Discrete-time Risk Measures with Time Consistency

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Abstract: According to the properties of general probability space, we propose the conception of acceptance set and capital requirement in the discrete-time risk measures framework. Related propositions are put forward and proved in the second part. Then we mainly focus on the time consistency properties shown during the course of discrete-time risk measures. Time consistency has been certified as one of the most important properties for discrete-time risk measures. In particular, the poor, middle and strong consistency based on the original recursiveness and consistency are proposed in the third section. These time consistency properties provide the mathematical description of dynamic risk measures on the general probability space. Finally, the validation of time consistency of existing methods including VaR, CVaR and ES is provided.

Keywords: Discrete-time risk measures, Risk measures, Time consistency

1 Introduction

The stability of global financial market depends on the capability of defending risk in financial institutes. This is especially true when those overwhelming accidents take place. Many possible definitions of risk have been proposed in the literature because different investors adopt different investment strategies in seeking to realize their investment objectives. Risk measures are, roughly speaking, methodologies that should tell us if a future random monetary position is sustainable or not and, more in general, help us rank different investment choices.

The variance or standard deviation of revenue or loss has been the most popular tools of risk measurement since Markowitz's seminal work on financial risk measurement[1]. However, because the variance ignores the "fat-tail" phenomenon of loss distribution and the difference between the loss and revenue[2], the variance has been criticized widely. Then the semi-variance and LPM become a reasonable substitution for variance and standard deviation[3][4]. Until 1990s, VaR has become the most popular method in the field of financial risk measurement[5]. There is no unified standard to tell which one is a better risk measurement before the famous papers about coherent measure of risk by Artner, Delbaen, Eber and Heath[6]. It is the first time to connect the analysis of investment portfolio with the asset values in the future[7]. All the investment portfolios before coherent measure of risk are constructed on the basis of investors' preference and utility function[8]. Since the coherent measure of risk was born, it has been proved that more and more drawbacks were related to VaR. In order to find a better solution, Rockafellar and Uryasev put forward another risk measurement-CVaR, which satisfies four axioms in coherent measure of risk, monotonicity, translation invariance, subadditivity and positive homogeneity[9].

All of risk measurements are on the basis of loss distribution of static investment portfolio choice, while most investors make their investment decision dynamically, which time structure is discrete-time process in practice. So it is necessary to supervise and control the risk in real time with the dynamic risk measures satisfying time consistency. In discrete-time multi-stage investment problems, investors make the investment decision in every stage in order to realize the risk limited under a fixed level not only in the whole investment process but also in the present stage. That is the conception of time consistency of dynamic risk measures.

The paper is structured as follows. We propose the concept of time consistency based on the optimal principle of dynamic programming. It is proved that both VaR/CVaR and ES don't follow the time consistency of dynamic optimal investment portfolio. To solve this problem, a method of risk measurement satisfying time consistency is proposed. We also state that traditional risk measurement could coexist with time consistency risk measurement.

2 Discrete-time risk measures frame

Dynamic risk measures lay great emphasis on continuous evaluation and control towards the final value of investment portfolio. If X stands for the value of asset and the fixed maturity is assigned to $T>0$, then the destination of risk measure should be the value of asset X on fixed future maturity T. Dynamic risk measures framework is trying to describe how to evaluate the risk of a process $X \in \mathcal{L}_1^\infty$ at each date t_n, with $n \le N-1$.

Since what matters in risk measure is future payoffs, the risk measure ρ_n at time t_n will only depend on X_m with $m \ge n+1$.

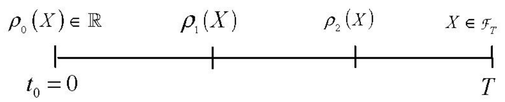

Fig.1 Dynamic risk measures

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We define that for each $n \geq 1$, the space

$$\mathcal{L}_n^\infty \triangleq \{\mathbf{X} = (X_n, ..., X_N) : X_m \in \mathcal{L}_m^\infty \quad \forall m\} \tag{1}$$

For $n=1$, this has been the ambient space in the previous sections. An element $\mathbf{X} \in \mathcal{L}_n^\infty$ represents a payoff steam starting at date t_n.

Suppose that at time 0 we put the sum $\alpha \geq 0$ in a bank account. At each subsequent time $t_1, .., t_N$, we "disinvest" the quantity Y_n from the bank account and add it to the payoff X_n. The quotes are necessary, since Y_n can be negative: in this case it is intended that we subtract the sum $-Y_n \geq 0$ from the payoff X_n and invest it in the bank account. In any case, the value of the bank account before time t_n is $\alpha - \sum_{i=1}^{n-1} Y_i$. We assume, for now, that it is possible to borrow money from a bank with no credit line and that every borrowed sum must be paid back only at final maturity t_N. In other words at each date t_n with $n \leq N-1$, Y_n can be greater than $\alpha - \sum_{i=1}^{n-1} Y_i$, i.e. the disinvested sum can be greater than the bank account value. However, at time t_N all debts must be paid back, so we impose that $Y_N \leq \alpha - \sum_{i=1}^{N-1} Y_i$, i.e. $\sum_{i=1}^{N} Y_i \leq \alpha$. If the sum α "invested" at time 0 is negative then are borrowing $-\alpha$ from the bank[10]. For every n, the set $\mathrm{S_n}(\alpha, X, \mathrm{A})$ collects all these "investment" stratifies Y that, in addition, make the composite payoff stream $X+Y$ acceptable and whose value after t_n is completely determined at that date. In particular, when $n=0$, the last requirement implies that every Y_n is real, i.e. its amount is decided today. Finally, $\rho_{n,A}(X)$ is the infimum of the initial investments that make possible such a strategy.

2.1 Acceptance sets

Let $\mathcal{X}$ be a linear space of functions on a given set Ω of possible scenarios. We assume that $\mathcal{X}$ contains all constant functions. Any risk measure $\rho : \mathcal{X} \to R \bigcup \{+\infty\}$ induces an acceptance set A_ρ defined as

$$A_\rho := \{X \in \mathcal{X} | (X) \leq 0\} \tag{2}$$

Conversely, for a given class A of acceptable positions, we can introduce an associated risk measure ρ_A by defining

$$\rho_A(X) := \inf\{m \in R | m + X \in A\} \tag{3}$$

The following two propositions summarize the relations between a convex measure of risk and its acceptance set A_ρ. They are similar to the ones found for coherent measure of risk.

Proposition 1 Suppose $\rho : \mathcal{X} \to R \bigcup \{+\infty\}$ is a convex measure of risk with associated acceptance set A_ρ. Then

$A := A_\rho$.

Moreover, $A := A_\rho$ enjoys the following properties.

(1) A is convex and non-empty.

(2) If $X \in A$ and $Y \in \mathcal{X}$ satisfies $Y \geq X$, then $Y \in A$.

(3) If $X \in A$ and $Y \in \mathcal{X}$, then $\{\lambda \in [0,1] \mid \lambda X + (1-\lambda) Y \in A\}$ is closed in $[0,1]$.

Proof: To show that $\rho_{A_\rho}(X) = \rho(X)$ for all X, note that the translation invariance of ρ implies that

$$\rho_{A_\rho}(X) = \inf\{m \mid m + X \in A_\rho\} = \inf\{m \mid \rho(x + X) \leq 0\}$$
$$= \inf\{m \mid \rho(X) \leq m\} = \rho(X)$$

The first two properties of $A = A_\rho$ are straightforward. As for the third one, note that the function $\lambda \to \rho(\lambda X + (1-\lambda) Y)$ is continuous, as it is convex and takes only finite values. Hence, the set of $\lambda \in [0,1]$ such that $\rho(\lambda X + (1-\lambda) Y) \leq 0$ is closed.

Proposition 2 Assume that $A \neq 0$ is a convex subset of $\mathcal{X}$ which satisfies positive homogeneity of Proposition 1, and denote by ρ_A the functional associated to A via (8). If $\rho_A(0) > -\infty$, then

(1) ρ_A is a convex measure of risk.

(2) A is a subset of $A_{\rho A}$. Moreover, if A satisfies property 3 of Proposition 1, then $A = A_{\rho A}$.

Proof: 1. It is straightforward to verify that ρ_A satisfies translation invariance and monotonicity. We show next that ρ_A takes only finite values. To this end, fix some element Y of the non-empty set A. For $X \in \mathcal{X}$ given, there exists a finite number m with $m + X > Y$, because X and Y are both bounded. Monotonicity, translation invariance, and the fact that $\rho_A(Y) \leq 0$ give $\rho_A(X) \leq m$. To show that $\rho_A(X) < -\infty$, we take m' such that $X + m' \leq 0$ and conclude that $\rho_A(X) \geq \rho_A(0) + m' > -\infty$.

As to the property of convexity, suppose that $X_1, X_2 \in \mathcal{X}$ and that $m_1, m_2 \in R$ are such that $m_i + X_i \in A$. If $\lambda \in [0,1]$, then the convexity of A implies that $\lambda(m_1 + X_1) + (1-\lambda)(m_2 + X_2) \in A$. Thus, by the translation invariance of ρ_A,

$$0 \geq \rho_A(\lambda(m_1 + X_1) + (1-\lambda)(m_2 + X_2))$$
$$= \rho_A(\lambda X_1 + (1-\lambda) X_2) - (\lambda m_1 + (1-\lambda) m_2) \tag{4}$$

and the convexity of ρ_A follows.

(3)The inclusion $A \subseteq A_{\rho A}$ is obvious. Now assume that A satisfies the third property of Proposition 1. We have to show that $X \notin A \Rightarrow \rho_A(X) > 0$. To this end, take $m > \rho_A(0)$. By property 3 of Proposition 1, there exists an $\varepsilon \in (0,1)$ such that $\varepsilon m + (1-\varepsilon)X \notin A$. Thus

$$\begin{aligned} \varepsilon m &\le \rho_A\left((1-\varepsilon)X\right) = \rho_A\left(\varepsilon \cdot 0 + (1-\varepsilon)X\right) \\ &\le \varepsilon \rho_A(0) + (1-\varepsilon)\rho_A(X) \end{aligned} \tag{5}$$

Hence $\rho_A(X) \ge \dfrac{\varepsilon(m-\rho_A(0))}{1-\varepsilon} > 0$.

2.2 Capital requirement

There is no unique natural notion for a capital requirement, when dealing with processes. In this part, we suppose that there is no credit line for the investors.

For any $\alpha \in \mathbb{R}$, $X \in \mathcal{L}^{\infty}$, and acceptance set A, we define:

$$\mathrm{S_n}(\alpha, X, \mathrm{A}) \triangleq \left\{ Y \in \mathcal{L}^{\infty}_{1,n} : X + Y \in \mathrm{A}, \sum_{\mathrm{n}=1}^{N} X_n \le \alpha \right\}$$

and the quantity

$$\rho_{\mathrm{A},n}(X) \triangleq \inf\left\{\alpha \in \mathbb{R} : \mathrm{S_n}(\alpha, X, \mathrm{A}) \ne 0\right\}.$$

Then for fixed $0 \le n \le N$ and acceptance A, the map $\rho_{\mathrm{A},n} : \mathcal{L}^{\infty}_{1} \to \mathbb{R}$ is called the dynamic capital requirement associated to A and n. Since we define the number of period is finite, our research is concentrated on the discrete-time process of risk measure.

3 Time consistency

Consider an n-stage portfolio optimization planning problem. Let a decision rule at time/stage t be denoted by π_t the column vector π_t whose entries represent the fractions of the total portfolio allocated to individual stocks. It is assumed that $\pi_t^T \cdot e = 1$, where e is the column vector of all ones of the dimension equal to the number m of available stocks. That is, we are assuming that all the resources are invested in these m stocks at each time t. A policy π will be defined as a sequence of decision rules, that is, $\pi = (\pi_1, \pi_2, ..., \pi_n)$.

Let the return/reward at stage t be denoted by the random variable r_t whose probability distribution function depends on the policy π. Let $R_t := g(r_1, ..., r_t)$, $t = 1, 2, ..., n$ be the aggregated return for the t-stage process. For all $t = 1, 2, ..., n$, let $Z_t^{\pi} = Z_t(\pi_1, ..., \pi_t)$ be the risk measure of t-stage value corresponding to the decision rules $\pi_1, ..., \pi_t$ and with respect to g.

3.1 Poor consistency

If the influence caused by the new information between time t_n and t_{n+1} on future position X is dedicated to $I_n(i)$, then risk measure ρ should follow:

$$\rho_{n+1}(X) = \rho_n(X) + I(i_n) \tag{6}$$

The poor consistency property means that: in the complete market, the change of risk measure between two neighborhood time t_n and t_{n+1} is caused by the influence of the new information during this period. If the new information is a positive kind, the risk measure ρ won't change. Otherwise, the risk measure ρ will be upgraded with $I(i_n)$. Since this property is constructed in the complete market, it is considered as poor.

Corollary 1 $\rho_N = \rho_0 + \sum_{n=0}^{N} I(i_n)$ for all n.

The proof is left out. This corollary states that the difference of risk measure ρ between the beginning and end of the period is caused by the information in the complete market. $\rho_N - \rho_0$ is the cumulative effect of information in the complete market.

3.2 Consistency

$$\rho_{n+1}(X) \le \rho_{n+1}(Y) \text{ implies } \rho_n(X) \le \rho_n(Y)$$

for every n.

The consistency property has a clear meaning: if today we are sure that tomorrow Y will be judged riskier than X, then the same conclusion must be drawn today, and this should be a natural behavior.

Corollary 2 $\rho_m(X) \le \rho_m(Y)$ implies $\rho_n(X) \le \rho_n(Y)$ for every $n \le m$.

We think it is easy to prove according to consistency property.

3.3 Strong consistency

$$\rho_n(-\rho_{n+1}(X)) = \rho_n(X) \text{ for every } n.$$

Strong consistency(recursiveness) states that ρ_n can be computed in a simple recursive way, by replacing X with the tomorrow risk measure $\rho_{n+1}(X)$, obviously changed in the sign.

Corollary 3 $\rho_n(-\rho_m(X)) = \rho_n(X)$, for every $n < m$

The proof is left out.

Time consistency of a risk measure means that if a decision-maker uses a risk measure minimizing policy for the n-stage problem, then the component of that multi-stage policy from the t^{th}-stage to the end should be a risk measure minimizing policy in the remaining $(n-t+1)$-stage problem, for every $t = 1, 2, ..., n$. In common sense terms, a decision maker needs to be constantly concerned about optimizing his or her decisions for the remaining portion of the time horizon. That is, current optimal decisions must look to the future, rather than the past.

4 Time consistency of existing risk measurement

As above, we assume that there are m stocks in the market. The investor will invest in these stocks, at the beginning of every time period t, $t=1,2,...,n$. he initial capital is given and there is no loss of generality in assuming it to be ￥1.

For $t=1,2,...,n$, let $Y_t=(Y_{t1},Y_{t2},...,Y_{tm})^T$, be the random percentage return vector of the m stocks under consideration by the investor. For notational convenience, we assume that Y_{tk}, $t=1,...,n$, $k=1,...,m$ are all continuous random variables. Now, define the $m\times n$ matrix Y to be the matrix whose columns are the percentage return vectors $Y_t=(Y_{t1},Y_{t2},...,Y_{tm})^T$, for every t.

Further, note that a policy π as defined in the previous section can also be regarded as an $m\times n$ matrix whose columns are the decision rules π_t for $t=1,...,n$.

Of course, if the initial capital were ￥1 and an investment policy π were used, then the total random return after n years will be simply:

$$R_n(\pi,Y):=\prod_{t=1}^{n}(1+\pi_t^TY_t) \tag{7}$$

A simple logarithmic transformation converts the above multiplicative return to a more convenient additive one[11]. Clearly, investors are most concerned about the initial capital going down and this corresponds to $R_n(\pi,Y)$ being strictly less than 1 which – after taking a logarithm – becomes negative. Consequently, it is natural to define the total random loss resulting from policy π as

$$L_n(\pi,Y):=-\log(R_n(\pi,Y))=-\log\prod_{t=1}^{n}(1+\pi_t^TY_t)=-\sum_{t=1}^{n}\log(1+\pi_t^TY_t) \tag{8}$$

Since $-\log(\cdot)$ is a convex function, it is easy to check that the total loss function $L_n(\pi,Y)$ is also a convex function of the policy π. Hence, following standard arguments (see, e.g. Rockafellar and Uryasev 2000) we can define VaR and CVaR; we simply substitute $L_n(\pi,Y)$ in place of $f(x,y)$ in Rockafellar and Uryasev (2000). That is, we have the following definitions.

Definition 1 The n-stage VaR, denoted by α-VaR and associated with the total loss $L_n(\pi,Y)$ is defined by:

$$\zeta_{n,\alpha}^{\pi}=\zeta_{n,\alpha}(\pi):=\min\{\zeta\mid P_Y(L_n(\pi,Y)\le\zeta)\ge\alpha\}$$

Definition 2 The n-stage CVaR, denoted by α-CVaR and associated with the total loss $L_n(\pi,Y)$ is defined by:

$$\phi_{n,\alpha}^{\pi}=\phi_{n,\alpha}(\pi):=E[L_n(\pi,Y)\mid L_n(\pi,Y)\ge\zeta_{n,\alpha}(\pi)]$$

where $E(\cdot)$ denotes the mathematical expectation.

For fixed α, if we choose $Z_n(\pi)=\zeta_{n,\alpha}(\pi)$, then we use VaR as our measure of risk; otherwise we choose $Z_n(\pi)=\phi_{n,\alpha}(\pi)$, then we use CVaR as our measure of risk. In what follows, we will show that both of VaR and CVaR need not to be time consistent in multi-stage setting.

From Theorem 1 in Rockafellar and Uryasev (2000), we know that $\phi_{n,\alpha}(\pi)$ is a convex function of π and can be calculated as

$$\phi_{n,\alpha}(\pi)=\min_{\zeta}F_{n,\alpha}(\pi,\zeta) \tag{9}$$

where $F_{n,\alpha}(\pi,\zeta)=\zeta+\frac{1}{1-\alpha}E_Y\{[L_n(\pi,Y)-\zeta]^+\}$.

Hence it is not surprising that by an argument analogous to that presented in Rockafellar and Uryasev (2000, 2002a) we can derive the following algorithm to minimize the n-stage CVaR $\phi_{n,\alpha}(\pi)$ by choosing appropriate decision rule at each stage.

For each stage t, we assume the distribution of the random return Y_t is known and given by $p(y)$. Hence, we can generate (vector-valued) samples y_t^k, $k=1,...,q$ for each t from the distribution $p(y)$. Thus we obtain a corresponding approximation for $F_{n,\alpha}(\pi,\zeta)$ as

$$\tilde{F}_{n,\alpha}(\pi,\zeta)=\zeta+\frac{1}{q(1-\alpha)}\sum_{k=1}^{q}[-\sum_{t=1}^{n}\log(1+\pi_t^Ty_t^k)-\zeta]^+ \tag{10}$$

Based on this, we have the following optimization algorithm for deriving an optimal policy (investment portfolio) in the n-stage problem:

$$\begin{cases}\min[\zeta+\frac{1}{q(1-\alpha)}\sum_{k=1}^{q}u_k]\\ s.t.\ \ \pi_{tj}\ge0,\ t=1,...,n,\ j=1,...,m;\ \sum_{j=1}^{m}\pi_{tj}=1,\ t=1,...,n;\\ \quad u_k\ge0,\ u_k+\sum_{t=1}^{n}\log(1+\pi_t^Ty_t^k)+\zeta\ge0,\ k=1,...,q.\end{cases} \tag{11}$$

Next, we want to calculate the optimal VaR, CVaR and corresponding optimal portfolios for some given stocks, time periods and distributions.

In this simple example (based on an example given in Rockafellar and Uryasev (2000)[12]), we set $m=3$ and $n=2$, so that we have three stocks and we invest in them, each year, for three consecutive years. Further, we set $\alpha=0.99$ and want to choose an optimal portfolio at the beginning of each year so that the 0.99-CVaR of the total random loss L_t, for $t=1,2$ is as small as possible. Finally, we assume that the distribution $p(y)$ of the return is the multi-normal distribution $N(\mu,\Sigma)$ with the mean vector μ and the variance-covariance matrix Σ.

After generating 100 samples – for each of the two stages – from the multinormal distribution $N(\mu,\Sigma)$ we are able to solve the mathematical program (5) to obtain an optimal portfolio and a corresponding value of 0.99-CVaR. This was done with n set to 1 and 2, respectively. Here, our measure of risk is $Z_t(\pi)=\phi_{t,\alpha}(\pi)$ for $t=1,2$ and $\alpha=0.99$. We used Lingo 8.0 to obtain the following numerical results and checked if this $Z_t(\pi)$ satisfied Definition 1 and Definition 2.

An optimal policy for the 1-stage problem (set $t=2$ in equation (1)) with respect to the samples, y_2^k, $k=1,2,...,q$ is $\pi^*=\pi_2^*$, where,

$$\pi_2^*=(0.101, 0.829, 0.070)^T$$

An optimal policy for the 2-stage problem ($n=2$) is $\hat{\pi}=(\hat{\pi}_1,\hat{\pi}_2)$, where,

$$\hat{\pi}_1=(0, 1, 0)^T,\ \hat{\pi}_2=(0, 0.994, 0.006)^T$$

However, setting $t=1$ in equation (1), by direct calculation, it can be checked that if we fix $\pi_2=\pi_2^*$ in (5) and minimize its objective with respect to π_1 only then we obtain $\pi_1^*=(0.154, 0.846, 0)$.

Unfortunately, it is easy to check that,

$$\phi_{2,\alpha}(\pi_1^*,\pi_2^*)=0.0394>0.0374=\phi_{2,\alpha}(\widehat{\pi_1},\widehat{\pi_2})$$

which contradicts (Definition 1).

Conversely we verify that with respect to the samples,

$$y_2^k,\ k=1,2,...,q$$

$$\phi_{1,\alpha}(\hat{\pi}_2)=0.0547>0.0312=\phi_{1,\alpha}(\pi_2^*)$$

So, $\hat{\pi}_2$ is not the optimal solution for $\phi_{1,\alpha}(\pi)$, that is equation (4) doesn't hold when $t=2$ and this contradicts (Definition 2).

The above calculation shows that the policy chosen from optimal action for each stage is not optimal for the total horizon. That means CVaR is not a time consistent risk measure.

Corollary 9 in Rockafellar and Uryasev (2002a) shows that for suitably chosen probability threshold α and sample size, VaR and CVaR(ES) coincide. In our example, $\alpha=0.99$ and sample size equal to 100 were chosen so as to satisfy the conditions of that corollary. Thus the above example also shows that VaR $\zeta_{n,\alpha}(\pi)$ and ES are not a time consistent risk measure.

5 Risk measures with time consistency

We shall propose a new risk measure that is not only time consistent in the multistage case but will also consider the decision-maker's target. We will use Markov decision processes with probability criteria to model such a risk measure.

Tab.1 Portfolio mean return

Instrument	Mean return
Option1	0.0101110
Option 2	0.0043532
Option 3	0.0137058

Tab.2 Portfolio variance–covariance matrix

	Option 1	Option 2	Option 3
Option 1	0.0324625	0.00022983	0.00420395
Option 2	0.00022983	0.00049937	0.00019247
Option 3	0.00420395	0.00019247	0.00764097

We consider the following discrete-time and stationary Markov decision process:

$$\Gamma=(S,A,R,P,\beta)$$

where the state space S is countable, the action space $A(i)$ in each state i is finite and the overall action space $A=\bigcup_{i\in S}A(i)$ is countable. The return set R is a bounded countable subset of $\mathcal{R}=(-\infty,+\infty)$. For each t from $t=1,...,$ let i_t, a_t and r_t denote the state of the system, the action taken by the decision maker, and the return received at stage t, respectively. The stationary, single-stage, conditional transition probabilities are defined by

$$p_{ijr}^{\alpha}:=P(i_{t+1}=j,r_t=r\,|\,i_t=i,\alpha_t=\alpha)\quad,\quad i,j\in S\quad,$$

$$\alpha\in A(i),\ r\in R,\ n\geq 1$$

$$\sum_{j\in S,r\in R}p_{ijr}^{\alpha}=1, i\in S, \alpha\in A(i)$$

We shall also assume that future costs are discounted by the discount factor $\beta\in(0,1]$.

In our formulation, when making a decision and taking an action at each stage, the decision maker considers not only the state of the original system but also his target. Effectively, this means that a new hybrid state $(i,x)\in S\times\mathcal{R}$ is introduced. Hence we expand MDP Γ by enlarging the state space. We refer to (i,x) as the hybrid state of the decision maker to distinguish it from the system's state i, where x is the target value. Note that if the initial state of the decision maker is (i,x) and an action α is taken according to (7), the decision-maker's new hybrid state transits from (i,x) to $(j,(x-r)/\beta)$ with probability p_α.

Thus, if we denote E as the extended (hybrid) state space, then the extended MDP Γ has the following structure:

$$\tilde{\Gamma}=(E,A,R,P,\beta)$$

where the state space $E=S\times\mathcal{R}$, the action space $A=\bigcup_{(i,x)\in E}A(i,x)=\bigcup_{i\in S}A(i)$. Note that $A(i,x)=A(i)$, $(i,x)\in E$, the extended transition probabilities are simply

$$P:P\left(e_{t+1}=\left(J,\frac{x-r}{\beta}\right)|\,e_t=(i,x),\,\alpha_t=\alpha\right)=p_{ijr}^{\alpha} \quad (9)$$

where $i,j\in S$, $\alpha\in A(i)$, $r\in R$, $x\in \mathcal{R}$. The return set R and the discount factor β are the same as in MDP Γ.

Since in the model (9), the target is important when making decisions we must define policies which depend both on the system's state and the target, that is on the hybrid state.

Let Π_m, Π_m^d, Π_S, Π_s^d denote the set of all Markov policies, all deterministic Markov policies, all stationary policies and all deterministic stationary policies in $\tilde{\Gamma}$ defined in the usual way.

6 Conclusion

This paper approaches the problem of discrete-time risk measures with time consistency. It begins with the discrete-time risk measures framework and conception of acceptance set and capital requirement. In the dynamic framework, a set of axioms aiming at the characters of dynamic risk measures on general probability space is proposed. The axioms of dynamic risk measures are not used to specify a certain risk measures. Instead, they characterize a class of risk measures. The strong, middle and poor consistency are the necessary properties which the dynamic risk measure should own. The consistency property provides a function for calculating the former risk measure according to the latter one. The risk measures between the neighborhood times could be calculated by means of recursive algorithm. In the last section, we provide an example to prove that the existing methods including VaR, CVaR and ES are not following time consistency. So a more exact method for risk measurement should be put forward. Scholars should work hard on it in the soon future.

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