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Tseng's extragradient algorithm for pseudomonotone variational inequalities on Hadamard manifolds

Jingjing Fan[a], Xiaolong Qin[b] and Bing Tan[a]

[a]Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, People's Republic of China; [b]Department of Mathematics, Zhejiang Normal University, Zhejiang, People's Republic of China

ABSTRACT

In this paper, we investigate the Tseng's extragradient algorithm for non-Lipschitzian variational inequalities with pseudomonotone vector fields on Hadamard manifolds. The convergence analysis of the proposed algorithm is discussed under mild assumptions. Two experiments are provided to illustrate the asymptotical behavior of the algorithm. The results presented in this paper generalize some known results presented in the literature.

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1. Introduction

The theory of variational inequalities has important applications in many fields, such as machine learning, network equilibrium problems, image reconstruction, signal restoration and so on. It has been extensively studied in finite or infinite dimensional linear spaces; see, e.g. [1–5] and the references therein.

Recently, in many practical applications, the natural structure of the data is modeled as constrained optimization problems, where the constraints are non-linear and non-convex. More specially, the constraints are Riemannian manifolds, see [6–8]. Many issues in nonlinear analysis, such as fixed point problems, variational inequality problems, and optimization problems have been magnified from the linear setting to nonlinear systems because the problems cannot be posted in linear spaces and require a manifold structure. Therefore, the extension of the concepts and techniques in variational inequalities and related topics from Euclidian spaces to Riemannian manifolds is natural. Indeed, the generalizations of optimization methods from Euclidean spaces to Riemannian manifolds also have some important advantages; see, e.g. [9–12].

In 2003, Németh [13] studied the variational inequality on Hadamard manifolds, which consists of finding $x \in C$ such that

$$\langle Ax, \exp_x^{-1} y\rangle \geq 0, \quad \forall y \in C, \tag{1}$$

CONTACT Xiaolong Qin qxlxajh@163.com

where C is a nonempty convex and closed set in Hadamard manifold $\mathcal{M}$, $A : C \to T\mathcal{M}$ is a vector field, that is, $Ax \in T_x\mathcal{M}$ for each $x \in C$, and $\exp^{-1}$ is the inverse of the exponential map. One denotes (1) by $VI(C, A)$ and it's the solution set by Ω, which is assumed to be nonempty. Variational inequality problem (1) on the Hadamard manifolds is an extension of the Hartman Stampacchia variational inequality in Euclidean spaces. More precisely, if $\mathcal{M} = \mathbb{R}^n$, then the vector field $A : C \to T\mathcal{M}$ collapses to the operator $A : C \to \mathbb{R}^n$. So problem (1) is reduced to the problem of finding $x \in C$ such that $\langle A(x), y - x\rangle \geq 0, \forall y \in C$. Actually, in the recent years, some algorithms to solve the variational inequalities which involve monotone operators have been extended from the framework of Hilbert spaces to the more general framework of the Riemannian manifolds; see, e.g. [14–16]. To the best of our knowledge, the more focused algorithms are Korpelevich's method [17] and proximal point algorithm [18]. Recall that Tseng [19] introduced the Tseng's extragradient method for solving pseudomonotone variational inequalities in finite and infinite dimensional linear spaces. It is known that the research on pseudomonotone variational inequalities is limited due to the conditions imposed on operators and the nonlinearity of manifolds.

Motivated by the results described above, the aim of this paper is to present an extragradient algorithm for variational inequalities associated with pseudomonotone vector fields in Hadamard manifolds and to study the convergence properties of the extragradient algorithm. We first incorporate the Tseng's extragradient method with a suitable linesearch to remove the dependence on the Lipschitz continuity modulus of A when choosing stepsize λ. In particular, we weaken the Lipschitz continuity of A to the uniform continuity, which is crucial when the operator is not Lipschitz continuous or the Lipschitz modulus is difficult to estimate in advance. To the best of our knowledge, this result has not been studied in Hadamard manifolds before. It is worth mentioning that our results can be seen as a generalization of the corresponding results presented by Thong and Vuong [20] in real Hilbert spaces.

The rest of this paper is presented by dividing several sections. In Section 2, we present some basic definitions and fundamental results from manifolds which will be needed in the sequel. In Section 3, we propose a Tseng's extragradient method for finding the solutions of the variational inequality problem (1) in the setting of Hadamard manifolds and study the convergence of the sequences generated by the proposed algorithm. In Section 4, we give two numerical experiments to illustrate the performance of the proposed algorithm. Finally, Section 5 concludes the paper with a brief summary.

2. Preliminaries

Let $\mathcal{M}$ be a finite dimensional differentiable manifold. The set of all tangents at $x \in \mathcal{M}$ is called a tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$, which forms a vector space of the same dimension as $\mathcal{M}$ and is denoted by $T_x\mathcal{M}$. The tangent bundle of $\mathcal{M}$ is denoted by $T\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$, which is naturally a manifold. We denote by $\langle\cdot,\cdot\rangle_x$ the scalar product on $T_x\mathcal{M}$ with the associated norm $\|\cdot\|_x$, where the subscript x is sometimes omitted. A differentiable manifold $\mathcal{M}$ with a Riemannian metric $\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. Let $\gamma : [a, b] \to \mathcal{M}$ be a piecewise differentiable curve joining $x = \gamma(a)$ to $y = \gamma(b)$ in $\mathcal{M}$, we can define the length of $l(\gamma) = \int_a^b \|\gamma'(t)\|\mathrm{d}t$. The minimal length of all such curves joining x to y is called the Riemannian distance and it is denoted by $\mathrm{d}(x, y)$.

Let ∇ be the Levi–Civita connection associated with the Riemannian metric. Let γ be a smooth curve in $\mathcal{M}$. A vector field X is said to be parallel along γ iff $\nabla_{\gamma'}X = 0$. If γ' is parallel along γ, i.e. $\nabla_{\gamma'}\gamma' = 0$, then γ is said to be geodesic, and in this case, $\|\gamma'\|$ is a constant. Furthermore, if $\|\gamma'\| = 1$, then γ is called normalized. A geodesic joining x to y in $\mathcal{M}$ is said to be minimal if its length equals $\mathrm{d}(x, y)$. Let $\gamma : \mathbb{R} \to \mathcal{M}$ be a geodesic and $P_\gamma[.,.]$ denote the parallel transport along γ with respect to V, which is defined by $P_{\gamma[\gamma(a),\gamma(b)]}(v) = V(\gamma(b))$ for all $a, b \in \mathbb{R}$ and $v \in T_{\gamma(a)}\mathcal{M}$, where V is the unique vector field satisfying $\nabla_{\gamma'(t)}V = 0$ and $V(\gamma(a)) = v$. Then, for any $a, b \in \mathbb{R}$, $P_{\gamma,[\gamma(b),\gamma(a)]}$ is an isometry from $T_{\gamma(a)}\mathcal{M}$ to $T_{\gamma(b)}\mathcal{M}$. We will write $P_{y,x}$ instead of $P_{\gamma,[y,x]}$ in the case that γ is a minimal geodesic joining x to y if this will avoid any confusion.

A Riemannian manifold is complete if for any $x \in \mathcal{M}$ all geodesics emanating from x are defined for all $-\infty < t < +\infty$. By the Hopf–Rinow Theorem [21], we know that if $\mathcal{M}$ is complete, then any pair of points in $\mathcal{M}$ can be joined by a minimal geodesic. Moreover, $(\mathcal{M}, d)$ is a complete metric space and bounded closed subsets are compact. If $\mathcal{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ at x is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_x\mathcal{M}$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v, that is, $\gamma(0) = x$ and $\gamma'(0) = v$. It is easy to see that $\exp_x tv = \gamma_v(t, x)$ for each real number t. Note that the mapping $\exp_x$ is differentiable on $T_x\mathcal{M}$ for any $x \in \mathcal{M}$. By the inverse mapping theorem, there exists an inverse exponential map $\exp_x^{-1} : \mathcal{M} \to T_x\mathcal{M}$. Moreover, the geodesic is the unique shortest path with $\| \exp_x^{-1} y\| = \| \exp_y^{-1} x\| = \mathrm{d}(x, y)$, where $\mathrm{d}(x, y)$ is the geodesic distance between x and y in $\mathcal{M}$. For further details, we refer [21].

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. If $\mathcal{M}$ is a Hadamard manifold, then $\exp_x^{-1} : \mathcal{M} \to T_x\mathcal{M}$ is a diffeomorphism for every $x \in \mathcal{M}$ and if $x, y \in \mathcal{M}$, then there exists a unique minimal geodesic joining x to y. The rest of the paper, one always assumes that $\mathcal{M}$ is a Hadamard manifold. The following result is known and will be useful.

Proposition 2.1 ([21]): *Let $\mathcal{M}$ be a Hadamard manifold and $p \in \mathcal{M}$. Then $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and for any two points $p, q \in \mathcal{M}$, there exists a unique normalized geodesic joining p to q, which is, in fact, a minimal geodesic.*

This proposition yields that $\mathcal{M}$ is diffeomorphic to space $\mathbb{R}^n$. Thus one sees that $\mathcal{M}$ has the same topology and differential structure as $\mathbb{R}^n$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties, and one of the most important proprieties is illustrated in the following proposition.

Proposition 2.2 ([21]): *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle in a Hadamard manifold $\mathcal{M}$. For each $i = 1, 2, 3 \pmod 3$, let $\gamma_i : [0, l_i] \to \mathcal{M}$ denote the geodesic joining p_i to p_{i+1}. Let $l_i = L(\gamma_i)$ and $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$ be the angle between tangent vectors $\gamma_i'(0)$ and $\gamma_{i-1}'(l_{i-1})$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \le \pi$;
(ii) $l_i^2 + l_{i+1}^2 - 2l_il_{i+1} \cos \alpha_{i+1} \le l_{i-1}^2$;
(iii) $l_{i+1} \cos \alpha_{i+2} + l_i \cos \alpha_i \ge l_{i+2}$.

In terms of the distance and the exponential map, Proposition 2.2 (ii) and (iii) can be rewritten as

$$\mathrm{d}^2(p_i, p_{i+1}) + \mathrm{d}^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}\rangle \le \mathrm{d}^2(p_{i-1}, p_i) \tag{2}$$

and

$$\mathrm{d}^2(p_i, p_{i+1}) \le \langle \exp_{p_i}^{-1} p_{i+2}, \exp_{p_i}^{-1} p_{i+1}\rangle + \langle \exp_{p_{i+1}}^{-1} p_{i+2}, \exp_{p_{i+1}}^{-1} p_i\rangle,$$

since $\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2}\rangle = \mathrm{d}(p_i, p_{i+1})\, \mathrm{d}(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}$. For further detail, one refers to [22].

Lemma 2.3 ([23]): *Let $\{x_n\}$ be a sequence in $\mathcal{M}$ such that $x_n \to x_0 \in \mathcal{M}$. Then the following assertions hold.*

(i) *For any $y \in \mathcal{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*
(ii) *If $v_n \in T_{x_n}\mathcal{M}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathcal{M}$.*
(iii) *Given $u_n, v_n \in T_{x_n}\mathcal{M}$ and $u_0, v_0 \in T_{x_0}\mathcal{M}$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$.*

(iv) *For any* $u \in T_{x_0}\mathcal{M}$, *the function* $A : \mathcal{M} \to T\mathcal{M}$ *defined by* $A(x) = P_{x.x_0}u$ *for each* $x \in \mathcal{M}$ *is continuous on* $\mathcal{M}$.

The following inequality is crucial in convergence analysis of our algorithm.

Lemma 2.4 ([24]): *Let* $\Delta(p,q,r)$ *be a geodesic triangle in Hadamard manifold* $\mathcal{M}$. *Then there exists a triangle* $\Delta(\bar{p},\bar{q},\bar{r})$ ($\bar{p},\bar{q},\bar{r} \in \mathbb{R}^2$) *for* $\Delta(p,q,r)$ *such that* $\mathrm{d}(p,q) = \|\bar{p}-\bar{q}\|$, $\mathrm{d}(q,r) = \|\bar{q}-\bar{r}\|$, *and* $\mathrm{d}(r,p) = \|\bar{r}-\bar{p}\|$.

The triangle $\Delta(\bar{p},\bar{q},\bar{r})$ is called the comparison triangle of the geodesic triangle $\Delta(p,q,r)$, which is unique up to the isometry of $\mathcal{M}$.

The next result describes the relationships between a geodesic triangle and its comparison triangle involving distances between points.

Lemma 2.5 ([25]): *Let* $\Delta(p,q,r)$ *be a geodesic triangle in a Hadamard manifold* $\mathcal{M}$ *and* $\Delta(\bar{p},\bar{q},\bar{r})$ *be its comparison triangle.*

(i) *Let* α,β,γ *(respectively,* $\bar{\alpha},\bar{\beta},\bar{\gamma}$*) be the angles of* $\Delta(p,q,r)$ *(respectively,* $\Delta(\bar{p},\bar{q},\bar{r})$*) at the vertices* p, q, r *(respectively,* $\bar{p},\bar{q},\bar{r}$*). Then, the following inequalities hold:* $\bar{\alpha} \ge \alpha$, $\bar{\beta} \ge \beta$, *and* $\bar{\gamma} \ge \gamma$.
(ii) *Let* z *be a point on the geodesic joining* p *to* q *and* $\bar{z}$ *be its comparison point in the interval* $[\bar{p},\bar{q}]$. *Suppose that* $\mathrm{d}(z,p) = \|\bar{z}-\bar{p}\|$ *and* $\mathrm{d}(z,q) = \|\bar{z}-\bar{q}\|$. *Then* $\mathrm{d}(z,r) \le \|\bar{z}-\bar{r}\|$.

Given C, a nonempty subset of $\mathcal{M}$, one uses $\mathfrak{X}\mathcal{M}$ to denote the set of all univalued vector fields $A : \mathcal{M} \to T\mathcal{M}$ such that $A(x) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$, and uses $D(A)$ to denote the domain of A, which is defined by $D(A) = \{x \in \mathcal{M} : A(x) \neq \emptyset\}$.

Definition 2.6 ([26, 27]): Let $\mathcal{M}$ be a Hadamard manifold. A vector field $A \in \mathfrak{X}\mathcal{M}$ is said to be

(i)monotone if, for any $x,y \in \mathcal{M}$, $\langle Ax, \exp_x^{-1} y\rangle \le \langle Ay, -\exp_y^{-1} x\rangle$;
(ii)pseudomonotone if, for any $x,y \in \mathcal{M}$, $\langle Ax, \exp_x^{-1} y\rangle \ge 0 \Rightarrow \langle Ay, \exp_y^{-1} x\rangle \le 0$.

The notion of the uniform continuity (cf. [5]) for operators in Banach spaces is extended in the following definition to the setting of Hadamard manifolds.

Definition 2.7: A vector field $A \in \mathfrak{X}\mathcal{M}$ is said to be uniformly continuous if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x,y \in \mathcal{M}$, $\mathrm{d}(x,y) < \delta \Rightarrow \mathrm{d}(Ax,Ay) < \varepsilon$.

Let P_C denote the projection onto $C \subset \mathcal{M}$ defined by

$$P_C(p) = \{p_0 \in C : \mathrm{d}(p,p_0) \le \mathrm{d}(p,q), \forall q \in C\}, \quad \forall p \in \mathcal{M}.$$

The next result gives a characterization of projection P_C.

Lemma 2.8 ([28]): *Given* $q \in \mathcal{M}$, *there exists a unique projection* $P_C(q)$. *Then, the following inequality holds:* $\langle \exp_{P_C(q)}^{-1} q, \exp_{P_C(q)}^{-1} p\rangle \le 0, \forall p \in C$.

The following lemmas are useful for the convergence of our proposed algorithm.

Lemma 2.9: *For* $x \in \mathcal{M}$ *and* $\alpha \ge \beta > 0$, *the following inequalities hold:*

$$\frac{\mathrm{d}(x, P_C(\exp_x(-\alpha Ax)))}{\alpha} \le \frac{\mathrm{d}(x, P_C(\exp_x(-\beta Ax)))}{\beta}$$

and

$$\mathrm{d}(x, P_C(\exp_x(-\beta Ax))) \le \mathrm{d}(x, P_C(\exp_x(-\alpha Ax))).$$

Proof: Suppose $x_\alpha = P_C(\exp_x(-\alpha Ax))$ and $x_\beta = P_C(\exp_x(-\beta Ax))$. From Lemma 2.8, it follows that

$$\left\langle \frac{x_\alpha - \exp_x(-\alpha Ax)}{\alpha}, \exp_{x_\alpha}^{-1} x_\beta \right\rangle \geq 0, \left\langle \frac{x_\beta - \exp_x(-\beta Ax)}{\beta}, \exp_{x_\beta}^{-1} x_\alpha \right\rangle \geq 0.$$

Adding the inequalities yields

$$\begin{aligned} 0 &\leq \langle \frac{\exp_{x_\alpha}^{-1} x}{\alpha} - \frac{\exp_{x_\beta}^{-1} x}{\beta}, \exp_{x_\beta}^{-1} x_\alpha \rangle \\ &\leq \frac{\| \exp_{x_\alpha}^{-1} x\| \| \exp_{x_\beta}^{-1} x_\alpha \|}{\alpha} - \frac{\| \exp_{x_\beta}^{-1} x\| \| \exp_{x_\beta}^{-1} x_\alpha \|}{\beta}. \end{aligned}$$

In view of Lemma 2.4, one obtains that

$$\begin{aligned} 0 &\leq \frac{1}{\alpha} \mathrm{d}(x, x_\alpha)(\mathrm{d}(x, x_\beta) - \mathrm{d}(x, x_\alpha)) - \frac{1}{\beta} \mathrm{d}(x, x_\beta)(\mathrm{d}(x, x_\beta) - \mathrm{d}(x, x_\alpha)) \\ &= -\mathrm{d}^2(x, x_\alpha) - \frac{\alpha}{\beta} \mathrm{d}^2(x, x_\beta) + \mathrm{d}(x, x_\alpha)\, \mathrm{d}(x, x_\beta) + \frac{\alpha}{\beta} \mathrm{d}(x, x_\alpha)\, \mathrm{d}(x, x_\beta). \end{aligned}$$

Hence,

$$0 \geq (\mathrm{d}(x, x_\alpha) - \frac{\alpha}{\beta} \mathrm{d}(x, x_\beta))(\mathrm{d}(x, x_\alpha) - \mathrm{d}(x, x_\beta)),$$

It follows that

$$\mathrm{d}(x, x_\alpha) - \frac{\alpha}{\beta} \mathrm{d}(x, x_\beta) \leq 0.$$

As was to be shown. ■

Lemma 2.10: *Suppose the function $A : \mathcal{M} \to T\mathcal{M}$ is uniformly continuous on bounded subsets of $\mathcal{M}$ and U is a bounded subset of $\mathcal{M}$. Then $A(U)$ is bounded.*

The proof is trivial. We omit it here.

Definition 2.11 ([22]): Let X be a complete metric space and $C \in X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejér convergent to C if, for all $y \in C$ and $n \geq 0$, $\mathrm{d}(x_{n+1}, y) \leq \mathrm{d}(x_n, y)$.

Lemma 2.12 ([23]): *Let X be a complete metric space and let $C \in X$ be a nonempty set. Let $x_n \subset X$ be Fejér convergent to C and suppose that any cluster point of $\{x_n\}$ belongs to C. Then $\{x_n\}$ converges to a point in C.*

3. Main results

In this section, we introduce a Tseng's extragradient algorithm for the variational inequality problem in Hadamard manifolds. Next, we make the following conditions:

(C1) The solution set of $VI(C, A)$ (1) is nonempty, that is, $\Omega \neq \emptyset$.
(C2) C is a nonempty convex and closed subset of Hadamard manifold $\mathcal{M}$, $A : C \to T\mathcal{M}$ is a vector field, that is, $A(x) \in T_x\mathcal{M}$ for each $x \in C$, and $\exp^{-1}$ is the inverse of exponential map.
(C3) The mapping A is a pseudomonotone and uniformly continuous on bounded subsets of $\mathcal{M}$.

The following lemmas will be useful in the proof of the convergence.

Lemma 3.1: *Assume that conditions* (C2) − (C3) *hold. Then Armijo-line-search rule* (3) *is well-defined and* $\lambda_n \le \gamma$.

Proof: If $x_n \in \Omega$, then $x_n = P_C(\exp_{x_n}(-\gamma A x_n))$. Therefore, (3) holds with $m = 0$. If $x_n \notin \Omega$, then, for all m,

$$\gamma l^m \,\mathrm{d}(Ax_n.AP_C(\exp_{x_n}(-\gamma l^m Ax_n))) > \mu \,\mathrm{d}(x_n, P_C(\exp_{x_n}(-\gamma l^m Ax_n))), \tag{4}$$

which is equivalent to

$$\mathrm{d}(Ax_n, AP_C(\exp_{x_n}(-\gamma l^m Ax_n))) > \mu \frac{\mathrm{d}(x_n, P_C(\exp_{x_n}(-\gamma l^m Ax_n)))}{\gamma l^m}. \tag{5}$$

One next considers two possibilities of x_n. First, if $x_n \in C$, then one obtains from the continuity of A and P_C that

$$\lim_{m\to\infty} \mathrm{d}(x_n, P_C(\exp_{x_n}(-\gamma l^m Ax_n))) = 0.$$

From the uniform continuity of operator A on bounded subsets of $\mathcal{M}$, one asserts that

$$\lim_{m\to\infty} \mathrm{d}(Ax_n, AP_C(\exp_{x_n}(-\gamma l^m Ax_n))) = 0. \tag{6}$$

Using (5) and (6), one has

$$\lim_{m\to\infty} \frac{\mathrm{d}(x_n, P_C(\exp_{x_n}(-\gamma l^m Ax_n)))}{\gamma l^m} = 0. \tag{7}$$

Setting $z_m = P_C(\exp_{x_n}(-\gamma l^m Ax_n))$, one gets

$$\langle \exp_{x_n}^{-1} z_m + \gamma l^m Ax_n, \exp_{z_m}^{-1} x\rangle \ge 0, \quad \forall x \in C.$$

This is equivalent to

$$\langle \frac{\exp_{x_n}^{-1} z_m}{\gamma l^m}, \exp_{z_m}^{-1} x\rangle + \langle Ax_n, \exp_{z_m}^{-1} x\rangle \ge 0, \quad \forall x \in C. \tag{8}$$

Taking the limit $m \to \infty$ in (8), one obtains $\langle Ax_n, \exp_{x_n}^{-1} x\rangle \ge 0, \forall x \in C$, which implies that $x_n \in \Omega$. This is a contradiction. Then, if $x_n \notin C$, then

$$\lim_{m\to\infty} \mathrm{d}(x_n, P_C(\exp_{x_n}(-\gamma l^m Ax_n))) = \mathrm{d}(x_n, P_C(x_n)) > 0 \tag{9}$$

and

$$\lim_{m\to\infty} \gamma l^m \,\mathrm{d}(Ax_n, AP_C(\exp_{x_n}(-\gamma l^m Ax_n))) = 0. \tag{10}$$

From (4), (9) and (10), one gets a contradiction. So λ_n is well defined and obviously, $\lambda_n \le \gamma$. ■

Lemma 3.2: *Let* $\{x_n\}$ *be a sequence generated by Algorithm 1. Then, for every* $p \in \Omega$, *it holds that*

$$\mathrm{d}^2(x_{n+1}, p) \le \mathrm{d}^2(x_n, p) - (1-\mu^2)\,\mathrm{d}^2(y_n, x_n),$$

and the sequence $\{x_n\}$ *is bounded.*

Algorithm 1 (A Tseng's extragradient algorithm)

Initialization: Give $\gamma > 0, l \in (0,1), \mu \in (0,1)$ and let $x_0 \in \mathcal{M}$ be an arbitrary starting point. Set $n = 0$.

Iterative Steps: Given the current iterate $x_n \in \mathcal{M}$, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(\exp_{x_n}(-\lambda_n A x_n)),$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \,\mathrm{d}(Ax_n, Ay_n) \le \mu \,\mathrm{d}(x_n, y_n). \tag{1}$$

If $y_n = x_n$, then stop and x_n is a solution of variational inequality (1). Otherwise,

Step 2. Compute

$$x_{n+1} = \exp_{y_n}(\lambda_n(Ax_n - Ay_n)).$$

Set $n := n+1$ and return to **Step 1**.

Proof: From the definition of y_n, one deduces from Lemma 2.8 that

$$\langle \exp_{x_n}^{-1} y_n + \lambda_n A x_n, \exp_p^{-1} y_n \rangle \ge 0,$$

which is equivalent to

$$\langle \exp_{x_n}^{-1} y_n, \exp_p^{-1} y_n \rangle \le -\lambda_n \langle Ax_n, \exp_p^{-1} y_n \rangle. \tag{11}$$

Since $p \in \Omega$, one has $\langle Ap, \exp_p^{-1} y_n \rangle \ge 0$. Also, from the pseudomonotonicity of A on $\mathcal{M}$, it follows that

$$\begin{aligned}\langle Ay_n - Ax_n, \exp_p^{-1} y_n \rangle &= \langle Ay_n, \exp_p^{-1} y_n \rangle - \langle Ax_n, \exp_p^{-1} y_n \rangle \\ &\ge -\langle Ax_n, \exp_p^{-1} y_n \rangle. \end{aligned} \tag{12}$$

Let $\Delta(x_n, y_n, p) \subseteq \mathcal{M}$ be a geodesic triangle with vertices x_n, y_n, and p, and let $\Delta(\overline{x_n}, \overline{y_n}, \bar{p}) \subseteq \mathbb{R}^2$ be a comparison triangle. By utilizing Lemma 2.4, one concludes that

$$\mathrm{d}(x_n, p) = \mathrm{d}(\overline{x_n}, \bar{p}), \quad \mathrm{d}(y_n, p) = \mathrm{d}(\overline{y_n}, \bar{p}), \quad \mathrm{d}(x_n, y_n) = \mathrm{d}(\overline{x_n}, \overline{y_n}).$$

From $x_{n+1} = \exp_{y_n}(\lambda_n(Ax_n - Ay_n))$, the comparison point of $\overline{x_{n+1}}$ is $\overline{y_n} + \lambda_n(A\overline{x_n} - A\overline{y_n})$. By use of Lemma 2.5, one has

$$\begin{aligned} \mathrm{d}^2(x_{n+1}, p) \le \mathrm{d}^2(\overline{x_{n+1}}, \bar{p}) &= \|\overline{y_n} + \lambda_n(A\overline{x_n} - A\overline{y_n}) - \bar{p}\|^2 \\ &= \|\overline{y_n} - \bar{p}\|^2 + \lambda_n^2 \|A\overline{x_n} - A\overline{y_n}\|^2 + 2\lambda_n \langle A\overline{x_n} - A\overline{y_n}, \overline{y_n} - \bar{p} \rangle \\ &= \|\overline{y_n} - \overline{x_n}\|^2 + \|\overline{x_n} - \bar{p}\|^2 + 2\langle \overline{y_n} - \overline{x_n}, \overline{x_n} - \bar{p} \rangle \\ &\quad + \lambda_n^2 \|A\overline{x_n} - A\overline{y_n}\|^2 + 2\lambda_n \langle A\overline{x_n} - A\overline{y_n}, \overline{y_n} - \bar{p} \rangle \\ &= \|\overline{x_n} - \bar{p}\|^2 + \|\overline{y_n} - \overline{x_n}\|^2 - 2\langle \overline{y_n} - \overline{x_n}, \overline{y_n} - \overline{x_n} \rangle + 2\langle \overline{y_n} - \overline{x_n}, \overline{y_n} - \bar{p} \rangle \\ &\quad + \lambda_n^2 \|A\overline{y_n} - A\overline{x_n}\|^2 + 2\lambda_n \langle A\overline{x_n} - A\overline{y_n}, \overline{y_n} - \bar{p} \rangle \\ &= \|\overline{x_n} - \bar{p}\|^2 - \|\overline{y_n} - \overline{x_n}\|^2 + \lambda_n^2 \|A\overline{y_n} - A\overline{x_n}\|^2 \\ &\quad + \langle 2\overline{y_n} - 2\overline{x_n} + 2\lambda_n A\overline{x_n} - 2\lambda_n A\overline{y_n}, \overline{y_n} - \bar{p} \rangle \end{aligned}$$

$$\leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\|A\overline{y_n} - A\overline{x_n}\|^2 + \langle 2\overline{y_n} - 2\overline{x_n} + 2\lambda_n A\overline{x_n} - 2\lambda_n A\overline{y_n}, \overline{y_n} - \bar{p}\rangle. \tag{13}$$

In the geodesic triangle $\Delta(Ax_n, Ay_n, x_{n+1})$ and its comparison triangle $\Delta(A\overline{x_n}, A\overline{y_n}, \overline{x_{n+1}})$, using Lemma 2.4 again, we have $\|A\overline{x_n} - A\overline{y_n}\| = \mathrm{d}(Ax_n, Ay_n)$. From (13), we obtain

$$\mathrm{d}^2(x_{n+1},p) \leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\,\mathrm{d}^2(Ay_n,Ax_n) + \langle 2\overline{y_n} - 2\overline{x_n} + 2\lambda_n A\overline{x_n} - 2\lambda_n A\overline{y_n}, \overline{y_n} - \bar{p}\rangle. \tag{14}$$

In view of the geodesic triangle $\Delta(a,b,c)$ and its comparison triangle $\Delta(\bar{a},\bar{b},\bar{c})$, one sets $a = 2\exp_{x_n}^{-1} y_n - 2\lambda_n(Ay_n - Ax_n)$ and $b = \exp_p^{-1} y_n$. The comparison point $\bar{a} = 2\overline{y_n} - 2\overline{x_n} + 2\lambda_n A\overline{x_n} - 2\lambda_n A\overline{y_n}$ and $\bar{b} = \overline{y_n} - \bar{p}$. Let β and $\bar{\beta}$ denote the angles at c and $\bar{c}$, respectively. Then by use of Lemma 2.5 (i), we have $\bar{\beta} \geq \beta$, so $\cos\bar{\beta} \leq \cos\beta$. Using Proposition 2.2 and Lemma 2.4, we get

$$\langle \bar{a},\bar{b}\rangle = \|\bar{a}\|\|\bar{b}\|\cos\bar{\beta} \leq \|\bar{a}\|\|\bar{b}\|\cos\beta = \|a\|\|b\|\cos\beta = \langle a,b\rangle,$$

and hence

$$\langle 2\overline{y_n} - 2\overline{x_n} + 2\lambda_n A\overline{x_n} - 2\lambda_n A\overline{y_n}, \overline{y_n} - \bar{p}\rangle \leq \langle 2\exp_{x_n}^{-1} y_n - 2\lambda_n(Ay_n - Ax_n), \exp_p^{-1} y_n\rangle. \tag{15}$$

Combining (14) and (15) yields that

$$\begin{aligned}
\mathrm{d}^2(x_{n+1},p) &\leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\,\mathrm{d}^2(Ay_n,Ax_n) \\
&\quad + \langle 2\overline{y_n} - 2\overline{x_n} + 2\lambda_n A\overline{x_n} - 2\lambda_n A\overline{y_n}, \overline{y_n} - \bar{p}\rangle \\
&\leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\,\mathrm{d}^2(Ay_n,Ax_n) \\
&\quad + \langle 2\exp_{x_n}^{-1} y_n - 2\lambda_n(Ay_n - Ax_n), \exp_p^{-1} y_n\rangle \\
&\leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\,\mathrm{d}^2(Ay_n,Ax_n) \\
&\quad - 2\lambda_n\langle Ay_n - Ax_n, \exp_p^{-1} y_n\rangle + 2\langle \exp_{x_n}^{-1} y_n, \exp_p^{-1} y_n\rangle.
\end{aligned} \tag{16}$$

Due to (11), (12) and (16), it follows that

$$\begin{aligned}
\mathrm{d}^2(x_{n+1},p) &\leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\,\mathrm{d}^2(Ay_n,Ax_n) \\
&\quad + 2\lambda_n\langle Ax_n, \exp_p^{-1} y_n\rangle - 2\lambda_n\langle Ax_n, \exp_p^{-1} y_n\rangle \\
&= \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \lambda_n^2\,\mathrm{d}^2(Ay_n,Ax_n).
\end{aligned} \tag{17}$$

From (3) and (17), we claim that

$$\begin{aligned}
\mathrm{d}^2(x_{n+1},p) &\leq \mathrm{d}^2(x_n,p) - \mathrm{d}^2(y_n,x_n) + \mu^2\,\mathrm{d}^2(y_n,x_n) \\
&= \mathrm{d}^2(x_n,p) - (1-\mu^2)\,\mathrm{d}^2(y_n,x_n).
\end{aligned} \tag{18}$$

This implies that $\mathrm{d}(x_{n+1},p) \leq \mathrm{d}(x_n,p)$. So, $\{x_n\}$ is bounded. ■

We are now ready to show the main result regarding convergence of the proposed algorithm.

Theorem 3.3: *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Assume that conditions* (C1)−(C3) *hold. Then $\{x_n\}$ is convergent to a solution of problem $VI(C,A)$* (1).

Proof: From Lemma 3.2 and Definition 2.11, we know that $\{x_n\}$ is Fejér convergent to Ω. Let x^* be a cluster point of x_n. Then there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x^*$. Also, adding $\mu \in (0,1)$ and (18) yields that

$$(1-\mu^2)\,\mathrm{d}^2(y_n,x_n) \le \mathrm{d}^2(x_n,p) - \mathrm{d}^2(x_{n+1},p),$$

which implies that $\lim_{n\to\infty} \mathrm{d}(x_n,y_n) = 0$. So the sequence $\{y_n\}$ is bounded and $y_{n_k} \to x^*$. Since $y_{n_k} = P_C(\exp_{x_{n_k}}(-\lambda_n A x_{n_k}))$, it holds

$$\langle \exp^{-1}_{x_{n_k}} y_{n_k} + \lambda_{n_k} A x_{n_k}, \exp^{-1}_{y_{n_k}} x\rangle \ge 0, \quad \forall x \in C,$$

that is,

$$\frac{1}{\lambda_{n_k}}\langle \exp^{-1}_{y_{n_k}} x_{n_k}, \exp^{-1}_{y_{n_k}} x\rangle \le \langle A x_{n_k}, \exp^{-1}_{y_{n_k}} x\rangle. \tag{19}$$

Let $\Delta(y_{n_k}, x_{n_k}, x)$ be a geodesic triangle. By use of (2), we have

$$\langle \exp^{-1}_{y_{n_k}} x_{n_k}, \exp^{-1}_{y_{n_k}} x\rangle \ge \frac{1}{2}(\mathrm{d}^2(y_{n_k},x_{n_k}) + \mathrm{d}^2(y_{n_k},x) - \mathrm{d}^2(x_{n_k},x)). \tag{20}$$

From (19) and (20), we have

$$\frac{1}{2\lambda_{n_k}}(\mathrm{d}^2(y_{n_k},x_{n_k}) + \mathrm{d}^2(y_{n_k},x) - \mathrm{d}^2(x_{n_k},x)) \le \langle A x_{n_k}, \exp^{-1}_{y_{n_k}} x\rangle. \tag{21}$$

Following Lemma 2.12, it remains to prove that every weak limit point of x_n belongs to Ω. Now, passing to the limit, we show that

$$\langle A x^*, \exp^{-1}_{x^*} x\rangle \ge 0, \quad \forall x \in C, \tag{22}$$

by considering two possible cases on sequence $\{\lambda_n\}$.

Case 1. Assume that $\lim_{k\to\infty} \lambda_{n_k} > 0$. Since $\{x_{n_k}\}$ is a bounded sequence, and A is uniformly continuous on bounded subsets of $\mathcal{M}$, one asserts from Lemma 2.10 that $\{A x_{n_k}\}$ is bounded. Taking $k \to \infty$ in (21), we get

$$\langle A x^*, \exp^{-1}_{x^*} x\rangle \ge 0, \quad \forall x \in C.$$

Case 2. Assume that $\lim_{k\to\infty} \lambda_{n_k} = 0$. Setting $z_{n_k} = P_C(\exp_{x_{n_k}}(-\lambda_{n_k} l^{-1} A x_{n_k}))$, we have $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Applying Lemma 2.9, we obtain $\mathrm{d}(x_{n_k}, z_{n_k}) \le 1/l\,\mathrm{d}(x_{n_k}, y_{n_k}) \to 0$ as $k \to \infty$. Then, $z_{n_k} \to x^*$, which implies that $\{z_{n_k}\}$ is bounded. Using Lemma 2.10 again, one concludes that

$$\lim_{k\to\infty} \mathrm{d}(A x_{n_k}, A z_{n_k}) = 0. \tag{23}$$

From (3), we get

$$\lambda_{n_k} l^{-1}\,\mathrm{d}(A x_{n_k}, A P_C(\exp_{x_{n_k}}(-\lambda_{n_k} l^{-1} A x_{n_k}))) > \mu\,\mathrm{d}(x_{n_k}, P_C(\exp_{x_{n_k}}(-\lambda_{n_k} l^{-1} A x_{n_k}))).$$

That is,

$$\frac{1}{\mu}\,\mathrm{d}(A x_{n_k}, A z_{n_k}) > \frac{\mathrm{d}(x_{n_k}, z_{n_k})}{\lambda_{n_k} l^{-1}}. \tag{24}$$

According to (23) and (24), we obtain

$$\lim_{k\to\infty} \frac{\mathrm{d}(x_{n_k}, z_{n_k})}{\lambda_{n_k} l^{-1}} = 0.$$

In view of $z_{n_k} = P_C(\exp_{x_{n_k}}(-\lambda_{n_k} l^{-1} A x_{n_k}))$ and Lemma 2.8, one arrives at

$$\langle \exp_{x_{n_k}}^{-1} z_{n_k} + \lambda_{n_k} l^{-1} A x_{n_k}, \exp_{z_{n_k}}^{-1} x \rangle \geq 0, \quad \forall x \in C. \tag{25}$$

Let $\Delta(z_{n_k}, x_{n_k}, x)$ be a geodesic triangle. It follows from (2) that

$$\langle \exp_{z_{n_k}}^{-1} x_{n_k}, \exp_{z_{n_k}}^{-1} x \rangle \geq \frac{1}{2}(\mathrm{d}^2(z_{n_k}, x_{n_k}) + \mathrm{d}^2(z_{n_k}, x) - \mathrm{d}^2(x_{n_k}, x)). \tag{26}$$

By use of (25) and (26), we have

$$\frac{1}{2\lambda_{n_k} l^{-1}}(\mathrm{d}^2(z_{n_k}, x_{n_k}) + \mathrm{d}^2(z_{n_k}, x) - \mathrm{d}^2(x_{n_k}, x)) \leq \langle A x_{n_k}, \exp_{z_{n_k}}^{-1} x \rangle. \tag{27}$$

Taking the limit $k \to \infty$ in (27), we get

$$\langle Ax^*, \exp_{x^*}^{-1} x \rangle \geq 0, \quad \forall x \in C.$$

Therefore, inequality (22) is proved. So, we have $x^* \in \Omega$. By use of Lemma 2.12, one completes the proof immediately. ■

Remark 3.4: If $\mathcal{M} = H$, a real Hilbert space, and A is pseudomonotone, then Algorithm 1 is reduced to the algorithm proposed by Thong and Vuong [[20], Algorithm 1].

4. Numerical examples

In this section, we provide two numerical examples in the setting of Hadamard manifolds to illustrate the convergence behavior of Algorithm 1. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60 GHz computer with RAM 8.00 GB.

Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathcal{M} = (\mathbb{R}_{++}, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold with the Riemannian metric

$$\langle u, v \rangle := \frac{1}{x^2} uv, \quad \text{for } x \in \mathcal{M}, \quad \text{and} \quad \forall u, v \in T_x\mathcal{M},$$

where $T_x\mathcal{M}$ denotes the tangent plane at $x \in \mathcal{M}$. For all $x \in \mathcal{M}$, the tangent plane $T_x\mathcal{M}$ at x equals to $\mathbb{R}$.

The Riemannian distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ is given by

$$\mathrm{d}(x, y) := |\ln(x/y)|, \quad \forall x, y \in \mathcal{M}.$$

Then $\mathcal{M}$ is a Hadamard manifold. Let $\gamma[0, 1] \to \mathcal{M}$ be a geodesic starting from $x = \gamma(0)$ with velocity $v = \gamma'(0) \in T_x\mathcal{M}$ defined by $\gamma(s) := x\,\mathrm{e}^{(v/x)s}$. Therefore,

$$\exp_x sv = x\,\mathrm{e}^{(v/x)s}.$$

Furthermore, for any $x, y \in \mathcal{M}$, we get

$$y = \exp_x\left(\mathrm{d}(x, y)\frac{\exp_x^{-1} y}{\mathrm{d}(x, y)}\right) = x\,\mathrm{e}^{\frac{\exp_x^{-1} y}{x\,\mathrm{d}(x,y)}\mathrm{d}(x,y)} = x\,\mathrm{e}^{\frac{\exp_x^{-1} y}{x}}.$$

Thus the inverse of exponential map is $\exp_x^{-1} y = x \ln(y/x)$.

Example 4.1: Let $C = [1, 2]$ be a closed geodesic convex subset of $\mathcal{M}$. We consider the single-valued vector field $A : \mathcal{M} \to T\mathcal{M}$ defined by

$$A(x) = -x, \quad \text{for all } x \in C.$$

It is easy to see that A is pseudomonotone on C. Indeed, for any $x, y \in C$, if $\langle A(x), \exp_x^{-1} y\rangle = 1/x^2(-x) \cdot x \ln(y/x) \geq 0$, that is, $\ln(y/x) \leq 0$. Consequently, we can get $\langle A(y), \exp_y^{-1} x\rangle = 1/y^2(-y) \cdot y \ln(x/y) = -\ln(x/y) \leq 0$.

Let x^* be the solution set of variational inequality (1). Then, $x \in x^*$ if and only if

$$\left\langle A(x), \exp_x^{-1} y\right\rangle \geq 0, \quad \text{for all } y \in C.$$

Equivalently,

$$\frac{1}{x^2}(-x) \cdot x \ln\left(y/x\right) \geq 0, \quad \text{for all } y \in [1, 2].$$

This is equivalent to $x = 2$. Hence, $x^* = 2$. In Algorithm 1, we set $\gamma = l = \mu = 0.5$. We test the numerical behavior of Algorithm 1 with two different initial point x_0. The numerical results are reported in Table 1 and Figure 1. One can see that the algorithm converges to 2 after a few iterations.

Example 4.2: In this example, let $C = [1, +\infty)$ be a closed geodesic convex subset of $\mathcal{M}$ and $A : C \to \mathbb{R}$ be a single-valued vector field defined by

$$A(x) := x \ln x, \quad \forall x \in C.$$

Note that A is pseudomonotone on C. Indeed, for any $x, y \in C$, if $\langle A(x), \exp_x^{-1} y\rangle = 1/x^2 \cdot x \ln x \cdot x \ln(y/x) \geq 0$, we have $\ln(y/x) \geq 0$. Thus we get immediately that $\langle Ay, \exp_y^{-1} x\rangle = 1/y^2 \cdot y \ln y \cdot y \ln(x/y) \leq 0$.

Clearly, variational inequality (1) has a unique solution. Indeed,

$$\left\langle A\left(x^*\right), \exp_{x^*}^{-1} y\right\rangle = \frac{1}{x^{*2}} \cdot x^* \ln x^* \cdot x^* \ln\left(\frac{y}{x^*}\right), \quad \forall y \in C$$

Table 1. The numerical result for Example 4.1.

Iterate n	x_n with initial guest $x_0 = 1$	x_n with initial guest $x_0 = 1.5$
0	1	1.5
1	2.0072	2.2663
2	1.9964	1.8712
3	2.0018	2.0655
4	1.9991	1.9675
5	2.0004	2.0163
6	1.9998	1.9919
7	2.0001	2.0041
8	1.9999	1.9980
9	2.0000	2.0010
10	2.0000	1.9995
11	2.0000	2.0003
12	2.0000	1.9999
13	2.0000	2.0001
14	2.0000	2.0000
15	2.0000	2.0000
16	2.0000	2.0000
17	2.0000	2.0000
18	2.0000	2.0000
19	2.0000	2.0000

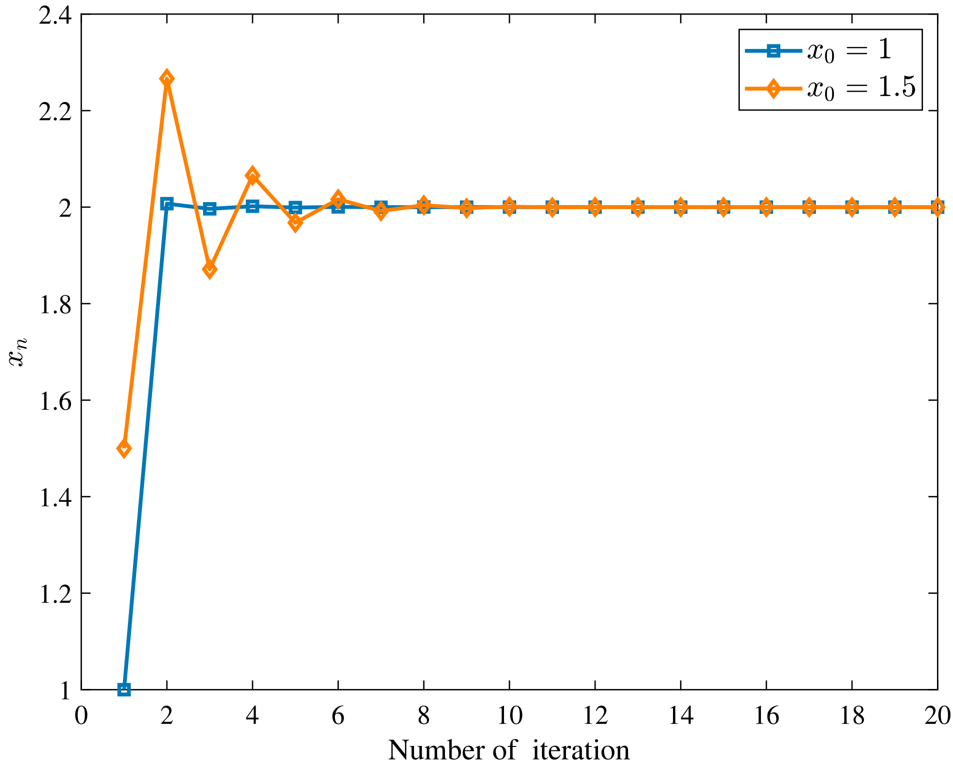

Figure 1. Iterative process of Example 4.1.

Table 2. The numerical result for Example 4.2.

Iterate n	x_n with initial guest $x_0 = 3$	x_n with initial guest $x_0 = 6$
0	3	6
1	2.0514	5.8658
5	1.6046	3.9438
9	1.3535	2.3305
13	1.2698	1.7534
17	1.2072	1.4417
21	1.1599	1.2943
25	1.1314	1.2256
29	1.1154	1.1738
33	1.1015	1.1384
37	1.0893	1.1215
41	1.0786	1.1068
45	1.0692	1.0939
49	1.0639	1.0826

$$= \ln x^* \ln\left(\frac{y}{x^*}\right) \geq 0, \quad \forall y \in C$$

$$\Leftrightarrow x^* = 1.$$

Therefore, solution set of variational inequality problem (1) is 1. We choose $\gamma = l = \mu = 0.5$ in Algorithm 1. Table 2 and Figure 2 show that the numerical behavior of Algorithm 1 with two different initial point x_0. We see that the iteration point converges to 1, which verifies the effectiveness of our algorithm.

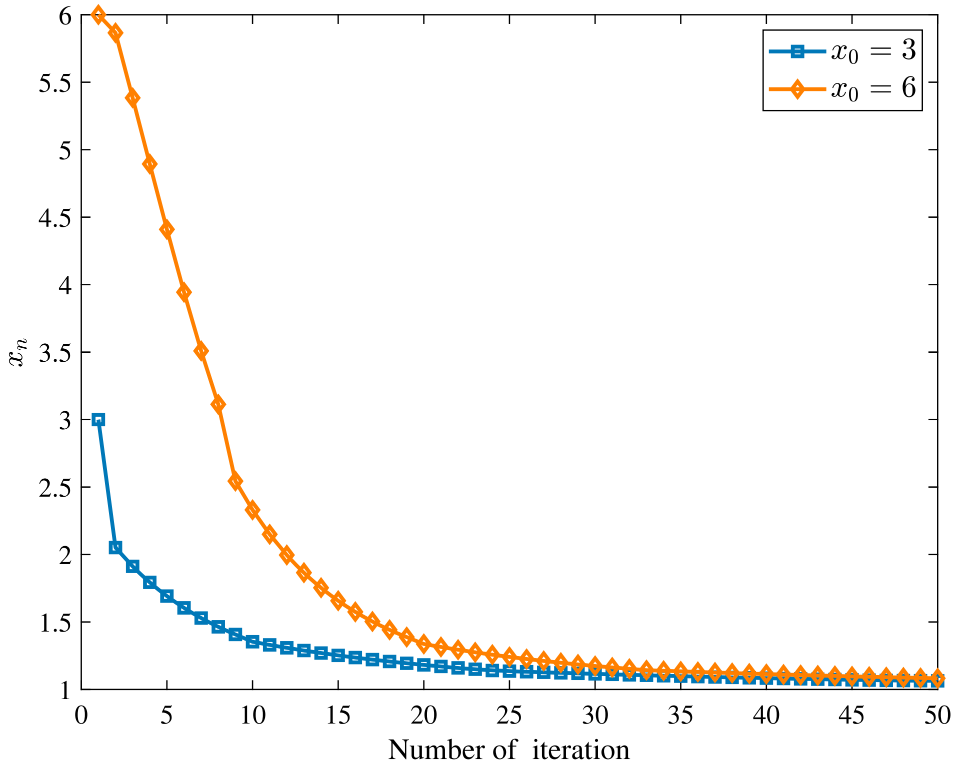

Figure 2. Iterative process of Example 4.2.

5. Concluding remarks

In this paper, we investigated the convergence of the Tseng's extragradient algorithm for pseudomonotone variational inequalities and provided a class of conditions of well-definedness for this algorithm in Hadamard manifolds. Our results were illustrated by several numerical experiments. To devise more effective algorithms for problem (1) on Hadamard manifolds, we will consider the geometric structure of manifolds in the future. Moreover, it is also of interest to do some numerical experiments and comparisons with other algorithms for practical problems on Riemannian manifolds.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Bing Tan http://orcid.org/0000-0003-1509-1809

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