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Visualization in the Multiple Objective Decision-Making Framework

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Abstract. In this paper we describe various visualization techniques which have been used or which might be useful in the multiple objective decision making framework. Several of the ideas originate from statistics, especially multivariate statistics. Some techniques are simply for illustrating snapshots of a single solution or a set of solutions. Others are used as an essential part of the human-computer interface.

8.1 Introduction

We describe various visualization techniques which have been proven useful or which we feel might prove useful in the multiple objective decision making framework. We focus on fundamental visualization techniques (see Chapter 9, for more specific techniques). Several of our ideas originate from statistics, especially multivariate statistics. Typically, in the multiple objectives framework, the decision maker (DM) is asked to evaluate a number of alternatives. Each alternative is characterized using an objective vector. From the perspective of visualization, the complexity of the decision problem depends on two dimensions: the number of objectives and the number of alternatives. A problem may be complex due to a large number of alternatives and a small number of objectives, or the other way round, although the nature of the complexity is different. Different visualization techniques are required for each case. The number of alternatives may also be uncountable, such as a subset of a feasible region in an objective space in multiobjective optimization.

In descriptive statistics, computer graphics is widely used to illustrate numerical information by producing standard visual representations (bar charts, line graphs, pie charts, etc.). More advanced visualization techniques, for example, Andrews (1972) curves and Chernoff (1973) faces have also been proposed. Especially Andrews curves and Chernoff faces were developed to illus-

Reviewed by: Julian Molina, University of Malaga, Spain
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J. Branke et al. (Eds.): Multiobjective Optimization, LNCS 5252, pp. 195–212, 2008.

trate multivariate data; a problem closely related to ours. These techniques have been developed for problems in which the main purpose is to obtain a holistic view of the data and/or to identify clusters, outliers, etc. In the multiple objective framework, an additional requirement is to provide the DM with information (value information) for articulating preferences.

In this chapter we review visualization techniques, which are useful for illustrating snapshots of a single solution or a set of solutions in discrete and continuous situations. The evaluation of alternatives is a key issue in multiple objective approaches. Although this book is devoted to continuous MCDM/EMO-problems, the set of alternatives which the DM is asked to evaluate is generally finite and its cardinality is small. An example of an exception is Lotov's Generalized Reachable Sets method (Lotov *et al.*, 2004). Therefore, graphical methods developed in statistics to illustrate discrete alternatives are essential, irrespective of whether the MCDM/EMO-problem is continuous or discrete. In addition, many continuous problems are approximated with discrete sets.

To facilitate DM's evaluations, graphical representation is an essential part of the human-computer interface. For more information, see Chapter 9. Interactive methods are described in Chapter 2. The organization of this chapter is as follows. In Section 8.2 we consider the use and misuse of standard statistical techniques for visual representation of numerical data. In Section 8.3 we describe visualization in the context of a multiple objective framework. Section 8.4 provides a discussion and conclusion.

8.2 Visual Representation of Numerical Data

Graphical techniques have been considered extremely useful by statisticians in analyzing data. However, they have not been utilized by the MCDM- or EMO-community to their full potential in, for example, interactive approaches. Statisticians have developed a number of graphical methods for data analysis. Standard graphical techniques, such as bar charts, value paths, line graphs, etc., have a common feature: they provide an alternative representation for numerical data and there is a one-to-one correspondence between a graphical representation and numerical data (see any basic textbook in statistics, for example, Levine *et al.* (2006).) The graphical representation can be transformed back into numerical form (with a certain accuracy) and conversely.

8.2.1 Standard Statistical Techniques

In this subsection we review some standard graphical techniques, such as bar charts, line graphs, and scatter plots, which are widely used to summarize information in statistical data sets. As an example, we use a small data set consisting of the unemployment rate (%) and the inflation rate (%) in nine countries.

The bar charts are a standard technique to summarize frequency data (see, e.g., Figure 8.10), and they are called histograms in this context. In Figure 8.1, we use a bar chart to represent the (values of) unemployment rate (%) and the inflation rate (%) for each of the nine countries. This is customary in a multiobjective framework, where instead of summary data the values of variables (objectives) of various alternatives are more interesting. Let us emphasize that in this subsection we follow the terminology of statistics and talk about variables, but in connection to multiobjective optimization variables correspond to objectives. In other words, we refer to the objective space and not to the variable space of multiobjective optimization.

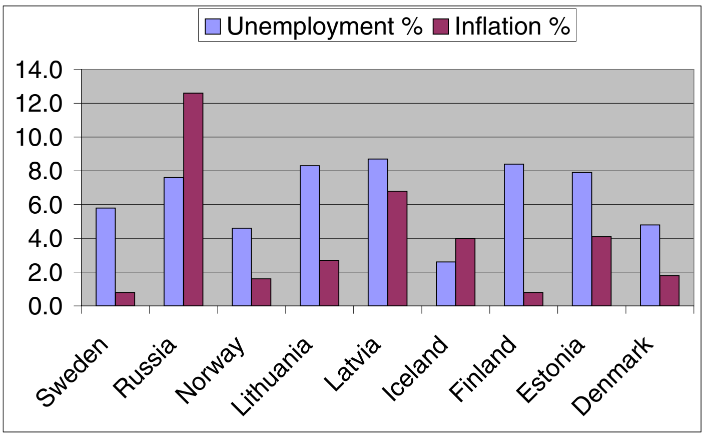


Fig. 8.1. Illustrating unemployment and inflation rates with a bar chart

Line graphs are another technique used like bar charts. (Line graphs (line charts) are called value paths in the multiple objective framework.) They are particularly appropriate, when the order of alternatives has a special meaning as in time series. In Figure 8.2, we ordered the alternatives in terms of decreasing inflation rates. This should make it easier to see how the unemployment rate and the inflation rate are related in different countries. The dependence between the unemployment rate and the inflation rate can alternatively be observed from the scatter diagram in Figure 8.3. Figure 8.3 is very useful in case where we are interested in recognizing the best (Pareto optimal) countries (Iceland, Norway, and Sweden).

There are also many other visualization techniques used in statistics, such as pie charts and boxplots, which may be useful in a specific multiple objective context. Pie charts are, for example, useful for visualizing probabilities and percentages. For further information, please consult any basic statistics textbook such as (Bowerman *et al.*, 2004).

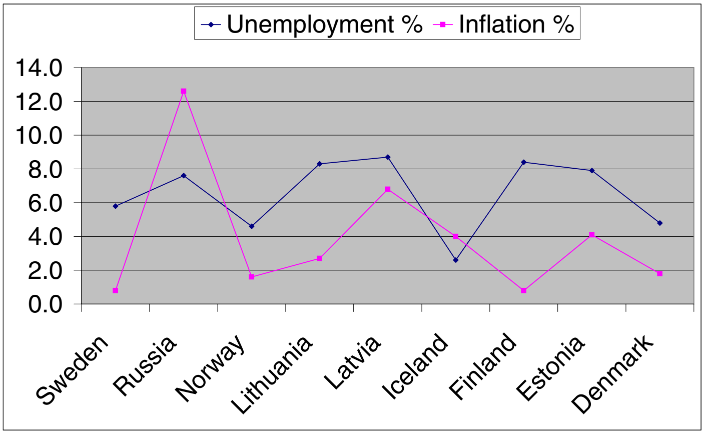


Fig. 8.2. Line graph

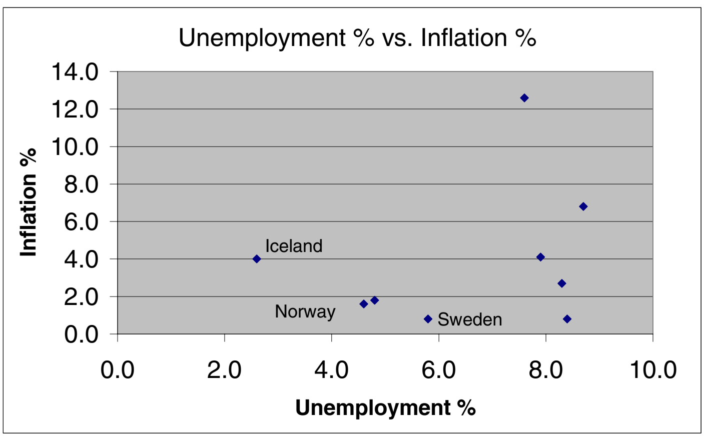


Fig. 8.3. Relationship between unemployment and inflation using a scatter diagram

8.2.2 Visualization of Multivariate Data

Visual representation is limited to two dimensions. Therefore, the main problem in visualizing multivariate data is to construct a two-dimensional representation, when the number of variables (i.e., objectives in the MCDM/EMO context) exceeds two. In statistics, two general principles have been applied to this problem:

1. reduce the dimensionality of a problem or
2. plot a multivariate observation as an object (an icon).

Principal component analysis and multidimensional scaling (MDS) (Mardia *et al.*, 1979) are two well-known techniques for obtaining a low-dimensional (specifically two-dimensional) representation of multivariate data, so that the data may be examined visually (see, for example, Everitt (1978)).

In principle, some standard graphical techniques, such as bar charts and line graphs can be used to illustrate more than two-variable data sets, but when the number of variables and/or alternatives increases the graphs quickly become unreadable. However, some standard techniques such as radar charts are more appropriate to illustrate multivariate data, provided the number of variables is reasonably small. As you can see from Figure 8.4, the radar chart is not very clear, even if we have only nine alternatives and two variables. However, the chart is readable if the number of variables is not large. A remedy to problems with a large number of variables is to represent them in different pictures.

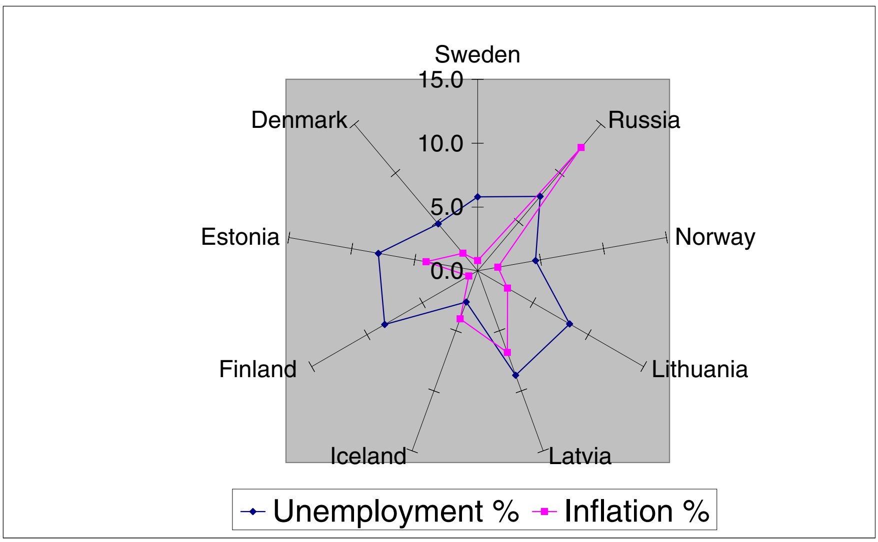


Fig. 8.4. Radar chart

In the early 1970's two promising techniques: Andrews (1972) curves and Chernoff (1973) faces were developed for visualizing multivariate data. Andrews plotted the following curve

$$g_i(t) = \frac{z_{i1}}{\sqrt{2}} + z_{i2} \sin t + z_{i3} \cos t + z_{i4} \sin 2t + \ldots$$

for each data point $z_i = (z_{i1}, z_{i2}, \ldots, z_{ip})$ over the interval -$\pi \leq t \leq \pi$, $i = 1, 2, \ldots, n$, where p refers to the number of variables (i.e., objectives in multiobjective optimization, denoted by k in this book). Thus, each observation is a harmonic curve in two dimensions. In this technique the number of variables is unlimited. The harmonic curves depend on the order in which the variables have been presented. Figure 8.5 reproduces the famous iris flower data (Fisher, 1936). The data set consists of three different species of iris flowers.

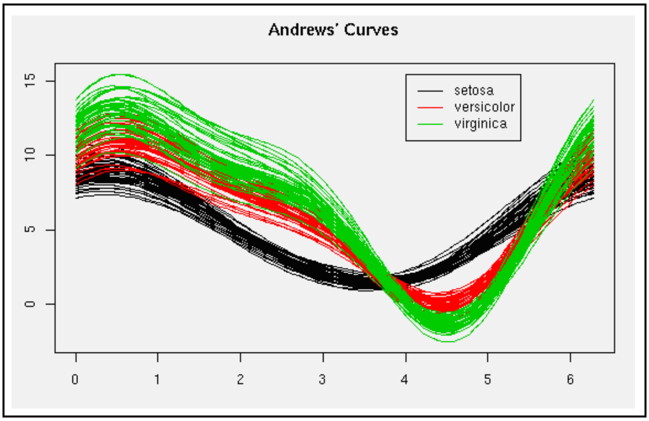


Fig. 8.5. Andrews curves (R Graph Gallery, 2007)

Chernoff used a human face to represent each observation graphically. The construction of Chernoff faces consists of geometrically well-defined elements, such as arcs of circles, arcs of ellipses, and straight lines. The values of variables are used as the parameters of these elements. Chernoff's original proposal consisted of 18 face parameters (Figure 8.6).

Fig. 8.6. Chernoff face

Andrews harmonic curves and Chernoff faces help us view similarities and dissimilarities between observations, identify clusters, outliers etc., but they are not very suitable for describing preference information. In Andrews curves, each curve stands for one observation, and the curves, which do not deviate much from each other, represent similar observations. However, Andrews curves are not good to illustrate the magnitude of variable values, and are thus not very practical to describing preference information. In a Chernoff face, it is easy to understand that a "smile" means something positive, but the length of the nose does not convey similar information. In addition, we have no knowledge about the joint effects of the face parameters. Big eyes and a long nose may make the face look silly in some user's mind, although big eyes are usually a positive feature.

In spite of the preceding disadvantages, the techniques provide us with new directions for developing visual techniques. Especially, there is a need for methods, which can also convey preference information.

8.2.3 A Careful Look at Using Graphical Illustrations

We should always present as truthful a representation of the data set as possible. However, it is possible to construct graphs that are misleading. We do not want that. One should always be aware of the ways statistical graphs and charts can be manipulated purposefully to distort the truth. In the multiple objective framework, where the DM is an essential part of the solution process, this may be even more important. For example, in interactive approaches, the DM is asked to react to graphical representations. A wrong illusion provided by a graph may lead to an undesirable final solution.

In the following, we present some common examples of (purposefully) misleading graphs. In Figure 8.7, we have described the development of traffic fatalities in Finland during selected years between 1980–2000 using histograms. The left-hand figure stands for the standard case, where the vertical axis starts from zero. In the right-hand figure, we present the same data, but start the vertical scale from 350 instead of 0. This makes the decrease in traffic fatalities appear more dramatic.

In Figure 8.8, we illustrate the per capita electricity consumption in Denmark and Finland. The consumption of electricity in Finland is about twice that of Denmark. It sounds appealing to illustrate the consumption by using an object somehow related to electricity such as light bulbs. Maintaining the shape of the light bulb in the left-hand figure makes the electricity consumption (height of the bulb) in Finland appear much larger than it actually is. In

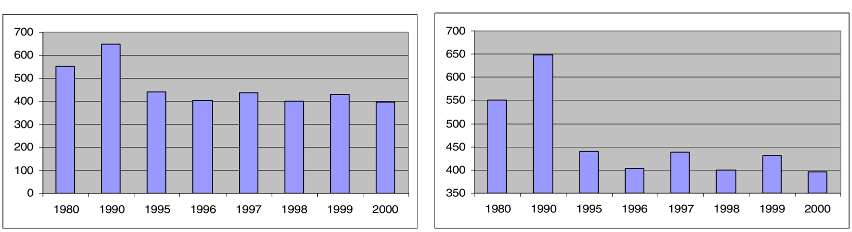
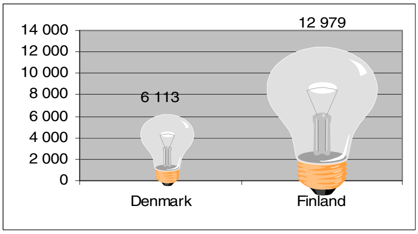


Fig. 8.7. Traffic fatalities in Finland: two representations

the right-hand figure, we have only stretched the height of Denmark's light bulb for Finland. This kind of illustration provides a correct impression.



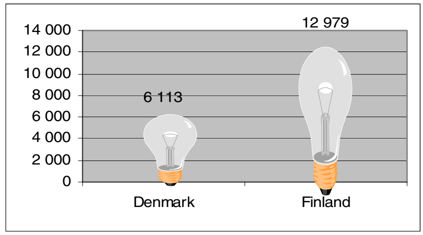
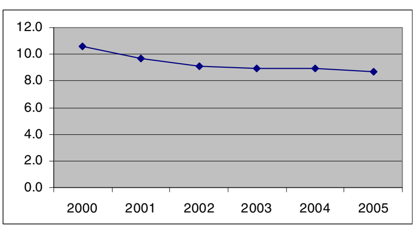


Fig. 8.8. Electricity consumption/capita in Denmark and Finland: two representations

In Figure 8.9, we illustrate the meaning of a stretched axis with a line graph. A neutral approach to describing, for example, the development of the unemployment rate by using a line graph is to start the percentage scale from zero and end it above the maximum, as is done in the left-hand figure. If we stretch the vertical axis and take the range roughly from the minimum to the maximum in the right-hand figure, it makes the downward trend look steeper, demonstrating "a dramatic improvement in the unemployment rate". Concerning the use of Figures 8.7–8.9 in an MCDM/EMO-context, it is not obvious which figure is better. For instance, sometimes it is useful to zoom in on some value range of objectives, especially, when we like to compare small differences in objective values. In other cases, a holistic figure is more desirable.



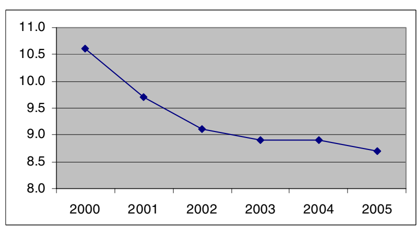


Fig. 8.9. Unemployment (%) in Finland during 2000-2005: two representations

Above, we have presented only some representative examples of misleading graphs. More information can be found in (Huff, 1954). In this classical book, the author shows how to take a graph and make it say anything you want. See also Wainer (1984).

8.3 Visualization in Multiple Objective Decision Making Approaches

Graphical techniques used with multivariate data have been of special interest for researchers working on MCDM problems, because of many similarities between these two problems. These techniques may also be used in MCDM/EMO problems to provide the DM with holistic information and to obtain a quick overall view of the relevant information, as well as detailed information for evaluation and comparison purposes.

Many authors have proposed the use of graphical techniques (bar charts, value paths, line graphs, etc.) to help evaluate alternatives (see, for example, Cohon (1978); Geoffrion *et al.* (1972); Grauer (1983); Grauer *et al.* (1984); Kok and Lootsma (1985); Korhonen and Laakso (1986); Korhonen and Wallenius (1988); Schilling *et al.* (1983); Silverman *et al.* (1985); Steuer (1986)). In Miettinen (1999), there is one chapter devoted to different visualization techniques. In most situations, the amount of information presented to the DM for evaluation may be considerable. A visual representation improves the readability of such information.

In statistics, the reduction of the dimensionality of a problem is a widely used technique to compress the information in multivariate data. If the number of objectives in the multiobjective context can be reduced to two (or three) without loosing essential information, then the data may be examined visually. Principal component analysis and multidimensional scaling (MDS) are also interesting from the point of view of MCDM. However, to our knowledge the principal component analysis has not been used for graphical purposes – with an exception of the paper by Mareschal and Brans (1988), in which they showed how to describe objectives and alternatives in the same picture. Korhonen *et al.* (1980) used MDS to reduce a four-objective problem into two dimensions and then described their search procedure in terms of a planar graph.

Graphical techniques have also been implemented as part of several computer systems developed for solving MCDM problems. Well known systems DIDASS (Dynamic Interactive Decision Analysis and Support Systems), Expert Choice, PREFCALC, NIMBUS, and Reachable Goals method are good examples. DIDASS has been developed by the System and Decision Sciences (SDS) research group at IIASA (Grauer *et al.*, 1984). Expert Choice has been developed to implement the AHP (the Analytic Hierarchy Process) (Saaty, 1980). PREFCALC has been proposed by Jacquet-Lagreze and Siskos (1982) for assessing a set of additive utility functions. Miettinen and Mäkelä (1995, 2000, 2006) developed the WWW-NIMBUS system, the first web-based MCDM decision support system, for nonlinear multiobjective optimization (http://nimbus.it.jyu.fi/). Lotov and his colleagues proposed the Reachable Goals method, in which the authors slice and visualize the Pareto optimal set (Lotov *et al.*, 2004). In addition, we would like to mention our systems VIG (Korhonen, 1987) and VIMDA (Korhonen, 1988; Korhonen and Karaivanova, 1999). For linear multiobjective optimization problems, VIG implements a free search in the Pareto optimal set by using a dynamic graphical interface called Pareto Race (Korhonen and Wallenius, 1988). VIMDA is a discrete version of the original Korhonen and Laakso (1986) method. The current version is able to deal with many millions of nondominated or Pareto optimal alternatives. In Figure 8.13 we display how a reference direction is projected into a set of randomly generated 500,000 alternatives.

8.3.1 Snapshots of a Single Solution

We can use graphical techniques to illustrate a single solution (objective vector, alternative). Classical techniques, such as bar charts, are mostly suitable for this purpose. The objectives are described on the x-axis and their values on the y-axis. Figure 8.10 illustrates unemployment rates in different population groups (total, male, female, youth, and long term) in Finland. This corresponds to a situation of one alternative and five objectives. When the number of objectives increases, the bar chart representation looses its ability to convey holistic information.

Chernoff faces (Figure 8.6) is one of the techniques, which makes it possible to provide information on an alternative with one icon. In the MCDM/EMO context, one has multiple alternatives to choose from. Sometimes, the choice is between two (or a few) alternatives (in what follows, we refer to such cases as discrete sets), sometimes the most preferred solution has to be chosen from an actually infinite number of alternatives (in what follows, we refer to such cases as infinite sets). In the following subsection we consider these situations.

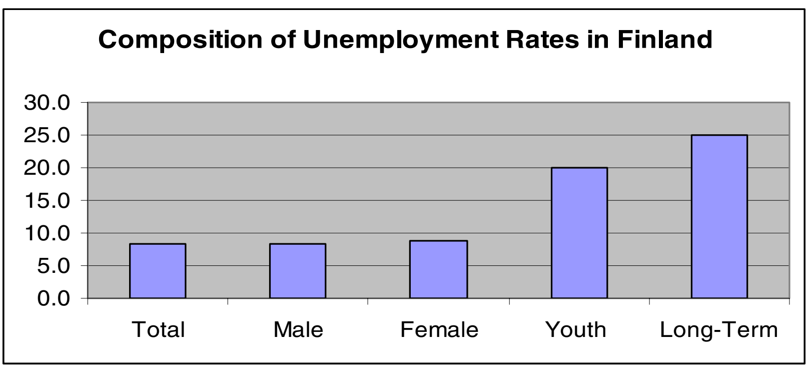


Fig. 8.10. Illustrating unemployment rates in different groups in Finland

8.3.2 Illustrating a Set of Solutions/Alternatives

We first consider finite sets and then extend the discussion to infinite sets (the continuous case).

Finite Sets

Bar charts, again, can be used to compare alternatives described with multiple objectives, when their number is small. In Figure 8.11 we compare the unemployment rates in five different population groups in Finland, Norway and Denmark. As we can see, the bar chart representation suits to objective-wise comparisons well. In this bar chart the country information is provided objective-wise. In fact one can think of Figure 8.11 as consisting of three separate bar charts presented in one graph. However, if one wants to compare across alternatives and choose the best one from the three countries, it is difficult to conclude whether the unemployment situation in Norway is better than in Denmark. (Finland is clearly the worst!)

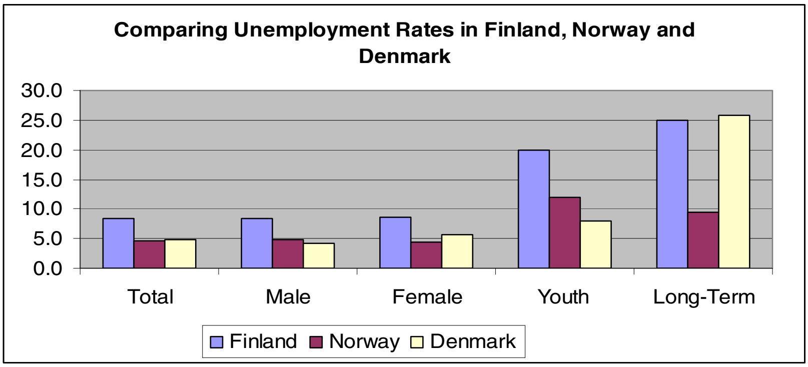


Fig. 8.11. Comparing unemployment rates in Finland, Norway, and Denmark

When the number of objectives is small, and we wish to compare alternatives, the bar chart representation is appropriate, but it would make sense to present the objectives pertaining to each alternative (country) together, as is done in Figure 8.1. A widely used alternative in MCDM is to use line graphs. Each line commonly stands for one objective. See Figure 8.2 for a typical two-objective example. The visual effect may be enhanced by ranking alternatives according to one objective, as is done in Figure 8.2.

A line graph may also be drawn in such a way that one line stands for one alternative, with the objectives on the x-axis. In the MCDM framework, this is often called a score profile (Belton and Stewart, 2001).

When standard graphical techniques (e.g., bar charts, value graphs, line graphs) are used to visualize alternative solutions, whether a solution is more or less preferred to another one in terms of some objective can be seen in the details. Short bars, lines with a negative slope, etc. stand for good values or improvements (in the minimization case). However, obtaining a holistic perception of these pieces of preference information, when there are a lot of details, is often impossible. On the other hand, advanced techniques, such as

Chernoff faces and Andrews harmonic curves, help the DM obtain a holistic perception of the alternatives, but do not provide a good basis for the purpose of evaluation. Therefore, Korhonen (1991) developed an approach, which transforms a vector into a picture in the spirit of Chernoff faces and Andrews curves, but which also enables a DM to see information that influences her/his preference on the basis of a visual representation. The underlying ideas are based on the use of two concepts: harmony and symmetry. For the icon, Korhonen chose a simple "harmonious house". The ideal (standard/normal) house is described in a harmonious (and symmetric) form. Deviations from this ideal are perceived as abnormal. The "degree" to which a house resembles the ideal serves as the basis for evaluation.

The first requirement for the icon is that it can be parametrized in such a way that by improving the value of an objective the icon becomes "better" or more "positive" in some sense. To convey this positive-negative information we can apply the concepts of harmony and symmetry.

The structure of the house is controlled by varying the positions of the corner points. Each corner point is in its default position and allowable moves of corner points are shown as squares. An objective can now be associated with the x- or y-coordinate of any corner point in such a way that the ideal value of the objective corresponds to the default value of the coordinate and the deviation from the ideal value is shown as a move in an x- or y-direction. The x- and y-coordinates of corner points are called house parameters. Two objectives can be associated with each corner point, one affecting the horizontal position and the other the vertical position of the corner point. When the value of the objective deviates much from the ideal, it has a dramatic effect on the position of a corner point.

A preliminary version of an interactive micro-computer decision support system known as VICO (A VIsual multiple criteria COmparison) has been developed to implement the above idea. A preliminary version of VICO has been used for experimental purposes with student subjects. The results were encouraging. Figure 8.12 refers to a test situation, where the subjects compared 20 companies using 11 objectives, consisting of detailed financial information. The purpose of the test was to identify the three companies, which had filed for bankruptcy. One of the companies (Yritys 19) in Figure 8.12 was one of the bankrupt companies. Note how disharmonious it is compared to a well-performing company (Yritys 1). In the system, the houses are compared in a pairwise fashion.

The method is dependent on how the house parameters are specified. It is important to associate key objectives with the house parameters that determine the structure of the house – thus influencing the degree of harmony. Of course, the directions of the changes also play an essential role. The method is subjective, and therefore it is important that a DM takes full advantage of this subjectivity and uses his/her knowledge of the problem and its relationships in the best possible way. VICO is quite suitable for MCDM problems, where the number of objectives is large and one uses pairwise comparisons to collect preference information.

An even more popular approach to visualizing a set of solutions is to use line graphs. This approach has been used, for example, in VIMDA (Korhonen, 1988). VIMDA is a "free search" type of approach that makes no assumptions, except monotonicity, about the properties of the DM's value function. It is a visual, interactive procedure for solving discrete multiple criteria decision problems. It is very suitable to continuous problems as well, when the number of objective vectors to be simultaneously evaluated by the DM is large. The search is controlled by varying the aspiration levels. The information is used to generate a set of discrete alternatives, which are provided for evaluation in a graphical form described in Figure 8.13.

The objective values in Figure 8.13 are shown on the ordinate. The current alternative is shown in the left-hand margin. The objective values of consecutive alternatives have been connected with lines using different colors and patterns. The cursor characterizes the alternative whose objective values are printed numerically at the top of the screen. The cursor moves to the right and to the left, and each time the objective values are updated. The DM is asked to choose his/her most preferred alternative from the screen by pointing the cursor.

Using this procedure, the DM is free to examine any Pareto optimal solution. Furthermore, this freedom is not restricted by previous choices. The currently implemented version of VIMDA does not include a stopping criterion based on a mathematical optimality test. The process is terminated when the DM is satisfied with the currently best solution.

Infinite Sets

The Geoffrion *et al.* (1972) interactive procedure was the first to present the idea of a (one dimensional) search in a projected direction in the context of continuous multiobjective optimization. They implemented the classic Frank-Wolfe (single objective) nonlinear programming algorithm for solving multiobjective problems. Korhonen and Laakso (1986), in their reference direction approach for solving continuous multiobjective optimization problems, adopted the idea of a visual line search from Geoffrion *et al.* (1972). In their approach, the DM provides his/her aspiration levels for the objectives, thereby defining a reference direction. This reference direction is projected onto the Pareto optimal set. Solutions along the projection are presented to the DM, who is assumed to choose his/her most preferred solution along the projection. The algorithm continues by updating the aspiration levels, forming a new reference direction, etc.

Lotov's "slices" represent an interesting visualization technique for continuous multiobjective problems (see, for example, Lotov *et al.* (1997, 2004)). The approach is based on a visualization of the feasible set in the objective space. For details, see Chapter 9.

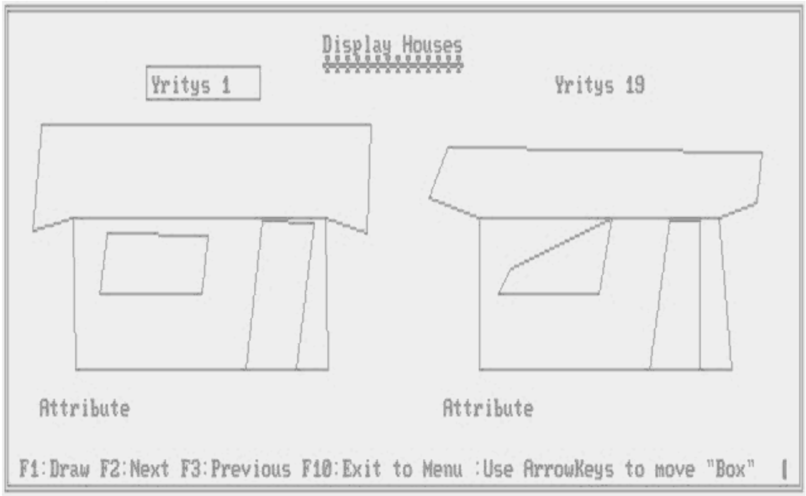


Fig. 8.12. Two harmonious houses

For other techniques providing a graphical illustration of Pareto optimal solutions, see, for example, Miettinen (2003).

8.3.3 Dynamic Representation of a Set of Solutions

Pareto Race (Korhonen and Wallenius, 1988) is a dynamic version of the Korhonen and Laakso (1986) reference direction approach. It enables a DM to move freely in the Pareto optimal set and, thus to work with the computer to find the most preferred values for the objectives (output variables). Figure 8.14 shows an example of the Pareto Race screen. In Pareto Race the DM sees the objective values on a display in numeric form and as bar graphs, as (s)he travels along the Pareto optimal set. The keyboard controls include an accelerator, gears and brakes. The search in the Pareto optimal set is analogous to driving an automobile. The DM can, for example, increase/decrease speed and brake at any time. It is also possible to change direction.

8.4 Discussion and Conclusion

In this chapter we have considered the use of graphics in the multiobjective decision making framework. The main issue is to enable the DM to view objective vectors (multidimensional alternatives) and to facilitate preference comparisons. The alternatives may be presented one at a time, two at a time, or many at a time for the DM's consideration. Based on the available information, the MCDM procedures ask the DM to choose the best (or the worst)

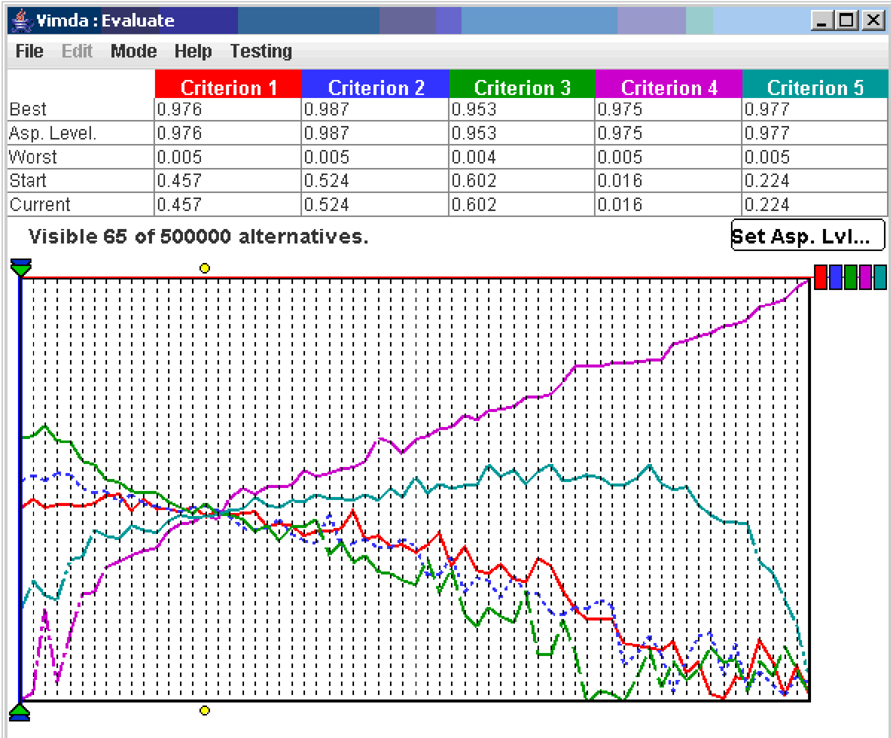


Fig. 8.13. VIMDA's visual interface

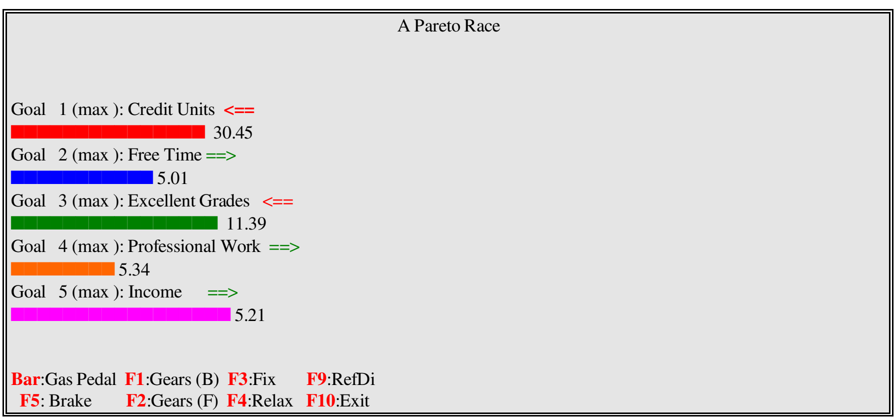


Fig. 8.14. Pareto Race interface

from the set of displayed alternatives, as well as to rank, or cluster the alternatives. In many cases one can use standard statistical techniques, or their variations, developed for visualizing numerical data. In statistics the main purpose is to provide a tool for classifying, clustering or identifying outliers – in short to obtain a holistic view of the data. In MCDM/EMO, the DM needs support in expressing preference information, based on displayed numerical data. Hence, in some cases more advanced techniques are needed for visualizing alternatives.

We have explored to what extent standard graphical techniques can be used in the MCDM/EMO framework, and reviewed some more advanced techniques specifically developed for the MCDM problem. Examples include the use of dynamic bar charts, such as the Pareto Race interface (Korhonen and Wallenius, 1988), and Korhonen's harmonious houses (Korhonen, 1991). Typically, the graphical representation is an essential part of the user interface in interactive MCDM procedures.

There exist several other ways to utilize graphics in MCDM procedures. For example, Salo and Hämäläinen (1992) developed a method allowing the DM to express approximate preference statements as interval judgments, which indicate a range for the relative importance of the objectives. The ranges are given as bar charts in an interactive manner. Moreover, Hämäläinen and his colleagues have implemented a system by name HIPRE 3+ (http://www.hipre.hut.fi/), which includes the above idea and other graphical interfaces. A typical example is a graphical assessment of a value function (Hämäläinen, 2004).

An important problem in MCDM/EMO is to provide a tool, with which one could obtain a holistic view of the Pareto optimal set. This is difficult in more than two or three dimensions in the objective space. Interestingly, in EMO the nondominated set evolves from one generation to the next; hence it would be important to visualize this process. The two-dimensional case is easy, but to obtain a good visualization in three dimensions is already challenging (for additional discussions, see Chapter 3).

There is clearly a need to develop advanced techniques, which help DMs evaluate and compare alternatives characterized by multiple objectives. Problems where the number of objectives is large are especially challenging. We need techniques which help DMs make preference comparisons between alternatives, which are characterized with tens of objectives (criteria, attributes). It is quite plausible that such techniques will be developed in the near future. To visualize search in the multiple objective framework is not easy, but even the current techniques provide useful tools for this case. In fact, when we developed Pareto Race (Korhonen and Wallenius, 1988), we noticed that it was very demanding to "drive a car" in ten dimensions, but using moving bar charts we were able to implement the idea. The visualization of the whole Pareto optimal set in more than three dimensions is a problem which may be too complicated to solve. However, we may surely invent partial solutions!

Acknowledgements

The authors wish to thank Kaisa Miettinen, Julian Molina, Alexander Lotov, and Mariana Vasileva for useful comments. Moreover, we would like to acknowledge the financial support of the Academy of Finland (grant 121980).

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