Tail Calls

Class outline:

- Lexical vs. dynamic scopes
- Recursion efficiency
- Tail recursive functions
- Tail call optimization

Scopes

Lexical scope

The standard way in which names are looked up in Scheme and Python.

Lexical (static) scope: The parent of a frame is the frame in which a procedure was defined

```
(define f (lambda (x) (+ x y)))
(define g (lambda (x y) (f (+ x x))))
(g 3 7)
```

Global frame

f	→ λ (x)
g	→ λ (x, y)

f1: g [parent=Global]

x	3
y	7

f2: f [parent=Global]

x	6

What happens when we run this code?

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              g |→      λ (x, y)

f1: g [parent=Global]
              x |3
              y |7

f2: f [parent=Global]
              x |6
```

What happens when we run this code?

Error: unknown identifier: y

Dynamic scope

An alternate approach to scoping supported by some languages.

Dynamic scope: The parent of a frame is the frame in which a procedure was called

Scheme includes the `mu` special form for dynamic scoping.

```
(define f (mu (x) (+ x y)))
(define g (lambda (x y) (f (+ x x))))
(g 3 7)
```

Global frame

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g	→ λ (x, y)

f1: g [parent=Global]

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f1: g [parent=Global]

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f2: f [parent=f1]

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What happens when we run this code?
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Recursion efficiency

Recursion and iteration in Python

Code	Time	Space

`def factorial(n, k):`
`    while n > 0:`
`        n = n -1`
`        k = k * n`
`    return k`		

`def factorial(n, k):`
`    if n == 0:`
`        return k`
`    else:`
`        return factorial(n-1, k*n)`		

Recursion and iteration in Python

Code	Time	Space

`def factorial(n, k):`
`    while n > 0:`
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`        k = k * n`
`    return k`
	Linear	

`def factorial(n, k):`
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Recursion and iteration in Python

<table>
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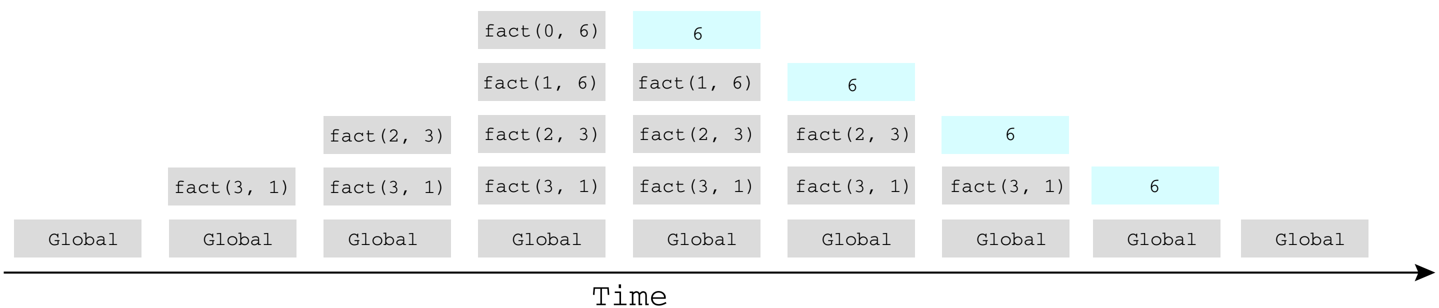
`def factorial(n, k):`
`    if n == 0:`
`        return k`
`    else:`
`        return factorial(n-1, k*n)`
	Linear	Linear

Recursion frames in Python

In Python, recursive calls always create new frames.

```
def factorial(n, k):
    if n == 0:
        return k
    else:
        return factorial(n-1, k*n)
```

Active frames over time:



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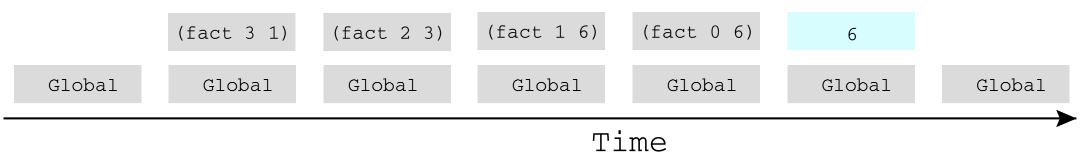
Recursion in Scheme

In Scheme interpreters, a tail-recursive function should only require a **constant** number of active frames.

```
(define (factorial n k)
    (if (= n 0)
        k
        (factorial (- n 1) (* k n))))
```

Active frames over time:



Tail recursive functions

Tail recursive functions

In a **tail recursive function**, every recursive call must be a tail call.

```
(define (factorial n k)
    (if (= n 0)
        k
        (factorial (- n 1) (* k n))))
```

A **tail call** is a call expression in a **tail context**:

- The last body sub-expression in a `lambda` expression
- Sub-expressions 2 & 3 in a tail context `if` expression
- All non-predicate sub-expressions in a tail context `cond`
- The last sub-expression in a tail context `and`, `or`, `begin`, or `let`

Example: Length of list

```
(define (length s)
     (if (null? s) 0
     (+ 1 (length (cdr s)) ) )
```

A call expression is not a tail call if more computation is still required in the calling procedure.

But linear recursive procedures can often be re-written to use tail calls...

Example: Length of list

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(define (length s)
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    (+ 1 (length (cdr s)) ) )
```

A call expression is not a tail call if more computation is still required in the calling procedure.

But linear recursive procedures can often be re-written to use tail calls...

```
(define (length-tail s)
    (define (length-iter s n)
        (if (null? s) n
        (length-iter (cdr s) (+ 1 n)) ) )
    (length-iter s 0) )
```

Is it tail recursive?

```
;; Compute the length of s.
(define (length s)
    (+ 1 (if (null? s)
        -1
        (length (cdr s))) ) )
```

```
;; Return whether s contains v.
(define (contains s v)
    (if (null? s)
    false
    (if (= v (car s))
        true
        (contains (cdr s) v))))
```

Is it tail recursive?

```
;; Compute the length of s.
(define (length s)
    (+ 1 (if (null? s)
        -1
        (length (cdr s))) ) )
```

✖ No, because `if` is not in a tail context.

```
;; Return whether s contains v.
(define (contains s v)
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✖ No, because `if` is not in a tail context.

```
;; Return whether s contains v.
(define (contains s v)
    (if (null? s)
    false
    (if (= v (car s))
        true
        (contains (cdr s) v))))
```

✅ Yes, because `contains` is in a tail context `if`.

Is it tail recursive? 2

```
;; Return whether s has any repeated elements.
(define (has-repeat s)
    (if (null? s)
    false
    (if (contains? (cdr s) (car s))
        true
        (has-repeat (cdr s))) ) )
```

```
;; Return the nth Fibonacci number.
(define (fib n)
    (define (fib-iter current k)
        (if (= k n)
            current
            (fib-iter (+ current
                            (fib (- k 1)))
                         (+ k 1)) ) )
    (if (= 1 n) 0 (fib-iter 1 2)))
```

Is it tail recursive? 2

```
;; Return whether s has any repeated elements.
(define (has-repeat s)
    (if (null? s)
    false
    (if (contains? (cdr s) (car s))
        true
        (has-repeat (cdr s))) ) )
```

✅ Yes, because `has-repeat` is in a tail context.

```
;; Return the nth Fibonacci number.
(define (fib n)
    (define (fib-iter current k)
        (if (= k n)
            current
            (fib-iter (+ current
                         (fib (- k 1)))
                      (+ k 1)) ) )
    (if (= 1 n) 0 (fib-iter 1 2)))
```

Is it tail recursive? 2

```
;; Return whether s has any repeated elements.
(define (has-repeat s)
    (if (null? s)
    false
    (if (contains? (cdr s) (car s))
        true
        (has-repeat (cdr s))) ) )
```

✅ Yes, because `has-repeat` is in a tail context.

```
;; Return the nth Fibonacci number.
(define (fib n)
    (define (fib-iter current k)
        (if (= k n)
            current
            (fib-iter (+ current
                            (fib (- k 1)))
                        (+ k 1)) ) )
    (if (= 1 n) 0 (fib-iter 1 2)))
```

✖ No, because `fib` is not in a tail context.

Example: Reduce

```
(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
```

Example: Reduce

```
(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
```

```
(define (reduce procedure s start)
    (if (null? s) start
        (reduce procedure
            (cdr s)
            (procedure start (car s)) ) ) )
```

Is it tail recursive?

Example: Reduce

```
(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
```

```
(define (reduce procedure s start)
    (if (null? s) start
        (reduce procedure
            (cdr s)
            (procedure start (car s)) ) ) )
```

Is it tail recursive?

✅ Yes, because `reduce` is in a tail context.

Example: Reduce

```
(reduce * '(3 4 5) 2) 120
(reduce (lambda (x y) (cons y x)) '(3 4 5) '(2)) (5 4 3 2)
```

```
(define (reduce procedure s start)
    (if (null? s) start
        (reduce procedure
            (cdr s)
            (procedure start (car s)) ) ) )
```

Is it tail recursive?

✅ Yes, because `reduce` is in a tail context.

However, if `procedure` is not tail recursive, then this may still require more than constant space for execution.

Example: Map

```
(map (lambda (x) (- 5 x)) (list 1 2))
```

Example: Map

```
(map (lambda (x) (- 5 x)) (list 1 2))
```

```
(define (map procedure s)
    (if (null? s)
        nil
        (cons (procedure (car s))
            (map procedure (cdr s))) ) )
```

Is it tail recursive?

Example: Map

```
(map (lambda (x) (- 5 x)) (list 1 2))
```

```
(define (map procedure s)
    (if (null? s)
        nil
        (cons (procedure (car s))
              (map procedure (cdr s))) ) )
```

Is it tail recursive?

✘ No, because `map` is not in a tail context.

Example: Map (Tail recursive)

```
(define (map procedure s)
    (define (map-reverse s m)
    (if (null? s)
        m
        (map-reverse (cdr s) (cons (procedure (car s)) m))))
    (reverse (map-reverse s nil)))

(define (reverse s)
    (define (reverse-iter s r)
    (if (null? s)
        r
        (reverse-iter (cdr s) (cons (car s) r))))
    (reverse-iter s nil))

(map (lambda (x) (- 5 x)) (list 1 2))
```

Tail call optimization with trampolining

What the thunk?

Thunk: An expression wrapped in an argument-less function.

Making thunks in Python:

```
thunk1 = lambda: 2 * (3 + 4)
thunk2 = lambda: add(2, 4)
```

Calling a thunk later:

```
thunk1()
thunk2()
```

Trampolining

Trampoline: A loop that iteratively invokes thunk-returning functions.

```
def trampoline(f, *args):
    v = f(*args)
    while callable(v):
        v = v()
    return v
```

The function needs to be thunk-returning! One possibility:

```
def factorial_thunked(n, k):
    if n == 0:
        return k
    else:
        return lambda: factorial_thunked(n - 1, k * n)
```

```
trampoline(factorial_thunked, 3, 1)
```

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Demo: Trampolined interpreter

The Scheme project EC is to implement trampolining.
Let's see how it improves the ability to call tail recursive functions...