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INERTIAL ITERATIVE METHOD FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS OF PSEUDO-MONOTONE OPERATORS AND FIXED POINT PROBLEMS OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we propose a new inertial viscosity iterative algorithm for solving the variational inequality problem with a pseudo-monotone operator and the fixed point problem involving a nonexpansive mapping in real Hilbert spaces. The advantage of the proposed algorithm is that it can work without the prior knowledge of the Lipschitz constant of the mapping. The strong convergence of the sequence generated by the proposed algorithm is proved under some suitable assumptions imposed on the parameters. Some numerical experiments are given to support our main results.

1. **Introduction.** Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $A : H \to H$ be a nonlinear operator. The aim of this paper is to study the classical variational inequality problem which is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall\, y \in C. \tag{1}$$

The solution set of (1) is denoted by $VI(C, A)$. In recent years, variational inequality theory has become an important tool in solving many problems appeared in some fields such as in transportation, economics, engineering mechanics, and many

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others ([23, 24, 16, 48, 40]). Many iterative methods have been constructed by authors for solving variational inequalities and their related optimization problems(see [25, 47, 7, 8, 9, 21, 29, 30, 35, 36, 41, 3, 4, 5, 13, 15, 34, 18, 42, 43, 38, 26, 37] and the references therein). For examples, Korpelevich[25] introduced the following double projection method in Euclidean space:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases}$$

here $\lambda \in (0, \frac{1}{L})$, A is monotone and L-Lipschitz continuous. It is noted that this method requires us to calculate two projections onto the closed convex subset C in each iteration. This may affect the efficiency if C is a general closed convex set in numerical experiments. To overcome this drawback, Censor et al. [7] studied the subgradient extragradient method:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H : \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \forall\, n \geq 0, \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$. We observe that the second projection onto C is replaced by a projection onto a specific constructible half-space. On the other hand, Tseng [47] proposed the following method for finding a zero of the sum of two maximal monotone operators:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \forall\, n \geq 0, \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$. Very recently, Gibali, Thong and Tuan [19] proposed the following viscosity projection type algorithm for monotone and Lipschitz continuous operator in Hilbert spaces:

Algorithm 1

Initialization: Given $\lambda > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$, $\gamma \in (0, 2)$. Let $x_0 \in C$ be arbitrary.

Iterative Steps: Given the current iterative x_n, calculate the next iterative x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \tau_n A x_n),$$

where τ_n is chosen to be the largest $\tau \in \left\{\lambda, \lambda l, \lambda l^2, \cdots\right\}$ satisfying

$$\tau \left\| A x_n - A y_n \right\| \leq \mu \left\| x_n - y_n \right\|.$$

If $x_n = y_n$, then stop and y_n is a solution of $VI(C, A)$. Otherwise

Step 2. Compute

$$z_n = x_n - \gamma \eta_n d_n,$$

where

$$\eta_n := (1 - \mu) \frac{\left\| x_n - y_n \right\|^2}{\left\| d_n \right\|^2},$$

and

$$d_n := x_n - y_n - \tau_n (A x_n - A y_n).$$

Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n.$$

Set $n := n+1$ and go to **Step 1**,

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where $f : H \to H$ is a contraction and $\alpha_n \in [0,1]$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Recently, the inertial methods have been studied by some authors (see[2, 1, 28, 14, 39] and the references therein). For instance, Alvarez and Attouch [2] proposed the following inertial proximal method for finding zero of a maximal monotone operator:

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})),$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n, θ_n satisfies $0 \le \theta_n \le \theta$, $\theta \in (0,1)$.

On the other hand, it is clear that the variational inequality problem (1) is equivalent to the following fixed point problem: find a point $x^* \in C$ such that

$$x^* = P_C(x^* - \lambda A x^*),$$

where λ is positive real number. Let $T : H \to H$ be a nonlinear mapping. We denoted by $F(T)$ the set of fixed points of T. There are some iterative algorithms for finding a common element of $F(T)$ and the solution set $VI(C,A)$ in Hilbert spaces or more general Banach spaces. For example, Nadezhkina and Takahashi[31] proposed the following iterative process:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T P_C(x_n - \lambda_n A y_n), \end{cases}$$

where $A : C \to H$ is monotone, L-Lipschitz continuous and $T : C \to C$ is nonexpansive, $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Moreover, they proved that the sequence $\{x_n\}$ generated by above proposed algorithm converges weakly to $z = \lim_{k\to\infty} P_{F(T)\cap VI(C,A)}(x_k)$.

In this paper, we study the classical variational inequality problem (1) for Lipschitz-continuous and pseudomonotone operators and fixed point problems of a nonexpansive mapping in a real Hilbert space. Precisely, we introduce a new inertial viscosity algorithm and obtain a strong convergence theorem under some suitable assumptions imposed on the parameters. Finally, we give some numerical examples to illustrate the performance of the proposed algorithm.

2. Preliminaries.

In what follows, the weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, and the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$. The fixed point set of T is denoted by $F(T)$, that is $F(T) := \{x \in C \,|\, Tx = x\}$. For each $x, y, z \in H$, it is well known that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \tag{2}$$

and

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 = {} & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 \\ & - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2, \end{aligned} \tag{3}$$

where $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Now, we recall the following concepts. Let $T : H \to H$ be an operator.

(a) The operator T is called L-Lipschitz continuous with $L > 0$ if
$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall\, x, y \in H.$$
If $L = 1$, then the operator T is called nonexpansive and if $L \in (0, 1)$, T is called a contraction.

(b) The operator T is called monotone if
$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall\, x, y \in H.$$

(c) The operator T is called pseudomonotone if
$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, x - y\rangle \leq 0, \quad \forall\, x, y \in H.$$

(d) The operator T is called sequentially weakly continuous if for each sequence $\{x_n\}$ satisfying $x_n \rightharpoonup x$, then we have that $Tx_n \rightharpoonup Tx$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

The following lemmas are very useful for proving our main results.

Lemma 2.1 ([17])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \geq 0$, $\forall\, y \in C$.*

Lemma 2.2 ([27])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*
$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}.$$
In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that $a_n \leq a_{n+1}$.

Lemma 2.3 ([49])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that:*
$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$
where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ is a sequence such that
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} b_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([32])**.** *Let $T : H \to H$ be a nonexpansive mapping and H be a real Hilbert space. Let $\{x_n\}$ be a sequence in H and x be a point in H. Suppose that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then $x \in F(T)$.*

Lemma 2.5 ([22])**.** *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ is bounded.*

Lemma 2.6 ([46])**.** *For $x \in H$ and $\alpha \geq \beta > 0$, the following inequalities hold:*
$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$
$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

Lemma 2.7 ([11],Lemma 2.1)**.** *Consider the $VI(C, A)$ with C being a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ being pseudomonotone and continuous. Then, x^* is an element of $VI(C, A)$ if and only if*
$$\langle Ax, x - x^*\rangle \geq 0, \forall\, x \in C.$$

3. **Main results.** In this section, let $T : H \to H$ be a nonexpansive mapping, $f : H \to H$ be a contraction with a constant $\rho \in [0, 1)$ and let $\{\beta_n\}$ and $\{\gamma_n\}$ be two sequences in $[0, 1)$ such that $\beta_n + \gamma_n < 1$ and $\{\alpha_n\}$ be a sequence in $[0, 1)$. In order to obtain the convergence of our proposed method, we need the following assumptions.

Condition 3.1 The feasible set C is nonempty closed and convex.

Condition 3.2 The operator $A : H \to H$ is uniformly continuous, pseudomonotone on H and sequentially weakly continuous on C.

Condition 3.3 $VI(C, A) \cap F(T) \neq \emptyset$.

Now, we introduce the following algorithm.

Algorithm 3.1

Initialization: Given $\lambda > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$, $\gamma \in (0, 2)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \tau_n A w_n),$$

where τ_n is chosen to be the largest $\tau \in \{\lambda, \lambda l, \lambda l^2, ...\}$ satisfying

$$\tau \|Aw_n - Ay_n\| \leq \mu \|w_n - y_n\|. \tag{4}$$

If $y_n = w_n$ or $Ay_n = 0$, then stop and y_n is an element of $VI(C, A)$. Otherwise, go to **Step 2**.

Step 2. Compute

$$z_n = w_n - \gamma \eta_n d_n,$$

where

$$d_n := w_n - y_n - \tau_n(Aw_n - Ay_n),$$

and

$$\eta_n := (1 - \mu)\frac{\|w_n - y_n\|^2}{\|d_n\|^2}.$$

Step 3. Compute

$$x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + (1 - \beta_n - \gamma_n)T z_n.$$

Set $n := n + 1$ and go to **Step 1**.

The following lemmas are very important for proving our main results in this section.

Lemma 3.1. *Assume that Conditions 3.1–3.3 hold, then Armijo line search rule (4) is well defined.*

Proof. When $w_n \in VI(C, A)$, then we have $w_n = P_C(w_n - \lambda A w_n)$. We deduce that $w_n = y_n$ and (4) holds. Now we consider the situation $w_n \notin VI(C, A)$ and assume that the contrary of (4) holds, then we have for all m

$$\lambda l^m \|Aw_n - AP_C(w_n - \lambda l^m A w_n)\| > \mu \|w_n - P_C(w_n - \lambda l^m A w_n)\|. \tag{5}$$

It follows that

$$\|AP_C(w_n - \lambda l^m A w_n) - Aw_n\| > \mu \frac{\|P_C(w_n - \lambda l^m A w_n) - w_n\|}{\lambda l^m}. \tag{6}$$

We consider two cases of w_n. First, if $w_n \in C$, since P_C is continuous, we have

$$\lim_{m\to\infty} \|w_n - P_C(w_n - \lambda l^m A w_n)\| = 0. \tag{7}$$

By the fact that the uniform continuity of the mapping A on H, we obtain

$$\lim_{m\to\infty} \|Aw_n - AP_C(w_n - \lambda l^m A w_n)\| = 0. \tag{8}$$

It follows from (6) and (8) that

$$\lim_{m\to\infty} \frac{\|w_n - P_C(w_n - \lambda l^m A w_n)\|}{\lambda l^m} = 0. \tag{9}$$

Let $t_m = P_C(w_n - \lambda l^m A w_n)$, by Lemma 2.1, we have

$$\langle t_m - w_n + \lambda l^m A w_n, x - t_m\rangle \geq 0,\ \forall\, x \in C,$$

which implies

$$\langle \frac{t_m - w_n}{\lambda l^m}, x - t_m\rangle + \langle Aw_n, w_n - t_m\rangle + \langle Aw_n, x - w_n\rangle \geq 0,\ \forall\, x \in C. \tag{10}$$

Taking the limit $m \to \infty$ in (10) and using (7) and (9), we get

$$\langle Aw_n, x - w_n\rangle \geq 0,\ \forall\, x \in C,$$

which implies that $w_n \in VI(C, A)$. This is a contradiction.

Second, if $w_n \notin C$, then we obtain

$$\lim_{m\to\infty} \|w_n - P_C(w_n - \lambda l^m A w_n)\| = \|w_n - P_C w_n\| > 0, \tag{11}$$

and

$$\lim_{m\to\infty} \lambda l^m \|Aw_n - AP_C(w_n - \lambda l^m A w_n)\| = 0. \tag{12}$$

From (5), (11) and (12), we obtain a contradiction. This finishes the proof. $\square$

Lemma 3.2. *Assume that Conditions 3.1–3.3 hold. Let $\{w_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3.1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to $z \in H$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, then $z \in VI(C, A)$.*

Proof. By $w_{n_k} \rightharpoonup z$ as $k \to \infty$, $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ and $\{y_n\} \subset C$, we obtain $z \in C$. From $y_{n_k} = P_C(w_{n_k} - \tau_{n_k} A w_{n_k})$, we have

$$\langle w_{n_k} - \tau_{n_k} A w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0,\ \forall\, x \in C,$$

which implies

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle Aw_{n_k}, x - y_{n_k}\rangle,\ \forall\, x \in C,$$

or equivalently

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Aw_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle Aw_{n_k}, x - w_{n_k}\rangle,\ \forall\, x \in C. \tag{13}$$

Now we show that

$$\liminf_{k\to\infty} \langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0. \tag{14}$$

We consider two possible cases. Firstly, suppose $\liminf_{k\to\infty} \tau_{n_k} > 0$. Since $\{w_{n_k}\}$ is a bounded sequence and A is uniformly continuous on H, it follows from Lemma 2.5 that $\{Aw_{n_k}\}$ is bounded. Taking $k \to \infty$ in (13) and by the boundedness of $\{y_{n_k}\}$, we have

$$\liminf_{k\to\infty} \langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0.$$

Secondly, we assume that $\liminf_{k\to\infty} \tau_{n_k} = 0$. Put

$$t_{n_k} = P_C(w_{n_k} - \tau_{n_k} l^{-1} A w_{n_k}),$$

we have $\tau_{n_k} l^{-1} > \tau_{n_k}$. By Lemma 2.6, we have

$$\|w_{n_k} - t_{n_k}\| \le \frac{1}{l} \|w_{n_k} - y_{n_k}\| \to 0, \text{ as } k \to \infty.$$

Therefore $t_{n_k} \rightharpoonup z \in C$, thus we get that $\{t_{n_k}\}$ is bounded. Noticing that A is uniformly continuous on H, we have

$$\|Aw_{n_k} - At_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{15}$$

By the Armijo linesearch rule (4), we have

$$\begin{aligned}&\tau_{n_k} l^{-1} \left\|AP_C(w_{n_k} - \tau_{n_k} l^{-1} Aw_{n_k}) - Aw_{n_k}\right\| \\ &> \mu \left\|w_{n_k} - P_C(w_{n_k} - \tau_{n_k} l^{-1} Aw_{n_k})\right\|,\end{aligned}$$

which implies

$$\frac{1}{\mu} \left\|AP_C(w_{n_k} - \tau_{n_k} l^{-1} Aw_{n_k}) - Aw_{n_k}\right\| > \frac{\left\|w_{n_k} - P_C(w_{n_k} - \tau_{n_k} l^{-1} Aw_{n_k})\right\|}{\tau_{n_k} l^{-1}}. \tag{16}$$

From (15) and (16), we have

$$\lim_{k\to\infty} \frac{\left\|w_{n_k} - P_C(w_{n_k} - \tau_{n_k} l^{-1} Aw_{n_k})\right\|}{\tau_{n_k} l^{-1}} = 0.$$

By the definition of t_{n_k} and Lemma 2.1, we get

$$\left\langle w_{n_k} - \tau_{n_k} l^{-1} Aw_{n_k} - t_{n_k}, x - t_{n_k} \right\rangle \le 0, \forall\, x \in C,$$

which implies that

$$\begin{aligned}&\frac{1}{\tau_{n_k} l^{-1}} \langle w_{n_k} - t_{n_k}, x - t_{n_k}\rangle + \langle Aw_{n_k}, t_{n_k} - w_{n_k}\rangle \\ &\le \langle Aw_{n_k}, x - w_{n_k}\rangle, \forall\, x \in C.\end{aligned} \tag{17}$$

Taking the limit $k \to \infty$ in (17), we have

$$\liminf_{k\to\infty} \langle Aw_{n_k}, x - w_{n_k}\rangle \ge 0.$$

It implies that the inequality (14) holds. On the other hand, we observe

$$\begin{aligned}&\langle Ay_{n_k}, x - y_{n_k}\rangle \\ &= \langle Ay_{n_k} - Aw_{n_k}, x - w_{n_k}\rangle + \langle Aw_{n_k}, x - w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{18}$$

By the uniformly continuity of A on H and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, we have

$$\lim_{k\to\infty} \|Aw_{n_k} - Ay_{n_k}\| = 0.$$

From (14) and (18), we obtain

$$\liminf_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle \ge 0.$$

Finally, we show that $z \in VI(C, A)$. Indeed, we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0 as $k \to \infty$. For every k, we denote N_k the smallest positive integer such that

$$\left\langle Ay_{n_j}, x - y_{n_j}\right\rangle + \epsilon_k \ge 0, \forall j \ge N_k.$$

Furthermore, for each k, since $\{y_{N_k}\} \subset C$, we can assume that $Ay_{N_k} \neq 0$ (otherwise, y_{N_k} belongs to $VI(C, A)$) and let $g_{N_k} = \frac{Ay_{N_k}}{\|Ay_{N_k}\|^2}$, thus we have $\langle Ay_{N_k}, g_{N_k}\rangle = 1$ for each k. Therefore we get

$$\langle Ay_{N_k}, x + \epsilon_k g_{N_k} - y_{N_k}\rangle \geq 0.$$

Noticing the fact that A is pseudo-monotone, we obtain

$$\langle A(x + \epsilon_k g_{N_k}), x + \epsilon_k g_{N_k} - y_{N_k}\rangle \geq 0,$$

which implies

$$\begin{aligned} &\langle Ax, x - y_{N_k}\rangle \\ &\geq \langle Ax - A(x + \epsilon_k g_{N_k}), x + \epsilon_k g_{N_k} - y_{N_k}\rangle - \langle Ax, \epsilon_k g_{N_k}\rangle . \end{aligned} \tag{19}$$

Next we prove that $\lim_{k\to\infty} \epsilon_k g_{N_k} = 0$. In fact, it follows from $w_{n_k} \rightharpoonup z$ as $k \to \infty$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ that $y_{N_k} \rightharpoonup z$ as $k \to \infty$. Since A is sequentially weakly continuous on C, then $\{Ay_{N_k}\}$ converges weakly to Az. We can assume that $Az \neq 0$(otherwise, z already belongs to $VI(C, A)$). Since the norm mapping is sequentially weakly lower semicontinuous, we get

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\| .$$

By $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we have

$$\begin{aligned} 0 \leq \limsup_{k\to\infty} \|\epsilon_k g_{N_k}\| &= \limsup_{k\to\infty} (\frac{\epsilon_k}{\|Ay_{n_k}\|}) \\ &\leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0. \end{aligned}$$

It implies that $\lim_{k\to\infty} \epsilon_k g_{N_k} = 0$. Letting $k \to \infty$, then the right hand side of (19) tends to zero since A is uniformly continuous, $\{y_{N_k}\}$ is bounded and $\lim_{k\to\infty} \epsilon_k g_{N_k} = 0$. Therefore, we obtain

$$\liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

Thus, for all $x \in C$, we get

$$\langle Ax, x - z\rangle = \lim_{k\to\infty} \langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

By Lemma 2.7, we have $z \in VI(C, A)$. This finishes the proof. □

Lemma 3.3. *Assume that Conditions 3.1–3.3 hold. Let $\{z_n\}$ be a sequence generated by Algorithm 3.1. Then*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \frac{2-\gamma}{\gamma}\|w_n - z_n\|^2, \quad \forall p \in VI(C, A).$$

Proof. Using (4), we have

$$\begin{aligned} \langle w_n - p, d_n\rangle &= \langle w_n - y_n, d_n\rangle + \langle y_n - p, d_n\rangle \\ &= \langle w_n - y_n, w_n - y_n - \tau_n(Aw_n - Ay_n)\rangle \\ &\quad + \langle y_n - p, w_n - y_n - \tau_n(Aw_n - Ay_n)\rangle \\ &\geq \|w_n - y_n\|^2 - \mu\|w_n - y_n\|^2 \\ &\quad + \langle y_n - p, w_n - y_n - \tau_n(Aw_n - Ay_n)\rangle \\ &= (1-\mu)\|w_n - y_n\|^2 + \langle y_n - p, w_n - y_n - \tau_n Aw_n + \tau_n Ay_n\rangle. \end{aligned} \tag{20}$$

Since $y_n = P_C(w_n - \tau_n A w_n)$, we get

$$\langle w_n - \tau_n A w_n - y_n, y_n - p\rangle \geq 0. \tag{21}$$

By $p \in VI(C, A)$ and $y_n \in C$, we obtain

$$\langle Ap, y_n - p\rangle \geq 0.$$

Noticing the fact that A is pseudomonotone on H, we have

$$\langle Ay_n, y_n - p\rangle \geq 0. \tag{22}$$

Combining (20), (21) and (22), we get

$$\langle w_n - p, d_n\rangle \geq (1-\mu)\|w_n - y_n\|^2. \tag{23}$$

It follows from (23) that

$$\begin{aligned}
\|z_n - p\|^2 &= \|w_n - \gamma\eta_n d_n - p\|^2 \\
&= \|w_n - p\|^2 + \|\gamma\eta_n d_n\|^2 - 2\gamma\eta_n\langle w_n - p, d_n\rangle \\
&\leq \|w_n - p\|^2 + \|\gamma\eta_n d_n\|^2 - 2\gamma\eta_n(1-\mu)\|w_n - y_n\|^2 \\
&= \|w_n - p\|^2 + \|\gamma\eta_n d_n\|^2 - 2\gamma\|\eta_n d_n\|^2 \\
&= \|w_n - p\|^2 - \frac{2-\gamma}{\gamma}\|w_n - z_n\|^2.
\end{aligned}$$

This completes the proof. □

Lemma 3.4. *Assume that Conditions 3.1–3.3 hold and let the sequence $\{w_n\}$ be generated by Algorithm 3.1. Then*

$$\|w_n - y_n\|^2 \leq \frac{(1+\mu)^2}{((1-\mu)\gamma)^2}\|w_n - z_n\|^2. \tag{24}$$

Proof. We have

$$\|w_n - y_n\|^2 = \frac{\eta_n}{1-\mu}\|d_n\|^2 = \frac{\|\gamma\eta_n d_n\|^2}{(1-\mu)\gamma^2\eta_n} = \frac{1}{(1-\mu)\gamma^2\eta_n}\|w_n - z_n\|^2. \tag{25}$$

It follows from (4) that

$$\begin{aligned}
\|d_n\| &= \|w_n - y_n - \tau_n(Aw_n - Ay_n)\| \\
&\leq \|w_n - y_n\| + \tau_n\|Aw_n - Ay_n\| \\
&\leq \|w_n - y_n\| + \mu\|w_n - y_n\| \\
&= (1+\mu)\|w_n - y_n\|.
\end{aligned} \tag{26}$$

Using (26) we have

$$\eta_n = (1-\mu)\frac{\|w_n - y_n\|^2}{\|d_n\|^2} \geq (1-\mu)\frac{\|w_n - y_n\|^2}{(1+\mu)^2\|w_n - y_n\|^2} = \frac{1-\mu}{(1+\mu)^2}. \tag{27}$$

Combining (25) and (27), we know that (24) holds. The proof is completed. □

Theorem 3.5. *Assume that Conditions 3.1 – 3.3 hold. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1)$ such that*

$$\lim_{n\to\infty}\beta_n = 0,\ \sum_{n=1}^{\infty}\beta_n = \infty,\ 0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$$

and $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. *If Algorithm 3.1 stops in Step 1, then* y_n *is an element of* $VI(C, A)$. *Otherwise, the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to an element* $p \in F(T) \cap VI(C, A)$, *where* $p = P_{F(T)\cap VI(C,A)} f(p)$.

Proof. **Claim 1.** We prove that $\{x_n\}$ is bounded. Indeed, by Lemma 3.3, we have

$$\|z_n - p\| \le \|w_n - p\|. \tag{28}$$

By (28), we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\beta_n f(x_n) + \gamma_n x_n + (1 - \beta_n - \gamma_n)Tz_n - p\| \\
&\le \beta_n\|f(x_n) - p\| + \gamma_n \|x_n - p\| + (1 - \beta_n - \gamma_n)\|z_n - p\| \\
&\le \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + \gamma_n \|x_n - p\| + (1 - \beta_n - \gamma_n)\|z_n - p\| \\
&\le \beta_n\rho\|x_n - p\| + \beta_n\|f(p) - p\| + \gamma_n \|x_n - p\| + (1 - \beta_n - \gamma_n)\|w_n - p\|.
\end{aligned} \tag{29}$$

We also have

$$\|w_n - p\| = \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \le \|x_n - p\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.$$

Since

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0,$$

there exists $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1, \quad \forall n \in \mathbb{N}.$$

Then

$$\|w_n - p\| \le \|x_n - p\| + \beta_n M_1. \tag{30}$$

Combining (29) and (30) we get

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\le \beta_n\rho\|x_n - p\| + \beta_n\|f(p) - p\| + \gamma_n\|x_n - p\| \\
&\quad + (1 - \beta_n - \gamma_n)\|x_n - p\| + \beta_n M_1 \\
&= \beta_n\rho\|x_n - p\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|x_n - p\| + \beta_n M_1 \\
&= [1 - \beta_n(1 - \rho)]\|x_n - p\| + \beta_n(1 - \rho)\frac{\|f(p) - p\| + M_1}{1 - \rho} \\
&\le \max\{\|x_n - p\|, \frac{\|f(p) - p\| + M_1}{1 - \rho}\}.
\end{aligned}$$

By induction, we obtain

$$\|x_{n+1} - p\| \le \max\{\|x_1 - p\|, \frac{\|f(p) - p\| + M_1}{1 - \rho}\}.$$

This implies that the sequence $\{x_n\}$ is bounded.

Claim 2. We prove that

$$\begin{aligned}
&(1 - \beta_n - \gamma_n)\frac{2 - \gamma}{\gamma}\|w_n - z_n\|^2 + \gamma_n(1 - \beta_n - \gamma_n)\|Tz_n - x_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n M_3,
\end{aligned}$$

for some $M_3 > 0$. Indeed, by (3) and Lemma 3.3, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\beta_n(f(x_n) - p) + \gamma_n(x_n - p) + (1 - \beta_n - \gamma_n)(Tz_n - p)\|^2 \\
&\leq \beta_n\|f(x_n) - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n - \gamma_n)\|Tz_n - p\|^2 \\
&\quad - \gamma_n(1 - \beta_n - \gamma_n)\|Tz_n - x_n\|^2 \\
&\leq \beta_n\|f(x_n) - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n - \gamma_n)\|z_n - p\|^2 \\
&\quad - \gamma_n(1 - \beta_n - \gamma_n)\|Tz_n - x_n\|^2 \\
&\leq \beta_n\|f(x_n) - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n - \gamma_n)[\|w_n - p\|^2 \\
&\quad - \frac{2-\gamma}{\gamma}\|w_n - z_n\|^2] - \gamma_n(1 - \beta_n - \gamma_n)\|Tz_n - x_n\|^2.
\end{aligned} \tag{31}$$

Owing to (30), we have

$$\begin{aligned}
\|w_n - p\|^2 &\leq (\|x_n - p\| + \beta_n M_1)^2 \\
&= \|x_n - p\|^2 + \beta_n(2M_1\|x_n - p\| + \beta_n M_1^2) \\
&\leq \|x_n - p\|^2 + \beta_n M_2,
\end{aligned} \tag{32}$$

where $M_2 := \sup_{n\geq 1}\{2M_1\|x_n - p\| + \beta_n M_1^2\}$. Substituting (32) into (31), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \beta_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 + \beta_n M_2 - (1 - \beta_n - \gamma_n)\frac{2-\gamma}{\gamma}\|w_n - z_n\|^2 \\
&\quad - \gamma_n(1 - \beta_n - \gamma_n)\|Tz_n - x_n\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&(1 - \beta_n - \gamma_n)\frac{2-\gamma}{\gamma}\|w_n - z_n\|^2 + \gamma_n(1 - \beta_n - \gamma_n)\|Tz_n - x_n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n M_3,
\end{aligned}$$

where $M_3 := \sup_{n\geq 1}\{\|f(x_n) - p\|^2 + M_2\}$.

Claim 3. We prove that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq [1 - \beta_n(1 - \rho)]\|x_n - p\|^2 + \beta_n(1 - \rho)[\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|\frac{M_4}{1-\rho} + \\
&\quad + \frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle],
\end{aligned} \tag{33}$$

where M_4 is a constant. We observe that

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n\|x_n - p\|\|x_n - x_{n-1}\| + \alpha_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|) \\
&\leq \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \beta_n M_1).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \langle\beta_n(f(x_n) - p) + \gamma_n(x_n - p) + (1 - \beta_n - \gamma_n)(Tz_n - p), x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \beta_n \langle f(x_n) - f(p), x_{n+1} - p\rangle + \gamma_n \langle x_n - p, x_{n+1} - p\rangle \\
&\quad + (1-\beta_n-\gamma_n)\langle Tz_n - p, x_{n+1} - p\rangle + \beta_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \beta_n\rho \|x_n - p\| \|x_{n+1} - p\| + \gamma_n \|x_n - p\| \|x_{n+1} - p\| \\
&\quad + (1-\beta_n-\gamma_n) \|w_n - p\| \|x_{n+1} - p\| + \beta_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \frac{1}{2}(\beta_n\rho + \gamma_n)[\|x_n - p\|^2 + \|x_{n+1} - p\|^2] + \frac{1}{2}(1-\beta_n-\gamma_n)[\|w_n - p\|^2 \\
&\quad + \|x_{n+1} - p\|^2] + \beta_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \frac{1}{2}(\beta_n\rho + \gamma_n)[\|x_n - p\|^2 + \|x_{n+1} - p\|^2] + \frac{1}{2}(1-\beta_n-\gamma_n)[\|x_n - p\|^2 \\
&\quad + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \beta_n M_1) + \|x_{n+1} - p\|^2] \\
&\quad + \beta_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le \frac{1}{2}[1-\beta_n(1-\rho)] \|x_n - p\|^2 + \frac{1}{2} \|x_{n+1} - p\|^2 \\
&\quad + \frac{1}{2}\alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \beta_n M_1) + \beta_n \langle f(p) - p, x_{n+1} - p\rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le [1-\beta_n(1-\rho)] \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \beta_n M_1) \\
&\quad + 2\beta_n \langle f(p) - p, x_{n+1} - p\rangle \\
&\le [1-\beta_n(1-\rho)] \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\| M_4 \\
&\quad + 2\beta_n \langle f(p) - p, x_{n+1} - p\rangle \\
&= [1-\beta_n(1-\rho)] \|x_n - p\|^2 + \beta_n(1-\rho)[\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|\frac{M_4}{1-\rho} + \\
&\quad + \frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle],
\end{aligned}
$$

where $M_4 = \sup_{n\ge 1}(2\|x_n - p\| + \beta_n M_1)$.

Claim 4. Now, we will prove that the sequence $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.

Case 1: There exists $N_2 \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \le \|x_n - p\|^2$ for all $n \ge N_2$. This implies that $\lim_{n\to\infty} \|x_n - p\|^2$ exists. By Claim 2, $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$, $\lim_{n\to\infty} \beta_n = 0$ and $\beta_n + \gamma_n < 1$, we get

$$\lim_{n\to\infty} \|w_n - z_n\| = 0, \tag{34}$$

and

$$\lim_{n\to\infty} \|Tz_n - x_n\| = 0. \tag{35}$$

From Lemma 3.4 and (34), we also get

$$\lim_{n\to\infty} \|w_n - y_n\| \le \lim_{n\to\infty} \frac{(1+\mu)}{(1-\mu)\gamma}\|w_n - z_n\| = 0. \tag{36}$$

Since

$$\|w_n - z_n\| = \|x_n + \alpha_n(x_n - x_{n-1}) - z_n\| \ge \|x_n - z_n\| - \alpha_n\|x_n - x_{n-1}\|,$$

that is

$$\|x_n - z_n\| \le \|w_n - z_n\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.$$

It follows from (34) and $\lim_{n\to\infty} \beta_n = 0$ that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{37}$$

By (35) and (37), we have

$$\begin{aligned} \|Tx_n - x_n\| &\le \|Tx_n - Tz_n\| + \|Tz_n - x_n\| \\ &\le \|x_n - z_n\| + \|Tz_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{38}$$

Since $\lim_{n\to\infty} \beta_n = 0$ and (35), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\beta_n f(x_n) + \gamma_n x_n + (1 - \beta_n - \gamma_n)Tz_n - x_n\| \\ &\le \beta_n\|f(x_n) - x_n\| + (1 - \beta_n - \gamma_n)\|Tz_n - x_n\| \\ &\le \beta_n\|f(x_n) - x_n\| + \|Tz_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{39}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that weakly converge to some $z \in H$ such that

$$\begin{aligned} \limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle &= \lim_{k\to\infty} \langle f(p) - p, x_{n_k} - p\rangle \\ &= \langle f(p) - p, z - p\rangle. \end{aligned}$$

We note that

$$\begin{aligned} \|w_n - x_n\| &= \|\alpha_n(x_n - x_{n-1})\| \\ &= \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Then we have $w_{n_k} \rightharpoonup z \in H$ as $k \to \infty$. From (36) and Lemma 3.2, we have $z \in VI(C, A)$. By (38) and Lemma 2.4, we get $z \in F(T)$. Hence $z \in F(T) \cap VI(C, A)$. By Lemma 2.1, we have

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{40}$$

By (39) and (40), we have

$$\begin{aligned} &\limsup_{n\to\infty} \langle f(p) - p, x_{n+1} - p\rangle \\ &\le \limsup_{n\to\infty} \langle f(p) - p, x_{n+1} - x_n\rangle + \limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \\ &= \langle f(p) - p, z - p\rangle \\ &\le 0. \end{aligned} \tag{41}$$

Using (41) and $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, we get

$$\limsup_{n\to\infty} \left(\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \frac{M_4}{1-\rho} + \frac{2}{1-\rho} \langle f(p) - p, x_{n+1} - p\rangle \right) \le 0.$$

Apply Lemma 2.3 to (33), we obtain $x_n \to p$ as $n \to \infty$.

Case 2: There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 \le \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.2 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all sufficiently large number $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2, \tag{42}$$

and

$$\|x_k - p\|^2 \leq \|x_{m_k+1} - p\|^2. \tag{43}$$

According to Claim 2 and (42), we have

$$\begin{aligned}&(1-\beta_{m_k}-\gamma_{m_k})\frac{2-\gamma}{\gamma}\|w_{m_k} - z_{m_k}\|^2 + \gamma_{m_k}(1-\beta_{m_k}-\gamma_{m_k})\|Tz_{m_k} - x_{m_k}\|^2 \\ &\leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \beta_{m_k} M_3.\end{aligned} \tag{44}$$

It follows that

$$\lim_{n\to\infty} \|w_{m_k} - z_{m_k}\| = 0, \quad \lim_{n\to\infty} \|Tz_{m_k} - x_{m_k}\| = 0.$$

By Lemma 3.4, we have

$$\lim_{n\to\infty} \|w_{m_k} - y_{m_k}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we obtain

$$\limsup_{k\to\infty} \langle f(p) - p, x_{m_k+1} - p\rangle \leq 0. \tag{45}$$

In the light of Claim 3, we have

$$\begin{aligned}&\|x_{m_k+1} - p\|^2 \\ &\leq [1-(1-\rho)\beta_{m_k}]\|x_{m_k} - p\|^2 + (1-\rho)\beta_{m_k}(\frac{M_4}{1-\rho}\frac{\alpha_{m_k}}{\beta_{m_k}}\|x_{m_k} - x_{m_k-1}\| \\ &\quad + \frac{2}{1-\rho}\langle f(p) - p, x_{m_k+1} - p\rangle) \\ &\leq [1-(1-\rho)\beta_{m_k}]\|x_{m_k+1} - p\|^2 + (1-\rho)\beta_{m_k}(\frac{M_4}{1-\rho}\frac{\alpha_{m_k}}{\beta_{m_k}}\|x_{m_k} - x_{m_k-1}\| \\ &\quad + \frac{2}{1-\rho}\langle f(p) - p, x_{m_k+1} - p\rangle),\end{aligned}$$

which implies

$$\|x_{m_k+1} - p\|^2 \leq \frac{M_4}{1-\rho}\frac{\alpha_{m_k}}{\beta_{m_k}}\|x_{m_k} - x_{m_k-1}\| + \frac{2}{1-\rho}\langle f(p) - p, x_{m_k+1} - p\rangle.$$

By (43), (45) and $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, we have

$$\limsup_{k\to\infty} \|x_k - p\| \leq \limsup_{k\to\infty} \|x_{m_k+1} - p\| \leq 0,$$

which implies $x_k \to p$ as $k \to \infty$. This completes the proof. □

Remark 3.6. It is easy to see that the condition $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$ of Theorem 3.5 can be implemented easily in the numerical computation as the value of $\|x_n - x_{n-1}\|$ is known before choosing α_n. Indeed, the parameter α_n can be chosen such that

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\theta_n}{\|x_n - x_{n-1}\|}, \alpha\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise,}\end{cases} \tag{46}$$

where α is a constant such that $0 < \alpha < 1$ and $\{\theta_n\}$ is a positive sequence such that $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0$.

Remark 3.7. Theorem 3.5 improves and extends Theorem 3.2 of Gibali, Thong and Tuan [19] in the following aspects.

(i) From monotone operator to more general psudomonotone operator.
(ii) From Lipschitz continuous operator to more general uniformly continuous operator.
(iii) We add an inertial term in our proposed algorithm which improve the convergence speed of the generated sequence.
(iv) Our iterative Algorithm 3.1 is more general than ones of Gibali, Thong and Tuan [19] since it can be applied to solve variational inequality problems and fixed point problems of nonexpansive mapping.

4. **Numerical experiments.** In this section, we provide some numerical examples occurring in finite- and infinite-dimensional spaces to demonstrate the efficiency of the proposed algorithm compared to some known ones in the literature. We update the inertia parameter α_n by (46). All the programs are implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

Example 4.1. Let the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $Ax = Gx$ and $G = BB^\mathsf{T} + S + E$, matrix $B \in \mathbb{R}^{m\times m}$, matrix $S \in \mathbb{R}^{m\times m}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are nonnegative (hence G is positive symmetric definite). The feasible set C is given by $C = \{x \in \mathbb{R}^m : -2 \le x_i \le 5, i = 1, 2, \cdots, m\}$. It is easy to see that A is monotone (hence it is pseudomonotone) Lipschitz continuous with its Lipschitz constant $L = \|G\|$. In this example, all entries of B, E are generated randomly in $[0, 2]$ and S is generated randomly in $[-2, 2]$. Then the solution set is $\{\mathbf{0}\}$. We compare the proposed algorithm with the Algorithm 3.1 introduced by Cholamjiak, Thong and Cho [10] (shortly, CTC Alg. 3.1) and the Algorithms 3.1 and 3.2 presented by Gibali, Thong and Tuan [19] (shortly, GTT Alg. 3.1 and GTT Alg. 3.2).

The parameters of all algorithms are set as follows.

- In the proposed Algorithm 3.1, we take $\theta_n = 100/(n+1)^2$, $\alpha = 0.4$ in (3.43), $\lambda = 0.5$, $l = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\beta_n = 1/(n+1)$, $\gamma_n = 0.5\beta_n$, $f(x) = 0.1x$ and $Tx = x$.
- In the CTC Alg. 3.1, we choose $\tau_n = 100/(n+1)^2$, $\alpha = 0.4$, $\lambda = 0.9/L$, $\beta_n = 1/(n+1)$ and $\theta_n = 0.5(1-\beta_n)$.
- In the GTT Alg. 3.1 and GTT Alg. 3.2, we pick $\lambda = 0.5$, $l = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 1/(n+1)$, $\beta_n = 0.5(1-\alpha_n)$ and $f(x) = 0.1x$.

The maximum number of iterations 200 as a common stopping criterion for all algorithms. We use $D_n = \|x_n - x^*\|$ to measure the iteration error at the n-th step. Fig. 1 and Table 1 show the numerical results of all algorithms with different dimensions, where "CPU" in Table 1 denotes the execution time in seconds for all algorithms.

Example 4.2. We consider an example in the Hilbert space $H = L^2([0, 1])$ with inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t, \quad \forall x, y \in H,$$

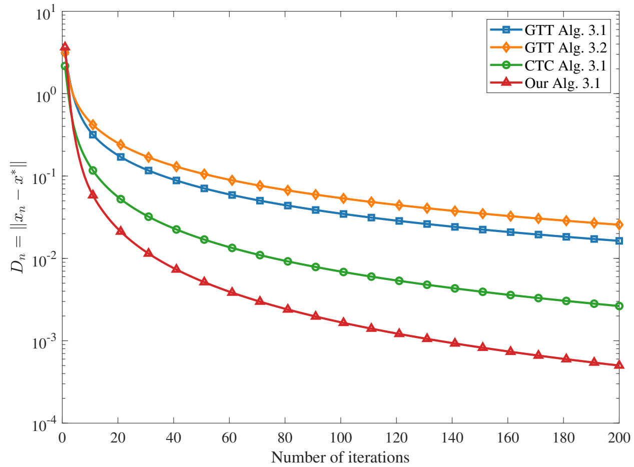

FIGURE 1. The behavior of our Algorithm 3.1 in Example 4.1 ($m = 200$)

TABLE 1. Numerical results of all algorithms with different dimensions in Example 4.1

Algorithms	$m = 20$		$m = 50$		$m = 100$		$m = 200$	
	D_n	*CPU*	D_n	*CPU*	D_n	*CPU*	D_n	*CPU*
Our Alg. 3.1	*4.47E-05*	*0.0303*	*1.70E-04*	*0.0316*	*3.17E-04*	*0.0707*	*5.00E-04*	*0.1119*
CTC Alg. 3.1	*3.36E-04*	*0.0227*	*9.95E-04*	*0.0227*	*1.69E-03*	*0.0250*	*2.64E-03*	*0.0273*
GTT Alg. 3.1	*2.80E-03*	*0.0310*	*6.86E-03*	*0.0361*	*1.08E-02*	*0.1044*	*1.63E-02*	*0.1393*
GTT Alg. 3.2	*3.38E-03*	*0.0483*	*9.78E-03*	*0.0379*	*1.64E-02*	*0.0704*	*2.56E-02*	*0.1027*

and induced norm

$$\|x\| := \Big(\int_0^1 |x(t)|^2 \,\mathrm{d}t\Big)^{1/2}, \quad \forall x \in H.$$

The feasible set is given by $C = \{x \in H : \|x\| \le 1\}$. Define an operator $A : C \to H$ by

$$(Ax)(t) = \int_0^1 \big(x(t) - G(t,s)g(x(s))\big)\,\mathrm{d}s + h(t), \quad t \in [0,1],\, x \in C,$$

where

$$G(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad g(x) = \cos x, \quad h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

It is known that A is monotone (hence it is pseudomonotone) and L-Lipschitz continuous with $L = 2$ (see [20] for more details), and $x^*(t) = \{\mathbf{0}\}$ is the solution of the corresponding variational inequality problem. The parameters of all algorithms are the same as those set in Example 4.1. We choose the maximum number of iterations 50 as the common stopping criterion for all algorithms and use $D_n = \|x_n(t) - x^*(t)\|$ to measure the iteration error of the n-th step. The numerical behaviors of all algorithms with four starting points $x_0(t) = x_1(t)$ are reported in Fig. 2 and Table 2.

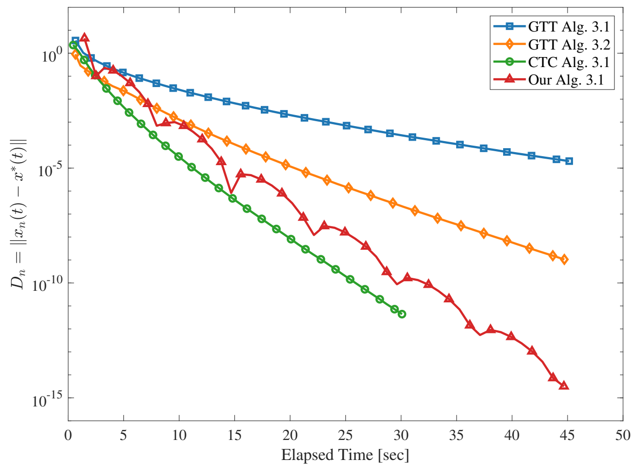

FIGURE 2. The behavior of our Algorithm 3.1 in Example 4.2 ($x_0 = x_1 = 10\exp(t)$)

TABLE 2. Numerical results of all algorithms with different initial values in Example 4.2

Algorithms	$x_1 = 10t^3$		$x_1 = 10\sin(2t)$		$x_1 = 10\log(t)$		$x_1 = 10\exp(t)$	
	D_n	*CPU*	D_n	*CPU*	D_n	*CPU*	D_n	*CPU*
Our Alg. 3.1	*1.47E-15*	*37.7392*	*4.04E-15*	*38.5818*	*7.26E-15*	*39.9932*	*3.16E-15*	*44.6711*
CTC Alg. 3.1	*3.16E-13*	*22.8652*	*2.55E-13*	*24.3823*	*3.36E-12*	*25.0381*	*4.39E-12*	*30.0801*
GTT Alg. 3.1	*9.04E-06*	*33.7345*	*1.54E-05*	*34.5784*	*2.31E-05*	*36.9545*	*2.01E-05*	*45.1488*
GTT Alg. 3.2	*6.25E-11*	*33.5898*	*7.40E-10*	*36.0191*	*1.04E-09*	*37.5059*	*1.06E-09*	*44.7099*

Example 4.3. Consider the Hilbert space $H = l_2 := \{x = (x_1, x_2, \ldots, x_i, \ldots) \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ with inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, $\forall x, y \in H$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$, $\forall x \in H$. Let $C := \{x \in H : |x_i| \leq 1/i\}$. Define an operator $A : C \to H$ by

$$Ax = (\|x\| + 1/\|x\| + \varphi)\, x, \quad , \varphi > 0$$

It can be verified that mapping A is pseudo-monotone on H, uniformly continuous and sequentially weakly continuous on C but not Lipschitz continuous on H (see [45]). In this example, we take $\varphi = 0.5$, $H = \mathbb{R}^m$ for different values of m. We compare the proposed Algorithm 3.1 with several convergent algorithms that can find an element of $VI(C, A)$ with uniformly continuous operators, which including the Algorithm 4 proposed by Reich et al. [33] (shortly, RTDLD Alg. 4), the Algorithm 3 suggested by Thong et al. [44] (shortly, TSI Alg. 3) and the Algorithm 3.1 introduced by Cai et al. [6] (shortly, CDP Alg. 3.1). The parameters of all algorithms are set as follows.

- The parameters of the proposed Algorithm 3.1 are the same as those set in Example 4.1.
- In the RTDLD Alg. 4, we take $l = 0.5$, $\mu = 0.4$, $\lambda = 0.5/\mu$, $\alpha_n = 1/(n+1)$ and $f(x) = 0.1x$.

- In the TSI Alg. 3, we pick $\gamma = 0.5$, $l = 0.5$, $\mu = 0.4$, $\alpha_n = 1/(n+1)$ and $f(x) = 0.1x$.
- In the CDP Alg. 3.1, we choose $\gamma = 0.5$, $l = 0.5$, $\mu = 0.4$, $\beta_n = 1/(n+1)$ and $f(x) = 0.1x$.

The maximum number of iterations 200 is used as a common stopping criterion. The numerical behaviors of $D_n = \|x_n - x_{n-1}\|$ of all algorithms with four different dimensions are reported in Fig. 3 and Table 3. Furthermore, this stopping criterion is applicable if the algorithm converges superlinearly(see [12] for more detail.)

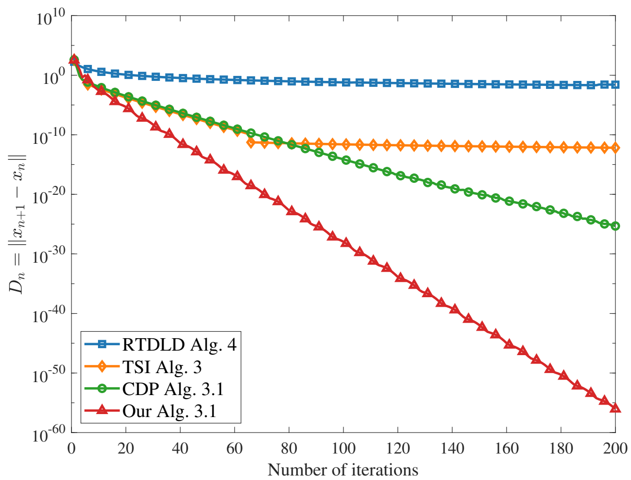

FIGURE 3. The behavior of our Algorithm 3.1 in Example 4.3 ($m = 500000$)

TABLE 3. Numerical results of all algorithms with different dimensions in Example 4.3

Algorithms	$m = 500$		$m = 5000$		$m = 50000$		$m = 500000$	
	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 3.1	*7.13E-57*	*0.0249*	*8.76E-57*	*0.1079*	*3.77E-57*	*0.4058*	*8.89E-57*	*13.8430*
CDP Alg. 3.1	*3.97E-27*	*0.0406*	*7.89E-27*	*0.1274*	*7.25E-27*	*0.5290*	*4.78E-26*	*13.9558*
TSI Alg. 3	*8.38E-13*	*0.0318*	*7.96E-13*	*0.1270*	*8.17E-13*	*0.4180*	*6.62E-13*	*15.0426*
RTDLD Alg. 4	*4.72E-10*	*0.0312*	*3.07E-07*	*0.1132*	*1.64E-03*	*0.4540*	*2.59E-02*	*19.2562*

Remark 4.4. From Figs. 1–3 and Tables 1–3, we know that our algorithm has a higher accuracy than some known methods in the literature [19, 10, 33, 44, 6] when performing the same stopping criterion, and this result is independent of the size of the dimension and the choice of the initial values. Therefore, our algorithm is efficient and robust.

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