# Dynamic Risk Measures for Discrete-time Process

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## Abstract

*In this paper, we establish a class of dynamic risk measures for evaluating discrete-time process. Our research is mostly concentrated on the properties of dynamic risk measures. The properties of risk measures in the static framework, are introduced into the dynamic risk measures framework. We present the conception of capital requirement for discrete-time process to measure the risk in the dynamic framework. In particular, four of axioms about dynamic risk measures have been proposed in the third section. We establish strong, middle and poor consistency properties to show our efforts in the mathematics description of the dynamic risk measure of discrete-time process.*

***Keywords:*** *General probability space; risk measure; dynamic risk measure.*

## 1. Introduction

Since the corruption of LTCM(Long-Term Capital Management) in 1998[1], both financial scholars and practitioners have been searching for methods to avoid the similar disasters in the existing financial system. Furthermore, risk measure has been the most important part in the setting of capital requirements in the recent Basel Accords on banking supervision.

Artzner, Delbaen, Eber, Heath finished their creative works in the axiomatic theory of risk measures in 1999[2]. Coherent measure of risk has been the most solid ground of further research on risk measures. Four properties which are considered the minimal requirement for methods of risk measures were proposed in a static framework[3]. The research of properties which a perfect method of risk measurement should include, has been going forward since the coherent measure of risk was born. Subsequently, Föllmer and Schied widened the scope of coherent risk measures, defining convex risk measures[4]. They proposed to replace the original subadditivity and translation invariance with convexity, while kept the original monotonicity and positive homogeneity.

Although great achievements have been made in the field of axiomatic theory of risk measures, they share the common restriction that they are all one-period risk measures. That is to say they are all constructed in the static framework. There are at least three reasons why single-period treatment of a multi-period problem can lead to financial disasters. Firstly, in many applications, risks are inherently multi-period due to intermediate cash flows. For example, a financial position in futures results in a cash flow everyday due to marking to market[5]. If the liquidation period of the position is more than one day, these cash flows occur over time until the position is liquidated. Most of practical cases illustrate how significant the risk from the intermediate cash flows can be even when the hedge is perfect. Having to account for intermediate cash flows is at the heart of the difference between a single-period and a multi-period measure. Dynamic risk measure is relevant when intertemporal aggregation of cash flows is important. Secondly, dynamic risk measure is the demand for consistency in meeting regulatory requirement. Reconciling the day to day computation of their risk exposures has been a major problem for financial institutions using single-period measures. The moving horizon problem can only be resolved by using a dynamic risk measure. Thirdly, dynamic risk measure is due to risk control requirement, risk management is often a component of larger optimization problem, such as portfolio management[6]. Since the optimization problem is dynamic, then, to integrate risk management into the dynamic problem in a consistent way[7], the risk measures must also be dynamic.

The purpose of this paper is to establish risk measures for process in the dynamic setting. Financial position is described by the discrete-time process. The reason for this extension is dictated by the undeniable fact that in current financial practice risky payoffs are usually spread over different dates. This aspect seems to be neglected by the common methodology to measure risk of payoff

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streams. We will introduce the conception of capital requirement for process which is the basis of dynamic risk measures, then establish the dynamic risk measures for discrete-time process.

The paper is structured as follows. In section 2, we briefly review the properties of coherent risk measures and convex risk measures, the simple capital requirement included. The great emphasis is laid on concept of capital requirement since it is the basis of risk measures. In section 3, some properties that dynamic risk measures for discrete-time process must include are stated. Moreover the mathematical notion of capital requirement and the axioms mentioned above are provided. We also propose some propositions and corresponding proofs. In section 4, we develop our dynamic risk measures for discrete process in the general probability space, and relate them to the axioms on acceptance sets. In section 5, we make our conclusion towards the dynamic risk measures.

## 2. Static risk measures

### 2.1. Preparation

Although some scholars provided definition of risk as changes in values between two dates, we still insist that it is better to consider future values only because risk is related to the variability of the future value of a position.

Static risk measures emphasize on current assessment of risk of a final position[8].

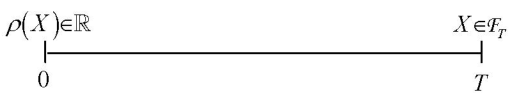


**Figure 1. Static risk measures**

The objects of our research should be the random variables on the set of states of nature at a future time, which stand for the possible future values of positions.

For simplicity, it is necessary to consider only one period of uncertainty $(0,T)$ between two times $0$ and $T$ firstly. The various investment objects are numbered by $i$, $1 \le i \le N$. For each of them, an "optional transaction" is provided, which carries one unit of time $0$ investment object $i$ into $r_i$ units of time $T$ investment object $i$. Default free zero coupon bonds with maturity at time $T$ could be chosen as simple optional transaction.

The period $(0,T)$ could be the period between hedging and rehedging, such as a fixed interval like two weeks, the period enough to liquidate a position, or the length of coverage ruled by an insurance contract[9].

We assume that the investor considers a portfolio in various investment instruments. On time $0$, the value of the portfolio are supposed to be one, while $e_i$ denotes the random number of units of investment 1 which one unit of investment $i$ buys at time $T$. An investor's initial portfolio consists of positions $A_i$, $1 \le i \le N$. The position $A_i$ provides $A_i(T)$ units of investment $i$ at time $T$. We will study the risk of the investor's future net worth $\sum_{1\le i\le N} e_i \cdot A_i(T)$.

The assumption of position being held during the whole period in the static risk measures will be relaxed substantially in the dynamic risk measures[10].

We suppose that the set of all possible states of the world at the end of the period is known, however the probabilities of various states occurring is unknown or not subject to common rules. If we need to deal with market risk, the state of the world might be described by a list of the prices of all investment objects, and the set of all possible such lists is supposed to be known. Of course, this assumes that markets at time $T$ are liquid(if the market is illiquid, the mapping from the former to the latter may not be linear and more complicated model is required[11]).

$\Omega$ is the set of states of nature, and assume it is finite. Considering $\Omega$ as the set of outcomes of an experiment, we compute the final net worth of a position for each element of $\Omega$. It is a random variable denoted by $X$. Its negative part, $\max(-X,0)$, is denoted by $X^-$ and the supremum of $X^-$ is denoted by $\|X^-\|$. The random variable identically equal to 1 is denoted by $\mathbf{1}$. The indicator function of state $\omega$ is denoted by $\mathbf{1}_{\{\omega\}}$.

Let $\varsigma$ be the set of all risks, which is the set of all real valued functions on $\Omega$. Since $\Omega$ is assumed to be finite, $\varsigma$ can be identified with $R^n$, where $n = card(\Omega)$. The cone of non-negative elements in $\varsigma$ shall be denoted by $L_+$, its negative by $L_-$.

$\mathcal{A}_{i,j}$, $j \in J_i$, stands for a set of final net worths, expressed in investment objects $i$, which is accepted by regulator $j$. We denote $\mathrm{A}_i$ the intersection $\bigcap_{j\in J_i} \mathrm{A}_{i,j}$ and use the generic notation $\mathrm{A}$ in the listing of axioms below.

### 2.2. Properties of static risk measures

We now show axioms for acceptance sets. They are the basis of the static risk measures.

**Axiom 2.1** *The acceptance set* $\mathrm{A}$ *contains* $L_+$.

**Axiom 2.2** *The acceptance set* $\mathrm{A}$ *does not intersect the set* $L_{--}$ *where*

$$L_{--} = \{X \mid for \quad each\ \omega \in \Omega, X(\omega) < 0\ \}$$

The two axioms above mean that a final net worth which is always nonnegative does not require extra capital, while a net worth which is always negative certainly does.

The next axiom is a stronger version of Axiom 2.2.

**Axiom 2.3** *The acceptance set* $\mathbb{A}$ *satisfies* $\mathbb{A} \cap L_- = \{0\}$.

**Axiom 2.4** *The acceptance set* $\mathbb{A}$ *is convex.*

This axiom reflects risk aversion on the part of the regulator, exchange director or trading room supervisor.

**Axiom 2.5** *The acceptance set* $\mathbb{A}$ *is a positively homogeneous cone.*

This is a less natural requirement on the set of acceptable final net worths.

Then we list the properties which standard static risk measures should follow.

Subadditivity: $\rho(x+y) \le \rho(x)+\rho(y)$

Monotonicity: $X \ge Y$ implies $\rho(X) \le \rho(Y)$

(Positive) Homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for every $\lambda \ge 0$

(Translation) Invariance: $\rho(X+\alpha) = \rho(X)-\alpha$ for every $\alpha \in \mathbb{R}$.

It is obvious that subadditivity is a questionable property according to the requirement of portfolio diversification. Subadditivity is originally proposed to show the idea that diversification promotes risk optimization. However, it can't include all the cases and needs to be improved. Four axioms above are the major content of coherent risk measures.

Convexity: $\rho(\lambda X+(1-\lambda)Y) \le \lambda\rho(X)+(1-\lambda)\rho(Y)$ for $\lambda \in [0,1]$.

It is easy to discover that convexity equals to subadditivity when $\lambda = 1/2$. So subadditivity is stricter than convexity. Then convexity, translation invariance and monotonicity make up of the convex risk measures.

Convexity can be extended as:

$$\rho(\lambda_1 X_1+\lambda_2 X_2+\ldots+\lambda_n X_n)$$
$$=\lambda_1\rho(X_1)+\lambda_2\rho(X_2)+\ldots+\lambda_n\rho(X_n)$$

for all $\lambda_1+\lambda_2+\ldots+\lambda_n = 1$.

It has been observed that VaR presents several theoretical lacks, especially subadditivity. So it's not a coherent risk measures. Since VaR fails to encourage diversification of risk, this is a fatal drawback of VaR[12].

Particularly, the translation invariance gives in a sense a monetary dimension to $\rho(X)$, since every coherent risk measure, but more in general every invariant map, is in fact a capital requirement associated to a suitable acceptance set. Roughly speaking, a capital requirement is the minimal sum $\alpha$ that we must invest in a reference, sure, instrument in order the overall position $X+\alpha$ be acceptable, i.e. belongs to an priori fixed set $\mathbb{A}$ of sustainable payoffs.

Our current results in this simplified setting represent the first step towards the dynamic risk measures in the following section.

## 3. Dynamic risk measures for process

### 3.1. Setup

Dynamic risk measures emphasize on continuous assessment of risk of a final position.

Let's consider a sequence of time periods $t = 0,\ldots,T$, a finite set of states of the world $\Omega$, and a sequence of random variables $X_t : \Omega \to R$, $t = 0,\ldots,T$, $X_t$ stands for the information available at time $t$. Accordingly, the corresponding information filtration is $F_t = \sigma(X_1,\ldots,X_t)$, $t = 1,\ldots T$ and $F_0 = \{0,\Omega\}$. A position $D = (D_t)$ is a $(F_t)$-adapted process, to be interpreted as a sequence of random payments $D_t$ at time $t$. The set of all position is denoted by $D$. Throughout, $r > -1$ is a fixed, exogenous interest rate.

A set of subsequent times $(t_n)_{n=1}^{N}$ is fixed, with $0 = t_0 < t_n < t_{n+1}$ for any $n \ge 1$ and $t_n = T$.

Suppose that a payoff can occur only at the final time $T > 0$. We are interested in performing a risk measurement of $X$ at each time of a fixed set of intermediate dates $0 = t_0 < \ldots < t_N = T$, except the last one, since in our models risk measures are concerned with future payoffs.

### 3.2. Dynamic risk measures properties

The assumption of position being held during the whole period in the static risk measures is relaxed substantially in the dynamic risk measures. In fact, financial positions frequently vary due to the agent's actions or those of counterparties[13].

The properties we have considered in the static framework are not sufficient in order to define dynamic risk measures. According to the special characters of dynamic risk measures, there must be some new properties to be proposed. Below we propose four possible properties for dynamics risk measures:

**3.2.1. Poor consistency.** If the influence caused by the new information between time $T_n$ and $T_{n+1}$ on future position $X$ is dedicated to $I_n(i)$, then risk measure $\rho$ should follow:

$$\rho_{n+1}(X) = \rho_n(X)+I(i_n)$$

for every $n = 0,\ldots,N$.

The poor consistency property means that: in the complete market, the change of risk measure between two neighborhood time $T_n$ and $T_{n+1}$ is caused by the influence of the new information during this period. If the new information is a positive kind, the risk measure $\rho$ won't change[14]. Otherwise, the risk measure $\rho$ will be upgraded with $I(i_n)$. Since this property is constructed in the complete market, it is considered as poor.

Corollary3.1 $\rho_N = \rho_0 + \sum_{n=0}^{N} I(i_n)$ for all $n$.

According to the poor consistency, it is easy to prove.

This corollary states that the difference of risk measure $\rho$ between the beginning and end of the period is caused by the information in the complete market. $\rho_N - \rho_0$ is the cumulative effect of information in the complete market.

**3.2.2. Consistency.** $\rho_{n+1}(X) \le \rho_{n+1}(Y)$ implies $\rho_n(X) \le \rho_n(Y)$ for every $n$.

The consistency property has a clear meaning: if today we are sure that tomorrow $Y$ will be judged riskier than $X$, then the same conclusion must be drawn today, and this should be a natural behavior.

Corollary 3.2 $\rho_m(X) \le \rho_m(Y)$ implies $\rho_n(X) \le \rho_n(Y)$ for every $n \le m$.

We think it is easy to prove according to consistency property.

**3.2.3. Recursiveness.** $\rho_n(-\rho_{n+1}(X)) = \rho_n(X)$ for every $n$.

Recursiveness(Strong consistency) states that $\rho_n$ can be computed in a simple recursive way, by replacing $X$ with the tomorrow risk measurement $\rho_{n+1}(X)$, obviously changed in the sign.

Corollary3.3 $\rho_n(-\rho_m(X)) = \rho_n(X)$ for every $n < m$.

Proof: $\rho_n(X) = \rho_n(-\rho_{n+1}(X))$

$= \rho_n(-\rho_{n+1}(-\rho_{n+2}(X)))$

$= \rho_n(-\rho_{n+2}(X))$

and the proof for $m \ge n+3$ is just the same.

**3.2.4. Supermartingale property.** If $Q \sim P$ is a reference probability, then another important requirement that we can ask to a dynamic risk measure is the following:

Supermartingale:

$(\rho_n(X))_{n=0}^{N-1}$ is a $Q$-supermartingale for every $X$.

The above property can be explained as following. As time goes by, the payoff $X$ is always the same, but the information about it increases. This should reasonably low the perceived risk, of course not almost surely, but in (conditioned) mean w.r.t. a "reference" (statistical or subjective) probability $Q \sim P$. This amounts exactly to ask for the process $(\rho_n(X))_{n=0}^{N-1}$ to be a $Q$-supermartingale.

### 3.3. Dynamic risk measures framework

**3.3.1. Dynamic capital requirements.** There is no unique natural notion for a capital requirement, when dealing with processes. In this part, we suppose that there is no credit line for the investors.

For any $\alpha \in \mathbb{R}$, $X \in \mathcal{L}^\infty$, and acceptance set $\mathrm{A}$, we define:

$\mathrm{S_n}(\alpha, X, \mathrm{A}) \triangleq \left\{ Y \in \mathcal{L}^\infty_{1,n} : X + Y \in \mathrm{A}, \sum_{n=1}^{N} X_n \le \alpha \right\}$ and the quantity

$\rho_{\mathrm{A},n}(X) \triangleq \inf\{\alpha \in \mathbb{R} : \mathrm{S_n}(\alpha, X, \mathrm{A}) \ne 0\}$.

Then for fixed $0 \le n \le N$ and acceptance $\mathrm{A}$, the map $\rho_{\mathrm{A},n} : \mathcal{L}^\infty_1 \to \mathbb{R}$ is called the dynamic capital requirement associated to $\mathrm{A}$ and $n$. Since we define the number of period is finite, our research is concentrated on the discrete-time process of risk measure.

**3.3.2. Dynamic risk measures process.** Dynamic risk measures framework is trying to describe how to evaluate the risk of a process $X \in \mathcal{L}^\infty_1$ at each date $t_n$, with $n \le N-1$.

Since what matters in risk measurement is future payoffs, the risk measure $\rho_n$ at time $t_n$ will only depend on $X_m$ with $m \ge n+1$.

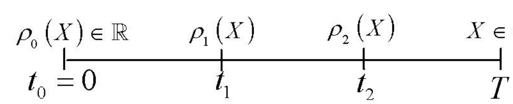


**Figure 2. Dynamics risk measures**

We define that for each $n \ge 1$, the space

$\mathcal{L}^\infty_n \triangleq \{\mathbf{X} = (X_n, ..., X_N) : X_m \in \mathcal{L}^\infty_m \forall m\}$;

For $n = 1$, this has been the ambient space in the previous sections. An element $\mathbf{X} \in \mathcal{L}^\infty_n$ represents a payoff steam starting at date $t_n$.

Suppose that at time 0 we put the sum $\alpha \ge 0$ in a bank account. At each subsequent time $t_1,..,t_N$, we “disinvest” the quantity $Y_n$ from the bank account and add it to the payoff $X_n$. The quotes are necessary, since $Y_n$ can be negative: in this case it is intended that we subtract the sum $-Y_n \geq 0$ from the payoff $X_n$ and invest it in the bank account. In any case, the value of the bank account before time $t_n$ is $\alpha - \sum_{i=1}^{n-1} Y_i$. We assume, for now, that it is possible to borrow money from a bank with no credit line and that every borrowed sum must be paid back only at final maturity $t_N$. In other words at each date $t_n$ with $n \leq N-1$, $Y_n$ can be greater than $\alpha - \sum_{i=1}^{n-1} Y_i$, i.e. the disinvested sum can be greater than the bank account value. However, at time $t_N$ all debts must be paid back, so we impose that $Y_N \leq \alpha - \sum_{i=1}^{N-1} Y_i$, i.e. $\sum_{i=1}^{N} Y_i \leq \alpha$. If the sum $\alpha$ “invested” at time 0 is negative then are borrowing $-\alpha$ from the bank. For every $n$, the set $\mathrm{S_n}(\alpha, X, \mathrm{A})$ collects all these “investment” stratifies $Y$ that, in addition, make the composite payoff stream $X+Y$ acceptable and whose value after $t_n$ is completely determined at that date. In particular, when $n=0$, the last requirement implies that every $Y_n$ is real, i.e. its amount is decided today. Finally, $\rho_{n,\mathrm{A}}(X)$ is the infimum of the initial investments that make possible such a strategy.

## 4. Conclusion

This paper has approached the problem of dynamic risk measures for discrete-time process. We lay our emphasis on the properties of dynamic risk measures instead of trying to find a specific method. We begin with the properties in the static setting, which is considered as the first step towards the latter dynamic risk measures. In the simplified situation, we setup a family of axioms of acceptance set and introduce the coherent requirement and convexity requirement for risk measure. We believe that all the properties in the static setting can be introduced into the dynamic setting.

In the dynamic framework, we propose a set of axioms aiming at the characters of dynamic risk measure. The axioms of dynamic risk measures we proposed are not restrictive enough to specify a certain risk measures. Instead, they characterize a class of risk measures. Among them, the three consistency properties are emphasized. To make their economic signification clear, the economic phenomena are related to the properties. We believe that the risk measure between consistent time are related. The function to calculate the former risk measure according to the latter one is provided by the consistency property. The risk measure at $t_{n+1}$ and position at $t_n$ could be replace each other in the calculation of risk measure at $t_n$. According to the supermartingale property, the risk measure will reduce its effect when it is close to future date. In the last section, a mathematic description of dynamic risk measure for discrete-time process is provided.

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