GLOBAL SENSITIVITY ANALYSIS

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In applications of operations research models, decision makers must assess the sensitivity of outputs to imprecise values for some of the model's parameters. Existing analytic approaches for classic optimization models rely heavily on duality properties for assessing the impact of local parameter variations, parametric programming for examining systematic variations in model coefficients, or stochastic programming for ascertaining a robust solution. This paper accommodates extensive simultaneous variations in any of an operations research model's parameters. For constrained optimization models, the paper demonstrates practical approaches for determining relative parameter sensitivity with respect to a model's optimal objective function value, decision variables, and other analytic functions of a solution. Relative sensitivity is assessed by assigning a portion of variation in an output value to each parameter that is imprecisely specified. The computing steps encompass optimization, Monte Carlo sampling, and statistical analyses, in addition to model specification. The required computations can be achieved with commercially available off-the-shelf software available for microcomputers and other platforms. The paper uses a broad set of test models to demonstrate the merit of the approaches. The results are easily put to use by a practitioner. The paper also outlines further research developments to extend the applicability of the approaches.

Of importance to practitioners is understanding the sensitivity of a model's outputs to *simultaneous* variations in several parameters. This article suggests apparently practical and effective ways to provide such sensitivity information to managers who request it. The paper's principal focus is the sensitivity of a mathematical programming model's optimal value. We also discuss how these sensitivity notions may be applied to the values of individual model variables, other functions of a solution, heuristics, and outcomes from spreadsheet models. All of the models analyzed in this paper are deterministic and the exposition is simplified by maintaining this view. At the end of the article, however, we indicate briefly the straightforward modifications required to treat several important classes of probabilistic models.

A typical situation occurs when an optimization model is being validated, prior to a decision-maker's taking action predicated on the model results. The initial solution may indicate that there is a substantial improvement opportunity in the enterprise's economic performance. The manager realizes, however, that the values of several model parameters have been *guesstimated*, and questions whether the optimal value of the objective function is sensitive to these values. Situations of this sort arise especially in planning models that include forecast customer demand for various products; future prices, costs, and interest rates; process yields; and supply and manufacturing lead times. If the manager finds that one or more of the parameter values have a significant impact on the model's outcome, an attempt may be made to reduce imprecision about these values, or seek a robust solution by means of a stochastic programming formulation. We observe that a similar set of issues applies with respect to optimal values of a model's decision variables.

Sensitivity assessment is easily done when only a single parameter is at issue; it requires a straightforward one-dimensional analysis, amenable to graphic presentation. When two or more parameters are under scrutiny, however, the issue is more complicated, because the several parameters may not have a separable monotonic influence on the optimal value of the objective function. In these instances, the core *concept* of sensitivity must be established first. Subsequently, algorithmic tools that can quantify the concept in specific applications must be developed.

We pose as an analytic challenge finding a way to provide a manager with a global assessment of parameter sensitivity for an initially specified deterministic optimization model. Depending on the outcome of the sensitivity analysis, the manager may want to take follow-on action. The manager may obtain more accurate parameter values, provided that it is possible to do so with a reasonable expenditure of effort. Alternatively, the manager may direct that the initial model be revised to explicitly accommodate the imprecise parameter values. In any event, a goal of this paper is to provide procedures that assess whether variations in particular parameters

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do or do not have significant impact on the optimal value of the objective function.

During the past four decades, several approaches to sensitivity analysis for mathematical programming models have appeared in the operations research literature. Classic duality results for linear and nonlinear programs (see Kuhn and Tucker 1950, Dantzig 1963, Gal 1979, Rubin and Wagner 1990, Ward and Wendell 1990, Jansen, Roos and Terlaky 1992, and Greenberg 1993) pertain to local variations. In most software packages, the duality analyses are designated as ranging, and usually are restricted to the objective-function and right-hand side coefficients. In linear programming, extensive simultaneous variations in several parameters, possibly including constraint coefficients, give rise to optimal basis changes, which in turn alter the dual values. Determination of parameter variations that maintain the current optimal basis is central to the 100% rule and tolerance approaches (see Bradley, Hax and Magnanti 1977, Ravi and Wendell 1985, 1989, Wendell 1984, 1985, 1992, and Wondolowski 1991). Parametric programming allows for more extensive coefficient variation, but also is restricted in scope (see Manne 1953, Gass and Saaty 1954, Gass and Saaty 1955, Gal and Nedoma 1972, Fiacco 1983, Jongen and Weber 1990, Camm and Burwell 1991, and Schenkerman 1993). We remark that there is a lacuna in commercial software for dealing with simultaneous variation in *constraint* coefficients. When the required multivariate sensitivity is too extensive to be encompassed by currently available algorithmic methods and software, practitioners usually rely on testing specified alternative scenarios.

This article suggests a different way to approach extensive sensitivity analysis of deterministic models, that usually, *but not necessarily*, encompass optimization—the approach is well suited to spreadsheet models. As we indicate later, the sensitivity notions also can be applied to certain simple simulation models and several kinds of models with probabilistic components. Statisticians, with some justification, may judge the approaches that are recommended in this paper to be old wine in a new bottle; I view them as old wine in an old bottle that is opened with a new corkscrew. We share with statisticians a common interest in deducing insights about multivariate relationships. The statistician postulates that for given input values, the outcome value typically varies among different observations due to a random error component. The statistician attempts to discern a multivariate relationship in the presence of the random disturbances, and may have relatively few observations available. In contrast, we are faced with an implicit but deterministic multivariate function, produced by a computer algorithm; consequently, we may be able to obtain thousands of observations, each of which can be replicated exactly.

In the next section we comment on why our procedures may be more effective than the others previously cited in providing global sensitivity information requested by managers. Nevertheless, we caution at the outset that the paper is a first checkpoint in the development of the ideas. We provide encouraging evidence of the approaches' practicality, but we counsel tempered optimism until additional experience accumulates. In this respect, the suggested approaches are kindred to familiar multivariate statistical techniques, such as stepwise regression, that have required empirical testing to establish their practical value. At the end of the paper, we suggest some promising avenues for further investigation that can improve on the approaches' generality and usefulness.

The article provides practitioners with a quick start in performing global sensitivity analyses using off-the-shelf software that is commercially available for microcomputers and other platforms. In addition, it supplies technical background for theorists who wish to pursue advances in the mathematical underpinnings of the notions.

In Section 1 we establish the context of multivariate sensitivity analysis, and offer a perspective on approaches developed by other researchers *vis à vis* the ideas put forth in this paper. We explain the fundamental concepts underlying our approaches in Section 2. In Section 3, a general Monte Carlo sampling procedure is described that ascertains the relative influence of model parameters. In Section 4 we examine a well-established statistical algorithm that provides alternatives to Monte Carlo sampling, reduces the computational burden, and yields other useful output; these alternatives are illustrated using a battery of classic OR models. In Section 5 we propose a different statistical approach that looks promising for difficult-to-analyze models. All the sensitivity concepts are applied to three industrial applications of linear programming in Section 6. Extensions to probabilistic models are treated in Section 7. In Section 8 we suggest a research agenda for extending the preliminary findings of this paper.

1. FAMILIAR APPROACHES TO SENSITIVITY ANALYSIS

We begin by exploring in more depth what we believe a manager seeks from global sensitivity analysis. For our discussion, we use the context of a deterministic optimization model. In industrial applications of mathematical programming models, there almost always are uncertain elements that are assumed away or suppressed in the formal description of the model. Many years of OR experience with deterministic models have demonstrated that these models yield results that can be very valuable to managers. It is not the case, however, that the perceived uncertain elements or imprecise inputs remain ignored altogether. In the testing phase of the model-building effort, the uncertainties and guesses are usually re-examined. In my experience, a manager reviewing a model is apt to express doubt regarding the precision of a few assumed parameter values. It is important to recognize at the outset that an individual parameter need

not be a model coefficient per se; it may be an input to complex calculations that result in one or more model coefficients.

The manager may believe, for example, that the actual parameter values of interest are within ±10% of the assumed values; a manager with extensive experience in applying deterministic models may even postulate a distinct percentage range of values for each imprecise parameter (see Eschenbach 1992). If interrogated further, the manager may be willing to give what OR professionals call a personal probability assessment of the likelihood of the possibilities circumscribed by the expressed ranges of parameter values. By way of historical precedent, we comment that this *weltanschauung* is consonant with early forays in stochastic programming (see Babbar 1955, Wagner 1958, Madansky 1960, 1962, Prekopa 1966, and Ewbank, Foote and Kumin 1974). The formulation sometimes is termed as *wait-and-see* to indicate that the optimization occurs *after* the random values are realized; thus, the variation in the objective function value is viewed as if the model were to be solved repeatedly, each time with different realized values for the parameters.

Prior to analysis, it is not always obvious whether the postulated variation in the parameters gives rise to significant variation in the optimal value of the objective function. For example, it may turn out that the implied variation in the optimal objective function value is less than ±1%. If the manager perceives the variation to be economically important, then it is necessary to probe further as to which parameters account for the variation. Do only a few parameters cause most of it? Do some have negligible impact, given the variation caused by others? Since the optimal value of the objective function is uniquely and completely determined by a model's parameters—a mathematical programming model, by its very nature, has no exogenous random disturbances—it is intuitively reasonable to view the variation in the optimal values as being *caused* by the variation in the parameter values.

An imprecise parameter value, then, may be used in practice because:

- insufficient effort was expended to obtain an exact value; an educated guess was taken but could be improved upon with added effort, if warranted;
- the value is not knowable until *after* the imminent decision is made; for example, the value may be the operating capacity of a competitor a few years from now;
- the value is *intrinsically* unknowable; for example, it may be a modeling construct that is meant to reflect an amalgam of factors that influence customers' decisions to replace or upgrade an old product.

Whatever the reason for the imprecision, the manager's apprehension may be reduced if the variation in a parameter value has negligible influence on the optimal objective function value. Contrariwise, if the variation in a parameter's value is the root cause of economically significant variation in the optimal value, then the manager may require closer scrutiny of the model before acting on an emergent optimal solution based on the *guesstimates*.

The mathematical material below shows why simultaneous variation in several parameters presents a tough analytic issue in determining the influence of separate parameters. Holding every parameter but one at its mean value, say, does not lead to a reliable assessment of the impact of varying the remaining parameter. The substantive issue is primarily conceptual—what is meant by sensitivity to a parameter when other parameters vary as well—and secondarily, but not unimportantly, algorithmic.

Consider sensitivity information that is provided by existing mathematical approaches. Duality, gradients, ranging, the 100% rule, and a tolerance region focus on the local properties of a particular optimal solution. In a linear program, these approaches provide formulas that prescribe limits on parameter values which ensure that the current basis remains optimal. Note that only in a few instances do the approaches provide limits in situations with variations in several constraint coefficients (see Ravi and Wendell 1985, 1989).

It is important to realize that even if the prescribed limits encompass the domain of imprecision expressed by the manager, the approaches do not indicate which of the uncertain parameters contribute the most to the variation in the optimal value, except in special cases (for example, only independent variations in objective-function coefficients of a linear program (see Dantzig 1955, Wagner 1975)). If the manager's domain of parameter imprecision exceeds the mathematically determined local limits, then little, if any, sensitivity information is provided. Drawing on a fractile analogy, classic sensitivity analyses examine a coast line from the perspective of a stroll along the shore; they indicate whether your feet will get wet if you walk in a particular direction from where you are. The objective of this paper is to describe the coast line as you would view if from an aerial photograph. Thus, the classic approaches do not meet the goal of providing extensive global sensitivity, although they are precise and informative in other important and well understood respects. In particular, they provide results concerning the current set of optimal variables.

Ad hoc scenario sensitivity testing can be viewed as a simplified factorial experiment. Typically, each scenario tests the combined impact of trial values for several of the parameters. To illustrate, a scenario may examine the outcome if high (or low) values are assumed for all the parameters, or a high (or low) value for one parameter and middle values for all the rest (see Eschenbach). In theory, we can test, say, k predetermined values for each of the N parameters. Using a full-factorial experiment with k^N combinations, there are 3,125 test combinations when k and N are both 5. For larger, more realistic, values of k or N, the number of combinations may be so big as to be prohibitive.

Parameter uncertainties are addressed explicitly in the formulation of a stochastic programming model to obtain an optimal solution (see Wets 1989, Infanger 1994, and Mulvey, Vanderbei and Zenios 1995). But stochastic programming sidesteps the issue of which uncertain parameters contribute most importantly to the uncertainty in the eventual outcome. We suggest that global sensitivity analysis, as proposed here, could be a useful precursor to formulating a stochastic programming model. The analysis could help narrow which parameters should be selected as probabilistic in the ensuing stochastic programming model.

In summary, the classic approaches to sensitivity analysis, which are predicated on local information, cannot be relied on to give global extensive information; limited ad hoc scenario testing, by its nature, falls short of providing general sensitivity conclusions; and stochastic programming does not address explicitly which of the uncertain parameters are the most significant.

2. GLOBAL SENSITIVITY METHODS

In this section, we formally define global sensitivity. The idea is a familiar statistical concept, and we elaborate on the analytic issues that need to be addressed to provide a theoretical foundation. We show a direct way for practitioners to apply the idea. To keep the discussion focused, we continue to present the notions in the context of a mathematical programming model.

2.1. *N* Parameter Model

We formulate the model as

$$\text{optimize } f(x; p) \equiv f^*(p) \tag{1}$$

subject to

$$g_i(x; p) \leq 0 \quad \text{for } i = 1, \ldots, m \quad \text{and } x \geq 0. \tag{2}$$

In (1), the term optimize usually denotes minimize or maximize, but in some of the models that we have tested, it specifies minmax or maxmin. The vector x comprises the decision variables. The vector p is N-dimensional and denotes model parameters whose values are imprecise. In typical applications, these affect only a subset of model coefficients. We emphasize that the parameters may be any of the coefficients themselves or may determine coefficient values functionally and possibly jointly. In the small-scale examples in Sections 3 and 4, each parameter will be one of the model coefficients. In the industrial examples in Section 6, the parameters are functionally related to families of model coefficients.

When a single parameter p_k enters a model only as $h(p_k)$, we recommend that the kth parameter be defined as $h(p_k)$. When p_k affects a model in a more complex fashion, we recommend that the sensitivity analysis be conducted using p_k. The reason is that doing otherwise causes the resultant *parameters*, the number of which may thereby increase, to be probabilistically dependent in situations where the possibly fewer original parameters are probabilistically independent. This view is pursued further in Section 6. We clarify later the conceptual complication that arises from probabilistically dependent parameters.

We can suitably generalize (1) to apply to the value of the objective function obtained from a heuristic, the optimal value of a particular variable, the value of a specified function of an optimal solution, and, in the case of a standard deterministic spreadsheet model, permit optimization to be absent altogether, provided that the output value of interest is unique for each p.

There are special cases where the mathematical properties of $f^*(p)$ are known, such as monotonicity. When the parameters are only the right-hand side coefficients in a linear program, the shape of $f^*(p)$ is known to be concave (maximization) or convex (minimization). Analogous properties hold when only objective-function coefficients vary. But even in simple examples, the shape of $f^*(p)$ may be irregular; to illustrate, consider the linear program maximize px subject to $x \leq \min(1, p)$ for $0 \leq p \leq 10$. The function $f^*(p)$ is neither convex nor concave. In this paper, we do not try to exploit special properties of $f^*(p)$; we return to this thought in Section 8.

We specify a domain P for managerially relevant values of p; in this section P may be a discrete set of points, if appropriate. We assume that a feasible solution exists and that the value $f^*(p)$ is finite for $p \in P$. For the approximation approach in Section 4 we impose an additional assumption that the output value is continuous for $p \in P$. These assumptions do not hold for all deterministic optimization models. The restrictions are likely to be met in real applications of linear and convex programming, and models in which the parameters affect only the objective function. There are familiar modeling approaches that can be applied to ensure feasible finite optimal solutions, such as including high-cost *artificial* variables and upper bounds on x. Note that a continuity assumption on the output value, which is imposed in Section 4, may not be satisfied if the parameters affect the constraints of a mixed-integer programming model, a dynamic program, an ill-conditioned nonlinear program, or if the output value is a decision variable.

It is convenient for expository purposes to characterize the variation in p as a joint probability density function $\pi(p)$. In many real applications, however, the separate parameters may be completely independent, and we may choose to let each parameter vary uniformly over a specified range. Attractive alternatives to a uniform distribution are normal and triangular distributions, and the latest versions of popular spreadsheet programs offer a wide selection of univariate densities. The shape of $\pi(p)$ should reflect a manager's assessment of parameter imprecision. There are some situations where the parameter distribution reflects a historical or empirical pattern.

We pointed out that complete probabilistic independence among the parameters does not necessarily imply that the model's coefficients are independent, because several model coefficients may depend on one or more of the parameters. *A fortiori* the specification of $\pi(p)$ is affected by whether the parameters are coefficients per se, or are variables used in functions that yield several coefficients. Technically, the choice of what is taken as a parameter determines $\pi(p)$ via the probability calculus and the mathematics of variable transformations. When there is leeway in this choice, I favor using fewer parameters and, where possible, maintaining probabilistic independence.

In practice, the specification of the domain P and the distribution $\pi(p)$ needs to be done with care. The unit and scale of measurement may differ among the parameters. A manager's predilection for using ±10%, say, for all parameter ranges may result in some parameters appearing to be more significant than is truly the case when further thought is given to specifying the extent of imprecision (see Eschenbach).

2.2. Definition of Global Sensitivity

We desire that our N-parameter sensitivity concept be intuitive and practical. Consider the variance of f^*

$$\sigma_{f^*}^2 = \underset{p_1 \ldots p_N}{E}[f^* - \mu_{f^*}]^2, \tag{3}$$

where

$$\mu_{f^*} \equiv \underset{p_1 \ldots p_N}{E} f^*(p_1, \ldots, p_N). \tag{4}$$

We propose that $\sigma_{f^*}^2$ be the measure of variation in f^*.

If all the parameters are completely independent and

$$f^*(p) = \sum_{k=1}^{N} F_k(p_k) \tag{5}$$

for known functions, then $\sigma_{f^*}^2$ is *exactly* the sum of the variances of the statistically independent separable functions. It immediately follows that if $F_k(p_k)$ has the largest variance, then no other variable causes more variation in f^* than p_k does.

Even when the parameters vary completely independently, however, the separable form (5) rarely holds exactly or even approximately. The N parameters do not usually influence the value of f^* in an additive way.

We propose two complementary sensitivity notions, both of which employ the familiar probability concept of conditional expectation (see Wilks 1962, and Kendall and Stuart 1979). In the first notion, for each parameter p_k, we consider the conditional expectation of f^* given values for all the other $N - 1$ parameters. We measure the variation in f^* about this conditional expectation, taking account of the fact that the other $N - 1$ parameters vary probabilistically. Denote the measured variation as σ_k^2. Using p_1 for illustration, this notion is stated formally as

$$\sigma_1^2 = \underset{p_2 \ldots p_N}{E} \underset{p_1 | p_2 \ldots p_N}{E}[f^* - f_1(p_2, \ldots, p_N)]^2, \tag{6}$$

where

$$f_1(p_2, \ldots, p_N) \equiv \underset{p_1 | p_2 \ldots p_N}{E} f^*(p_1, \ldots, p_N). \tag{7}$$

This measure of unexplained or remaining variation is evaluated for each p_k.

Using classic statistical methodology, we express sensitivity in terms of R square, which is the fraction of variation in f^* attributable to the *conditioning* variables,

$$R_{\sigma k}^2 = 1 - (\sigma_k^2 / \sigma_{f^*}^2). \tag{8}$$

The value of $1 - R_{\sigma k}^2$ is a measure of variation in f^* attributable to p_k.

If $1 - R_{\sigma k}^2$ is nearly 1, then p_k has a very large influence on the variation in f^* because very little of the variation in f^* is attributable to the other parameters. The larger the value of $1 - R_{\sigma k}^2$, the more p_k influences the variation in f^*, and if $1 - R_{\sigma k}^2$ is larger than $1 - R_{\sigma j}^2$, then we say that f^* is more sensitive to p_k than p_j.

This first notion attributes to a parameter the variation in f^* under the assumption that all other parameter values but this one are known. Later in the paper we refer to this sensitivity notion as the *all-save-one* version, and σ_k^2 as the *sigma measure*.

In the second notion, for each parameter p_k, we consider the conditional expectation of f^* given only the value of p_k. We measure the variation in f^*—due to the joint variation in all the other $N - 1$ parameters—about this conditional expectation, taking account of the fact that p_k varies probabilistically. Denote the measured variation as τ_k^2. Using p_1 for illustration, this notion is stated formally as

$$\tau_1^2 = \underset{p_1}{E} \underset{p_2 \ldots p_N | p_1}{E}[f^* - g_1(p_1)]^2, \tag{9}$$

where

$$g_1(p_1) \equiv \underset{p_2 \ldots p_N | p_1}{E} f^*(p_1, \ldots, p_N). \tag{10}$$

The value of τ_k^2 indicates the variation caused by all parameters other than p_k.

Here we calculate R square as the fraction of variation in f^* attributable to the conditioning variable p_k,

$$R_{\tau k}^2 = 1 - (\tau_k^2 / \sigma_{f^*}^2). \tag{11}$$

The larger the value of $R_{\tau k}^2$, the more p_k influences the variation in f^*. If $R_{\tau k}^2$ is larger than $R_{\tau j}^2$, then we say that f^* is more sensitive to p_k than p_j.

This second notion attributes to a parameter the variation in f^* under the assumption that only this one parameter value is known. We refer to this sensitivity notion as the *one-at-a-time* version, and τ_k^2 as the *tau measure*.

Thus, both notions provide a global assessment of sensitivity and they differ on the hypothesized conditioning of f^*. In our experimental tests, the sigma measure is marginally better at discerning which parameters have a

relatively weak influence on f^*, and the tau measure is marginally better at discerning which parameters have a relatively strong influence.

The suggested concept of explained variation is essentially a variant of R-square methodology frequently used in multivariate statistics. The measure is ordinal and relative. As is typical in multivariate statistical procedures, the behavior of a complex function of N variables—in the case of a deterministic optimization, a function without a stochastic noise component—is being assessed by a one-dimensional measure. Because R-square methodology has been applied widely in situations of statistical uncertainty, its strengths and limitations have been explored extensively, and caveats about its use are well recognized (see Kendall and Stuart 1979, Draper and Smith 1981, Hamilton 1987, Kruskal 1987, Inman and Helton 1988, Kruskal and Majors 1989, and Cuadras 1993). I think that the concept goes a long way to satisfying the main objective of global sensitivity analysis in that it indicates which imprecise parameters are important *vis à vis* insignificant. The empirical evidence for this claim will be presented later.

We point out that statisticians have proposed further refinements, involving standardized regression coefficients, for the special case when $f^*(p)$ is correctly specified as the right-hand side of (5), with $F_k(p_k) \equiv F_k p_k$, plus a random error that is uncorrelated with the parameter values; the parameters need not be independent (see Pratt 1987, and Bring 1994). This special separable form is rarely an appropriate approximation for the output from a deterministic optimization model.

2.3. Introductory Examples

Two examples further clarify the global sensitivity concept. The first reveals the nonlinear impact of multiparameter variation on f^*. The second examines the situation when the parameters are not probabilistically independent.

Consider this artificial example:

$$f^*(p) = p_1 \times p_2, \tag{12}$$

and assume that the two parameters vary independently. The variance of f^* is

$$\sigma_{f^*}^2 = \mathrm{Var}(p_1) \times \mathrm{Var}(p_2) + \mathrm{Var}(p_1) \times \{E(p_2)\}^2 + \mathrm{Var}(p_2) \times \{E(p_1)\}^2. \tag{13}$$

The two sensitivity notions result in the same *assessment* of sensitivity because the two variables enter the model symmetrically. Given only the value of p_2, the variance remaining in f^* caused by p_1 is

$$\sigma_1^2 = \tau_2^2 = \mathrm{Var}(p_1) \times [\mathrm{Var}(p_2) + \{E(p_2)\}^2], \tag{14}$$

and similarly, given only p_1, the variance caused by p_2 is

$$\sigma_2^2 = \tau_1^2 = \mathrm{Var}(p_2) \times [\mathrm{Var}(p_1) + \{E(p_1)\}^2]. \tag{15}$$

The R squares can be calculated by (8) and (11).

Observe that the values of (14) and (15) are equal if the mean (that is, the expected value) of each parameter is 0. In that case, f^* is equally sensitive to each parameter irrespective of the variances of the parameters. Such equivalence would not arise if (12) were a sum instead of a product; then f^* would be more sensitive to the parameter with the larger variance. Also, observe that if the variances of the parameters are equal, then the values of (14) and (15) differ only if their means differ. In this case, f^* is more sensitive to the parameter with the smaller squared mean. In contrast, if (12) were a sum, relative sensitivity would not be affected by the parameter means.

This example, in which the impact of parameter variation is nonadditive, shows why standard ad hoc scenario testing is problematic in presenting a complete and reliable sensitivity picture. In analyzing the variation in $f^*(p)$, we find that we are no longer in Kansas, so to speak.

Illustrative values of R squares are shown in Table I for example (12), where each parameter value can be either -5, 5, or 10, with the associated probabilities given in the table. Due to the simple form of (12), the R squares in Table I agree with the results of ordinary linear regression using probability weights for the joint outcomes. Each sample point $\{f_t^*, x_t, y_t\}$, where $x_t, y_t \in \{-5, 5, 10\}$, occurs with joint probability $\pi(x_t, y_t)$. The value of $R_{\tau k}^2$ is found by regressing f_t^* on x_t or y_t using $\pi(x_t, y_t)$ as the weight for sample point t (see SPSS® for Windows™ 1993). Note that the R squares are unaffected by scale changes in either or both of the parameters.

The next artificial example illustrates the issue that arises when parameters co-vary probabilistically, and points up the differences between the sigma and tau notions. Consider

$$f^*(p) = p_1 + p_2 + p_3. \tag{16}$$

Assume that p_1 and p_2, as they vary, have nearly identical values, but p_3 is independent and has slightly smaller variance than the variance of p_1 and p_2. Applying the one-at-a-time sensitivity notion, two-thirds of the variation in f^* can be explained by knowing p_1, or alternatively p_2, because having the value of one gives accurate information about the other. Only about one-third of the variation in f^* is explained by knowing p_3. This result accords with our intuition about how parameter variation in (16) influences f^*. Contrariwise, the all-save-one sensitivity notion is not so useful. Given p_1 (or p_2) and p_3, almost no variation in f^* about the conditional expectation remains, whereas given p_1 and p_2, a third of the variation in f^* about the conditional expectation remains and is attributed to p_3. Hence, p_3 appears to be the most influential variable, which does not accord with our intuition. We suggest that when the parameters vary jointly, it is better to use the *one-at-a-time* notion as an indicator of global sensitivity. It is important to realize that the variability ascribed to p_k in the tau measure reflects both

Table I
Illustrations of R Squares for $f^*(p) = p_1 \times p_2$

	p_1			p_2			p_1		p_2		R^2_τ	
Case	−5	5	10	−5	5	10	EV	Var	EV	Var	p_1	p_2
1	1/3	1/3	1/3	1/3	1/3	1/3	3.3	38.9	3.3	38.9	0.18	0.18
2	1/2		1/2	1/2		1/2	2.5	56.3	2.5	56.3	0.09	0.09
3	1/2		1/2	1/2	1/2		2.5	56.3	0.0	25.0	0.00	0.10
4	1/2		1/2		1/2	1/2	2.5	56.3	7.5	6.3	0.89	0.01
5	1/2	1/2		1/2	1/2		0.0	25.0	0.0	25.0	0.00	0.00
6	1/2	1/2			1/2	1/2	0.0	25.0	7.5	6.3	0.90	0.00
7		1/2	1/2		1/2	1/2	7.5	6.3	7.5	6.3	0.47	0.47
8	1/5		4/5	1/5		4/5	7.0	36.0	7.0	36.0	0.37	0.37
9	1/5		4/5	4/5		1/5	7.0	36.0	−2.0	36.0	0.04	0.55
10	4/5		1/5	4/5		1/5	−2.0	36.0	−2.0	36.0	0.09	0.09
11	1/5		4/5	1/5	4/5		7.0	36.0	3.0	16.0	0.19	0.47
12	4/5		1/5	1/5	4/5		−2.0	36.0	3.0	16.0	0.34	0.07
13	1/5		4/5		1/5	4/5	7.0	36.0	9.0	4.0	0.90	0.06
14	1/5		4/5		4/5	1/5	7.0	36.0	6.0	4.0	0.79	0.12
15	4/5		1/5		1/5	4/5	−2.0	36.0	9.0	4.0	0.95	0.01
16	4/5		1/5		4/5	1/5	−2.0	36.0	6.0	4.0	0.89	0.01
17	1/5	4/5		1/5	4/5		3.0	16.0	3.0	16.0	0.26	0.26
18	1/5	4/5			1/5	4/5	3.0	16.0	9.0	4.0	0.93	0.03
19	1/5	4/5			4/5	1/5	3.0	16.0	6.0	4.0	0.85	0.05
20		1/5	4/5		1/5	4/5	9.0	4.0	9.0	4.0	0.49	0.49
21		1/5	4/5		4/5	1/5	9.0	4.0	6.0	4.0	0.30	0.67
22		4/5	1/5		4/5	1/5	6.0	4.0	6.0	4.0	0.47	0.47
23	1/2	1/2		2/3		1/3	0.0	25.0	0.0	50.0	0.00	0.00
24	1/2		1/2	2/3		1/3	2.5	56.3	0.0	50.0	0.00	0.10
25		1/2	1/2	2/3		1/3	7.5	6.3	0.0	50.0	0.00	0.90
26	1/2	1/2			2/3	1/3	0.0	25.0	6.7	5.6	0.89	0.00
27	1/2	1/2			1/3	2/3	0.0	25.0	8.3	5.6	0.93	0.00
28	1/2	1/2		1/3	2/3		0.0	25.0	1.7	22.2	0.11	0.00
29	2/3		1/3	2/3		1/3	0.0	50.0	0.0	50.0	0.00	0.00

its own direct influence as well as the implied impact due to the probabilistic dependence among the parameters. This discussion is what motivates our reluctance in subsection 2.1 to choose the parameters in a way that results in covariation among them when it is possible, instead, to select parameters so that they vary independently.

The sigma and tau measures are first-order assessments (see Kendall and Stuart 1979, Kruskal 1987, Theil and Chung 1988, and Chevan and Sutherland 1991). They evaluate a parameter's relative influence on the variation in f^* in the presence (or absence) of all other parameter values. If an uncertain parameter that most influences the variation in f^* is subsequently specified precisely, then the relative assessment of the *remaining* parameters may differ from the first-order relative assessment. A related aspect is assessing the sensitivity of n-tuples, such as pairs of parameters. Many of the computational approaches that we will examine can be applied to n-tuple sensitivity analysis. In statistics literature, this general topic is referred to as partial correlation (see Wilks). Although it is my opinion that higher-order refinements will be of limited use to managers, future research may prove this opinion wrong.

2.4. Computational Strategies

In the next three sections, alternative algorithmic methods are offered for arriving at numerical assessments of global sensitivity. Each technique requires randomly generating parameter values using $\pi(p)$. This aspect is common to recent developments for obtaining numerical solutions in stochastic programming models (see Ermoliev and Wets 1988, Dantzig and Glynn 1990, Gasswmann 1994, and Infanger 1994).

The first approach is a Monte Carlo experiment that directly estimates sigma or tau measures along with $\sigma^2_{f^*}$, thereby providing estimates of $R^2_{\sigma k}$ or $R^2_{\tau k}$. The second approach uses least-squares regression (or an analogous technique) in a way that reveals the global sensitivity of each parameter. The third approach applies a classic statistical technique that appears to be effective for analyzing situations that prove difficult for the other two methods. Our empirical evidence indicates there is considerable consistency among these three approaches in distinguishing among significant and unimportant parameters.

We all are aware that in today's environment of widely available, fast, desktop PCs, it is possible for practitioners to run analyses that a decade ago would have been

Table II
3-Parameter Linear Program

Parameter	Monte Carlo $1 - R_\sigma^2$	Monte Carlo R_τ^2	Approximation $0.96 - R_\sigma^2$	Approximation R_τ^2	Paired $1 - Q_\sigma^2$	Chi-Square Significance
RHS(a_{20})	0.58	0.56	0.59	0.52	0.87	262
OBJFN(a_{01})	0.33	0.35	0.34	0.30	0.60	123
MATRIX(a_{12})	0.14	0.15	0.13	0.05	0.29	46

impractical or too expensive. This paper explores approaches that plainly take advantage of cheap computing.

3. MONTE CARLO ESTIMATION OF PARAMETER SENSITIVITY

Sigma and tau measures can be estimated directly by Monte Carlo sampling (see Sacks et al. 1989). In the case of σ_k^2, the values of the conditioning parameters are sampled S times, using the appropriate *marginal* distribution (see Gelfand and Smith 1990, and Casella and George 1992). When the parameters are completely independent, each conditioning parameter value can be sampled from its own marginal distribution. For each sample point s, T values for parameter p_k are sampled, using the appropriate *conditional* distribution; when the parameters are completely independent, p_k is sampled from its own marginal distribution. Each value of f_{st}^* is calculated and these values are averaged for $t = 1, \ldots, T$, giving $avgf_s^*$. Then the ST residuals ($f_{st}^* - avgf_s^*$) are calculated. The sample variance of these residuals is an estimate for σ_k^2. The value of τ_k^2 is estimated analogously. The results are conveyed in terms R squares, as given by (8) and (11); for this purpose, $\sigma_{f^*}^2$ is estimated from the same f_{st}^* values that are used to estimate σ_k^2 and τ_k^2, respectively.

When the distribution of f^* exhibits what would be viewed as outliers in a statistical context, it can be helpful to apply a monotonic transformation such as the square root or logarithm. Doing so will help to stabilize the Monte Carlo estimates for R squares.

3.1. Linear Programming Examples

Consider

$$\text{maximize} \sum_{j=1}^{n} a_{0j}x_j \tag{17}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq a_{i0} \quad \text{for } i = 1, \ldots, m \quad \text{and } x_j \geq 0. \tag{18}$$

In the first experimental example, n and m are 2. The imprecise parameters a_{01}, a_{12}, and a_{20} are completely independent, and each varies according to a uniform distribution between 0.5 and 1.5; the other five parameters equal 1. The value of S is 200 and T is 10. The resulting R squares are shown in the Monte Carlo columns of Table II. The variation in f^* is most influenced by the right-hand side coefficient, and least influenced by the constraint-matrix coefficient. We will refer to this example as the 2×2 *LP model*.

The second example differs from the first in that n and m are 3, and all 15 parameters vary randomly as before. The results are similar, as shown in Table III; here the entries are averages for the comparable parameters. The larger number of parameters attenuates the impact of the variation in each, but the pattern of relative importance persists. We will refer to this example as the 3×3 *LP model*.

3.2. Longest Path Example

Consider an *acyclic network model* with five nodes, indexed $i = 1, 2, \ldots, 5$, and ten directed arcs, where each node j is connected to every node i if $i < j$. The value of f^* is the length of the longest (a.k.a. critical) path, and each p_k is an arc length. The model may be viewed as a network linear program or a dynamic program. Each p_k is uniformly distributed between 0 and $2n/8$, where $n = j - i$ for arc(i, j). Consequently, every arc is on a critical path of total length 1/2 when all arcs lengths are at their mean $n/8$.

The R squares are shown in Table IV. As before, we let S be 200 and T be 10. Observe that arc (1, 5) accounts for the largest amount of variation in f^*, whereas the interior arcs (2, 3), (3, 4), and (2, 4) have the smallest impact.

Table III
15-Parameter Linear Program

Parameter	Monte Carlo $1 - R_\sigma^2$	Monte Carlo R_τ^2	Approximation $0.84 - R_\sigma^2$	Approximation R_τ^2	Paired $1 - Q_\sigma^2$	Chi-Square Significance	Chi-Square Stress Test
RHS(a_{i0})	0.20	0.20	0.131	0.111	0.43	39.3	77.0
OBJFN(a_{0j})	0.13	0.17	0.076	0.073	0.27	16.7	11.0
MATRIX(a_{ij})	0.06	0.11	0.005	0.015	0.11	3.8	2.6

Table IV
Acyclic Network

Arc Length	Monte Carlo $1 - R^2_\sigma$	Monte Carlo R^2_τ	Approximation $0.87 - R^2_\sigma$	Approximation R^2_τ	Paired $1 - Q^2_\sigma$	Chi-Square Significance
(1, 5)	0.425	0.378	0.420	0.332	0.810	140.0
(1, 4)(2, 5)	0.165	0.186	0.130	0.086	0.365	24.0
(1, 2)(1, 3)(3, 5)(4, 5)	0.115	0.140	0.098	0.053	0.220	13.0
(2, 3)(3, 4)(2, 4)	0.032	0.095	0.020	0.006	0.073	1.3

3.3. Computational Considerations

Monte Carlo sampling provides a universally applicable and direct way to obtain the global sensitivity of imprecise parameters in deterministic models. The computational burden is due to generating and solving NST sampled versions of the model; a reasonable value for ST is 2,000, although a smaller value may be sufficient. The examples in this section were analyzed on a 486 microcomputer by using a spreadsheet template to generate the results for each sample s, and a utility (see Palisade 1991) to provide the samples. Each sensitivity analysis experiment required no more than a day's effort.

3.4. Short-Cut Method

A practitioner who is comfortable with statistical methodology can apply a simplification of the all-save-one Monte Carlo procedure. The sensitivity result are analogous, but the approach does not provide an estimate of $R^2_{\sigma k}$.

In this method, T equals 2 and paired comparisons are analyzed. In our experiments, we set S equal to 1,000. For each parameter, the approach obtains the associated R square between f^*_{s1} and f^*_{s2}—this is easily done with any software containing a linear regression procedure. To avoid notational confusion, we designate the resulting R square as $Q^2_{\sigma k}$. Since all *other* parameters are the same in each sample pair, the R square will be relatively high for an unimportant parameter and low for a significant one. As before, we express sensitivity with respect to p_k in terms of $1 - Q^2_{\sigma k}$.

The values of $1 - Q^2_{\sigma k}$ for the examples in this section are displayed in the columns labeled *Paired* in Tables II, III, and IV. The results are relationally similar to $1 - R^2_{\sigma k}$ in these tables.

4. APPROXIMATION APPROACH

In this section, $R^2_{\sigma k}$ and $R^2_{\tau k}$ are estimated from standard least-squares regression fits using S sample observations. The analytic fits are linear in the regression coefficients of the mostly nonlinear terms involving the parameters (see Box and Draper 1959, Griffin 1977, and Wagner, Kathuria and Vargas 1992). The method, in effect, substitutes the power of statistical calculations for brute force estimation through Monte Carlo sampling—brain for brawn.

Here we impose the added assumption that the output value is continuous for $p \in P$. Although this is a restrictive technical assumption, it holds for many situations of practical importance, as the illustrations will demonstrate.

For the all-save-one notion, as a preliminary step we find a good approximation for f^* that contains nonlinear terms of the form

$$p_1^{r_1} \cdot p_2^{r_2} \cdots p_N^{r_N}. \tag{19}$$

In some situations, the terms also may be algebraic functions involving expressions of the form (19). Then, for each k, we set $p_k = 1$ and refit the approximation to obtain an estimate of $R^2_{\sigma k}$. For the one-at-a-time notion, we fit f^* by using *only* p_k. An estimate of $R^2_{\tau k}$ is obtained directly from a polynomial fit that allows terms p_k^r, where $r = -6, \ldots, 5, 6$. In effect, the regressions are estimating the functions (7) and (10).

Least squares can be viewed as an application of nonlinear programming. It is recognized in NLP that transformations of variables can impact the obtained solution. Likewise, statisticians find it useful to apply a monotonic transformation to random variables if their distribution is badly skewed. If a term in (19) has a negative exponent and the corresponding parameter has values that are near, or equal, to 0, then the fit may be improved by adding a sufficiently large constant to the parameter value so that the term in the fit is bounded away from 0.

First we illustrate the regression fits with the models given in Section 3. Subsequently, we summarize our success with other models and provide general guidelines for obtaining approximations.

4.1. Global Sensitivity Analysis of Linear Programming Models

A good numerical approximation to f^* for the 2 × 2 LP model is shown in Table V. The entire set of nonlinear terms allowed in the approximate fit for the 3 × 3 LP model is provided in Table VI. In both experiments we sample 2,200 sets of parameter values, fit the approximations for f^* with 2,000 cases, and use the remaining 200 cases as a holdout sample to assess a fit's accuracy. The fit for the 2 × 2 LP model has an adjusted R square (see Draper and Smith) of 0.96 in the holdout sample. For the 3 × 3 LP model, the least-squares model uses 122 terms

Table V
Approximation for Three Coefficients

Term	Coefficient
Intercept	−1.017102
a_{01}	−1.920974
a_{20}	3.460295
a_{01}^2	0.773771
a_{20}^2	−0.995632
a_{12}^2	0.071258
$1/a_{12}$	0.772993
$a_{01} \cdot a_{12}$	0.753313
$a_{20} \cdot a_{12}$	−0.777219
$a_{01} \cdot a_{20}$	0.053099
$1/a_{12}^2$	−0.176509
$1/(a_{01} \cdot a_{20})^2$	0.021800
$1/(a_{01} \cdot a_{12})^2$	−0.034013
$1/(a_{20} \cdot a_{12})^2$	0.009146

selected from those in Table VI, and has an adjusted R square of 0.84 in the holdout sample.

We obtained these statistical fits with SPSS® for Windows™, and used backward stepwise regression (see Mantel 1970, Draper and Smith 1981, Hamilton 1987, and Cuadras 1993), keeping the software's default criterion 0.10 for removal of a term. We caution that in our context this criterion should be viewed as an algorithmic choice, and not as a true probability criterion (no normality whatsoever is being assumed).

Notwithstanding a campus bumper sticker that proclaims *statisticians do it smoother with noise*, we emphasize that the discrepancy in the resulting approximations is due to misspecifying the implicit function that determines f^*, and not to random error. In contrast to typical statistical applications of regression, the dependent value f^* is determinate for a given p. Because misspecification occurs, there is a limit on how good the fit can ever be, irrespective of a large sample size.

The sensitivity results are displayed in the *Approximation* columns of Tables II and III. For the all-save-one fits, we subtract the adjusted R square from 0.96 in the 2 × 2 LP model, and from 0.84 in the 3 × 3 LP model, because the fits for f^* involving *all* parameters have these R squares. This gives values that are comparable to the R squares for the *one-at-a-time* fits. We observe that the sensitivity results using least squares agree with those from Monte Carlo sampling: the right-hand side coefficients have the largest influence on the variation in f^*, and the constraint-matrix coefficients have the smallest.

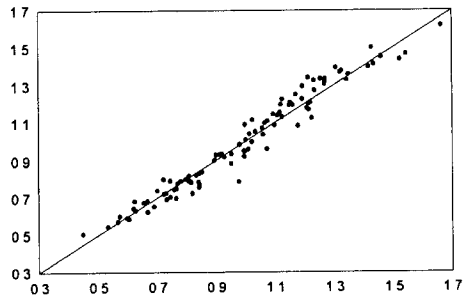
Scatter plots of 100 random samples of the approximations for the 2 × 2 LP model appear in Figure 1. Each point represents a value of f^* (shown on the vertical axis) corresponding to a fit value (shown on the horizontal axis). Observe that the constraint-matrix coefficient scatter for the all-save-one approximation does not differ much from the scatter for the overall approximation, whereas little covariation is in evidence in the comparable one-at-a-time scatter. Contrariwise, for the right-hand side coefficient.

The one-at-a-time approximations for $g_k(p_k)$ in the 2 × 2 LP model are illustrated in Figure 2. These functions are the probabilistic analog of classic parametric programming analyses (see Manne 1953, and Gass and Saaty 1954, 1955). Each approximates the expected value of f^* as a function of one of the parameters. The standard error of the estimate, which is provided by the regression software, summarizes the variation in f^* about this function. The regression for an unimportant variable gives rise to a nearly constant-value function—approximately the mean of f^*—and a standard error of approximately $\sigma_{f^*}^2$, whereas a very significant variable produces a relatively small standard error, and consequently, a function that varies as extensively as f^* does.

We summarize two analogous linear programming experiments in which m and n are 3. In one, only the six objective-function and right-hand side coefficients vary as before, and in the other, only the nine constraint-matrix coefficients vary. In both experiments we obtain adjusted R squares of about 0.91 with approximations for f^* that make use of all uncertain parameters. In these approximations, only a subset of terms in Table VI are pertinent.

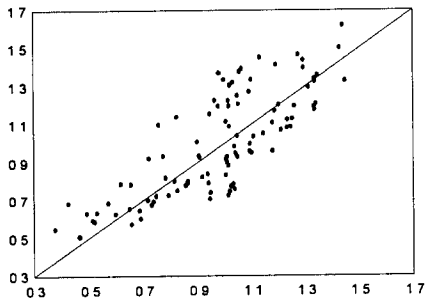
Table VI
Terms Used in LP Model Approximation

Term	Expression	Indices	Count
1	a_{i0}^2	$i = 1, 2, 3$	3
2	$(a_{0j} \cdot a_{i0})^2$	$j, i = 1, 2, 3$	9
3	$(a_{i0} \cdot a_{i+1,0})^2$	$i = 1, 2$	2
4	$1/(a_{i0} \cdot a_{i+1,0})^2$	$i = 1, 2$	2
5	$1/(a_{i0} \cdot a_{kj})^2$	$i, k, j = 1, 2, 3$	27
6	$1/(a_{0j} \cdot a_{ik})^2$	$j, i, k = 1, 2, 3$	27
7	$(a_{0j} \cdot a_{ij})^2$	$i, j = 1, 2, 3$	9
8	$(a_{i0} \cdot a_{ij})^2$	$i, j = 1, 2, 3$	9
9	$(a_{ij} \cdot a_{hk})$	Unique pairs	105
10	$1/(a_{ij} \cdot a_{ik})$ $1/(a_{ij} \cdot a_{kj})$	Unique pairs	18
11	$1/(a_{ij} \cdot a_{ik})^2$ $1/(a_{ij} \cdot a_{kj})^2$	Unique pairs	18
12	(a_{ij}/a_{ik}) (a_{ij}/a_{kj})	Unique pairs	18
13	$(a_{ij}/a_{ik})^2$ $(a_{ij}/a_{kj})^2$	Unique pairs	18
14	$(a_{ik}/a_{ij})^2$ $(a_{kj}/a_{ij})^2$	Unique pairs	18
15	$(a_{ij}/a_{i3})^2$	$i = 1, 2, 3$ $j = 1, 2$	6
	$(a_{k3}/a_{i3})^2$	$k < i$	3
16	$(a_{i1} \cdot a_{i2} \cdot a_{i3})$	$i = 0, \ldots, 3$	4
17	$(a_{1j} \cdot a_{2j} \cdot a_{3j})$	$j = 0, \ldots, 3$	4
18	$1/(a_{i1} \cdot a_{i2} \cdot a_{i3})^2$	$i = 0, \ldots, 3$	4
19	$1/(a_{1j} \cdot a_{2j} \cdot a_{3j})^2$	$j = 0, \ldots, 3$	4
20	$(a_{0j} \cdot a_{i0}/a_{ij})^2$	$i, j = 1, 2, 3$	9
21	$(a_{0j} \cdot a_{10} \cdot a_{20} \cdot a_{30})$	$j = 1, 2, 3$	3
22	$(a_{i0} \cdot a_{01} \cdot a_{02} \cdot a_{03})$	$i = 1, 2, 3$	3
23	$1/(a_{0j} \cdot a_{10} \cdot a_{20} \cdot a_{30})^2$	$j = 1, 2, 3$	3
24	$1/(a_{i0} \cdot a_{01} \cdot a_{02} \cdot a_{03})^2$	$i = 1, 2, 3$	3

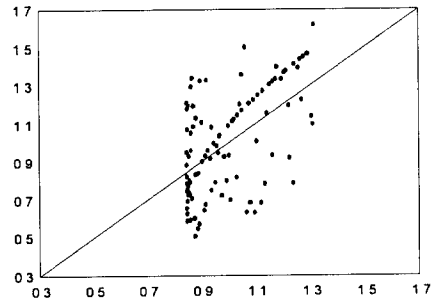


Regression Fit (R-square .95)

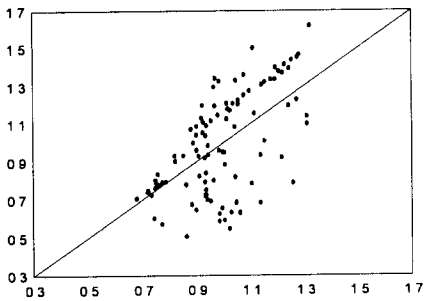
All save one

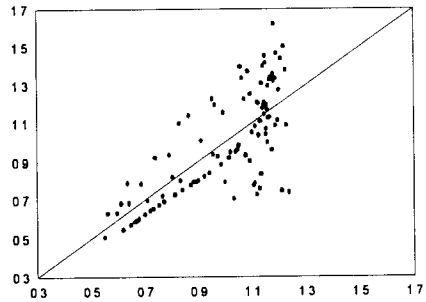


One at a time

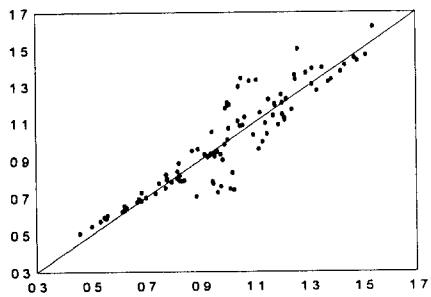


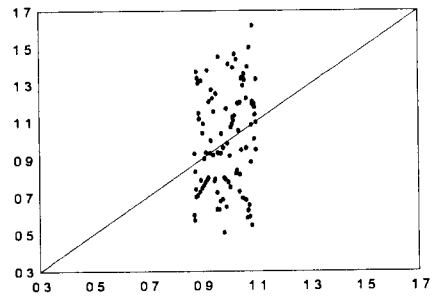
Objective-Function Coefficient





Right-hand-side Coefficient





Constraint-Matrix Coefficient

Figure 1. Scatter plots for the 2 × 2 LP model.

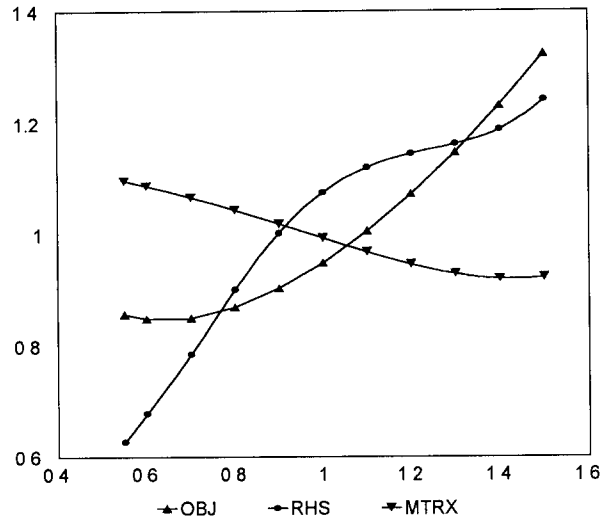


Figure 2. One-at-a-time approximations for the 2 × 2 LP model.

4.2. Acyclic Network Model

A good approximation to f^* in the 10 arc acyclic network is obtained with terms of the form c_i, c_i^2, c_i^3, c_ic_j, and $c_i^2c_j^2$, where c_i is an arc length. We use the same sampling strategy as in the linear programming examples. We obtain an approximation with adjusted R square of about 0.87 for the holdout sample. The companion global sensitivity analyses are shown in the *Approximation* columns of Table IV. The findings also parallel those in the Monte Carlo analysis.

We experimented with four variations of this network structure: the shortest path, triangular distributions for arc lengths, and the maximum of the minimum arc lengths among all paths from node 1 to node 5. The approximations to f^* in these tests have R squares between 0.83 and 0.92.

4.3. Test Results From Additional Small Examples

In presentations of these concepts, my colleagues have expressed surprise that least-squares fits with multinomial terms encompassing all parameters produce good approximations for f^* from optimization models. I used a battery of some 40 models to explore the robustness of the idea and to gain experience in selecting nonlinear terms to make available for the fits. All the models in the collection include only linear constraints. The models can be described in several equivalent ways (see Wagner 1975). The examples are small-scale tests—ten to twenty parameters—but the parameter distributions are selected, I believe, so as *not* to guarantee favorable results. In real applications, it is often known a priori that some parameters essentially have a larger influence than others on the optimal value. The distributions have been selected so as not to predispose particular variables to be present or particular constraints to be binding in the optimal solutions. This means that a good approximation does not result a priori from a narrow focus on only a subset of parameters.

The examples tested are

- knapsack model;
- dynamic economic lot-size model;
- linear assignment model;
- quadratic assignment model;
- traveling salesman model.

We assumed that the parameters are completely independent. We sampled 2,000 observations to obtain the fit and another 200 to test the fit. In these experiments, typically, the adjusted R squares exceed 0.90 for the holdout samples.

The *knapsack model* that we used is

$$\text{maximize} \sum_{k=1}^{N} p_k x_k \tag{20}$$

subject to

$$\sum_{k=1}^{N} x_k \leq 1 \quad \text{and} \quad x_k \geq 0. \tag{21}$$

We assume that each parameter p_k is distributed uniformly between 0 and 1, and let $N = 10$; an equivalent formulation for f^* is maximum $(p_1, p_2, \ldots, p_{10})$. The approximation for f^* has an adjusted R square of 0.96.

Because each parameter influences the model symmetrically, the challenge is to find a reasonable number of nonlinear terms that produce a close fit. We were successful with terms that use the expressions

$$a_j \equiv \sum_{k=1}^{N} p_k^j \quad \text{for} \quad j = 1, 2, \ldots, 5. \tag{22}$$

For the fit, we used terms a_ja_h, a_j^2, $a_j^2a_h^2$, $a_j^3a_h^3$, $1/a_j$, $1/a_ja_h$, and a_j/a_h.

We also tested six other variations that encompassed triangular distributions for the parameters, $N = 20$, and the constraint $\sum_{k=1}^{10} q_kx_k \leq z$, where, in addition to p_k, we also let q_k and z vary uniformly between 0.5 and 1.5. In the last mentioned case, we transform the model so that the kth parameter is $(p_k/q_k)z$. In these tests the adjusted R squares are between 0.90 and 0.99.

The dynamic economic lot-size model (see Wagner and Whitin 1957, 1958, and Manne 1958) can be viewed as the least-cost route in an acyclic network or as a dynamic program with special properties. It offers an example in which only a few parameters, namely, a holding cost h, a setup cost K, and a repeating pattern of demand requirements d_j, functionally determine all the arc costs—over 1,000 in the 48 period model that we tested. We assume that there is no initial inventory, and denote the nodes as $i = 1, 2, \ldots, 49$. The arc costs can be specified recursively as

$$c(i, i+1) = K$$
$$c(i, j) = c(i, j-1) + h(j-i-1)d_{j-1} \quad \text{for} \quad i+1 < j \leq 49. \tag{23}$$

The first six demands are specified as model parameters, and the sequence of six demands is repeated so that $d_{j+6} = d_j$. We assume that h is uniformly distributed between 0.5 and 1.5, and that K is uniformly distributed between 35 and 55. We tested four sets of probability distributions for generating the demands; they are lognormal and uniform distributions with differing patterns of means. Motivated by classic lot-size formulas, we included in the least-squares fits terms in which the exponents are 0.5. Specifically, the terms made available for the fit include c_k^2, $\sqrt{c_k}$, $c_k c_g$, $c_k d_j$, d_j, $\sqrt{d_j}$, d_j^2, $d_j d_g$, $\sqrt{c_k c_g d_j}$, $\sqrt{c_k c_g d}$, where $c_k \in \{h, K\}$ and d is the average of the d_j. The adjusted R squares for the approximations are at least 0.96.

We tested seven cases of the classic assignment model for $n = 3$ and 5. In these cases, the number of objective-function coefficients was 9 or 10, the distributions for the parameters were uniform or triangular between 0 and 1, and the sense of optimization was minimum or maximum. The terms made available for the fits include c_k, c_k^3, $c_k^2 c_g^2$, s_i^3, $s_i s_j$, $s_i^2 s_j^2$, $s_i^3 s_j^3$, where each c_k is an objective-function parameter, and each s_i is the sum of objective-function coefficients for either row i or column i of the n-dimensional assignment matrix. The adjusted R squares are at least 0.89.

We tested two cases of the quadratic assignment model with $n = 3$. This model differs from the standard one with respect to its objective function

$$\text{minimize} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{h=1}^{3} c(i, j, k, h) x_{ij} x_{kh} \qquad (24)$$

in that each coefficient is multiplied by the product of two decision variables. Without loss of generality, we have $c(i, j, k, h) = c(k, h, i, j)$, and for our test we set $c(i, j, i, j) = 0$. The constraints of the assignment model also allow us to set $c(i, j, i, h) = c(i, j, k, j) = 0$. When $n = 3$, there are six feasible assignments, which means that only 18 distinct $c(i, j, k, h)$ need to be specified.

In the two cases that we tested, either 9 or 18 coefficients vary uniformly between 0 and 1. The terms made available for the fits include s_j, s_j^3, t_j for $j = 1, 2, 3$, where $s_j = \sum_1^N p_k^j$ and $t_j = \sum_{k,l} (p_k p_l)^j$, and three other complex expressions explained next. The imprecise parameters can be arranged in a symmetric $n^2 \times n^2$ matrix. For $n = 3$, there are 9 unique row and column sums in this matrix. We let h denote an index of these sums, and P_h the corresponding set of coefficients. The three additional terms are $\sum_h w_h^2$, $\sum_h w_h^3$, and $\sum_{h,l} (w_h w_l)^3$, where $w_h = \sum_{k \in P_h} p_k$. The adjusted R squares are 0.88 for the smaller model and 0.83 for the larger.

We tested a five-city symmetric traveling salesman model in which each of the ten arc lengths is distributed uniformly between 0 and 1. We examined five cases that differed with respect to the sense of optimization, namely, shortest and longest tours, and uniform and triangular distributions for the parameters. The terms made available for the fits include $c_l c_k$, $c_l^2 c_k^2$, a_j, a_j^3, $1/a_j$, $1/a_j a_g$, a_j/a_g, where each c_l is one of the 10 imprecise arc lengths and $a_j = \sum_1^{10} a_l^j$ for $j = 1, 2, 3$. The adjusted R squares are at least 0.96.

In a few additional experiments, we altered the tests by letting several of the parameters co-vary jointly. The accuracy of the fit using all parameters improved.

We also tested several models, however, in which we were not able to obtain satisfactory approximations: the adjusted R squares were less than 0.80. In the knapsack model, using minimize instead of maximize in (20), the accuracy of the fit degraded to 0.77; the issue here can be addressed by transforming the model into an equivalent maximize problem. Similarly, when a constant was added to the uniform variates the accuracy of the fit degraded to 0.60. Again, a transformation of variables eliminates the issue. Using a minmax criterion in the acyclic network resulted in a poor fit; the adjusted R square was 0.73. For the bottleneck assignment and traveling salesman models—we tested both minmax and maxmin optimization in the two models—we obtained fits with adjusted R squares between 0.74 and 0.79. For eight experiments in which the adjusted R squares were below 0.90, a neural network produced fits with at least a 0.1 increase in the R squares, which represent a substantial improvement. We did not get satisfactory neural net fits, however, for the minmax route in an acyclic network and the minmax and maxmin optimizations in the traveling salesman model. Saddle-point models apparently are hard to approximate.

4.4. Obtaining Reliable Fits for f^* Using All Parameters

The previous examples and the industrial applications in Section 6 demonstrate that it is not particularly difficult, when there are 10 or so parameters that vary, to obtain a least-squares multinomial approximation for f^* having an adjusted R square of 0.85 or better when all the parameters are used. We pointed out, however, that there is no guarantee that this approach yields a good fit for any specific deterministic model. In referencing the classic Bolzanno–Weierstrass theorem (see Boor 1986), we are reminded of the old quip about a drunk and a lamp post: The theorem provides us with more support than illumination when it comes to obtaining close fits. At present, we believe that for some models a practitioner will not succeed in obtaining a good initial fit that is required for the all-save-one approximation, and, consequently, will have to resort to one of the other global sensitivity approaches suggested in this paper.

We have positioned the goal of obtaining a close fit for f^* as principally a means to an end, namely, global sensitivity analyses—scenario testing is a more effective way to get an accurate solution for *particular* parameter values. In discussing the fitting process with other researchers and practitioners, I have been urged to provide

a managerial interpretation for the resulting coefficients of the regression terms. While it may be possible in some applications to attempt such an interpretation, I am persuaded that usually the fit should be viewed in its entirety as an engineered result. Using an automotive analogy, you steer a car to the right or left by giving the wheel a sufficiently large turn clockwise or counterclockwise; you rapidly learn the feel of the wheel and easily transfer this knowledge skill to other cars. Accurate approximations also move in the appropriate direction, given a sufficient shift in the parameter values. We will explain how an R square value indicates the responsiveness in f^* to parameter shifts. I believe that this view—it is better to be approximately right than precisely wrong—is consonant with the thrust of current research in fitting-algorithms and neural networks, namely, to get well-engineered response functions.

4.4.1. Approximation Algorithms

There are a variety of alternatives to ordinary least squares for obtaining fits. The GMDH algorithm (see Farlow 1981, 1984) is a variant of least squares. This method iteratively generates terms of the form (19). Briefly described, the method repeatedly augments and then selects, using least-squares regression, multinomial terms to include in the fit. The GMDH approach resembles a nested algorithm for computing a power-series expansion. A very promising approach with respect to approximation accuracy is neural networks (see Talon 1992). Frequently, neural fits for the samples in this paper have R squares above 0.95 in both the training and holdout samples. Theoretical underpinnings establish the validity of approximating multivariate functions with neural networks (see Hornik, Stinchcombe and White 1989). Seminal work on multivariate spline fits also seems promising (see Breiman 1991, and Friedman 1991).

Compared to ordinary least squares, however, the preceding alternatives are far more computationally intense—remember that they must be applied not only to approximate f^*, but also to obtain the N sensitivity fits. Furthermore, they are not yet standard options in widely available commercial statistics software. I suspect that the approaches examined in this paper are presently the most practical for applied efforts.

4.4.2. Inferential Considerations

For assurance that the experimental estimates of $R^2_{\sigma k}$ are reliable assessments of global parameter sensitivity, we seek a large adjusted R square for the approximation of f^* using *all* parameters. Then the differences between this R square and the estimates of $R^2_{\sigma k}$ are good indications of the relative influence of p_k. I have found that adjusted R squares of 0.85 or larger when all parameters are used will give reliable sensitivity results.

There are some circumstances in which the approximation of f^* containing all parameters is useful per se to a manager. I am grateful to one referee who posed a situation where a parameter value actually represents a decision made *outside* the model; the suggested example was a budget limit. In this situation, a close fit can indicate the functional impact of this exogenous decision variable. A similar circumstance arises when f^* is an input to a different model. This circumstance often occurs in a set of cascaded spreadsheets.

For another example, consider a manager who wants to know whether a model's optimal value for a hypothesized set of parameter values is larger (say) than that for an alternative set. The manager can use the approximation of f^* for this purpose, rather than re-solving the model. The issue, then, is whether the sign of the difference between the two approximate values correctly predicts the sign of the difference between the corresponding true values. We have run Monte Carlo experiments with a variety of distributions for approximation errors and observed that the signs agree about 75% of the time when an R square is 0.50; the agreement increases to 83% for an R square of 0.75, to 85% for 0.80, to 90% for 0.90, 93% for 0.95, and to 97% for 0.99. We mention for the interest of researchers that there is considerable experimental evidence based on the many examples tested in this paper that the residuals from an accurate least-squares fit using all parameters resemble a normal distribution (see Prekopa). We have found that confidence intervals based on normal distribution tables are quite reliable and conservative.

Note that even when an R square is high, there can be a sizable *relative* error in the approximation f^*. The coefficient of variation, which is the ratio of the standard deviation of the fit error to the sample mean of f^*, is a commonly used statistic that summarizes this aspect of the fit.

The statistics literature contains extensive discussions regarding the usefulness and limitations of R square as the primary measure of fit accuracy (see Kendall and Stuart, and Draper and Smith). The reader is best advised to consult texts in multivariate analysis for further information on this topic. It is fair to say, however, that there is no alternative to R square that is as widely adopted by statisticians and also available in commercial software.

4.4.3. Local Perspective

We caution that a small perturbation in the components of p at an observation point may induce an increase in f^* but a decrease in the approximation, even when an approximation for f^* has a high R square. This implies that the partial derivatives of the approximation are not useful indicators of *local* sensitivity. It is better to use classic duality formulas when they apply to evaluate small parameter variations. This disparity in local behavior between the actual function and its approximation is also likely to arise when a complex method such as neural networks is used to obtain the fit. In special situations, it may be possible to limit the terms made available to the

fitting process to ensure a correspondence between the shape of $f^*(p)$ and its approximation; limited experience suggests that this is a challenging avenue for future research.

4.4.4. Experimental Design

On the basis of the results reported in this paper, we suggest that for 10 to 20 parameters, 2,000 sample observations are ample for a fitting experiment; we use 10% additional observations to validate a fit. We have tested several models using as many as 8,000 observations to obtain a fit and found no significant improvement in the accuracy with the larger sample size.

In a few situations, we have improved the accuracy by deriving a two-stage approximation (see Breiman and Meisel 1976, Breiman 1991, and Friedman 1991). This approach may add as much as 0.04 to the adjusted R square. Quartile values for the first fit are used to stratify the observations into four groups, each having about 500 observations. The final approximation for f^* is found by calculating the first fit, and on the basis of the quartile in which it falls, calculating the second fit. The quartiles so obtained are used in the sigma measure sensitivity fits as well. The two-stage method is a way to accommodate interactions that are not identified by all the terms included in the first fit. We experimented with increasing the number of observations and/or strata, and found that there is little if any improvement over using 2,000 observations separated into four nearly equal strata.

Evidence from the statistics literature (see Mantel 1970, Draper and Smith 1981, Hamilton 1987, and Cuadras 1993) indicates that selecting the backward option for stepwise regression is likely to yield better approximations. Whenever possible, this method uses all variables for an initial approximation, and then, to avoid overfitting, the algorithm sequentially drops variables according to a statistical criterion. There is little if any downside to being generous in the number of variables to make available for the fit, aside from adding to the computational burden. The danger of overfitting also is mitigated by testing the resulting approximation with a holdout sample. For models that give rise to outliers—specific observations that are very poorly fit—using standard outlier removal techniques to drop a few observations (less than 1% of the sample) and then refitting can add precision to the subsequent parameter sensitivity analyses.

In a linear programming model, such as (17) and (18), where the parameters are the model coefficients, consideration of the mathematical form of an optimal solution suggests terms such as those shown in Table VI. In many real applications, however, the parameters do not necessarily appear directly as model coefficients. Rather, they are contained in formulas that produce model coefficients. We can think of no universal prescription for the specific terms to make available to the least-squares fits. We suggest applying parallel reasoning: If several parameters affect objective-function coefficients, we would include terms analogous to those with a_{0j} in Table VI. We used this strategy in the industrial applications below.

4.4.5. Parameters That Co-Vary

The least-squares fit estimating the tau measure reflects the *combined* direct and indirect influence of a parameter when the parameters are not completely independent. This is an attractive property of the one-at-a-time notion *vis à vis* the all-save-one idea. In contrast, a transformation of variables is required prior to applying least squares to estimate the sigma measure. If the procedure just described is used without modification in this circumstance, then the sigma measure for a parameter will be underestimated because the least-squares fit will use probabilistic information about p_k that is contained in the values of the other parameters that remain in the fit. A reasonable approach is to regress the values of each p_j on p_k, and define P_j as the residual. Then use the set of P_j for $j \neq k$ as the parameters in fitting f^*. The approach is analogous to using instrumental variables in econometric modeling. Although I have had encouraging results with this transformation, I caution that the suggestion has not yet been thoroughly researched. We defer to Section 6 a detailed explanation of another approach for modeling covariation; the approach is analogous to calculating so-called beta coefficients in finance theory.

5. CHI-SQUARE ANALYSIS

In this section, we apply a classic multivariate statistical tool that is useful for situations where Monte Carlo and approximation approaches may not be effective. These instances include models in which the sampled distribution of the output measure exhibits discontinuities or unusually large (or small) values. For example, the distribution of optimal values of a decision variable in a linear programming model typically exhibits discontinuities if the variable is positive in some, but not all, optimal solutions. We reported in subsection 4.3 that we were unable to obtain accurate approximations for several small models. For experimental purposes, we will use as a *stress-test model* the linear program (17) and (18) with n and m equal to 3 and all 15 coefficients varying uniformly between 0 and 1. This test model does not yield useful approximations when fit with the terms in Table VI: The holdout sample gives a low R square even when a sample of 8,000 observations is used to obtain the fits. (We remark that a monotonic transformation of the input and output variables can be applied to yield a usable approximation.)

The approach, which is familiar in statistics literature (see Agresti 1990), recasts the data that we have used for one-at-a-time experiments in terms of crosstabulations.

The output measure is the row variable of each crosstabulation and an imprecise parameter is the column variable. A table is constructed for each parameter. For a continuous output variable, we generally recommend using 3-tiles (tertiles). In the case of the output being the optimal value of a linear programming variable that is continuous but not always in an optimal solution, we recommend that one row represent the 0 outcome, and the other two rows be constructed to have nearly equal frequency. A two-row table can be used for a dichotomous output variable to indicate, for example, whether or not an arc is included in a longest path of an acyclic network. There may be situations in which a practitioner wants to assess sensitivity with respect to, say, particularly large values of an output variable. Then a two-row table can be used with the row variable defined dichotomously. We generally recommend that the columns be defined as deciles for each parameter's distribution, if that is possible. Care in model formulation and solution must be exercised to ensure uniqueness of the output measure.

Values of the Pearson chi-square contingency statistic for crosstabulations yield measures of covariation analogous to values of $R^2_{\tau k}$ for assessing the global sensitivity of imprecise parameters. The chi-square statistic reflects nonlinear covariation as well as shifts in distributional shape. I think that a practitioner will find interpretation easier if the chi-square value is converted to a significance probability. A smaller probability is indicative of a more sensitive parameter. The computational steps that are required to construct crosstabulations and obtain chi-squares are easily accomplished with standard statistics software (see SPSS® for Windows™) or current versions of popular spreadsheets. Transforming chi-square values into significance probabilities for samples that are as large as those in this paper requires the direct computation of appropriate formulas (see Zelen and Severo 1964, and SPSS Statistical Algorithms 1991) or the application of tailor-made software (see Hawkins 1990).

Chi-square contingency analysis is a so-called nonparametric or distribution free statistical procedure. I would not recommend using another such procedure, the Spearman rank-order correlation, for determining relative importance. This method may perform poorly if any of the parameters has a nonmonotone influence on the output measure.

5.1. Sensitivity of Objective Function Value in the Small Linear Programs and Acyclic Network

The chi-square significance levels for the 2 × 2 LP, 3 × 3 LP, and acyclic network models in subsections 3.1 and 3.2 are shown in Tables II, III, and IV, respectively. The corresponding stress-test model values are also shown in the right-most column of Table III. We use 2,000 sample observations. The actual probabilities, which, due to the sample sizes, are very small, are expressed in scientific notation, and the values displayed in these tables are averages of the exponents, where the leading negative signs have been suppressed. A large (negative) exponent indicates a small probability. The significance levels reveal a pattern of relative sensitivity that is similar to that for the R squares. The stress-test model results agree directionally with those for 3 × 3 LP model, but show increased relative sensitivity to the right-hand side coefficients.

All of these experiments trichotomize the objective function value into tertiles and used deciles for each parameter. Hence, each crosstabulation has $(3 - 1) \times (10 - 1) = 18$ degrees of freedom. The sample observations are the same as those that were used for the corresponding approximation analyses.

5.2. Sensitivity of Decision Variables

The global sensitivity of x_j, $j = 1, 2, 3$, in (17) and (18) is assessed for the 3 × 3 LP and the stress-test models in Table VII. The first row of each crosstabulation contains the observations in which $x_j = 0$ in the optimal solution; in the other two rows, the positive values are dichotomized such that the row frequencies are nearly equal.

The results show that a variable's own objective-function coefficient is the most important in influencing its optimal value. For the 3 × 3 LP model, the right-hand side coefficients and constraint-matrix coefficients in other columns are unimportant. In the stress-test model, however, the right-hand side coefficients are as important as the other variables' objective-function coefficients and the variable's own constraint-matrix coefficients.

Global sensitivity results for the acyclic network model present a more complicated picture. A decision variable corresponds to whether or not an arc is included in a longest path. Accordingly, the crosstabulations for each decision arc have two rows. Because the frequency of an arc being used in an optimal solution varies among the arcs, the chi-square significance levels should not be compared among different decision arcs (except for the symmetric decision). Table VIII exhibits those arc length parameters that have a significant influence on whether an arc is used in an optimal path. Observe in Table IV that the length of arc(1, 5) has the most influence on the length of the optimal path, and in Table VIII that using arc(1, 5) is significantly influenced only by its own length. The length of arc(1, 5) also has an influence on whether other arcs are used, and sometimes the influence

Table VII
Three Decision Variables Chi-Square Sensitivity Analysis for 15-Parameter Linear Programs

Parameter	3 × 3 LP	Stress Test
OWN OBJFN	91.3	96.0
OTHER OBJFN	14.2	14.3
RHS	4.1	15.3
OWN MATRIX	12.9	15.0
OTHER MATRIX	2.1	1.4

is greater than the other arc's own length, such as occurs for arc(1, 2). Arc(2, 4) has negligible impact on the optimal path length, and its length is significant only for its own use. The length of arc(1, 3) has some significant impact on the optimal path, an important influence on whether its own arc is used, and a moderate influence on whether other arcs are used, such as arc(3, 5).

If two random variables are truly independently distributed as chi-square, both with k degrees of freedom, then their ratio is distributed as an F distribution with k, k degrees of freedom (see Wilks). This leads to a *rough* guideline that allows practitioners to compare whether chi-square *values* for two parameters are likely to be different rather than due to chance. To illustrate, suppose that the crosstabulations have 3 rows and 10 columns. Then the ratio of the larger chi-square value to the smaller should be at least as large as 1.85 [$= F(18, 18)$ at 0.10 level of significance] to consider them as different.

6. INDUSTRIAL MODELS

In this section, we examine three actual linear programming applications in manufacturing companies. By today's standards, the models are small. Nevertheless, the applications are two orders of magnitude larger with respect to the numbers of variables and constraints than the battery of test models that we examined previously, so they afford more realistic tests of the proposed sensitivity analyses. Practitioners will not be surprised that only a few parameters in these applications emerge as significantly influencing the variation in the objective function. It is not evident a priori, however, which parameters they are, or their relative impact. The all-save-one approximations are excellent and were easy to obtain. The one-at-a-time fits use the polynomial approach described in subsection 4.1. The chi-square significance probabilities are obtained with 3×10 crosstabulations, as suggested. The three approaches give virtually identical results with respect to assessing the influence of separate parameters.

Table VIII
Decision Variable Sensitivity in Acyclic Network

Decision Variable	Arc Length	Decision Variable	Arc Length	Chi-Square Significance
(1, 5)	(1, 5)			269
(1, 4)	(1, 4)	(2, 5)	(2, 5)	163
(1, 4)	(1, 5)	(2, 5)	(1, 5)	16
(1, 3)	(1, 3)	(3, 5)	(3, 5)	115
(1, 3)	(1, 5)	(3, 5)	(1, 5)	20
(1, 3)	(3, 5)	(3, 5)	(1, 3)	16
(1, 3)	(1, 2)	(3, 5)	(4, 5)	8
(2, 4)	(2, 4)			64
(1, 2)	(1, 5)	(4, 5)	(1, 5)	59
(1, 2)	(1, 3)	(4, 5)	(3, 5)	36
(1, 2)	(1, 2)	(4, 5)	(4, 5)	34
(1, 2)	(2, 5)	(4, 5)	(1, 4)	32
(1, 2)	(1, 4)	(4, 5)	(2, 5)	31
(2, 3)	(2, 3)	(3, 4)	(3, 4)	21
(2, 3)	(1, 3)	(3, 4)	(3, 5)	20
(2, 3)	(1, 5)	(3, 4)	(1, 5)	15
(2, 3)	(2, 5)	(3, 4)	(1, 4)	14
(2, 3)	(3, 5)	(3, 4)	(1, 3)	13
(2, 3)	(1, 2)	(3, 4)	(4, 5)	12
(2, 3)	(1, 4)	(3, 4)	(2, 5)	11

In applications such as these, subsets of model coefficients may be identical. For example, a process yield may appear in several time periods, or coefficients may be contextually related, such as the costs of similar raw materials. We have used an indexing approach to vary the coefficients in a family. This approach has become popular for financial portfolio model analyses. To illustrate, suppose that each c_i^k, $i = 1, 2, \ldots, I$, is a coefficient value in the kth family; then we assume that the parameter p_k is distributed, say, uniformly between 0.5 and 1.5, and use $p_k c_i^k$, $i = 1, 2, \ldots, I$, as the model coefficients. We perform the sensitivity analyses using parameter p_k.

6.1. Integrated Petroleum Model

This is a small-scale planning model for a southeast Asia state-owned petroleum company. It is comprised of one large integrated refinery, minor subsidiary processing facilities, and contractual arrangements with privately-owned refineries in neighboring countries. The company produces most of the country's consumption requirements, and also engages in imports and exports of crudes and refined products.

The model has 126 variables and 84 equality constraints, along with lower or upper bounds on 80 variables. The objective function is maximum contribution to profit, and contains costs and revenues pertaining to purchasing, storage, processing, swaps of crudes and final products, blending, transportation, distribution, and sales.

The parameters that vary are only in the constraint matrix, and not in the objective function, right-hand side, or bounds. They represent process inputs and yields. Given the functional relationships among these coefficients, we select and group 37 of the more than 700 nonzero model coefficients into ten families. Each of the parameters p_k, $k = 1, 2, \ldots, 10$, is distributed uniformly between 0 and a_k, where a_k differs among the ten groups. The 37 coefficients affect 24 variables and 8 constraints.

We fit the approximation with all parameters using 2,500 observations, and validate it with 200. Within this sample, the objective function varies between 78% and 115% of the overall mean. The terms made available to the fit include p_i, p_i^2, $1/(1 + p_i^2)$, $p_i p_j$, and $1/(1 + p_i^2 p_j^2)$. The fit has an adjusted R square of 0.95, and contains 47 terms.

The two approaches to sensitivity analysis reveal that one parameter which affects six variables and six constraints produces values for .95 $- R_\sigma^2$ and R_τ^2 of 0.40 and 0.32 (115); the number in parentheses is the exponent of the chi-square significance probability. The corresponding values for two other parameters are 0.25 and 0.17 (61), two more are 0.05 and 0.08 (10), and the remaining are 0.01 and 0.02 (1.6). This outcome is a broad classification of the yield characteristics as they influence profit contribution. The findings have managerial implications with respect to crude purchases and swaps, as well as refinery operations.

6.2. Consumer Semidurable Product Model

This model guides production planning for a U.S. manufacturer of highly seasonal semidurable items. The four principal products are refillable supplies for recreational devices, and they retail for less than $20. They are sold domestically and abroad.

The model includes seven stages in the product's manufacture; the make span of an item is at least two months, due to materials-aging requirements. The model variables are monthly amounts which are time staged over an entire year. Some of the variables designate materials, and others specify regular and overtime hours at the different stages of production. The manufacturing flows interrelate according to a parts explosion, which also specifies the necessary time delays to allow the semifinished components to age. The model coefficients reflect monthly sales requirements, recipes and yields, physical and labor capacity limits, as well as limits on overtime.

The model has about 400 variables and nearly 300 constraints. Total costs, due to labor use and inventory investment, are minimized. There are more than 1,600 nonzero coefficients in the model, and we select about 350 to vary. We group them into nine families, each with a parameter p_k that is distributed uniformly between $1 \pm a_k$. The nine groups span the objective-function, right-hand side, and constraint-matrix coefficients. They relate to sales requirements, capacity limits, overtime limits, product yields, and overtime costs. A few of the model coefficients are simultaneously affected by two of the parameters.

We fit the approximation with all parameters using 1,200 observations, and validate it with 200. Within this sample, the objective function varies between 5% and 500% of its overall mean. The terms made available to the fit include c^2, $1/c^2$, b_i^2, $1/b_i^2$, ac, b_ic, b_ia, $1/b_i^2c^2$, $1/b_i^2a^2$, b_ib_j, $1/b_i^2b_j^2$, b_i/b_j, b_i/a, c/a, cb_i/a, where c is the objective-function parameter, b_k ($k = 1, \ldots, 7$) is a right-hand side parameter, and a is the constraint-matrix parameter. The fit has an adjusted R square 0.98 and contains 53 terms.

The two approaches to sensitivity analysis reveal that the overall sales-requirement level has the largest impact; the resulting values for 0.98 $- R_\sigma^2$ and R_τ^2 are 0.68 and 0.56 (233). There is a gradual progression of sensitivity among the remaining parameters. Physical capacity and overtime cost have the next largest influence, with corresponding values of 0.15 and 0.10 (13), and 0.12 and 0.07 (12), respectively, Yields have an intermediate impact, and the overtime limits have only a slight impact (1.7).

These findings provide useful information to guide strategic decisions relating to improving the company's competitive cost position. The results lend perspective to the importance of the sales target *vis à vis* specific operational factors in the determination of labor and inventory costs, given that stock must be made considerably in advance of the seasonal peak.

6.3. Commodity Distribution Model

This application deals with a consortium of livestock feed producers in five European countries. The enterprise imports, exports, and processes agricultural products. There are three primary raw materials that are supplied by 23 continental producing areas. The materials are first processed in one of 58 plants, and some of the output is sold immediately to one or more of 23 demand locations in several countries; the remainder of the output is transferred to one of 18 plants for additional processing; the final product is sold as a higher-grade feed.

The model balances available supplies and anticipated demands, given processing limitations and national restrictions on importing and exporting. The objective function maximizes net contribution, which takes account of sales revenues, transportation charges, import/export duties, and processing costs.

The model contains nearly 700 variables, of which 145 have upper bounds, and more than 270 constraints. There are about 1960 nonzero coefficients, and we select 233 to vary. These are combined into nine groups: one is related to demand requirements, one to revenues, three to costs, three to processing capacities, and one to yields. We use the index approach to vary related coefficients, where the indices have uniform distributions.

We fit the approximation with all parameters using 700 observations, and validate it with 700. Within this sample, the objective function varies between 30% and 300% of its overall mean. The terms made available to the fit include c_i, c_i^2, c_ic_j, c_ib_j, c_ia, $1/c_i^2c_j^2$, b_i, b_i^2, $1/b_i^2$, b_ib_j, $1/b_i^2b_j^2$, b_ia, where c_k ($k = 1, \ldots, 4$) is an objective-function parameter, b_k ($k = 1, \ldots, 4$) is a right-hand side parameter and a is the constraint-matrix parameter. The fit has an adjusted R square of 0.99 and contains 33 terms.

The two approaches to sensitivity analysis indicate that the variation in the net contribution is mostly influenced by the revenue parameter, which when removed, gives values for 0.99 $- R_\sigma^2$ and R_τ^2 of 0.52 and 0.51 (148). One of the three capacity parameters and one of the three cost parameters are the next most influential, with

corresponding values of 0.24 and 0.22 (42), and 0.27 and 0.20 (39), respectively. All the remaining parameters have virtually no impact on R square (1.2).

These findings suggest that managerial attention be narrowed to a couple of factors that are geographically focused. Thus, despite the apparent diversity of influences on net contribution, there are in fact only a very few swing factors.

6.4. Software Considerations

All the industrial illustrations were analyzed on a 486/50 microcomputer. The generation of sample points was achieved using @RISK (see Palisade), a Lotus add-in.

The linear programming optimizations were done with XA (see Sunset Software 1992), because it is well suited for this sort of sensitivity analysis. The program permits compiling the entire model into an input file (for a base case) that can be read rapidly by the software. The software provides an option that permits the revision of specified model coefficients after the compiled version has been read; the software also provides a basis for an advanced start.

The sample points, that is, sets of model coefficients, are stored in spreadsheet files (and identified by range names). By means of a DOS batch file, the sample models are solved by XA in succession. The output from XA is saved in ASCII files that are later combined and read by the statistics software (see SPSS® for Windows™). The processing time is reduced if all the files reside on a RAM drive for fast access. For each of the industrial models above, only a few hours were required to optimize the entire set of sample models; the slowest of the applications required 12 seconds to solve each sample model.

An analyst with intermediate skills in using worksheets and DOS will find this setup straightforward. There are other widely available software packages that can achieve the same results at each step.

7. EXTENSIONS TO PROBABILISTIC MODELS

Methods for global sensitivity analysis have been described in the context of deterministic models. Probabilities only have been introduced to represent imprecision about model parameter values. These global sensitivity procedures also can be applied to many stochastic applications if we cast the models so that their stochastic components are represented as uncertain parameters of a deterministic model. In this section, we indicate briefly how to do this for several kinds of probabilistic models. Taking this view, we sample observations of parameter values and obtain the corresponding output values; then these observations are analyzed by the methods described above.

The simplest stochastic situations arise in models such as queuing and inventory replenishment in which model outputs can be calculated using one or more algebraic formulas that encompass a model's parameters, possibly including the moments of specified probability distributions. These situations are equivalent to deterministic spreadsheet models.

More complicated circumstances are spreadsheet models that contain random variables that take on different values with each recalculation of the spreadsheet. If there are not too many distinct random variables, then each can be viewed as an uncertain parameter along with the other model parameters of interest.

In some dynamic models, random variables are used repeatedly, and the outputs of interest are averages (possibly time weighted) of stochastic outcomes. A straightforward procedure can be applied for some of these situations. We illustrate the approach with a standard discrete-time inventory policy simulation, such as might be constructed with a spreadsheet model. In this application, the model simulates the average performance of a replenishment rule for Z periods. Assume that the model is for a single product, so that each period a random selection is made from a hypothesized demand distribution. We make Z sufficiently large so that there is negligible sampling error in the selected output averages. We randomly draw the demands only once, and then treat them as *deterministic* (that is, fixed) for the purpose of global sensitivity analysis. The uncertain parameters may include economic inputs, capacity limits, and numeric values that are part of the replenishment rule. Moments of the demand distribution also can be parameters if demands are determined using uniform variates and the inverse transform of the distribution function (see Wagner 1975); here we treat the uniform variates, once drawn, as deterministic.

More complex simulations can be analyzed similarly. Each sample observation required for global sensitivity indicates, for a given a set of parameter values, an output average from a simulation experiment in which the run length is long enough to ensure a sufficiently accurate assessment of the outcome. Thus, the output average is considered a deterministic function of the parameters (which may include the moments of underlying random elements). For this approach to sensitivity analysis to be practical, there must be only a small number of parameters and each experimental observation must not require more than a few minutes of computing time.

Component random variables that co-vary would not be unusual in probabilistic models. In such situations the one-at-a-time approaches to global sensitivity analyses may be more useful than the all-save-one approaches.

8. CONCLUSIONS AND RESEARCH AGENDA

This paper proposes global sensitivity analyses that can be widely applied to many deterministic and probabilistic models. The approaches build on tools that have been developed by statisticians for the analysis of complex multivariate relations. The ideas have broad conceptual

applicability encompassing optimization, heuristic, and spreadsheet models. In this paper, we have concentrated mostly on variation in the optimal value of the objective function of a mathematical programming model comprised of linear constraints. We have not yet studied models containing nonlinear constraints, and we have tested only small-scale combinatorial models. With a few exceptions, our examples assumed that the parameters varied independently, and we did not thoroughly investigate situations with probabilistic dependence among the parameters. We also have not probed in depth the extent to which sensitivity insights, such as those in Tables II, III, and IV, are robust to variations in $\pi(p)$. In several cases, we have found that terms that produce a good fit when a uniform distribution is assumed also produce a good fit when triangular distributions are used instead. This suggests investigating whether success with a uniform distribution is indicative of a fundamental robustness of the insights. These extensions need additional study to further flesh out the approaches.

The biggest limitation to the suggested sampling experiments may be excessive computation to obtain the sample points to analyze. For example, sensitivity analysis of an industrial-sized mixed-integer programming model is likely to present a challenge. Research is needed as to how many sample points are necessary to provide reliable results for the various approaches. A related aspect is the number of parameters that practically can be accommodated. Both the sampling technique and the experimental design strategy are subjects for further investigation (see Owen 1992).

When the distribution function for the output value varies smoothly as a function of the parameters, so that an approximation method can be used to capture sensitivity, then fitting methods other than ordinary least squares may prove attractive. We have cited our preliminary findings using neural networks to obtain a close fit. The derivation of effective approximation algorithms is a lively subject of research in the field of statistics, and methods that are successful can also be used for deterministic models (see Breiman and Meisel 1976, Farlow 1981, 1984, Breiman 1991, and Friedman 1991). In a similar research vein, there is the challenging task of finding a close approximation that has the same shape characteristics (e.g., convexity or monotonicity) as the actual underlying function.

In Section 1 we mentioned the possibility of using global sensitivity analysis as a precursor to solving a stochastic programming model, which provides a robust *solution* to an optimization model that contains imprecise parameters (see Dantzig 1963, Ermoliev and Wets 1988, Wets 1989, Dantzig and Glynn 1990, Gassman 1990, Infanger 1994, Mulvey, Vanderbei, and Zenios 1995). If an imprecise parameter is viewed as uncertain but knowable before any decisions are made, then a robust solution can be sought for any possible value of the parameter. A comparison between the expected value associated with the initial robust solution and the *expected value* of a parameter-induced robust solution provides the value of perfect information (VPI) for this parameter. An interesting avenue of research is to test whether there is a close correspondence between the VPI's and the measures in this paper.

Cox (1985) applied the Shapley value to a well-behaved multivariate response function to assign parameter influence. The computations involved averaging the function's partial derivatives. We cautioned in subsection 4.4.3 that the partial derivatives to the approximation for f^* are not reliable indicators of local response. Nevertheless, it may be possible to apply Shapley value axiomatics to a different formulation so as to obtain a reasonable assessment of parameter influence.

Several recent contributions by statisticians on the assignment of relative influence (see Chevan and Sutherland 1991, Kruskal 1987, Theil and Chung 1988, and Kruskal and Majors 1989) go beyond using only first-order approaches, as we have in this paper. These authors consider situations in which a model contains random noise. In their articles, the parameters in the regressions typically enter separably (and linearly), but all the parameters are allowed to co-vary. The algorithms consider fits with *sets* of n-tuples of parameters. We suggest, as an avenue for further research, finding practical situations where combining the results from sets of approximations may afford a clearer indication of parameter influence.

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