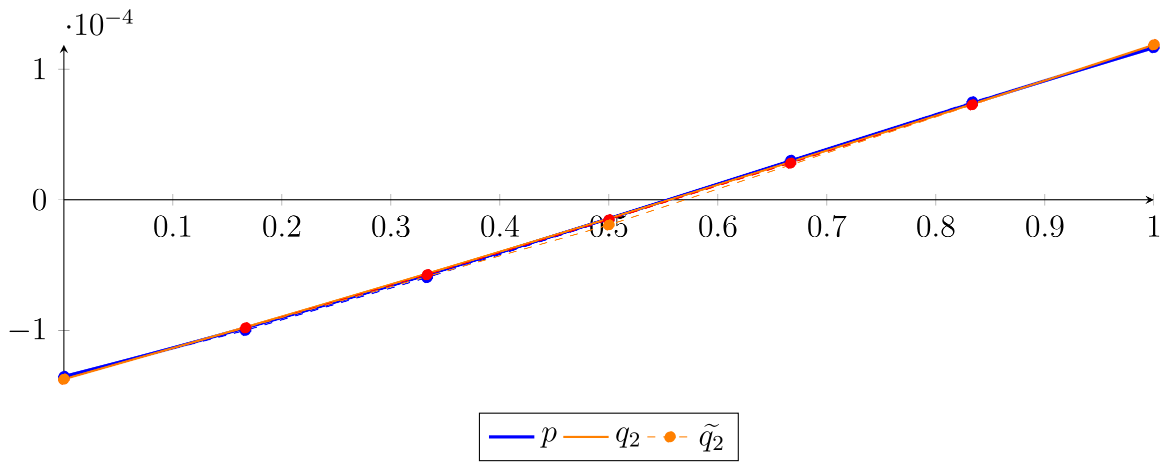
$$\begin{aligned} p &= 1.57383 \cdot 10^{-07} X^6 + 9.06439 \cdot 10^{-07} X^5 - 6.93413 \cdot 10^{-06} X^4 - 3.21926 \\ &\quad \cdot 10^{-05} X^3 + 7.81836 \cdot 10^{-05} X^2 + 0.000211476 X - 0.000135154 \\ &= -0.000135154 B_{0,6}(X) - 9.99079 \cdot 10^{-05} B_{1,6}(X) - 5.94496 \cdot 10^{-05} B_{2,6}(X) - 1.53887 \\ &\quad \cdot 10^{-05} B_{3,6}(X) + 3.02029 \cdot 10^{-05} B_{4,6}(X) + 7.49421 \cdot 10^{-05} B_{5,6}(X) + 0.000116443 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.99073 \cdot 10^{-05} X^2 + 0.000235981 X - 0.000137242 \\ &= -0.000137242 B_{0,2} - 1.92518 \cdot 10^{-05} B_{1,2} + 0.000118646 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.55865 \cdot 10^{-19} X^6 - 2.05781 \cdot 10^{-18} X^5 + 2.89288 \cdot 10^{-18} X^4 - 1.92753 \\ &\quad \cdot 10^{-18} X^3 + 1.99073 \cdot 10^{-05} X^2 + 0.000235981 X - 0.000137242 \\ &= -0.000137242 B_{0,6} - 9.7912 \cdot 10^{-05} B_{1,6} - 5.72547 \cdot 10^{-05} B_{2,6} - 1.52703 \\ &\quad \cdot 10^{-05} B_{3,6} + 2.80412 \cdot 10^{-05} B_{4,6} + 7.26799 \cdot 10^{-05} B_{5,6} + 0.000118646 B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.26215 \cdot 10^{-06}$.
Bounding polynomials M and m:

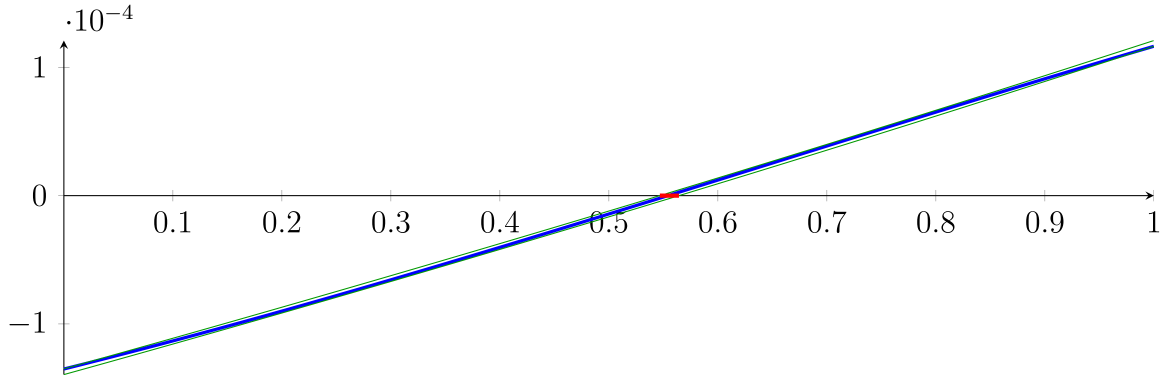
$$M = 1.99073 \cdot 10^{-05} X^2 + 0.000235981 X - 0.00013498$$

$$m = 1.99073 \cdot 10^{-05} X^2 + 0.000235981 X - 0.000139504$$

Root of M and m:

$$N(M) = \{-12.4007, 0.546775\} \qquad N(m) = \{-12.4183, 0.564305\}$$

Intersection intervals:



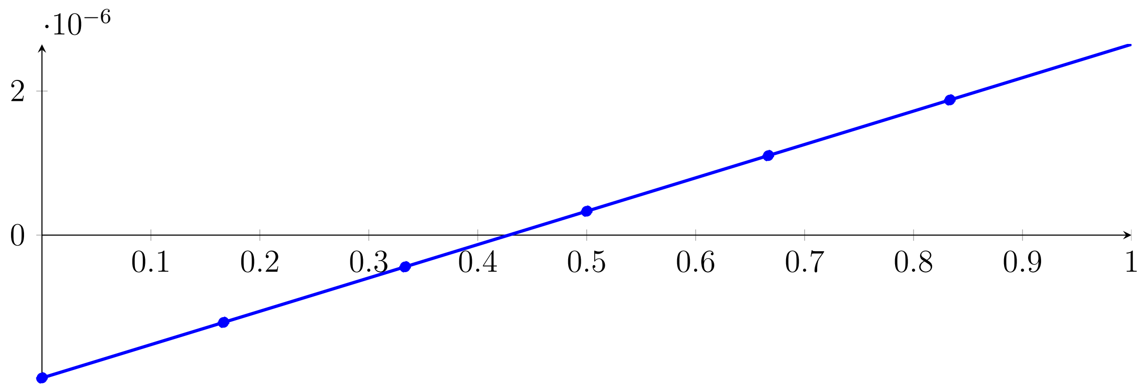
$$[0.546775, 0.564305]$$

Longest intersection interval: 0.0175293
$\Longrightarrow$ Selective recursion: interval 1: $[0.610709, 0.611997]$,

11.19 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.610709, 0.611997]$

Normalized monomial und Bézier representations and the Bézier polygon:

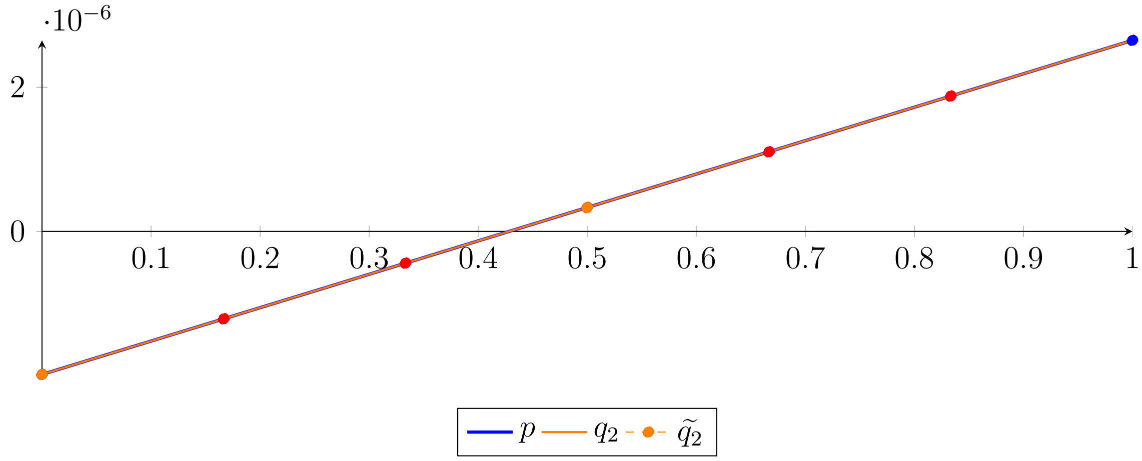
$$\begin{aligned} p &= 4.56614 \cdot 10^{-18} X^6 + 2.35482 \cdot 10^{-15} X^5 - 3.54097 \cdot 10^{-13} X^4 - 2.37721 \\ &\quad \cdot 10^{-10} X^3 + 4.49598 \cdot 10^{-09} X^2 + 4.62806 \cdot 10^{-06} X - 1.98354 \cdot 10^{-06} \\ &= -1.98354 \cdot 10^{-06} B_{0,6}(X) - 1.2122 \cdot 10^{-06} B_{1,6}(X) - 4.40553 \cdot 10^{-07} B_{2,6}(X) + 3.31378 \\ &\quad \cdot 10^{-07} B_{3,6}(X) + 1.10358 \cdot 10^{-06} B_{4,6}(X) + 1.87606 \cdot 10^{-06} B_{5,6}(X) + 2.64878 \cdot 10^{-06} B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 4.13879 \cdot 10^{-09} X^2 + 4.6282 \cdot 10^{-06} X - 1.98355 \cdot 10^{-06} \\ &= -1.98355 \cdot 10^{-06} B_{0,2} + 3.3055 \cdot 10^{-07} B_{1,2} + 2.64879 \cdot 10^{-06} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.5039 \cdot 10^{-20} X^6 + 4.05732 \cdot 10^{-20} X^5 - 3.83305 \cdot 10^{-20} X^4 + 1.39545 \\ &\quad \cdot 10^{-20} X^3 + 4.13879 \cdot 10^{-09} X^2 + 4.6282 \cdot 10^{-06} X - 1.98355 \cdot 10^{-06} \\ &= -1.98355 \cdot 10^{-06} B_{0,6} - 1.21218 \cdot 10^{-06} B_{1,6} - 4.40541 \cdot 10^{-07} B_{2,6} + 3.31378 \\ &\quad \cdot 10^{-07} B_{3,6} + 1.10357 \cdot 10^{-06} B_{4,6} + 1.87604 \cdot 10^{-06} B_{5,6} + 2.64879 \cdot 10^{-06} B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.19327 \cdot 10^{-11}$.

Bounding polynomials M and m:

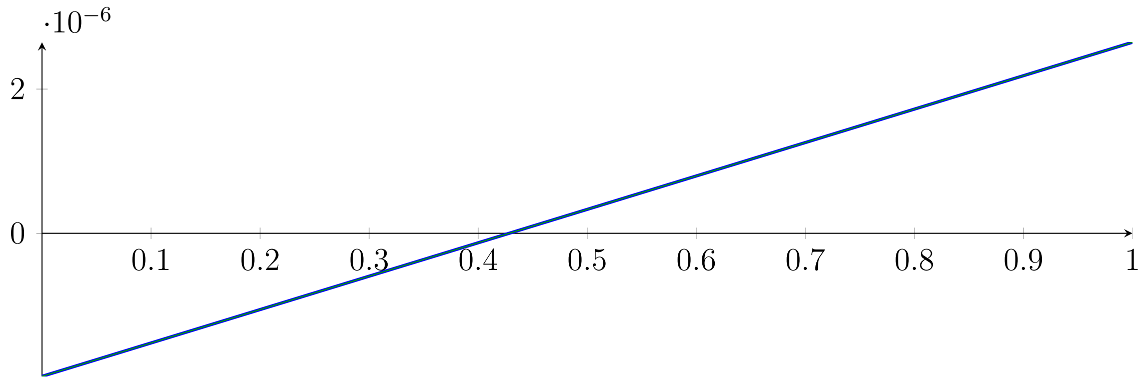
$$M = 4.13879 \cdot 10^{-09} X^2 + 4.6282 \cdot 10^{-06} X - 1.98354 \cdot 10^{-06}$$

$$m = 4.13879 \cdot 10^{-09} X^2 + 4.6282 \cdot 10^{-06} X - 1.98356 \cdot 10^{-06}$$

Root of M and m:

$$N(M) = \{-1118.68, 0.428412\} \qquad N(m) = \{-1118.68, 0.428418\}$$

Intersection intervals:



$$[0.428412, 0.428418]$$

Longest intersection interval: $5.15256 \cdot 10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.61126, 0.61126]$,

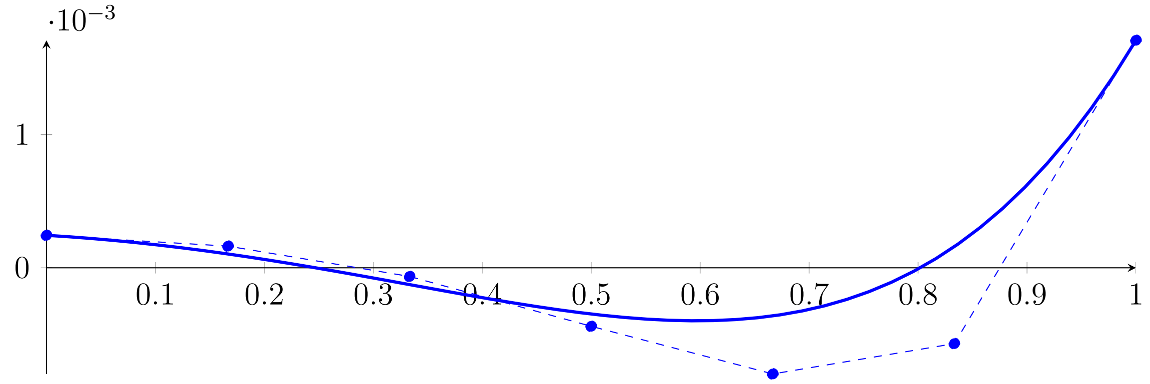
11.20 Recursion Branch 1 2 1 1 1 1 in Interval 1: $[0.61126, 0.61126]$

Found root in interval $[0.61126, 0.61126]$ at recursion depth 6!

11.21 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

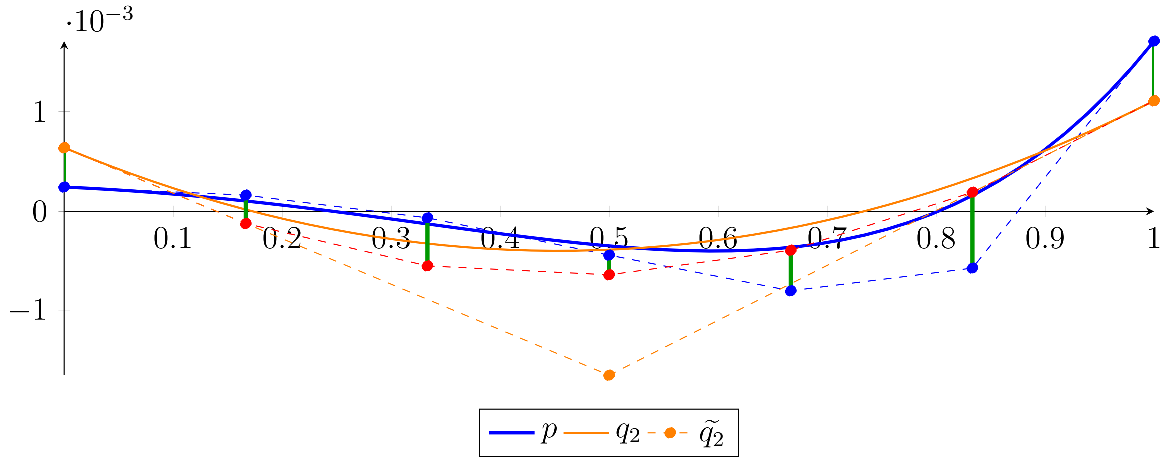
$$\begin{aligned} p &= 0.000244141 X^6 + 0.00146484 X^5 + 0.00244141 X^4 - 1.07997 \\ &\quad \cdot 10^{-19} X^3 - 0.00219727 X^2 - 0.000488281 X + 0.000244141 \\ &= 0.000244141 B_{0,6}(X) + 0.00016276 B_{1,6}(X) - 6.51042 \cdot 10^{-05} B_{2,6}(X) \\ &\quad - 0.000439453 B_{3,6}(X) - 0.000797526 B_{4,6}(X) - 0.000569661 B_{5,6}(X) + 0.00170898 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00503976X^2 - 0.00456892X + 0.000639416 \\ &= 0.000639416B_{0,2} - 0.00164504B_{1,2} + 0.00111026B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 3.58893\cdot 10^{-17}X^6 - 1.09783\cdot 10^{-16}X^5 + 1.26926\cdot 10^{-16}X^4 - 6.84371 \\ &\quad \cdot 10^{-17}X^3 + 0.00503976X^2 - 0.00456892X + 0.000639416 \\ &= 0.000639416B_{0,6} - 0.00012207B_{1,6} - 0.000547573B_{2,6} - 0.000637091B_{3,6} \\ &\quad - 0.000390625B_{4,6} + 0.000191825B_{5,6} + 0.00111026B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000761486$.

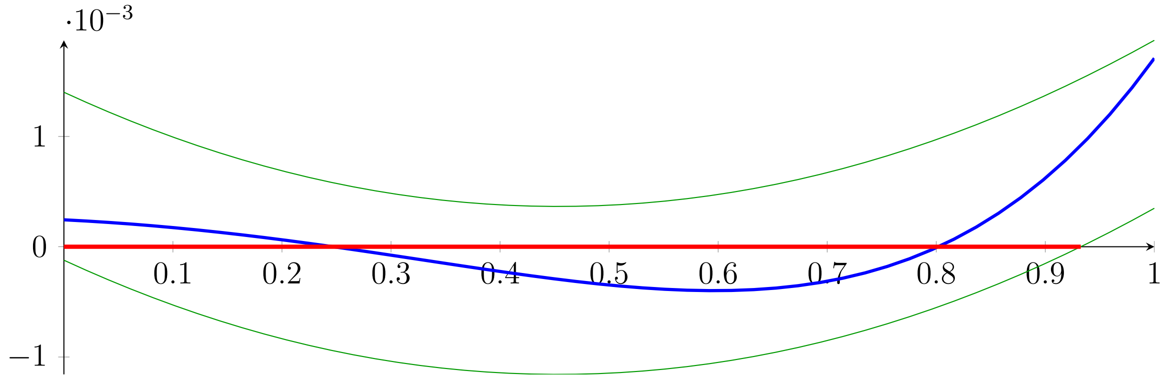
Bounding polynomials M and m:

$$M = 0.00503976X^2 - 0.00456892X + 0.0014009$$
$$m = 0.00503976X^2 - 0.00456892X - 0.00012207$$

Root of M and m:

$$N(M) = \{\} \qquad\qquad N(m) = \{-0.0259734, 0.932548\}$$

Intersection intervals:



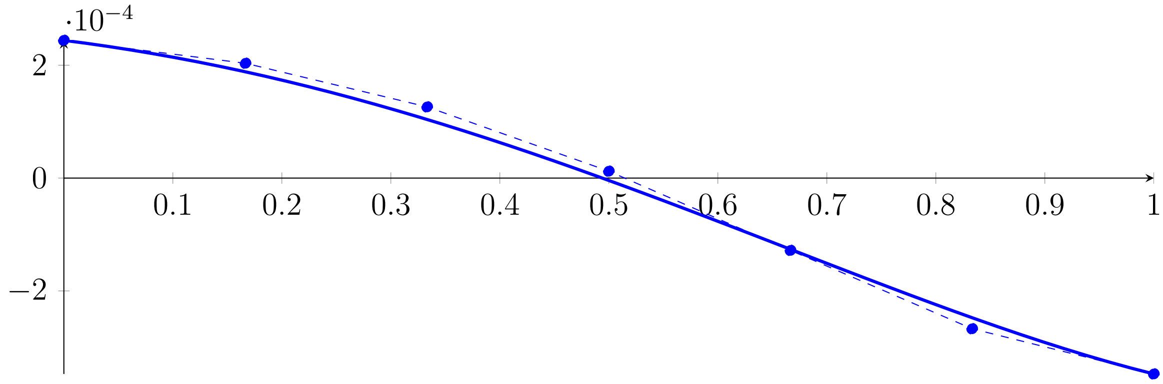
$$[0, 0.932548]$$

Longest intersection interval: 0.932548
$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$
Bisection point is very near to a root?!?

11.22 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

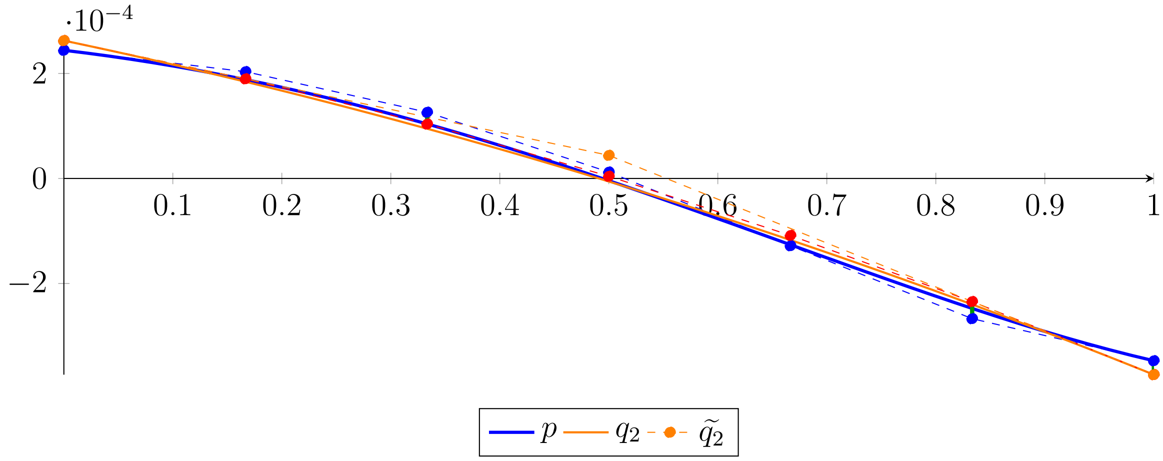
$$\begin{aligned}p &= 3.8147{\cdot}10^{-06}X^6 + 4.57764{\cdot}10^{-05}X^5 + 0.000152588X^4 - 1.37643\\ &\quad {\cdot}10^{-20}X^3 - 0.000549316X^2 - 0.000244141X + 0.000244141\\ &= 0.000244141B_{0,6}(X) + 0.000203451B_{1,6}(X) + 0.000126139B_{2,6}(X) + 1.2207\\ &\quad {\cdot}10^{-05}B_{3,6}(X) - 0.000128174B_{4,6}(X) - 0.000267029B_{5,6}(X) - 0.000347137B_{6,6}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -0.000199182X^2 - 0.000437055X + 0.000262578\\ &= 0.000262578B_{0,2} + 4.40507{\cdot}10^{-05}B_{1,2} - 0.000373659B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -3.42492{\cdot}10^{-19}X^6 + 1.81673{\cdot}10^{-18}X^5 - 3.27683{\cdot}10^{-18}X^4 + 2.66127\\ &\quad {\cdot}10^{-18}X^3 - 0.000199182X^2 - 0.000437055X + 0.000262578\\ &= 0.000262578B_{0,6} + 0.000189736B_{1,6} + 0.000103614B_{2,6} + 4.21433\\ &\quad {\cdot}10^{-06}B_{3,6} - 0.000108465B_{4,6} - 0.000234422B_{5,6} - 0.000373659B_{6,6}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.26066{\cdot}10^{-05}$.
Bounding polynomials M and m:

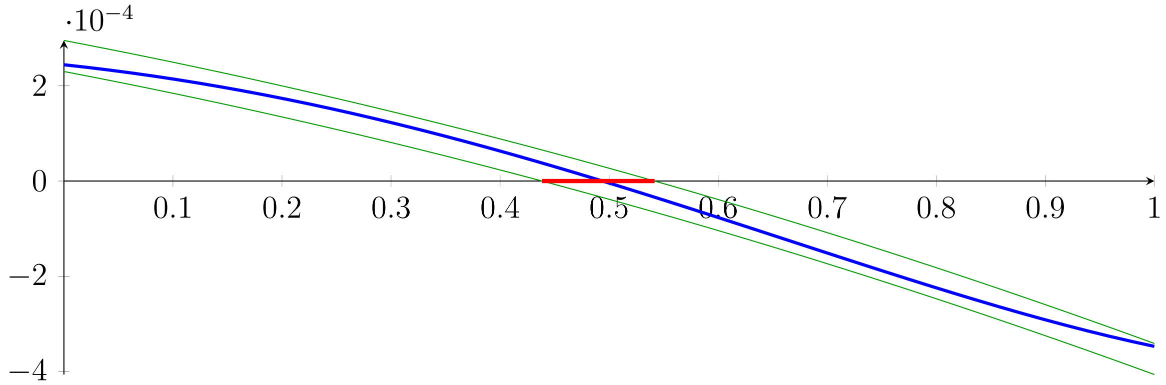
$$M = -0.000199182X^2 - 0.000437055X + 0.000295185$$

$$m = -0.000199182X^2 - 0.000437055X + 0.000229972$$

Root of M and m:

$$N(M) = \{-2.73593, 0.541676\} \qquad N(m) = \{-2.63279, 0.438539\}$$

Intersection intervals:



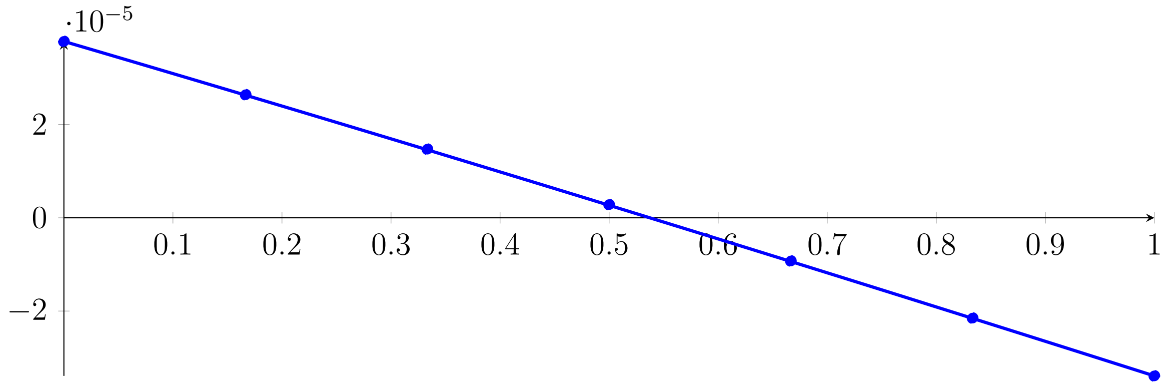
$$[0.438539, 0.541676]$$

Longest intersection interval: 0.103137
$\implies$ Selective recursion: interval 1: $[0.804817, 0.81771]$,

11.23 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.804817, 0.81771]$

Normalized monomial und Bézier representations and the Bézier polygon:

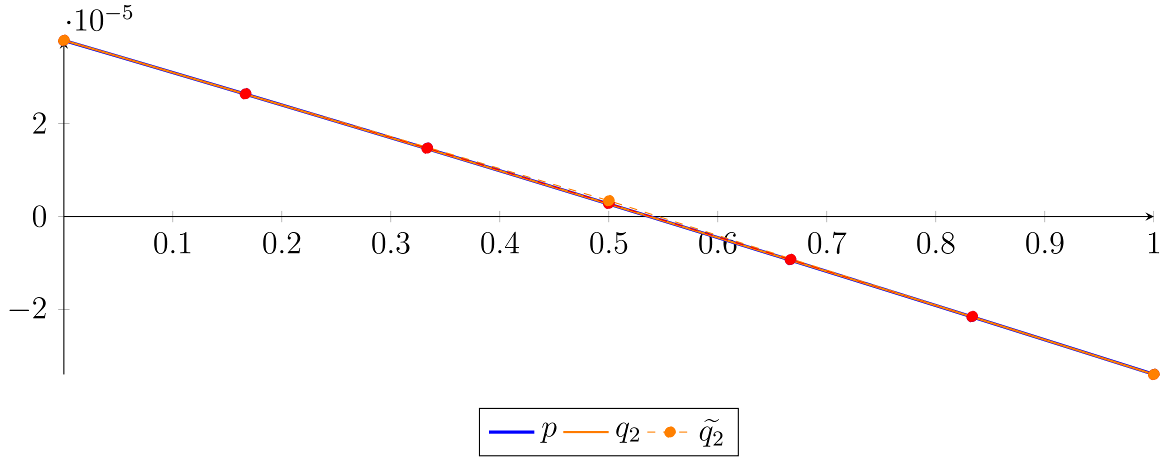
$$\begin{aligned} p = {} & 4.59143{\cdot}10^{-12}X^6 + 6.5135{\cdot}10^{-10}X^5 + 2.9868{\cdot}10^{-08}X^4 + 3.97294 \\ & {\cdot}10^{-07}X^3 - 3.53712{\cdot}10^{-06}X^2 - 6.86502{\cdot}10^{-05}X + 3.7846{\cdot}10^{-05} \\ = {} & 3.7846{\cdot}10^{-05}B_{0,6}(X) + 2.64043{\cdot}10^{-05}B_{1,6}(X) + 1.47268{\cdot}10^{-05}B_{2,6}(X) + 2.83332 \\ & {\cdot}10^{-06}B_{3,6}(X) - 9.25422{\cdot}10^{-06}B_{4,6}(X) - 2.15119{\cdot}10^{-05}B_{5,6}(X) - 3.39135{\cdot}10^{-05}B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.8888{\cdot}10^{-06}X^2 - 6.89166{\cdot}10^{-05}X + 3.78685{\cdot}10^{-05} \\ &= 3.78685{\cdot}10^{-05}B_{0,2} + 3.41018{\cdot}10^{-06}B_{1,2} - 3.39369{\cdot}10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = {} & -9.20917{\cdot}10^{-20}X^6 + 3.80232{\cdot}10^{-19}X^5 - 5.82517{\cdot}10^{-19}X^4 + 4.17693 \\ & {\cdot}10^{-19}X^3 - 2.8888{\cdot}10^{-06}X^2 - 6.89166{\cdot}10^{-05}X + 3.78685{\cdot}10^{-05} \\ = {} & 3.78685{\cdot}10^{-05}B_{0,6} + 2.63824{\cdot}10^{-05}B_{1,6} + 1.47037{\cdot}10^{-05}B_{2,6} + 2.83242 \\ & {\cdot}10^{-06}B_{3,6} - 9.23144{\cdot}10^{-06}B_{4,6} - 2.14879{\cdot}10^{-05}B_{5,6} - 3.39369{\cdot}10^{-05}B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.40038{\cdot}10^{-08}$.
Bounding polynomials M and m:

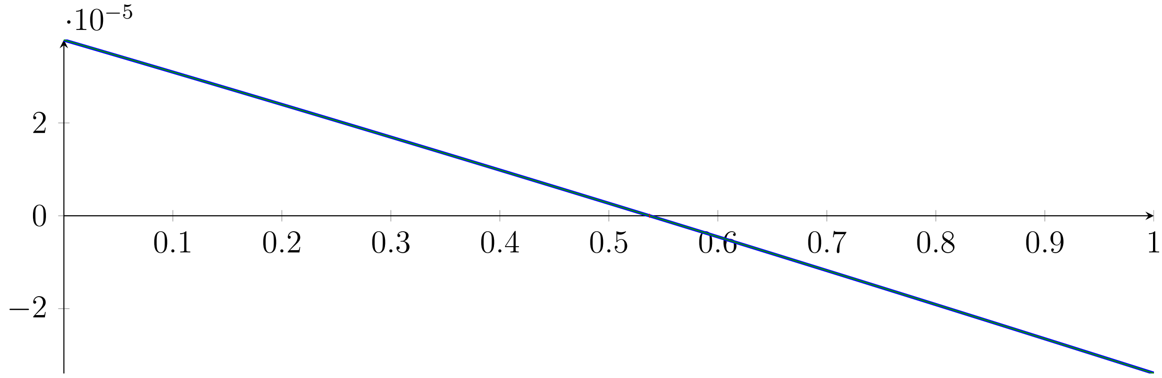
$$M = -2.8888{\cdot}10^{-06}X^2 - 6.89166{\cdot}10^{-05}X + 3.78925{\cdot}10^{-05}$$

$$m = -2.8888{\cdot}10^{-06}X^2 - 6.89166{\cdot}10^{-05}X + 3.78445{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-24.3942, 0.537711\} \qquad N(m) = \{-24.3935, 0.537045\}$$

Intersection intervals:



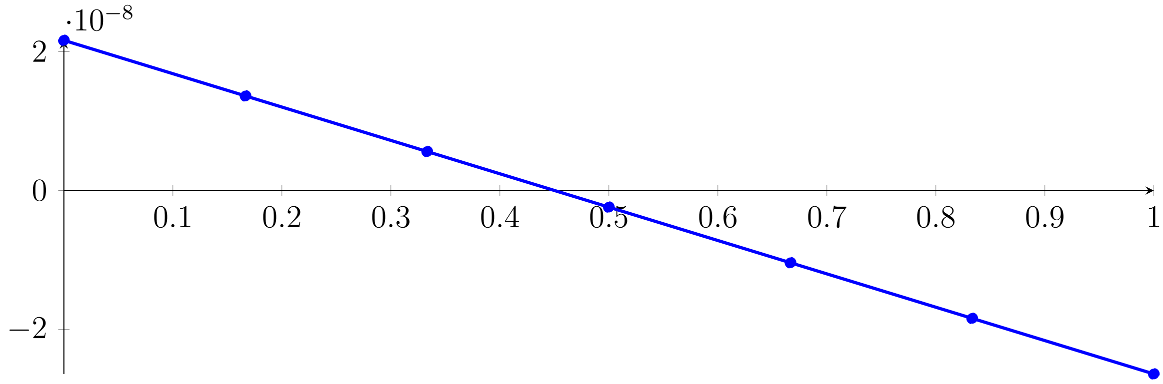
$$[0.537045, 0.537711]$$

Longest intersection interval: 0.000666575
$\Longrightarrow$ Selective recursion: interval 1: $[0.811741, 0.81175]$,

11.24 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.811741, 0.81175]$

Normalized monomial und Bézier representations and the Bézier polygon:

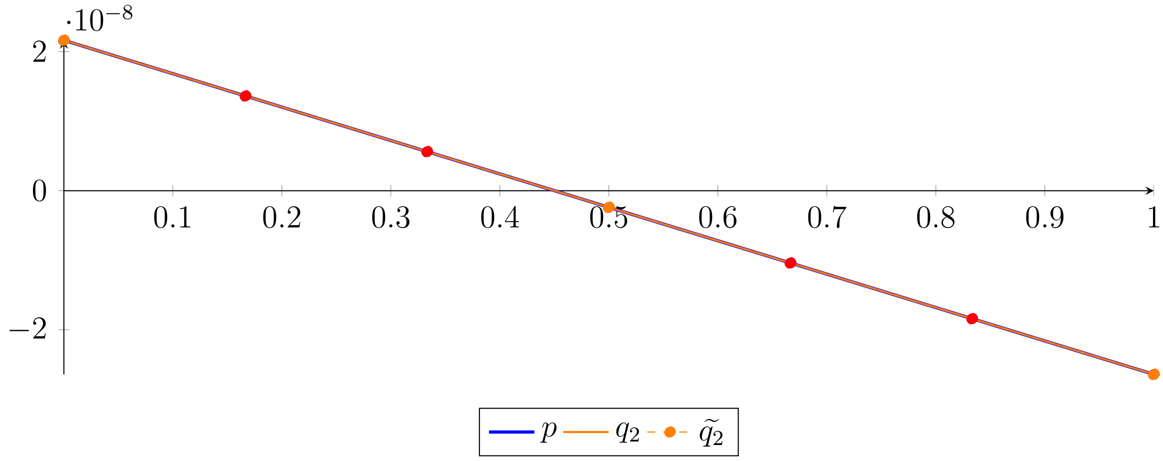
$$\begin{aligned}p &= 4.52364{\cdot}10^{-26}X^6 - 7.75482{\cdot}10^{-26}X^5 + 6.24578{\cdot}10^{-21}X^4 + 1.37232\\ &\quad {\cdot}10^{-16}X^3 - 1.2638{\cdot}10^{-12}X^2 - 4.80513{\cdot}10^{-08}X + 2.16374{\cdot}10^{-08}\\ &= 2.16374{\cdot}10^{-08}B_{0,6}(X) + 1.36289{\cdot}10^{-08}B_{1,6}(X) + 5.62022{\cdot}10^{-09}B_{2,6}(X) - 2.3885\\ &\quad {\cdot}10^{-09}B_{3,6}(X) - 1.03973{\cdot}10^{-08}B_{4,6}(X) - 1.84062{\cdot}10^{-08}B_{5,6}(X) - 2.64152{\cdot}10^{-08}B_{6,6}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= -1.26359{\cdot}10^{-12}X^2 - 4.80513{\cdot}10^{-08}X + 2.16374{\cdot}10^{-08}\\
&= 2.16374{\cdot}10^{-08}B_{0,2} - 2.38824{\cdot}10^{-09}B_{1,2} - 2.64152{\cdot}10^{-08}B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_2 &= 1.14052{\cdot}10^{-22}X^6 - 2.90525{\cdot}10^{-22}X^5 + 2.45464{\cdot}10^{-22}X^4 - 6.27171\\
&\quad {\cdot}10^{-23}X^3 - 1.26359{\cdot}10^{-12}X^2 - 4.80513{\cdot}10^{-08}X + 2.16374{\cdot}10^{-08}\\
&= 2.16374{\cdot}10^{-08}B_{0,6} + 1.36289{\cdot}10^{-08}B_{1,6} + 5.62022{\cdot}10^{-09}B_{2,6} - 2.3885\\
&\quad {\cdot}10^{-09}B_{3,6} - 1.03973{\cdot}10^{-08}B_{4,6} - 1.84062{\cdot}10^{-08}B_{5,6} - 2.64152{\cdot}10^{-08}B_{6,6}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.86243{\cdot}10^{-18}$.

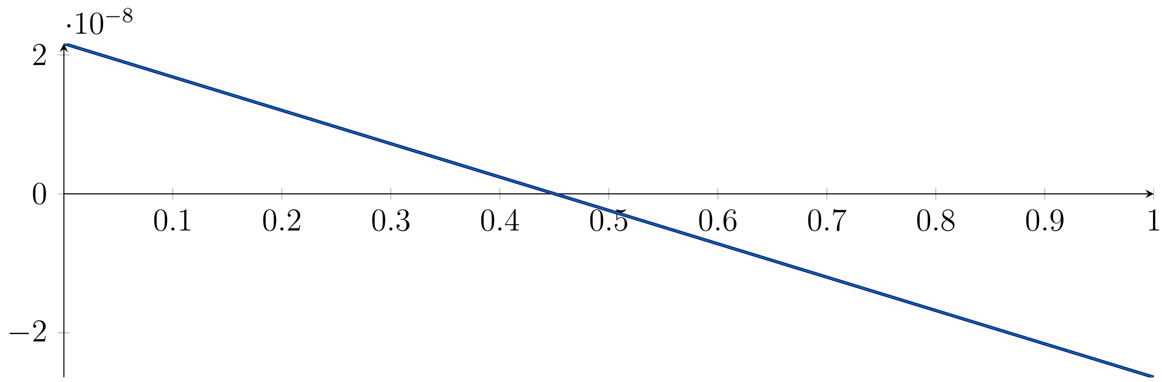
Bounding polynomials M and m:

$$M = -1.26359{\cdot}10^{-12}X^2 - 4.80513{\cdot}10^{-08}X + 2.16374{\cdot}10^{-08}$$
$$m = -1.26359{\cdot}10^{-12}X^2 - 4.80513{\cdot}10^{-08}X + 2.16374{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{-38028, 0.450293\} \qquad N(m) = \{-38028, 0.450293\}$$

Intersection intervals:



$$[0.450293, 0.450293]$$

Longest intersection interval: $2.85622 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.811745, 0.811745]$,

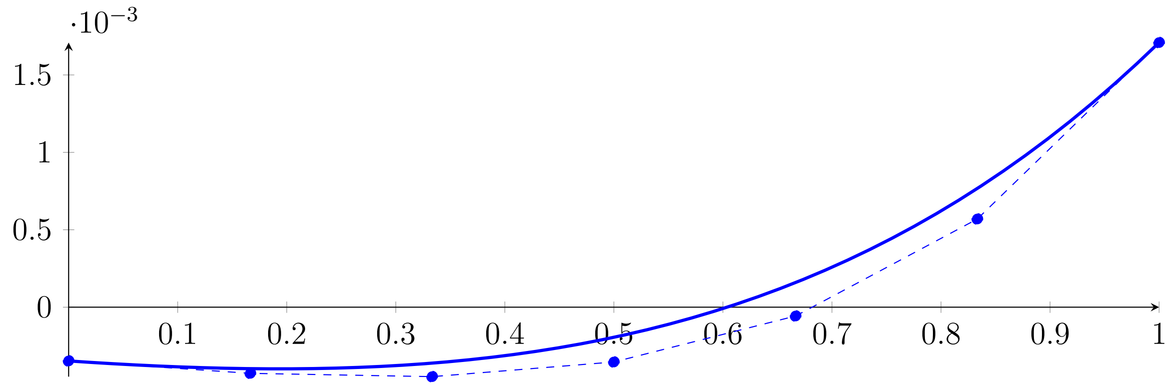
11.25 Recursion Branch 1 2 2 1 1 1 1 in Interval 1: $[0.811745, 0.811745]$

Found root in interval $[0.811745, 0.811745]$ at recursion depth 7!

11.26 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

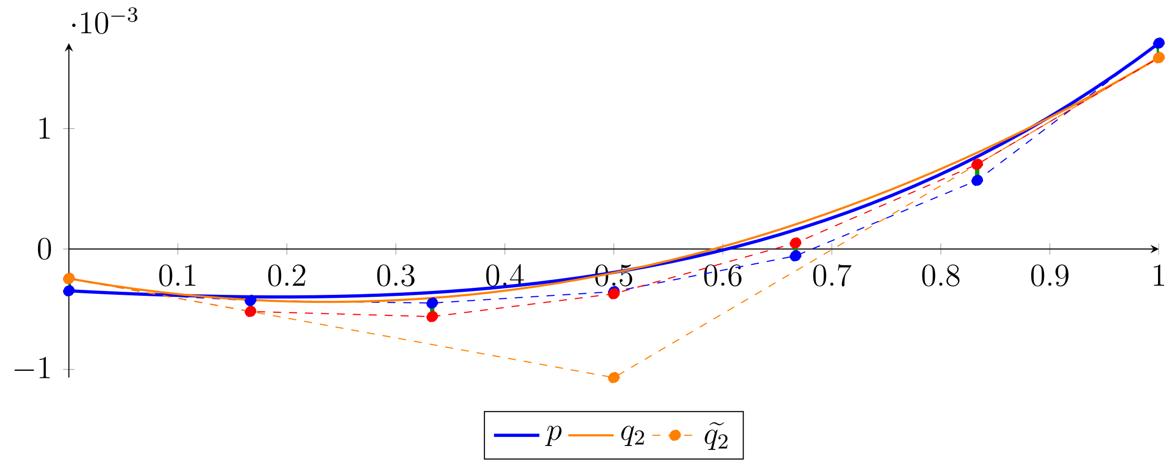
$$\begin{aligned} p &= 3.8147 \cdot 10^{-06} X^6 + 6.86646 \cdot 10^{-05} X^5 + 0.00043869 X^4 + 0.00114441 X^3 \\ &\quad + 0.000881195 X^2 - 0.000480652 X - 0.000347137 \\ &= -0.000347137 B_{0,6}(X) - 0.000427246 B_{1,6}(X) - 0.000448608 B_{2,6}(X) \\ &\quad - 0.000354004 B_{3,6}(X) - 5.69661 \cdot 10^{-05} B_{4,6}(X) + 0.000569661 B_{5,6}(X) + 0.00170898 B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00347928 X^2 - 0.00164631 X - 0.000244504 \\ &= -0.000244504 B_{0,2} - 0.00106766 B_{1,2} + 0.00158846 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.14766 \cdot 10^{-17} X^6 - 6.75427 \cdot 10^{-17} X^5 + 8.09348 \cdot 10^{-17} X^4 - 4.58255 \\ &\quad \cdot 10^{-17} X^3 + 0.00347928 X^2 - 0.00164631 X - 0.000244504 \\ &= -0.000244504 B_{0,6} - 0.00051889 B_{1,6} - 0.000561324 B_{2,6} - 0.000371806 B_{3,6} \\ &\quad + 4.96637 \cdot 10^{-05} B_{4,6} + 0.000703085 B_{5,6} + 0.00158846 B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000133424$.

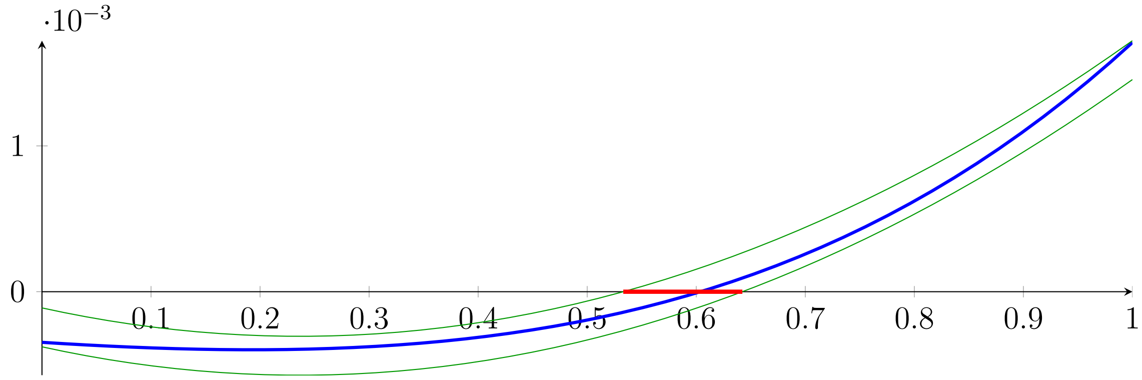
Bounding polynomials M and m:

$$M = 0.00347928X^2 - 0.00164631X - 0.00011108$$
$$m = 0.00347928X^2 - 0.00164631X - 0.000377928$$

Root of M and m:

$$N(M) = \{-0.0598915, 0.533069\} \qquad N(m) = \{-0.169116, 0.642294\}$$

Intersection intervals:



$$[0.533069, 0.642294]$$

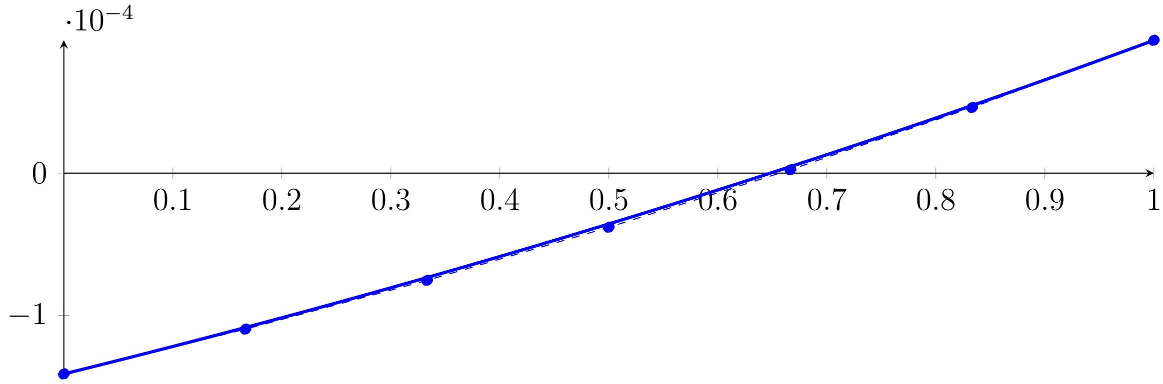
Longest intersection interval: 0.109225

$\Longrightarrow$ Selective recursion: interval 1: $[0.941634, 0.955287]$,

11.27 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.941634, 0.955287]$

Normalized monomial und Bézier representations and the Bézier polygon:

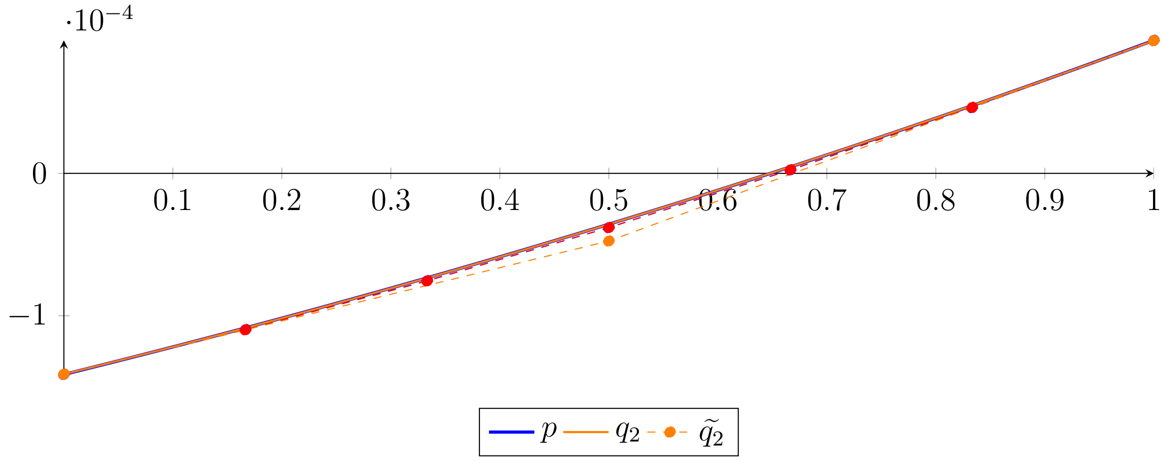
$$\begin{aligned} p &= 6.47727\cdot 10^{-12}X^6 + 1.25711\cdot 10^{-09}X^5 + 9.07997\cdot 10^{-08}X^4 + 2.97945 \\ &\quad \cdot 10^{-06}X^3 + 4.25657\cdot 10^{-05}X^2 + 0.000188843X - 0.000141136 \\ &= -0.000141136B_{0,6}(X) - 0.000109662B_{1,6}(X) - 7.53509\cdot 10^{-05}B_{2,6}(X) - 3.80527 \\ &\quad \cdot 10^{-05}B_{3,6}(X) + 2.38719\cdot 10^{-06}B_{4,6}(X) + 4.61301\cdot 10^{-05}B_{5,6}(X) + 9.33437\cdot 10^{-05}B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 4.71928\cdot 10^{-05}X^2 + 0.000186971X - 0.000140979 \\ &= -0.000140979B_{0,2} - 4.74939\cdot 10^{-05}B_{1,2} + 9.31842\cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 1.53411\cdot 10^{-18}X^6 - 5.0552\cdot 10^{-18}X^5 + 6.33726\cdot 10^{-18}X^4 - 3.74944 \\ &\quad \cdot 10^{-18}X^3 + 4.71928\cdot 10^{-05}X^2 + 0.000186971X - 0.000140979 \\ &= -0.000140979B_{0,6} - 0.000109817B_{1,6} - 7.55095\cdot 10^{-05}B_{2,6} - 3.80554 \\ &\quad \cdot 10^{-05}B_{3,6} + 2.54494\cdot 10^{-06}B_{4,6} + 4.62915\cdot 10^{-05}B_{5,6} + 9.31842\cdot 10^{-05}B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.61381 \cdot 10^{-07}$.
Bounding polynomials M and m:

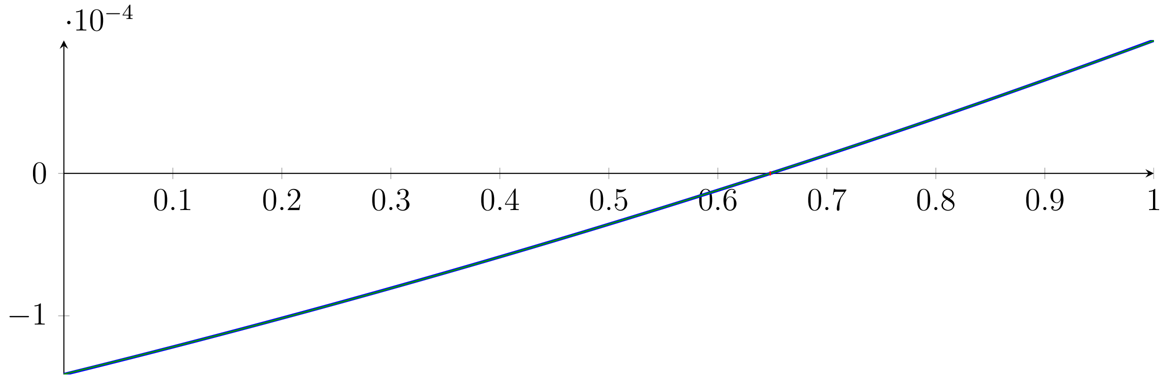
$$M = 4.71928 \cdot 10^{-05} X^2 + 0.000186971 X - 0.000140818$$

$$m = 4.71928 \cdot 10^{-05} X^2 + 0.000186971 X - 0.000141141$$

Root of M and m:

$$N(M) = \{-4.60922, 0.647373\} \qquad N(m) = \{-4.61052, 0.648674\}$$

Intersection intervals:



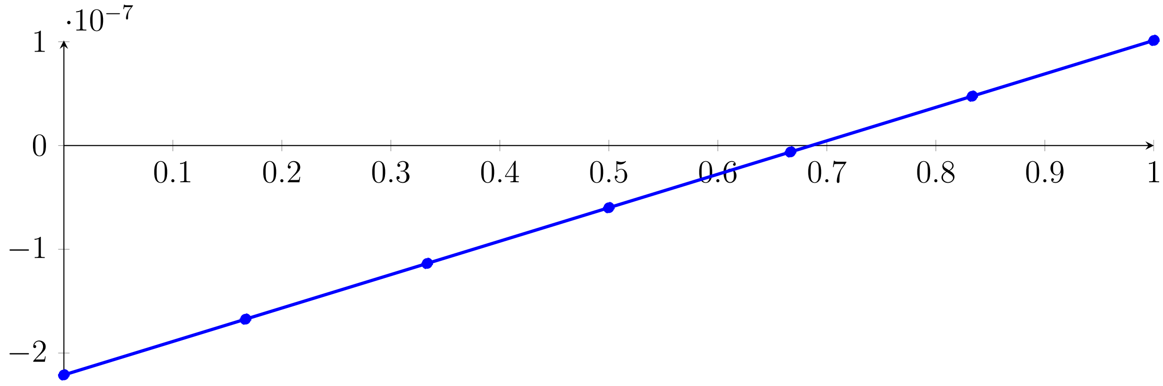
$$[0.647373, 0.648674]$$

Longest intersection interval: 0.00130075
$\Longrightarrow$ Selective recursion: interval 1: $[0.950472, 0.95049]$,

11.28 Recursion Branch 1 2 2 2 1 1 in Interval 1: $[0.950472, 0.95049]$

Normalized monomial und Bézier representations and the Bézier polygon:

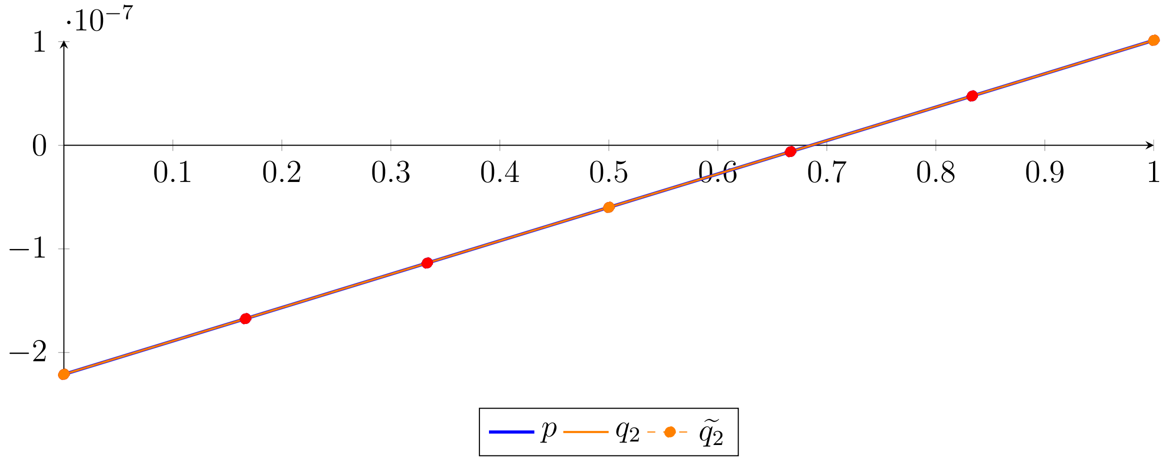
$$\begin{aligned} p &= 7.23783 \cdot 10^{-25} X^6 + 4.96308 \cdot 10^{-24} X^5 + 2.71698 \cdot 10^{-19} X^4 + 7.08633 \\ &\quad \cdot 10^{-15} X^3 + 8.22017 \cdot 10^{-11} X^2 + 3.22326 \cdot 10^{-07} X - 2.21095 \cdot 10^{-07} \\ &= -2.21095 \cdot 10^{-07} B_{0,6}(X) - 1.67374 \cdot 10^{-07} B_{1,6}(X) - 1.13648 \cdot 10^{-07} B_{2,6}(X) - 5.99157 \\ &\quad \cdot 10^{-08} B_{3,6}(X) - 6.17822 \cdot 10^{-09} B_{4,6}(X) + 4.75647 \cdot 10^{-08} B_{5,6}(X) + 1.01313 \cdot 10^{-07} B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.22123\cdot 10^{-11}X^2 + 3.22326\cdot 10^{-07}X - 2.21095\cdot 10^{-07} \\ &= -2.21095\cdot 10^{-07}B_{0,2} - 5.99321\cdot 10^{-08}B_{1,2} + 1.01313\cdot 10^{-07}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.27786\cdot 10^{-21}X^6 - 7.50581\cdot 10^{-21}X^5 + 9.38895\cdot 10^{-21}X^4 - 5.52893 \\ &\quad \cdot 10^{-21}X^3 + 8.22123\cdot 10^{-11}X^2 + 3.22326\cdot 10^{-07}X - 2.21095\cdot 10^{-07} \\ &= -2.21095\cdot 10^{-07}B_{0,6} - 1.67374\cdot 10^{-07}B_{1,6} - 1.13648\cdot 10^{-07}B_{2,6} - 5.99157 \\ &\quad \cdot 10^{-08}B_{3,6} - 6.17822\cdot 10^{-09}B_{4,6} + 4.75647\cdot 10^{-08}B_{5,6} + 1.01313\cdot 10^{-07}B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.54353\cdot 10^{-16}$.

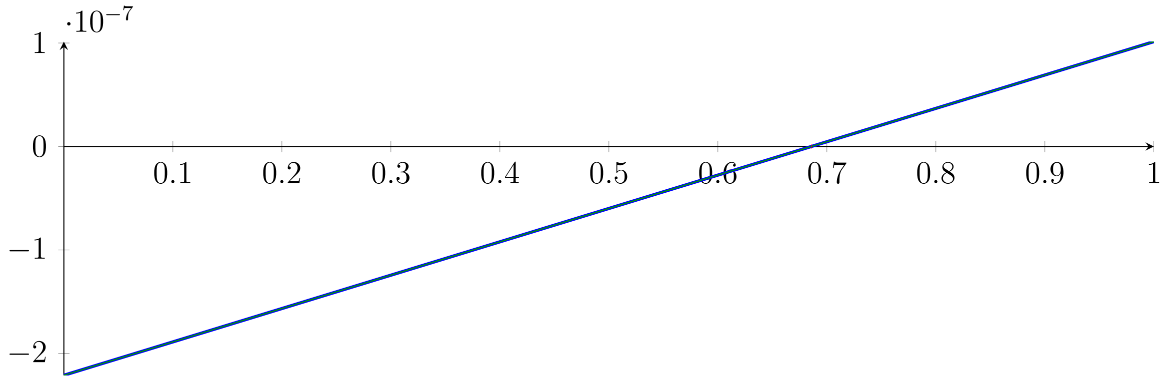
Bounding polynomials M and m:

$$M = 8.22123\cdot 10^{-11}X^2 + 3.22326\cdot 10^{-07}X - 2.21095\cdot 10^{-07}$$
$$m = 8.22123\cdot 10^{-11}X^2 + 3.22326\cdot 10^{-07}X - 2.21095\cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{-3921.34, 0.685816\} \qquad N(m) = \{-3921.34, 0.685816\}$$

Intersection intervals:



$$[0.685816, 0.685816]$$

Longest intersection interval: $2.19795 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.950484, 0.950484]$,

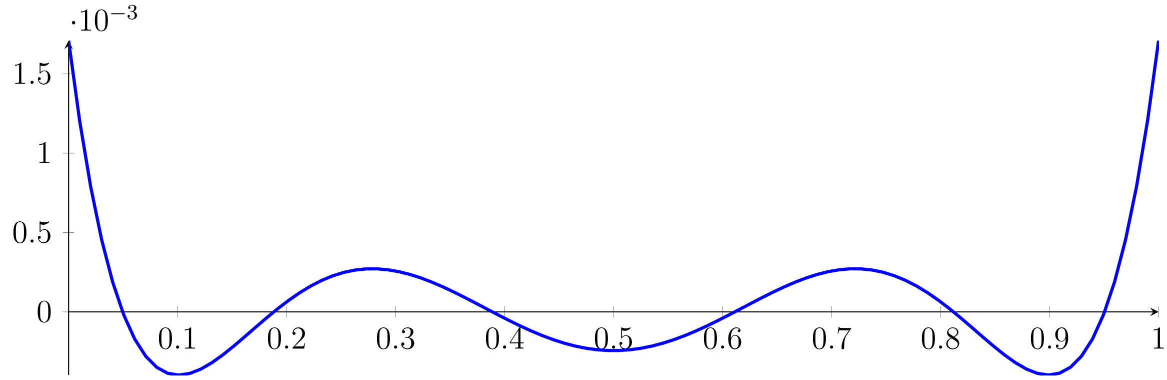
11.29 Recursion Branch 1 2 2 2 1 1 1 in Interval 1: $[0.950484, 0.950484]$

Found root in interval $[0.950484, 0.950484]$ at recursion depth 7!

11.30 Result: 6 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$



Result: Root Intervals

$$[0.0495156, 0.0495156],\ [0.188255, 0.188255],\ [0.38874, 0.38874],\ [0.61126, 0.61126],\ [0.811745, 0.811745],\ [0.950484, 0.950484]$$

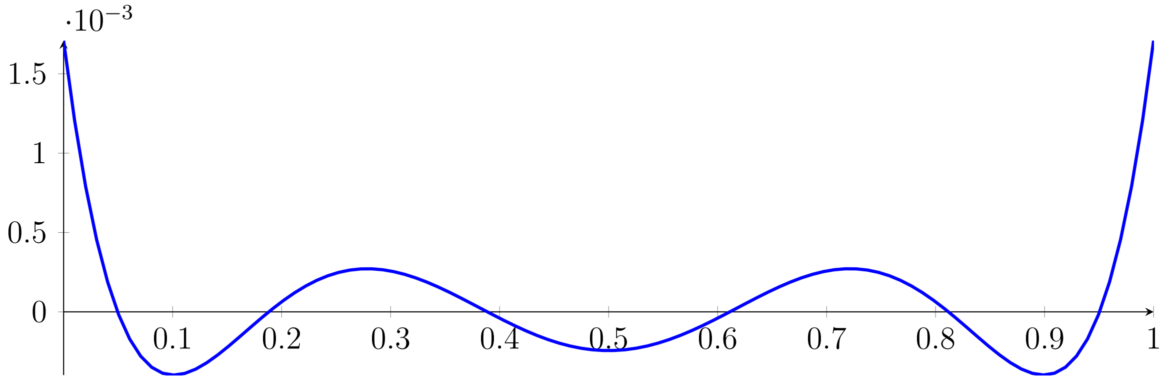
with precision $\varepsilon = 1 \cdot 10^{-06}$.

12 Running CubeClip on p6 with epsilon 6

$1X^6-3X^5+3.4375X^4-1.875X^3+0.492188X^2-0.0546875X+0.00170898$

Called CubeClip with input polynomial on interval $[0, 1]$:

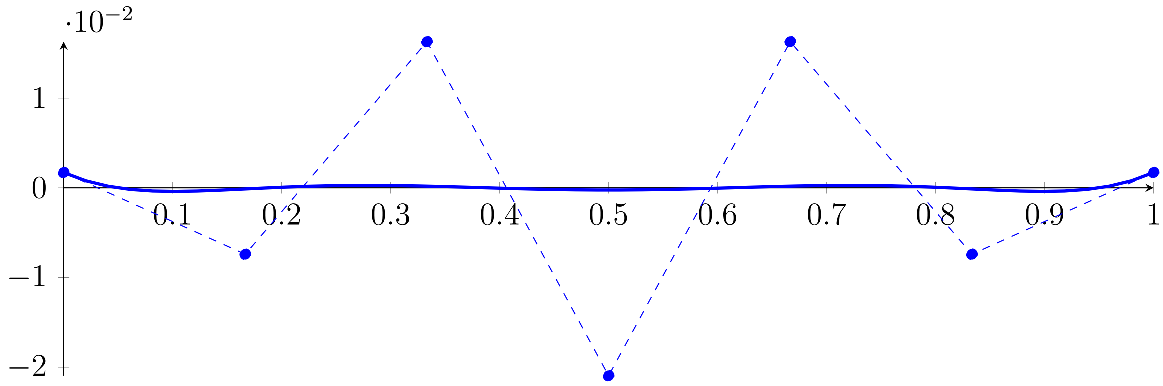
$$p = 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$



12.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

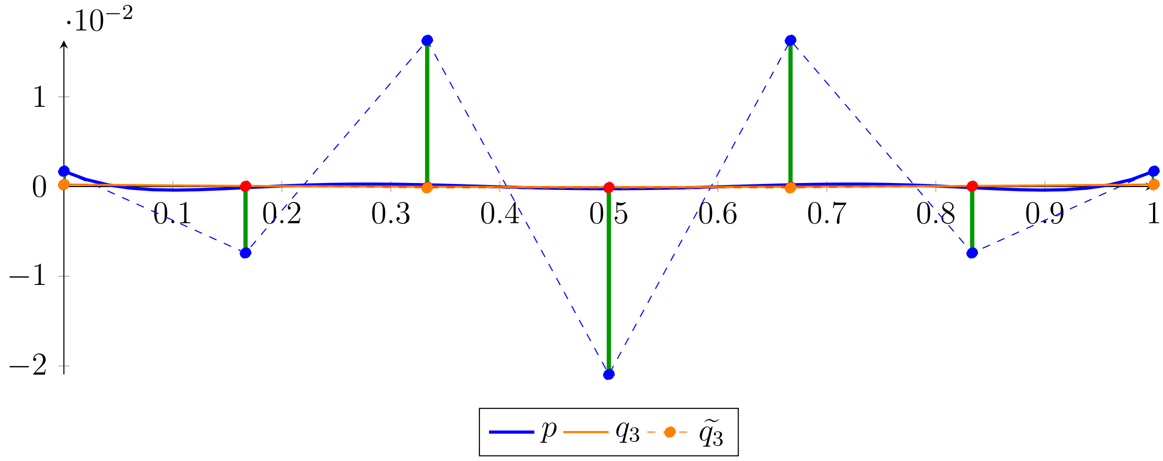
$$\begin{aligned} p &= 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898 \\ &= 0.00170898B_{0,6}(X) - 0.0074056B_{1,6}(X) + 0.0162923B_{2,6}(X) - 0.0209473B_{3,6}(X) \\ &\quad + 0.0162923B_{4,6}(X) - 0.0074056B_{5,6}(X) + 0.00170898B_{6,6}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.7513\cdot 10^{-18}X^3 + 0.00111607X^2 - 0.00111607X + 0.000220889 \\ &= 0.000220889B_{0,3} - 0.000151135B_{1,3} - 0.000151135B_{2,3} + 0.000220889B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 8.2902\cdot 10^{-18}X^6 - 2.50648\cdot 10^{-17}X^5 + 2.88093\cdot 10^{-17}X^4 - 1.83719 \\ &\quad \cdot 10^{-17}X^3 + 0.00111607X^2 - 0.00111607X + 0.000220889 \\ &= 0.000220889B_{0,6} + 3.48772\cdot 10^{-05}B_{1,6} - 7.67299\cdot 10^{-05}B_{2,6} - 0.000113932B_{3,6} \\ &\quad - 7.67299\cdot 10^{-05}B_{4,6} + 3.48772\cdot 10^{-05}B_{5,6} + 0.000220889B_{6,6} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0208333$.
Bounding polynomials M and m:

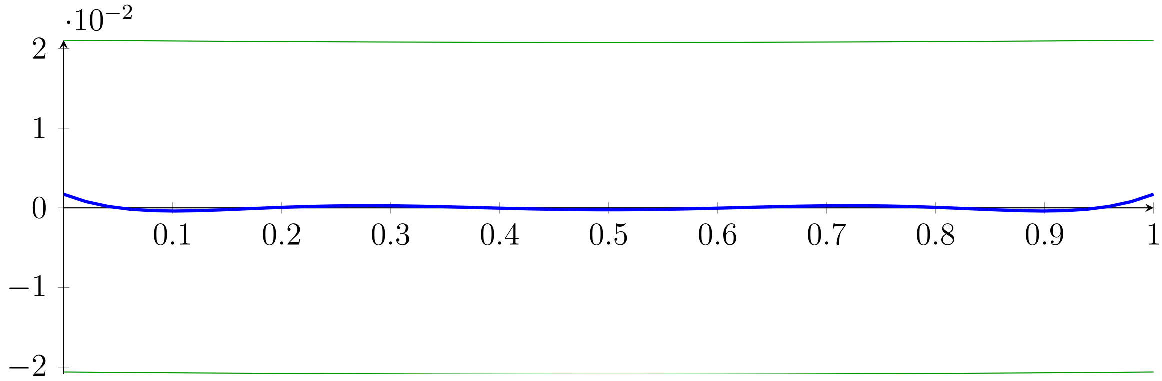
$$M = -2.75455 \cdot 10^{-18} X^3 + 0.00111607 X^2 - 0.00111607 X + 0.0210542$$

$$m = -2.75116 \cdot 10^{-18} X^3 + 0.00111607 X^2 - 0.00111607 X - 0.0206124$$

Root of M and m:

$$N(M) = \{-25674.7, 25675.7, 4.05174 \cdot 10^{14}\} \qquad N(m) = \{-25706.3, 25707.3, 4.05673 \cdot 10^{14}\}$$

Intersection intervals:

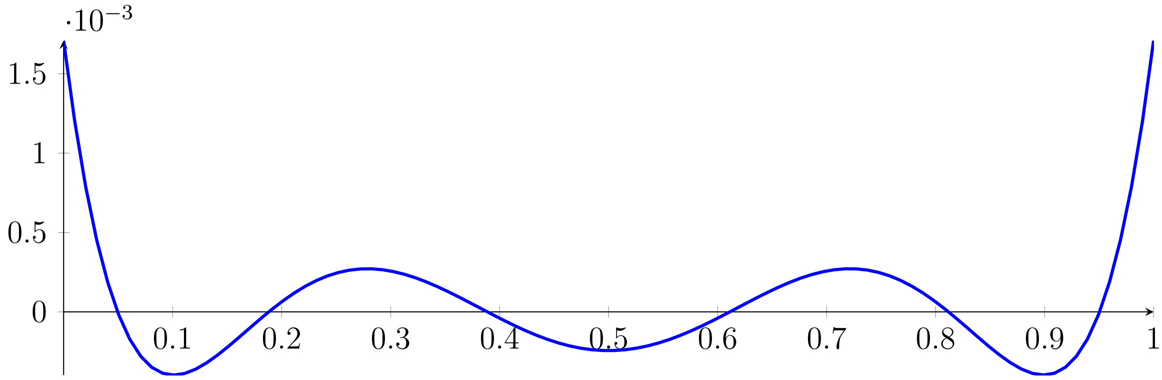


No intersection intervals with the x axis.

12.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^6 - 3X^5 + 3.4375X^4 - 1.875X^3 + 0.492188X^2 - 0.0546875X + 0.00170898$$



Result: Root Intervals

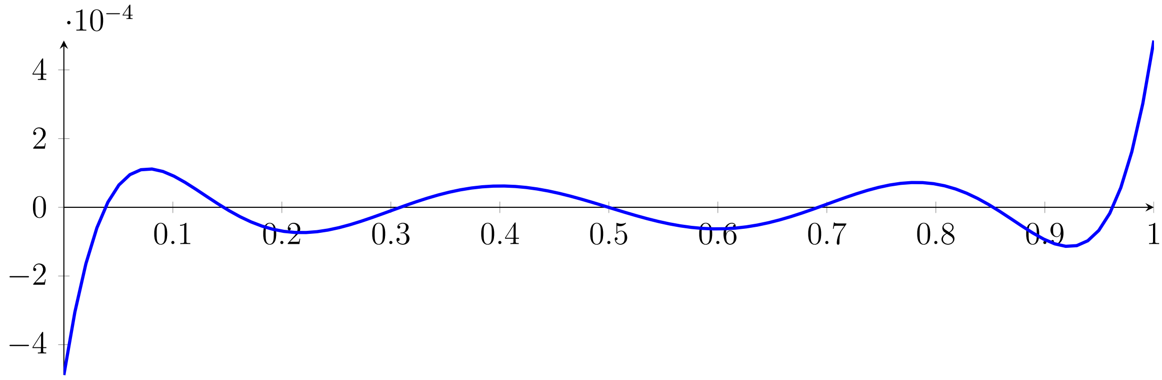
with precision $\varepsilon = 1 \cdot 10^{-06}$.

13 Running `BezClip` on p7 with epsilon 6

$$1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$

Called `BezClip` with input polynomial on interval $[0,1]$:

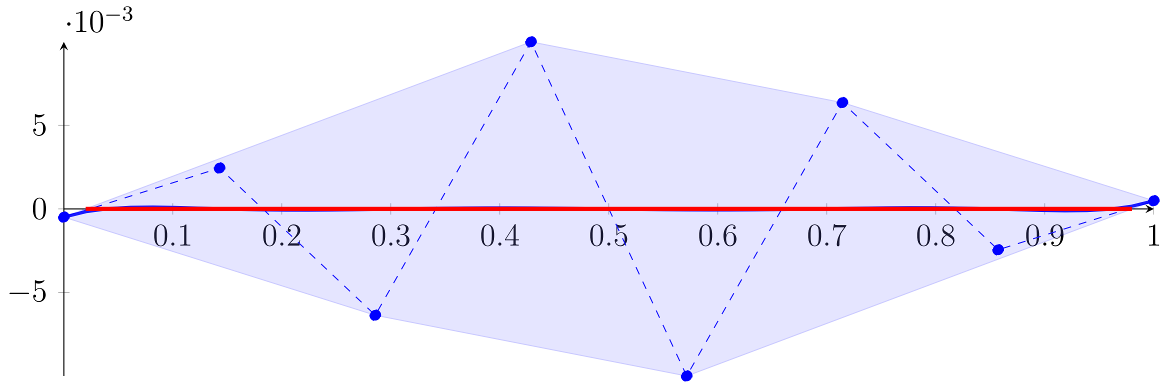
$$p = 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$



13.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.00244141B_{1,7}(X) - 0.00634766B_{2,7}(X) + 0.00997489B_{3,7}(X) \\ &\quad - 0.00997489B_{4,7}(X) + 0.00634766B_{5,7}(X) - 0.00244141B_{6,7}(X) + 0.000488281B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.02, 0.98\}$$

Intersection intervals with the x axis:

$$[0.02, 0.98]$$

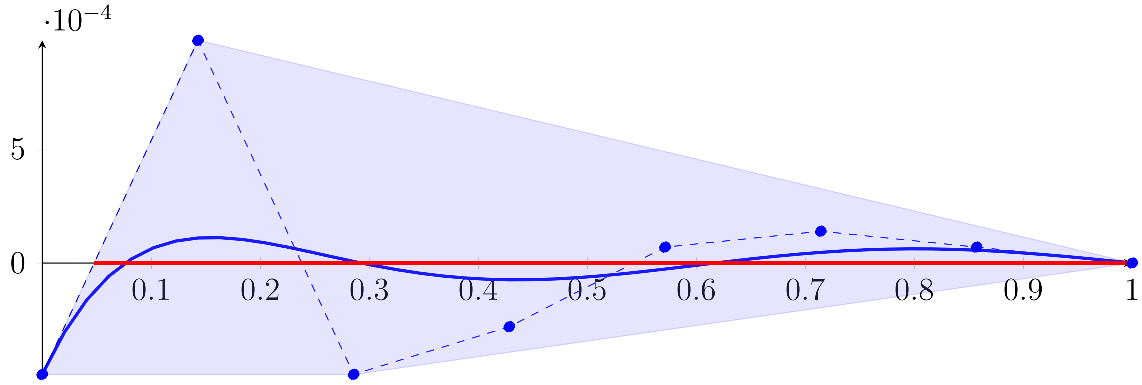
Longest intersection interval: 0.96

$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

13.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.0078125X^7 - 0.0546875X^6 + 0.152344X^5 - 0.214844X^4 \\ &\quad + 0.161133X^3 - 0.0615234X^2 + 0.0102539X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.000976562B_{1,7}(X) - 0.000488281B_{2,7}(X) - 0.000279018B_{3,7}(X) \\ &\quad + 6.97545 \cdot 10^{-05}B_{4,7}(X) + 0.000139509B_{5,7}(X) + 6.97545 \cdot 10^{-05}B_{6,7}(X) - 2.05803 \cdot 10^{-21}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.047619, 1\}$$

Intersection intervals with the x axis:

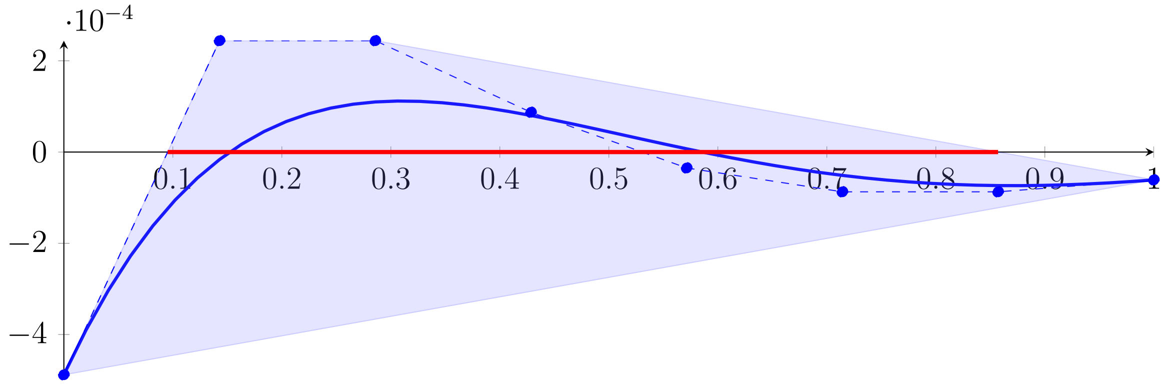
$$[0.047619, 1]$$

Longest intersection interval: 0.952381
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

13.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 6.10352 \cdot 10^{-05}X^7 - 0.000854492X^6 + 0.00476074X^5 - 0.0134277X^4 \\ &\quad + 0.0201416X^3 - 0.0153809X^2 + 0.00512695X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.000244141B_{1,7}(X) + 0.000244141B_{2,7}(X) + 8.71931 \cdot 10^{-05}B_{3,7}(X) \\ &\quad - 3.48772 \cdot 10^{-05}B_{4,7}(X) - 8.71931 \cdot 10^{-05}B_{5,7}(X) - 8.71931 \cdot 10^{-05}B_{6,7}(X) - 6.10352 \cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0952381, 0.857143\}$$

Intersection intervals with the x axis:

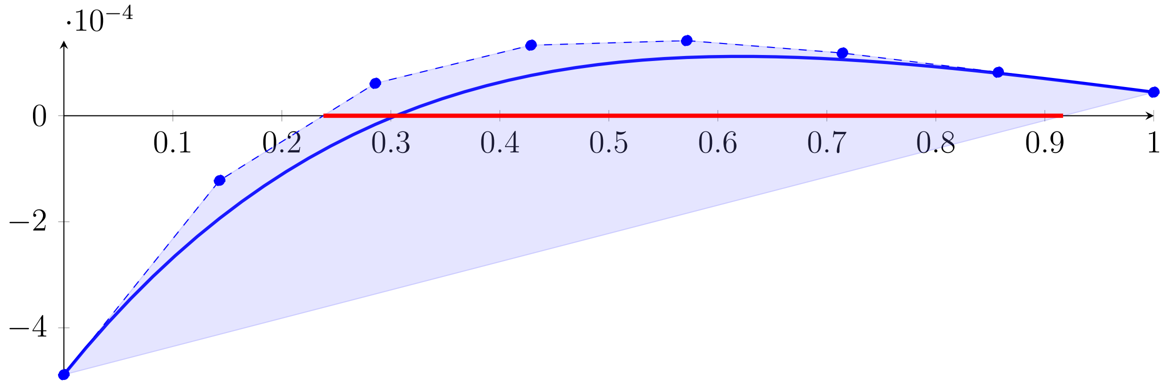
$$[0.0952381, 0.857143]$$

Longest intersection interval: 0.761905
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$

13.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 4.76837\cdot 10^{-07}X^7 - 1.33514\cdot 10^{-05}X^6 + 0.000148773X^5 - 0.000839233X^4 + 0.0025177X^3 - 0.00384521X^2 + 0.00256348X - 0.000488281$$
$$= -0.000488281B_{0,7}(X) - 0.00012207B_{1,7}(X) + 6.10352\cdot 10^{-05}B_{2,7}(X) + 0.000132969B_{3,7}(X) + 0.000141689B_{4,7}(X) + 0.000118256B_{5,7}(X) + 8.2016\cdot 10^{-05}B_{6,7}(X) + 4.43459\cdot 10^{-05}B_{7,7}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.238095, 0.916741\}$$

Intersection intervals with the x axis:

$$[0.238095, 0.916741]$$

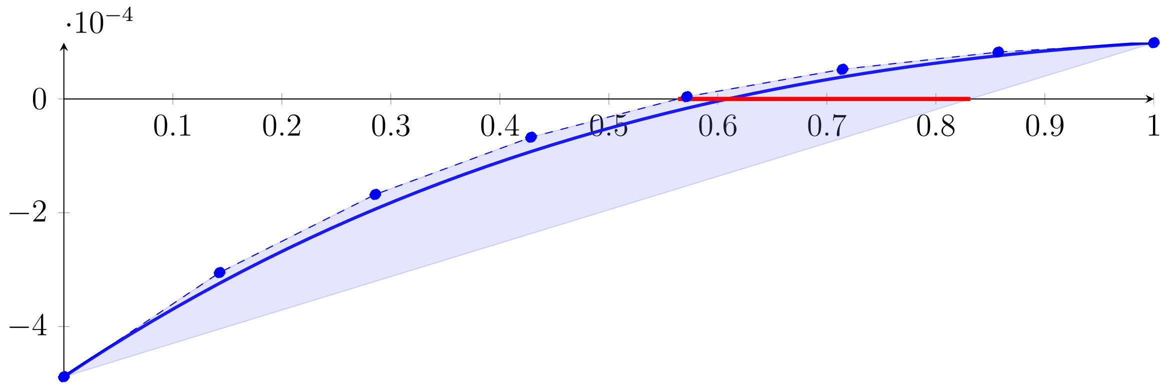
Longest intersection interval: 0.678646

$\Longrightarrow$ Bisection: first half $[0, 0.0625]$ und second half $[0.0625, 0.125]$

13.5 Recursion Branch 1 1 1 1 1 on the First Half $[0, 0.0625]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 3.72529\cdot 10^{-09}X^7 - 2.08616\cdot 10^{-07}X^6 + 4.64916\cdot 10^{-06}X^5 - 5.24521\cdot 10^{-05}X^4 + 0.000314713X^3 - 0.000961304X^2 + 0.00128174X - 0.000488281$$
$$= -0.000488281B_{0,7}(X) - 0.000305176B_{1,7}(X) - 0.000167847B_{2,7}(X) - 6.73022\cdot 10^{-05}B_{3,7}(X) + 3.95094\cdot 10^{-06}B_{4,7}(X) + 5.21285\cdot 10^{-05}B_{5,7}(X) + 8.23608\cdot 10^{-05}B_{6,7}(X) + 9.8858\cdot 10^{-05}B_{7,7}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.563507, 0.831628\}$$

Intersection intervals with the x axis:

$$[0.563507, 0.831628]$$

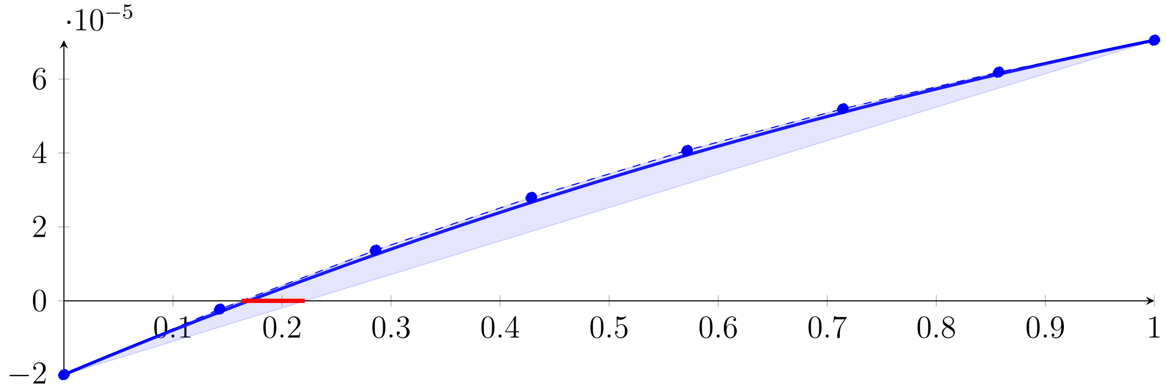
Longest intersection interval: 0.26812

$\Longrightarrow$ Selective recursion: interval 1: $[0.0352192, 0.0519767]$,

13.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0352192, 0.0519767]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.71081\cdot10^{-13}X^7 - 7.20453\cdot10^{-11}X^6 + 5.49912\cdot10^{-09}X^5 - 2.08389\cdot10^{-07}X^4\\ &\quad + 4.0576\cdot10^{-06}X^3 - 3.74684\cdot10^{-05}X^2 + 0.000124104X - 1.9983\cdot10^{-05}\\ &= -1.9983\cdot10^{-05}B_{0,7}(X) - 2.25381\cdot10^{-06}B_{1,7}(X) + 1.36912\cdot10^{-05}B_{2,7}(X) + 2.79679\cdot10^{-05}B_{3,7}(X)\\ &\quad + 4.06863\cdot10^{-05}B_{4,7}(X) + 5.19507\cdot10^{-05}B_{5,7}(X) + 6.18599\cdot10^{-05}B_{6,7}(X) + 7.05076\cdot10^{-05}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.16305, 0.22083\}$$

Intersection intervals with the x axis:

$$[0.16305, 0.22083]$$

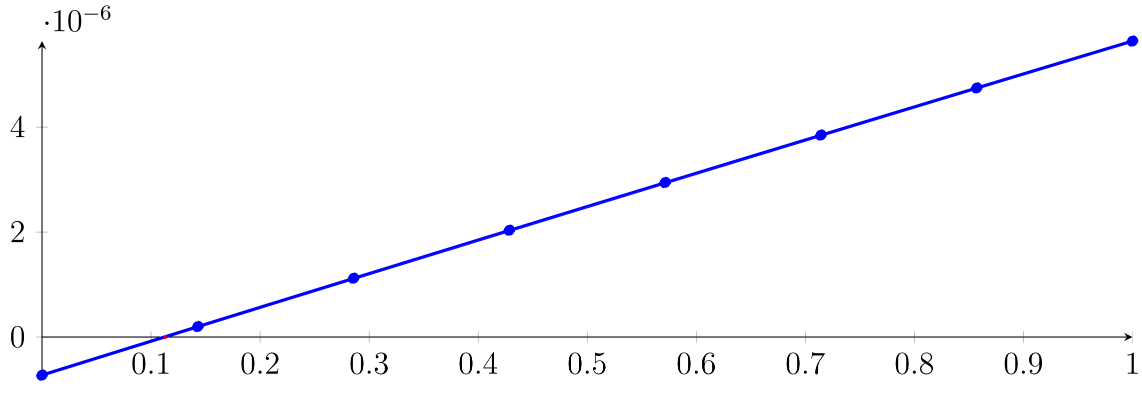
Longest intersection interval: 0.0577797

$\Longrightarrow$ Selective recursion: interval 1: $[0.0379515, 0.0389198]$,

13.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.0379515, 0.0389198]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 7.73827\cdot10^{-22}X^7 - 2.66522\cdot10^{-18}X^6 + 3.49611\cdot10^{-15}X^5 - 2.27296\cdot10^{-12}X^4\\ &\quad + 7.56765\cdot10^{-10}X^3 - 1.18572\cdot10^{-07}X^2 + 6.48323\cdot10^{-06}X - 7.26469\cdot10^{-07}\\ &= -7.26469\cdot10^{-07}B_{0,7}(X) + 1.99707\cdot10^{-07}B_{1,7}(X) + 1.12024\cdot10^{-06}B_{2,7}(X) + 2.03514\cdot10^{-06}B_{3,7}(X)\\ &\quad + 2.94444\cdot10^{-06}B_{4,7}(X) + 3.84816\cdot10^{-06}B_{5,7}(X) + 4.74632\cdot10^{-06}B_{6,7}(X) + 5.63894\cdot10^{-06}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.112054, 0.114128\}$$

Intersection intervals with the x axis:

$$[0.112054, 0.114128]$$

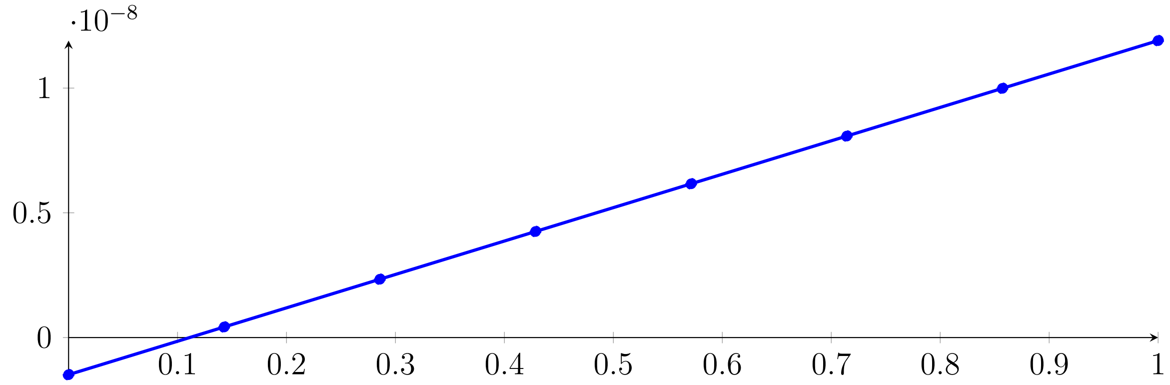
Longest intersection interval: 0.002074

$\Longrightarrow$ Selective recursion: interval 1: $[0.03806, 0.038062]$,

13.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.03806, 0.038062]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -6.40176 \cdot 10^{-26} X^7 - 3.71787 \cdot 10^{-25} X^6 + 2.41732 \cdot 10^{-25} X^5 - 4.20345 \cdot 10^{-23} X^4 \\ &\quad + 6.74224 \cdot 10^{-18} X^3 - 5.08943 \cdot 10^{-13} X^2 + 1.33912 \cdot 10^{-08} X - 1.48773 \cdot 10^{-09} \\ &= -1.48773 \cdot 10^{-09} B_{0,7}(X) + 4.253 \cdot 10^{-10} B_{1,7}(X) + 2.3383 \cdot 10^{-09} B_{2,7}(X) + 4.25128 \cdot 10^{-09} B_{3,7}(X) \\ &\quad + 6.16423 \cdot 10^{-09} B_{4,7}(X) + 8.07716 \cdot 10^{-09} B_{5,7}(X) + 9.99007 \cdot 10^{-09} B_{6,7}(X) + 1.1903 \cdot 10^{-08} B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.111097, 0.111102\}$$

Intersection intervals with the x axis:

$$[0.111097, 0.111102]$$

Longest intersection interval: $4.22246 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0380602, 0.0380602]$,

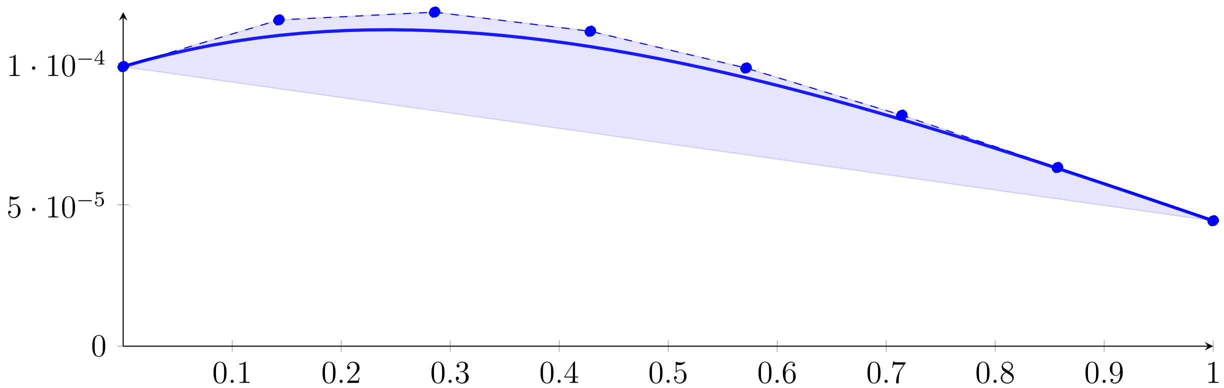
13.9 Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.0380602, 0.0380602]$

Found root in interval $[0.0380602, 0.0380602]$ at recursion depth 9!

13.10 Recursion Branch 1 1 1 1 2 on the Second Half $[0.0625, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.72529 \cdot 10^{-09} X^7 - 1.82539 \cdot 10^{-07} X^6 + 3.4757 \cdot 10^{-06} X^5 - 3.22051 \cdot 10^{-05} X^4 \\ &\quad + 0.000147354 X^3 - 0.000288438 X^2 + 0.00011548 X + 9.8858 \cdot 10^{-05} \\ &= 9.8858 \cdot 10^{-05} B_{0,7}(X) + 0.000115355 B_{1,7}(X) + 0.000118117 B_{2,7}(X) + 0.000111354 B_{3,7}(X) \\ &\quad + 9.83562 \cdot 10^{-05} B_{4,7}(X) + 8.16584 \cdot 10^{-05} B_{5,7}(X) + 6.31809 \cdot 10^{-05} B_{6,7}(X) + 4.43459 \cdot 10^{-05} B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

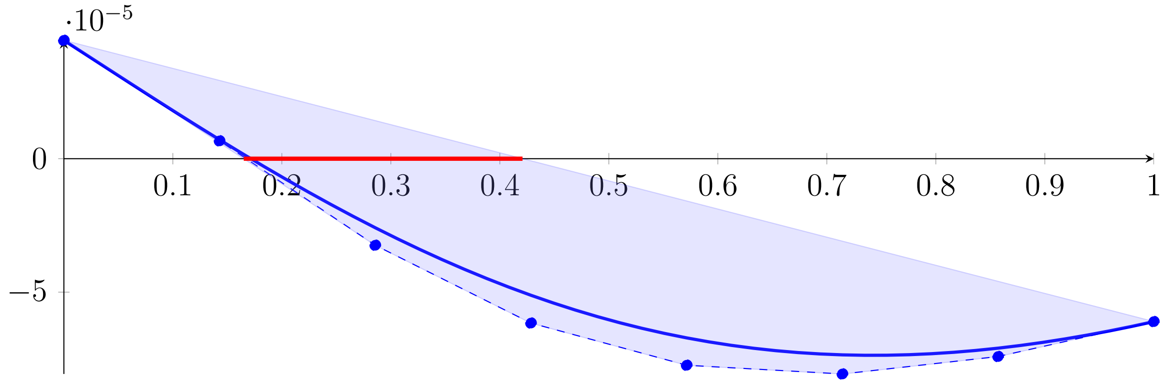
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

13.11 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 4.76837\cdot10^{-07}X^7 - 1.00136\cdot10^{-05}X^6 + 7.86781\cdot10^{-05}X^5 - 0.00027895X^4 \\ &\quad + 0.000398159X^3 - 3.00407\cdot10^{-05}X^2 - 0.000263691X + 4.43459\cdot10^{-05} \\ &= 4.43459\cdot10^{-05}B_{0,7}(X) + 6.67572\cdot10^{-06}B_{1,7}(X) - 3.24249\cdot10^{-05}B_{2,7}(X) - 6.15801\cdot10^{-05}B_{3,7}(X) \\ &\quad - 7.73839\cdot10^{-05}B_{4,7}(X) - 8.06536\cdot10^{-05}B_{5,7}(X) - 7.41141\cdot10^{-05}B_{6,7}(X) - 6.10352\cdot10^{-05}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.16504, 0.420814\}$$

Intersection intervals with the x axis:

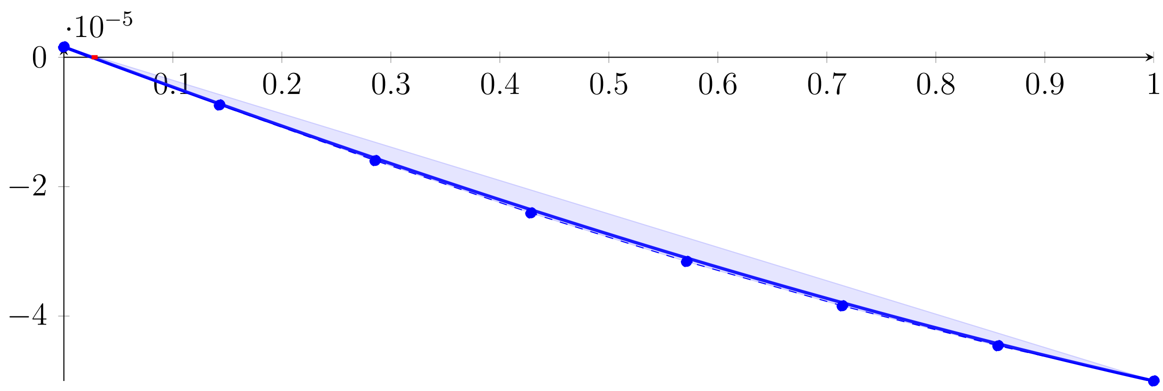
$$[0.16504, 0.420814]$$

Longest intersection interval: 0.255775
$\Longrightarrow$ Selective recursion: interval 1: $[0.14563, 0.177602]$,

13.12 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.14563, 0.177602]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.41485\cdot10^{-11}X^7 - 2.64947\cdot10^{-09}X^6 + 7.55712\cdot10^{-08}X^5 - 9.33184\cdot10^{-07}X^4 \\ &\quad + 3.92471\cdot10^{-06}X^3 + 8.17325\cdot10^{-06}X^2 - 6.28701\cdot10^{-05}X + 1.60042\cdot10^{-06} \\ &= 1.60042\cdot10^{-06}B_{0,7}(X) - 7.38102\cdot10^{-06}B_{1,7}(X) - 1.59733\cdot10^{-05}B_{2,7}(X) - 2.40642\cdot10^{-05}B_{3,7}(X) \\ &\quad - 3.15683\cdot10^{-05}B_{4,7}(X) - 3.84231\cdot10^{-05}B_{5,7}(X) - 4.45862\cdot10^{-05}B_{6,7}(X) - 5.00319\cdot10^{-05}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.025456, 0.0309964\}$$

Intersection intervals with the x axis:

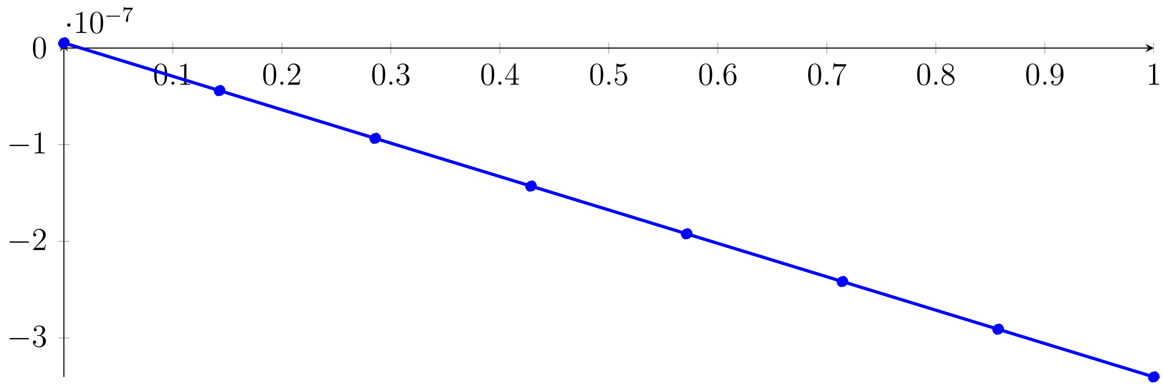
$$[0.025456, 0.0309964]$$

Longest intersection interval: 0.00554047
$\Longrightarrow$ Selective recursion: interval 1: $[0.146444, 0.146621]$,

13.13 Recursion Branch 1 1 1 2 1 1 in Interval 1: $[0.146444, 0.146621]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.2537{\cdot}10^{-24}X^7 - 6.23358{\cdot}10^{-23}X^6 + 3.92419{\cdot}10^{-19}X^5 - 8.70292{\cdot}10^{-16}X^4\\ &\quad + 6.51416{\cdot}10^{-13}X^3 + 2.59982{\cdot}10^{-10}X^2 - 3.45982{\cdot}10^{-07}X + 5.36067{\cdot}10^{-09}\\ &= 5.36067{\cdot}10^{-09}B_{0,7}(X) - 4.40654{\cdot}10^{-08}B_{1,7}(X) - 9.3479{\cdot}10^{-08}B_{2,7}(X) - 1.4288{\cdot}10^{-07}B_{3,7}(X)\\ &\quad - 1.92269{\cdot}10^{-07}B_{4,7}(X) - 2.41646{\cdot}10^{-07}B_{5,7}(X) - 2.9101{\cdot}10^{-07}B_{6,7}(X) - 3.40361{\cdot}10^{-07}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0154941, 0.0155057\}$$

Intersection intervals with the x axis:

$$[0.0154941, 0.0155057]$$

Longest intersection interval: $1.16806{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.146447, 0.146447]$,

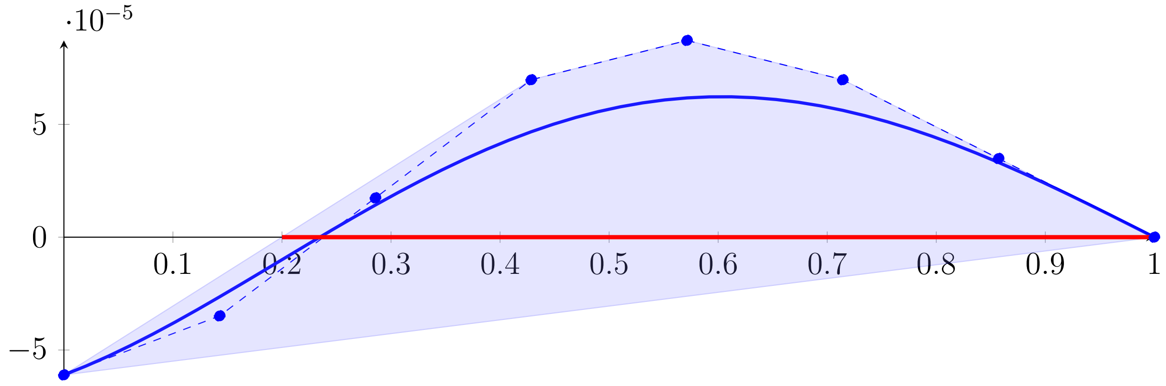
13.14 Recursion Branch 1 1 1 2 1 1 1 in Interval 1: $[0.146447, 0.146447]$

Found root in interval $[0.146447, 0.146447]$ at recursion depth 7!

13.15 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 6.10352{\cdot}10^{-05}X^7 - 0.000427246X^6 + 0.000915527X^5 - 0.000305176X^4\\ &\quad - 0.000915527X^3 + 0.000549316X^2 + 0.000183105X - 6.10352{\cdot}10^{-05}\\ &= -6.10352{\cdot}10^{-05}B_{0,7}(X) - 3.48772{\cdot}10^{-05}B_{1,7}(X) + 1.74386{\cdot}10^{-05}B_{2,7}(X) + 6.97545{\cdot}10^{-05}B_{3,7}(X)\\ &\quad + 8.71931{\cdot}10^{-05}B_{4,7}(X) + 6.97545{\cdot}10^{-05}B_{5,7}(X) + 3.48772{\cdot}10^{-05}B_{6,7}(X) - 2.05803{\cdot}10^{-21}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.2, 1\}$$

Intersection intervals with the x axis:

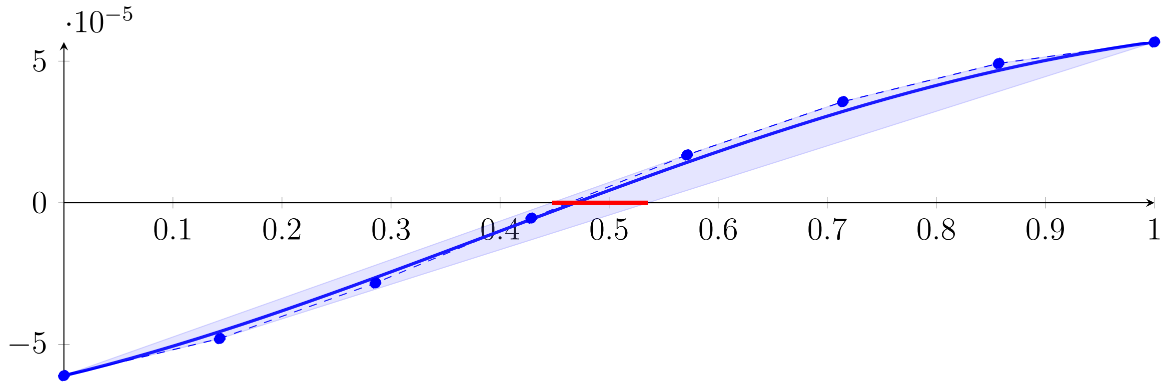
$$[0.2, 1]$$

Longest intersection interval: 0.8
$\Longrightarrow$ Bisection: first half $[0.25, 0.375]$ und second half $[0.375, 0.5]$

13.16 Recursion Branch 1 1 2 1 on the First Half $[0.25, 0.375]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 4.76837{\cdot}10^{-07}X^7 - 6.67572{\cdot}10^{-06}X^6 + 2.86102{\cdot}10^{-05}X^5 - 1.90735{\cdot}10^{-05}X^4 \\ &\quad - 0.000114441X^3 + 0.000137329X^2 + 9.15527{\cdot}10^{-05}X - 6.10352{\cdot}10^{-05} \\ &= -6.10352{\cdot}10^{-05}B_{0,7}(X) - 4.79562{\cdot}10^{-05}B_{1,7}(X) - 2.83378{\cdot}10^{-05}B_{2,7}(X) - 5.44957{\cdot}10^{-06}B_{3,7}(X) \\ &\quad + 1.68937{\cdot}10^{-05}B_{4,7}(X) + 3.56947{\cdot}10^{-05}B_{5,7}(X) + 4.91823{\cdot}10^{-05}B_{6,7}(X) + 5.67436{\cdot}10^{-05}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.447552, 0.535459\}$$

Intersection intervals with the x axis:

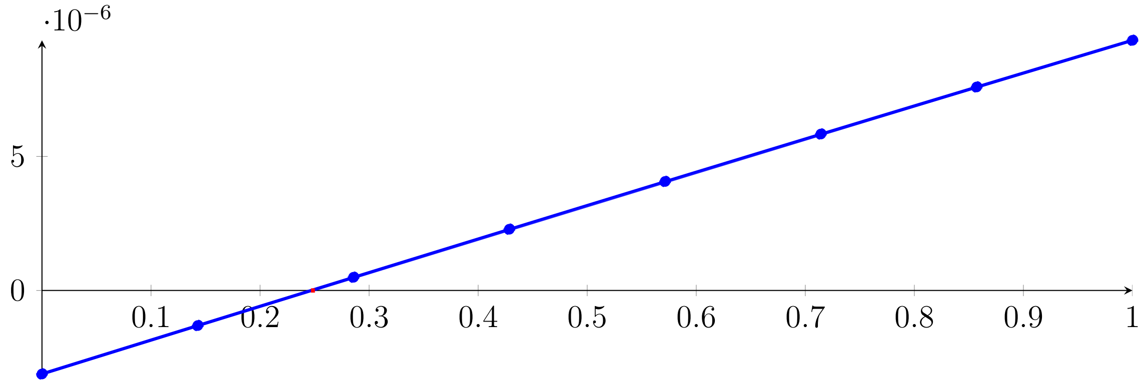
$$[0.447552, 0.535459]$$

Longest intersection interval: 0.0879062
$\Longrightarrow$ Selective recursion: interval 1: $[0.305944, 0.316932]$,

13.17 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.305944, 0.316932]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.93423\cdot10^{-14}X^7 - 2.39113\cdot10^{-12}X^6 + 6.66109\cdot10^{-11}X^5 + 1.57574\cdot10^{-09}X^4\\ &\quad - 6.96813\cdot10^{-08}X^3 - 1.34756\cdot10^{-07}X^2 + 1.26511\cdot10^{-05}X - 3.11569\cdot10^{-06}\\ &= -3.11569\cdot10^{-06}B_{0,7}(X) - 1.30839\cdot10^{-06}B_{1,7}(X) + 4.92485\cdot10^{-07}B_{2,7}(X) + 2.28496\cdot10^{-06}B_{3,7}(X)\\ &\quad + 4.06708\cdot10^{-06}B_{4,7}(X) + 5.83694\cdot10^{-06}B_{5,7}(X) + 7.59271\cdot10^{-06}B_{6,7}(X) + 9.33259\cdot10^{-06}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.246647, 0.250291\}$$

Intersection intervals with the x axis:

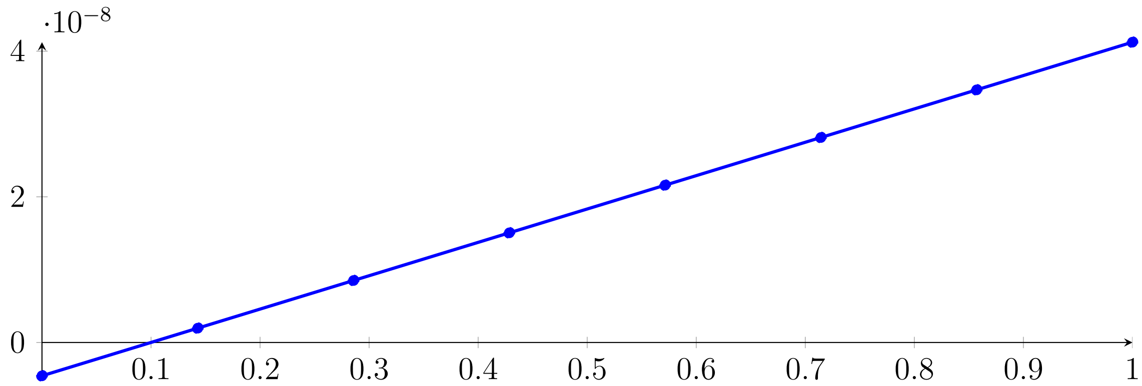
$$[0.246647, 0.250291]$$

Longest intersection interval: 0.00364366
$\Longrightarrow$ Selective recursion: interval 1: $[0.308654, 0.308694]$,

13.18 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.308654, 0.308694]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.97909\cdot10^{-25}X^7 - 1.61155\cdot10^{-24}X^6 + 4.14762\cdot10^{-23}X^5 + 2.91836\cdot10^{-19}X^4\\ &\quad - 3.29366\cdot10^{-15}X^3 - 2.46582\cdot10^{-12}X^2 + 4.58081\cdot10^{-08}X - 4.57542\cdot10^{-09}\\ &= -4.57542\cdot10^{-09}B_{0,7}(X) + 1.96859\cdot10^{-09}B_{1,7}(X) + 8.51249\cdot10^{-09}B_{2,7}(X) + 1.50563\cdot10^{-08}B_{3,7}(X)\\ &\quad + 2.15999\cdot10^{-08}B_{4,7}(X) + 2.81435\cdot10^{-08}B_{5,7}(X) + 3.46869\cdot10^{-08}B_{6,7}(X) + 4.12302\cdot10^{-08}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998823, 0.0998877\}$$

Intersection intervals with the x axis:

$$[0.0998823, 0.0998877]$$

Longest intersection interval: $5.38408\cdot10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.308658, 0.308658]$,

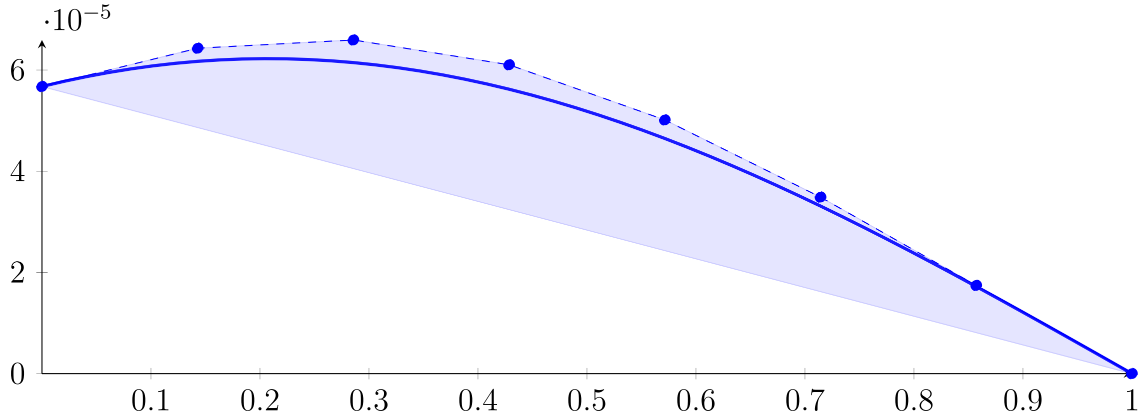
13.19 Recursion Branch 1 1 2 1 1 1 1 in Interval 1: $[0.308658, 0.308658]$

Found root in interval $[0.308658, 0.308658]$ at recursion depth 7!

13.20 Recursion Branch 1 1 2 2 on the Second Half $[0.375, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 4.76837\cdot 10^{-07}X^7 - 3.33786\cdot 10^{-06}X^6 - 1.43051\cdot 10^{-06}X^5 + 4.05312\cdot 10^{-05}X^4 \\
&\quad - 2.14577\cdot 10^{-05}X^3 - 0.000124454X^2 + 5.29289\cdot 10^{-05}X + 5.67436\cdot 10^{-05} \\
&= 5.67436\cdot 10^{-05}B_{0,7}(X) + 6.43049\cdot 10^{-05}B_{1,7}(X) + 6.59398\cdot 10^{-05}B_{2,7}(X) + 6.10352\cdot 10^{-05}B_{3,7}(X) \\
&\quad + 5.0136\cdot 10^{-05}B_{4,7}(X) + 3.48772\cdot 10^{-05}B_{5,7}(X) + 1.74386\cdot 10^{-05}B_{6,7}(X) - 2.05803\cdot 10^{-21}B_{7,7}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{1, 1\}$$

Intersection intervals with the x axis:

$$[1, 1]$$

Longest intersection interval: $2.81893\cdot 10^{-17}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

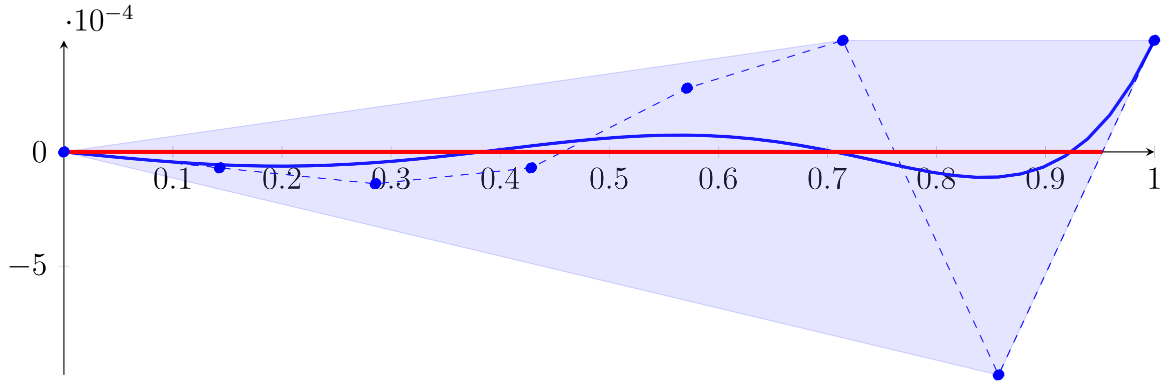
13.21 Recursion Branch 1 1 2 2 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

13.22 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 0.0078125X^7 - 1.49667\cdot 10^{-19}X^6 - 0.0117188X^5 - 7.52966\cdot 10^{-19}X^4 \\
&\quad + 0.00488281X^3 - 2.68622\cdot 10^{-19}X^2 - 0.000488281X - 2.05803\cdot 10^{-21} \\
&= -2.05803\cdot 10^{-21}B_{0,7}(X) - 6.97545\cdot 10^{-05}B_{1,7}(X) - 0.000139509B_{2,7}(X) - 6.97545\cdot 10^{-05}B_{3,7}(X) \\
&\quad + 0.000279018B_{4,7}(X) + 0.000488281B_{5,7}(X) - 0.000976563B_{6,7}(X) + 0.000488281B_{7,7}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{3.0106e - 18, 0.952381\}$$

Intersection intervals with the x axis:

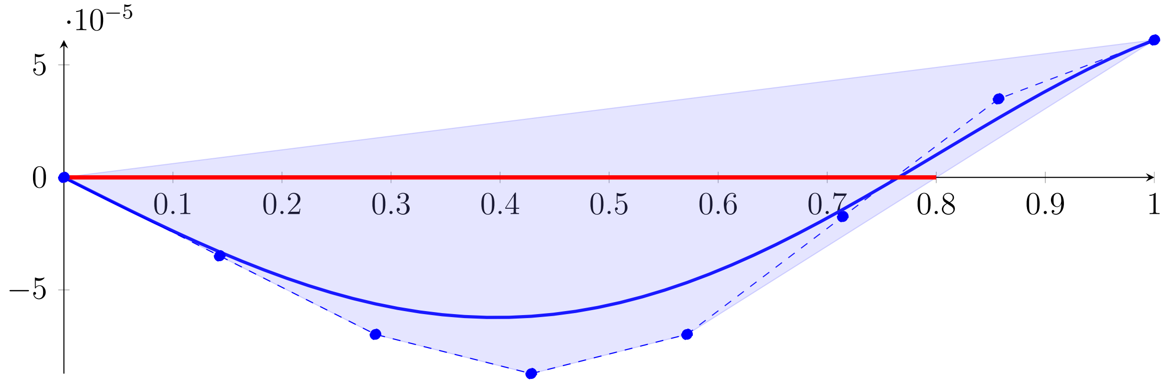
$$[3.0106e - 18, 0.952381]$$

Longest intersection interval: 0.952381
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

13.23 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 6.10352\cdot 10^{-05} X^7 + 4.12267\cdot 10^{-21} X^6 - 0.000366211 X^5 - 4.70169\cdot 10^{-20} X^4 \\ &\quad + 0.000610352 X^3 - 6.71207\cdot 10^{-20} X^2 - 0.000244141 X - 2.05803\cdot 10^{-21} \\ &= -2.05803\cdot 10^{-21} B_{0,7}(X) - 3.48772\cdot 10^{-05} B_{1,7}(X) - 6.97545\cdot 10^{-05} B_{2,7}(X) - 8.71931\cdot 10^{-05} B_{3,7}(X) \\ &\quad - 6.97545\cdot 10^{-05} B_{4,7}(X) - 1.74386\cdot 10^{-05} B_{5,7}(X) + 3.48772\cdot 10^{-05} B_{6,7}(X) + 6.10352\cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{3.37187e - 17, 0.8\}$$

Intersection intervals with the x axis:

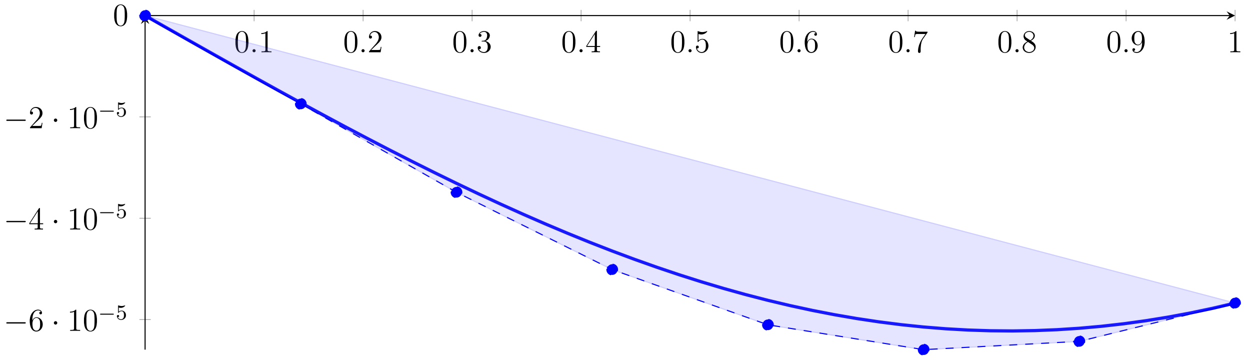
$$[3.37187e - 17, 0.8]$$

Longest intersection interval: 0.8
$\Longrightarrow$ Bisection: first half $[0.5, 0.625]$ und second half $[0.625, 0.75]$

13.24 Recursion Branch 1 2 1 1 on the First Half $[0.5, 0.625]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 4.76837\cdot10^{-07}X^7 + 3.24255\cdot10^{-21}X^6 - 1.14441\cdot10^{-05}X^5 - 3.01094\cdot10^{-21}X^4\\ &\quad + 7.62939\cdot10^{-05}X^3 - 1.68149\cdot10^{-20}X^2 - 0.00012207X - 2.05803\cdot10^{-21}\\ &= -2.05803\cdot10^{-21}B_{0,7}(X) - 1.74386\cdot10^{-05}B_{1,7}(X) - 3.48772\cdot10^{-05}B_{2,7}(X) - 5.0136\cdot10^{-05}B_{3,7}(X)\\ &\quad - 6.10352\cdot10^{-05}B_{4,7}(X) - 6.59398\cdot10^{-05}B_{5,7}(X) - 6.43049\cdot10^{-05}B_{6,7}(X) - 5.67436\cdot10^{-05}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

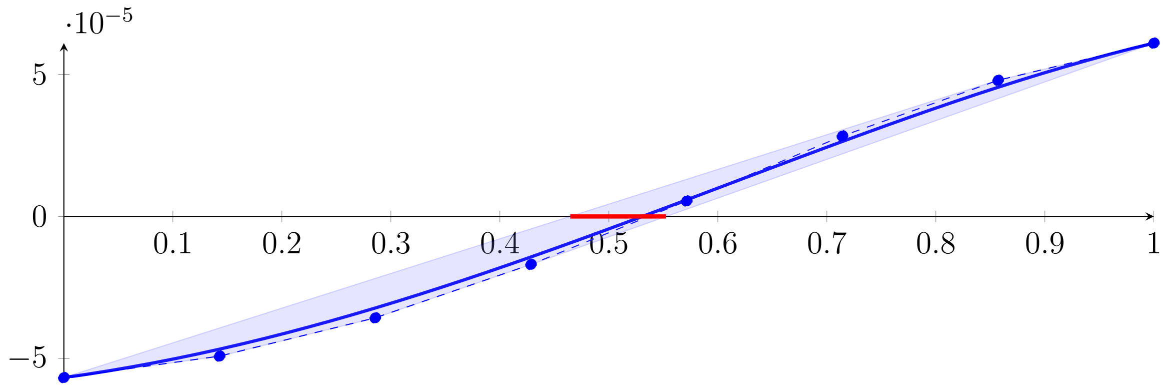
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

13.25 Recursion Branch 1 2 1 2 on the Second Half $[0.625, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 4.76837\cdot10^{-07}X^7 + 3.33786\cdot10^{-06}X^6 - 1.43051\cdot10^{-06}X^5 - 4.05312\cdot10^{-05}X^4\\ &\quad - 2.14577\cdot10^{-05}X^3 + 0.000124454X^2 + 5.29289\cdot10^{-05}X - 5.67436\cdot10^{-05}\\ &= -5.67436\cdot10^{-05}B_{0,7}(X) - 4.91823\cdot10^{-05}B_{1,7}(X) - 3.56947\cdot10^{-05}B_{2,7}(X) - 1.68937\cdot10^{-05}B_{3,7}(X)\\ &\quad + 5.44957\cdot10^{-06}B_{4,7}(X) + 2.83378\cdot10^{-05}B_{5,7}(X) + 4.79562\cdot10^{-05}B_{6,7}(X) + 6.10352\cdot10^{-05}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.464541, 0.552448\}$$

Intersection intervals with the x axis:

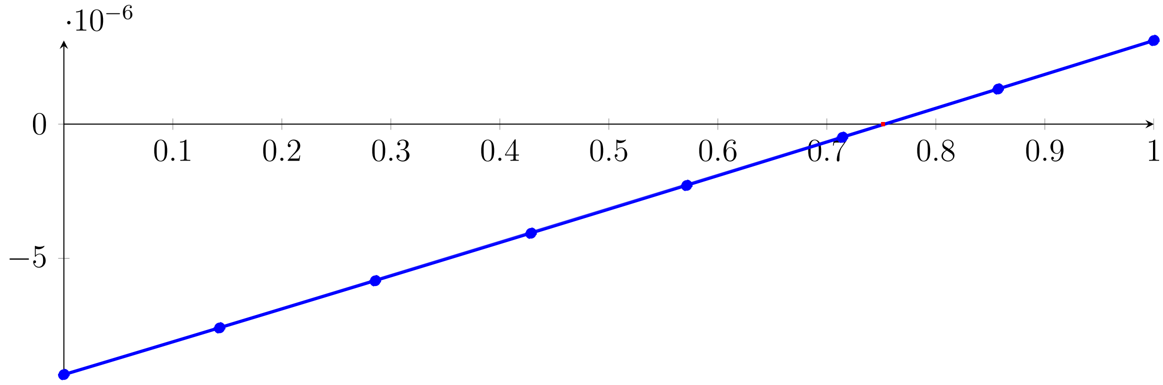
$$[0.464541, 0.552448]$$

Longest intersection interval: 0.0879062
$\Longrightarrow$ Selective recursion: interval 1: $[0.683068, 0.694056]$,

13.26 Recursion Branch 1 2 1 2 1 in Interval 1: $[0.683068, 0.694056]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.93423 \cdot 10^{-14} X^7 + 2.25573 \cdot 10^{-12} X^6 + 5.26703 \cdot 10^{-11} X^5 - 1.87361 \cdot 10^{-09} X^4 \\ &\quad - 6.27594 \cdot 10^{-08} X^3 + 3.33715 \cdot 10^{-07} X^2 + 1.21791 \cdot 10^{-05} X - 9.33259 \cdot 10^{-06} \\ &= -9.33259 \cdot 10^{-06} B_{0,7}(X) - 7.59271 \cdot 10^{-06} B_{1,7}(X) - 5.83694 \cdot 10^{-06} B_{2,7}(X) - 4.06708 \cdot 10^{-06} B_{3,7}(X) \\ &\quad - 2.28496 \cdot 10^{-06} B_{4,7}(X) - 4.92485 \cdot 10^{-07} B_{5,7}(X) + 1.30839 \cdot 10^{-06} B_{6,7}(X) + 3.11569 \cdot 10^{-06} B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.749709, 0.753353\}$$

Intersection intervals with the x axis:

$$[0.749709, 0.753353]$$

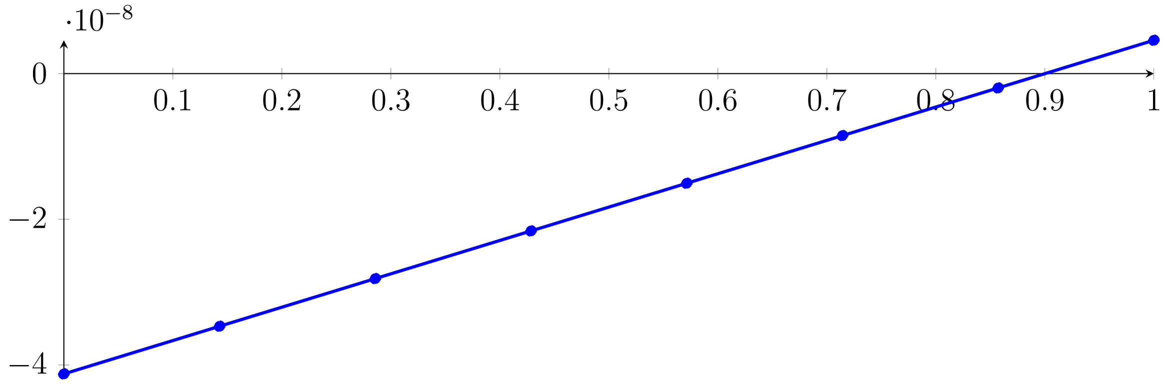
Longest intersection interval: 0.00364366

$\Longrightarrow$ Selective recursion: interval 1: $[0.691306, 0.691346]$,

13.27 Recursion Branch 1 2 1 2 1 1 in Interval 1: $[0.691306, 0.691346]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2.84343 \cdot 10^{-25} X^7 + 1.9904 \cdot 10^{-24} X^6 + 4.017 \cdot 10^{-23} X^5 - 2.92039 \cdot 10^{-19} X^4 \\ &\quad - 3.29249 \cdot 10^{-15} X^3 + 2.4757 \cdot 10^{-12} X^2 + 4.58031 \cdot 10^{-08} X - 4.12302 \cdot 10^{-08} \\ &= -4.12302 \cdot 10^{-08} B_{0,7}(X) - 3.46869 \cdot 10^{-08} B_{1,7}(X) - 2.81435 \cdot 10^{-08} B_{2,7}(X) - 2.15999 \cdot 10^{-08} B_{3,7}(X) \\ &\quad - 1.50563 \cdot 10^{-08} B_{4,7}(X) - 8.51249 \cdot 10^{-09} B_{5,7}(X) - 1.96859 \cdot 10^{-09} B_{6,7}(X) + 4.57542 \cdot 10^{-09} B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.900112, 0.900118\}$$

Intersection intervals with the x axis:

$$[0.900112, 0.900118]$$

Longest intersection interval: $5.38408 \cdot 10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.691342, 0.691342]$,

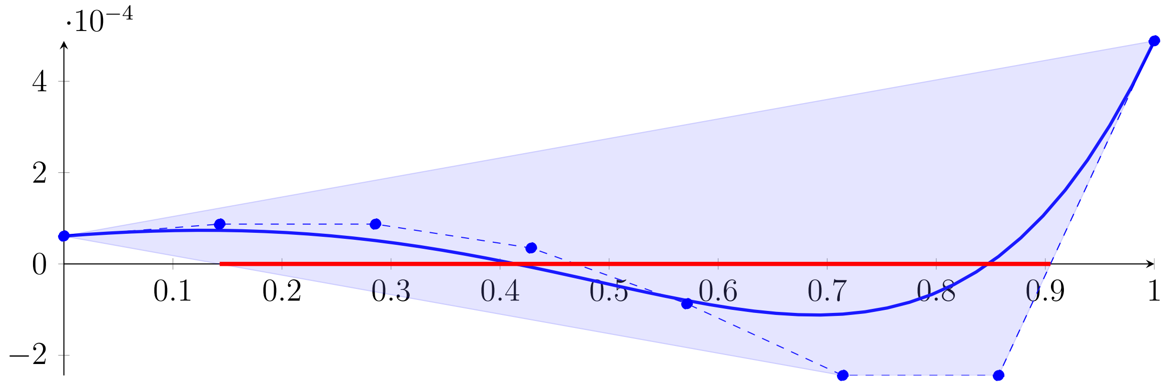
13.28 Recursion Branch 1 2 1 2 1 1 1 in Interval 1: $[0.691342, 0.691342]$

Found root in interval $[0.691342, 0.691342]$ at recursion depth 7!

13.29 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 6.10352\cdot 10^{-05}X^7 + 0.000427246X^6 + 0.000915527X^5 + 0.000305176X^4 \\ &\quad - 0.000915527X^3 - 0.000549316X^2 + 0.000183105X + 6.10352\cdot 10^{-05} \\ &= 6.10352\cdot 10^{-05}B_{0,7}(X) + 8.71931\cdot 10^{-05}B_{1,7}(X) + 8.71931\cdot 10^{-05}B_{2,7}(X) + 3.48772\cdot 10^{-05}B_{3,7}(X) \\ &\quad - 8.71931\cdot 10^{-05}B_{4,7}(X) - 0.000244141B_{5,7}(X) - 0.000244141B_{6,7}(X) + 0.000488281B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.142857, 0.904762\}$$

Intersection intervals with the x axis:

$$[0.142857, 0.904762]$$

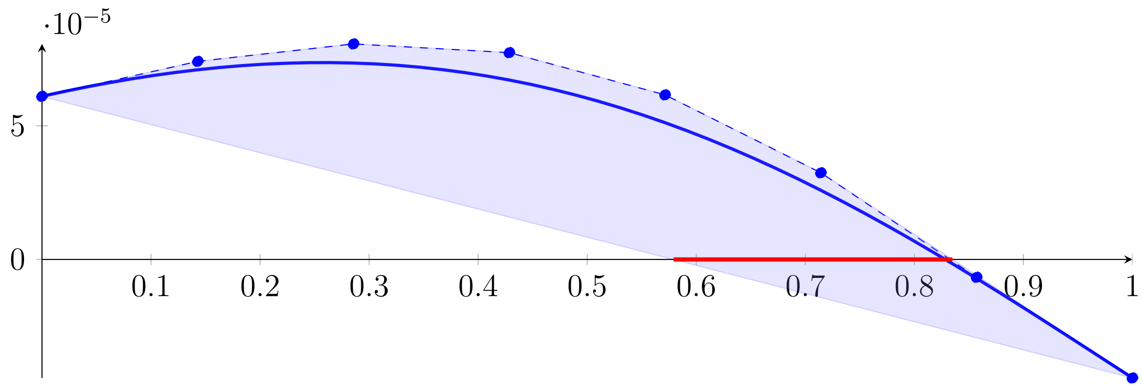
Longest intersection interval: 0.761905

$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$

13.30 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 4.76837\cdot 10^{-07}X^7 + 6.67572\cdot 10^{-06}X^6 + 2.86102\cdot 10^{-05}X^5 + 1.90735\cdot 10^{-05}X^4 \\ &\quad - 0.000114441X^3 - 0.000137329X^2 + 9.15527\cdot 10^{-05}X + 6.10352\cdot 10^{-05} \\ &= 6.10352\cdot 10^{-05}B_{0,7}(X) + 7.41141\cdot 10^{-05}B_{1,7}(X) + 8.06536\cdot 10^{-05}B_{2,7}(X) + 7.73839\cdot 10^{-05}B_{3,7}(X) \\ &\quad + 6.15801\cdot 10^{-05}B_{4,7}(X) + 3.24249\cdot 10^{-05}B_{5,7}(X) - 6.67572\cdot 10^{-06}B_{6,7}(X) - 4.43459\cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.579186, 0.83496\}$$

Intersection intervals with the x axis:

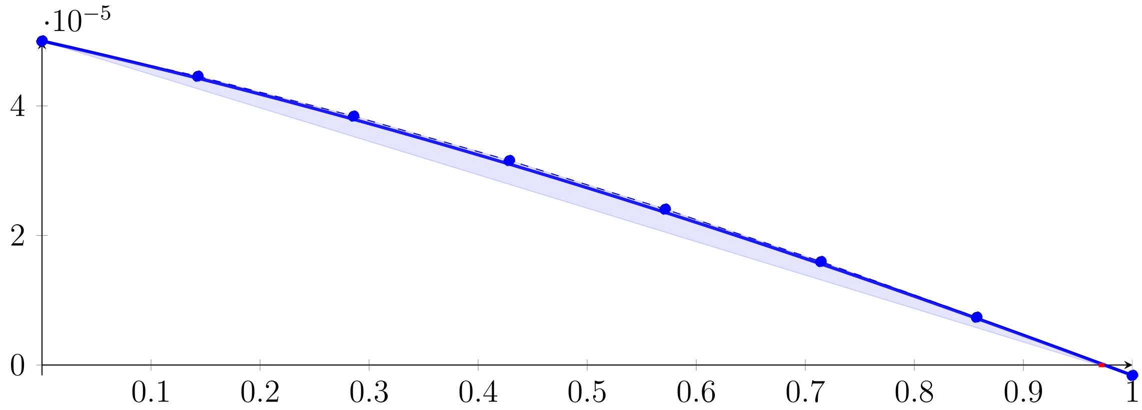
$$[0.579186, 0.83496]$$

Longest intersection interval: 0.255775
$\Longrightarrow$ Selective recursion: interval 1: $[0.822398, 0.85437]$,

13.31 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.822398, 0.85437]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.41485\cdot10^{-11}X^7 + 2.41043\cdot10^{-09}X^6 + 6.03915\cdot10^{-08}X^5 + 5.93875\cdot10^{-07}X^4 \\ &\quad + 8.95895\cdot10^{-07}X^3 - 1.5065\cdot10^{-05}X^2 - 3.812\cdot10^{-05}X + 5.00319\cdot10^{-05} \\ &= 5.00319\cdot10^{-05}B_{0,7}(X) + 4.45862\cdot10^{-05}B_{1,7}(X) + 3.84231\cdot10^{-05}B_{2,7}(X) + 3.15683\cdot10^{-05}B_{3,7}(X) \\ &\quad + 2.40642\cdot10^{-05}B_{4,7}(X) + 1.59733\cdot10^{-05}B_{5,7}(X) + 7.38102\cdot10^{-06}B_{6,7}(X) - 1.60042\cdot10^{-06}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.969004, 0.974544\}$$

Intersection intervals with the x axis:

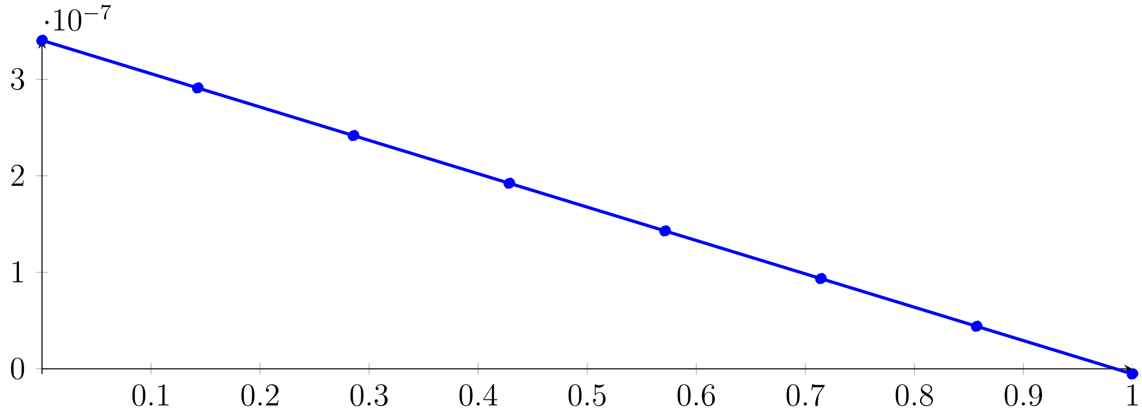
$$[0.969004, 0.974544]$$

Longest intersection interval: 0.00554047
$\Longrightarrow$ Selective recursion: interval 1: $[0.853379, 0.853556]$,

13.32 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.853379, 0.853556]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.2489\cdot10^{-24}X^7 + 5.9893\cdot10^{-23}X^6 + 3.91968\cdot10^{-19}X^5 + 8.68331\cdot10^{-16}X^4 \\ &\quad + 6.47939\cdot10^{-13}X^3 - 2.61931\cdot10^{-10}X^2 - 3.4546\cdot10^{-07}X + 3.40361\cdot10^{-07} \\ &= 3.40361\cdot10^{-07}B_{0,7}(X) + 2.9101\cdot10^{-07}B_{1,7}(X) + 2.41646\cdot10^{-07}B_{2,7}(X) + 1.92269\cdot10^{-07}B_{3,7}(X) \\ &\quad + 1.4288\cdot10^{-07}B_{4,7}(X) + 9.3479\cdot10^{-08}B_{5,7}(X) + 4.40654\cdot10^{-08}B_{6,7}(X) - 5.36067\cdot10^{-09}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.984494, 0.984506\}$$

Intersection intervals with the x axis:

$$[0.984494, 0.984506]$$

Longest intersection interval: $1.16806 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.853553, 0.853553]$,

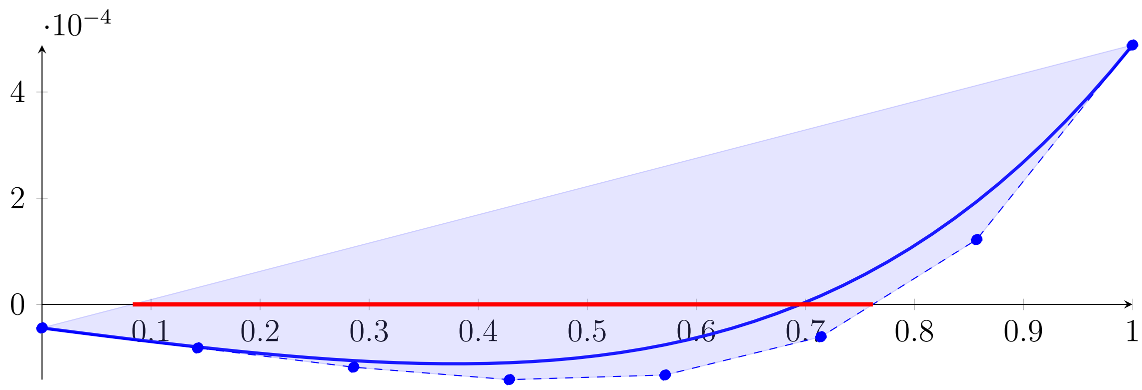
13.33 Recursion Branch 1 2 2 1 1 1 1 in Interval 1: $[0.853553, 0.853553]$

Found root in interval $[0.853553, 0.853553]$ at recursion depth 7!

13.34 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 4.76837 \cdot 10^{-07} X^7 + 1.00136 \cdot 10^{-05} X^6 + 7.86781 \cdot 10^{-05} X^5 + 0.00027895 X^4 \\
&\quad + 0.000398159 X^3 + 3.00407 \cdot 10^{-05} X^2 - 0.000263691 X - 4.43459 \cdot 10^{-05} \\
&= -4.43459 \cdot 10^{-05} B_{0,7}(X) - 8.2016 \cdot 10^{-05} B_{1,7}(X) - 0.000118256 B_{2,7}(X) - 0.000141689 B_{3,7}(X) \\
&\quad - 0.000132969 B_{4,7}(X) - 6.10352 \cdot 10^{-05} B_{5,7}(X) + 0.00012207 B_{6,7}(X) + 0.000488281 B_{7,7}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0832587, 0.761905\}$$

Intersection intervals with the x axis:

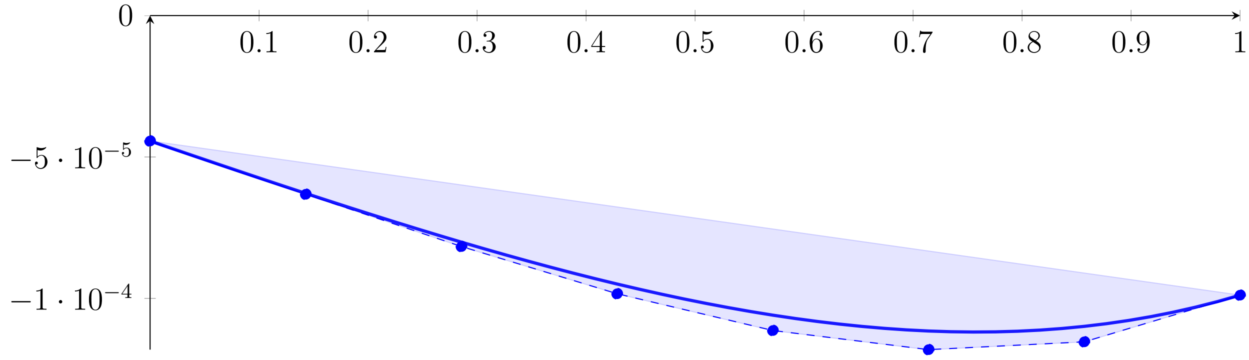
$$[0.0832587, 0.761905]$$

Longest intersection interval: 0.678646
$\Longrightarrow$ Bisection: first half $[0.875, 0.9375]$ und second half $[0.9375, 1]$

13.35 Recursion Branch 1 2 2 2 1 on the First Half $[0.875, 0.9375]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.72529\cdot10^{-09}X^7 + 1.56462\cdot10^{-07}X^6 + 2.45869\cdot10^{-06}X^5 + 1.74344\cdot10^{-05}X^4 \\ &\quad + 4.97699\cdot10^{-05}X^3 + 7.51019\cdot10^{-06}X^2 - 0.000131845X - 4.43459\cdot10^{-05} \\ &= -4.43459\cdot10^{-05}B_{0,7}(X) - 6.31809\cdot10^{-05}B_{1,7}(X) - 8.16584\cdot10^{-05}B_{2,7}(X) - 9.83562\cdot10^{-05}B_{3,7}(X) \\ &\quad - 0.000111354B_{4,7}(X) - 0.000118117B_{5,7}(X) - 0.000115355B_{6,7}(X) - 9.8858\cdot10^{-05}B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

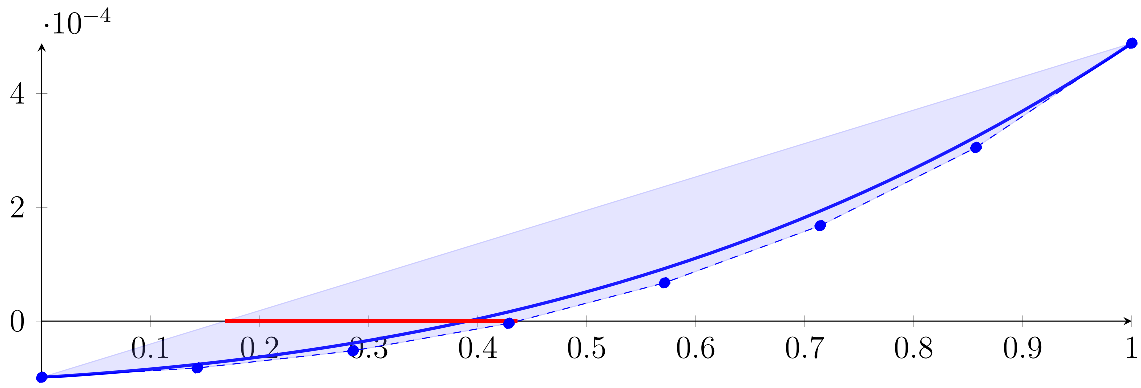
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

13.36 Recursion Branch 1 2 2 2 2 on the Second Half $[0.9375, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 3.72529\cdot10^{-09}X^7 + 1.82539\cdot10^{-07}X^6 + 3.4757\cdot10^{-06}X^5 + 3.22051\cdot10^{-05}X^4 \\ &\quad + 0.000147354X^3 + 0.000288438X^2 + 0.00011548X - 9.8858\cdot10^{-05} \\ &= -9.8858\cdot10^{-05}B_{0,7}(X) - 8.23608\cdot10^{-05}B_{1,7}(X) - 5.21285\cdot10^{-05}B_{2,7}(X) - 3.95094\cdot10^{-06}B_{3,7}(X) \\ &\quad + 6.73022\cdot10^{-05}B_{4,7}(X) + 0.000167847B_{5,7}(X) + 0.000305176B_{6,7}(X) + 0.000488281B_{7,7}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.168372, 0.436493\}$$

Intersection intervals with the x axis:

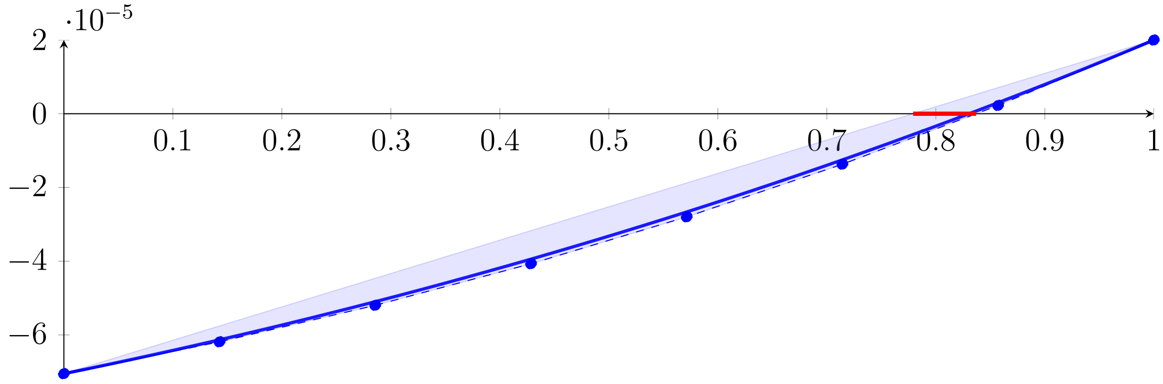
$$[0.168372, 0.436493]$$

Longest intersection interval: 0.26812
$\Longrightarrow$ Selective recursion: interval 1: $[0.948023, 0.964781]$,

13.37 Recursion Branch 1 2 2 2 2 1 in Interval 1: $[0.948023, 0.964781]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 3.71081\cdot 10^{-13}X^7 + 6.94477\cdot 10^{-11}X^6 + 5.07465\cdot 10^{-09}X^5 + 1.81961\cdot 10^{-07}X^4 \\ &\quad + 3.27761\cdot 10^{-06}X^3 + 2.6492\cdot 10^{-05}X^2 + 6.05339\cdot 10^{-05}X - 7.05076\cdot 10^{-05} \\ &= -7.05076\cdot 10^{-05}B_{0,7}(X) - 6.18599\cdot 10^{-05}B_{1,7}(X) - 5.19507\cdot 10^{-05}B_{2,7}(X) - 4.06863\cdot 10^{-05}B_{3,7}(X) \\ &\quad - 2.79679\cdot 10^{-05}B_{4,7}(X) - 1.36912\cdot 10^{-05}B_{5,7}(X) + 2.25381\cdot 10^{-06}B_{6,7}(X) + 1.9983\cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.77917, 0.83695\}$$

Intersection intervals with the x axis:

$$[0.77917, 0.83695]$$

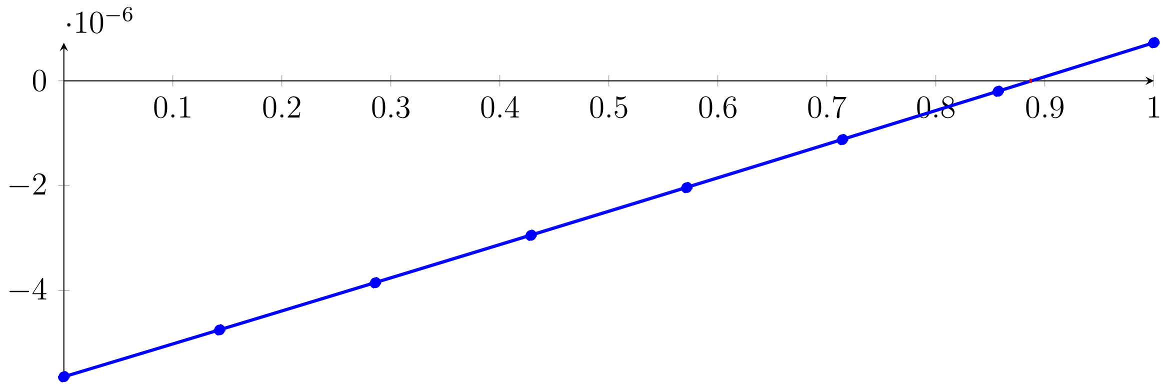
Longest intersection interval: 0.0577797

$\Longrightarrow$ Selective recursion: interval 1: $[0.96108, 0.962048]$,

13.38 Recursion Branch 1 2 2 2 2 1 1 in Interval 1: $[0.96108, 0.962048]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 8.34212\cdot 10^{-22}X^7 + 2.65965\cdot 10^{-18}X^6 + 3.48014\cdot 10^{-15}X^5 + 2.25552\cdot 10^{-12}X^4 \\ &\quad + 7.47708\cdot 10^{-10}X^3 + 1.16315\cdot 10^{-07}X^2 + 6.24835\cdot 10^{-06}X - 5.63894\cdot 10^{-06} \\ &= -5.63894\cdot 10^{-06}B_{0,7}(X) - 4.74632\cdot 10^{-06}B_{1,7}(X) - 3.84816\cdot 10^{-06}B_{2,7}(X) - 2.94444\cdot 10^{-06}B_{3,7}(X) \\ &\quad - 2.03514\cdot 10^{-06}B_{4,7}(X) - 1.12024\cdot 10^{-06}B_{5,7}(X) - 1.99707\cdot 10^{-07}B_{6,7}(X) + 7.26469\cdot 10^{-07}B_{7,7}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.885872, 0.887946\}$$

Intersection intervals with the x axis:

$$[0.885872, 0.887946]$$

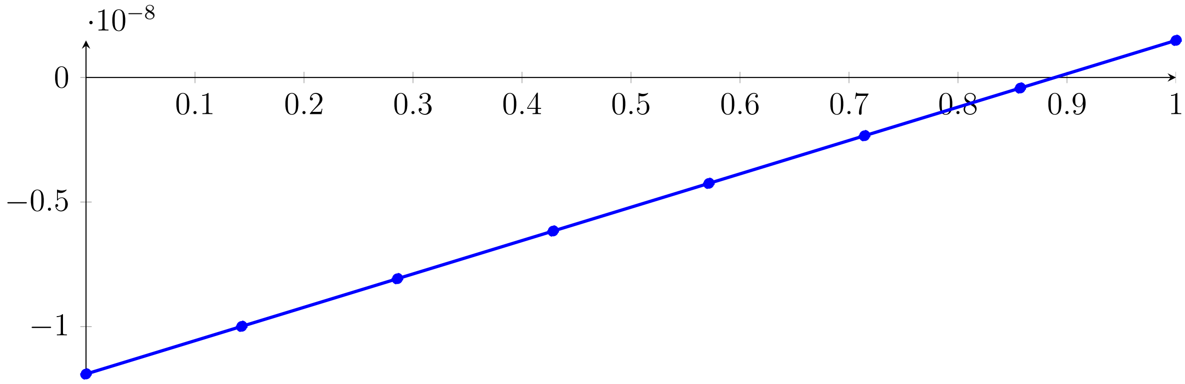
Longest intersection interval: 0.002074

$\Longrightarrow$ Selective recursion: interval 1: $[0.961938, 0.96194]$,

13.39 Recursion Branch 1 2 2 2 2 1 1 1 in Interval 1: $[0.961938, 0.96194]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 7.75482\cdot 10^{-26}X^7 + 5.88074\cdot 10^{-25}X^6 - 1.35709\cdot 10^{-25}X^5 + 4.21264\cdot 10^{-23}X^4 \\
&\quad + 6.74207\cdot 10^{-18}X^3 + 5.08923\cdot 10^{-13}X^2 + 1.33902\cdot 10^{-08}X - 1.1903\cdot 10^{-08} \\
&= -1.1903\cdot 10^{-08}B_{0,7}(X) - 9.99007\cdot 10^{-09}B_{1,7}(X) - 8.07716\cdot 10^{-09}B_{2,7}(X) - 6.16423\cdot 10^{-09}B_{3,7}(X) \\
&\quad - 4.25128\cdot 10^{-09}B_{4,7}(X) - 2.3383\cdot 10^{-09}B_{5,7}(X) - 4.253\cdot 10^{-10}B_{6,7}(X) + 1.48773\cdot 10^{-09}B_{7,7}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.888898, 0.888903\}$$

Intersection intervals with the x axis:

$$[0.888898, 0.888903]$$

Longest intersection interval: $4.22246\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.96194, 0.96194]$,

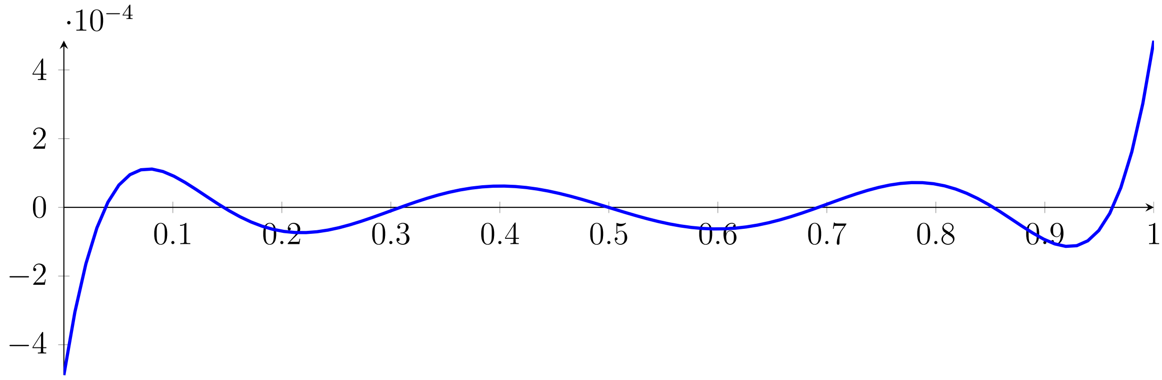
13.40 Recursion Branch 1 2 2 2 2 1 1 1 1 in Interval 1: $[0.96194, 0.96194]$

Found root in interval $[0.96194, 0.96194]$ at recursion depth 9!

13.41 Result: 7 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$



Result: Root Intervals

$$[0.0380602, 0.0380602],\ [0.146447, 0.146447],\ [0.308658, 0.308658],\ [0.5, 0.5],\ [0.691342, 0.691342],\ [0.853553, 0.853553],\ [0.96194, 0.96194]$$

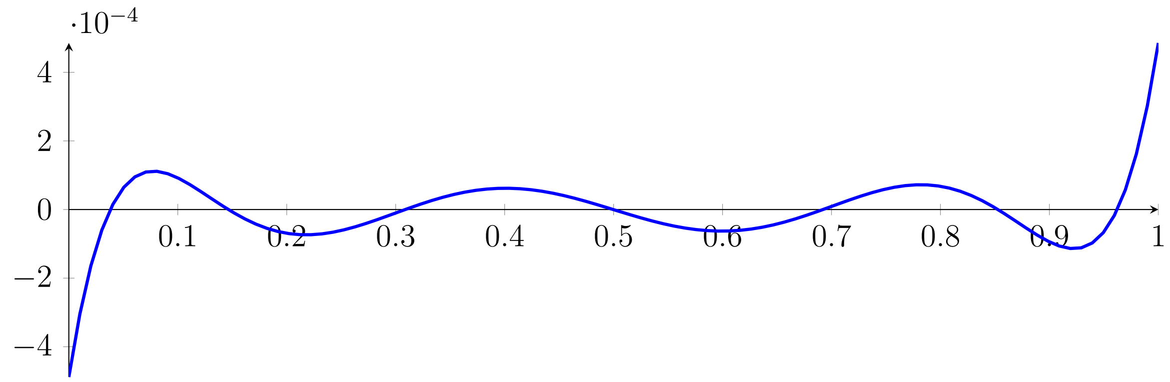
with precision $\varepsilon = 1 \cdot 10^{-06}$.

14 Running QuadClip on p7 with epsilon 6

$$1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$

Called QuadClip with input polynomial on interval $[0, 1]$:

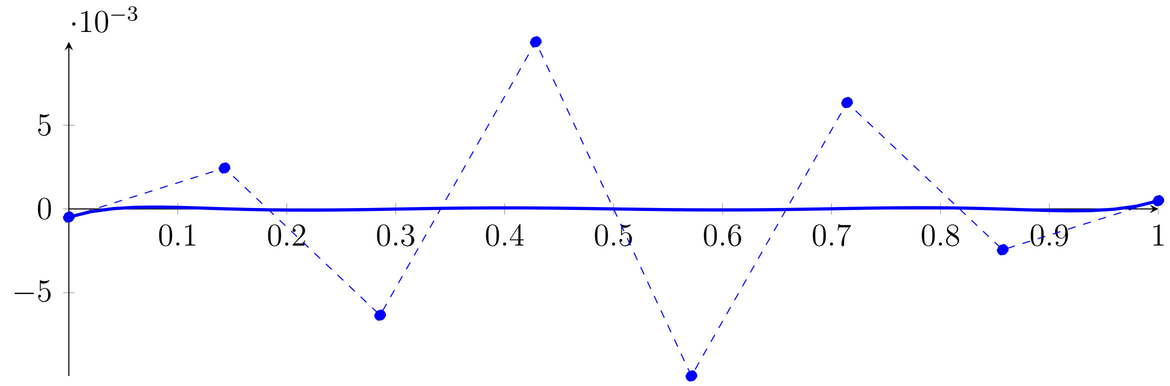
$$p = 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$



14.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

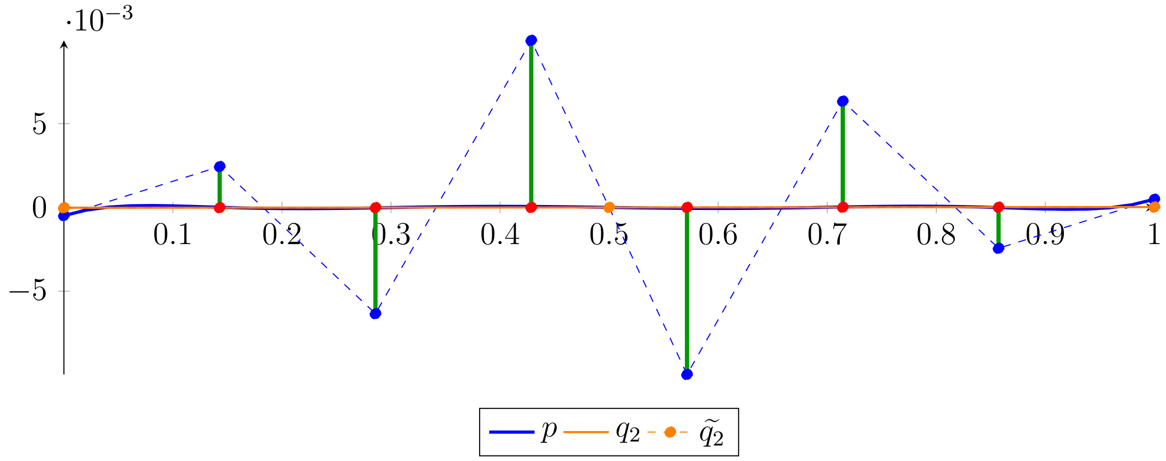
$$\begin{aligned} p &= 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.00244141B_{1,7}(X) - 0.00634766B_{2,7}(X) + 0.00997489B_{3,7}(X) \\ &\quad - 0.00997489B_{4,7}(X) + 0.00634766B_{5,7}(X) - 0.00244141B_{6,7}(X) + 0.000488281B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -4.04838 \cdot 10^{-18}X^2 + 4.6503 \cdot 10^{-05}X - 2.32515 \cdot 10^{-05} \\ &= -2.32515 \cdot 10^{-05}B_{0,2} + 1.08746 \cdot 10^{-18}B_{1,2} + 2.32515 \cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 4.34387 \cdot 10^{-18}X^7 - 1.50932 \cdot 10^{-17}X^6 + 2.08503 \cdot 10^{-17}X^5 - 1.45586 \cdot 10^{-17}X^4 \\ &\quad + 5.37302 \cdot 10^{-18}X^3 - 5.03492 \cdot 10^{-18}X^2 + 4.6503 \cdot 10^{-05}X - 2.32515 \cdot 10^{-05} \\ &= -2.32515 \cdot 10^{-05}B_{0,7} - 1.66082 \cdot 10^{-05}B_{1,7} - 9.96492 \cdot 10^{-06}B_{2,7} - 3.32164 \cdot 10^{-06}B_{3,7} \\ &\quad + 3.32164 \cdot 10^{-06}B_{4,7} + 9.96492 \cdot 10^{-06}B_{5,7} + 1.66082 \cdot 10^{-05}B_{6,7} + 2.32515 \cdot 10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00997821$.
Bounding polynomials M and m:

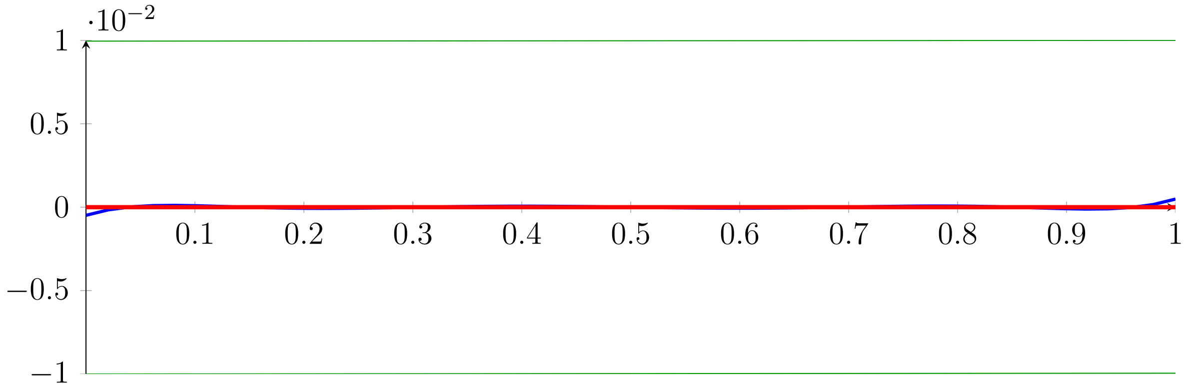
$$M = -4.04882 \cdot 10^{-18} X^2 + 4.6503 \cdot 10^{-05} X + 0.00995496$$

$$m = -4.04882 \cdot 10^{-18} X^2 + 4.6503 \cdot 10^{-05} X - 0.0100015$$

Root of M and m:

$$N(M) = \{-214.064, 1.14856 \cdot 10^{13}\} \qquad N(m) = \{215.074, 1.14856 \cdot 10^{13}\}$$

Intersection intervals:



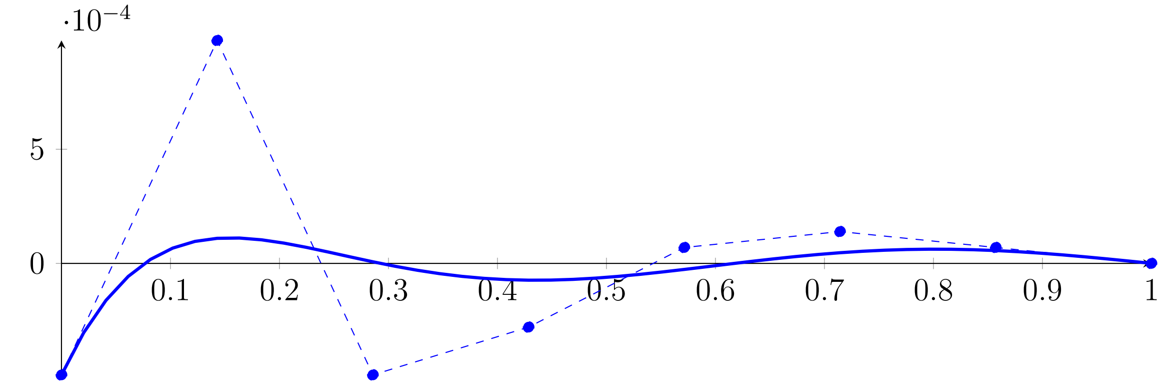
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$
Bisection point is very near to a root?!?

14.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

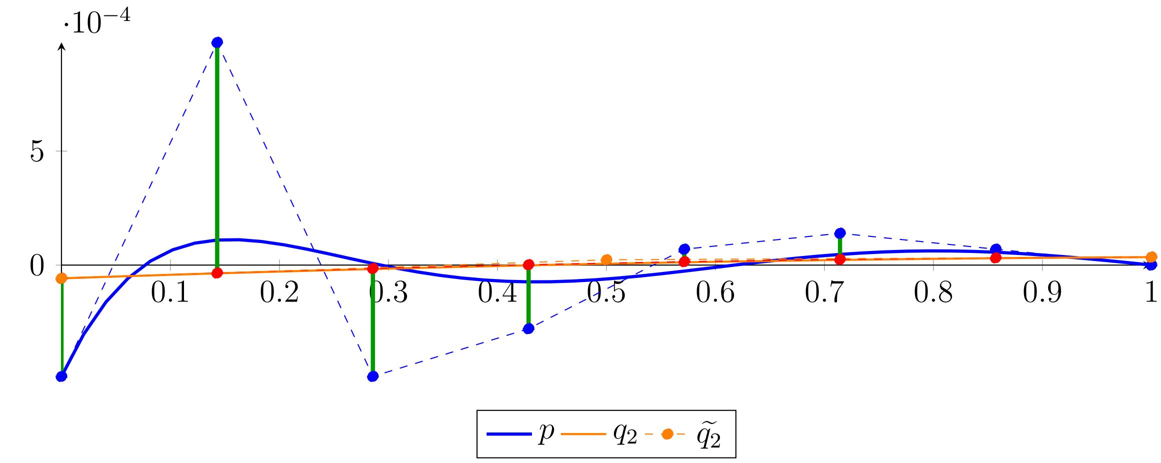
$$\begin{aligned} p &= 0.0078125X^7 - 0.0546875X^6 + 0.152344X^5 - 0.214844X^4 \\ &\quad + 0.161133X^3 - 0.0615234X^2 + 0.0102539X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.000976562B_{1,7}(X) - 0.000488281B_{2,7}(X) - 0.000279018B_{3,7}(X) \\ &\quad + 6.97545 \cdot 10^{-05}B_{4,7}(X) + 0.000139509B_{5,7}(X) + 6.97545 \cdot 10^{-05}B_{6,7}(X) - 2.05803 \cdot 10^{-21}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -6.97545 \cdot 10^{-05} X^2 + 0.00016276 X - 5.81287 \cdot 10^{-05} \\ &= -5.81287 \cdot 10^{-05} B_{0,2} + 2.32515 \cdot 10^{-05} B_{1,2} + 3.48772 \cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6.93363 \cdot 10^{-18} X^7 - 2.44492 \cdot 10^{-17} X^6 + 3.4378 \cdot 10^{-17} X^5 - 2.45123 \cdot 10^{-17} X^4 \\ &\quad + 9.2628 \cdot 10^{-18} X^3 - 6.97545 \cdot 10^{-05} X^2 + 0.00016276 X - 5.81287 \cdot 10^{-05} \\ &= -5.81287 \cdot 10^{-05} B_{0,7} - 3.48772 \cdot 10^{-05} B_{1,7} - 1.49474 \cdot 10^{-05} B_{2,7} + 1.66082 \cdot 10^{-06} B_{3,7} \\ &\quad + 1.49474 \cdot 10^{-05} B_{4,7} + 2.49123 \cdot 10^{-05} B_{5,7} + 3.15556 \cdot 10^{-05} B_{6,7} + 3.48772 \cdot 10^{-05} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00101144$.

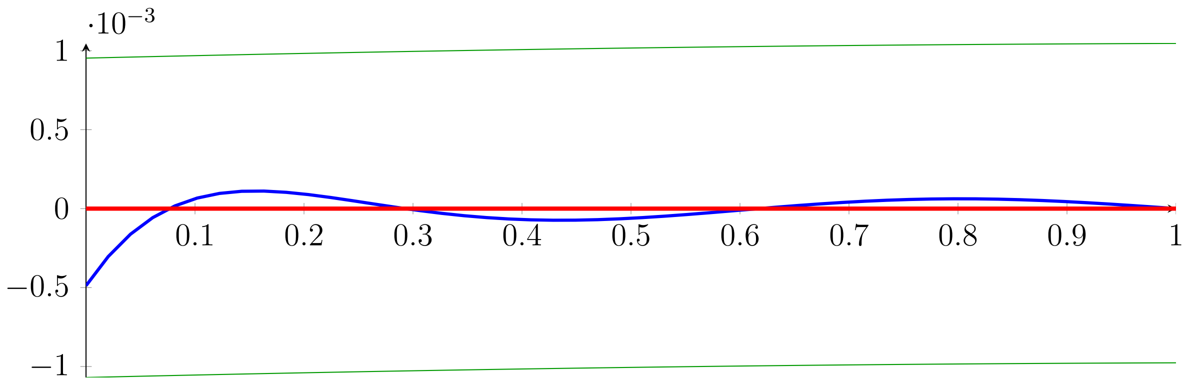
Bounding polynomials M and m:

$$M = -6.97545 \cdot 10^{-05} X^2 + 0.00016276 X + 0.000953311$$
$$m = -6.97545 \cdot 10^{-05} X^2 + 0.00016276 X - 0.00106957$$

Root of M and m:

$$N(M) = \{-2.7099, 5.04323\} \qquad N(m) = \{\}$$

Intersection intervals:



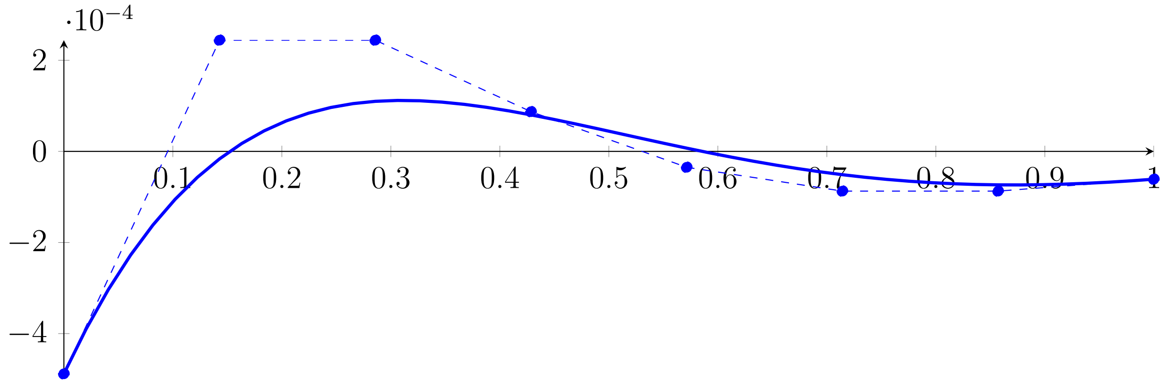
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$
Bisection point is very near to a root?!?

14.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

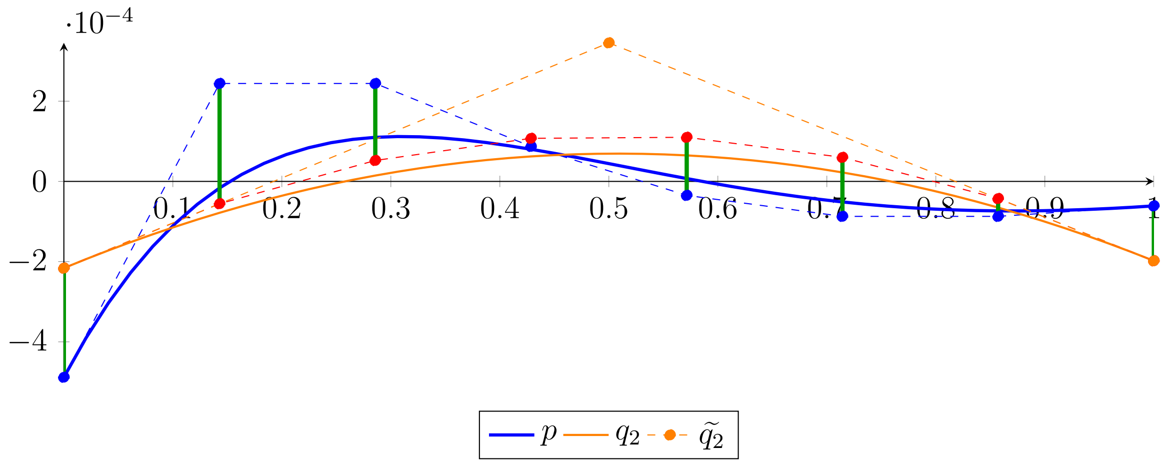
$$\begin{aligned} p &= 6.10352\cdot 10^{-05}X^7 - 0.000854492X^6 + 0.00476074X^5 - 0.0134277X^4 \\ &\quad + 0.0201416X^3 - 0.0153809X^2 + 0.00512695X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.000244141B_{1,7}(X) + 0.000244141B_{2,7}(X) + 8.71931\cdot 10^{-05}B_{3,7}(X) \\ &\quad - 3.48772\cdot 10^{-05}B_{4,7}(X) - 8.71931\cdot 10^{-05}B_{5,7}(X) - 8.71931\cdot 10^{-05}B_{6,7}(X) - 6.10352\cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00110517X^2 + 0.00112334X - 0.000216166 \\ &= -0.000216166B_{0,2} + 0.000345503B_{1,2} - 0.000198001B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -2.19785\cdot 10^{-17}X^7 + 6.96767\cdot 10^{-17}X^6 - 8.50065\cdot 10^{-17}X^5 + 4.98769 \\ &\quad \cdot 10^{-17}X^4 - 1.43177\cdot 10^{-17}X^3 - 0.00110517X^2 + 0.00112334X - 0.000216166 \\ &= -0.000216166B_{0,7} - 5.56894\cdot 10^{-05}B_{1,7} + 5.21601\cdot 10^{-05}B_{2,7} + 0.000107382B_{3,7} \\ &\quad + 0.000109977B_{4,7} + 5.99452\cdot 10^{-05}B_{5,7} - 4.27142\cdot 10^{-05}B_{6,7} - 0.000198001B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00029983$.
Bounding polynomials M and m:

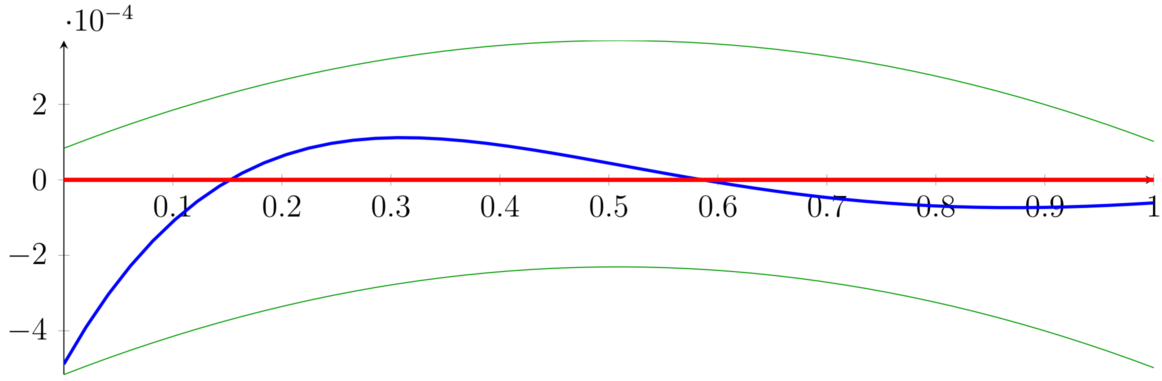
$$M = -0.00110517X^2 + 0.00112334X + 8.36638\cdot 10^{-05}$$

$$m = -0.00110517X^2 + 0.00112334X - 0.000515996$$

Root of M and m:

$$N(M) = \{-0.0696986, 1.08614\} \qquad N(m) = \{\}$$

Intersection intervals:



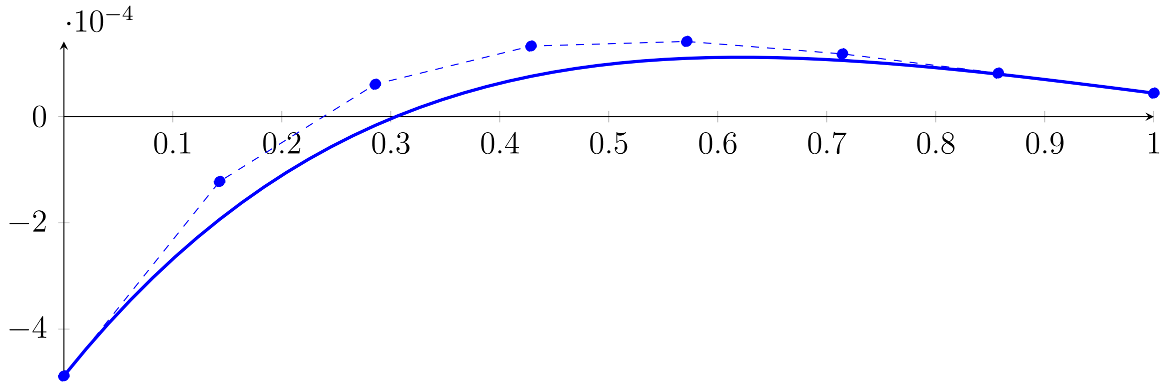
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$

14.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

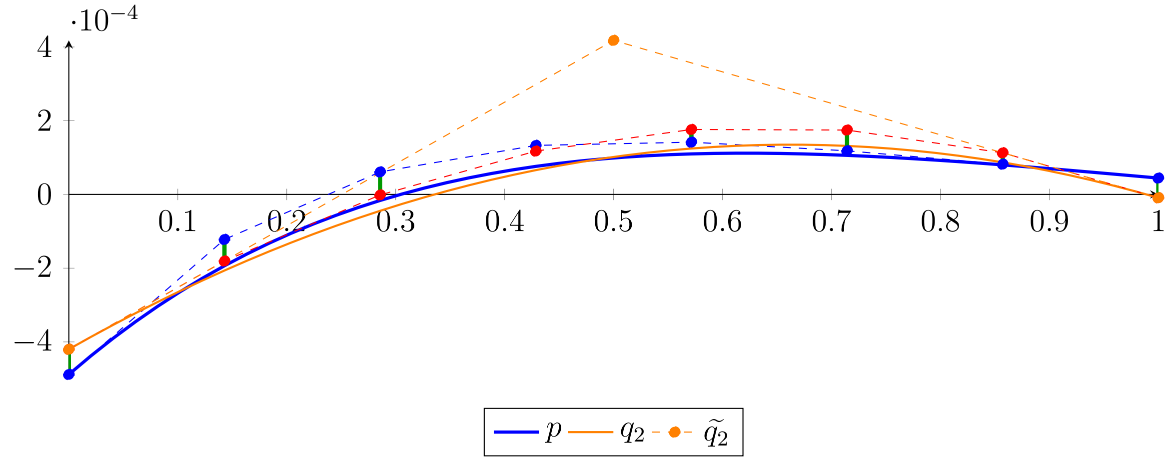
$$\begin{aligned} p &= 4.76837 \cdot 10^{-07}X^7 - 1.33514 \cdot 10^{-05}X^6 + 0.000148773X^5 - 0.000839233X^4 \\ &\quad + 0.0025177X^3 - 0.00384521X^2 + 0.00256348X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) - 0.00012207B_{1,7}(X) + 6.10352 \cdot 10^{-05}B_{2,7}(X) + 0.000132969B_{3,7}(X) \\ &\quad + 0.000141689B_{4,7}(X) + 0.000118256B_{5,7}(X) + 8.2016 \cdot 10^{-05}B_{6,7}(X) + 4.43459 \cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.00126469X^2 + 0.00167546X - 0.00041992 \\ &= -0.00041992B_{0,2} + 0.000417809B_{1,2} - 9.15357 \cdot 10^{-06}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 7.24091 \cdot 10^{-18}X^7 - 3.18114 \cdot 10^{-17}X^6 + 5.51552 \cdot 10^{-17}X^5 - 4.79508 \cdot 10^{-17}X^4 \\ &\quad + 2.17223 \cdot 10^{-17}X^3 - 0.00126469X^2 + 0.00167546X - 0.00041992 \\ &= -0.00041992B_{0,7} - 0.000180569B_{1,7} - 1.44146 \cdot 10^{-06}B_{2,7} + 0.000117463B_{3,7} \\ &\quad + 0.000176144B_{4,7} + 0.000174601B_{5,7} + 0.000112836B_{6,7} - 9.15357 \cdot 10^{-06}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.83609 \cdot 10^{-05}$.

Bounding polynomials M and m:

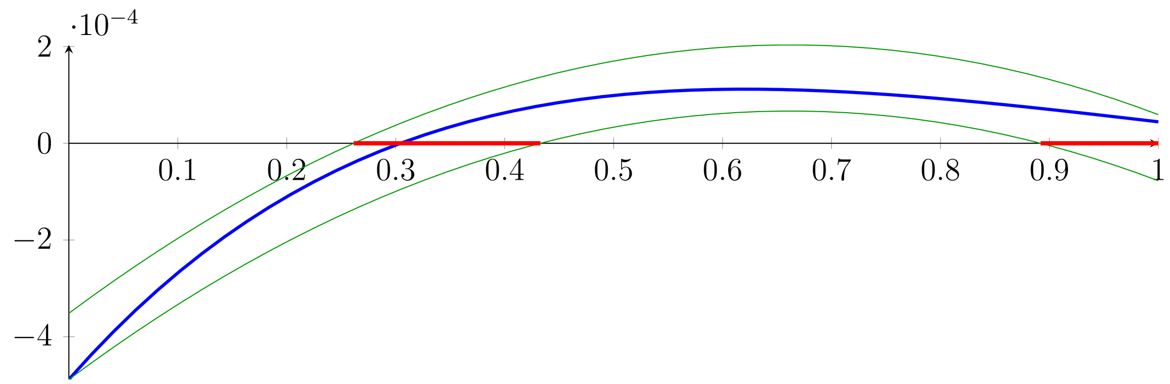
$$M = -0.00126469X^2 + 0.00167546X - 0.00035156$$

$$m = -0.00126469X^2 + 0.00167546X - 0.000488281$$

Root of M and m:

$$N(M) = \{0.261411, 1.06339\} \qquad N(m) = \{0.432868, 0.891928\}$$

Intersection intervals:



$$[0.261411, 0.432868], \; [0.891928, 1]$$

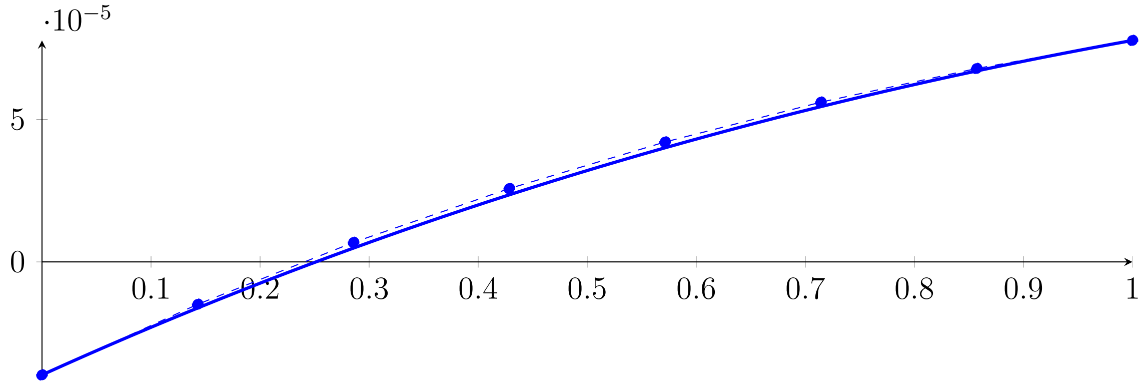
Longest intersection interval: 0.171457

$\Longrightarrow$ Selective recursion: interval 1: $[0.0326764, 0.0541085]$, interval 2: $[0.111491, 0.125]$,

14.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0326764, 0.0541085]$

Normalized monomial und Bézier representations and the Bézier polygon:

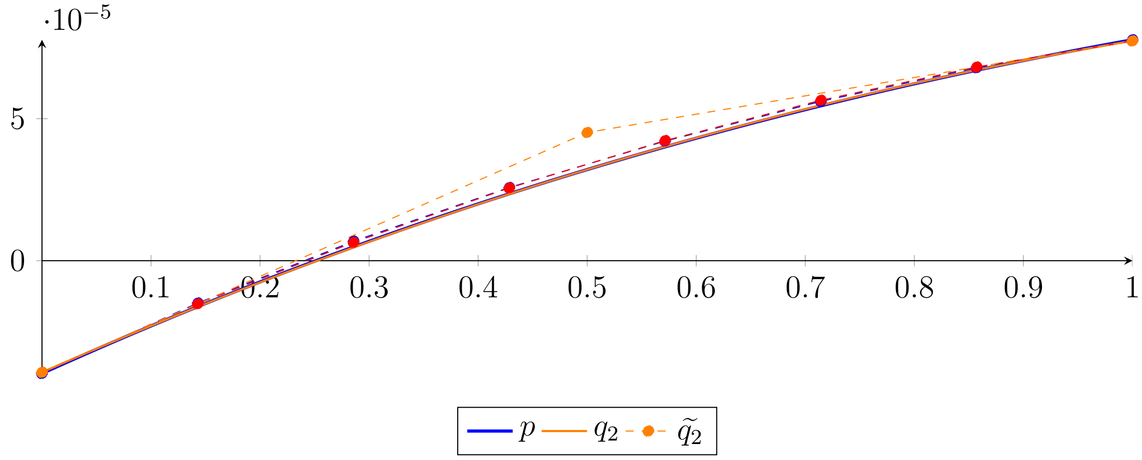
$$\begin{aligned} p &= 2.07713 \cdot 10^{-12} X^7 - 3.17039 \cdot 10^{-10} X^6 + 1.90432 \cdot 10^{-08} X^5 - 5.68801 \cdot 10^{-07} X^4 \\ &\quad + 8.75592 \cdot 10^{-06} X^3 - 6.43572 \cdot 10^{-05} X^2 + 0.00017363X - 3.9692 \cdot 10^{-05} \\ &= -3.9692 \cdot 10^{-05} B_{0,7}(X) - 1.48878 \cdot 10^{-05} B_{1,7}(X) + 6.85187 \cdot 10^{-06} B_{2,7}(X) + 2.5777 \cdot 10^{-05} B_{3,7}(X) \\ &\quad + 4.21217 \cdot 10^{-05} B_{4,7}(X) + 5.61043 \cdot 10^{-05} B_{5,7}(X) + 6.79291 \cdot 10^{-05} B_{6,7}(X) + 7.77864 \cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_2 &= -5.21649{\cdot}10^{-05}X^2 + 0.000168876X - 3.9301{\cdot}10^{-05} \\
&= -3.9301{\cdot}10^{-05}B_{0,2} + 4.51371{\cdot}10^{-05}B_{1,2} + 7.74103{\cdot}10^{-05}B_{2,2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_2 &= 4.59873{\cdot}10^{-18}X^7 - 1.50025{\cdot}10^{-17}X^6 + 1.91232{\cdot}10^{-17}X^5 - 1.19912{\cdot}10^{-17}X^4 \\
&\quad + 3.77907{\cdot}10^{-18}X^3 - 5.21649{\cdot}10^{-05}X^2 + 0.000168876X - 3.9301{\cdot}10^{-05} \\
&= -3.9301{\cdot}10^{-05}B_{0,7} - 1.51758{\cdot}10^{-05}B_{1,7} + 6.46534{\cdot}10^{-06}B_{2,7} + 2.56224{\cdot}10^{-05}B_{3,7} \\
&\quad + 4.22955{\cdot}10^{-05}B_{4,7} + 5.64845{\cdot}10^{-05}B_{5,7} + 6.81894{\cdot}10^{-05}B_{6,7} + 7.74103{\cdot}10^{-05}B_{7,7}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 3.91045{\cdot}10^{-07}$.

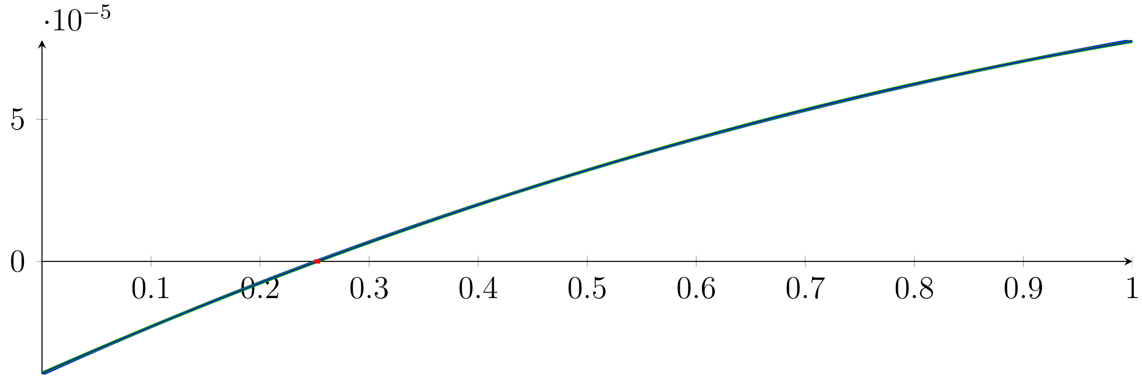
Bounding polynomials M and m:

$$
\begin{aligned}
M &= -5.21649{\cdot}10^{-05}X^2 + 0.000168876X - 3.89099{\cdot}10^{-05} \\
m &= -5.21649{\cdot}10^{-05}X^2 + 0.000168876X - 3.9692{\cdot}10^{-05}
\end{aligned}
$$

Root of M and m:

$$N(M) = \{0.249658, 2.98769\} \qquad N(m) = \{0.255145, 2.98221\}$$

Intersection intervals:



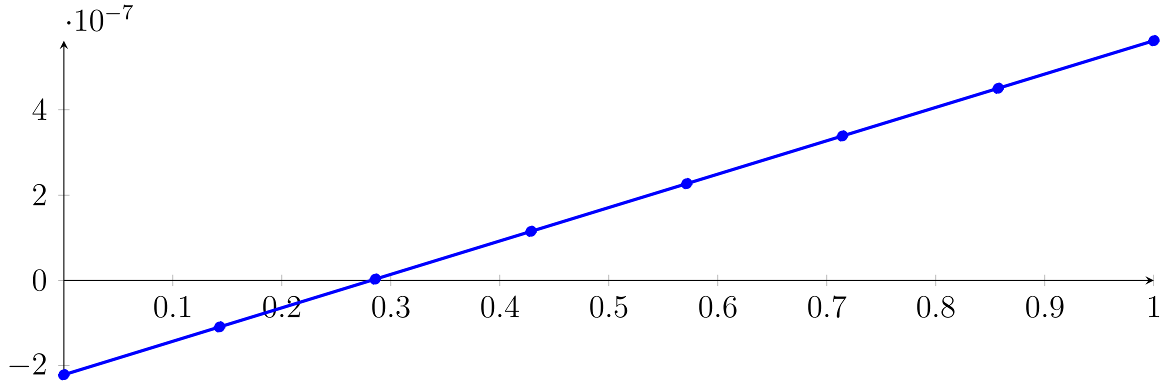
$$[0.249658, 0.255145]$$

Longest intersection interval: 0.00548668
$\Longrightarrow$ Selective recursion: interval 1: $[0.0380271, 0.0381447]$,

14.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0380271, 0.0381447]$

Normalized monomial und Bézier representations and the Bézier polygon:

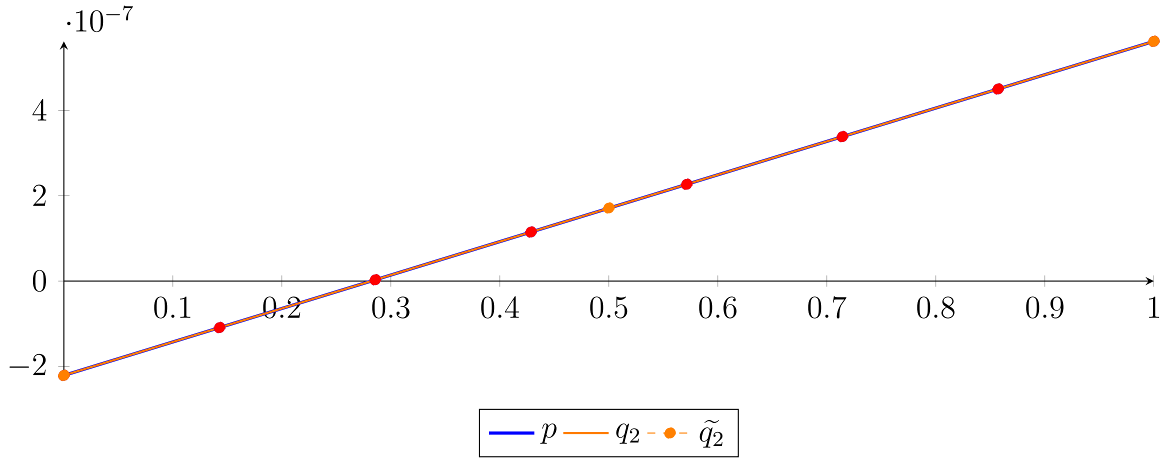
$$\begin{aligned}p &= -1.39587\cdot 10^{-24}X^7 - 2.26182\cdot 10^{-23}X^6 + 9.23496\cdot 10^{-20}X^5 - 4.94191\cdot 10^{-16}X^4 \\ &\quad + 1.35433\cdot 10^{-12}X^3 - 1.74628\cdot 10^{-09}X^2 + 7.8513\cdot 10^{-07}X - 2.212\cdot 10^{-07} \\ &= -2.212\cdot 10^{-07}B_{0,7}(X) - 1.09038\cdot 10^{-07}B_{1,7}(X) + 3.04007\cdot 10^{-09}B_{2,7}(X) + 1.15035\cdot 10^{-07}B_{3,7}(X) \\ &\quad + 2.26947\cdot 10^{-07}B_{4,7}(X) + 3.38776\cdot 10^{-07}B_{5,7}(X) + 4.50522\cdot 10^{-07}B_{6,7}(X) + 5.62185\cdot 10^{-07}B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -1.74425\cdot 10^{-09}X^2 + 7.85129\cdot 10^{-07}X - 2.212\cdot 10^{-07} \\ &= -2.212\cdot 10^{-07}B_{0,2} + 1.71365\cdot 10^{-07}B_{1,2} + 5.62185\cdot 10^{-07}B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= 4.24108\cdot 10^{-20}X^7 - 1.39719\cdot 10^{-19}X^6 + 1.80394\cdot 10^{-19}X^5 - 1.15255\cdot 10^{-19}X^4 \\ &\quad + 3.7568\cdot 10^{-20}X^3 - 1.74425\cdot 10^{-09}X^2 + 7.85129\cdot 10^{-07}X - 2.212\cdot 10^{-07} \\ &= -2.212\cdot 10^{-07}B_{0,7} - 1.09038\cdot 10^{-07}B_{1,7} + 3.04\cdot 10^{-09}B_{2,7} + 1.15035\cdot 10^{-07}B_{3,7} \\ &\quad + 2.26947\cdot 10^{-07}B_{4,7} + 3.38776\cdot 10^{-07}B_{5,7} + 4.50522\cdot 10^{-07}B_{6,7} + 5.62185\cdot 10^{-07}B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.76744\cdot 10^{-14}$.
Bounding polynomials M and m:

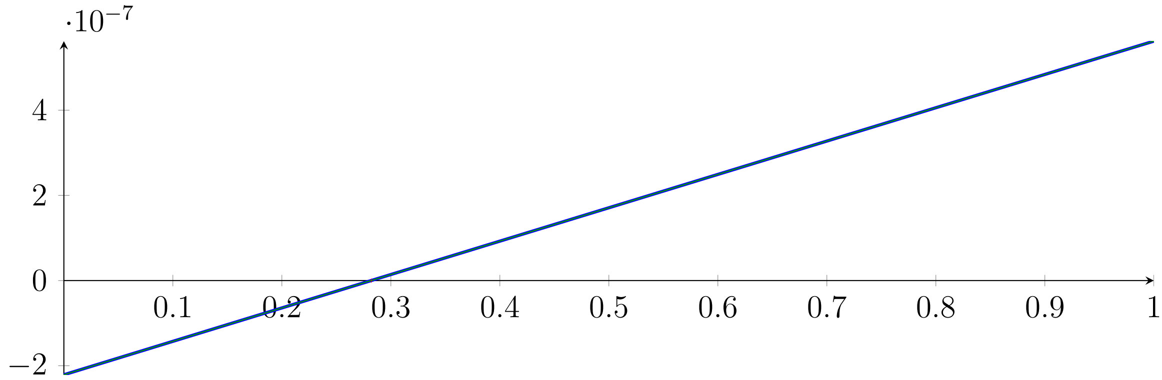
$$M = -1.74425\cdot 10^{-09}X^2 + 7.85129\cdot 10^{-07}X - 2.21199\cdot 10^{-07}$$

$$m = -1.74425{\cdot}10^{-09}X^2 + 7.85129{\cdot}10^{-07}X - 2.212{\cdot}10^{-07}$$

Root of M and m:

$$N(M) = \{0.281913, 449.841\} \qquad N(m) = \{0.281913, 449.841\}$$

Intersection intervals:



$$[0.281913, 0.281913]$$

Longest intersection interval: $1.72607{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0380602, 0.0380602]$,

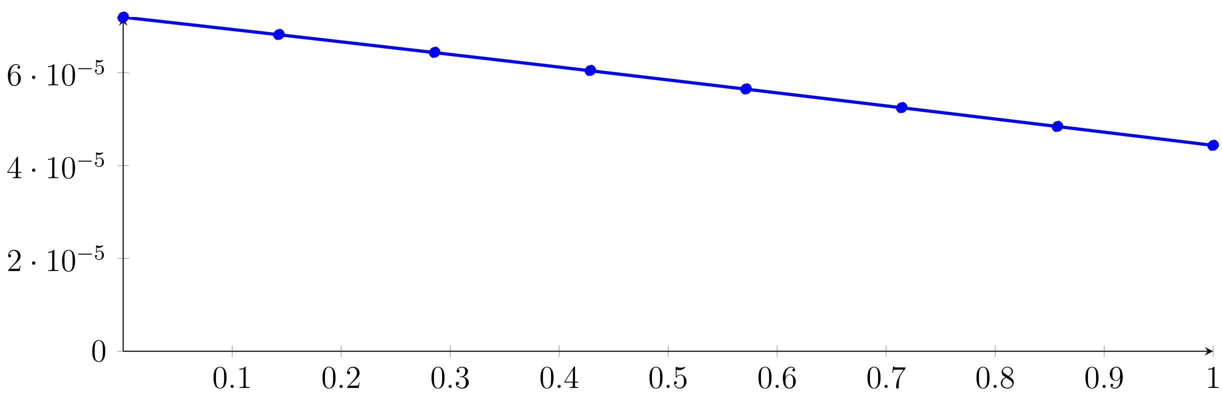
14.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.0380602, 0.0380602]$

Found root in interval $[0.0380602, 0.0380602]$ at recursion depth 7!

14.8 Recursion Branch 1 1 1 1 2 in Interval 2: $[0.111491, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

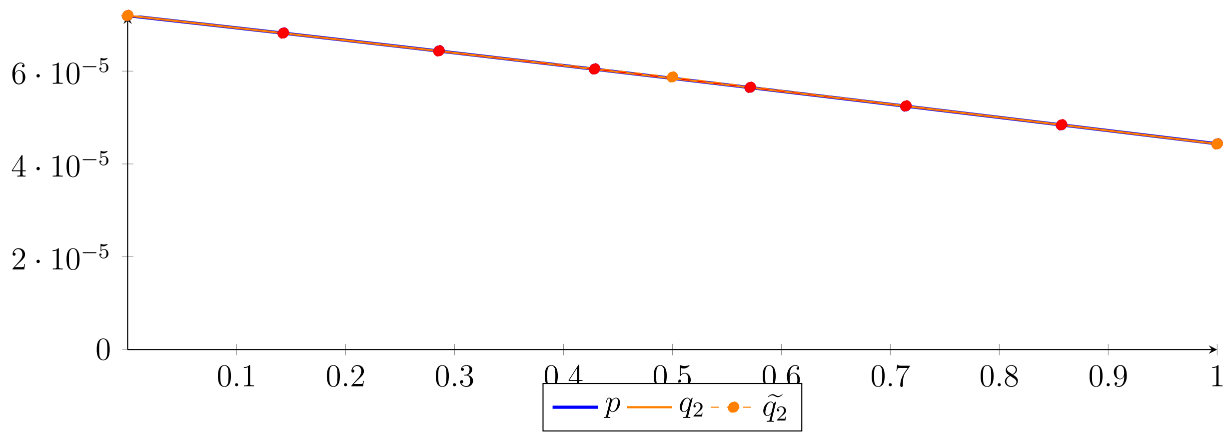
$$\begin{aligned} p &= 8.21048{\cdot}10^{-14}X^7 - 1.65289{\cdot}10^{-11}X^6 + 1.25736{\cdot}10^{-09}X^5 - 4.40941{\cdot}10^{-08}X^4 \\ &\quad + 6.66703{\cdot}10^{-07}X^3 - 2.09874{\cdot}10^{-06}X^2 - 2.61301{\cdot}10^{-05}X + 7.19509{\cdot}10^{-05} \\ &= 7.19509{\cdot}10^{-05}B_{0,7}(X) + 6.8218{\cdot}10^{-05}B_{1,7}(X) + 6.43852{\cdot}10^{-05}B_{2,7}(X) + 6.04715{\cdot}10^{-05}B_{3,7}(X) \\ &\quad + 5.64947{\cdot}10^{-05}B_{4,7}(X) + 5.24713{\cdot}10^{-05}B_{5,7}(X) + 4.8417{\cdot}10^{-05}B_{6,7}(X) + 4.43459{\cdot}10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.17206{\cdot}10^{-06}X^2 - 2.64911{\cdot}10^{-05}X + 7.19805{\cdot}10^{-05} \\ &= 7.19805{\cdot}10^{-05}B_{0,2} + 5.8735{\cdot}10^{-05}B_{1,2} + 4.43173{\cdot}10^{-05}B_{2,2} \end{aligned}$$

$$
\begin{aligned}
\widetilde{q}_2 &= -1.31084{\cdot}10^{-17}X^7 + 4.81555{\cdot}10^{-17}X^6 - 7.08293{\cdot}10^{-17}X^5 + 5.31095{\cdot}10^{-17}X^4 \\
&\quad - 2.12959{\cdot}10^{-17}X^3 - 1.17206{\cdot}10^{-06}X^2 - 2.64911{\cdot}10^{-05}X + 7.19805{\cdot}10^{-05} \\
&= 7.19805{\cdot}10^{-05}B_{0,7} + 6.81961{\cdot}10^{-05}B_{1,7} + 6.43558{\cdot}10^{-05}B_{2,7} + 6.04598{\cdot}10^{-05}B_{3,7} \\
&\quad + 5.65079{\cdot}10^{-05}B_{4,7} + 5.25002{\cdot}10^{-05}B_{5,7} + 4.84367{\cdot}10^{-05}B_{6,7} + 4.43173{\cdot}10^{-05}B_{7,7}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 2.96884{\cdot}10^{-08}$.

Bounding polynomials M and m:

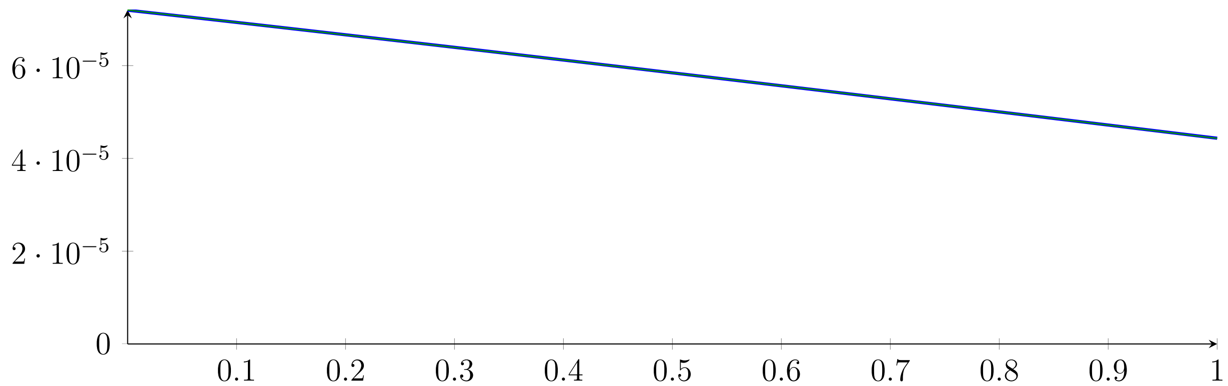
$$M = -1.17206{\cdot}10^{-06}X^2 - 2.64911{\cdot}10^{-05}X + 7.20102{\cdot}10^{-05}$$

$$m = -1.17206{\cdot}10^{-06}X^2 - 2.64911{\cdot}10^{-05}X + 7.19509{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-25.0545, 2.45222\} \qquad N(m) = \{-25.0527, 2.45038\}$$

Intersection intervals:

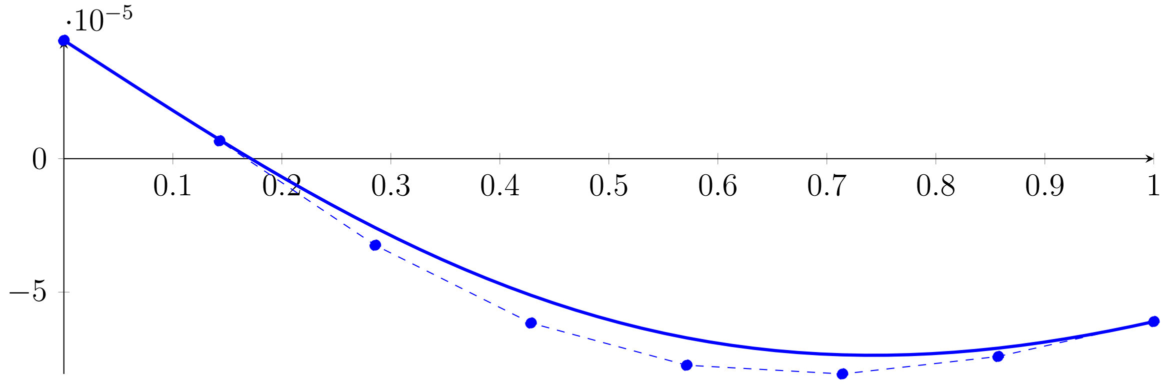


No intersection intervals with the x axis.

14.9 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

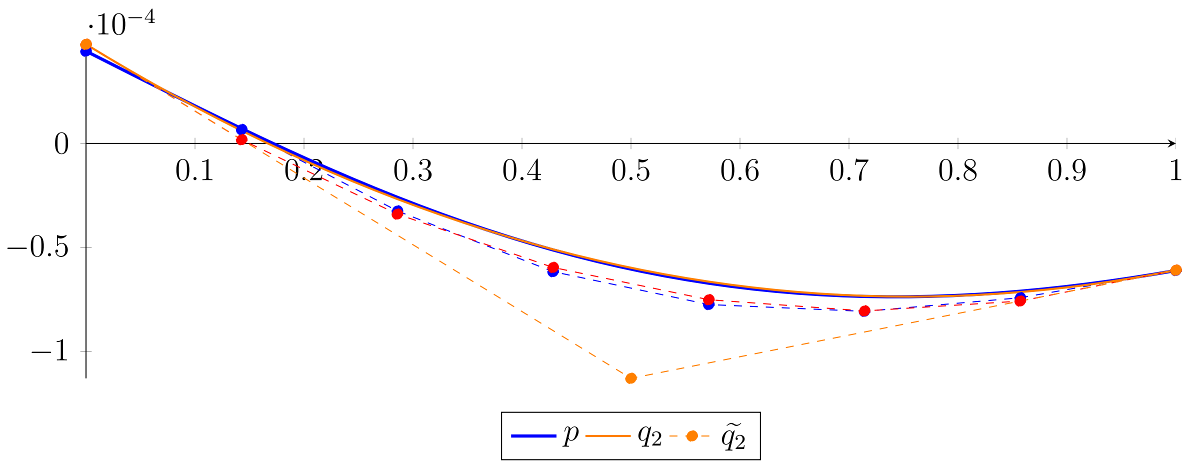
$$
\begin{aligned}
p &= 4.76837{\cdot}10^{-07}X^7 - 1.00136{\cdot}10^{-05}X^6 + 7.86781{\cdot}10^{-05}X^5 - 0.00027895X^4 \\
&\quad + 0.000398159X^3 - 3.00407{\cdot}10^{-05}X^2 - 0.000263691X + 4.43459{\cdot}10^{-05} \\
&= 4.43459{\cdot}10^{-05}B_{0,7}(X) + 6.67572{\cdot}10^{-06}B_{1,7}(X) - 3.24249{\cdot}10^{-05}B_{2,7}(X) - 6.15801{\cdot}10^{-05}B_{3,7}(X) \\
&\quad - 7.73839{\cdot}10^{-05}B_{4,7}(X) - 8.06536{\cdot}10^{-05}B_{5,7}(X) - 7.41141{\cdot}10^{-05}B_{6,7}(X) - 6.10352{\cdot}10^{-05}B_{7,7}(X)
\end{aligned}
$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.000212448X^2 - 0.000320957X + 4.76411\cdot 10^{-05} \\ &= 4.76411\cdot 10^{-05}B_{0,2} - 0.000112837B_{1,2} - 6.08677\cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.76564\cdot 10^{-18}X^7 - 1.04041\cdot 10^{-17}X^6 + 1.56443\cdot 10^{-17}X^5 - 1.20148\cdot 10^{-17}X^4 \\ &\quad + 4.99966\cdot 10^{-18}X^3 + 0.000212448X^2 - 0.000320957X + 4.76411\cdot 10^{-05} \\ &= 4.76411\cdot 10^{-05}B_{0,7} + 1.79017\cdot 10^{-06}B_{1,7} - 3.39442\cdot 10^{-05}B_{2,7} - 5.95621\cdot 10^{-05}B_{3,7} \\ &\quad - 7.50633\cdot 10^{-05}B_{4,7} - 8.0448\cdot 10^{-05}B_{5,7} - 7.57161\cdot 10^{-05}B_{6,7} - 6.08677\cdot 10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.88555\cdot 10^{-06}$.

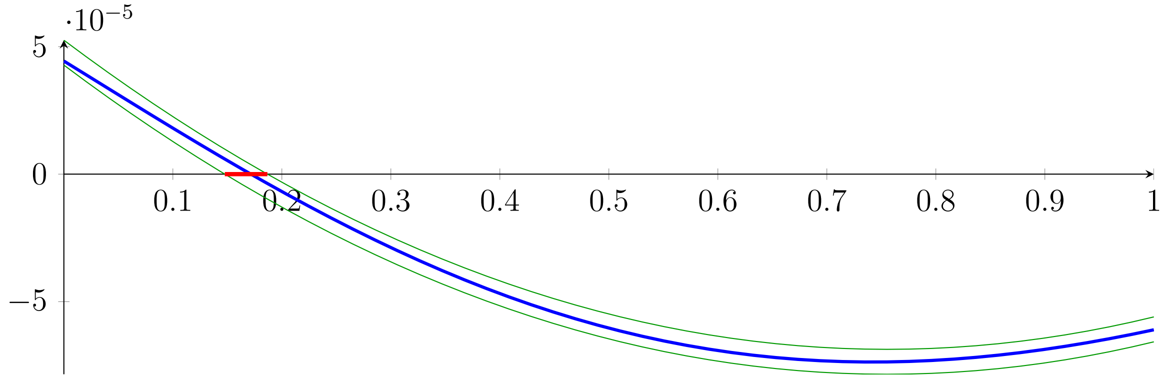
Bounding polynomials M and m:

$$M = 0.000212448X^2 - 0.000320957X + 5.25267\cdot 10^{-05}$$
$$m = 0.000212448X^2 - 0.000320957X + 4.27556\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.186739, 1.32402\} \qquad N(m) = \{0.147641, 1.36311\}$$

Intersection intervals:



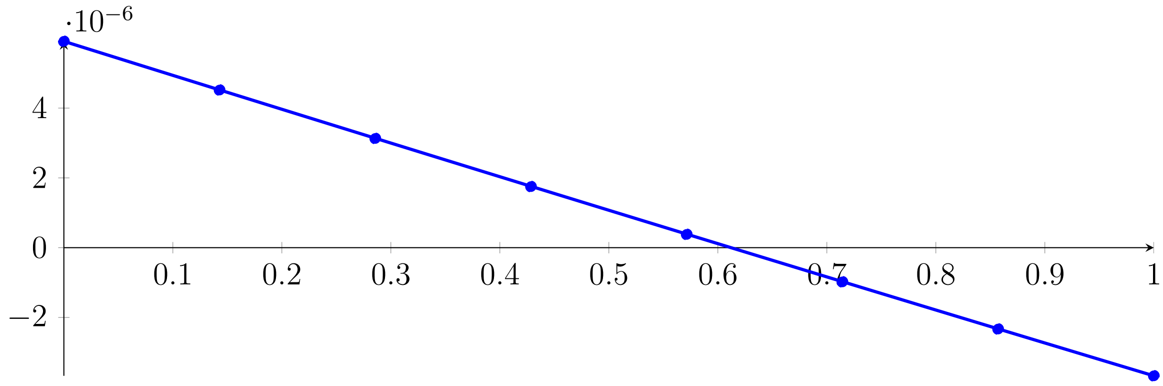
$$[0.147641, 0.186739]$$

Longest intersection interval: 0.0390972
$\Longrightarrow$ Selective recursion: interval 1: $[0.143455, 0.148342]$,

14.10 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.143455, 0.148342]$

Normalized monomial und Bézier representations and the Bézier polygon:

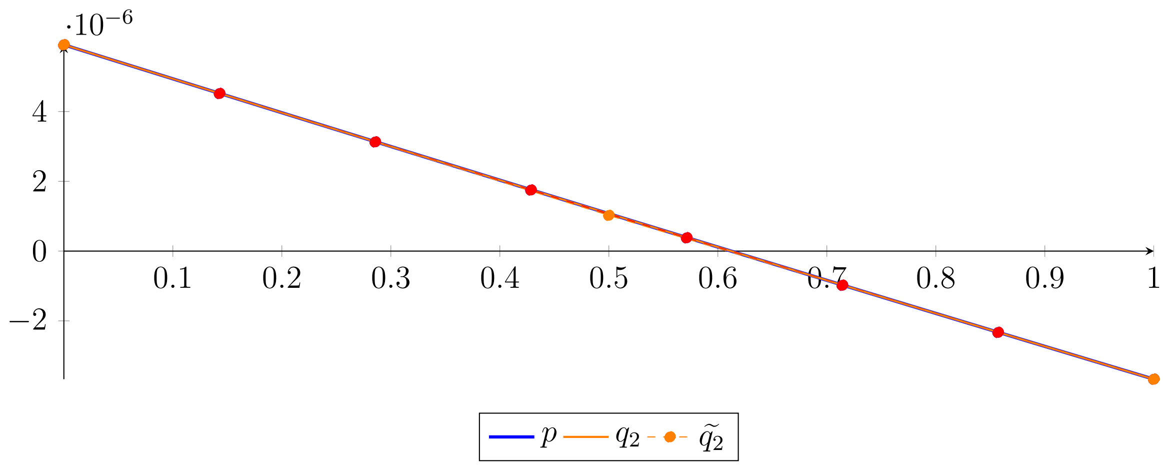
$$\begin{aligned} p &= 6.65868\cdot 10^{-17}X^7 - 3.40052\cdot 10^{-14}X^6 + 6.39714\cdot 10^{-12}X^5 - 5.23604\cdot 10^{-10}X^4 \\ &\quad + 1.4937\cdot 10^{-08}X^3 + 1.71648\cdot 10^{-07}X^2 - 9.77166\cdot 10^{-06}X + 5.91358\cdot 10^{-06} \\ &= 5.91358\cdot 10^{-06}B_{0,7}(X) + 4.51763\cdot 10^{-06}B_{1,7}(X) + 3.12985\cdot 10^{-06}B_{2,7}(X) + 1.75067\cdot 10^{-06}B_{3,7}(X) \\ &\quad + 3.8051\cdot 10^{-07}B_{4,7}(X) - 9.80245\cdot 10^{-07}B_{5,7}(X) - 2.33121\cdot 10^{-06}B_{6,7}(X) - 3.67201\cdot 10^{-06}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.93167\cdot 10^{-07}X^2 - 9.78015\cdot 10^{-06}X + 5.91428\cdot 10^{-06} \\ &= 5.91428\cdot 10^{-06}B_{0,2} + 1.02421\cdot 10^{-06}B_{1,2} - 3.6727\cdot 10^{-06}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -1.0865\cdot 10^{-18}X^7 + 3.82494\cdot 10^{-18}X^6 - 5.3661\cdot 10^{-18}X^5 + 3.81657\cdot 10^{-18}X^4 \\ &\quad - 1.44082\cdot 10^{-18}X^3 + 1.93167\cdot 10^{-07}X^2 - 9.78015\cdot 10^{-06}X + 5.91428\cdot 10^{-06} \\ &= 5.91428\cdot 10^{-06}B_{0,7} + 4.51712\cdot 10^{-06}B_{1,7} + 3.12915\cdot 10^{-06}B_{2,7} + 1.75039\cdot 10^{-06}B_{3,7} \\ &\quad + 3.80817\cdot 10^{-07}B_{4,7} - 9.79553\cdot 10^{-07}B_{5,7} - 2.33072\cdot 10^{-06}B_{6,7} - 3.6727\cdot 10^{-06}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.0265\cdot 10^{-10}$.
Bounding polynomials M and m:

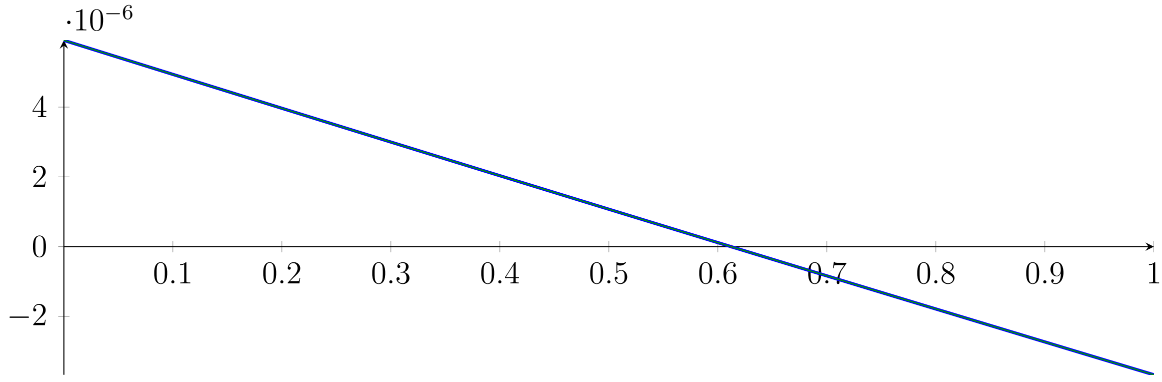
$$M = 1.93167\cdot 10^{-07}X^2 - 9.78015\cdot 10^{-06}X + 5.91498\cdot 10^{-06}$$

$$m = 1.93167{\cdot}10^{-07}X^2 - 9.78015{\cdot}10^{-06}X + 5.91358{\cdot}10^{-06}$$

Root of M and m:

$$N(M) = \{0.612197, 50.0183\} \qquad N(m) = \{0.61205, 50.0184\}$$

Intersection intervals:



$$[0.61205, 0.612197]$$

Longest intersection interval: 0.000147249
$\Longrightarrow$ Selective recursion: interval 1: $[0.146446, 0.146447]$,

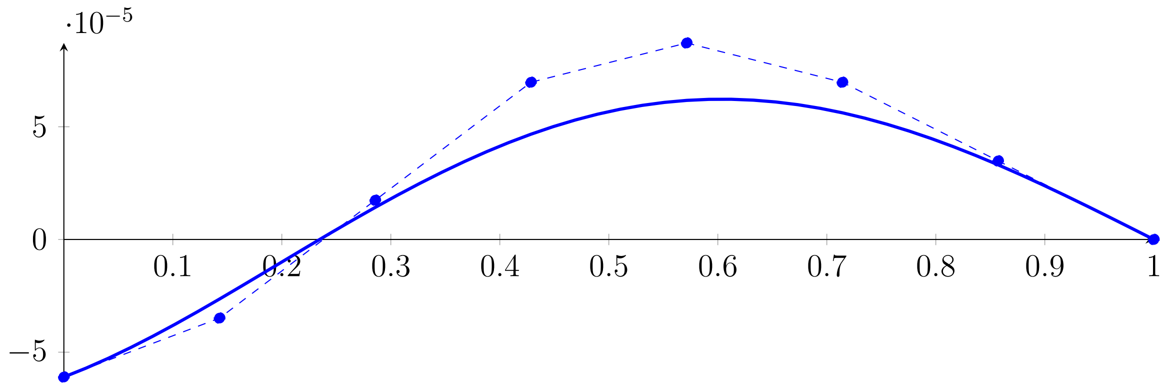
14.11 Recursion Branch 1 1 1 2 1 1 in Interval 1: $[0.146446, 0.146447]$

Found root in interval $[0.146446, 0.146447]$ at recursion depth 6!

14.12 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

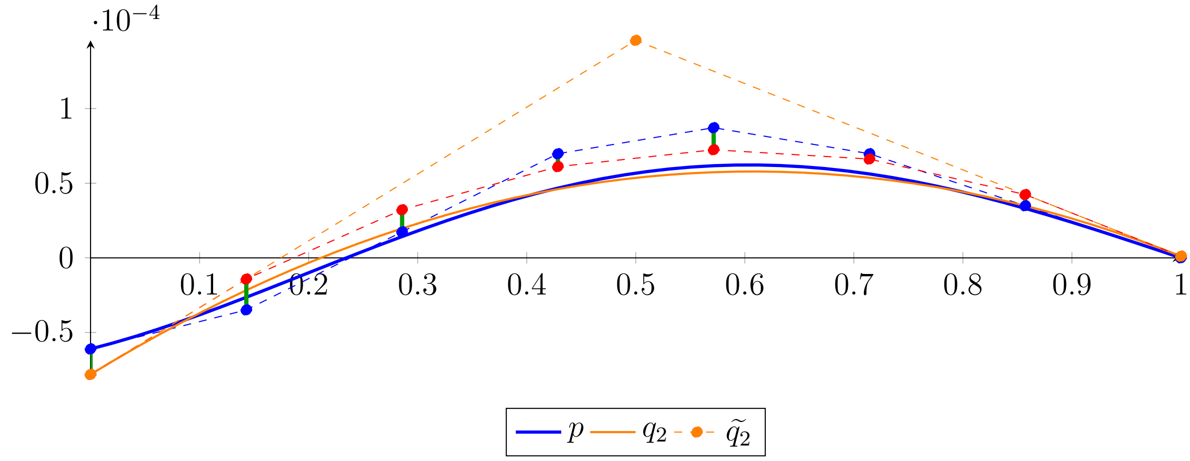
$$\begin{aligned} p &= 6.10352{\cdot}10^{-05}X^7 - 0.000427246X^6 + 0.000915527X^5 - 0.000305176X^4 \\ &\quad - 0.000915527X^3 + 0.000549316X^2 + 0.000183105X - 6.10352{\cdot}10^{-05} \\ &= -6.10352{\cdot}10^{-05}B_{0,7}(X) - 3.48772{\cdot}10^{-05}B_{1,7}(X) + 1.74386{\cdot}10^{-05}B_{2,7}(X) + 6.97545{\cdot}10^{-05}B_{3,7}(X) \\ &\quad + 8.71931{\cdot}10^{-05}B_{4,7}(X) + 6.97545{\cdot}10^{-05}B_{5,7}(X) + 3.48772{\cdot}10^{-05}B_{6,7}(X) - 2.05803{\cdot}10^{-21}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.000368391X^2 + 0.000447591X - 7.81105{\cdot}10^{-05} \\ &= -7.81105{\cdot}10^{-05}B_{0,2} + 0.000145685B_{1,2} + 1.08991{\cdot}10^{-06}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -5.98329{\cdot}10^{-18}X^7 + 1.99272{\cdot}10^{-17}X^6 - 2.60362{\cdot}10^{-17}X^5 + 1.69358{\cdot}10^{-17}X^4 \\ &\quad - 5.7759{\cdot}10^{-18}X^3 - 0.000368391X^2 + 0.000447591X - 7.81105{\cdot}10^{-05} \\ &= -7.81105{\cdot}10^{-05}B_{0,7} - 1.41689{\cdot}10^{-05}B_{1,7} + 3.22303{\cdot}10^{-05}B_{2,7} + 6.10871{\cdot}10^{-05}B_{3,7} \\ &\quad + 7.24014{\cdot}10^{-05}B_{4,7} + 6.61733{\cdot}10^{-05}B_{5,7} + 4.24028{\cdot}10^{-05}B_{6,7} + 1.08991{\cdot}10^{-06}B_{7,7} \end{aligned}$$



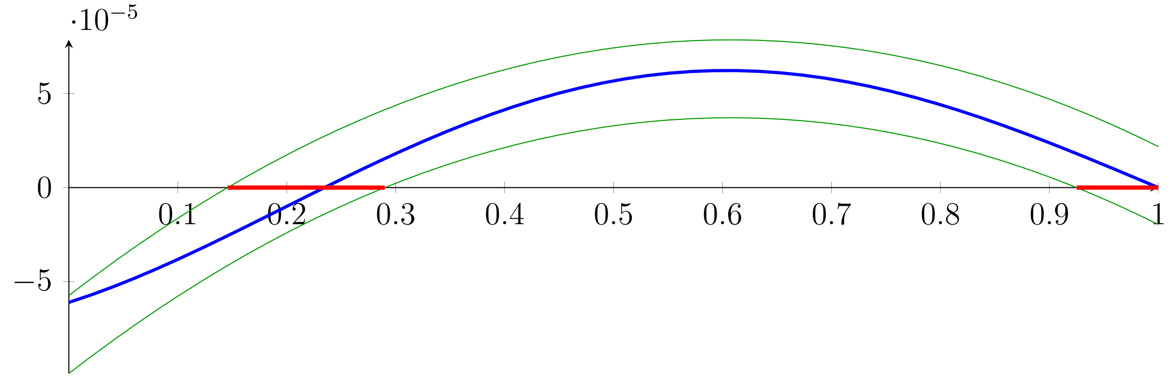
The maximum difference of the Bézier coefficients is $\delta = 2.07084{\cdot}10^{-05}$.
Bounding polynomials M and m:

$$M = -0.000368391X^2 + 0.000447591X - 5.74021{\cdot}10^{-05}$$
$$m = -0.000368391X^2 + 0.000447591X - 9.88188{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.145725, 1.06927\} \qquad N(m) = \{0.289996, 0.924994\}$$

Intersection intervals:



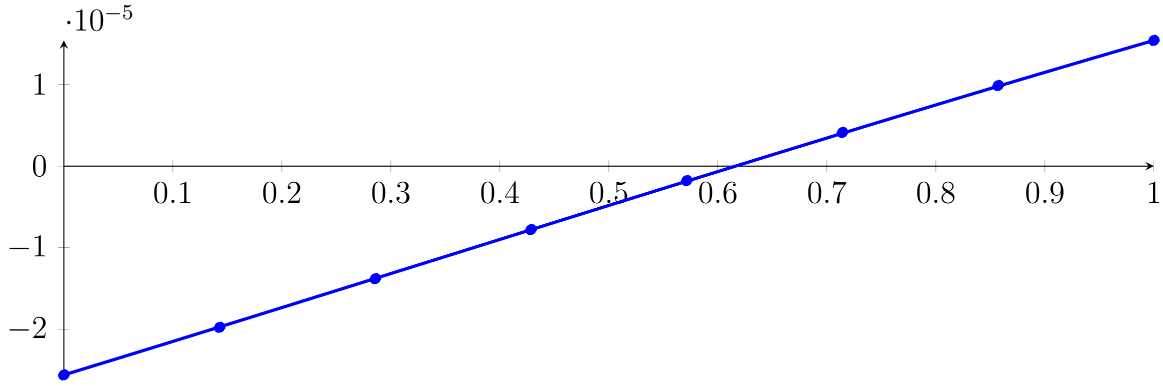
$$[0.145725, 0.289996],\ [0.924994, 1]$$

Longest intersection interval: 0.144271
$\Longrightarrow$ Selective recursion: interval 1: $[0.286431, 0.322499]$, interval 2: $[0.481249, 0.5]$,

14.13 Recursion Branch 1 1 2 1 in Interval 1: $[0.286431, 0.322499]$

Normalized monomial und Bézier representations and the Bézier polygon:

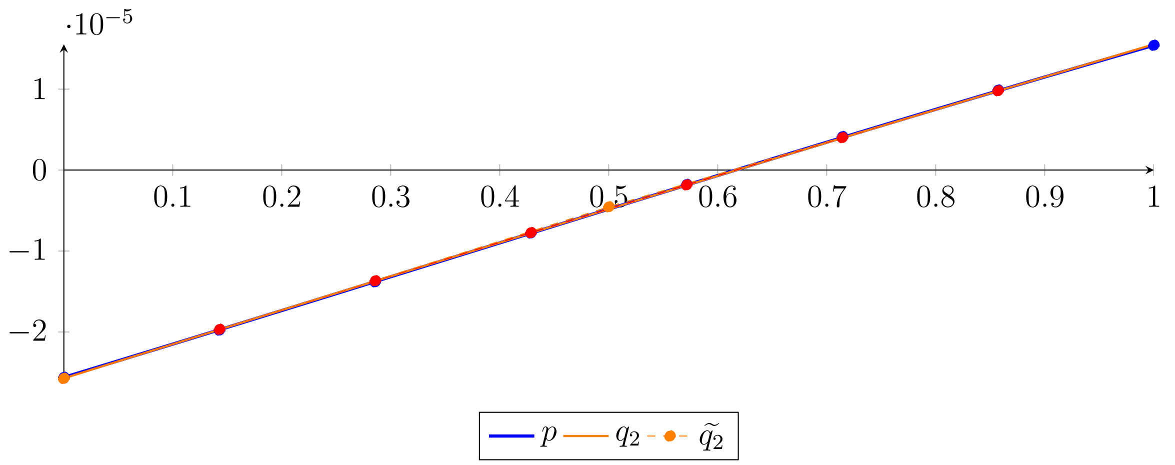
$$\begin{aligned}p &= 7.94027{\cdot}10^{-11}X^7 - 3.29118{\cdot}10^{-09}X^6 + 3.55753{\cdot}10^{-08}X^5 + 1.0069{\cdot}10^{-07}X^4 \\ &\quad - 2.77609{\cdot}10^{-06}X^3 + 2.82475{\cdot}10^{-06}X^2 + 4.08288{\cdot}10^{-05}X - 2.56016{\cdot}10^{-05} \\ &= -2.56016{\cdot}10^{-05}B_{0,7}(X) - 1.97689{\cdot}10^{-05}B_{1,7}(X) - 1.38018{\cdot}10^{-05}B_{2,7}(X) - 7.77937{\cdot}10^{-06}B_{3,7}(X) \\ &\quad - 1.77823{\cdot}10^{-06}B_{4,7}(X) + 4.1298{\cdot}10^{-06}B_{5,7}(X) + 9.87865{\cdot}10^{-06}B_{6,7}(X) + 1.54089{\cdot}10^{-05}B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_2 &= -1.10898{\cdot}10^{-06}X^2 + 4.23679{\cdot}10^{-05}X - 2.57284{\cdot}10^{-05} \\
&= -2.57284{\cdot}10^{-05}B_{0,2} - 4.54442{\cdot}10^{-06}B_{1,2} + 1.55306{\cdot}10^{-05}B_{2,2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_2 &= 4.71011{\cdot}10^{-18}X^7 - 1.65916{\cdot}10^{-17}X^6 + 2.32932{\cdot}10^{-17}X^5 - 1.65809{\cdot}10^{-17}X^4 \\
&\quad + 6.26576{\cdot}10^{-18}X^3 - 1.10898{\cdot}10^{-06}X^2 + 4.23679{\cdot}10^{-05}X - 2.57284{\cdot}10^{-05} \\
&= -2.57284{\cdot}10^{-05}B_{0,7} - 1.96758{\cdot}10^{-05}B_{1,7} - 1.36761{\cdot}10^{-05}B_{2,7} - 7.72912{\cdot}10^{-06}B_{3,7} \\
&\quad - 1.83499{\cdot}10^{-06}B_{4,7} + 4.00633{\cdot}10^{-06}B_{5,7} + 9.79485{\cdot}10^{-06}B_{6,7} + 1.55306{\cdot}10^{-05}B_{7,7}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 1.26744{\cdot}10^{-07}$.

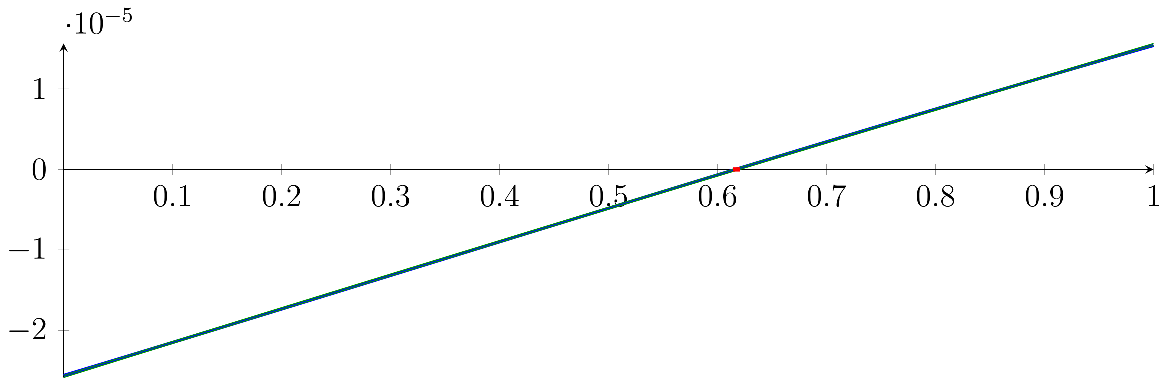
Bounding polynomials M and m:

$$
\begin{aligned}
M &= -1.10898{\cdot}10^{-06}X^2 + 4.23679{\cdot}10^{-05}X - 2.56016{\cdot}10^{-05} \\
m &= -1.10898{\cdot}10^{-06}X^2 + 4.23679{\cdot}10^{-05}X - 2.58551{\cdot}10^{-05}
\end{aligned}
$$

Root of M and m:

$$N(M) = \{0.614142, 37.5904\} \qquad N(m) = \{0.620325, 37.5842\}$$

Intersection intervals:



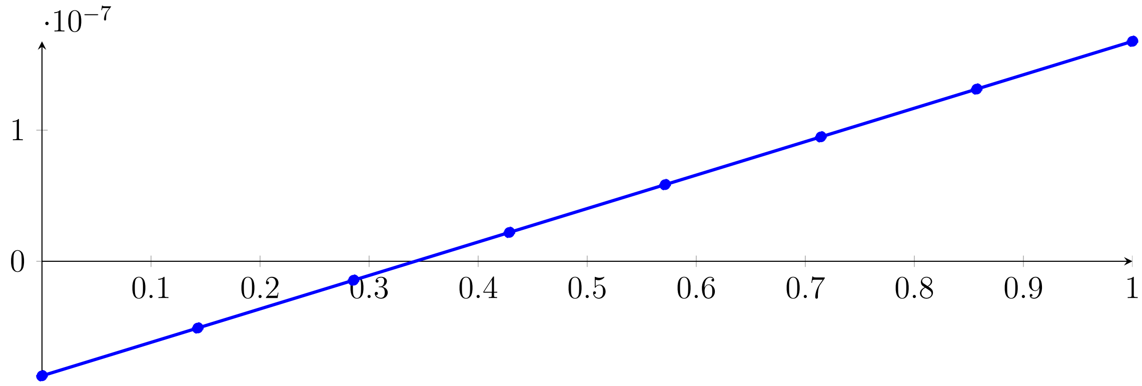
$$[0.614142, 0.620325]$$

Longest intersection interval: 0.0061828
$\Longrightarrow$ Selective recursion: interval 1: $[0.308582, 0.308805]$,

14.14 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.308582, 0.308805]$

Normalized monomial und Bézier representations and the Bézier polygon:

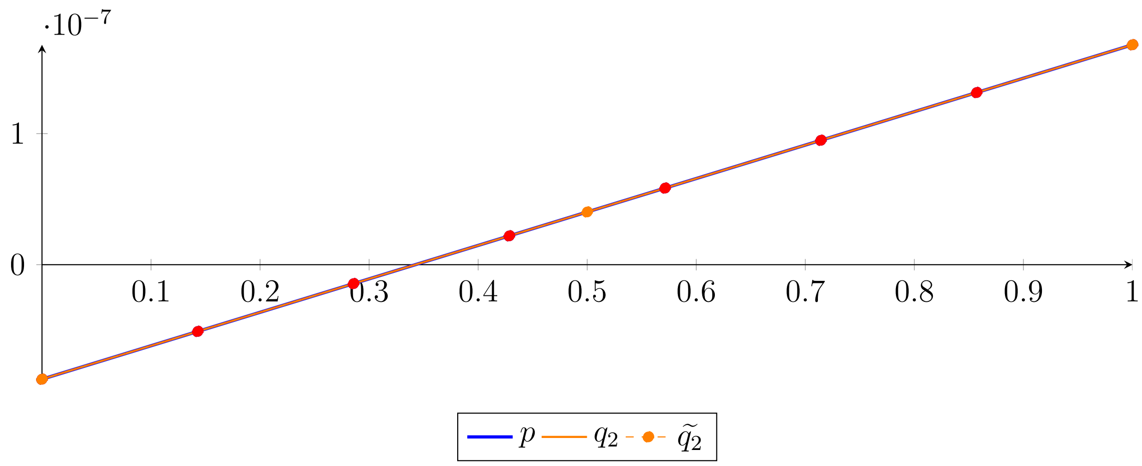
$$\begin{aligned}p = &-7.75482 \cdot 10^{-26} X^7 - 1.67465 \cdot 10^{-22} X^6 + 2.17535 \cdot 10^{-19} X^5 + 2.80505 \cdot 10^{-16} X^4 \\ &- 5.69463 \cdot 10^{-13} X^3 - 7.59413 \cdot 10^{-11} X^2 + 2.55189 \cdot 10^{-07} X - 8.73514 \cdot 10^{-08} \\ = &-8.73514 \cdot 10^{-08} B_{0,7}(X) - 5.08958 \cdot 10^{-08} B_{1,7}(X) - 1.44438 \cdot 10^{-08} B_{2,7}(X) + 2.20046 \cdot 10^{-08} B_{3,7}(X) \\ &+ 5.84493 \cdot 10^{-08} B_{4,7}(X) + 9.48904 \cdot 10^{-08} B_{5,7}(X) + 1.31328 \cdot 10^{-07} B_{6,7}(X) + 1.67761 \cdot 10^{-07} B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -7.6795 \cdot 10^{-11} X^2 + 2.5519 \cdot 10^{-07} X - 8.73514 \cdot 10^{-08} \\ &= -8.73514 \cdot 10^{-08} B_{0,2} + 4.02434 \cdot 10^{-08} B_{1,2} + 1.67762 \cdot 10^{-07} B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= 1.65938 \cdot 10^{-20} X^7 - 5.58559 \cdot 10^{-20} X^6 + 7.41869 \cdot 10^{-20} X^5 - 4.92774 \cdot 10^{-20} X^4 \\ &\quad + 1.70159 \cdot 10^{-20} X^3 - 7.6795 \cdot 10^{-11} X^2 + 2.5519 \cdot 10^{-07} X - 8.73514 \cdot 10^{-08} \\ &= -8.73514 \cdot 10^{-08} B_{0,7} - 5.08958 \cdot 10^{-08} B_{1,7} - 1.44437 \cdot 10^{-08} B_{2,7} + 2.20046 \cdot 10^{-08} B_{3,7} \\ &\quad + 5.84493 \cdot 10^{-08} B_{4,7} + 9.48904 \cdot 10^{-08} B_{5,7} + 1.31328 \cdot 10^{-07} B_{6,7} + 1.67762 \cdot 10^{-07} B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84491 \cdot 10^{-14}$.
Bounding polynomials M and m:

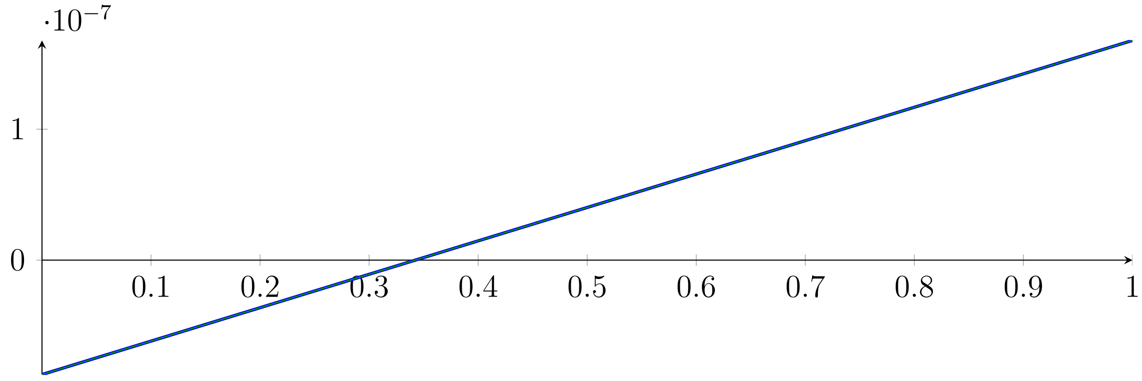
$$M = -7.6795 \cdot 10^{-11} X^2 + 2.5519 \cdot 10^{-07} X - 8.73514 \cdot 10^{-08}$$

$$m = -7.6795{\cdot}10^{-11}X^2 + 2.5519{\cdot}10^{-07}X - 8.73515{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{0.342335, 3322.66\} \qquad N(m) = \{0.342335, 3322.66\}$$

Intersection intervals:



$$[0.342335, 0.342335]$$

Longest intersection interval: $2.2301{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.308658, 0.308658]$,

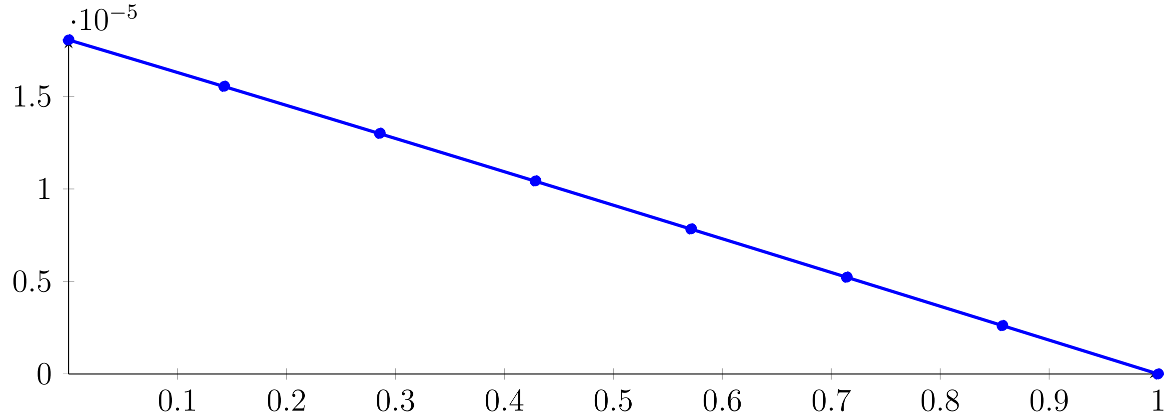
14.15 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.308658, 0.308658]$

Found root in interval $[0.308658, 0.308658]$ at recursion depth 6!

14.16 Recursion Branch 1 1 2 2 in Interval 2: $[0.481249, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

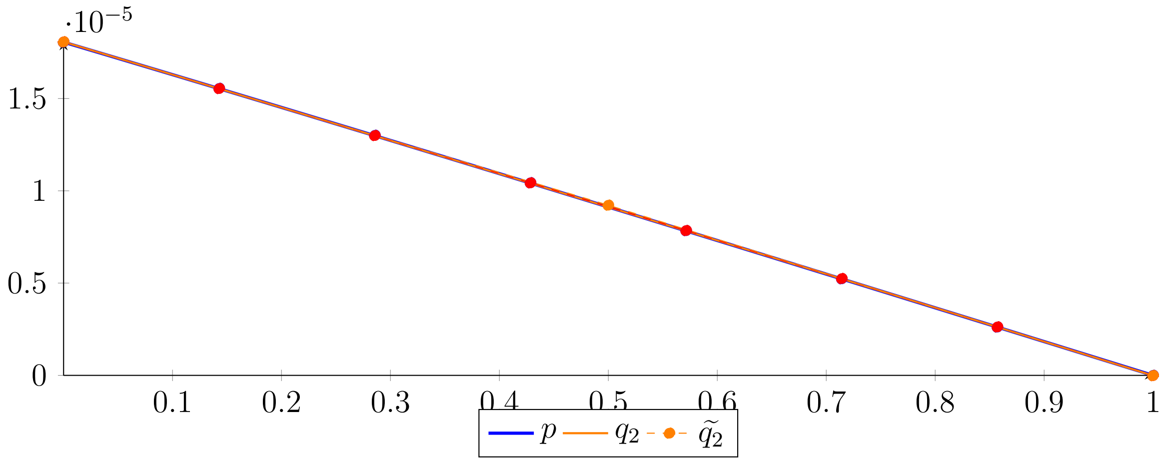
$$\begin{aligned} p &= 8.15164{\cdot}10^{-13}X^7 - 5.70615{\cdot}10^{-12}X^6 - 8.52255{\cdot}10^{-10}X^5 + 4.31834{\cdot}10^{-09}X^4 \\ &\quad + 2.48887{\cdot}10^{-07}X^3 - 7.6398{\cdot}10^{-07}X^2 - 1.75437{\cdot}10^{-05}X + 1.80553{\cdot}10^{-05} \\ &= 1.80553{\cdot}10^{-05}B_{0,7}(X) + 1.5549{\cdot}10^{-05}B_{1,7}(X) + 1.30064{\cdot}10^{-05}B_{2,7}(X) + 1.04345{\cdot}10^{-05}B_{3,7}(X) \\ &\quad + 7.84063{\cdot}10^{-06}B_{4,7}(X) + 5.23199{\cdot}10^{-06}B_{5,7}(X) + 2.616{\cdot}10^{-06}B_{6,7}(X) - 2.05803{\cdot}10^{-21}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.84777{\cdot}10^{-07}X^2 - 1.7696{\cdot}10^{-05}X + 1.8068{\cdot}10^{-05} \\ &= 1.8068{\cdot}10^{-05}B_{0,2} + 9.22{\cdot}10^{-06}B_{1,2} - 1.27846{\cdot}10^{-08}B_{2,2} \end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= -3.33408{\cdot}10^{-18}X^7 + 1.19901{\cdot}10^{-17}X^6 - 1.72328{\cdot}10^{-17}X^5 + 1.26004{\cdot}10^{-17}X^4 \\ &\quad - 4.91379{\cdot}10^{-18}X^3 - 3.84777{\cdot}10^{-07}X^2 - 1.7696{\cdot}10^{-05}X + 1.8068{\cdot}10^{-05} \\ &= 1.8068{\cdot}10^{-05}B_{0,7} + 1.554{\cdot}10^{-05}B_{1,7} + 1.29937{\cdot}10^{-05}B_{2,7} + 1.0429{\cdot}10^{-05}B_{3,7} \\ &\quad + 7.84606{\cdot}10^{-06}B_{4,7} + 5.24477{\cdot}10^{-06}B_{5,7} + 2.62515{\cdot}10^{-06}B_{6,7} - 1.27846{\cdot}10^{-08}B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.27846{\cdot}10^{-08}$.

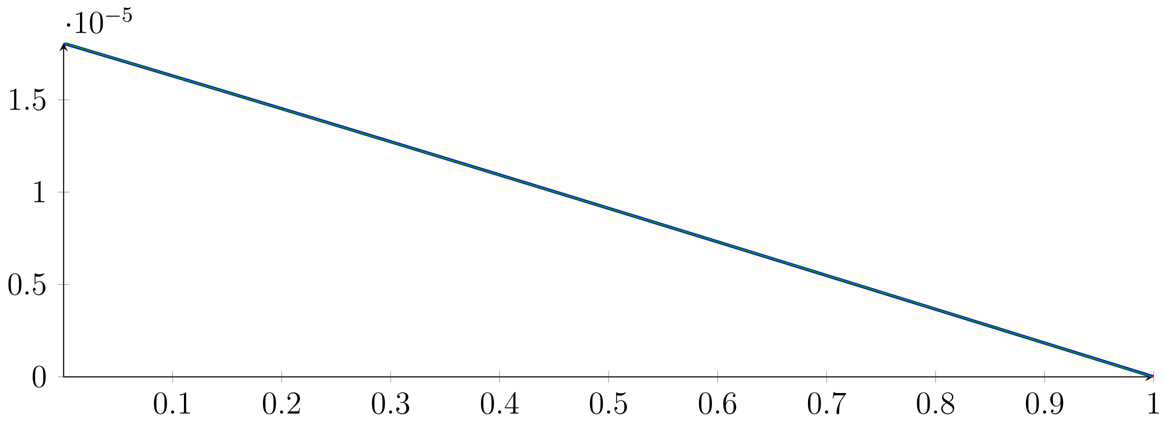
Bounding polynomials M and m:

$$M = -3.84777{\cdot}10^{-07}X^2 - 1.7696{\cdot}10^{-05}X + 1.80808{\cdot}10^{-05}$$
$$m = -3.84777{\cdot}10^{-07}X^2 - 1.7696{\cdot}10^{-05}X + 1.80552{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-46.9903, 1\} \qquad N(m) = \{-46.9889, 0.998615\}$$

Intersection intervals:



$[0.998615, 1]$

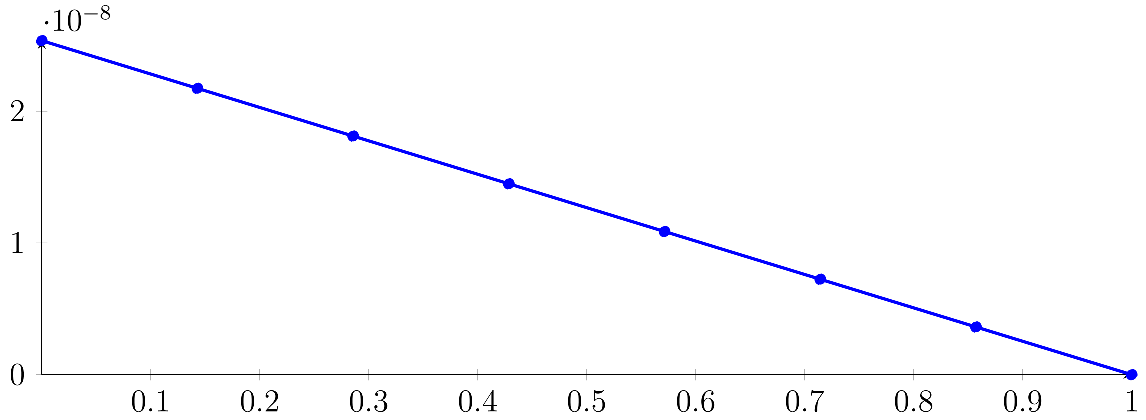
Longest intersection interval: 0.00138473

$\Longrightarrow$ Selective recursion: interval 1: $[0.499974, 0.5]$,

14.17 Recursion Branch 1 1 2 2 1 in Interval 1: $[0.499974, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

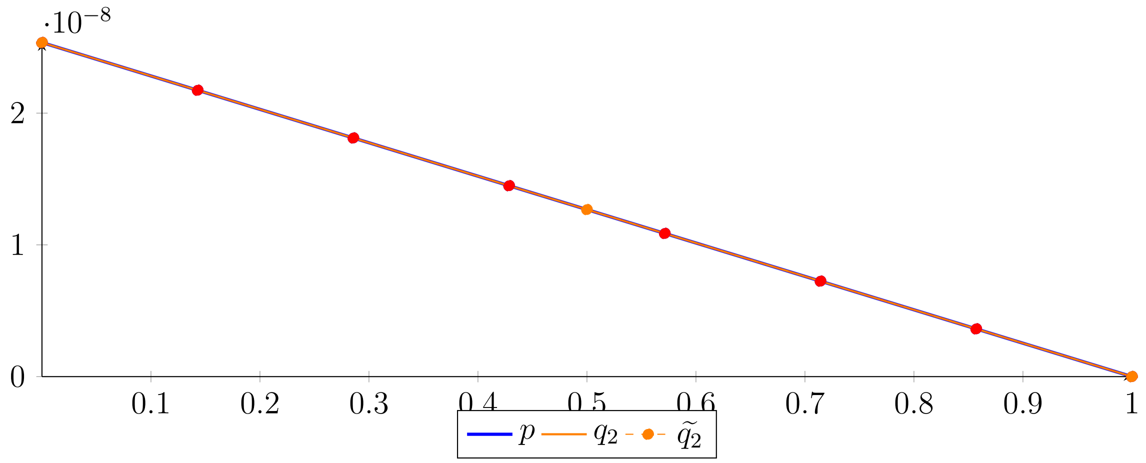
$$\begin{aligned}p &= -7.59326{\cdot}10^{-26}X^7 - 1.82077{\cdot}10^{-24}X^6 - 3.2231{\cdot}10^{-24}X^5 + 2.14308{\cdot}10^{-23}X^4 \\ &\quad + 6.83851{\cdot}10^{-16}X^3 - 2.05155{\cdot}10^{-15}X^2 - 2.53572{\cdot}10^{-08}X + 2.53572{\cdot}10^{-08} \\ &= 2.53572{\cdot}10^{-08}B_{0,7}(X) + 2.17347{\cdot}10^{-08}B_{1,7}(X) + 1.81123{\cdot}10^{-08}B_{2,7}(X) + 1.44898{\cdot}10^{-08}B_{3,7}(X) \\ &\quad + 1.08674{\cdot}10^{-08}B_{4,7}(X) + 7.2449{\cdot}10^{-09}B_{5,7}(X) + 3.62245{\cdot}10^{-09}B_{6,7}(X) - 8.7672{\cdot}10^{-22}B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.02578 \cdot 10^{-15} X^2 - 2.53572 \cdot 10^{-08} X + 2.53572 \cdot 10^{-08} \\ &= 2.53572 \cdot 10^{-08} B_{0,2} + 1.26786 \cdot 10^{-08} B_{1,2} - 3.41934 \cdot 10^{-17} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -4.64932 \cdot 10^{-21} X^7 + 1.67228 \cdot 10^{-20} X^6 - 2.40393 \cdot 10^{-20} X^5 + 1.75811 \cdot 10^{-20} X^4 \\ &\quad - 6.85773 \cdot 10^{-21} X^3 - 1.02578 \cdot 10^{-15} X^2 - 2.53572 \cdot 10^{-08} X + 2.53572 \cdot 10^{-08} \\ &= 2.53572 \cdot 10^{-08} B_{0,7} + 2.17347 \cdot 10^{-08} B_{1,7} + 1.81123 \cdot 10^{-08} B_{2,7} + 1.44898 \cdot 10^{-08} B_{3,7} \\ &\quad + 1.08674 \cdot 10^{-08} B_{4,7} + 7.2449 \cdot 10^{-09} B_{5,7} + 3.62245 \cdot 10^{-09} B_{6,7} - 3.41934 \cdot 10^{-17} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.41926 \cdot 10^{-17}$.

Bounding polynomials M and m:

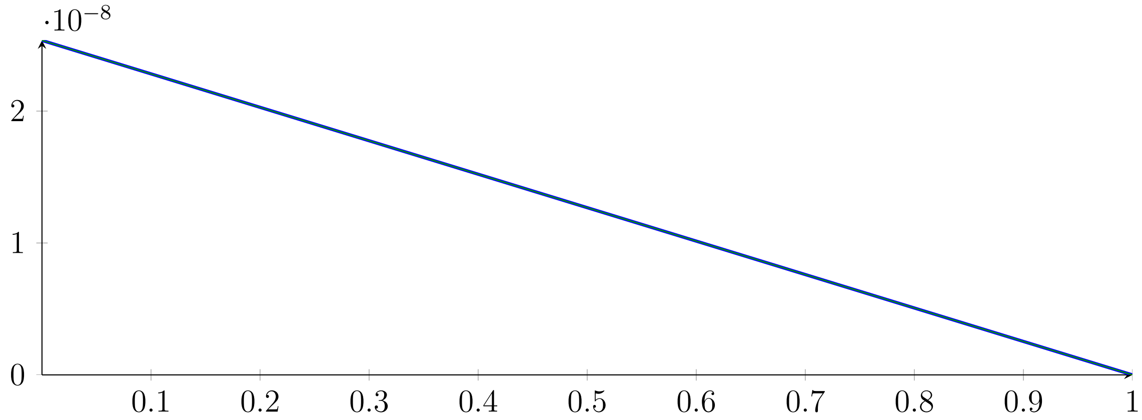
$$M = -1.02578 \cdot 10^{-15} X^2 - 2.53572 \cdot 10^{-08} X + 2.53572 \cdot 10^{-08}$$

$$m = -1.02578 \cdot 10^{-15} X^2 - 2.53572 \cdot 10^{-08} X + 2.53572 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{-2.472 \cdot 10^{7}, 1\} \qquad N(m) = \{-2.472 \cdot 10^{7}, 1\}$$

Intersection intervals:



$[1, 1]$

Longest intersection interval: $2.69717 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

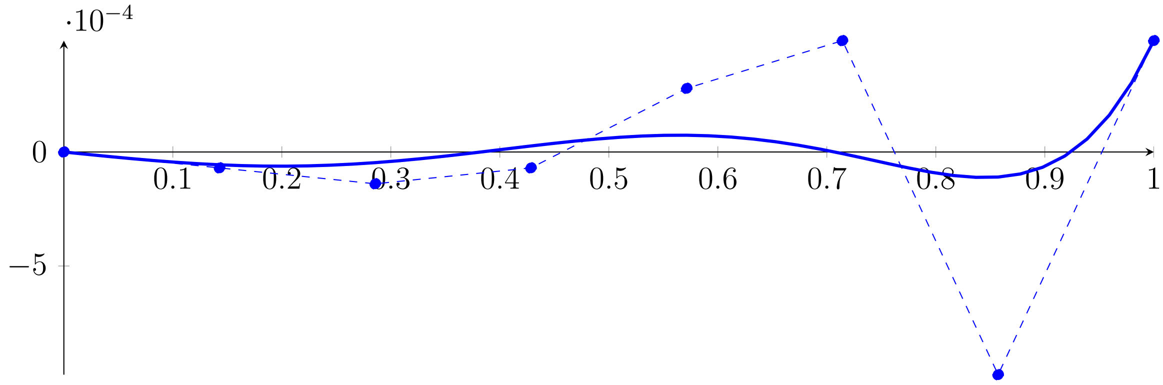
14.18 Recursion Branch 1 1 2 2 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 6!

14.19 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

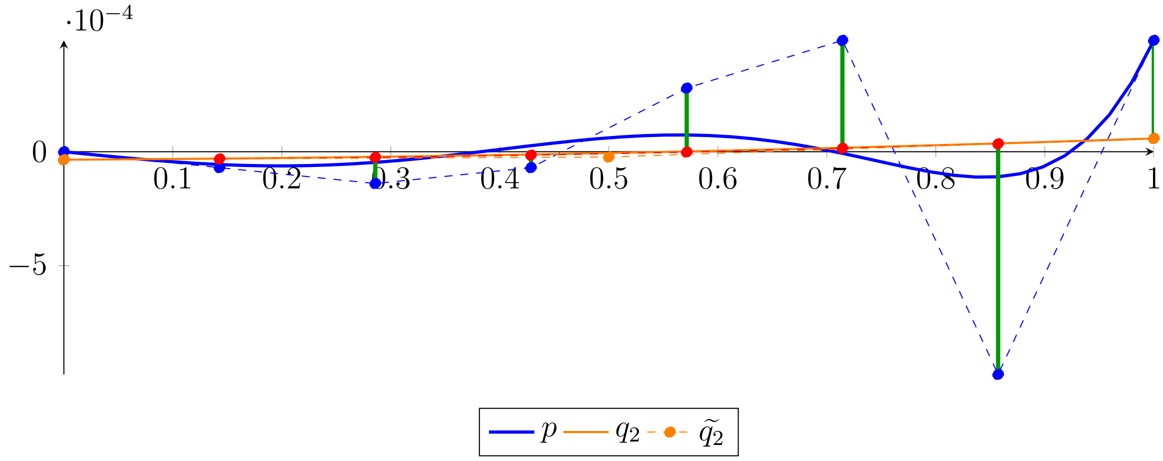
$$\begin{aligned} p &= 0.0078125X^7 - 1.49667 \cdot 10^{-19} X^6 - 0.0117188X^5 - 7.52966 \cdot 10^{-19} X^4 \\ &\quad + 0.00488281X^3 - 2.68622 \cdot 10^{-19} X^2 - 0.000488281X - 2.05803 \cdot 10^{-21} \\ &= -2.05803 \cdot 10^{-21} B_{0,7}(X) - 6.97545 \cdot 10^{-05} B_{1,7}(X) - 0.000139509 B_{2,7}(X) - 6.97545 \cdot 10^{-05} B_{3,7}(X) \\ &\quad + 0.000279018 B_{4,7}(X) + 0.000488281 B_{5,7}(X) - 0.000976563 B_{6,7}(X) + 0.000488281 B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 6.97545 \cdot 10^{-05} X^2 + 2.32515 \cdot 10^{-05} X - 3.48772 \cdot 10^{-05} \\ &= -3.48772 \cdot 10^{-05} B_{0,2} - 2.32515 \cdot 10^{-05} B_{1,2} + 5.81287 \cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.04419 \cdot 10^{-17} X^7 - 3.59241 \cdot 10^{-17} X^6 + 4.90231 \cdot 10^{-17} X^5 - 3.37226 \cdot 10^{-17} X^4 \\ &\quad + 1.22298 \cdot 10^{-17} X^3 + 6.97545 \cdot 10^{-05} X^2 + 2.32515 \cdot 10^{-05} X - 3.48772 \cdot 10^{-05} \\ &= -3.48772 \cdot 10^{-05} B_{0,7} - 3.15556 \cdot 10^{-05} B_{1,7} - 2.49123 \cdot 10^{-05} B_{2,7} - 1.49474 \cdot 10^{-05} B_{3,7} \\ &\quad - 1.66082 \cdot 10^{-06} B_{4,7} + 1.49474 \cdot 10^{-05} B_{5,7} + 3.48772 \cdot 10^{-05} B_{6,7} + 5.81287 \cdot 10^{-05} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00101144$.

Bounding polynomials M and m:

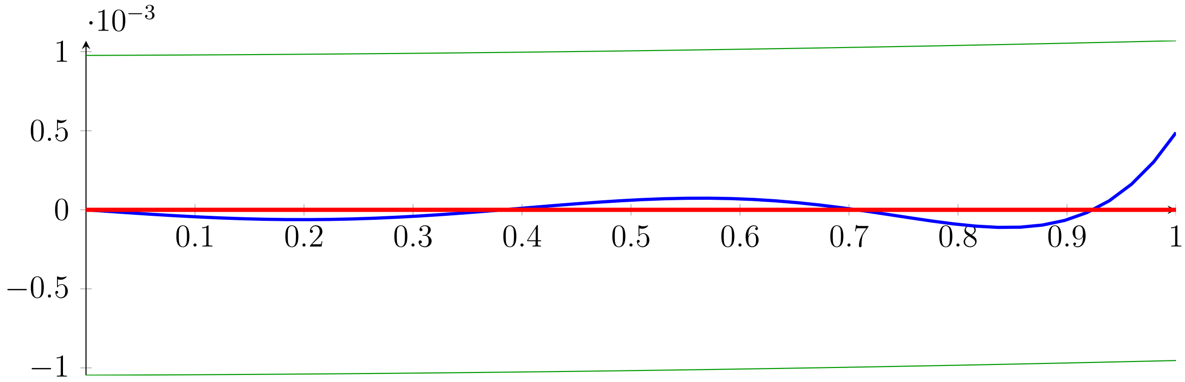
$$M = 6.97545 \cdot 10^{-05} X^2 + 2.32515 \cdot 10^{-05} X + 0.000976562$$

$$m = 6.97545 \cdot 10^{-05} X^2 + 2.32515 \cdot 10^{-05} X - 0.00104632$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-4.04323, 3.7099\}$$

Intersection intervals:



$[0, 1]$

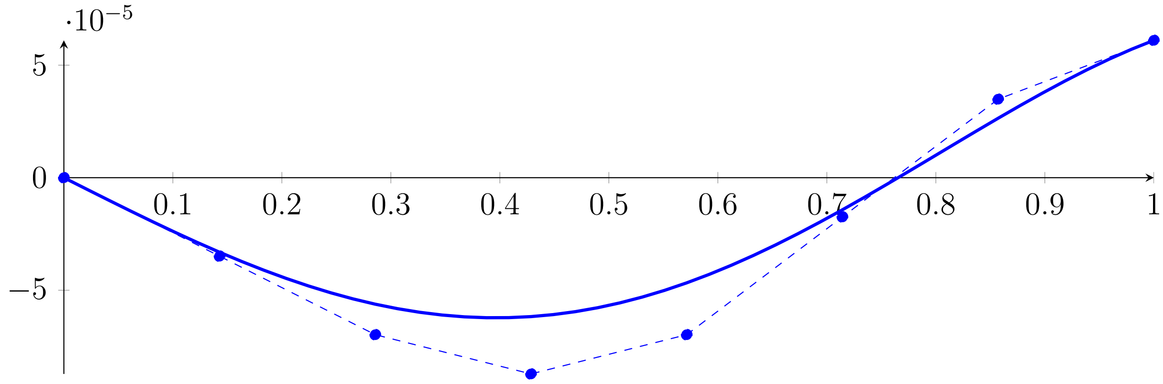
Longest intersection interval: 1

$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

14.20 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

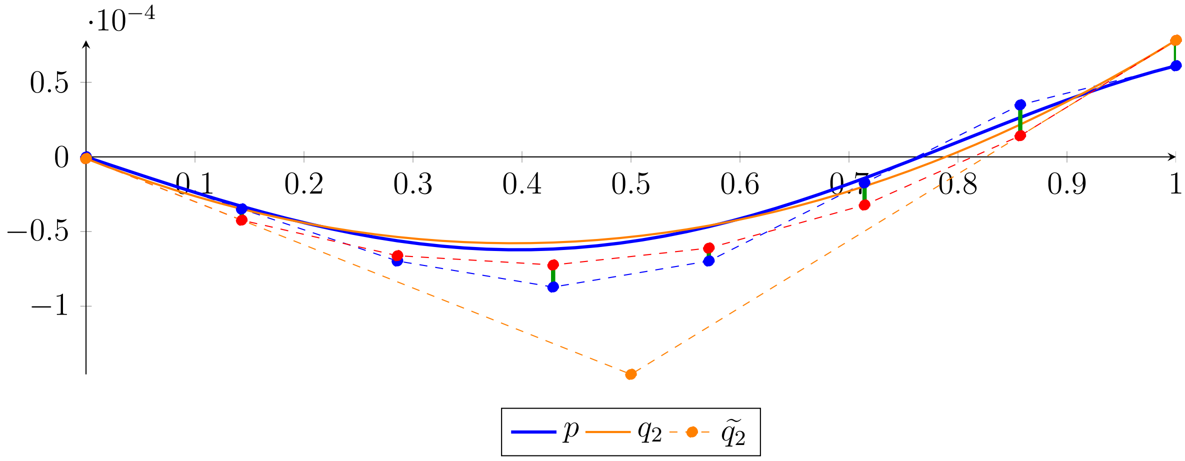
$$\begin{aligned} p &= 6.10352 \cdot 10^{-05} X^7 + 4.12267 \cdot 10^{-21} X^6 - 0.000366211 X^5 - 4.70169 \cdot 10^{-20} X^4 \\ &\quad + 0.000610352 X^3 - 6.71207 \cdot 10^{-20} X^2 - 0.000244141 X - 2.05803 \cdot 10^{-21} \\ &= -2.05803 \cdot 10^{-21} B_{0,7}(X) - 3.48772 \cdot 10^{-05} B_{1,7}(X) - 6.97545 \cdot 10^{-05} B_{2,7}(X) - 8.71931 \cdot 10^{-05} B_{3,7}(X) \\ &\quad - 6.97545 \cdot 10^{-05} B_{4,7}(X) - 1.74386 \cdot 10^{-05} B_{5,7}(X) + 3.48772 \cdot 10^{-05} B_{6,7}(X) + 6.10352 \cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.000368391X^2 - 0.00028919X - 1.08991\cdot 10^{-06} \\ &= -1.08991\cdot 10^{-06}B_{0,2} - 0.000145685B_{1,2} + 7.81105\cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.07796\cdot 10^{-17}X^7 - 7.13383\cdot 10^{-17}X^6 + 9.70555\cdot 10^{-17}X^5 - 6.65246\cdot 10^{-17}X^4 \\ &\quad + 2.40764\cdot 10^{-17}X^3 + 0.000368391X^2 - 0.00028919X - 1.08991\cdot 10^{-06} \\ &= -1.08991\cdot 10^{-06}B_{0,7} - 4.24028\cdot 10^{-05}B_{1,7} - 6.61733\cdot 10^{-05}B_{2,7} - 7.24014\cdot 10^{-05}B_{3,7} \\ &\quad - 6.10871\cdot 10^{-05}B_{4,7} - 3.22303\cdot 10^{-05}B_{5,7} + 1.41689\cdot 10^{-05}B_{6,7} + 7.81105\cdot 10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.07084\cdot 10^{-05}$.

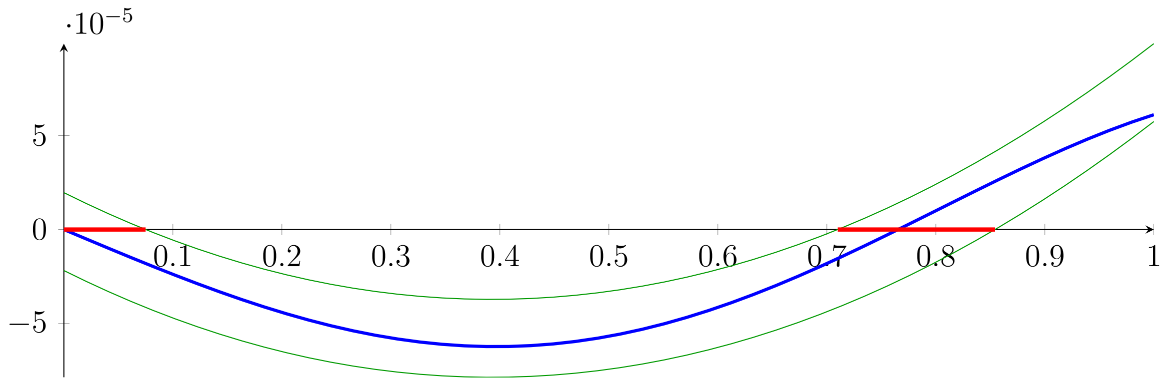
Bounding polynomials M and m:

$$\begin{aligned} M &= 0.000368391X^2 - 0.00028919X + 1.96184\cdot 10^{-05} \\ m &= 0.000368391X^2 - 0.00028919X - 2.17983\cdot 10^{-05} \end{aligned}$$

Root of M and m:

$$N(M) = \{0.0750058, 0.710004\} \qquad\qquad N(m) = \{-0.0692653, 0.854275\}$$

Intersection intervals:



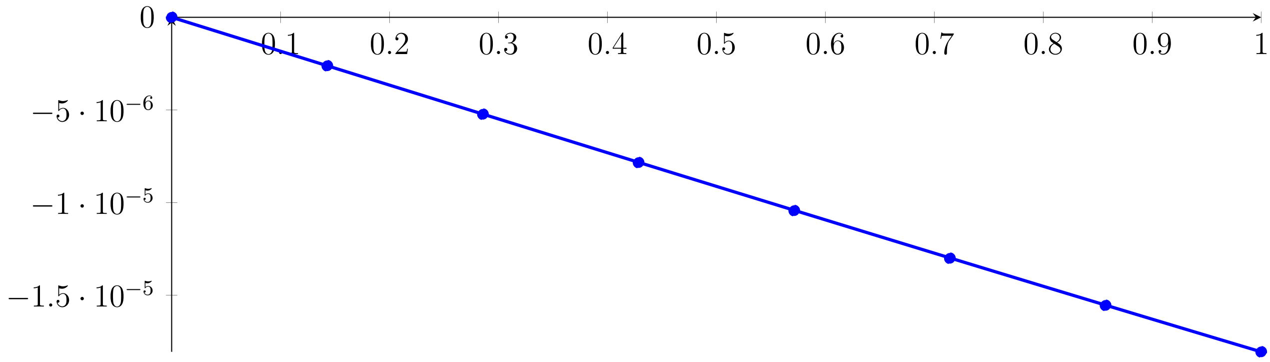
$$[0, 0.0750058], [0.710004, 0.854275]$$

Longest intersection interval: 0.144271
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.518751]$, interval 2: $[0.677501, 0.713569]$,

14.21 Recursion Branch 1 2 1 1 in Interval 1: $[0.5, 0.518751]$

Normalized monomial und Bézier representations and the Bézier polygon:

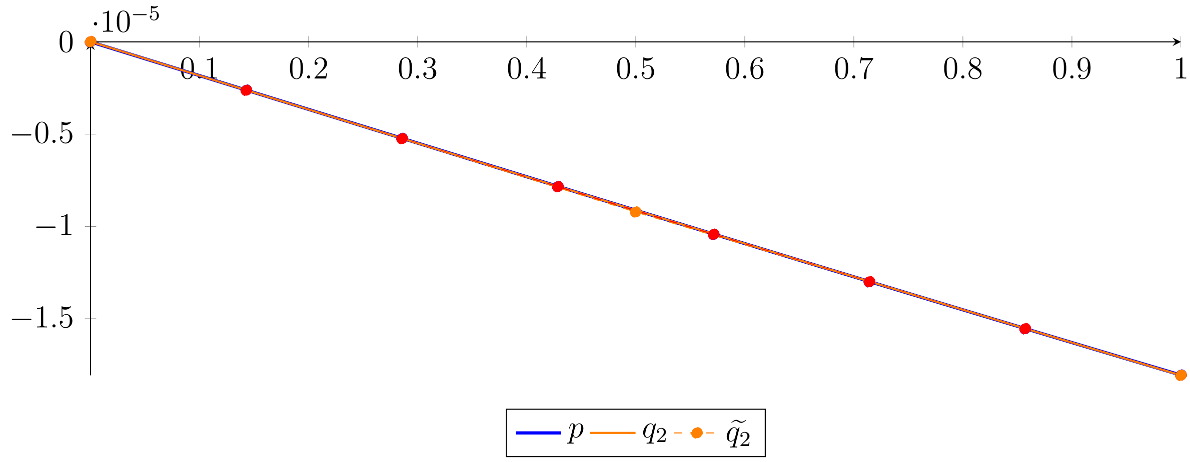
$$\begin{aligned} p = {} & 8.15164 \cdot 10^{-13} X^7 + 7.75895 \cdot 10^{-22} X^6 - 8.69373 \cdot 10^{-10} X^5 + 5.79026 \cdot 10^{-23} X^4 \\ & + 2.57552 \cdot 10^{-07} X^3 - 3.73472 \cdot 10^{-22} X^2 - 1.8312 \cdot 10^{-05} X - 2.05803 \cdot 10^{-21} \\ = {} & -2.05803 \cdot 10^{-21} B_{0,7}(X) - 2.616 \cdot 10^{-06} B_{1,7}(X) - 5.23199 \cdot 10^{-06} B_{2,7}(X) - 7.84063 \cdot 10^{-06} B_{3,7}(X) \\ & - 1.04345 \cdot 10^{-05} B_{4,7}(X) - 1.30064 \cdot 10^{-05} B_{5,7}(X) - 1.5549 \cdot 10^{-05} B_{6,7}(X) - 1.80553 \cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 3.84777 \cdot 10^{-07} X^2 - 1.84656 \cdot 10^{-05} X + 1.27846 \cdot 10^{-08} \\ &= 1.27846 \cdot 10^{-08} B_{0,2} - 9.22 \cdot 10^{-06} B_{1,2} - 1.8068 \cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = {} & -4.38589 \cdot 10^{-20} X^7 - 2.53126 \cdot 10^{-19} X^6 + 1.0189 \cdot 10^{-18} X^5 - 1.27909 \cdot 10^{-18} X^4 \\ & + 7.35648 \cdot 10^{-19} X^3 + 3.84777 \cdot 10^{-07} X^2 - 1.84656 \cdot 10^{-05} X + 1.27846 \cdot 10^{-08} \\ = {} & 1.27846 \cdot 10^{-08} B_{0,7} - 2.62515 \cdot 10^{-06} B_{1,7} - 5.24477 \cdot 10^{-06} B_{2,7} - 7.84606 \cdot 10^{-06} B_{3,7} \\ & - 1.0429 \cdot 10^{-05} B_{4,7} - 1.29937 \cdot 10^{-05} B_{5,7} - 1.554 \cdot 10^{-05} B_{6,7} - 1.8068 \cdot 10^{-05} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.27846 \cdot 10^{-08}$.
Bounding polynomials M and m:

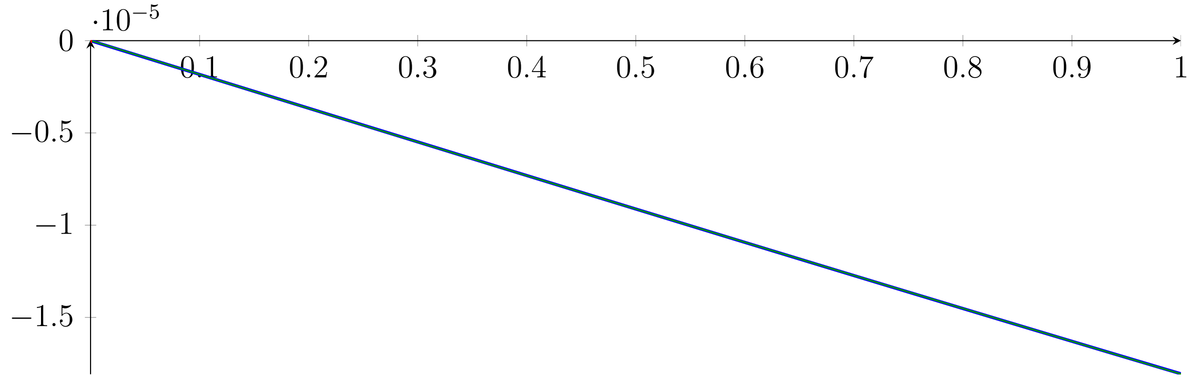
$$M = 3.84777 \cdot 10^{-07} X^2 - 1.84656 \cdot 10^{-05} X + 2.55691 \cdot 10^{-08}$$

$$m = 3.84777{\cdot}10^{-07}X^2 - 1.84656{\cdot}10^{-05}X - 1.65493{\cdot}10^{-21}$$

Root of M and m:

$$N(M) = \{0.00138473, 47.9889\} \qquad N(m) = \{-9.02902{\cdot}10^{-17}, 47.9903\}$$

Intersection intervals:



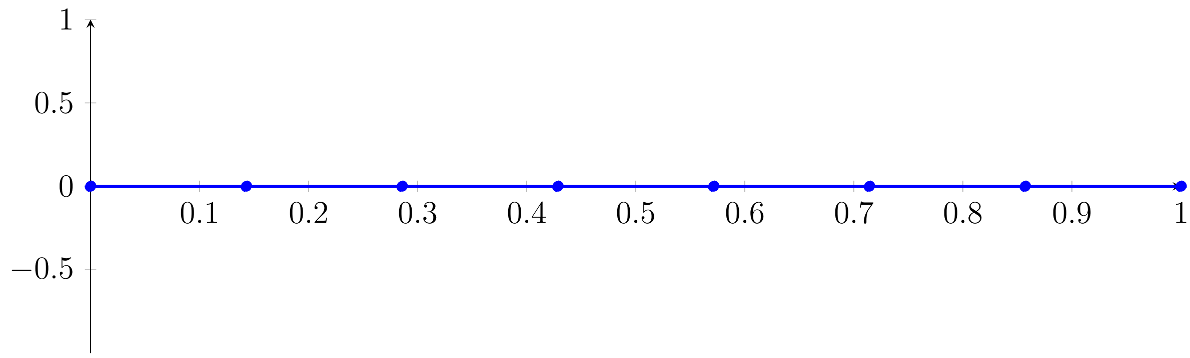
$[0, 0.00138473]$

Longest intersection interval: 0.00138473
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.500026]$,

14.22 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.5, 0.500026]$

Normalized monomial und Bézier representations and the Bézier polygon:

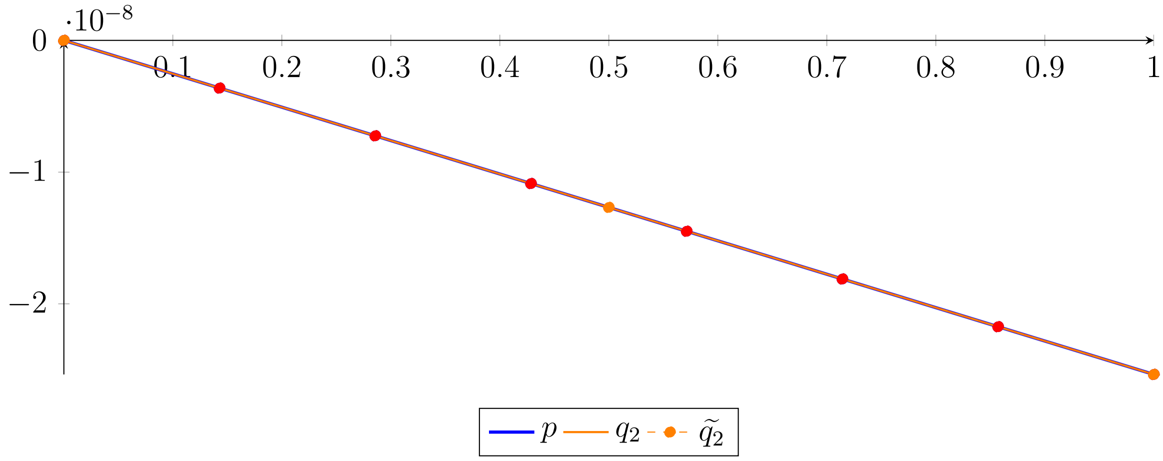
$$\begin{aligned} p &= 1.244{\cdot}10^{-25}X^7 + 1.02913{\cdot}10^{-24}X^6 - 4.95339{\cdot}10^{-24}X^5 + 1.13091 \\ &\quad {\cdot}10^{-25}X^4 + 6.83851{\cdot}10^{-16}X^3 - 2.53572{\cdot}10^{-08}X - 2.05803{\cdot}10^{-21} \\ &= -2.05803{\cdot}10^{-21}B_{0,7}(X) - 3.62245{\cdot}10^{-09}B_{1,7}(X) - 7.2449{\cdot}10^{-09}B_{2,7}(X) - 1.08674{\cdot}10^{-08}B_{3,7}(X) \\ &\quad - 1.44898{\cdot}10^{-08}B_{4,7}(X) - 1.81123{\cdot}10^{-08}B_{5,7}(X) - 2.17347{\cdot}10^{-08}B_{6,7}(X) - 2.53572{\cdot}10^{-08}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.02578{\cdot}10^{-15}X^2 - 2.53572{\cdot}10^{-08}X + 3.41905{\cdot}10^{-17} \\ &= 3.41905{\cdot}10^{-17}B_{0,2} - 1.26786{\cdot}10^{-08}B_{1,2} - 2.53572{\cdot}10^{-08}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -8.79497{\cdot}10^{-23}X^7 - 2.62448{\cdot}10^{-22}X^6 + 1.30063{\cdot}10^{-21}X^5 - 1.70365{\cdot}10^{-21}X^4 \\ &\quad + 9.97902{\cdot}10^{-22}X^3 + 1.02578{\cdot}10^{-15}X^2 - 2.53572{\cdot}10^{-08}X + 3.41905{\cdot}10^{-17} \\ &= 3.41905{\cdot}10^{-17}B_{0,7} - 3.62245{\cdot}10^{-09}B_{1,7} - 7.2449{\cdot}10^{-09}B_{2,7} - 1.08674{\cdot}10^{-08}B_{3,7} \\ &\quad - 1.44898{\cdot}10^{-08}B_{4,7} - 1.81123{\cdot}10^{-08}B_{5,7} - 2.17347{\cdot}10^{-08}B_{6,7} - 2.53572{\cdot}10^{-08}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.41926 \cdot 10^{-17}$.

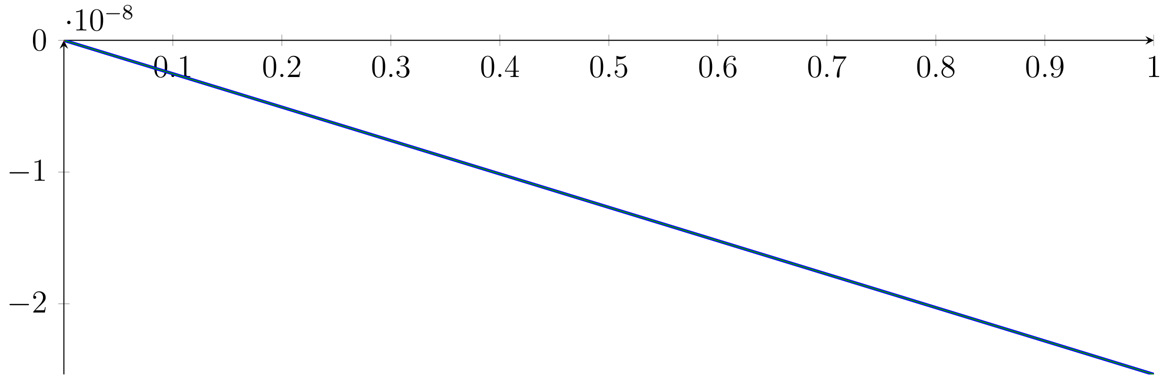
Bounding polynomials M and m:

$$M = 1.02578 \cdot 10^{-15} X^2 - 2.53572 \cdot 10^{-08} X + 6.83831 \cdot 10^{-17}$$
$$m = 1.02578 \cdot 10^{-15} X^2 - 2.53572 \cdot 10^{-08} X - 2.06283 \cdot 10^{-21}$$

Root of M and m:

$$N(M) = \{2.69717 \cdot 10^{-09}, 2.472 \cdot 10^{7}\} \qquad N(m) = \{0, 2.472 \cdot 10^{7}\}$$

Intersection intervals:



$$[0, 2.69717e - 09]$$

Longest intersection interval: $2.69717 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

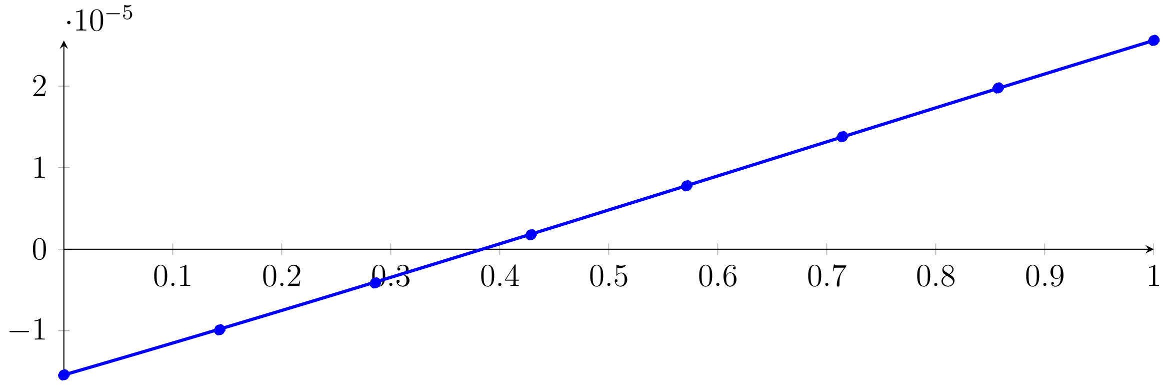
14.23 Recursion Branch 1 2 1 1 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 6!

14.24 Recursion Branch 1 2 1 2 in Interval 2: $[0.677501, 0.713569]$

Normalized monomial und Bézier representations and the Bézier polygon:

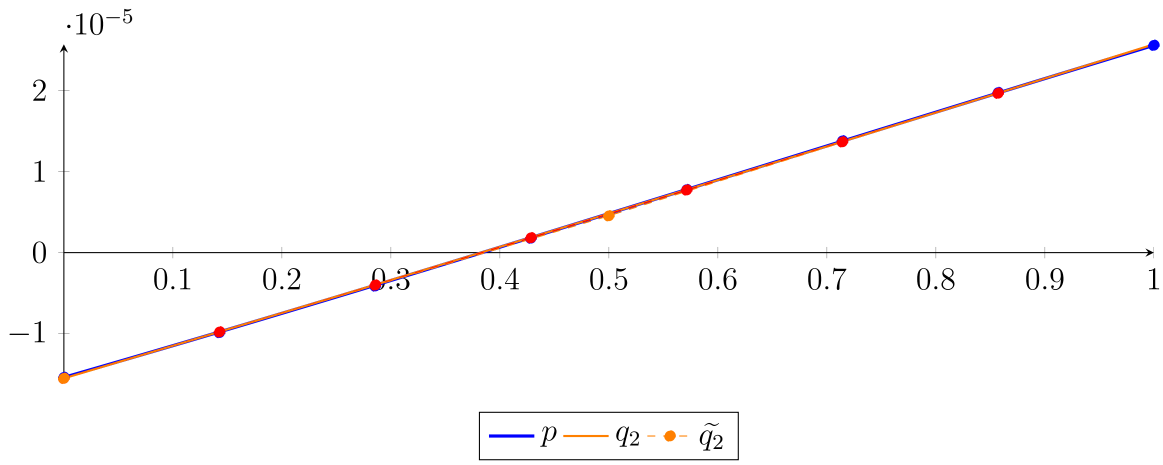
$$\begin{aligned} p &= 7.94027 \cdot 10^{-11} X^7 + 2.73536 \cdot 10^{-09} X^6 + 1.74957 \cdot 10^{-08} X^5 - 2.31978 \cdot 10^{-07} X^4 \\ &\quad - 2.08062 \cdot 10^{-06} X^3 + 4.59131 \cdot 10^{-06} X^2 + 3.87115 \cdot 10^{-05} X - 1.54089 \cdot 10^{-05} \\ &= -1.54089 \cdot 10^{-05} B_{0,7}(X) - 9.87865 \cdot 10^{-06} B_{1,7}(X) - 4.1298 \cdot 10^{-06} B_{2,7}(X) + 1.77823 \cdot 10^{-06} B_{3,7}(X) \\ &\quad + 7.77937 \cdot 10^{-06} B_{4,7}(X) + 1.38018 \cdot 10^{-05} B_{5,7}(X) + 1.97689 \cdot 10^{-05} B_{6,7}(X) + 2.56016 \cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.10898{\cdot}10^{-06}X^2 + 4.015{\cdot}10^{-05}X - 1.55306{\cdot}10^{-05} \\ &= -1.55306{\cdot}10^{-05}B_{0,2} + 4.54442{\cdot}10^{-06}B_{1,2} + 2.57284{\cdot}10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.99794{\cdot}10^{-18}X^7 - 1.01908{\cdot}10^{-17}X^6 + 1.37048{\cdot}10^{-17}X^5 - 9.25296{\cdot}10^{-18}X^4 \\ &\quad + 3.26831{\cdot}10^{-18}X^3 + 1.10898{\cdot}10^{-06}X^2 + 4.015{\cdot}10^{-05}X - 1.55306{\cdot}10^{-05} \\ &= -1.55306{\cdot}10^{-05}B_{0,7} - 9.79485{\cdot}10^{-06}B_{1,7} - 4.00633{\cdot}10^{-06}B_{2,7} + 1.83499{\cdot}10^{-06}B_{3,7} \\ &\quad + 7.72912{\cdot}10^{-06}B_{4,7} + 1.36761{\cdot}10^{-05}B_{5,7} + 1.96758{\cdot}10^{-05}B_{6,7} + 2.57284{\cdot}10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.26744{\cdot}10^{-07}$.

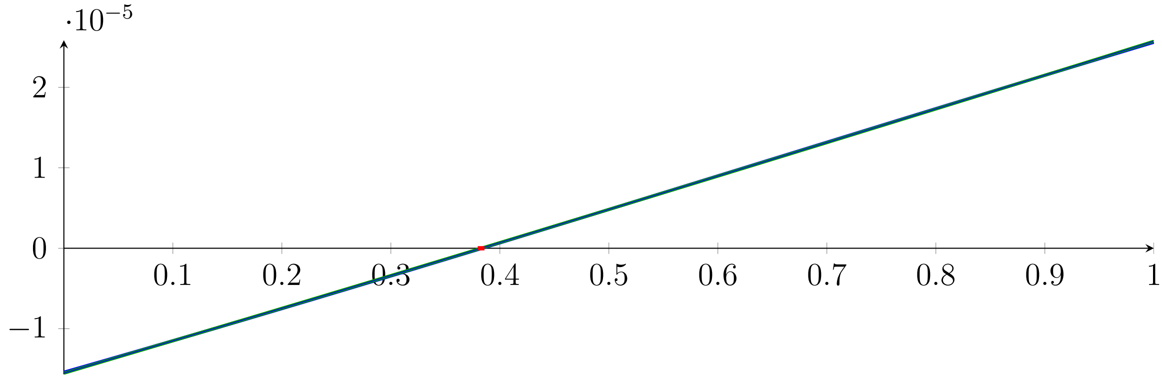
Bounding polynomials M and m:

$$M = 1.10898{\cdot}10^{-06}X^2 + 4.015{\cdot}10^{-05}X - 1.54038{\cdot}10^{-05}$$
$$m = 1.10898{\cdot}10^{-06}X^2 + 4.015{\cdot}10^{-05}X - 1.56573{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-36.5842, 0.379675\} \qquad N(m) = \{-36.5904, 0.385858\}$$

Intersection intervals:



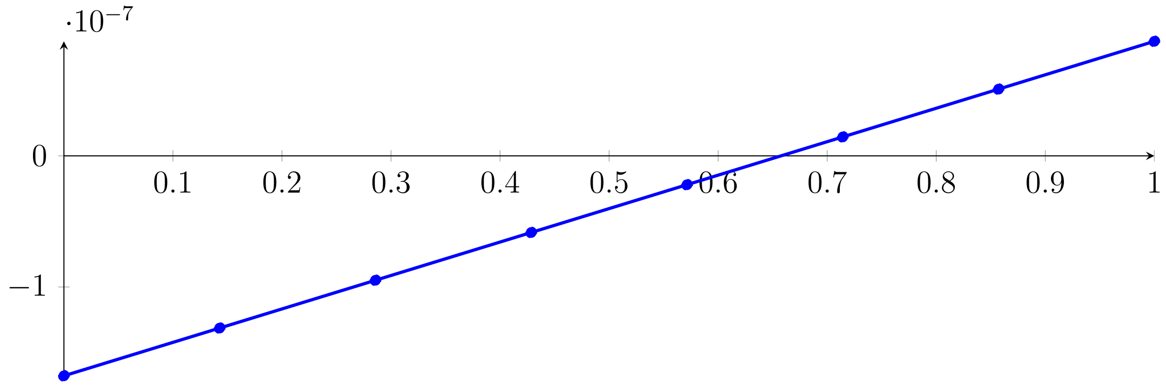
$$[0.379675, 0.385858]$$

Longest intersection interval: 0.0061828
$\Longrightarrow$ Selective recursion: interval 1: $[0.691195, 0.691418]$,

14.25 Recursion Branch 1 2 1 2 1 in Interval 1: $[0.691195, 0.691418]$

Normalized monomial und Bézier representations and the Bézier polygon:

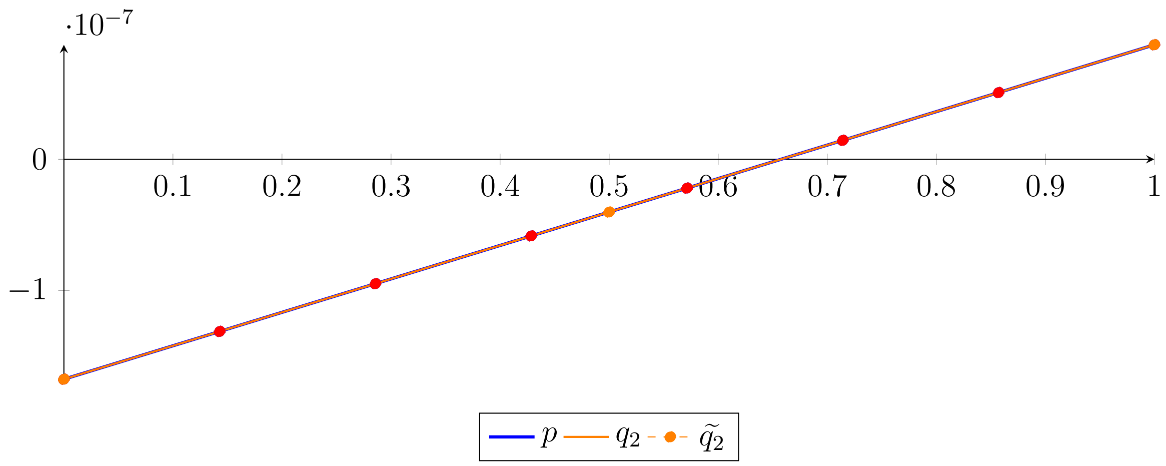
$$\begin{aligned} p &= 7.75482\cdot 10^{-25}X^7 + 1.69365\cdot 10^{-22}X^6 + 2.16544\cdot 10^{-19}X^5 - 2.8159\cdot 10^{-16}X^4 \\ &\quad - 5.68339\cdot 10^{-13}X^3 + 7.7648\cdot 10^{-11}X^2 + 2.55036\cdot 10^{-07}X - 1.67761\cdot 10^{-07} \\ &= -1.67761\cdot 10^{-07}B_{0,7}(X) - 1.31328\cdot 10^{-07}B_{1,7}(X) - 9.48904\cdot 10^{-08}B_{2,7}(X) - 5.84493\cdot 10^{-08}B_{3,7}(X) \\ &\quad - 2.20046\cdot 10^{-08}B_{4,7}(X) + 1.44438\cdot 10^{-08}B_{5,7}(X) + 5.08958\cdot 10^{-08}B_{6,7}(X) + 8.73514\cdot 10^{-08}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 7.6795\cdot 10^{-11}X^2 + 2.55036\cdot 10^{-07}X - 1.67762\cdot 10^{-07} \\ &= -1.67762\cdot 10^{-07}B_{0,2} - 4.02434\cdot 10^{-08}B_{1,2} + 8.73514\cdot 10^{-08}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 3.10668\cdot 10^{-20}X^7 - 1.09747\cdot 10^{-19}X^6 + 1.54583\cdot 10^{-19}X^5 - 1.10458\cdot 10^{-19}X^4 \\ &\quad + 4.19355\cdot 10^{-20}X^3 + 7.6795\cdot 10^{-11}X^2 + 2.55036\cdot 10^{-07}X - 1.67762\cdot 10^{-07} \\ &= -1.67762\cdot 10^{-07}B_{0,7} - 1.31328\cdot 10^{-07}B_{1,7} - 9.48904\cdot 10^{-08}B_{2,7} - 5.84493\cdot 10^{-08}B_{3,7} \\ &\quad - 2.20046\cdot 10^{-08}B_{4,7} + 1.44437\cdot 10^{-08}B_{5,7} + 5.08958\cdot 10^{-08}B_{6,7} + 8.73514\cdot 10^{-08}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84491\cdot 10^{-14}$.
Bounding polynomials M and m:

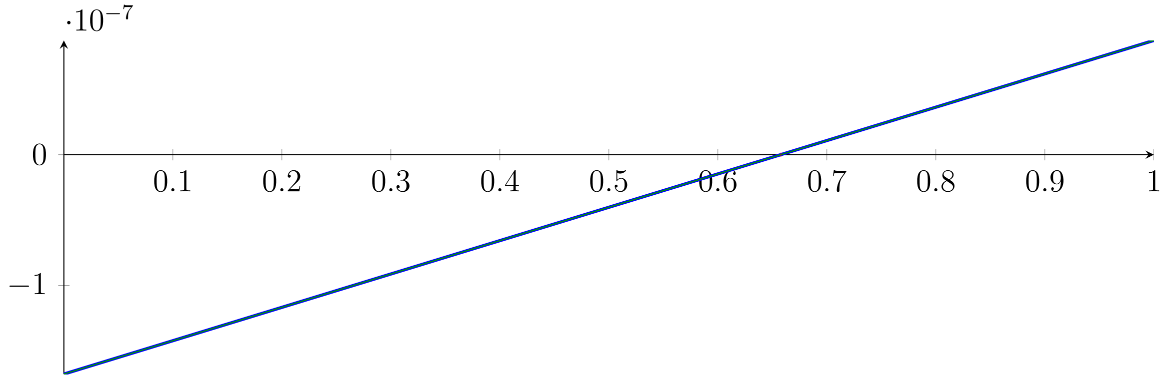
$$M = 7.6795\cdot 10^{-11}X^2 + 2.55036\cdot 10^{-07}X - 1.67761\cdot 10^{-07}$$

$$m = 7.6795 \cdot 10^{-11} X^2 + 2.55036 \cdot 10^{-07} X - 1.67762 \cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{-3321.66, 0.657665\} \qquad N(m) = \{-3321.66, 0.657665\}$$

Intersection intervals:



$[0.657665, 0.657665]$

Longest intersection interval: $2.2301 \cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.691342, 0.691342]$,

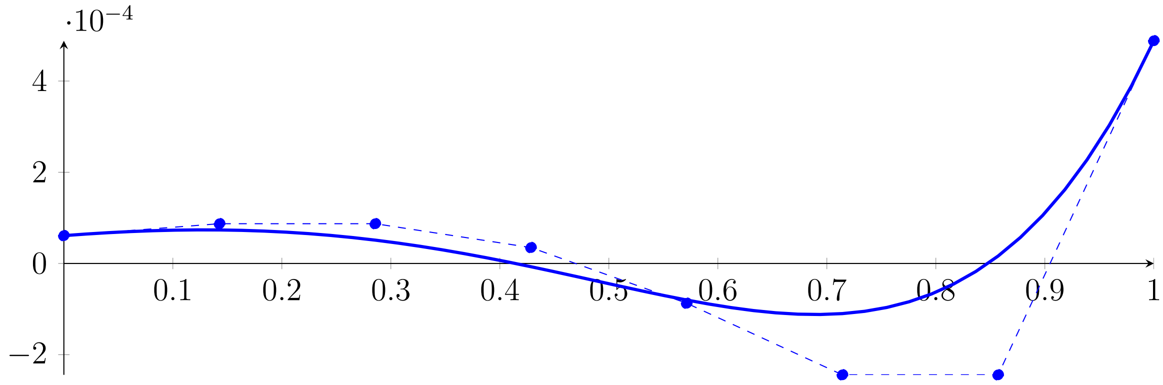
14.26 Recursion Branch 1 2 1 2 1 1 in Interval 1: $[0.691342, 0.691342]$

Found root in interval $[0.691342, 0.691342]$ at recursion depth 6!

14.27 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

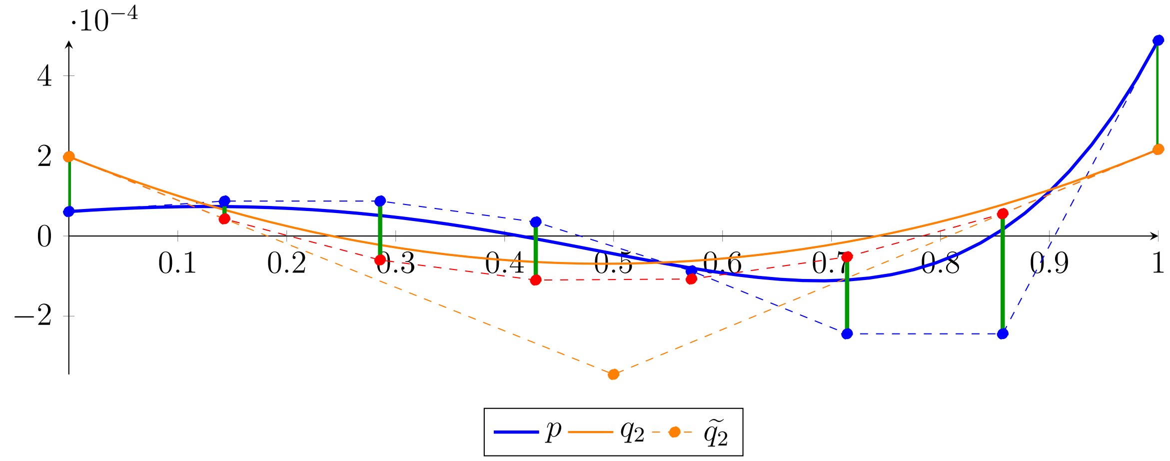
$$\begin{aligned} p &= 6.10352 \cdot 10^{-05} X^7 + 0.000427246 X^6 + 0.000915527 X^5 + 0.000305176 X^4 \\ &\quad - 0.000915527 X^3 - 0.000549316 X^2 + 0.000183105 X + 6.10352 \cdot 10^{-05} \\ &= 6.10352 \cdot 10^{-05} B_{0,7}(X) + 8.71931 \cdot 10^{-05} B_{1,7}(X) + 8.71931 \cdot 10^{-05} B_{2,7}(X) + 3.48772 \cdot 10^{-05} B_{3,7}(X) \\ &\quad - 8.71931 \cdot 10^{-05} B_{4,7}(X) - 0.000244141 B_{5,7}(X) - 0.000244141 B_{6,7}(X) + 0.000488281 B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00110517 X^2 - 0.00108701 X + 0.000198001 \\ &= 0.000198001 B_{0,2} - 0.000345503 B_{1,2} + 0.000216166 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.53722 \cdot 10^{-17} X^7 - 8.14674 \cdot 10^{-17} X^6 + 1.01292 \cdot 10^{-16} X^5 - 6.12471 \cdot 10^{-17} X^4 \\ &\quad + 1.8514 \cdot 10^{-17} X^3 + 0.00110517 X^2 - 0.00108701 X + 0.000198001 \\ &= 0.000198001 B_{0,7} + 4.27142 \cdot 10^{-05} B_{1,7} - 5.99452 \cdot 10^{-05} B_{2,7} - 0.000109977 B_{3,7} \\ &\quad - 0.000107382 B_{4,7} - 5.21601 \cdot 10^{-05} B_{5,7} + 5.56894 \cdot 10^{-05} B_{6,7} + 0.000216166 B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00029983$.
Bounding polynomials M and m:

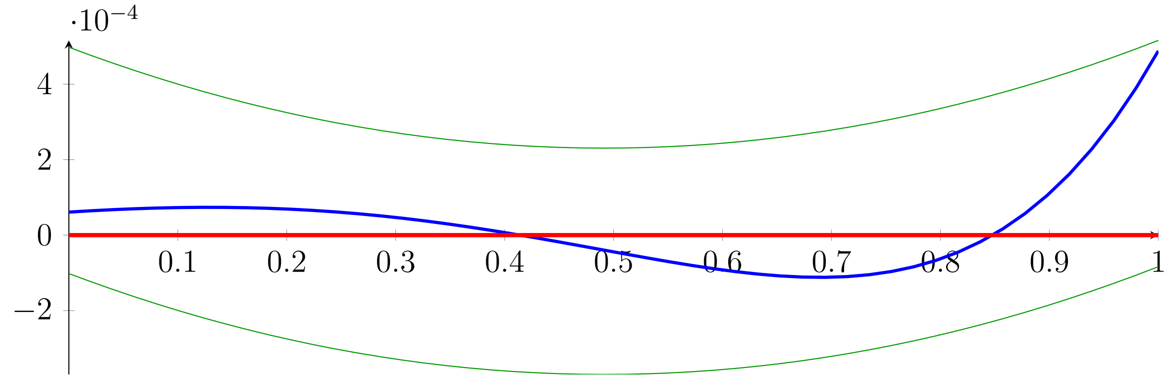
$$M = 0.00110517X^2 - 0.00108701X + 0.000497831$$

$$m = 0.00110517X^2 - 0.00108701X - 0.000101829$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-0.0861351, 1.0697\}$$

Intersection intervals:



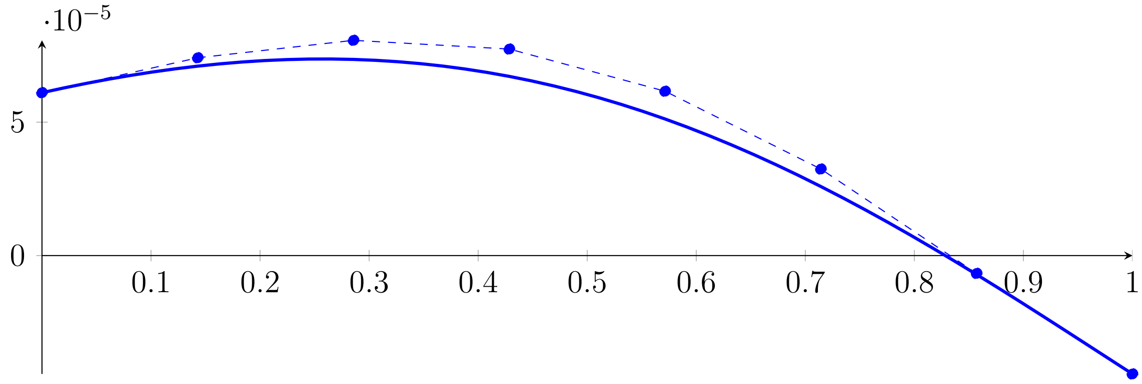
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$
Bisection point is very near to a root?!?

14.28 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

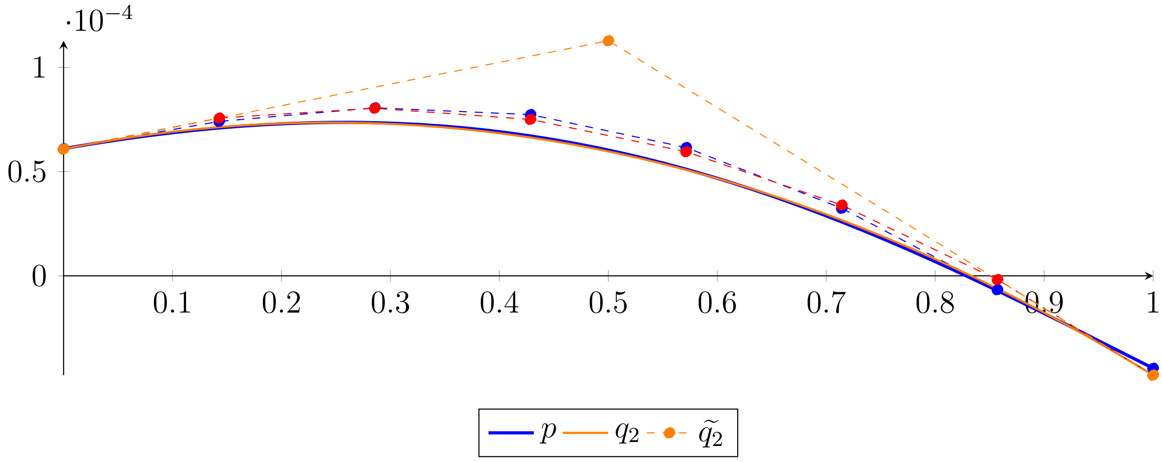
$$\begin{aligned}
p &= 4.76837\cdot 10^{-07}X^7 + 6.67572\cdot 10^{-06}X^6 + 2.86102\cdot 10^{-05}X^5 + 1.90735\cdot 10^{-05}X^4 \\
&\quad - 0.000114441X^3 - 0.000137329X^2 + 9.15527\cdot 10^{-05}X + 6.10352\cdot 10^{-05} \\
&= 6.10352\cdot 10^{-05}B_{0,7}(X) + 7.41141\cdot 10^{-05}B_{1,7}(X) + 8.06536\cdot 10^{-05}B_{2,7}(X) + 7.73839\cdot 10^{-05}B_{3,7}(X) \\
&\quad + 6.15801\cdot 10^{-05}B_{4,7}(X) + 3.24249\cdot 10^{-05}B_{5,7}(X) - 6.67572\cdot 10^{-06}B_{6,7}(X) - 4.43459\cdot 10^{-05}B_{7,7}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -0.000212448X^2 + 0.000103939X + 6.08677\cdot 10^{-05}\\ &= 6.08677\cdot 10^{-05}B_{0,2} + 0.000112837B_{1,2} - 4.76411\cdot 10^{-05}B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= -2.30374\cdot 10^{-17}X^7 + 8.08398\cdot 10^{-17}X^6 - 1.12948\cdot 10^{-16}X^5 + 7.99559\cdot 10^{-17}X^4\\ &\quad - 3.00744\cdot 10^{-17}X^3 - 0.000212448X^2 + 0.000103939X + 6.08677\cdot 10^{-05}\\ &= 6.08677\cdot 10^{-05}B_{0,7} + 7.57161\cdot 10^{-05}B_{1,7} + 8.0448\cdot 10^{-05}B_{2,7} + 7.50633\cdot 10^{-05}B_{3,7}\\ &\quad + 5.95621\cdot 10^{-05}B_{4,7} + 3.39442\cdot 10^{-05}B_{5,7} - 1.79017\cdot 10^{-06}B_{6,7} - 4.76411\cdot 10^{-05}B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.88555\cdot 10^{-06}$.

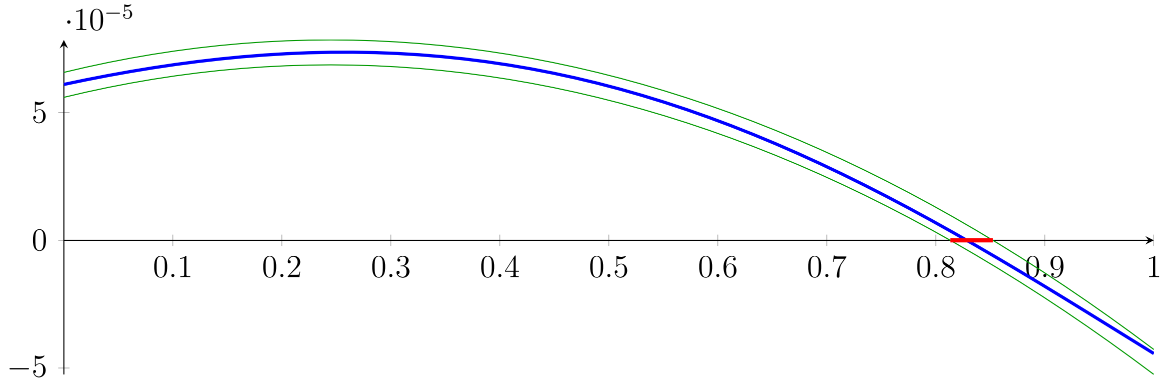
Bounding polynomials M and m:

$$\begin{aligned}M &= -0.000212448X^2 + 0.000103939X + 6.57532\cdot 10^{-05}\\ m &= -0.000212448X^2 + 0.000103939X + 5.59821\cdot 10^{-05}\end{aligned}$$

Root of M and m:

$$N(M) = \{-0.363113, 0.852359\} \qquad N(m) = \{-0.324016, 0.813261\}$$

Intersection intervals:



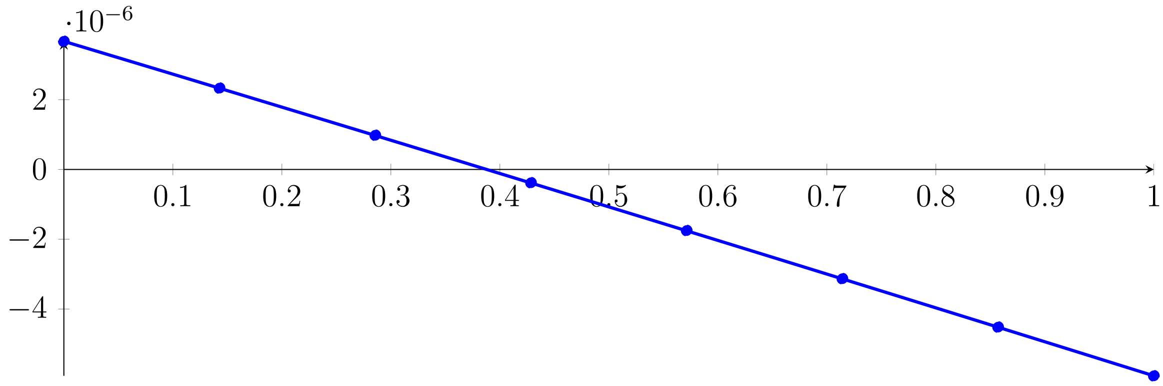
$$[0.813261, 0.852359]$$

Longest intersection interval: 0.0390972
$\Longrightarrow$ Selective recursion: interval 1: $[0.851658, 0.856545]$,

14.29 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.851658, 0.856545]$

Normalized monomial und Bézier representations and the Bézier polygon:

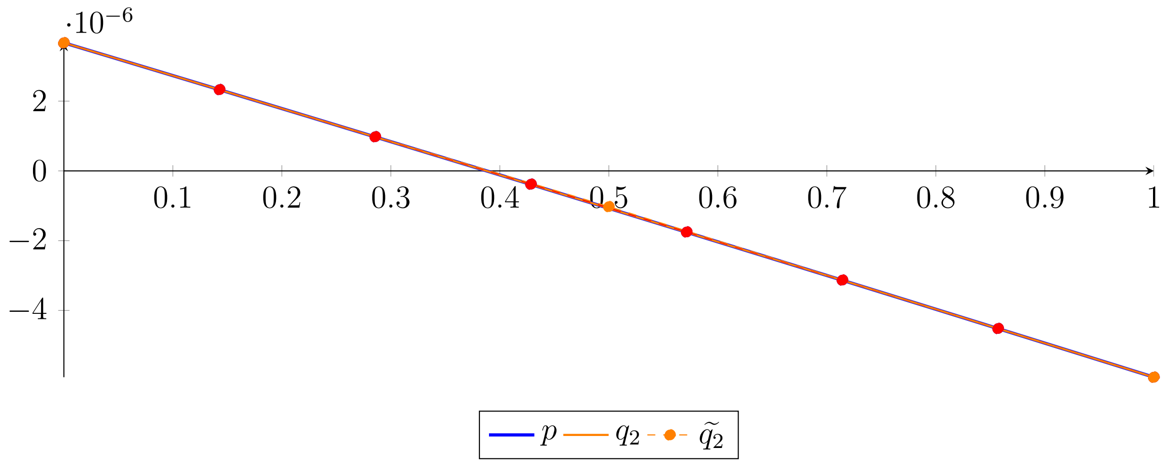
$$\begin{aligned}p &= 6.65868\cdot10^{-17}X^7 + 3.35391\cdot10^{-14}X^6 + 6.19451\cdot10^{-12}X^5 + 4.92126\cdot10^{-10}X^4\\ &\quad + 1.29058\cdot10^{-08}X^3 - 2.13381\cdot10^{-07}X^2 - 9.38561\cdot10^{-06}X + 3.67201\cdot10^{-06}\\ &= 3.67201\cdot10^{-06}B_{0,7}(X) + 2.33121\cdot10^{-06}B_{1,7}(X) + 9.80245\cdot10^{-07}B_{2,7}(X) - 3.8051\cdot10^{-07}B_{3,7}(X)\\ &\quad - 1.75067\cdot10^{-06}B_{4,7}(X) - 3.12985\cdot10^{-06}B_{5,7}(X) - 4.51763\cdot10^{-06}B_{6,7}(X) - 5.91358\cdot10^{-06}B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -1.93167\cdot10^{-07}X^2 - 9.39381\cdot10^{-06}X + 3.6727\cdot10^{-06}\\ &= 3.6727\cdot10^{-06}B_{0,2} - 1.02421\cdot10^{-06}B_{1,2} - 5.91428\cdot10^{-06}B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -7.04564\cdot10^{-19}X^7 + 2.39831\cdot10^{-18}X^6 - 3.2309\cdot10^{-18}X^5 + 2.18627\cdot10^{-18}X^4\\ &\quad - 7.74564\cdot10^{-19}X^3 - 1.93167\cdot10^{-07}X^2 - 9.39381\cdot10^{-06}X + 3.6727\cdot10^{-06}\\ &= 3.6727\cdot10^{-06}B_{0,7} + 2.33072\cdot10^{-06}B_{1,7} + 9.79553\cdot10^{-07}B_{2,7} - 3.80817\cdot10^{-07}B_{3,7}\\ &\quad - 1.75039\cdot10^{-06}B_{4,7} - 3.12915\cdot10^{-06}B_{5,7} - 4.51712\cdot10^{-06}B_{6,7} - 5.91428\cdot10^{-06}B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.0265\cdot10^{-10}$.
Bounding polynomials M and m:

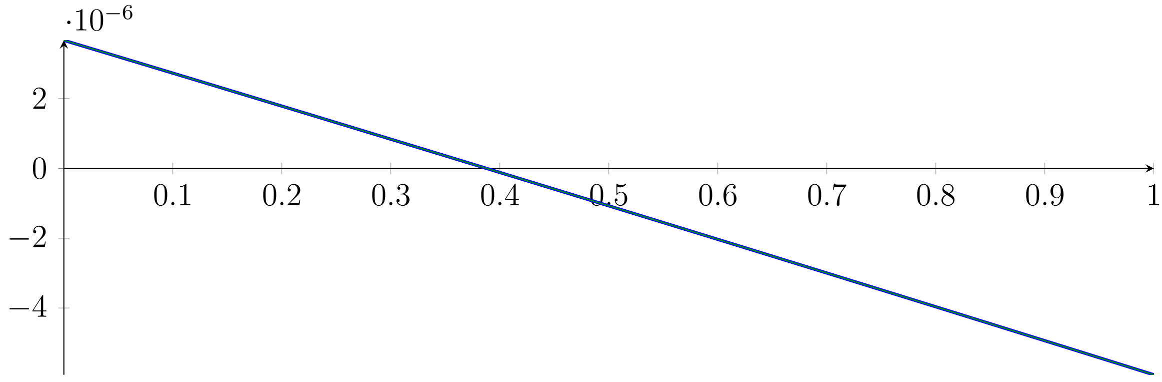
$$M = -1.93167\cdot10^{-07}X^2 - 9.39381\cdot10^{-06}X + 3.6734\cdot10^{-06}$$

$$m = -1.93167{\cdot}10^{-07}X^2 - 9.39381{\cdot}10^{-06}X + 3.67199{\cdot}10^{-06}$$

Root of M and m:

$$N(M) = \{-49.0184, 0.38795\} \qquad N(m) = \{-49.0183, 0.387803\}$$

Intersection intervals:



$[0.387803, 0.38795]$

Longest intersection interval: 0.000147249
$\Longrightarrow$ Selective recursion: interval 1: $[0.853553, 0.853554]$,

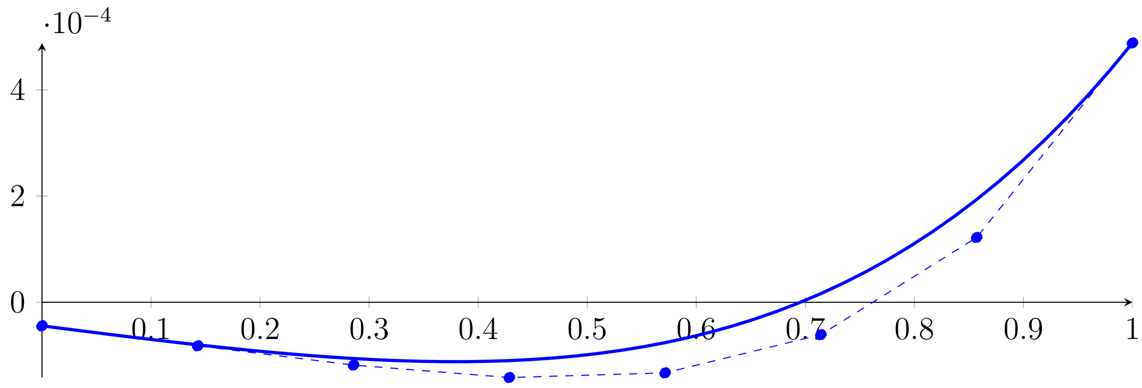
14.30 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.853553, 0.853554]$

Found root in interval $[0.853553, 0.853554]$ at recursion depth 6!

14.31 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

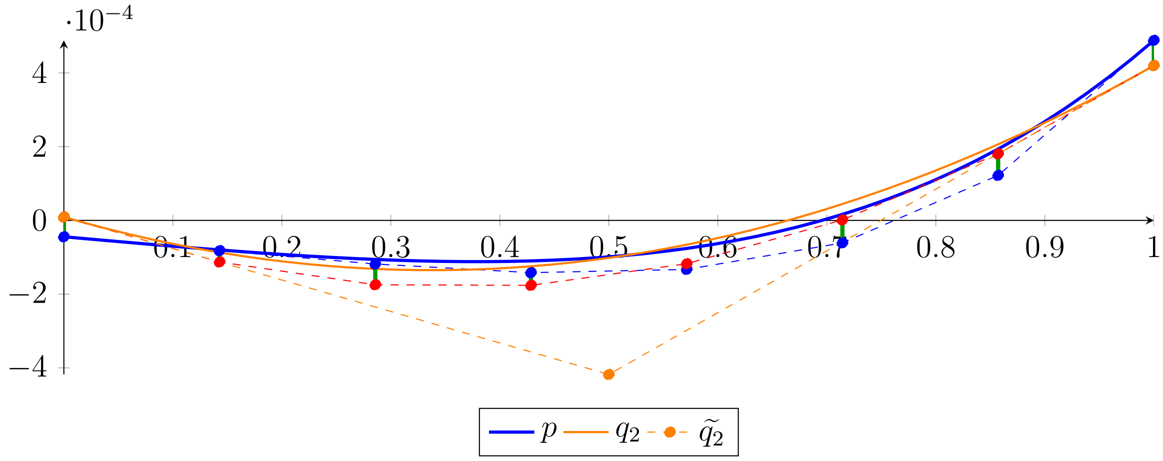
$$\begin{aligned} p &= 4.76837{\cdot}10^{-07}X^7 + 1.00136{\cdot}10^{-05}X^6 + 7.86781{\cdot}10^{-05}X^5 + 0.00027895X^4 \\ &\quad + 0.000398159X^3 + 3.00407{\cdot}10^{-05}X^2 - 0.000263691X - 4.43459{\cdot}10^{-05} \\ &= -4.43459{\cdot}10^{-05}B_{0,7}(X) - 8.2016{\cdot}10^{-05}B_{1,7}(X) - 0.000118256B_{2,7}(X) - 0.000141689B_{3,7}(X) \\ &\quad - 0.000132969B_{4,7}(X) - 6.10352{\cdot}10^{-05}B_{5,7}(X) + 0.00012207B_{6,7}(X) + 0.000488281B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00126469X^2 - 0.000853925X + 9.15357{\cdot}10^{-06} \\ &= 9.15357{\cdot}10^{-06}B_{0,2} - 0.000417809B_{1,2} + 0.00041992B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6.94995{\cdot}10^{-17}X^7 - 2.34833{\cdot}10^{-16}X^6 + 3.13194{\cdot}10^{-16}X^5 - 2.0925{\cdot}10^{-16}X^4 \\ &\quad + 7.31985{\cdot}10^{-17}X^3 + 0.00126469X^2 - 0.000853925X + 9.15357{\cdot}10^{-06} \\ &= 9.15357{\cdot}10^{-06}B_{0,7} - 0.000112836B_{1,7} - 0.000174601B_{2,7} - 0.000176144B_{3,7} \\ &\quad - 0.000117463B_{4,7} + 1.44146{\cdot}10^{-06}B_{5,7} + 0.000180569B_{6,7} + 0.00041992B_{7,7} \end{aligned}$$



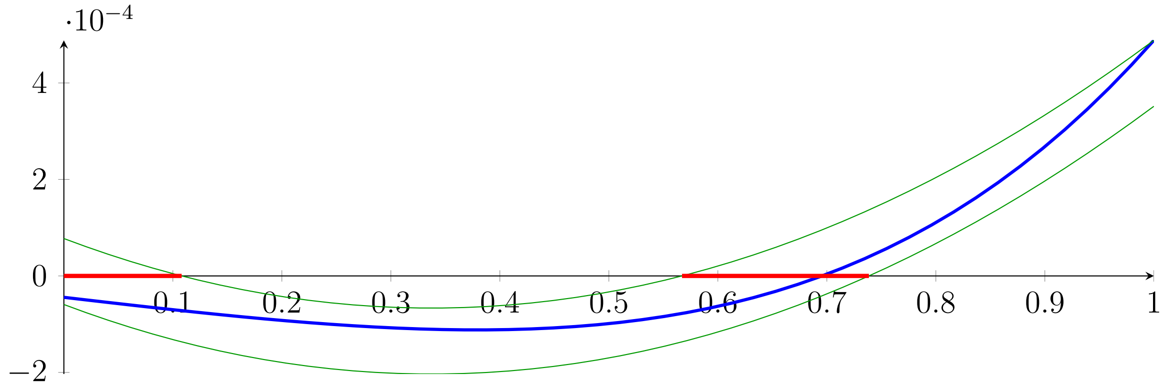
The maximum difference of the Bézier coefficients is $\delta = 6.83609 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = 0.00126469X^2 - 0.000853925X + 7.75144 \cdot 10^{-05}$$
$$m = 0.00126469X^2 - 0.000853925X - 5.92073 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.108072, 0.567132\} \qquad N(m) = \{-0.0633852, 0.738589\}$$

Intersection intervals:



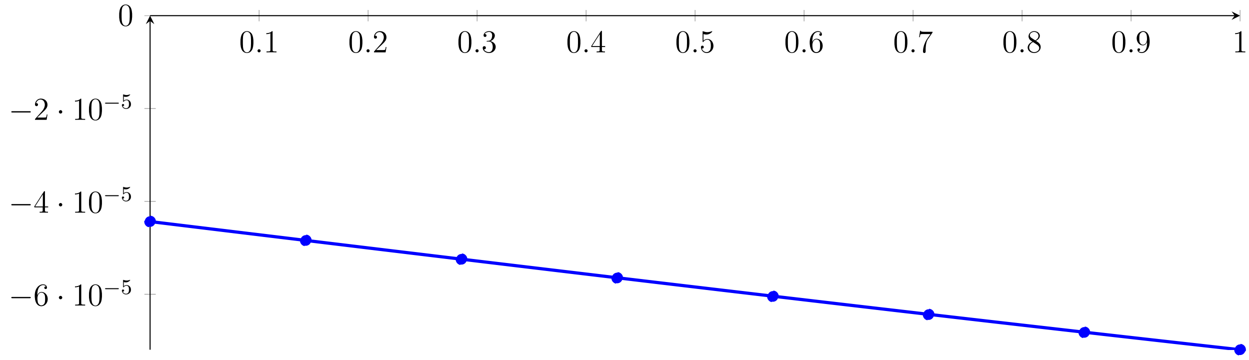
$$[0, 0.108072],\ [0.567132, 0.738589]$$

Longest intersection interval: 0.171457
$\Longrightarrow$ Selective recursion: interval 1: $[0.875, 0.888509]$, interval 2: $[0.945891, 0.967324]$,

14.32 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.875, 0.888509]$

Normalized monomial und Bézier representations and the Bézier polygon:

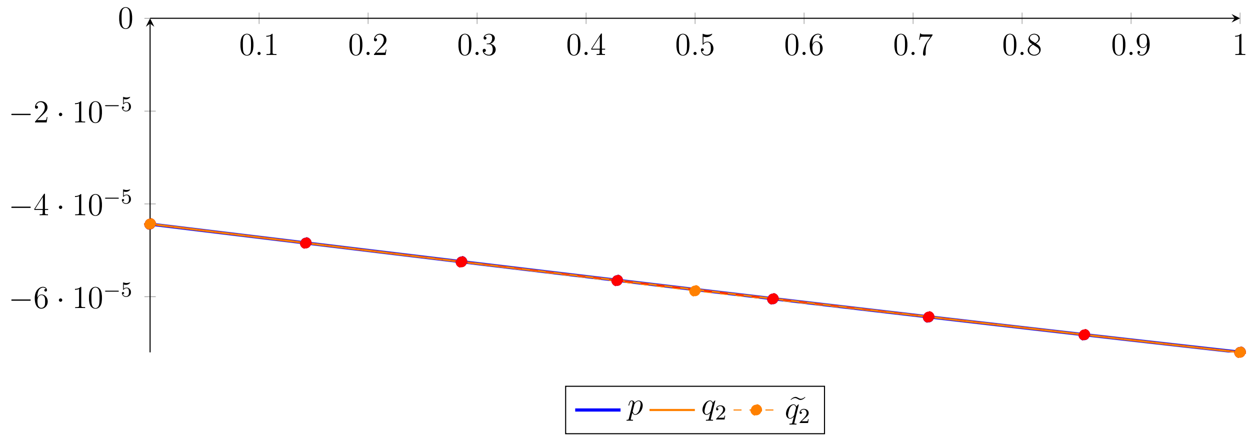
$$\begin{aligned}p &= 8.21048 \cdot 10^{-14}X^7 + 1.59542 \cdot 10^{-11}X^6 + 1.15991 \cdot 10^{-09}X^5 + 3.80524 \cdot 10^{-08}X^4 \\ &\quad + 5.02573 \cdot 10^{-07}X^3 + 3.50864 \cdot 10^{-07}X^2 - 2.84977 \cdot 10^{-05}X - 4.43459 \cdot 10^{-05} \\ &= -4.43459 \cdot 10^{-05}B_{0,7}(X) - 4.8417 \cdot 10^{-05}B_{1,7}(X) - 5.24713 \cdot 10^{-05}B_{2,7}(X) - 5.64947 \cdot 10^{-05}B_{3,7}(X) \\ &\quad - 6.04715 \cdot 10^{-05}B_{4,7}(X) - 6.43852 \cdot 10^{-05}B_{5,7}(X) - 6.8218 \cdot 10^{-05}B_{6,7}(X) - 7.19509 \cdot 10^{-05}B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.17206 \cdot 10^{-06} X^2 - 2.88353 \cdot 10^{-05} X - 4.43173 \cdot 10^{-05} \\ &= -4.43173 \cdot 10^{-05} B_{0,2} - 5.8735 \cdot 10^{-05} B_{1,2} - 7.19805 \cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 7.94051 \cdot 10^{-18} X^7 - 3.01985 \cdot 10^{-17} X^6 + 4.60231 \cdot 10^{-17} X^5 - 3.57879 \cdot 10^{-17} X^4 \\ &\quad + 1.49031 \cdot 10^{-17} X^3 + 1.17206 \cdot 10^{-06} X^2 - 2.88353 \cdot 10^{-05} X - 4.43173 \cdot 10^{-05} \\ &= -4.43173 \cdot 10^{-05} B_{0,7} - 4.84367 \cdot 10^{-05} B_{1,7} - 5.25002 \cdot 10^{-05} B_{2,7} - 5.65079 \cdot 10^{-05} B_{3,7} \\ &\quad - 6.04598 \cdot 10^{-05} B_{4,7} - 6.43558 \cdot 10^{-05} B_{5,7} - 6.81961 \cdot 10^{-05} B_{6,7} - 7.19805 \cdot 10^{-05} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.96884 \cdot 10^{-08}$.

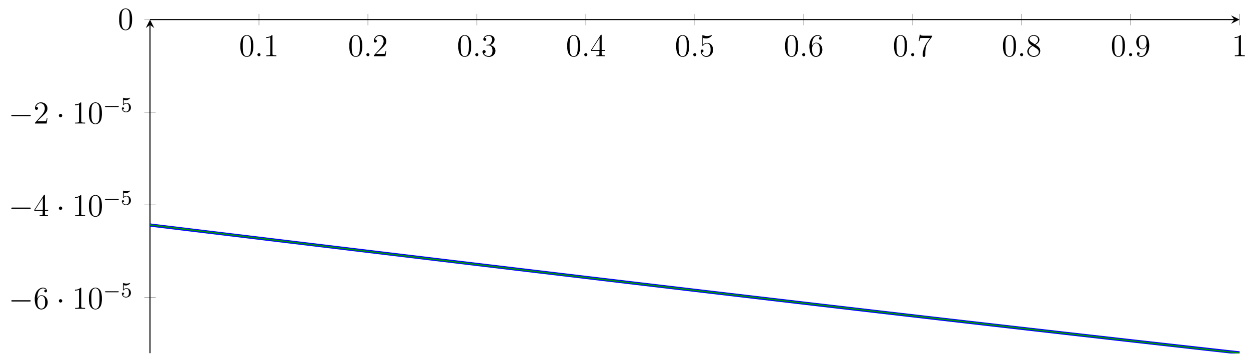
Bounding polynomials M and m:

$$\begin{aligned} M &= 1.17206 \cdot 10^{-06} X^2 - 2.88353 \cdot 10^{-05} X - 4.42877 \cdot 10^{-05} \\ m &= 1.17206 \cdot 10^{-06} X^2 - 2.88353 \cdot 10^{-05} X - 4.4347 \cdot 10^{-05} \end{aligned}$$

Root of M and m:

$$N(M) = \{-1.45038, 26.0527\} \qquad N(m) = \{-1.45222, 26.0545\}$$

Intersection intervals:

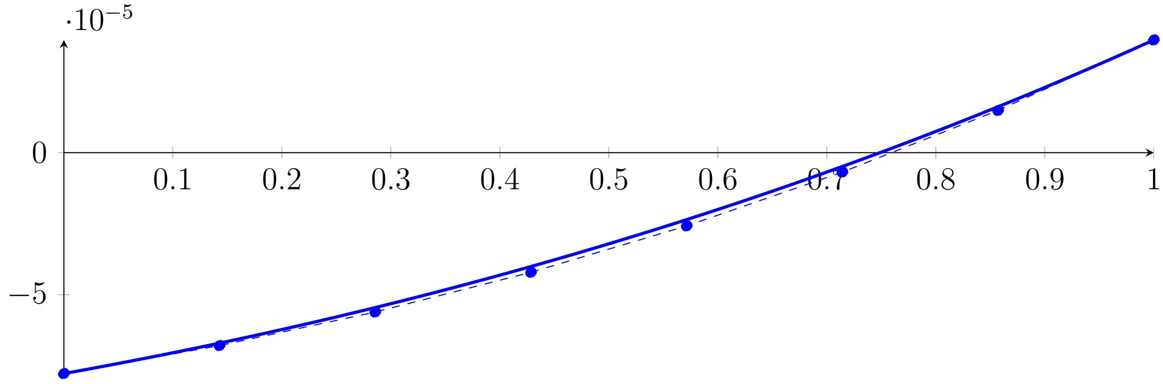


No intersection intervals with the x axis.

14.33 Recursion Branch 1 2 2 2 2 in Interval 2: $[0.945891, 0.967324]$

Normalized monomial und Bézier representations and the Bézier polygon:

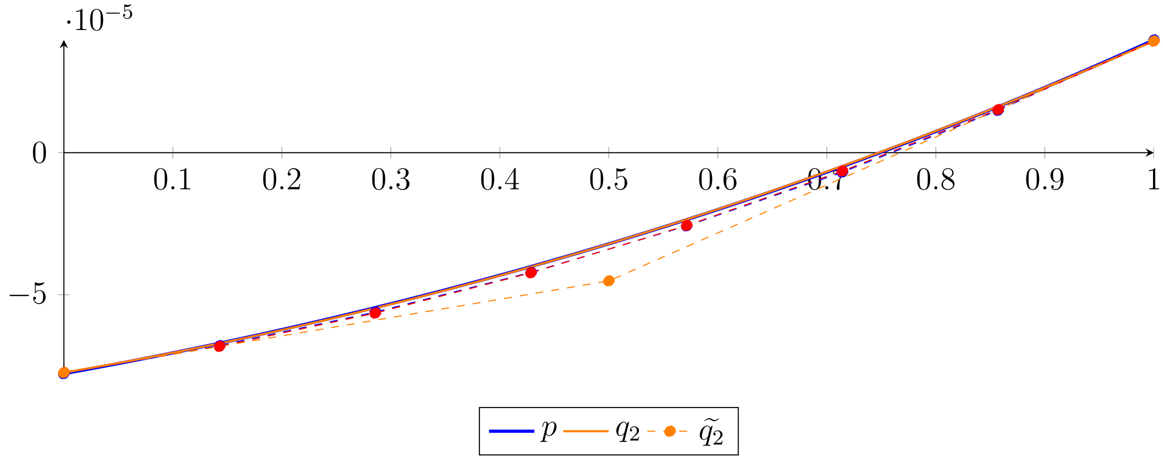
$$\begin{aligned}p &= 2.07713 \cdot 10^{-12} X^7 + 3.02499 \cdot 10^{-10} X^6 + 1.71845 \cdot 10^{-08} X^5 + 4.78268 \cdot 10^{-07} X^4 \\ &\quad + 6.66488 \cdot 10^{-06} X^3 + 4.13165 \cdot 10^{-05} X^2 + 6.90013 \cdot 10^{-05} X - 7.77864 \cdot 10^{-05} \\ &= -7.77864 \cdot 10^{-05} B_{0,7}(X) - 6.79291 \cdot 10^{-05} B_{1,7}(X) - 5.61043 \cdot 10^{-05} B_{2,7}(X) - 4.21217 \cdot 10^{-05} B_{3,7}(X) \\ &\quad - 2.5777 \cdot 10^{-05} B_{4,7}(X) - 6.85187 \cdot 10^{-06} B_{5,7}(X) + 1.48878 \cdot 10^{-05} B_{6,7}(X) + 3.9692 \cdot 10^{-05} B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= 5.21649 \cdot 10^{-05} X^2 + 6.45463 \cdot 10^{-05} X - 7.74103 \cdot 10^{-05} \\ &= -7.74103 \cdot 10^{-05} B_{0,2} - 4.51371 \cdot 10^{-05} B_{1,2} + 3.9301 \cdot 10^{-05} B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= 1.72055 \cdot 10^{-17} X^7 - 6.07579 \cdot 10^{-17} X^6 + 8.55343 \cdot 10^{-17} X^5 - 6.1085 \cdot 10^{-17} X^4 \\ &\quad + 2.31898 \cdot 10^{-17} X^3 + 5.21649 \cdot 10^{-05} X^2 + 6.45463 \cdot 10^{-05} X - 7.74103 \cdot 10^{-05} \\ &= -7.74103 \cdot 10^{-05} B_{0,7} - 6.81894 \cdot 10^{-05} B_{1,7} - 5.64845 \cdot 10^{-05} B_{2,7} - 4.22955 \cdot 10^{-05} B_{3,7} \\ &\quad - 2.56224 \cdot 10^{-05} B_{4,7} - 6.46534 \cdot 10^{-06} B_{5,7} + 1.51758 \cdot 10^{-05} B_{6,7} + 3.9301 \cdot 10^{-05} B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.91045 \cdot 10^{-07}$.

Bounding polynomials M and m:

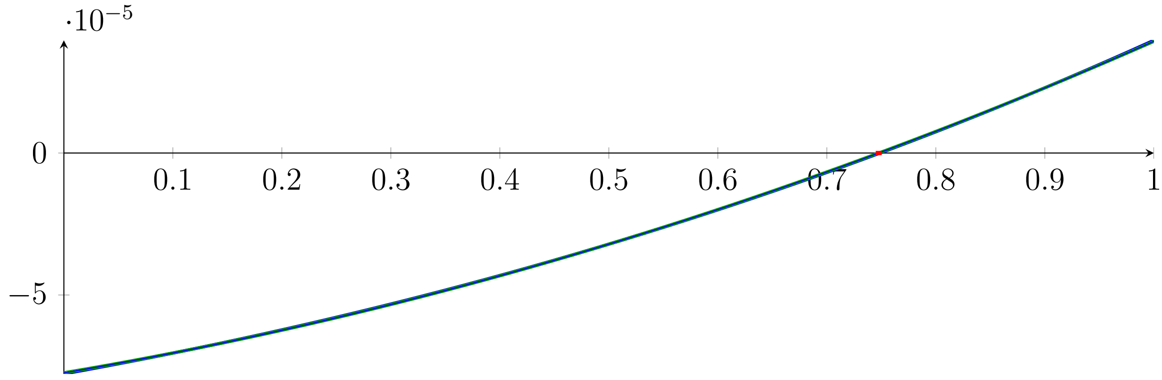
$$M = 5.21649 \cdot 10^{-05} X^2 + 6.45463 \cdot 10^{-05} X - 7.70193 \cdot 10^{-05}$$

$$m = 5.21649 \cdot 10^{-05} X^2 + 6.45463 \cdot 10^{-05} X - 7.78014 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{-1.98221, 0.744855\} \qquad N(m) = \{-1.98769, 0.750342\}$$

Intersection intervals:



$[0.744855, 0.750342]$

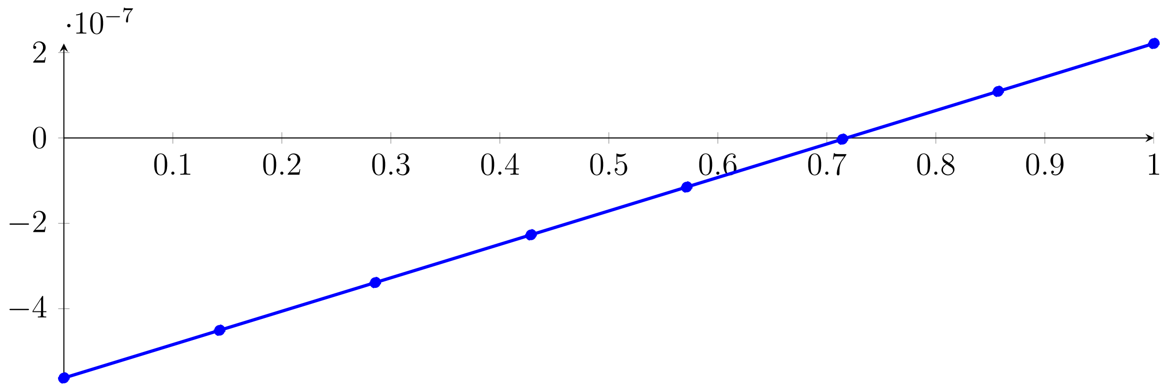
Longest intersection interval: 0.00548668

$\Longrightarrow$ Selective recursion: interval 1: $[0.961855, 0.961973]$,

14.34 Recursion Branch 1 2 2 2 2 1 in Interval 1: $[0.961855, 0.961973]$

Normalized monomial und Bézier representations and the Bézier polygon:

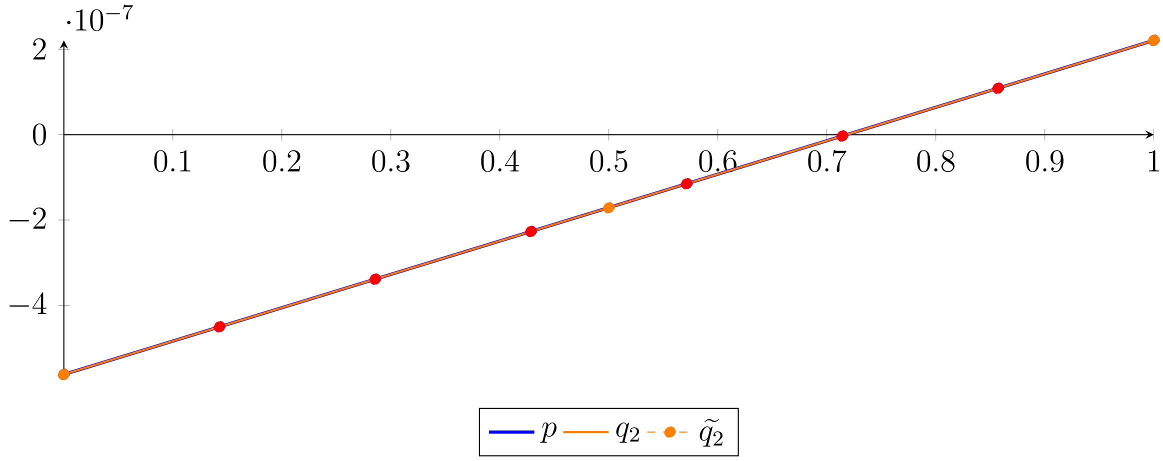
$$\begin{aligned} p &= 3.10193 \cdot 10^{-24} X^7 + 2.46086 \cdot 10^{-23} X^6 + 9.22889 \cdot 10^{-20} X^5 + 4.93729 \cdot 10^{-16} X^4 \\ &\quad + 1.35236 \cdot 10^{-12} X^3 + 1.74222 \cdot 10^{-09} X^2 + 7.81641 \cdot 10^{-07} X - 5.62185 \cdot 10^{-07} \\ &= -5.62185 \cdot 10^{-07} B_{0,7}(X) - 4.50522 \cdot 10^{-07} B_{1,7}(X) - 3.38776 \cdot 10^{-07} B_{2,7}(X) - 2.26947 \cdot 10^{-07} B_{3,7}(X) \\ &\quad - 1.15035 \cdot 10^{-07} B_{4,7}(X) - 3.04007 \cdot 10^{-09} B_{5,7}(X) + 1.09038 \cdot 10^{-07} B_{6,7}(X) + 2.212 \cdot 10^{-07} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.74425 \cdot 10^{-09} X^2 + 7.81641 \cdot 10^{-07} X - 5.62185 \cdot 10^{-07} \\ &= -5.62185 \cdot 10^{-07} B_{0,2} - 1.71365 \cdot 10^{-07} B_{1,2} + 2.212 \cdot 10^{-07} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.03941 \cdot 10^{-19} X^7 - 3.68796 \cdot 10^{-19} X^6 + 5.22088 \cdot 10^{-19} X^5 - 3.75245 \cdot 10^{-19} X^4 \\ &\quad + 1.43456 \cdot 10^{-19} X^3 + 1.74425 \cdot 10^{-09} X^2 + 7.81641 \cdot 10^{-07} X - 5.62185 \cdot 10^{-07} \\ &= -5.62185 \cdot 10^{-07} B_{0,7} - 4.50522 \cdot 10^{-07} B_{1,7} - 3.38776 \cdot 10^{-07} B_{2,7} - 2.26947 \cdot 10^{-07} B_{3,7} \\ &\quad - 1.15035 \cdot 10^{-07} B_{4,7} - 3.04 \cdot 10^{-09} B_{5,7} + 1.09038 \cdot 10^{-07} B_{6,7} + 2.212 \cdot 10^{-07} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.76744 \cdot 10^{-14}$.
Bounding polynomials M and m:

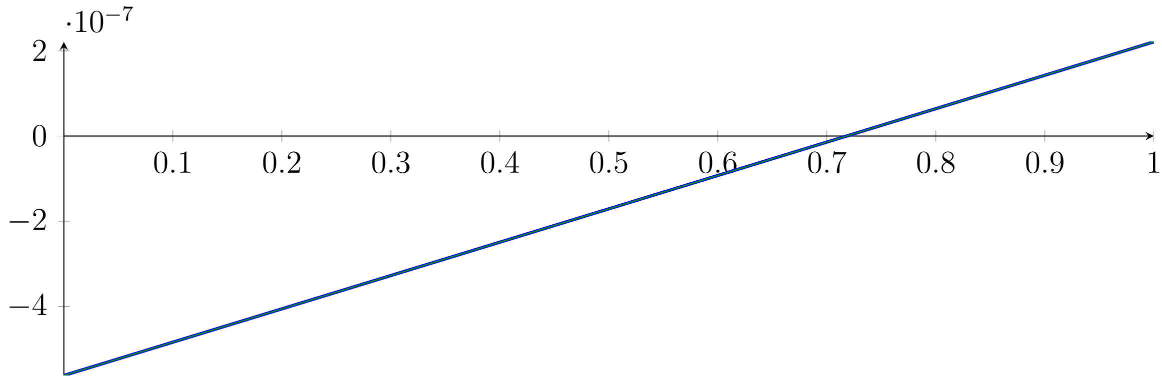
$$M = 1.74425 \cdot 10^{-09} X^2 + 7.81641 \cdot 10^{-07} X - 5.62185 \cdot 10^{-07}$$

$$m = 1.74425 \cdot 10^{-09} X^2 + 7.81641 \cdot 10^{-07} X - 5.62185 \cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{-448.841, 0.718087\} \qquad N(m) = \{-448.841, 0.718087\}$$

Intersection intervals:



$$[0.718087, 0.718087]$$

Longest intersection interval: $1.72607 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.96194, 0.96194]$,

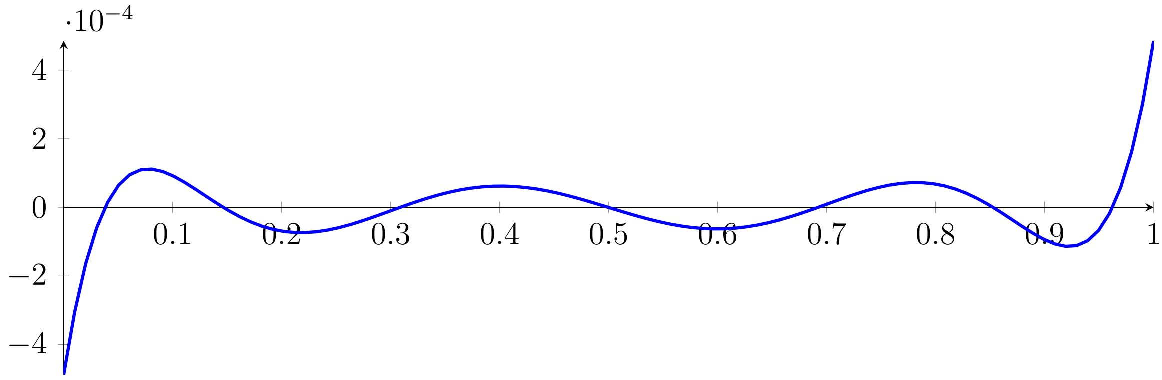
14.35 Recursion Branch 1 2 2 2 2 1 1 in Interval 1: $[0.96194, 0.96194]$

Found root in interval $[0.96194, 0.96194]$ at recursion depth 7!

14.36 Result: 8 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$



Result: Root Intervals

$$[0.0380602, 0.0380602],\ [0.146446, 0.146447],\ [0.308658, 0.308658],\ [0.5, 0.5],\ [0.5, 0.5],$$
$$[0.691342, 0.691342],\ [0.853553, 0.853554],\ [0.96194, 0.96194]$$

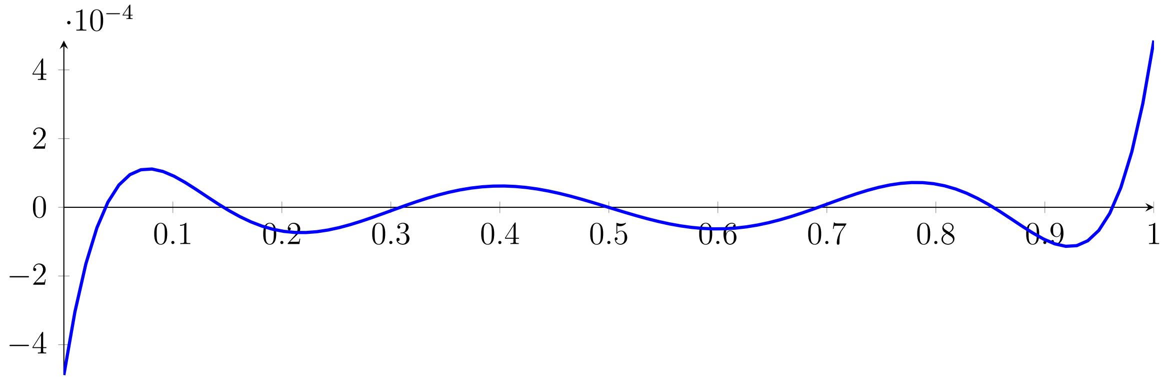
with precision $\varepsilon = 1 \cdot 10^{-06}$.

15 Running CubeClip on p7 with epsilon 6

$$1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$

Called CubeClip with input polynomial on interval $[0, 1]$:

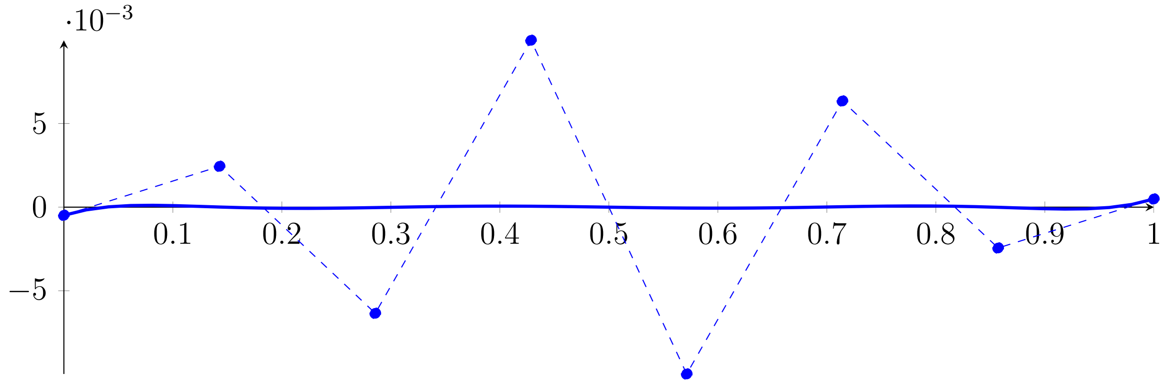
$$p = 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$



15.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

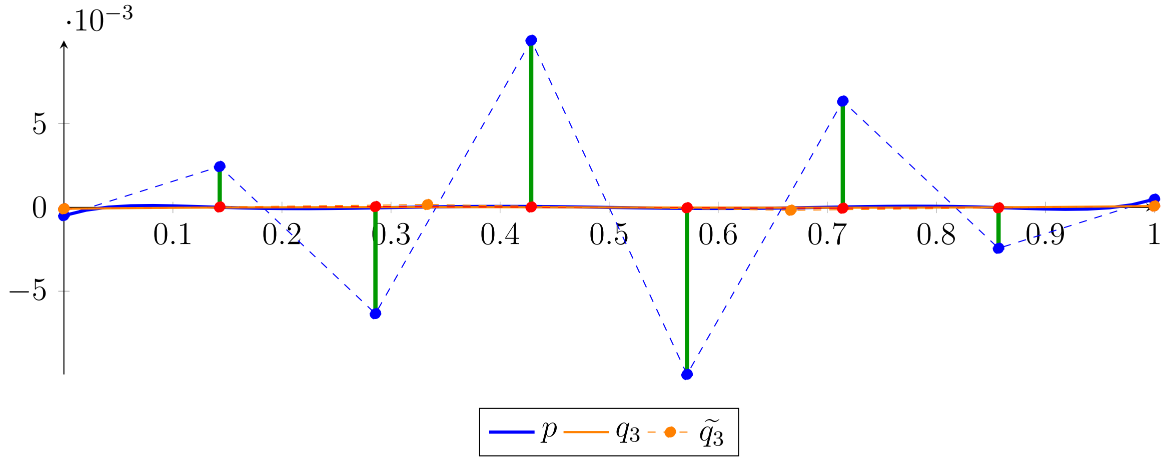
$$\begin{aligned}p &= 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281\\ &= -0.000488281B_{0,7}(X) + 0.00244141B_{1,7}(X) - 0.00634766B_{2,7}(X) + 0.00997489B_{3,7}(X)\\ &\quad - 0.00997489B_{4,7}(X) + 0.00634766B_{5,7}(X) - 0.00244141B_{6,7}(X) + 0.000488281B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 0.00118371X^3 - 0.00177557X^2 + 0.00075673X - 8.24371\cdot 10^{-05}\\ &= -8.24371\cdot 10^{-05}B_{0,3} + 0.000169806B_{1,3} - 0.000169806B_{2,3} + 8.24371\cdot 10^{-05}B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= -1.52814\cdot 10^{-17}X^7 + 5.36579\cdot 10^{-17}X^6 - 7.48506\cdot 10^{-17}X^5 + 5.26666\\ &\quad \cdot 10^{-17}X^4 + 0.00118371X^3 - 0.00177557X^2 + 0.00075673X - 8.24371\cdot 10^{-05}\\ &= -8.24371\cdot 10^{-05}B_{0,7} + 2.56672\cdot 10^{-05}B_{1,7} + 4.92207\cdot 10^{-05}B_{2,7} + 2.20436\cdot 10^{-05}B_{3,7}\\ &\quad - 2.20436\cdot 10^{-05}B_{4,7} - 4.92207\cdot 10^{-05}B_{5,7} - 2.56672\cdot 10^{-05}B_{6,7} + 8.24371\cdot 10^{-05}B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00995284$.
Bounding polynomials M and m:

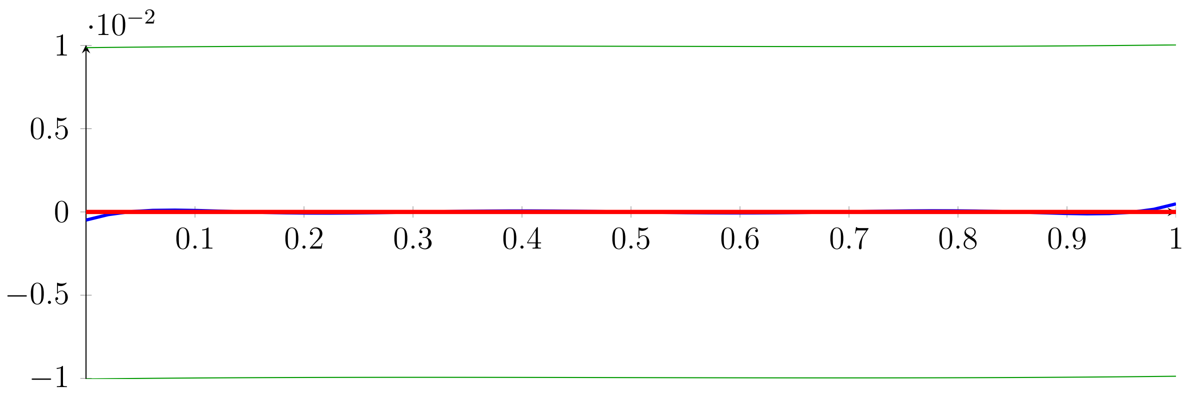
$$M = 0.00118371X^3 - 0.00177557X^2 + 0.00075673X + 0.00987041$$

$$m = 0.00118371X^3 - 0.00177557X^2 + 0.00075673X - 0.0100353$$

Root of M and m:

$$N(M) = \{-1.5516\} \qquad N(m) = \{2.5516\}$$

Intersection intervals:



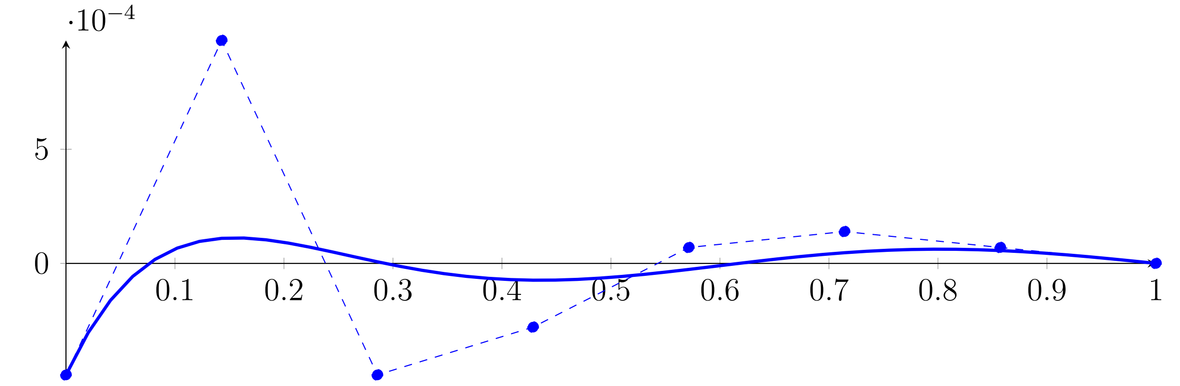
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$
Bisection point is very near to a root?!?

15.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

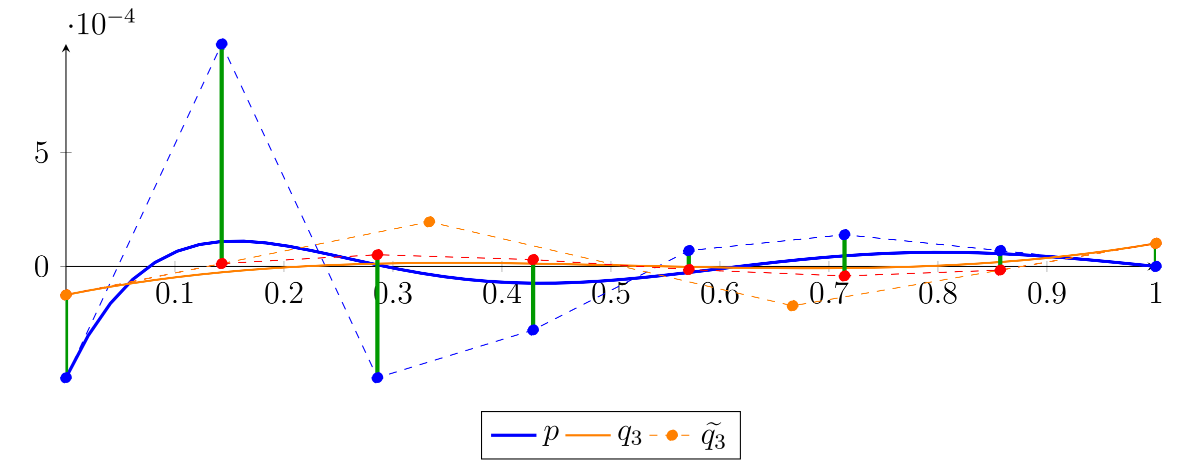
$$\begin{aligned} p &= 0.0078125X^7 - 0.0546875X^6 + 0.152344X^5 - 0.214844X^4 \\ &\quad + 0.161133X^3 - 0.0615234X^2 + 0.0102539X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.000976562B_{1,7}(X) - 0.000488281B_{2,7}(X) - 0.000279018B_{3,7}(X) \\ &\quad + 6.97545\cdot 10^{-05}B_{4,7}(X) + 0.000139509B_{5,7}(X) + 6.97545\cdot 10^{-05}B_{6,7}(X) - 2.05803\cdot 10^{-21}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.00133168X^3 - 0.00206727X^2 + 0.000961766X - 0.000124713 \\ &= -0.000124713B_{0,3} + 0.000195876B_{1,3} - 0.000172625B_{2,3} + 0.000101461B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.09638{\cdot}10^{-17}X^7 + 7.39904{\cdot}10^{-17}X^6 - 1.03696{\cdot}10^{-16}X^5 + 7.32649 \\ &\quad {\cdot}10^{-17}X^4 + 0.00133168X^3 - 0.00206727X^2 + 0.000961766X - 0.000124713 \\ &= -0.000124713B_{0,7} + 1.26826{\cdot}10^{-05}B_{1,7} + 5.16364{\cdot}10^{-05}B_{2,7} + 3.01967{\cdot}10^{-05}B_{3,7} \\ &\quad - 1.35885{\cdot}10^{-05}B_{4,7} - 4.16715{\cdot}10^{-05}B_{5,7} - 1.60043{\cdot}10^{-05}B_{6,7} + 0.000101461B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00096388$.

Bounding polynomials M and m:

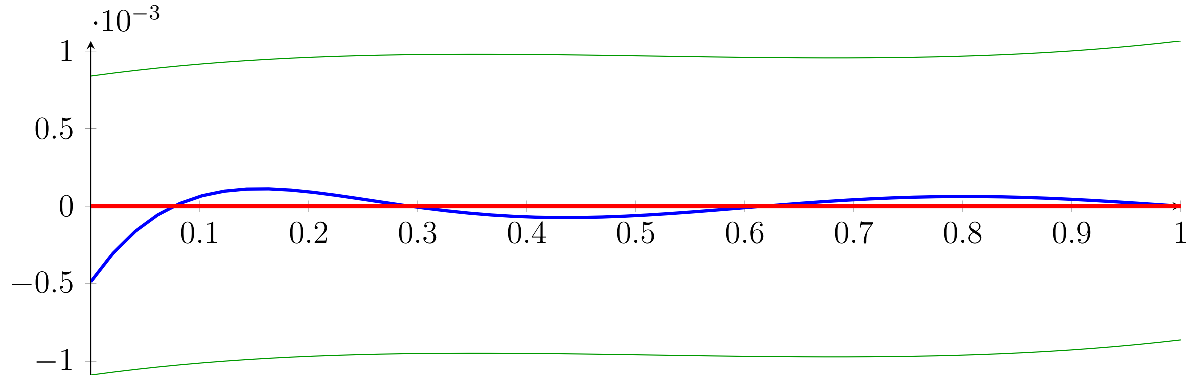
$$M = 0.00133168X^3 - 0.00206727X^2 + 0.000961766X + 0.000839167$$

$$m = 0.00133168X^3 - 0.00206727X^2 + 0.000961766X - 0.00108859$$

Root of M and m:

$$N(M) = \{-0.411666\} \qquad N(m) = \{1.44422\}$$

Intersection intervals:



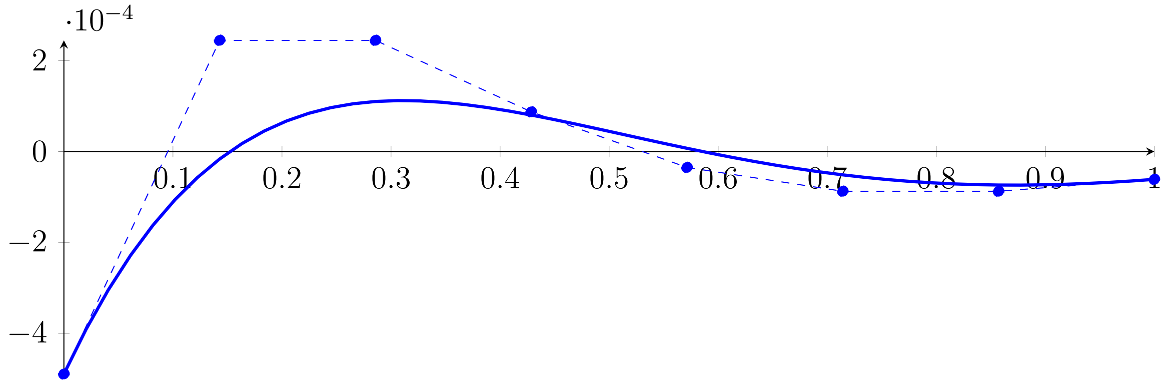
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$
Bisection point is very near to a root?!?

15.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

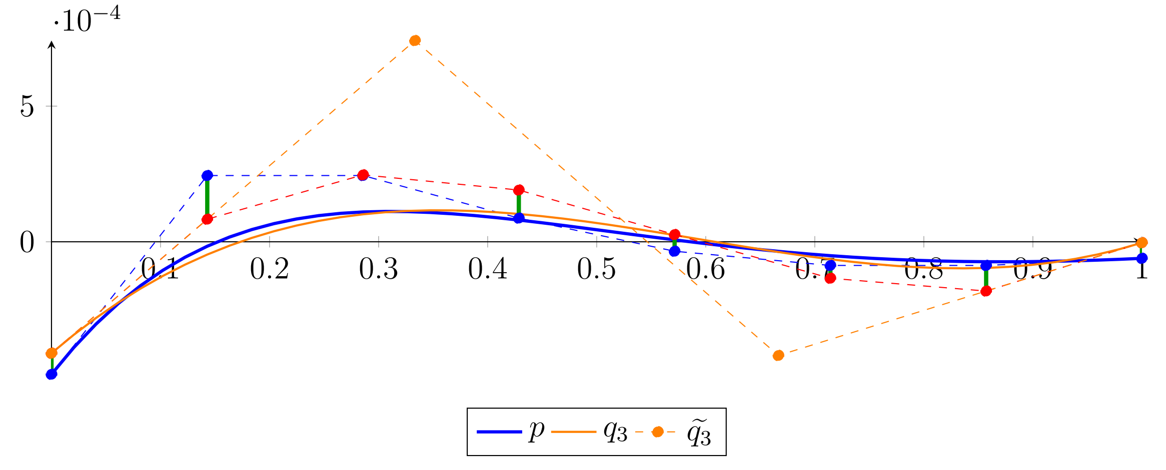
$$\begin{aligned} p &= 6.10352\cdot 10^{-05}X^7 - 0.000854492X^6 + 0.00476074X^5 - 0.0134277X^4 \\ &\quad + 0.0201416X^3 - 0.0153809X^2 + 0.00512695X - 0.000488281 \\ &= -0.000488281B_{0,7}(X) + 0.000244141B_{1,7}(X) + 0.000244141B_{2,7}(X) + 8.71931\cdot 10^{-05}B_{3,7}(X) \\ &\quad - 3.48772\cdot 10^{-05}B_{4,7}(X) - 8.71931\cdot 10^{-05}B_{5,7}(X) - 8.71931\cdot 10^{-05}B_{6,7}(X) - 6.10352\cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.00388868X^3 - 0.00693819X^2 + 0.00345655X - 0.0004106 \\ &= -0.0004106B_{0,3} + 0.000741582B_{1,3} - 0.000418967B_{2,3} - 3.56699\cdot 10^{-06}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -4.5167\cdot 10^{-17}X^7 + 1.62939\cdot 10^{-16}X^6 - 2.34472\cdot 10^{-16}X^5 + 1.71165 \\ &\quad \cdot 10^{-16}X^4 + 0.00388868X^3 - 0.00693819X^2 + 0.00345655X - 0.0004106 \\ &= -0.0004106B_{0,7} + 8.3192\cdot 10^{-05}B_{1,7} + 0.000246594B_{2,7} + 0.000190711B_{3,7} \\ &\quad + 2.66486\cdot 10^{-05}B_{4,7} - 0.000134489B_{5,7} - 0.000181596B_{6,7} - 3.56699\cdot 10^{-06}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000160949$.
Bounding polynomials M and m:

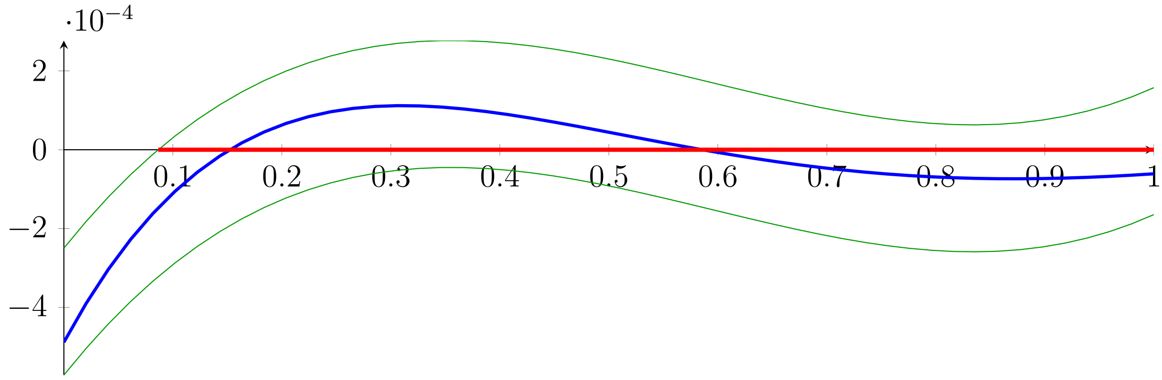
$$M = 0.00388868X^3 - 0.00693819X^2 + 0.00345655X - 0.000249652$$

$$m = 0.00388868X^3 - 0.00693819X^2 + 0.00345655X - 0.000571549$$

Root of M and m:

$$N(M) = \{0.0865243\} \qquad N(m) = \{1.09505\}$$

Intersection intervals:



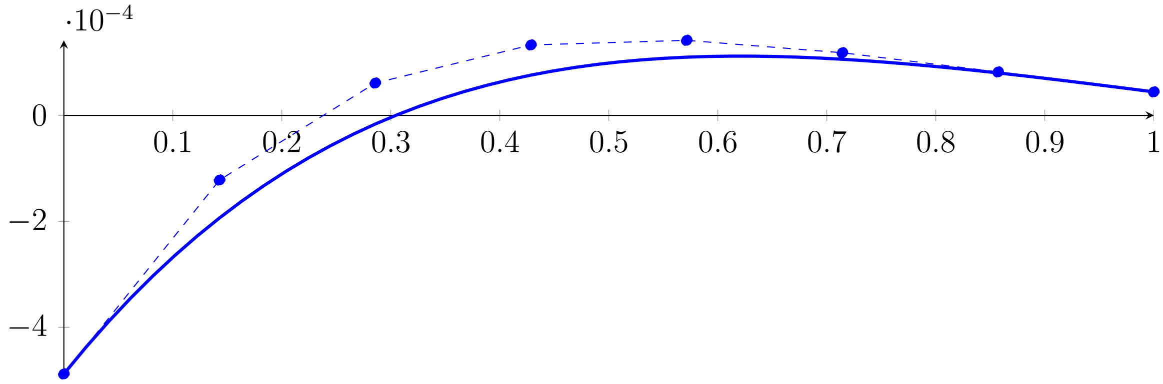
$$[0.0865243, 1]$$

Longest intersection interval: 0.913476
$\Longrightarrow$ Bisection: first half $[0, 0.125]$ und second half $[0.125, 0.25]$

15.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

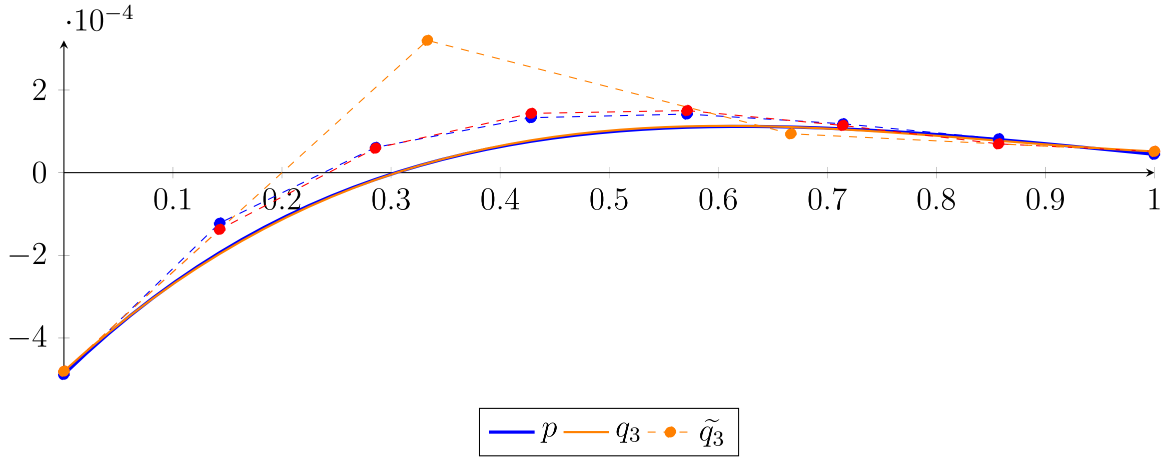
$$\begin{aligned}
p = {} & 4.76837\cdot 10^{-07}X^7 - 1.33514\cdot 10^{-05}X^6 + 0.000148773X^5 - 0.000839233X^4 \\
& + 0.0025177X^3 - 0.00384521X^2 + 0.00256348X - 0.000488281 \\
= {} & -0.000488281B_{0,7}(X) - 0.00012207B_{1,7}(X) + 6.10352\cdot 10^{-05}B_{2,7}(X) + 0.000132969B_{3,7}(X) \\
& + 0.000141689B_{4,7}(X) + 0.000118256B_{5,7}(X) + 8.2016\cdot 10^{-05}B_{6,7}(X) + 4.43459\cdot 10^{-05}B_{7,7}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= 0.00120976X^3 - 0.00307933X^2 + 0.00240131X - 0.000480408 \\
&= -0.000480408B_{0,3} + 0.000320029B_{1,3} + 9.40243\cdot 10^{-05}B_{2,3} + 5.13343\cdot 10^{-05}B_{3,3}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_3 &= -5.41402\cdot 10^{-17}X^7 + 1.96299\cdot 10^{-16}X^6 - 2.82556\cdot 10^{-16}X^5 + 2.05143 \\
&\quad \cdot 10^{-16}X^4 + 0.00120976X^3 - 0.00307933X^2 + 0.00240131X - 0.000480408 \\
&= -0.000480408B_{0,7} - 0.000137364B_{1,7} + 5.90464\cdot 10^{-05}B_{2,7} + 0.000143386B_{3,7} \\
&\quad + 0.000150221B_{4,7} + 0.000114114B_{5,7} + 6.963\cdot 10^{-05}B_{6,7} + 5.13343\cdot 10^{-05}B_{7,7}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.52933 \cdot 10^{-05}$.
Bounding polynomials M and m:

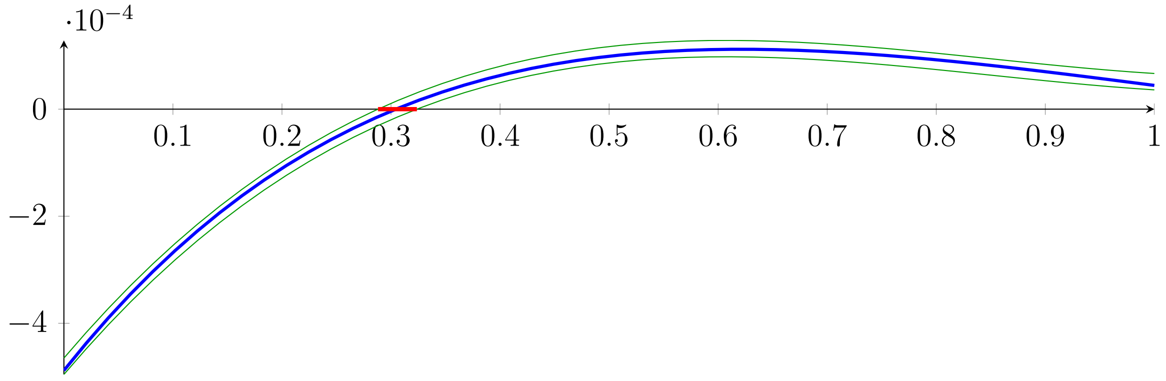
$$M = 0.00120976X^3 - 0.00307933X^2 + 0.00240131X - 0.000465115$$

$$m = 0.00120976X^3 - 0.00307933X^2 + 0.00240131X - 0.000495702$$

Root of M and m:

$$N(M) = \{0.288054\} \qquad N(m) = \{0.323727\}$$

Intersection intervals:



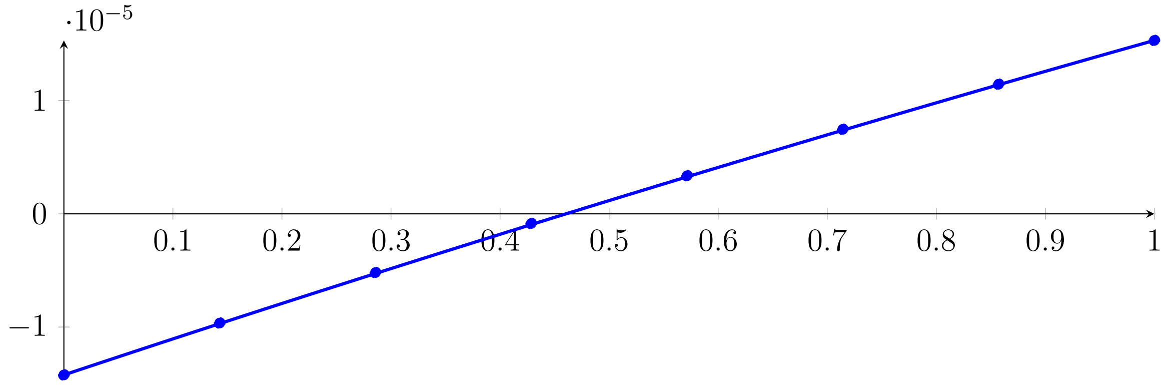
$$[0.288054, 0.323727]$$

Longest intersection interval: 0.0356732
$\Longrightarrow$ Selective recursion: interval 1: $[0.0360068, 0.0404659]$,

15.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0360068, 0.0404659]$

Normalized monomial und Bézier representations and the Bézier polygon:

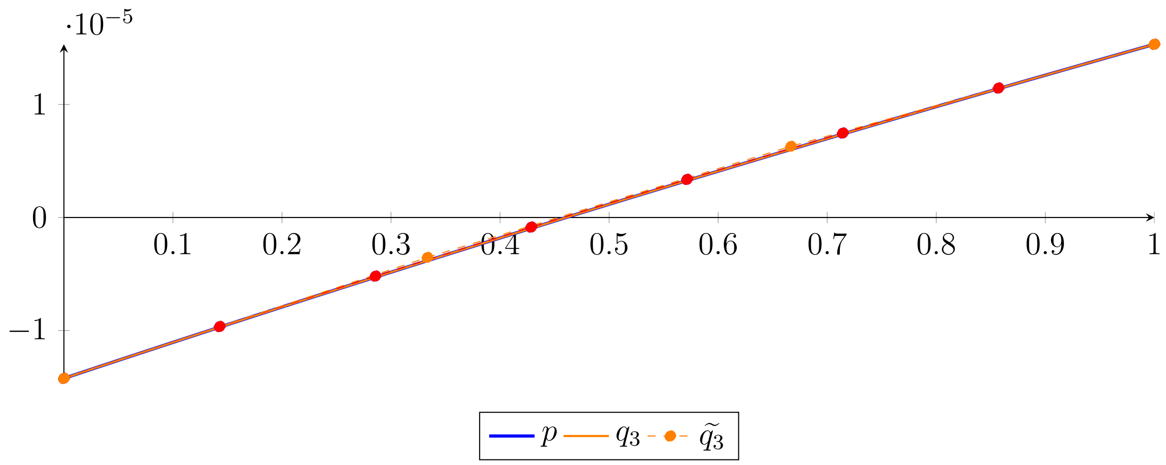
$$\begin{aligned} p = & \; 3.50563 \cdot 10^{-17} X^7 - 2.55342 \cdot 10^{-14} X^6 + 7.3097 \cdot 10^{-12} X^5 - 1.03836 \cdot 10^{-09} X^4 \\ & + 7.57175 \cdot 10^{-08} X^3 - 2.61276 \cdot 10^{-06} X^2 + 3.2094 \cdot 10^{-05} X - 1.42329 \cdot 10^{-05} \\ = & -1.42329 \cdot 10^{-05} B_{0,7}(X) - 9.648 \cdot 10^{-06} B_{1,7}(X) - 5.18756 \cdot 10^{-06} B_{2,7}(X) - 8.4938 \cdot 10^{-07} B_{3,7}(X) \\ & + 3.36868 \cdot 10^{-06} B_{4,7}(X) + 7.46873 \cdot 10^{-06} B_{5,7}(X) + 1.14528 \cdot 10^{-05} B_{6,7}(X) + 1.5323 \cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 7.3661 \cdot 10^{-08} X^3 - 2.61145 \cdot 10^{-06} X^2 + 3.20937 \cdot 10^{-05} X - 1.42328 \cdot 10^{-05} \\ &= -1.42328 \cdot 10^{-05} B_{0,3} - 3.53494 \cdot 10^{-06} B_{1,3} + 6.29247 \cdot 10^{-06} B_{2,3} + 1.53231 \cdot 10^{-05} B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.49577 \cdot 10^{-18} X^7 + 8.83978 \cdot 10^{-18} X^6 - 1.23688 \cdot 10^{-17} X^5 + 8.66684 \cdot 10^{-18} X^4 \\ &\quad + 7.3661 \cdot 10^{-08} X^3 - 2.61145 \cdot 10^{-06} X^2 + 3.20937 \cdot 10^{-05} X - 1.42328 \cdot 10^{-05} \\ &= -1.42328 \cdot 10^{-05} B_{0,7} - 9.64803 \cdot 10^{-06} B_{1,7} - 5.18757 \cdot 10^{-06} B_{2,7} - 8.49361 \cdot 10^{-07} B_{3,7} \\ &\quad + 3.3687 \cdot 10^{-06} B_{4,7} + 7.46872 \cdot 10^{-06} B_{5,7} + 1.14528 \cdot 10^{-05} B_{6,7} + 1.53231 \cdot 10^{-05} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.71607 \cdot 10^{-11}$.

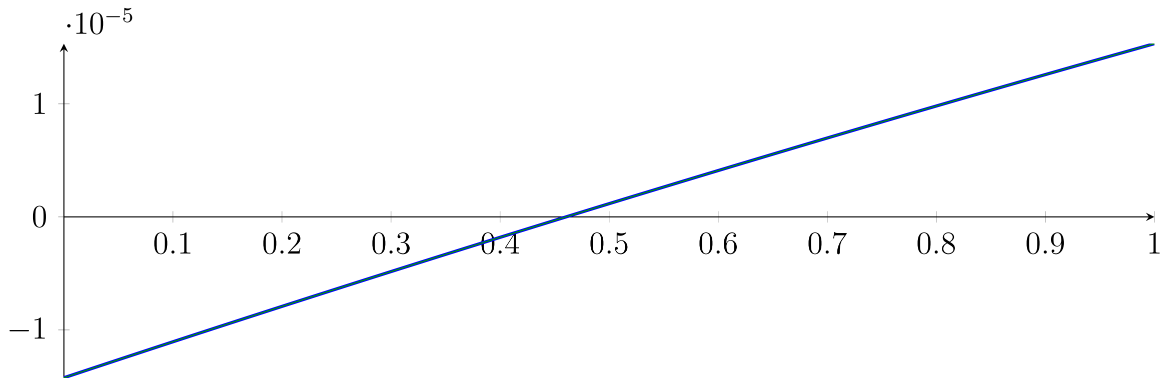
Bounding polynomials M and m:

$$\begin{aligned} M &= 7.3661 \cdot 10^{-08} X^3 - 2.61145 \cdot 10^{-06} X^2 + 3.20937 \cdot 10^{-05} X - 1.42328 \cdot 10^{-05} \\ m &= 7.3661 \cdot 10^{-08} X^3 - 2.61145 \cdot 10^{-06} X^2 + 3.20937 \cdot 10^{-05} X - 1.42329 \cdot 10^{-05} \end{aligned}$$

Root of M and m:

$$N(M) = \{0.460509\} \qquad N(m) = \{0.460511\}$$

Intersection intervals:



$$[0.460509, 0.460511]$$

Longest intersection interval: $1.82683 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0380602, 0.0380602]$,

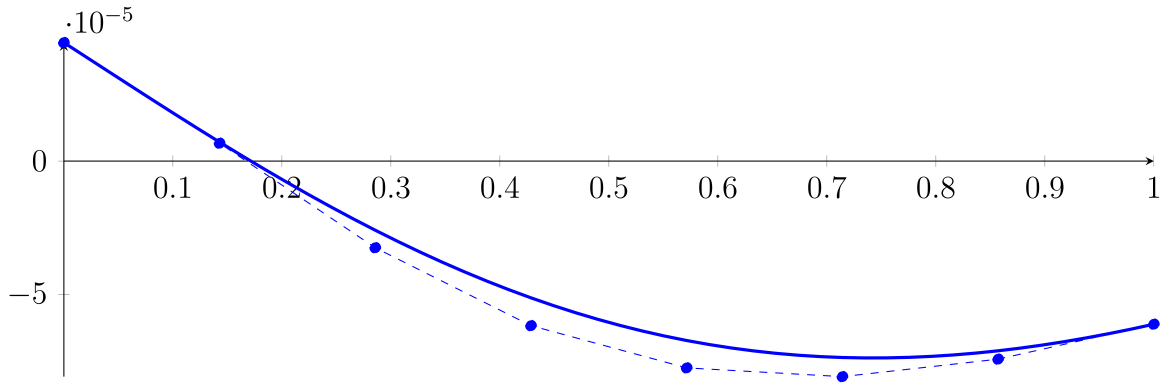
15.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0380602, 0.0380602]$

Found root in interval $[0.0380602, 0.0380602]$ at recursion depth 6!

15.7 Recursion Branch 1 1 1 2 on the Second Half $[0.125, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

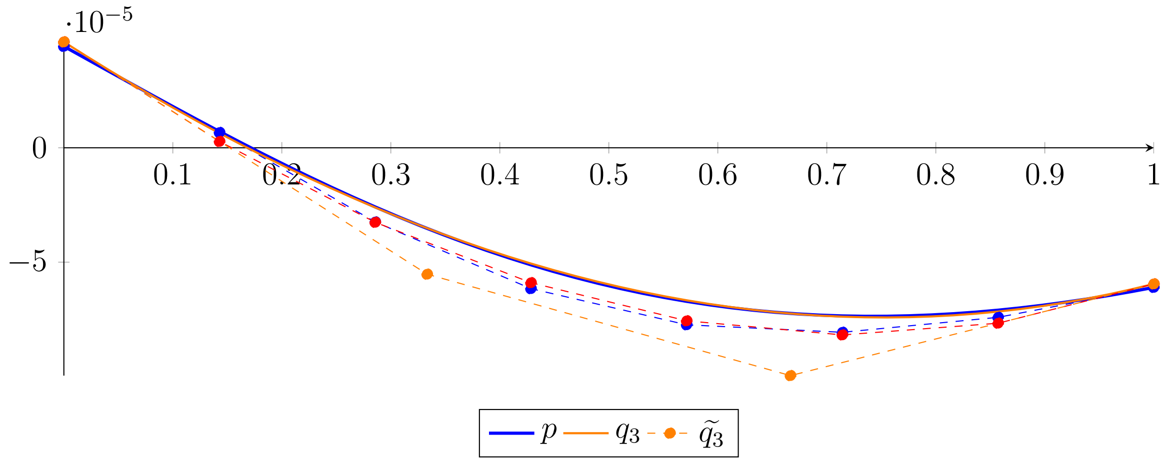
$$\begin{aligned}p &= 4.76837 \cdot 10^{-07} X^7 - 1.00136 \cdot 10^{-05} X^6 + 7.86781 \cdot 10^{-05} X^5 - 0.00027895 X^4 \\ &\quad + 0.000398159 X^3 - 3.00407 \cdot 10^{-05} X^2 - 0.000263691 X + 4.43459 \cdot 10^{-05} \\ &= 4.43459 \cdot 10^{-05} B_{0,7}(X) + 6.67572 \cdot 10^{-06} B_{1,7}(X) - 3.24249 \cdot 10^{-05} B_{2,7}(X) - 6.15801 \cdot 10^{-05} B_{3,7}(X) \\ &\quad - 7.73839 \cdot 10^{-05} B_{4,7}(X) - 8.06536 \cdot 10^{-05} B_{5,7}(X) - 7.41141 \cdot 10^{-05} B_{6,7}(X) - 6.10352 \cdot 10^{-05} B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 2.72014 \cdot 10^{-05} X^3 + 0.000171646 X^2 - 0.000304636 X + 4.62811 \cdot 10^{-05} \\ &= 4.62811 \cdot 10^{-05} B_{0,3} - 5.52643 \cdot 10^{-05} B_{1,3} - 9.95943 \cdot 10^{-05} B_{2,3} - 5.95076 \cdot 10^{-05} B_{3,3}\end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= 1.73797 \cdot 10^{-17} X^7 - 6.28258 \cdot 10^{-17} X^6 + 9.02668 \cdot 10^{-17} X^5 - 6.55408 \cdot 10^{-17} X^4 \\ &\quad + 2.72014 \cdot 10^{-05} X^3 + 0.000171646 X^2 - 0.000304636 X + 4.62811 \cdot 10^{-05} \\ &= 4.62811 \cdot 10^{-05} B_{0,7} + 2.76165 \cdot 10^{-06} B_{1,7} - 3.25842 \cdot 10^{-05} B_{2,7} - 5.89792 \cdot 10^{-05} B_{3,7} \\ &\quad - 7.56462 \cdot 10^{-05} B_{4,7} - 8.18081 \cdot 10^{-05} B_{5,7} - 7.66876 \cdot 10^{-05} B_{6,7} - 5.95076 \cdot 10^{-05} B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.91408 \cdot 10^{-06}$.

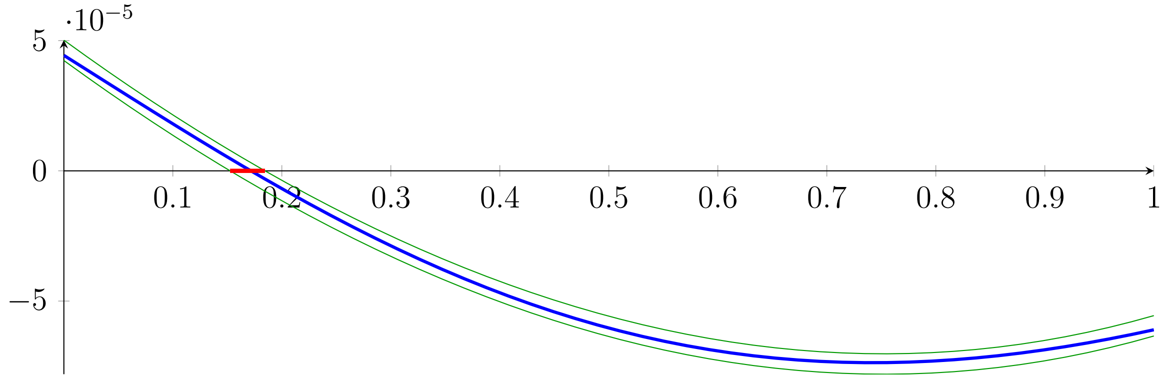
Bounding polynomials M and m:

$$M = 2.72014 \cdot 10^{-05} X^3 + 0.000171646 X^2 - 0.000304636 X + 5.01951 \cdot 10^{-05}$$
$$m = 2.72014 \cdot 10^{-05} X^3 + 0.000171646 X^2 - 0.000304636 X + 4.2367 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{-7.78014, 0.184515, 1.28544\} \qquad N(m) = \{-7.77615, 0.152493, 1.31347\}$$

Intersection intervals:



$$[0.152493, 0.184515]$$

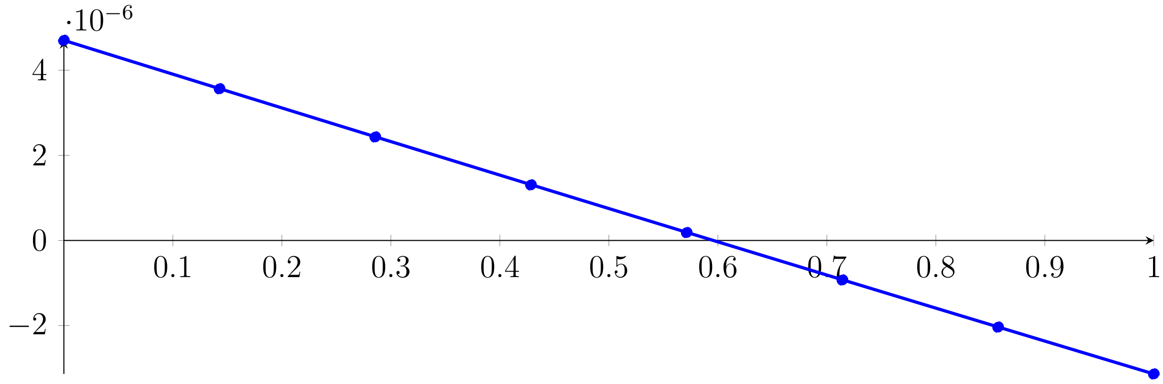
Longest intersection interval: 0.0320214

$\Longrightarrow$ Selective recursion: interval 1: $[0.144062, 0.148064]$,

15.8 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.144062, 0.148064]$

Normalized monomial und Bézier representations and the Bézier polygon:

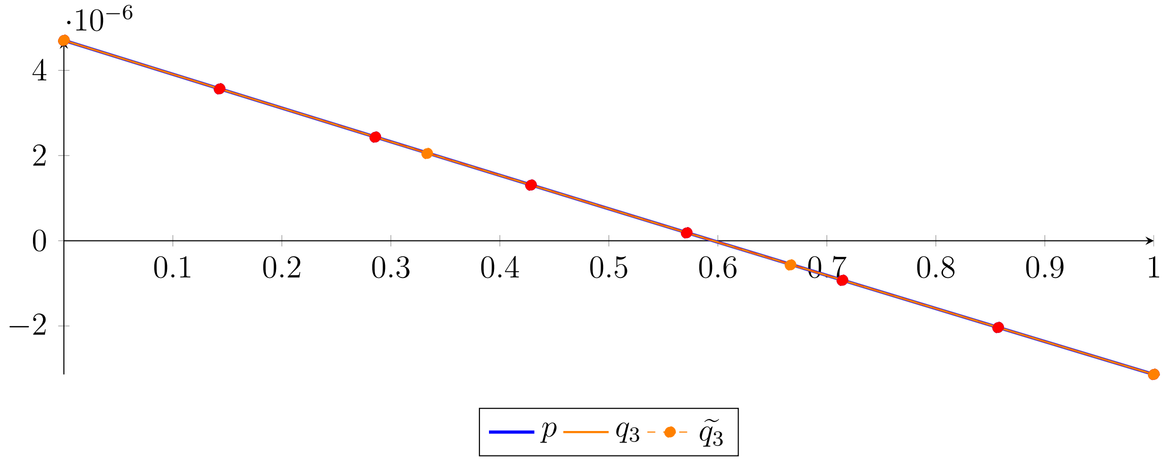
$$\begin{aligned} p &= 1.6461 \cdot 10^{-17} X^7 - 1.02466 \cdot 10^{-14} X^6 + 2.34824 \cdot 10^{-12} X^5 - 2.33822 \cdot 10^{-10} X^4 \\ &\quad + 8.06407 \cdot 10^{-09} X^3 + 1.18839 \cdot 10^{-07} X^2 - 7.96774 \cdot 10^{-06} X + 4.70362 \cdot 10^{-06} \\ &= 4.70362 \cdot 10^{-06} B_{0,7}(X) + 3.56537 \cdot 10^{-06} B_{1,7}(X) + 2.43278 \cdot 10^{-06} B_{2,7}(X) + 1.30608 \cdot 10^{-06} B_{3,7}(X) \\ &\quad + 1.85489 \cdot 10^{-07} B_{4,7}(X) - 9.28769 \cdot 10^{-07} B_{5,7}(X) - 2.03648 \cdot 10^{-06} B_{6,7}(X) - 3.13746 \cdot 10^{-06} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 7.60291 \cdot 10^{-09} X^3 + 1.19134 \cdot 10^{-07} X^2 - 7.96781 \cdot 10^{-06} X + 4.70362 \cdot 10^{-06} \\ &= 4.70362 \cdot 10^{-06} B_{0,3} + 2.04768 \cdot 10^{-06} B_{1,3} - 5.68542 \cdot 10^{-07} B_{2,3} - 3.13745 \cdot 10^{-06} B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 4.79433 \cdot 10^{-19} X^7 - 1.68518 \cdot 10^{-18} X^6 + 2.32907 \cdot 10^{-18} X^5 - 1.60024 \cdot 10^{-18} X^4 \\ &\quad + 7.60291 \cdot 10^{-09} X^3 + 1.19134 \cdot 10^{-07} X^2 - 7.96781 \cdot 10^{-06} X + 4.70362 \cdot 10^{-06} \\ &= 4.70362 \cdot 10^{-06} B_{0,7} + 3.56536 \cdot 10^{-06} B_{1,7} + 2.43278 \cdot 10^{-06} B_{2,7} + 1.30608 \cdot 10^{-06} B_{3,7} \\ &\quad + 1.85493 \cdot 10^{-07} B_{4,7} - 9.2877 \cdot 10^{-07} B_{5,7} - 2.03649 \cdot 10^{-06} B_{6,7} - 3.13745 \cdot 10^{-06} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.07909 \cdot 10^{-12}$.

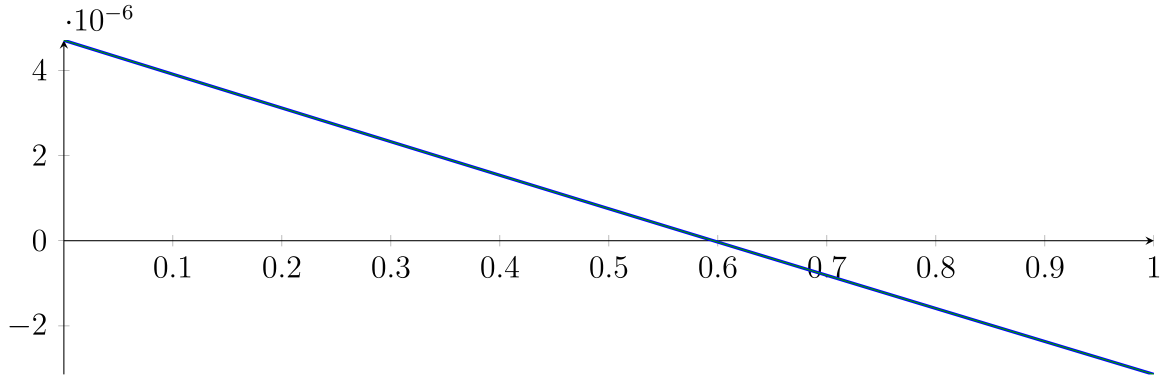
Bounding polynomials M and m:

$$M = 7.60291 \cdot 10^{-09} X^3 + 1.19134 \cdot 10^{-07} X^2 - 7.96781 \cdot 10^{-06} X + 4.70363 \cdot 10^{-06}$$
$$m = 7.60291 \cdot 10^{-09} X^3 + 1.19134 \cdot 10^{-07} X^2 - 7.96781 \cdot 10^{-06} X + 4.70361 \cdot 10^{-06}$$

Root of M and m:

$$N(M) = \{-41.3658, 0.595839, 25.1005\} \qquad N(m) = \{-41.3658, 0.595837, 25.1005\}$$

Intersection intervals:



$$[0.595837, 0.595839]$$

Longest intersection interval: $1.5552 \cdot 10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.146447, 0.146447]$,

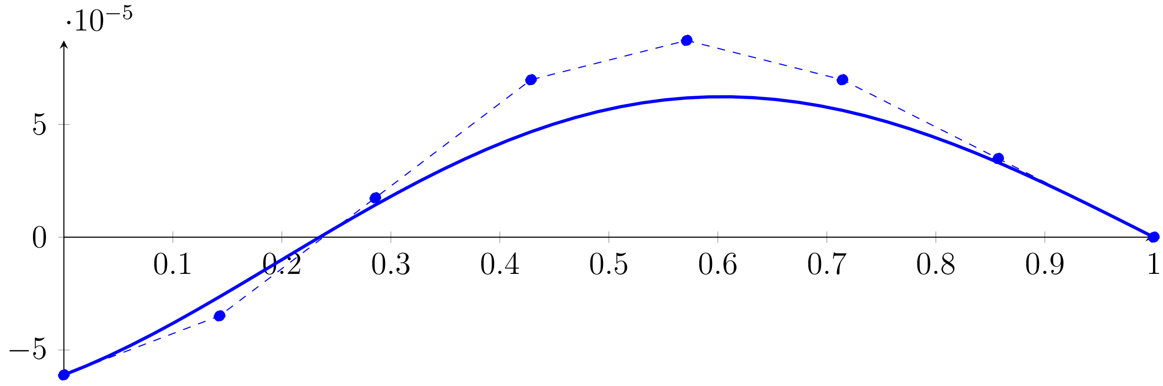
15.9 Recursion Branch 1 1 1 2 1 1 in Interval 1: $[0.146447, 0.146447]$

Found root in interval $[0.146447, 0.146447]$ at recursion depth 6!

15.10 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

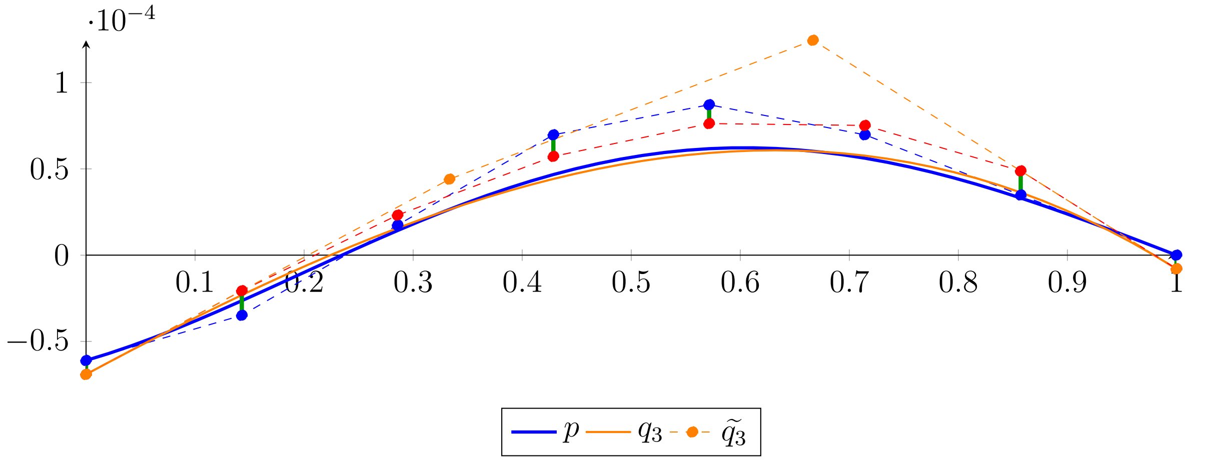
$$\begin{aligned} p &= 6.10352 \cdot 10^{-05} X^7 - 0.000427246 X^6 + 0.000915527 X^5 - 0.000305176 X^4 \\ &\quad - 0.000915527 X^3 + 0.000549316 X^2 + 0.000183105 X - 6.10352 \cdot 10^{-05} \\ &= -6.10352 \cdot 10^{-05} B_{0,7}(X) - 3.48772 \cdot 10^{-05} B_{1,7}(X) + 1.74386 \cdot 10^{-05} B_{2,7}(X) + 6.97545 \cdot 10^{-05} B_{3,7}(X) \\ &\quad + 8.71931 \cdot 10^{-05} B_{4,7}(X) + 6.97545 \cdot 10^{-05} B_{5,7}(X) + 3.48772 \cdot 10^{-05} B_{6,7}(X) - 2.05803 \cdot 10^{-21} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_3 &= -0.000180331X^3 - 9.7894\cdot 10^{-05}X^2 + 0.000339392X - 6.90939\cdot 10^{-05} \\
&= -6.90939\cdot 10^{-05}B_{0,3} + 4.40369\cdot 10^{-05}B_{1,3} + 0.000124536B_{2,3} - 7.92664\cdot 10^{-06}B_{3,3}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q_3} &= -1.15711\cdot 10^{-17}X^7 + 4.27826\cdot 10^{-17}X^6 - 6.28842\cdot 10^{-17}X^5 + 4.6752\cdot 10^{-17}X^4 \\
&\quad - 0.000180331X^3 - 9.7894\cdot 10^{-05}X^2 + 0.000339392X - 6.90939\cdot 10^{-05} \\
&= -6.90939\cdot 10^{-05}B_{0,7} - 2.06093\cdot 10^{-05}B_{1,7} + 2.32137\cdot 10^{-05}B_{2,7} + 5.72228\cdot 10^{-05}B_{3,7} \\
&\quad + 7.62656\cdot 10^{-05}B_{4,7} + 7.51899\cdot 10^{-05}B_{5,7} + 4.88432\cdot 10^{-05}B_{6,7} - 7.92664\cdot 10^{-06}B_{7,7}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 1.4268\cdot 10^{-05}$.

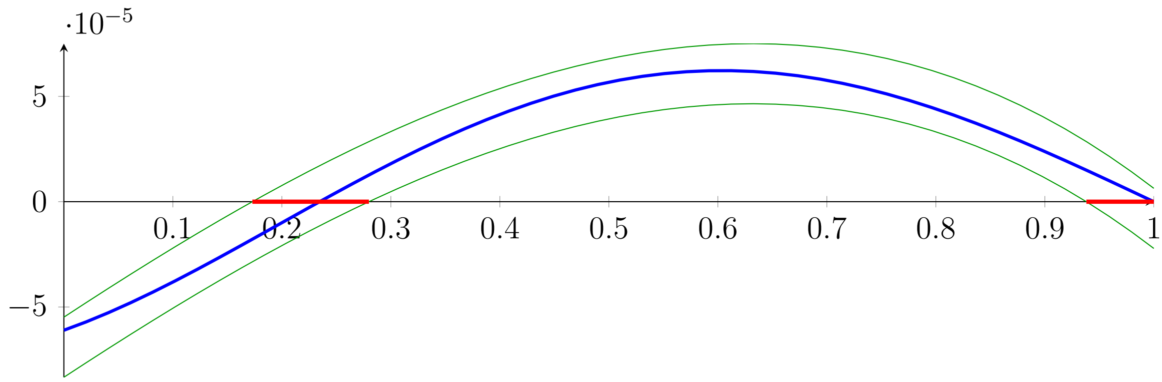
Bounding polynomials M and m:

$$
\begin{aligned}
M &= -0.000180331X^3 - 9.7894\cdot 10^{-05}X^2 + 0.000339392X - 5.4826\cdot 10^{-05} \\
m &= -0.000180331X^3 - 9.7894\cdot 10^{-05}X^2 + 0.000339392X - 8.33619\cdot 10^{-05}
\end{aligned}
$$

Root of M and m:

$$
N(M) = \{-1.73134, 0.172912, 1.01557\} \qquad N(m) = \{-1.76081, 0.279858, 0.938096\}
$$

Intersection intervals:



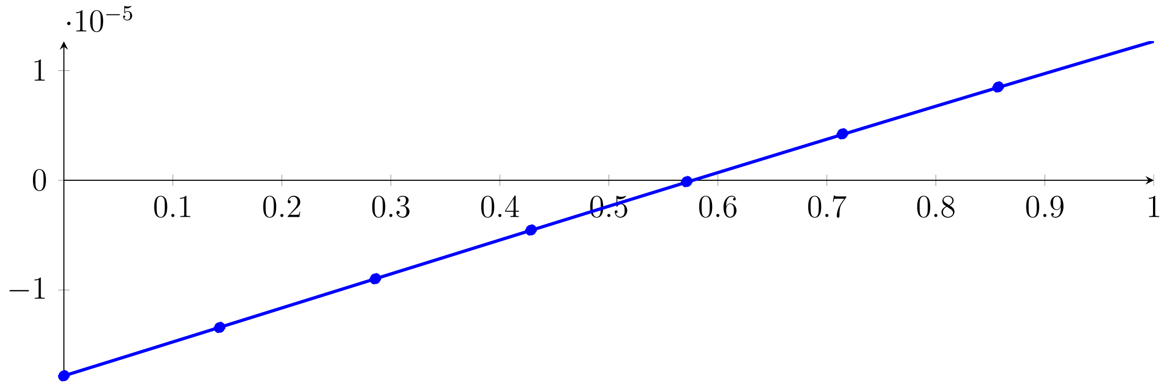
$$[0.172912, 0.279858],\ [0.938096, 1]$$

Longest intersection interval: 0.106945
$\Longrightarrow$ Selective recursion: interval 1: $[0.293228, 0.319964]$, interval 2: $[0.484524, 0.5]$,

15.11 Recursion Branch 1 1 2 1 in Interval 1: $[0.293228, 0.319964]$

Normalized monomial und Bézier representations and the Bézier polygon:

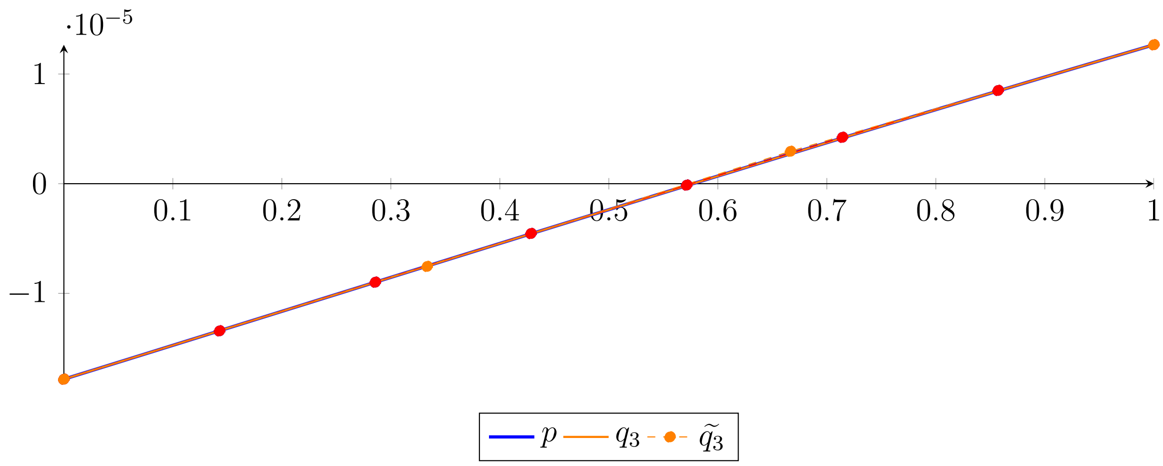
$$\begin{aligned} p &= 9.76586\cdot 10^{-12} X^7 - 5.28687\cdot 10^{-10} X^6 + 7.14302\cdot 10^{-09} X^5 + 4.00004\cdot 10^{-08} X^4 \\ &\quad - 1.0949\cdot 10^{-06} X^3 + 7.02855\cdot 10^{-07} X^2 + 3.08376\cdot 10^{-05} X - 1.78257\cdot 10^{-05} \\ &= -1.78257\cdot 10^{-05} B_{0,7}(X) - 1.34203\cdot 10^{-05} B_{1,7}(X) - 8.98147\cdot 10^{-06} B_{2,7}(X) - 4.54044\cdot 10^{-06} B_{3,7}(X) \\ &\quad - 1.27359\cdot 10^{-07} B_{4,7}(X) + 4.22911\cdot 10^{-06} B_{5,7}(X) + 8.50206\cdot 10^{-06} B_{6,7}(X) + 1.26665\cdot 10^{-05} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.96783\cdot 10^{-07} X^3 + 6.36081\cdot 10^{-07} X^2 + 3.08529\cdot 10^{-05} X - 1.78265\cdot 10^{-05} \\ &= -1.78265\cdot 10^{-05} B_{0,3} - 7.54218\cdot 10^{-06} B_{1,3} + 2.95414\cdot 10^{-06} B_{2,3} + 1.26657\cdot 10^{-05} B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -1.95709\cdot 10^{-18} X^7 + 6.88892\cdot 10^{-18} X^6 - 9.54209\cdot 10^{-18} X^5 + 6.57896\cdot 10^{-18} X^4 \\ &\quad - 9.96783\cdot 10^{-07} X^3 + 6.36081\cdot 10^{-07} X^2 + 3.08529\cdot 10^{-05} X - 1.78265\cdot 10^{-05} \\ &= -1.78265\cdot 10^{-05} B_{0,7} - 1.34189\cdot 10^{-05} B_{1,7} - 8.98107\cdot 10^{-06} B_{2,7} - 4.54142\cdot 10^{-06} B_{3,7} \\ &\quad - 1.28434\cdot 10^{-07} B_{4,7} + 4.2294\cdot 10^{-06} B_{5,7} + 8.50361\cdot 10^{-06} B_{6,7} + 1.26657\cdot 10^{-05} B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.54959\cdot 10^{-09}$.
Bounding polynomials M and m:

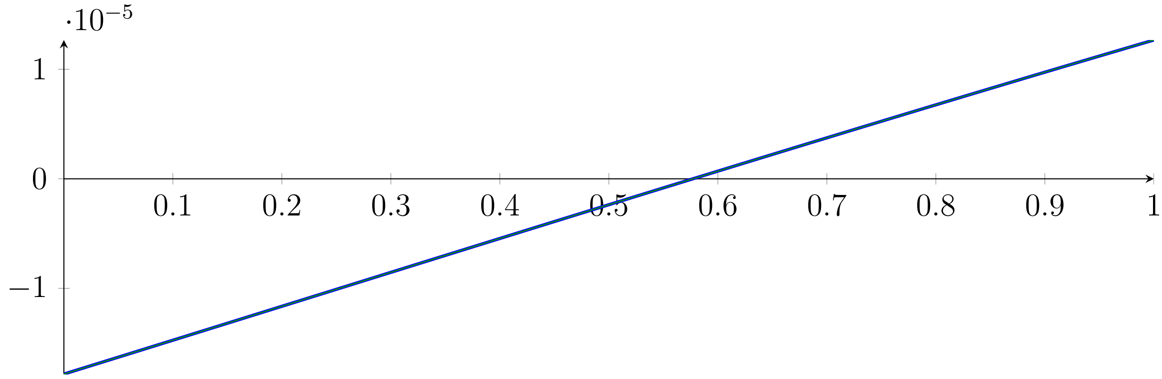
$$M = -9.96783\cdot 10^{-07} X^3 + 6.36081\cdot 10^{-07} X^2 + 3.08529\cdot 10^{-05} X - 1.78249\cdot 10^{-05}$$

$$m = -9.96783{\cdot}10^{-07}X^3 + 6.36081{\cdot}10^{-07}X^2 + 3.08529{\cdot}10^{-05}X - 1.7828{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-5.53622, 0.577082, 5.59727\} \qquad N(m) = \{-5.53626, 0.577184, 5.59721\}$$

Intersection intervals:



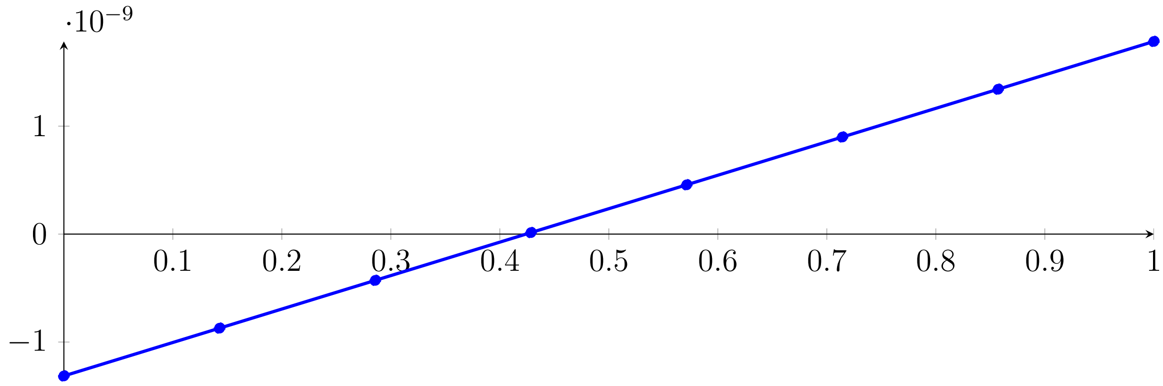
$[0.577082, 0.577184]$

Longest intersection interval: 0.00010131
$\Longrightarrow$ Selective recursion: interval 1: $[0.308657, 0.30866]$,

15.12 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.308657, 0.30866]$

Normalized monomial und Bézier representations and the Bézier polygon:

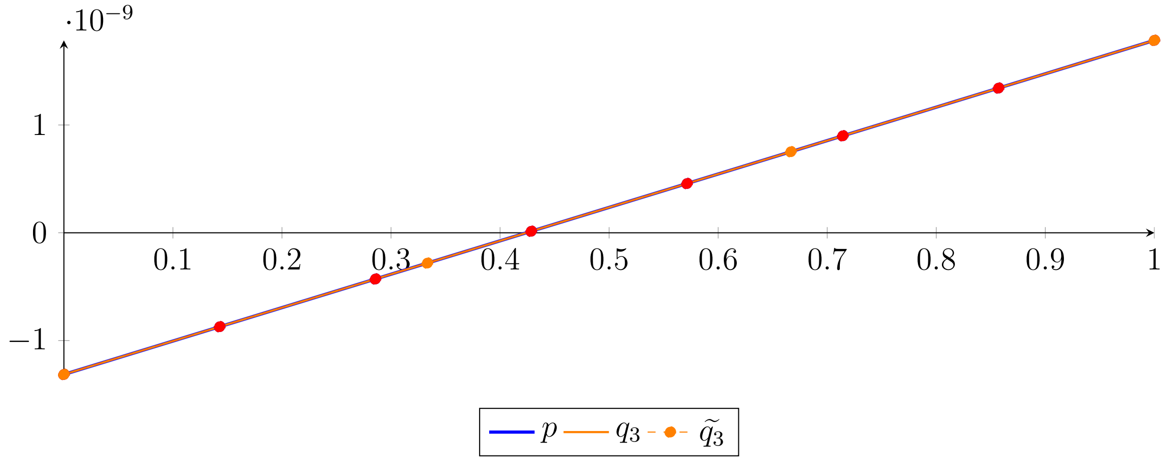
$$\begin{aligned} p = &-8.07794{\cdot}10^{-28}X^7 - 8.48183{\cdot}10^{-27}X^6 + 1.69637{\cdot}10^{-26}X^5 + 6.12812{\cdot}10^{-24}X^4 \\ &- 1.01982{\cdot}10^{-18}X^3 - 1.1289{\cdot}10^{-14}X^2 + 3.09902{\cdot}10^{-09}X - 1.31412{\cdot}10^{-09} \\ = &-1.31412{\cdot}10^{-09}B_{0,7}(X) - 8.71399{\cdot}10^{-10}B_{1,7}(X) - 4.28683{\cdot}10^{-10}B_{2,7}(X) + 1.40329{\cdot}10^{-11}B_{3,7}(X) \\ &+ 4.56748{\cdot}10^{-10}B_{4,7}(X) + 8.99462{\cdot}10^{-10}B_{5,7}(X) + 1.34218{\cdot}10^{-09}B_{6,7}(X) + 1.78489{\cdot}10^{-09}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.01981{\cdot}10^{-18}X^3 - 1.1289{\cdot}10^{-14}X^2 + 3.09902{\cdot}10^{-09}X - 1.31412{\cdot}10^{-09} \\ &= -1.31412{\cdot}10^{-09}B_{0,3} - 2.8111{\cdot}10^{-10}B_{1,3} + 7.51892{\cdot}10^{-10}B_{2,3} + 1.78489{\cdot}10^{-09}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 = &-2.8299{\cdot}10^{-22}X^7 + 1.00263{\cdot}10^{-21}X^6 - 1.40444{\cdot}10^{-21}X^5 + 9.8631{\cdot}10^{-22}X^4 \\ &- 1.02017{\cdot}10^{-18}X^3 - 1.1289{\cdot}10^{-14}X^2 + 3.09902{\cdot}10^{-09}X - 1.31412{\cdot}10^{-09} \\ = &-1.31412{\cdot}10^{-09}B_{0,7} - 8.71399{\cdot}10^{-10}B_{1,7} - 4.28683{\cdot}10^{-10}B_{2,7} + 1.40329{\cdot}10^{-11}B_{3,7} \\ &+ 4.56748{\cdot}10^{-10}B_{4,7} + 8.99462{\cdot}10^{-10}B_{5,7} + 1.34218{\cdot}10^{-09}B_{6,7} + 1.78489{\cdot}10^{-09}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06968 \cdot 10^{-24}$.

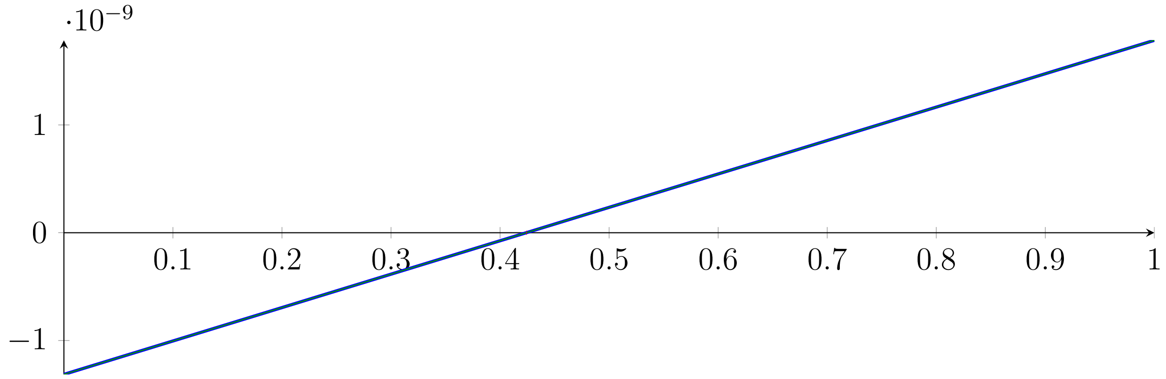
Bounding polynomials M and m:

$$M = -1.01981 \cdot 10^{-18} X^3 - 1.1289 \cdot 10^{-14} X^2 + 3.09902 \cdot 10^{-09} X - 1.31412 \cdot 10^{-09}$$
$$m = -1.01981 \cdot 10^{-18} X^3 - 1.1289 \cdot 10^{-14} X^2 + 3.09902 \cdot 10^{-09} X - 1.31412 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{-60937.8, 0.424043, 49867.6\} \qquad N(m) = \{-60937.8, 0.424043, 49867.6\}$$

Intersection intervals:



$$[0.424043, 0.424043]$$

Longest intersection interval: $1.9984 \cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.308658, 0.308658]$,

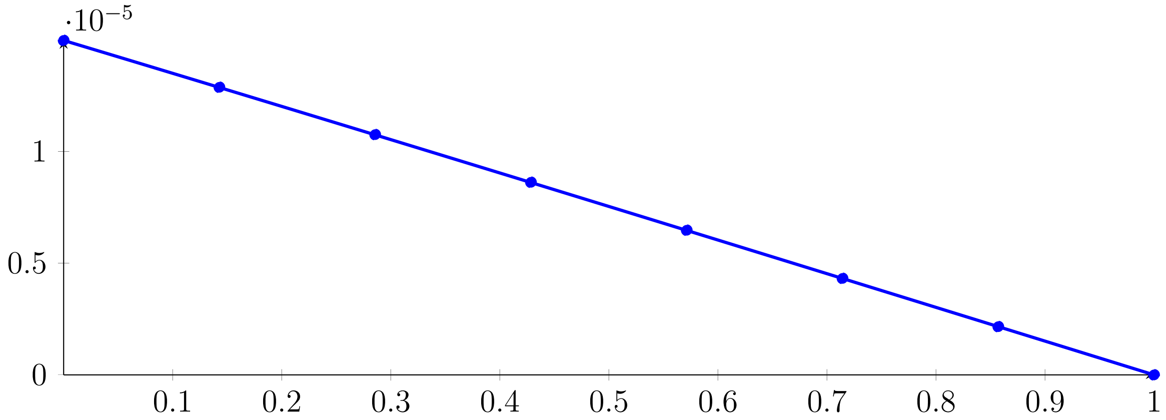
15.13 Recursion Branch 1 1 2 1 1 1 in Interval 1: $[0.308658, 0.308658]$

Found root in interval $[0.308658, 0.308658]$ at recursion depth 6!

15.14 Recursion Branch 1 1 2 2 in Interval 2: $[0.484524, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

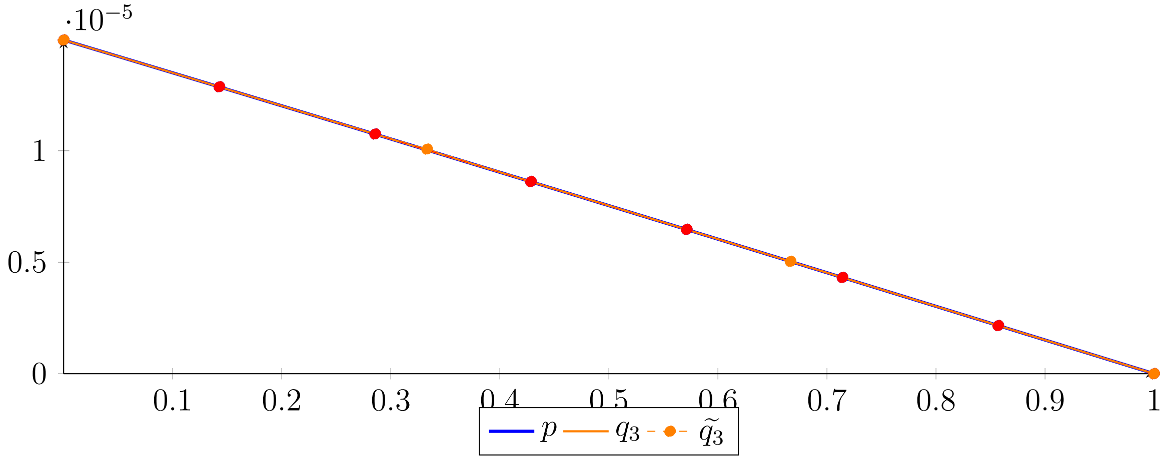
$$\begin{aligned} p = {} & 2.12634 \cdot 10^{-13} X^7 - 1.48843 \cdot 10^{-12} X^6 - 3.28454 \cdot 10^{-10} X^5 + 1.65716 \cdot 10^{-09} X^4 \\ & + 1.4147 \cdot 10^{-07} X^3 - 4.31052 \cdot 10^{-07} X^2 - 1.46807 \cdot 10^{-05} X + 1.49689 \cdot 10^{-05} \\ = {} & 1.49689 \cdot 10^{-05} B_{0,7}(X) + 1.28717 \cdot 10^{-05} B_{1,7}(X) + 1.07539 \cdot 10^{-05} B_{2,7}(X) + 8.61967 \cdot 10^{-06} B_{3,7}(X) \\ & + 6.47303 \cdot 10^{-06} B_{4,7}(X) + 4.31811 \cdot 10^{-06} B_{5,7}(X) + 2.15905 \cdot 10^{-06} B_{6,7}(X) - 2.05803 \cdot 10^{-21} B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.43868\cdot 10^{-07}X^3 - 4.32396\cdot 10^{-07}X^2 - 1.46804\cdot 10^{-05}X + 1.49689\cdot 10^{-05} \\ &= 1.49689\cdot 10^{-05}B_{0,3} + 1.00754\cdot 10^{-05}B_{1,3} + 5.03785\cdot 10^{-06}B_{2,3} - 1.0556\cdot 10^{-11}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.51822\cdot 10^{-19}X^7 + 6.28064\cdot 10^{-19}X^6 - 1.08764\cdot 10^{-18}X^5 + 9.96418\cdot 10^{-19}X^4 \\ &\quad + 1.43868\cdot 10^{-07}X^3 - 4.32396\cdot 10^{-07}X^2 - 1.46804\cdot 10^{-05}X + 1.49689\cdot 10^{-05} \\ &= 1.49689\cdot 10^{-05}B_{0,7} + 1.28717\cdot 10^{-05}B_{1,7} + 1.07539\cdot 10^{-05}B_{2,7} + 8.61965\cdot 10^{-06}B_{3,7} \\ &\quad + 6.47302\cdot 10^{-06}B_{4,7} + 4.31812\cdot 10^{-06}B_{5,7} + 2.15907\cdot 10^{-06}B_{6,7} - 1.0556\cdot 10^{-11}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.63707\cdot 10^{-11}$.
Bounding polynomials M and m:

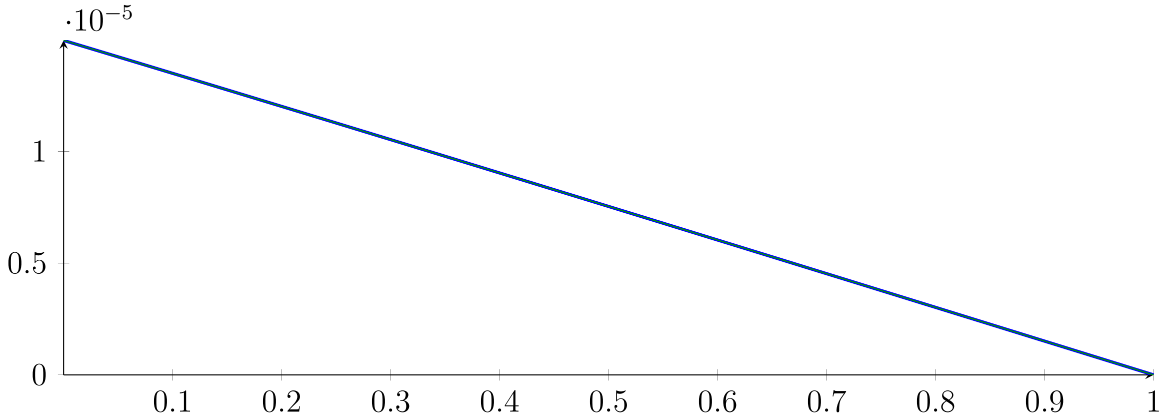
$$M = 1.43868\cdot 10^{-07}X^3 - 4.32396\cdot 10^{-07}X^2 - 1.46804\cdot 10^{-05}X + 1.49689\cdot 10^{-05}$$

$$m = 1.43868\cdot 10^{-07}X^3 - 4.32396\cdot 10^{-07}X^2 - 1.46804\cdot 10^{-05}X + 1.49689\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{-9.24672, 1, 11.2522\} \qquad N(m) = \{-9.24672, 0.999998, 11.2522\}$$

Intersection intervals:



$[0.999998, 1]$

Longest intersection interval: $2.44328 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

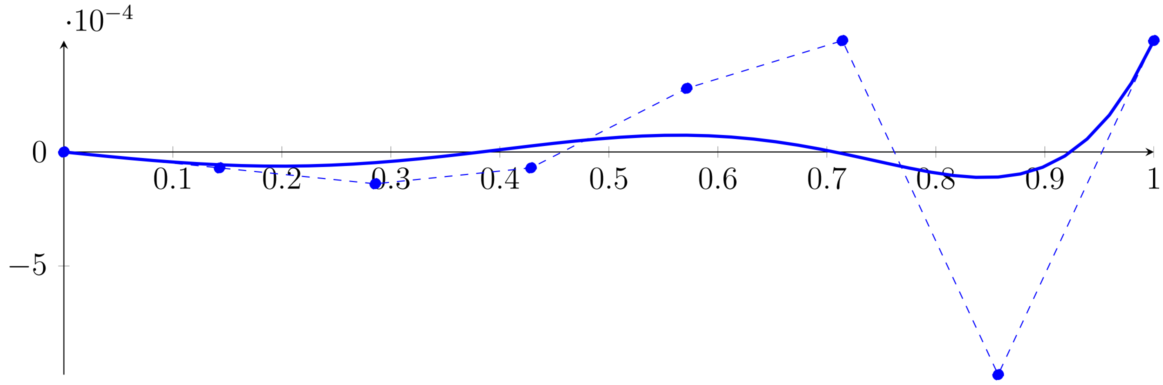
15.15 Recursion Branch 1 1 2 2 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

15.16 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

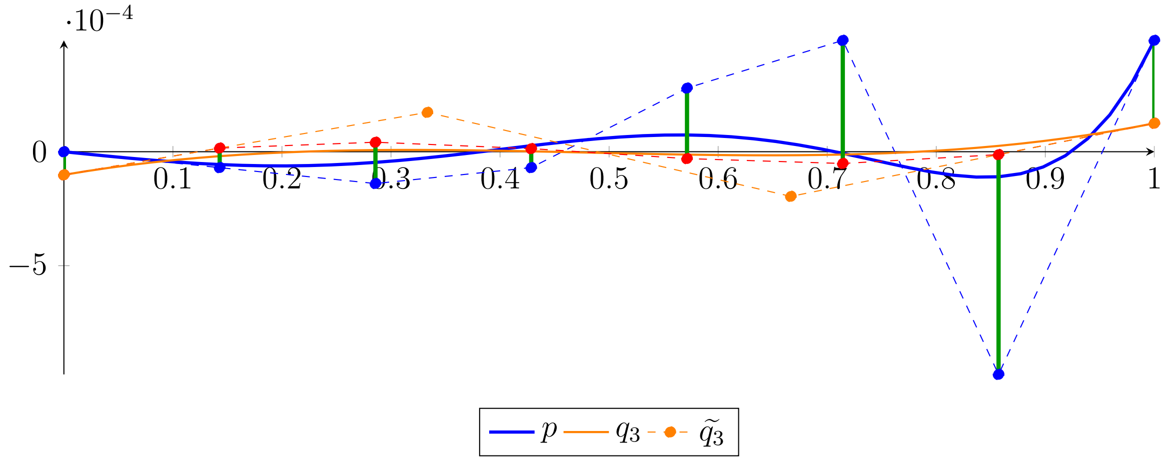
$$\begin{aligned} p &= 0.0078125X^7 - 1.49667 \cdot 10^{-19}X^6 - 0.0117188X^5 - 7.52966 \cdot 10^{-19}X^4 \\ &\quad + 0.00488281X^3 - 2.68622 \cdot 10^{-19}X^2 - 0.000488281X - 2.05803 \cdot 10^{-21} \\ &= -2.05803 \cdot 10^{-21}B_{0,7}(X) - 6.97545 \cdot 10^{-05}B_{1,7}(X) - 0.000139509B_{2,7}(X) - 6.97545 \cdot 10^{-05}B_{3,7}(X) \\ &\quad + 0.000279018B_{4,7}(X) + 0.000488281B_{5,7}(X) - 0.000976563B_{6,7}(X) + 0.000488281B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.00133168X^3 - 0.00192776X^2 + 0.000822257X - 0.000101461 \\ &= -0.000101461B_{0,3} + 0.000172625B_{1,3} - 0.000195876B_{2,3} + 0.000124713B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.97678 \cdot 10^{-17}X^7 + 6.9169 \cdot 10^{-17}X^6 - 9.59875 \cdot 10^{-17}X^5 + 6.70295 \\ &\quad \cdot 10^{-17}X^4 + 0.00133168X^3 - 0.00192776X^2 + 0.000822257X - 0.000101461 \\ &= -0.000101461B_{0,7} + 1.60043 \cdot 10^{-05}B_{1,7} + 4.16715 \cdot 10^{-05}B_{2,7} + 1.35885 \cdot 10^{-05}B_{3,7} \\ &\quad - 3.01967 \cdot 10^{-05}B_{4,7} - 5.16364 \cdot 10^{-05}B_{5,7} - 1.26826 \cdot 10^{-05}B_{6,7} + 0.000124713B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00096388$.
Bounding polynomials M and m:

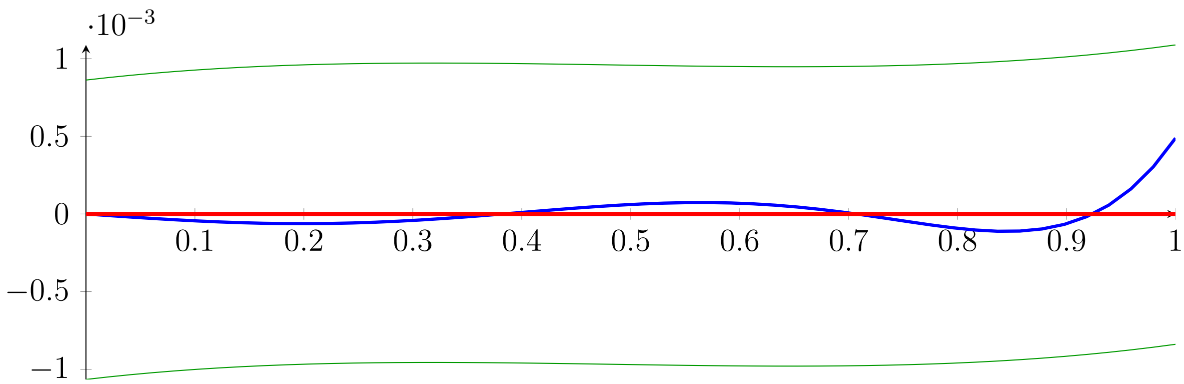
$$M = 0.00133168X^3 - 0.00192776X^2 + 0.000822257X + 0.000862419$$

$$m = 0.00133168X^3 - 0.00192776X^2 + 0.000822257X - 0.00106534$$

Root of M and m:

$$N(M) = \{-0.444225\} \qquad N(m) = \{1.41167\}$$

Intersection intervals:



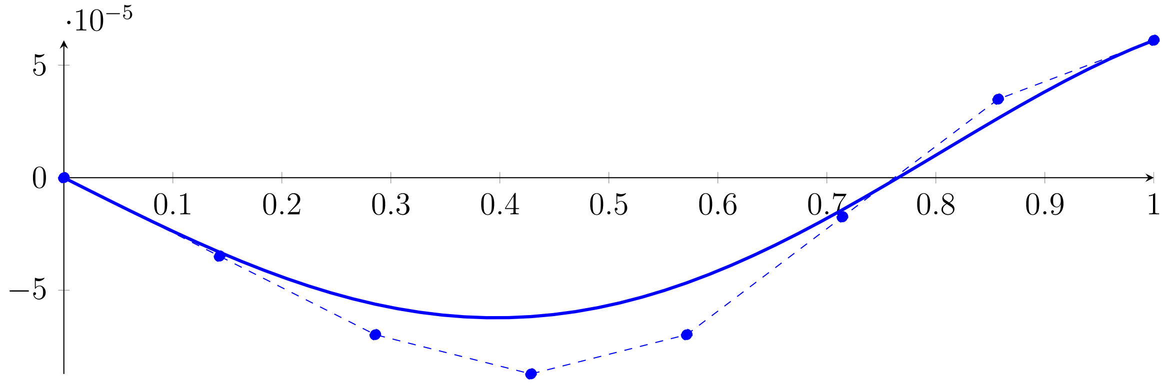
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0.5, 0.75]$ und second half $[0.75, 1]$

15.17 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

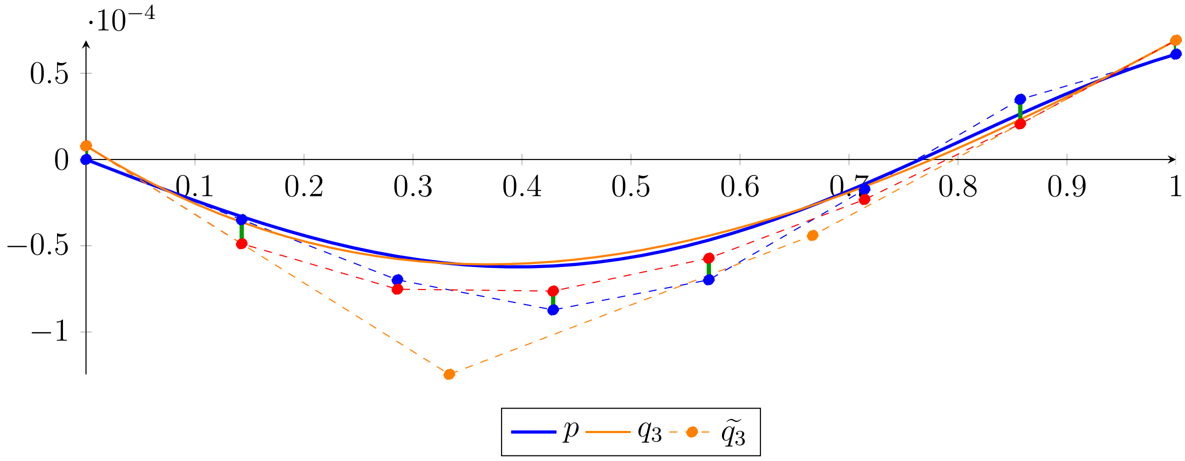
$$\begin{aligned} p &= 6.10352{\cdot}10^{-05}X^7 + 4.12267{\cdot}10^{-21}X^6 - 0.000366211X^5 - 4.70169{\cdot}10^{-20}X^4 \\ &\quad + 0.000610352X^3 - 6.71207{\cdot}10^{-20}X^2 - 0.000244141X - 2.05803{\cdot}10^{-21} \\ &= -2.05803{\cdot}10^{-21}B_{0,7}(X) - 3.48772{\cdot}10^{-05}B_{1,7}(X) - 6.97545{\cdot}10^{-05}B_{2,7}(X) - 8.71931{\cdot}10^{-05}B_{3,7}(X) \\ &\quad - 6.97545{\cdot}10^{-05}B_{4,7}(X) - 1.74386{\cdot}10^{-05}B_{5,7}(X) + 3.48772{\cdot}10^{-05}B_{6,7}(X) + 6.10352{\cdot}10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000180331X^3 + 0.000638887X^2 - 0.000397389X + 7.92664{\cdot}10^{-06} \\ &= 7.92664{\cdot}10^{-06}B_{0,3} - 0.000124536B_{1,3} - 4.40369{\cdot}10^{-05}B_{2,3} + 6.90939{\cdot}10^{-05}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.76516{\cdot}10^{-18}X^7 - 1.15701{\cdot}10^{-17}X^6 + 1.93835{\cdot}10^{-17}X^5 - 1.65814{\cdot}10^{-17}X^4 \\ &\quad - 0.000180331X^3 + 0.000638887X^2 - 0.000397389X + 7.92664{\cdot}10^{-06} \\ &= 7.92664{\cdot}10^{-06}B_{0,7} - 4.88432{\cdot}10^{-05}B_{1,7} - 7.51899{\cdot}10^{-05}B_{2,7} - 7.62656{\cdot}10^{-05}B_{3,7} \\ &\quad - 5.72228{\cdot}10^{-05}B_{4,7} - 2.32137{\cdot}10^{-05}B_{5,7} + 2.06093{\cdot}10^{-05}B_{6,7} + 6.90939{\cdot}10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.4268{\cdot}10^{-05}$.

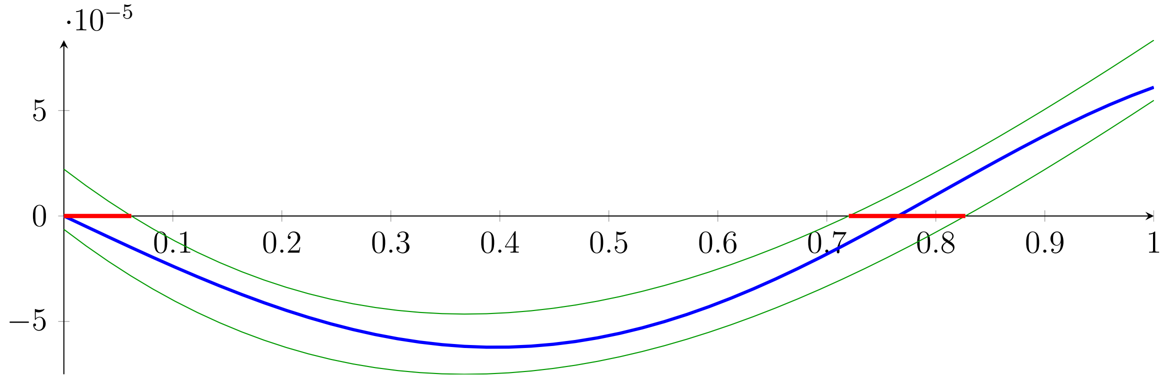
Bounding polynomials M and m:

$$\begin{aligned} M &= -0.000180331X^3 + 0.000638887X^2 - 0.000397389X + 2.21946{\cdot}10^{-05} \\ m &= -0.000180331X^3 + 0.000638887X^2 - 0.000397389X - 6.34131{\cdot}10^{-06} \end{aligned}$$

Root of M and m:

$$N(M) = \{0.0619044, 0.720142, 2.76081\} \qquad N(m) = \{-0.0155662, 0.827088, 2.73134\}$$

Intersection intervals:



$$[0, 0.0619044], [0.720142, 0.827088]$$

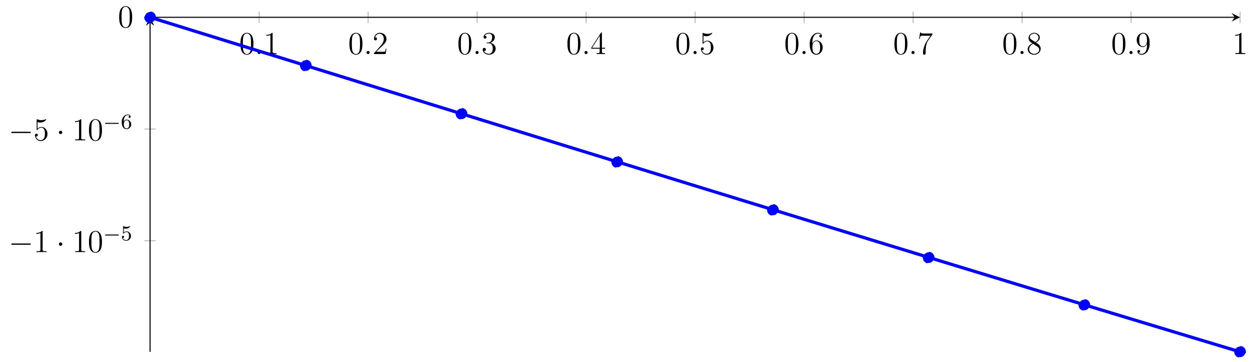
Longest intersection interval: 0.106945

$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.515476]$, interval 2: $[0.680036, 0.706772]$,

15.18 Recursion Branch 1 2 1 1 in Interval 1: $[0.5, 0.515476]$

Normalized monomial und Bézier representations and the Bézier polygon:

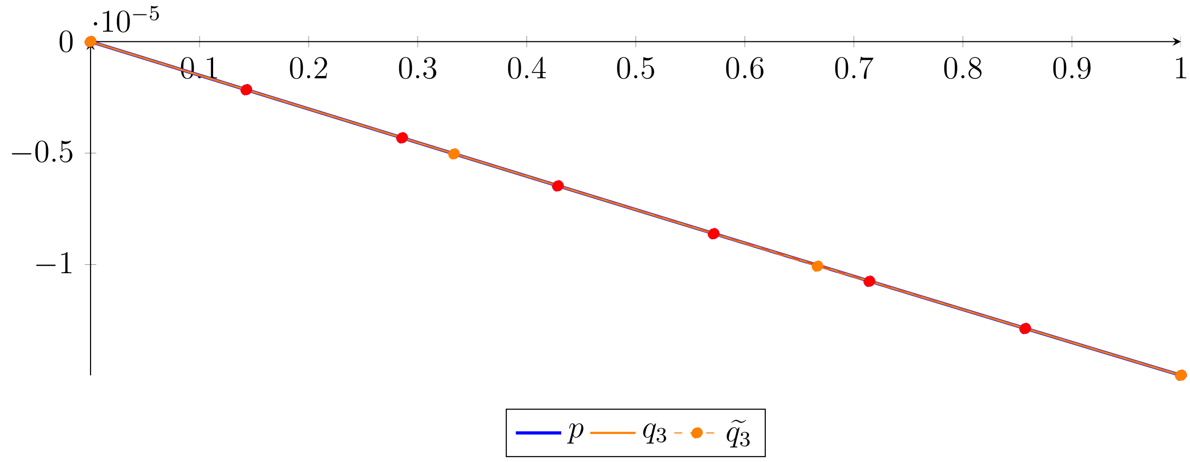
$$\begin{aligned} p &= 2.12634{\cdot}10^{-13}X^7 + 5.03753{\cdot}10^{-22}X^6 - 3.3292{\cdot}10^{-10}X^5 + 2.89513{\cdot}10^{-23}X^4 \\ &\quad + 1.44792{\cdot}10^{-07}X^3 - 2.60562{\cdot}10^{-22}X^2 - 1.51134{\cdot}10^{-05}X - 2.05803{\cdot}10^{-21} \\ &= -2.05803{\cdot}10^{-21}B_{0,7}(X) - 2.15905{\cdot}10^{-06}B_{1,7}(X) - 4.31811{\cdot}10^{-06}B_{2,7}(X) - 6.47303{\cdot}10^{-06}B_{3,7}(X) \\ &\quad - 8.61967{\cdot}10^{-06}B_{4,7}(X) - 1.07539{\cdot}10^{-05}B_{5,7}(X) - 1.28717{\cdot}10^{-05}B_{6,7}(X) - 1.49689{\cdot}10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.43868{\cdot}10^{-07}X^3 + 7.91854{\cdot}10^{-10}X^2 - 1.51136{\cdot}10^{-05}X + 1.0556{\cdot}10^{-11} \\ &= 1.0556{\cdot}10^{-11}B_{0,3} - 5.03785{\cdot}10^{-06}B_{1,3} - 1.00754{\cdot}10^{-05}B_{2,3} - 1.49689{\cdot}10^{-05}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.48671{\cdot}10^{-18}X^7 - 8.87719{\cdot}10^{-18}X^6 + 1.25885{\cdot}10^{-17}X^5 - 9.01236{\cdot}10^{-18}X^4 \\ &\quad + 1.43868{\cdot}10^{-07}X^3 + 7.91854{\cdot}10^{-10}X^2 - 1.51136{\cdot}10^{-05}X + 1.0556{\cdot}10^{-11} \\ &= 1.0556{\cdot}10^{-11}B_{0,7} - 2.15907{\cdot}10^{-06}B_{1,7} - 4.31812{\cdot}10^{-06}B_{2,7} - 6.47302{\cdot}10^{-06}B_{3,7} \\ &\quad - 8.61965{\cdot}10^{-06}B_{4,7} - 1.07539{\cdot}10^{-05}B_{5,7} - 1.28717{\cdot}10^{-05}B_{6,7} - 1.49689{\cdot}10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.63707{\cdot}10^{-11}$.

Bounding polynomials M and m:

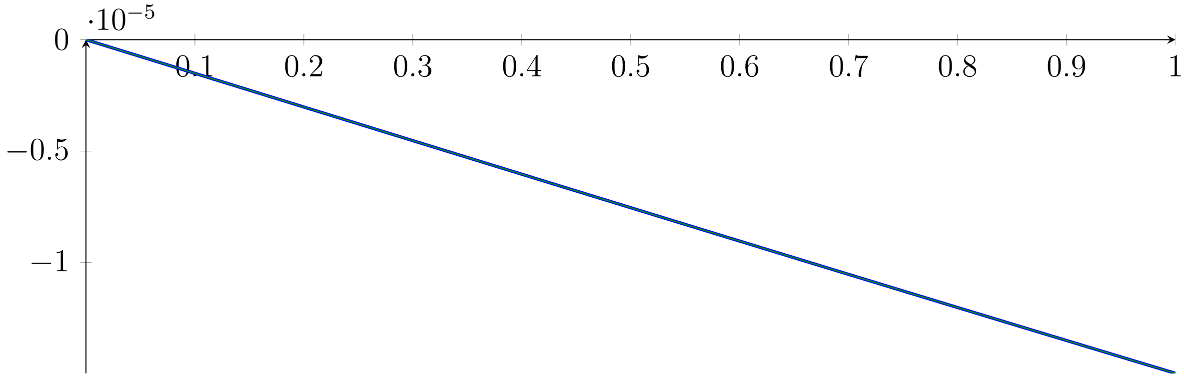
$$M = 1.43868{\cdot}10^{-07}X^3 + 7.91854{\cdot}10^{-10}X^2 - 1.51136{\cdot}10^{-05}X + 3.69267{\cdot}10^{-11}$$

$$m = 1.43868{\cdot}10^{-07}X^3 + 7.91854{\cdot}10^{-10}X^2 - 1.51136{\cdot}10^{-05}X - 1.58147{\cdot}10^{-11}$$

Root of M and m:

$$N(M) = \{-10.2522, 2.44328{\cdot}10^{-06}, 10.2467\} \qquad N(m) = \{-10.2522, -1.04639{\cdot}10^{-06}, 10.2467\}$$

Intersection intervals:



$[0, 2.44328e-06]$

Longest intersection interval: $2.44328{\cdot}10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.5]$,

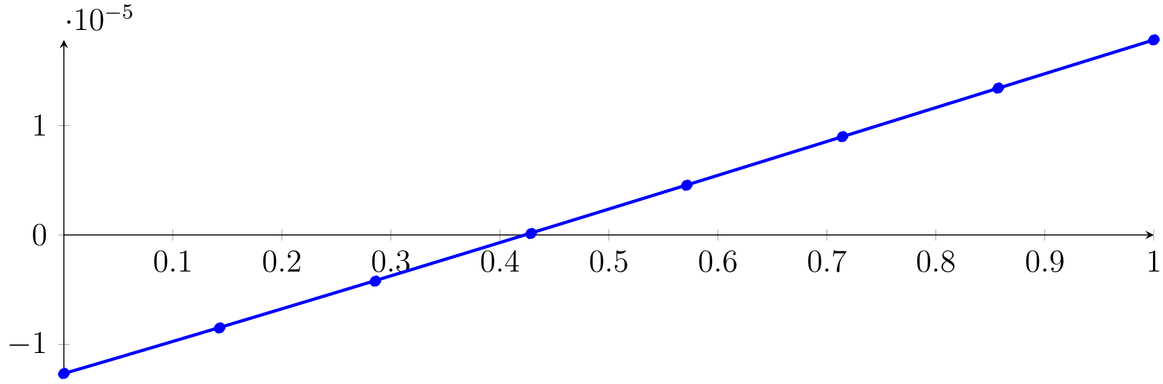
15.19 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.5, 0.5]$

Found root in interval $[0.5, 0.5]$ at recursion depth 5!

15.20 Recursion Branch 1 2 1 2 in Interval 2: $[0.680036, 0.706772]$

Normalized monomial und Bézier representations and the Bézier polygon:

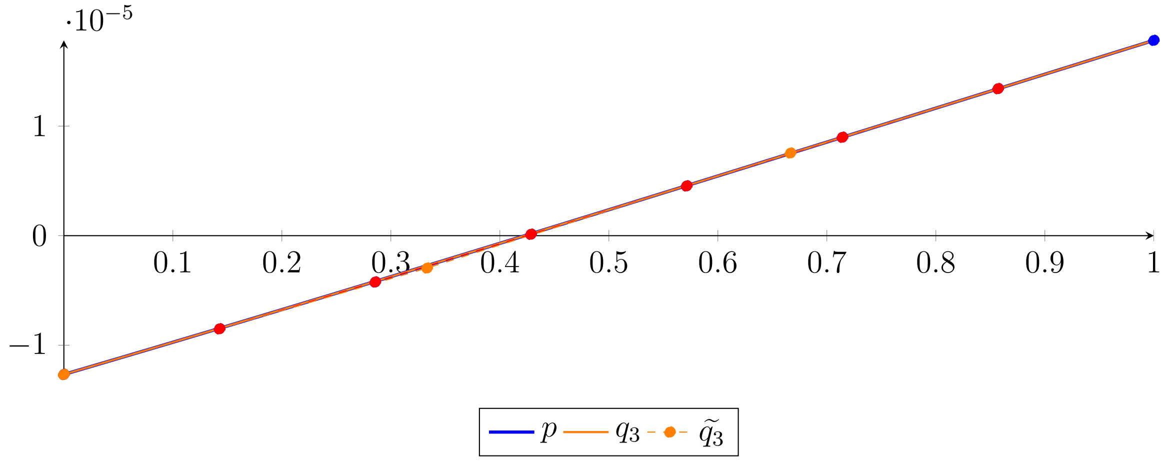
$$\begin{aligned} p &= 9.76586{\cdot}10^{-12}X^7 + 4.60326{\cdot}10^{-10}X^6 + 4.17599{\cdot}10^{-09}X^5 - 6.8127{\cdot}10^{-08}X^4 \\ &\quad - 8.737{\cdot}10^{-07}X^3 + 2.27814{\cdot}10^{-06}X^2 + 2.91513{\cdot}10^{-05}X - 1.26665{\cdot}10^{-05} \\ &= -1.26665{\cdot}10^{-05}B_{0,7}(X) - 8.50206{\cdot}10^{-06}B_{1,7}(X) - 4.22911{\cdot}10^{-06}B_{2,7}(X) + 1.27359{\cdot}10^{-07}B_{3,7}(X) \\ &\quad + 4.54044{\cdot}10^{-06}B_{4,7}(X) + 8.98147{\cdot}10^{-06}B_{5,7}(X) + 1.34203{\cdot}10^{-05}B_{6,7}(X) + 1.78257{\cdot}10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.96783{\cdot}10^{-07}X^3 + 2.35427{\cdot}10^{-06}X^2 + 2.91347{\cdot}10^{-05}X - 1.26657{\cdot}10^{-05} \\ &= -1.26657{\cdot}10^{-05}B_{0,3} - 2.95414{\cdot}10^{-06}B_{1,3} + 7.54218{\cdot}10^{-06}B_{2,3} + 1.78265{\cdot}10^{-05}B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.79626{\cdot}10^{-18}X^7 + 9.90508{\cdot}10^{-18}X^6 - 1.38716{\cdot}10^{-17}X^5 + 9.73966{\cdot}10^{-18}X^4 \\ &\quad - 9.96783{\cdot}10^{-07}X^3 + 2.35427{\cdot}10^{-06}X^2 + 2.91347{\cdot}10^{-05}X - 1.26657{\cdot}10^{-05} \\ &= -1.26657{\cdot}10^{-05}B_{0,7} - 8.50361{\cdot}10^{-06}B_{1,7} - 4.2294{\cdot}10^{-06}B_{2,7} + 1.28434{\cdot}10^{-07}B_{3,7} \\ &\quad + 4.54142{\cdot}10^{-06}B_{4,7} + 8.98107{\cdot}10^{-06}B_{5,7} + 1.34189{\cdot}10^{-05}B_{6,7} + 1.78265{\cdot}10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.54959{\cdot}10^{-09}$.

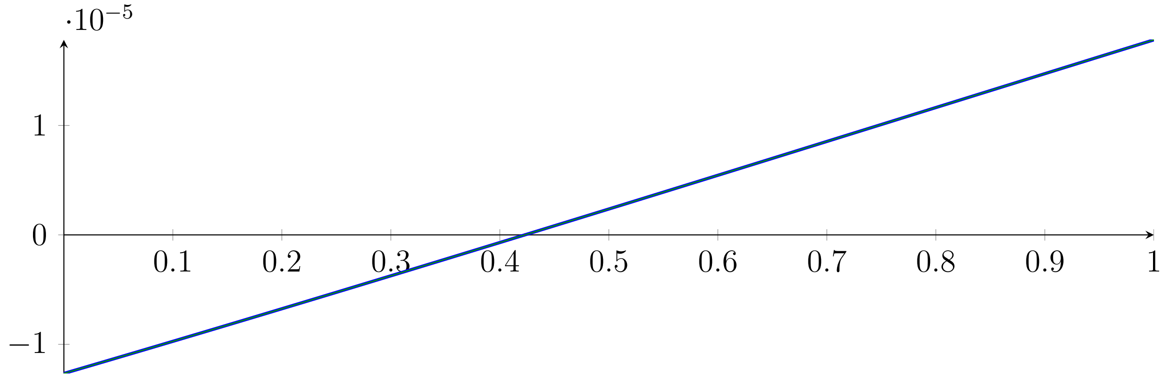
Bounding polynomials M and m:

$$M = -9.96783{\cdot}10^{-07}X^3 + 2.35427{\cdot}10^{-06}X^2 + 2.91347{\cdot}10^{-05}X - 1.26642{\cdot}10^{-05}$$
$$m = -9.96783{\cdot}10^{-07}X^3 + 2.35427{\cdot}10^{-06}X^2 + 2.91347{\cdot}10^{-05}X - 1.26673{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{-4.59721, 0.422816, 6.53626\} \qquad N(m) = \{-4.59727, 0.422918, 6.53622\}$$

Intersection intervals:



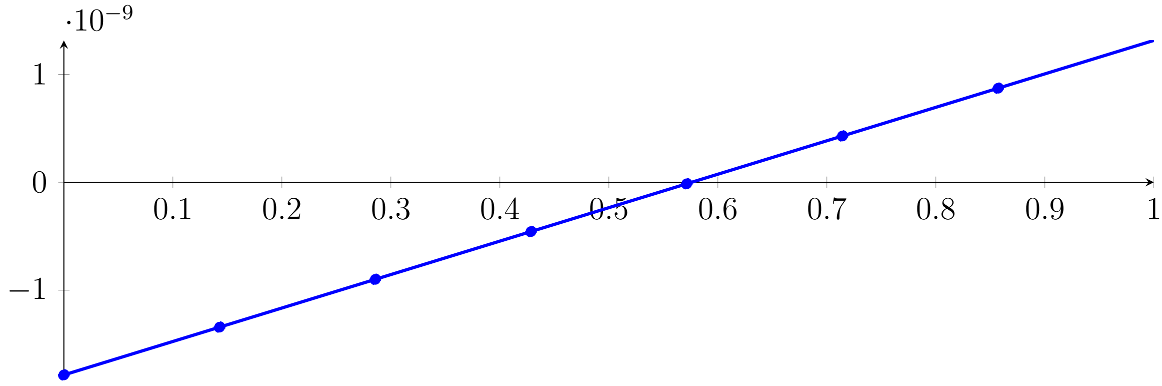
$$[0.422816, 0.422918]$$

Longest intersection interval: 0.00010131
$\Longrightarrow$ Selective recursion: interval 1: $[0.69134, 0.691343]$,

15.21 Recursion Branch 1 2 1 2 1 in Interval 1: $[0.69134, 0.691343]$

Normalized monomial und Bézier representations and the Bézier polygon:

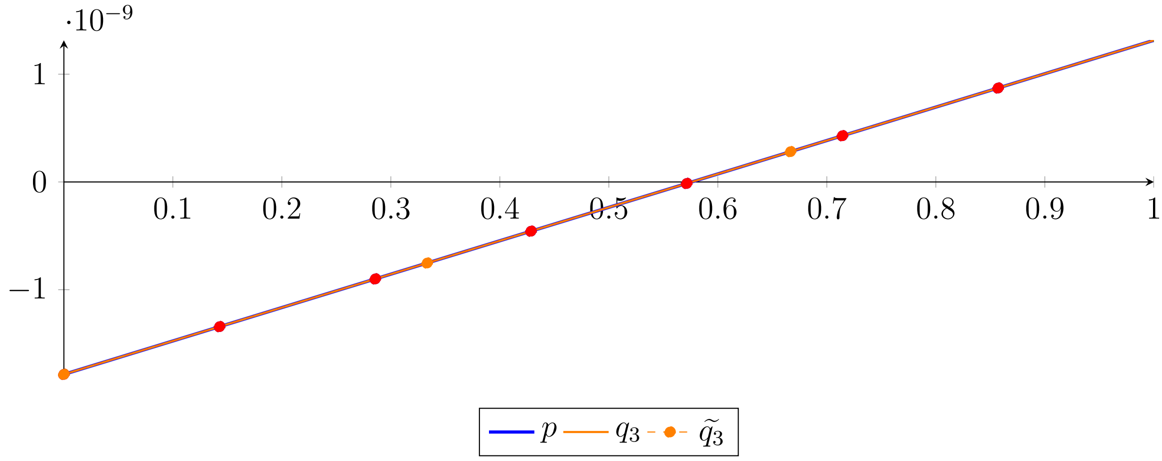
$$\begin{aligned} p = {} & 5.65455{\cdot}10^{-27}X^7 + 2.82728{\cdot}10^{-26}X^6 + 2.54455{\cdot}10^{-26}X^5 - 6.12106{\cdot}10^{-24}X^4 \\ & - 1.01979{\cdot}10^{-18}X^3 + 1.12921{\cdot}10^{-14}X^2 + 3.09899{\cdot}10^{-09}X - 1.78489{\cdot}10^{-09} \\ = {} & -1.78489{\cdot}10^{-09}B_{0,7}(X) - 1.34218{\cdot}10^{-09}B_{1,7}(X) - 8.99462{\cdot}10^{-10}B_{2,7}(X) - 4.56748{\cdot}10^{-10}B_{3,7}(X) \\ & - 1.40329{\cdot}10^{-11}B_{4,7}(X) + 4.28683{\cdot}10^{-10}B_{5,7}(X) + 8.71399{\cdot}10^{-10}B_{6,7}(X) + 1.31412{\cdot}10^{-09}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_3 &= -1.01981\cdot 10^{-18}X^3 + 1.12921\cdot 10^{-14}X^2 + 3.09899\cdot 10^{-09}X - 1.78489\cdot 10^{-09} \\
&= -1.78489\cdot 10^{-09}B_{0,3} - 7.51892\cdot 10^{-10}B_{1,3} + 2.8111\cdot 10^{-10}B_{2,3} + 1.31412\cdot 10^{-09}B_{3,3}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_3 &= -2.00522\cdot 10^{-22}X^7 + 7.05596\cdot 10^{-22}X^6 - 9.77135\cdot 10^{-22}X^5 + 6.73652\cdot 10^{-22}X^4 \\
&\quad - 1.02005\cdot 10^{-18}X^3 + 1.12921\cdot 10^{-14}X^2 + 3.09899\cdot 10^{-09}X - 1.78489\cdot 10^{-09} \\
&= -1.78489\cdot 10^{-09}B_{0,7} - 1.34218\cdot 10^{-09}B_{1,7} - 8.99462\cdot 10^{-10}B_{2,7} - 4.56748\cdot 10^{-10}B_{3,7} \\
&\quad - 1.40329\cdot 10^{-11}B_{4,7} + 4.28683\cdot 10^{-10}B_{5,7} + 8.71399\cdot 10^{-10}B_{6,7} + 1.31412\cdot 10^{-09}B_{7,7}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 2.26513\cdot 10^{-24}$.

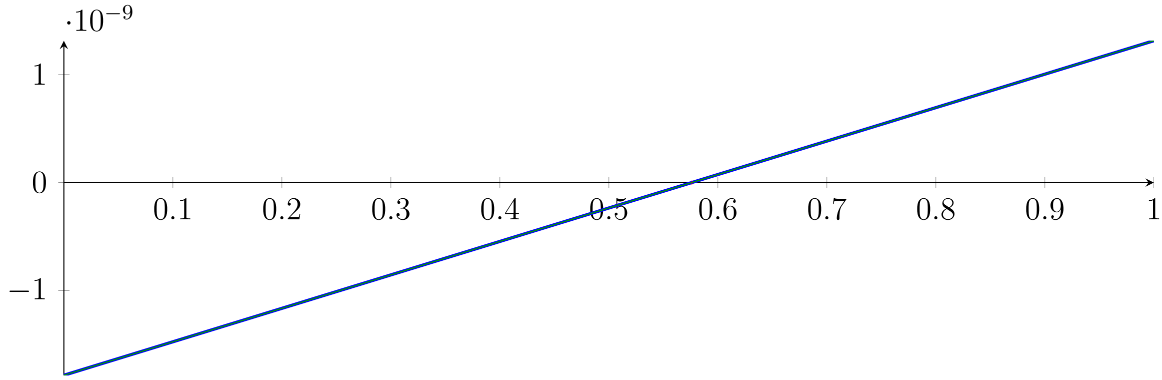
Bounding polynomials M and m:

$$
\begin{aligned}
M &= -1.01981\cdot 10^{-18}X^3 + 1.12921\cdot 10^{-14}X^2 + 3.09899\cdot 10^{-09}X - 1.78489\cdot 10^{-09} \\
m &= -1.01981\cdot 10^{-18}X^3 + 1.12921\cdot 10^{-14}X^2 + 3.09899\cdot 10^{-09}X - 1.78489\cdot 10^{-09}
\end{aligned}
$$

Root of M and m:

$$N(M) = \{-49866.6, 0.575957, 60938.8\} \qquad N(m) = \{-49866.6, 0.575957, 60938.8\}$$

Intersection intervals:



$$[0.575957, 0.575957]$$

Longest intersection interval: $9.10383 \cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.691342, 0.691342]$,

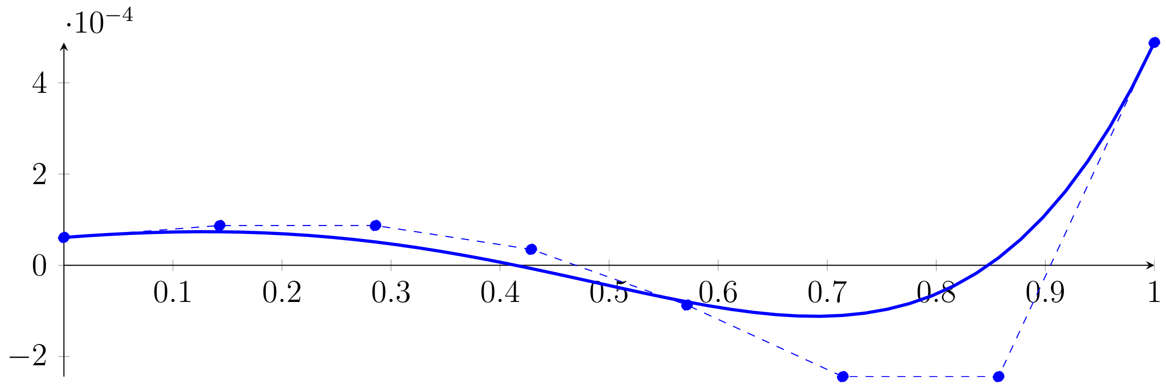
15.22 Recursion Branch 1 2 1 2 1 1 in Interval 1: $[0.691342, 0.691342]$

Found root in interval $[0.691342, 0.691342]$ at recursion depth 6!

15.23 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

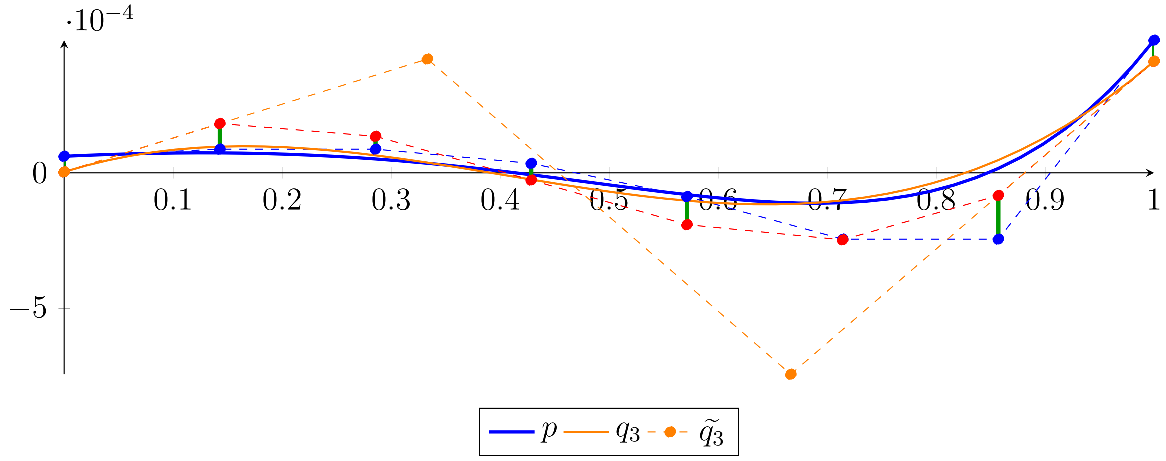
$$\begin{aligned}p &= 6.10352 \cdot 10^{-05} X^7 + 0.000427246 X^6 + 0.000915527 X^5 + 0.000305176 X^4 \\ &\quad - 0.000915527 X^3 - 0.000549316 X^2 + 0.000183105 X + 6.10352 \cdot 10^{-05} \\ &= 6.10352 \cdot 10^{-05} B_{0,7}(X) + 8.71931 \cdot 10^{-05} B_{1,7}(X) + 8.71931 \cdot 10^{-05} B_{2,7}(X) + 3.48772 \cdot 10^{-05} B_{3,7}(X) \\ &\quad - 8.71931 \cdot 10^{-05} B_{4,7}(X) - 0.000244141 B_{5,7}(X) - 0.000244141 B_{6,7}(X) + 0.000488281 B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 0.00388868 X^3 - 0.00472785 X^2 + 0.0012462 X + 3.56699 \cdot 10^{-06} \\ &= 3.56699 \cdot 10^{-06} B_{0,3} + 0.000418967 B_{1,3} - 0.000741582 B_{2,3} + 0.0004106 B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= -3.42349 \cdot 10^{-17} X^7 + 1.15416 \cdot 10^{-16} X^6 - 1.53874 \cdot 10^{-16} X^5 + 1.02775 \\ &\quad \cdot 10^{-16} X^4 + 0.00388868 X^3 - 0.00472785 X^2 + 0.0012462 X + 3.56699 \cdot 10^{-06} \\ &= 3.56699 \cdot 10^{-06} B_{0,7} + 0.000181596 B_{1,7} + 0.000134489 B_{2,7} - 2.66486 \cdot 10^{-05} B_{3,7} \\ &\quad - 0.000190711 B_{4,7} - 0.000246594 B_{5,7} - 8.3192 \cdot 10^{-05} B_{6,7} + 0.0004106 B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.000160949$.

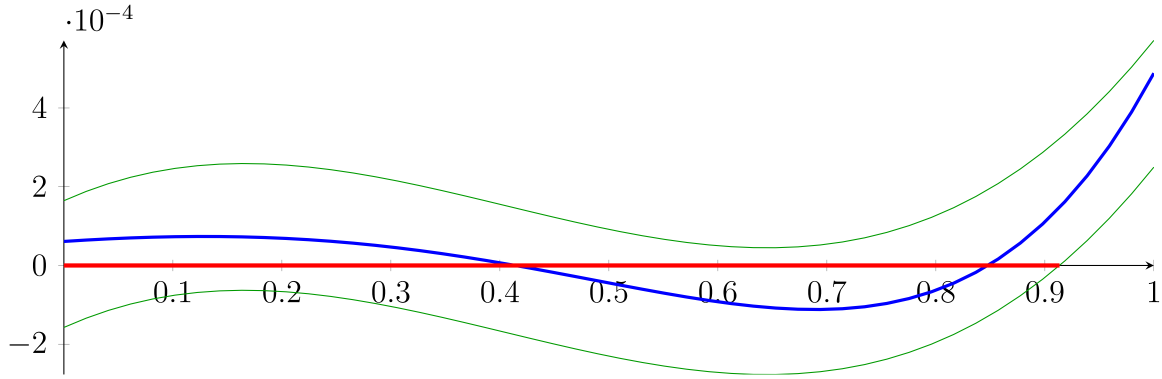
Bounding polynomials M and m:

$$M = 0.00388868X^3 - 0.00472785X^2 + 0.0012462X + 0.000164516$$
$$m = 0.00388868X^3 - 0.00472785X^2 + 0.0012462X - 0.000157382$$

Root of M and m:

$$N(M) = \{-0.095055\} \qquad N(m) = \{0.913476\}$$

Intersection intervals:



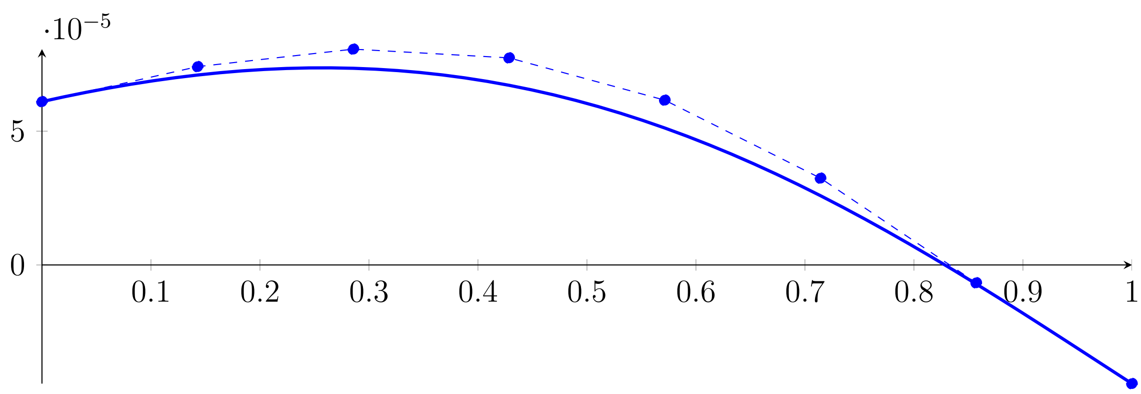
$$[0, 0.913476]$$

Longest intersection interval: 0.913476
$\Longrightarrow$ Bisection: first half $[0.75, 0.875]$ und second half $[0.875, 1]$
Bisection point is very near to a root?!?

15.24 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

Normalized monomial und Bézier representations and the Bézier polygon:

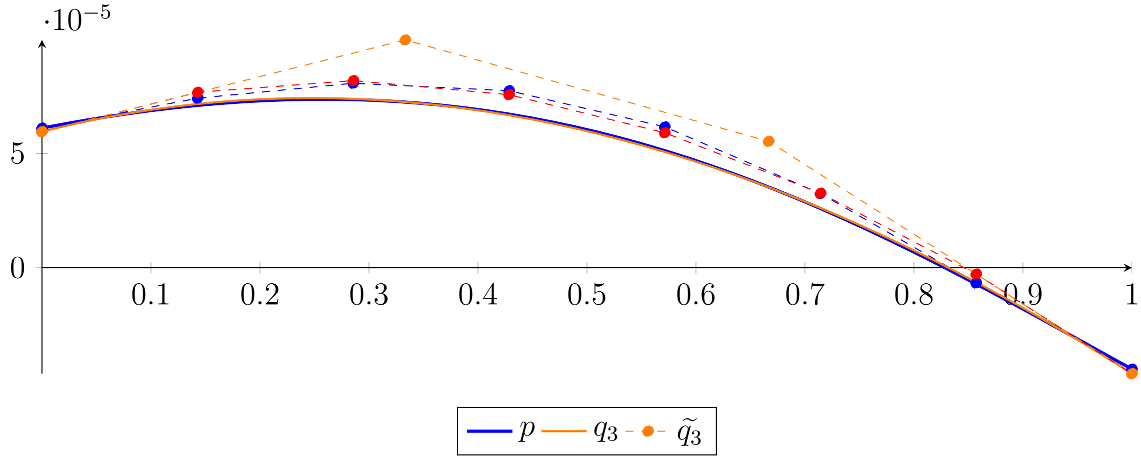
$$\begin{aligned} p &= 4.76837\cdot 10^{-07}X^7 + 6.67572\cdot 10^{-06}X^6 + 2.86102\cdot 10^{-05}X^5 + 1.90735\cdot 10^{-05}X^4 \\ &\quad - 0.000114441X^3 - 0.000137329X^2 + 9.15527\cdot 10^{-05}X + 6.10352\cdot 10^{-05} \\ &= 6.10352\cdot 10^{-05}B_{0,7}(X) + 7.41141\cdot 10^{-05}B_{1,7}(X) + 8.06536\cdot 10^{-05}B_{2,7}(X) + 7.73839\cdot 10^{-05}B_{3,7}(X) \\ &\quad + 6.15801\cdot 10^{-05}B_{4,7}(X) + 3.24249\cdot 10^{-05}B_{5,7}(X) - 6.67572\cdot 10^{-06}B_{6,7}(X) - 4.43459\cdot 10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.72014\cdot 10^{-05}X^3 - 0.00025325X^2 + 0.00012026X + 5.95076\cdot 10^{-05} \\ &= 5.95076\cdot 10^{-05}B_{0,3} + 9.95943\cdot 10^{-05}B_{1,3} + 5.52643\cdot 10^{-05}B_{2,3} - 4.62811\cdot 10^{-05}B_{3,3} \end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= -9.85586\cdot 10^{-19}X^7 + 4.89039\cdot 10^{-18}X^6 - 9.49606\cdot 10^{-18}X^5 + 9.26662\cdot 10^{-18}X^4 \\ &\quad + 2.72014\cdot 10^{-05}X^3 - 0.00025325X^2 + 0.00012026X + 5.95076\cdot 10^{-05} \\ &= 5.95076\cdot 10^{-05}B_{0,7} + 7.66876\cdot 10^{-05}B_{1,7} + 8.18081\cdot 10^{-05}B_{2,7} + 7.56462\cdot 10^{-05}B_{3,7} \\ &\quad + 5.89792\cdot 10^{-05}B_{4,7} + 3.25842\cdot 10^{-05}B_{5,7} - 2.76165\cdot 10^{-06}B_{6,7} - 4.62811\cdot 10^{-05}B_{7,7}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.91408\cdot 10^{-06}$.

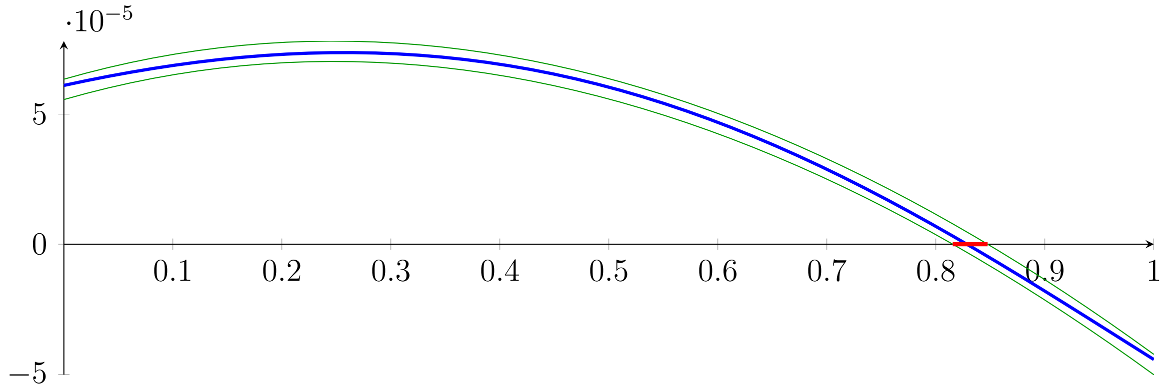
Bounding polynomials M and m:

$$M = 2.72014\cdot 10^{-05}X^3 - 0.00025325X^2 + 0.00012026X + 6.34217\cdot 10^{-05}$$
$$m = 2.72014\cdot 10^{-05}X^3 - 0.00025325X^2 + 0.00012026X + 5.55936\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{-0.313472, 0.847507, 8.77615\} \qquad N(m) = \{-0.28544, 0.815485, 8.78014\}$$

Intersection intervals:



$$[0.815485, 0.847507]$$

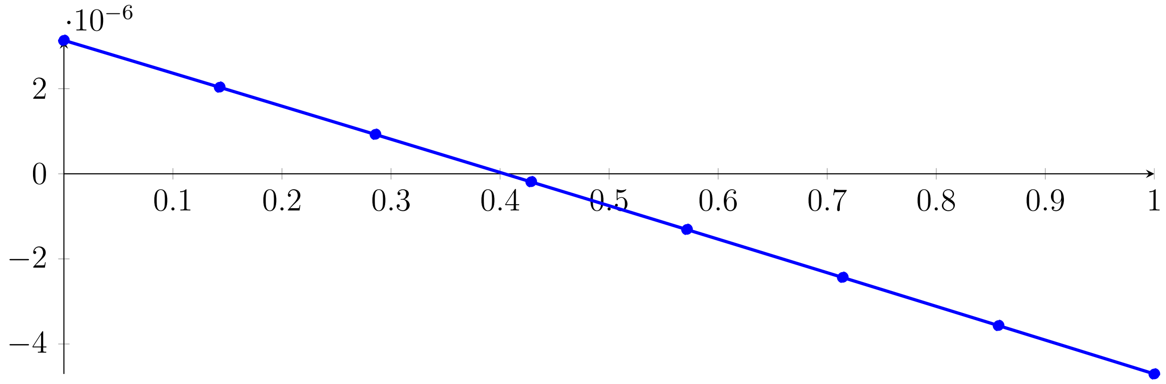
Longest intersection interval: 0.0320214

$\Longrightarrow$ Selective recursion: interval 1: $[0.851936, 0.855938]$,

15.25 Recursion Branch 1 2 2 1 1 in Interval 1: $[0.851936, 0.855938]$

Normalized monomial und Bézier representations and the Bézier polygon:

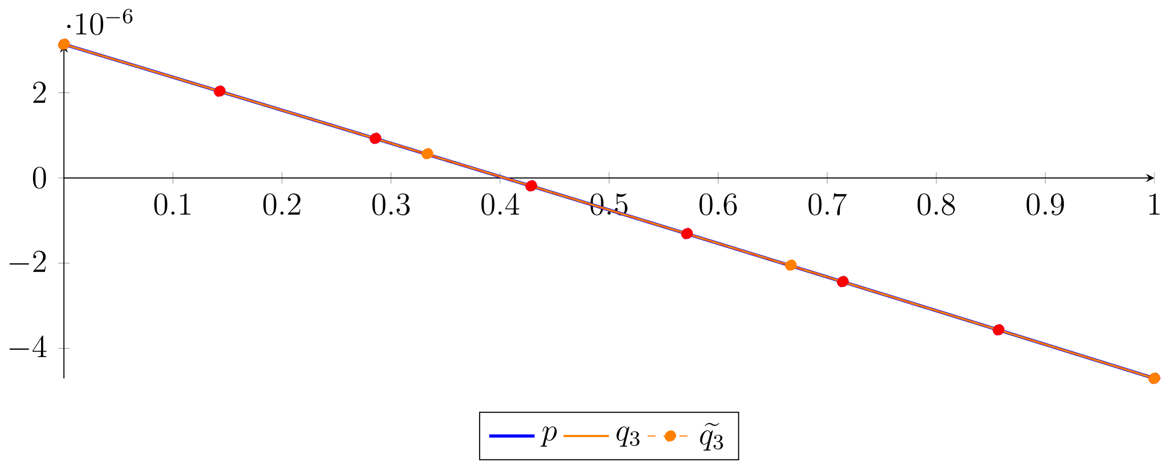
$$\begin{aligned}p &= 1.6461\cdot 10^{-17}X^7 + 1.01313\cdot 10^{-14}X^6 + 2.2871\cdot 10^{-12}X^5 + 2.22234\cdot 10^{-10}X^4 \\ &\quad + 7.15206\cdot 10^{-09}X^3 - 1.41651\cdot 10^{-07}X^2 - 7.7068\cdot 10^{-06}X + 3.13746\cdot 10^{-06} \\ &= 3.13746\cdot 10^{-06}B_{0,7}(X) + 2.03648\cdot 10^{-06}B_{1,7}(X) + 9.28769\cdot 10^{-07}B_{2,7}(X) - 1.85489\cdot 10^{-07}B_{3,7}(X) \\ &\quad - 1.30608\cdot 10^{-06}B_{4,7}(X) - 2.43278\cdot 10^{-06}B_{5,7}(X) - 3.56537\cdot 10^{-06}B_{6,7}(X) - 4.70362\cdot 10^{-06}B_{7,7}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 7.60291{\cdot}10^{-09}X^3 - 1.41943{\cdot}10^{-07}X^2 - 7.70673{\cdot}10^{-06}X + 3.13745{\cdot}10^{-06} \\ &= 3.13745{\cdot}10^{-06}B_{0,3} + 5.68542{\cdot}10^{-07}B_{1,3} - 2.04768{\cdot}10^{-06}B_{2,3} - 4.70362{\cdot}10^{-06}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 7.43917{\cdot}10^{-19}X^7 - 2.63686{\cdot}10^{-18}X^6 + 3.69664{\cdot}10^{-18}X^5 - 2.59971{\cdot}10^{-18}X^4 \\ &\quad + 7.60291{\cdot}10^{-09}X^3 - 1.41943{\cdot}10^{-07}X^2 - 7.70673{\cdot}10^{-06}X + 3.13745{\cdot}10^{-06} \\ &= 3.13745{\cdot}10^{-06}B_{0,7} + 2.03649{\cdot}10^{-06}B_{1,7} + 9.2877{\cdot}10^{-07}B_{2,7} - 1.85493{\cdot}10^{-07}B_{3,7} \\ &\quad - 1.30608{\cdot}10^{-06}B_{4,7} - 2.43278{\cdot}10^{-06}B_{5,7} - 3.56536{\cdot}10^{-06}B_{6,7} - 4.70362{\cdot}10^{-06}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.07909{\cdot}10^{-12}$.

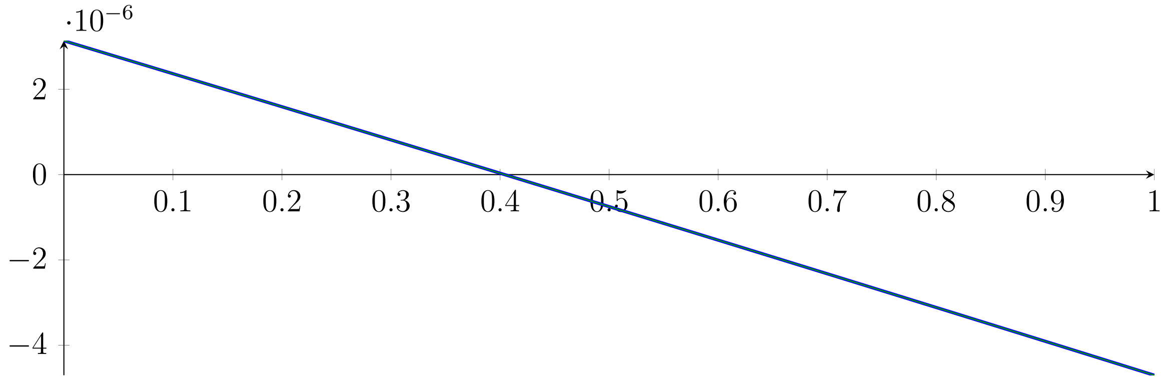
Bounding polynomials M and m:

$$\begin{aligned} M &= 7.60291{\cdot}10^{-09}X^3 - 1.41943{\cdot}10^{-07}X^2 - 7.70673{\cdot}10^{-06}X + 3.13746{\cdot}10^{-06} \\ m &= 7.60291{\cdot}10^{-09}X^3 - 1.41943{\cdot}10^{-07}X^2 - 7.70673{\cdot}10^{-06}X + 3.13745{\cdot}10^{-06} \end{aligned}$$

Root of M and m:

$$N(M) = \{-24.1005, 0.404163, 42.3658\} \qquad N(m) = \{-24.1005, 0.404161, 42.3658\}$$

Intersection intervals:



$$[0.404161, 0.404163]$$

Longest intersection interval: $1.5552 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.853553, 0.853553]$,

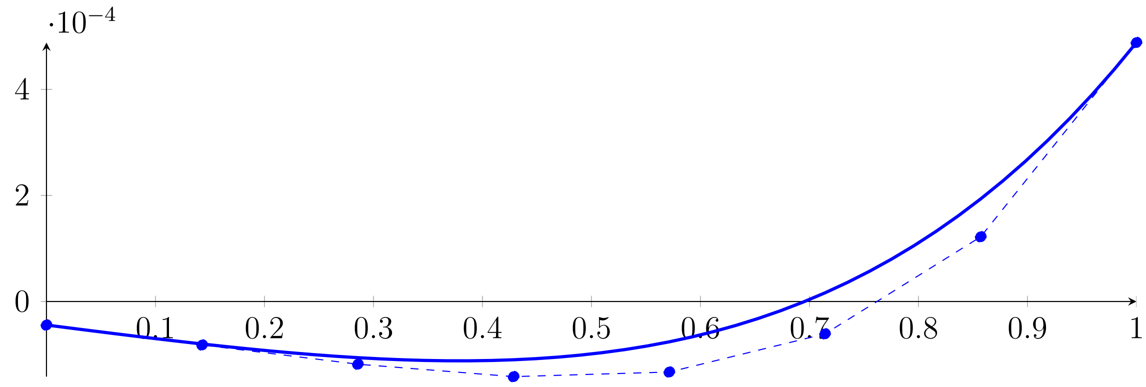
15.26 Recursion Branch 1 2 2 1 1 1 in Interval 1: $[0.853553, 0.853553]$

Found root in interval $[0.853553, 0.853553]$ at recursion depth 6!

15.27 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

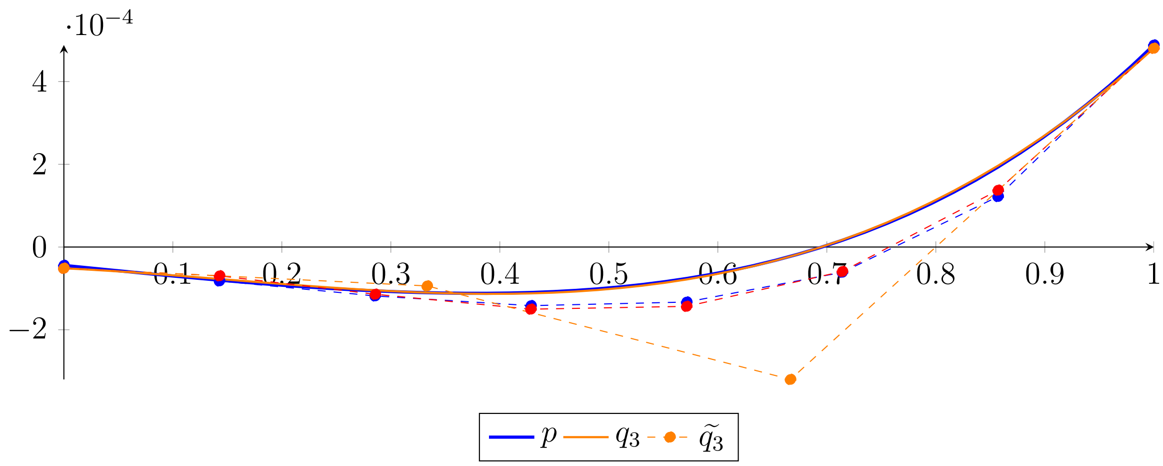
$$\begin{aligned} p &= 4.76837 \cdot 10^{-07} X^7 + 1.00136 \cdot 10^{-05} X^6 + 7.86781 \cdot 10^{-05} X^5 + 0.00027895 X^4 \\ &\quad + 0.000398159 X^3 + 3.00407 \cdot 10^{-05} X^2 - 0.000263691 X - 4.43459 \cdot 10^{-05} \\ &= -4.43459 \cdot 10^{-05} B_{0,7}(X) - 8.2016 \cdot 10^{-05} B_{1,7}(X) - 0.000118256 B_{2,7}(X) - 0.000141689 B_{3,7}(X) \\ &\quad - 0.000132969 B_{4,7}(X) - 6.10352 \cdot 10^{-05} B_{5,7}(X) + 0.00012207 B_{6,7}(X) + 0.000488281 B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 0.00120976 X^3 - 0.000549945 X^2 - 0.00012807 X - 5.13343 \cdot 10^{-05} \\ &= -5.13343 \cdot 10^{-05} B_{0,3} - 9.40243 \cdot 10^{-05} B_{1,3} - 0.000320029 B_{2,3} + 0.000480408 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -3.37688 \cdot 10^{-17} X^7 + 1.13604 \cdot 10^{-16} X^6 - 1.49587 \cdot 10^{-16} X^5 + 9.70744 \\ &\quad \cdot 10^{-17} X^4 + 0.00120976 X^3 - 0.000549945 X^2 - 0.00012807 X - 5.13343 \cdot 10^{-05} \\ &= -5.13343 \cdot 10^{-05} B_{0,7} - 6.963 \cdot 10^{-05} B_{1,7} - 0.000114114 B_{2,7} - 0.000150221 B_{3,7} \\ &\quad - 0.000143386 B_{4,7} - 5.90464 \cdot 10^{-05} B_{5,7} + 0.000137364 B_{6,7} + 0.000480408 B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.52933{\cdot}10^{-05}$.

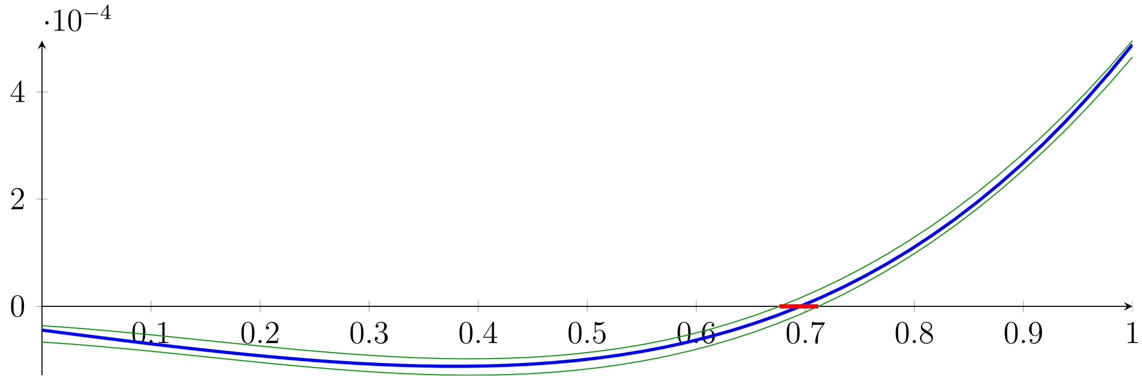
Bounding polynomials M and m:

$$M = 0.00120976X^3 - 0.000549945X^2 - 0.00012807X - 3.6041{\cdot}10^{-05}$$
$$m = 0.00120976X^3 - 0.000549945X^2 - 0.00012807X - 6.66276{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.676273\} \qquad N(m) = \{0.711946\}$$

Intersection intervals:



$$[0.676273, 0.711946]$$

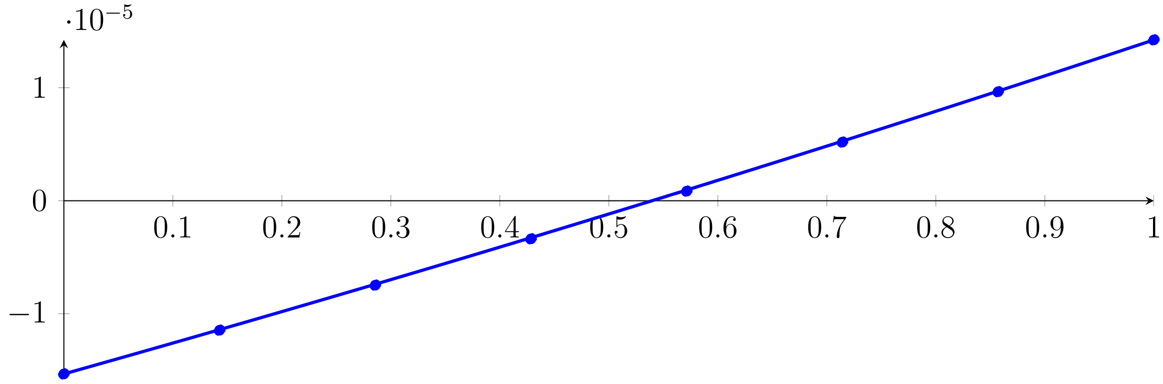
Longest intersection interval: 0.0356732

$\Longrightarrow$ Selective recursion: interval 1: $[0.959534, 0.963993]$,

15.28 Recursion Branch 1 2 2 2 1 in Interval 1: $[0.959534, 0.963993]$

Normalized monomial und Bézier representations and the Bézier polygon:

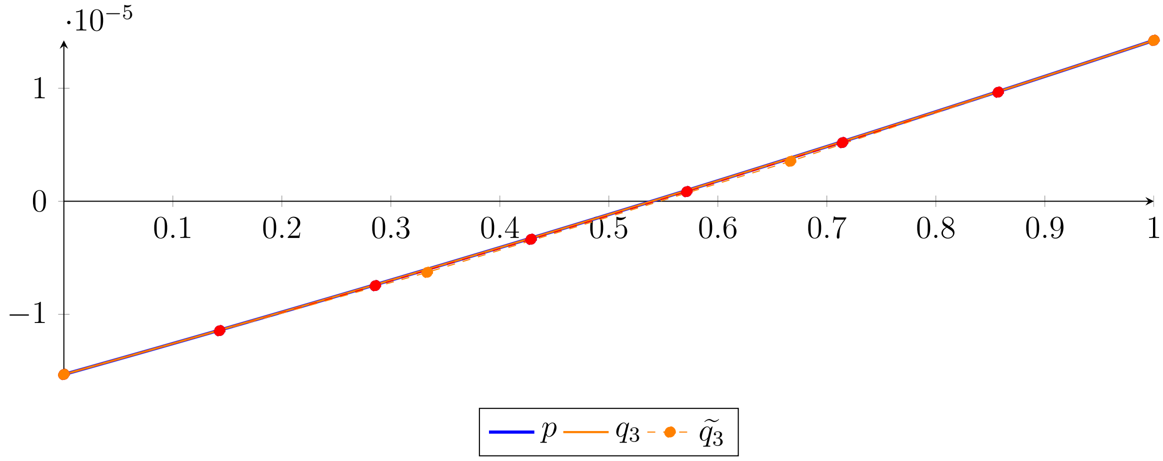
$$\begin{aligned} p &= 3.50563{\cdot}10^{-17}X^7 + 2.52889{\cdot}10^{-14}X^6 + 7.15723{\cdot}10^{-12}X^5 + 1.00219{\cdot}10^{-09}X^4 \\ &\quad + 7.16367{\cdot}10^{-08}X^3 + 2.39177{\cdot}10^{-06}X^2 + 2.70915{\cdot}10^{-05}X - 1.5323{\cdot}10^{-05} \\ &= -1.5323{\cdot}10^{-05}B_{0,7}(X) - 1.14528{\cdot}10^{-05}B_{1,7}(X) - 7.46873{\cdot}10^{-06}B_{2,7}(X) - 3.36868{\cdot}10^{-06}B_{3,7}(X) \\ &\quad + 8.4938{\cdot}10^{-07}B_{4,7}(X) + 5.18756{\cdot}10^{-06}B_{5,7}(X) + 9.648{\cdot}10^{-06}B_{6,7}(X) + 1.42329{\cdot}10^{-05}B_{7,7}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 7.3661{\cdot}10^{-08}X^3 + 2.39046{\cdot}10^{-06}X^2 + 2.70918{\cdot}10^{-05}X - 1.53231{\cdot}10^{-05} \\ &= -1.53231{\cdot}10^{-05}B_{0,3} - 6.29247{\cdot}10^{-06}B_{1,3} + 3.53494{\cdot}10^{-06}B_{2,3} + 1.42328{\cdot}10^{-05}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.11589{\cdot}10^{-18}X^7 + 7.453{\cdot}10^{-18}X^6 - 1.03462{\cdot}10^{-17}X^5 + 7.16563{\cdot}10^{-18}X^4 \\ &\quad + 7.3661{\cdot}10^{-08}X^3 + 2.39046{\cdot}10^{-06}X^2 + 2.70918{\cdot}10^{-05}X - 1.53231{\cdot}10^{-05} \\ &= -1.53231{\cdot}10^{-05}B_{0,7} - 1.14528{\cdot}10^{-05}B_{1,7} - 7.46872{\cdot}10^{-06}B_{2,7} - 3.3687{\cdot}10^{-06}B_{3,7} \\ &\quad + 8.49361{\cdot}10^{-07}B_{4,7} + 5.18757{\cdot}10^{-06}B_{5,7} + 9.64803{\cdot}10^{-06}B_{6,7} + 1.42328{\cdot}10^{-05}B_{7,7} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.71607{\cdot}10^{-11}$.
Bounding polynomials M and m:

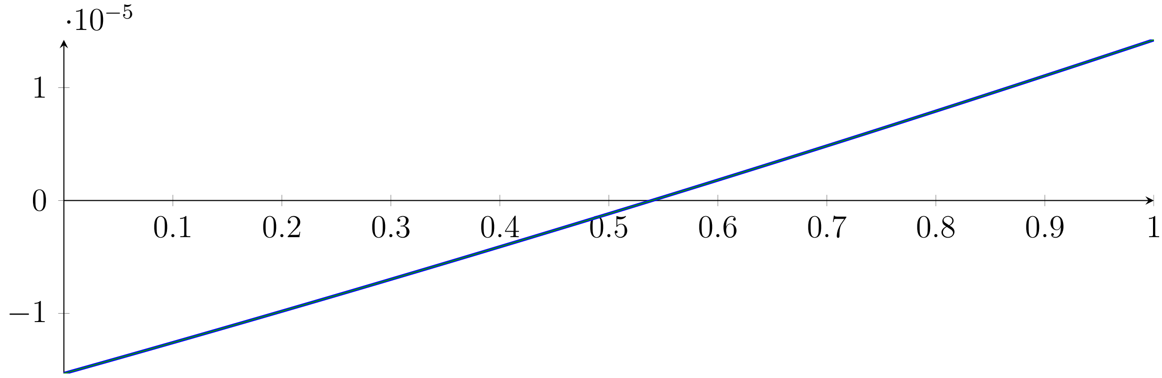
$$M = 7.3661{\cdot}10^{-08}X^3 + 2.39046{\cdot}10^{-06}X^2 + 2.70918{\cdot}10^{-05}X - 1.5323{\cdot}10^{-05}$$

$$m = 7.3661{\cdot}10^{-08}X^3 + 2.39046{\cdot}10^{-06}X^2 + 2.70918{\cdot}10^{-05}X - 1.53231{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.539489\} \qquad N(m) = \{0.539491\}$$

Intersection intervals:



$$[0.539489, 0.539491]$$

Longest intersection interval: $1.82683{\cdot}10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.96194, 0.96194]$,

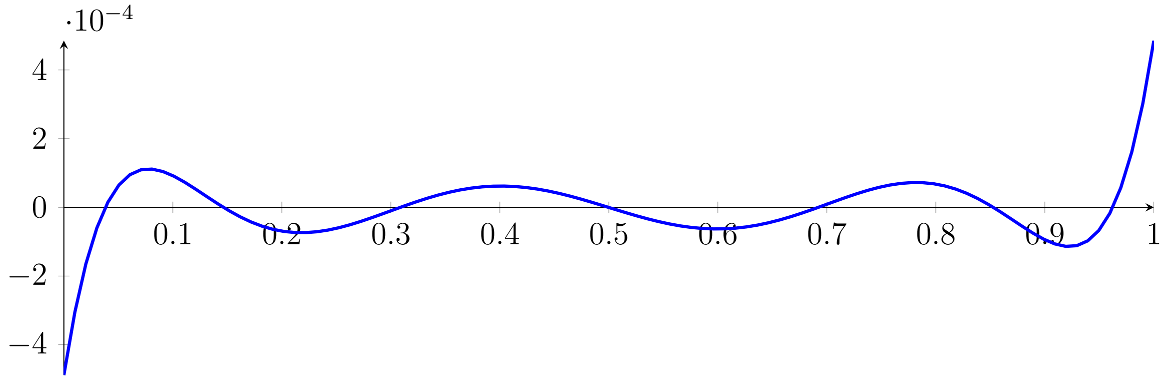
15.29 Recursion Branch 1 2 2 2 1 1 in Interval 1: $[0.96194, 0.96194]$

Found root in interval $[0.96194, 0.96194]$ at recursion depth 6!

15.30 Result: 8 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 1X^7 - 3.5X^6 + 4.875X^5 - 3.4375X^4 + 1.28906X^3 - 0.246094X^2 + 0.0205078X - 0.000488281$$



Result: Root Intervals

$$[0.0380602, 0.0380602], [0.146447, 0.146447], [0.308658, 0.308658], [0.5, 0.5], [0.5, 0.5], [0.691342, 0.691342], [0.853553, 0.853553], [0.96194, 0.96194]$$

with precision $\varepsilon = 1 \cdot 10^{-06}$.