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WEAK AND STRONG CONVERGENCE RESULTS OF FORWARD-BACKWARD SPLITTING METHODS FOR SOLVING INCLUSION PROBLEMS IN BANACH SPACES

YINGLIN LUO

Abstract. The purposes of this paper is to study the problem of the sum of two accretive operators in the framework of uniformly convex and uniformly smooth Banach spaces. Base on a viscosity-like type forward-backward splitting method, we prove two convergence theorems for solving zero points of the sum of two accretive operators. Numerical examples are given to illustrate the convergence of the methods.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and dual space E^*. The symbols $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively. In this paper, we consider the following variational inclusion problem: find $x \in E$ such that

$$0 \in Ax + Bx. \tag{1.1}$$

Inclusion problems play an important role in many fields, such as, economics, mechanics, optimization problems, signal processing, and image recovery. Many convex optimization problems can be reduced to a inclusion problem to deal; see, e.g., [1, 4, 6, 8–10, 20, 25, 28, 32, 33, 35], and the references therein. Since the wide applications of inclusion problems, they have became an important research area in the past several decades, and have attracted much attention from scholars; see, e.g., [2, 5, 17, 18, 23, 26, 27], and the references therein.

A classical method to solve the problem (1.1) is the forward-backward splitting algorithm, which proposed by Lions and Mercier [16]. This algorithm defined by the following iteration scheme: $\forall x_1 \in E$ and $x_{n+1} = (I+rA)^{-1}(I-rB)x_n$, where $r > 0$. A nice feature of this algorithm is that the iteration scheme involves only with A as the forward step and B as the backward step. There have been many works utilizing the forward-backward splitting algorithm to solve the problem (1.1). In 2012, López et al. [18] introduced the Halpern-type forward-backward method. Then, motivated by the results of [18] and [34], Cholamjiak [12] propose a general type of splitting methods for accretive operators in uniformly convex and q-uniformly smooth Banach spaces. Later, under some mild conditions, Shehu and Cai [30] presented the following algorithm which involves viscosity approximation method in the framework of uniformly convex and uniformly smooth Banach spaces:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J^A_{r_n}(x_n - r_n Bx_n),$$

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where $f(x)$ is a contraction, A is an m-accretive operator and B is an α-inverse strongly accretive mapping.

In this paper, we studied weak and strong convergence of splitting algorithms for solving variational inclusion problem (1.1) in the framework of uniformly convex and uniformly smooth Banach spaces. We propose a weak convergence algorithm involving viscosity approximation method with nonexpansive mappings, which extends the results of [30]. Also, under some mild assumptions as in [30], we present a strong convergence theorem of a general type of splitting method for solving problem (1.1), which extends the results of [12] from uniformly convex and q-uniformly smooth Banach spaces to the uniformly convex and uniformly smooth Banach spaces. Our results also partially extend the corresponding results in [7, 11, 22, 31] from Hilbert spaces to Banach spaces.

The paper is organized as follow. In Section 2, we present some related notations and definitions from accretive operators theory. Moreover, we also give some lemmas which will be used in the paper. In Section 3, in the framework of uniformly convex and uniformly smooth Banach spaces, we propose a weak convergence theorem involve viscosity approximation methods with nonexpansive mappings for solving problem (1.1). Also, a strong convergence theorem of a general type splitting method is introduced in this section. Finally, in Section 4, some numerical examples are given to illustrate the convergence of the considered iterative methods.

2. Preliminaries

Let E be a real Banach space. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\Big\}.$$

E is said to be uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0, 2]$. E is called p-uniformly smooth if the exists a constant c_p such that $\delta_E(\varepsilon) \geq c_p\varepsilon^p$ for any $\varepsilon \in (0, 2]$. The modulus of smoothness of E is the function $\rho_E(\epsilon) : R^+ \to R^+$ is defined by

$$\rho_E(\epsilon) := \sup\Big\{\frac{\|x - \epsilon y\| + \|x + \epsilon y\|}{2} - 1 : \|x\| = \|y\| = 1\Big\}.$$

E is called uniformly smooth if and only if $\lim_{t\to\infty} \frac{\rho_E(t)}{t} = 0$. If $1 < q \leq 2$, then E is called q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho_E(\epsilon) \leq c_q\epsilon^q$ for any $\epsilon > 0$. It is obvious that q-uniformly smooth Banach spaces must be uniformly smooth, but not vice versa. The norm of E is said to be the Fréchet differentiable, if for each $x \in E$, $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists and attained uniformly for all $\|y\| = 1$. It is easy to deduce that a uniformly smooth Banach space has a Fréchet differentiable norm.

The normalized duality mapping $J : E \to 2^{E^*}$ is the function defined by

$$Jx := \{f \in E^* : \|x\|^2 = \langle x, f\rangle = \|f\|^2\}, \forall x \in E.$$

It is known that if E is a real smooth and uniformly convex Banach space, then J is single-valued and norm-to-norm uniformly continuous on each bounded subsets of E. If E is a Hilbert space, then $J \equiv I$, where I is the identity mapping.

Lemma 2.1. *If E be a real Banach space, then $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$, $\forall x, y \in E$.*

Definition 2.2. Let E be a Banach space. A set-value operator $A : E \to 2^E$

(i) is said to be accretive, if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that $\langle u-v, j(x-y)\rangle \geq 0$, $u \in Ax$, $v \in Ay$;
(ii) is said m-accretive if $R(I+rA) = X$ for all $r > 0$, where $R(I+rA)$ is the range of $I + rA$;
(iii) is said to be α-inverse strongly accretive (α-ISA) if for each $x, y \in D(A)$, there exists $\alpha > 0$ and $j(x-y) \in J(x-y)$ such that $\langle u-v, j(x-y)\rangle \geq \alpha\|u-v\|^2$, $u \in Ax, v \in Ay$.

Let $A : E \to 2^E$ be a set-value operator, $D(A) = \{x \in E : Ax \neq \emptyset\}$ denote the domain of A and $R(A) = \bigcup\{Az : z \in D(A)\}$ denote the range of A, respectively. The inverse of A is defined by $x \in A^{-1}y$ if and only if $y \in Ax$, and it denoted by A^{-1}. Assume that A is an m-accretive operator, for each $r > 0$, we define the resolvent $J_r^A : R(I+rA) \to D(A)$ of A by $J_r^A := (I+rA)^{-1}$. The zero set of A, denote by $A^{-1}(0)$, that is $A^{-1}(0) := \{x \in D(A) : 0 \in Ax\}$. It is well know that J_r^A is single-valued and nonexpansive, and $F(J_r^A) = A^{-1}(0)$ for each $x \in R(I+rA)$, where $\mathrm{Fix}(J_r^A) := \{x \in R(I+rA) : J_r^A x = x\}$; see [5, 18, 21, 23].

Lemma 2.3 ([18])**.** *Let E be a real Banach space, $A : E \to 2^E$ be an m-accretive operator and $B : E \to E$ be an α-inverse strongly accretive mapping on E. Then we have*

(i) *for all $r > 0, F(T_r) = (A+B)^{-1}(0)$,*
(ii) *for all $0 < s \leq r$ and $x \in E, \|x - T_s x\| \leq \|x - T_r x\|$,*

where $T_\varepsilon = (I + \epsilon A)^{-1}(I - \epsilon B)$.

Lemma 2.4 ([13])**.** *Let E be a real Banach spaces with Fréchet differentiable norm, for $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by*

$$\beta^*(t) = \sup\Big\{\Big|\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x)\rangle\Big| : \|y\| = 1\Big\}.$$

Then, $\lim_{t\to 0}\beta^*(t) = 0$ and $\|x+h\|^2 \leq \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|)$ for all $h \in E\backslash\{0\}$.

In real Hilbert spaces, $\beta^*(t) = t$. As in [30], throughout this paper we assume that $\beta^*(t) \leq ct, t > 0$ for some $c > 1$.

The following immediate three lemmas can be found in [24].

Lemma 2.5. *Let E be a real uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^E$ be an m-accretive mapping and $B : E \to E$ be an α-inverse strongly accretive mapping. Let $T : E \to E$ be a nonexpansive mapping and assume that $(A+B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Defined a mapping $U : E \to E$ by $Ux = \lambda Tx + (1-\lambda)J_r^A(I - rB)x$, where r is a real number in $(0, \frac{2\alpha}{c})$ and λ is a real number in (0,1). Then U is nonexpansive and $Fix(U) = Fix(T) \cap (A+B)^{-1}(0)$.*

Lemma 2.6. *Let E be a real uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^E$ be an m-accretive mapping and $B : E \to E$ be an α-inverse strongly* *accretive mapping. Let $T : E \to E$ be a nonexpansive mapping and assume that $(A+B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Then there exists a sunny nonexpansive retraction from E to $Fix(T) \cap (A+B)^{-1}(0)$.*

Lemma 2.7. *Let E be a real uniformly convex Banach space. Let C ba nonempty closed convex subset of E and let $T : C \to E$ be a continuous peseudocontractive. Then $I - T$ is demiclosed at zero.*

Lemma 2.8 ([3])**.** *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex and bounded subset of E. Then there is a strictly increasing and continuous convex function $\psi : [0,\infty) \to 0,\infty)$ with $\psi(0) = 0$ such that, for every nonexpansive mapping $T : C \to E$ and, for all $x, y \in C$ and $t \in [0,1]$, the following inequality holds:*

$$\|(tTx + (1-t)Ty) - T(tx + (1-t)y)\| \leq \psi^{-1}(\|x-y\| - (Tx - Ty)).$$

Lemma 2.9 ([14])**.** *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} \|(1-t)p_1 - p_2 + tx_n\|$ exists for all $t \in [0,1]$ and $p_1, p_2 \in \omega_w(x_n)$, where $\omega_w(x_n) : \{x : \exists x)n_i \rightharpoonup x\}$ denotes the weak $w-$ limit set of $\{x_n\}$. Then $\omega_w(x_n)$ is a singleton.*

Lemma 2.10 ([30])**.** *Let E be a real uniformly convex with Fréchet differentiable norm. Let $A : E \to 2^E$ be an m-accretive mapping and $B : E \to E$ be an α-inverse strongly accretive mapping. Then, given $s > 0$, there exists a continuous, strictly increasing and convex function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that for all $x, y \in E$,*

$$\begin{aligned}\|J_r^A(I-rB)x - J_r^A(I-rB)y\|^2 &\leq \|x-y\|^2 - r(2\alpha - rc)\|Bx - By\|^2 \\ &\quad - \varphi(\|(I-J_r^A)(I-rB)x - (I-J_r^A)(I-rB)y)\|).\end{aligned}$$

Lemma 2.11 ([21])**.** *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E and $T : C \to C$ be a nonexpansive mapping with a fixed point. Let $f : E \to E$ be a fixed contraction with coefficient $k \in (0,1)$. If there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, and $p = \lim_{t\to 0} z_t$ exists, where z_t is defined by $z_t = tf(t) + (1-t)Tz_t$. Then $\lim_{n\to\infty}\langle f(p) - p, j(x_n - p)\rangle \leq 0$.*

Lemma 2.12 ([19])**.** *Let $\{s_n\}$ be a real sequence that non-decreasing at infinity, that is there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \leq s_{n_k+1}$ for all $k \geq 1$. For each $n \geq n_0$, define an integer sequence $\{s_{n_k}\}$ as $\tau_{(n)} = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}$. Then $\tau_{(n)} \to \infty$ as $n \to \infty$ and $\max\{s_{\tau_{(n)}}, s_n\} \leq s_{\tau_{(n)}+1}$ for all $n > n_0$.*

Lemma 2.13 ([23])**.** *Let E be a uniformly convex Banach space. Then there exists a strictly increasing continuous convex function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|ax + by\|^p \leq a\|x\|^p + b\|y\|^p - \frac{a^p b + b^p a}{(a+b)^p}\varphi(\|x-y\|),$$

where $p > 1$ is any real number, for all $x, y \in B_r(0) := \{x \in E : \|x\| \leq r\}$ and $a, b \in [0,1]$ such that $a + b = 1$.

Lemma 2.14 ([29]). *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and for all $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T . Define $Q : C \to D$ by $Qu = \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D.*

3. Main results

3.1. **Weak convergence theorem.** In this subsection, we prove a weak convergence theorem for solving inclusion problem (1.1) in framework of uniformly convex and uniformly smooth Banach spaces.

Theorem 3.1. *Let E be a real uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^E$ be an m-accretive mapping and $B : E \to E$ be an α-inverse strongly accretive mapping. Let $T : E \to E$ be a nonexpansive mapping and assume that $(A + B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{r_n\}$ be a sequence of positive real numbers and $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2\alpha}{c}$.

Let the sequence $\{x_n\}$ be generated by Algorithm 3.1:

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n) J^A_{r_n}(x_n - r_n B x_n), n \leq 1, \tag{3.1}$$

where $x_1 \in E$ and $J^A_{r_n} = (I + r_n A)^{-1}$. Then $\{x_n\}$ is converges weakly to some point in $(A + B)^{-1}(0) \cap Fix(T)$.

Proof. The proof is split into three steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $J_n = J^A_{r_n}(I - r_n B)$ and $x^* \in (A + B)^{-1}(0) \cap \mathrm{Fix}(T)$. Then, for all $x, y \in E$, from Lemma 2.5, we have J_n is nonexpansive and $\mathrm{Fix}(J_n) = (A + B)^{-1}(0)$. Thus,

$$\|x_{n+1} - x^*\| \leq \alpha_n \|T x_n - x^*\| + (1-\alpha_n)\|J_n x_n - x^*\| \leq \|x_n - x^*\|.$$

Hence, $\{x_n\}$ is bounded.

Step 2. We show that $\omega_w(x_n) \subset (A + B)^{-1}(0) \cap \mathrm{Fix}(T)$, where $\omega_w(x_n)$ denotes the weak accumulation point set of $\{x_n\}$.

By Lemma 2.1, we get

$$\begin{aligned} &\|(I - r_n B)x_n - (I - r_n B)x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - 2r_n\langle Bx_n - Bx^*, j(x_n - x^*)\rangle + cr_n^2\|Bx_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2 + (cr_n - 2\alpha)r_n\|Bx_n - Bx^*\|^2. \end{aligned} \tag{3.2}$$

Let $z_n = J^A_{r_n}(x_n - r_n B x_n)$. By Lemma 2.10 we have

$$\begin{aligned} \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 &- (2\alpha - cr_n)r_n\|Bx_n - Bx^*\|^2 \\ &- \varphi(\|((I - r_n B)x_n - (I - r_n B)x^*) - (z_n - x^*)\|). \end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|Tx_n-x^*\|^2+(1-\alpha_n)\|z_n-x^*\|^2\\ &\le \|x_n-x^*\|^2-(1-\alpha_n)(2\alpha-cr_n)r_n\|Bx_n-Bx^*\|^2\\ &\quad -(1-\alpha_n)\varphi(\|((I-r_nB)x_n-(I-r_nB)x^*)-(z_n-x^*)\|).\end{aligned}\tag{3.4}$$

Hence, we have

$$(1-\alpha_n)(2\alpha-cr_n)r_n\|Bx_n-Bx^*\|^2 \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2.$$

Since $\alpha_n \in (0,1)$, $0<r_n<\frac{2\alpha}{c}$ and $\{x_n\}$ is bounded, we have $\lim_{n\to\infty}\|Bx_n-Bx^*\|=0$. From (3.4), we also have

$$\begin{aligned}&(1-\alpha_n)\varphi(\|((I-r_nB)x_n-(I-r_nB)x^*)-(z_n-x^*)\|)\\ &\qquad\qquad \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2.\end{aligned}$$

Hence, we have $\lim_{n\to\infty}\|x_n-z_n-r_nBx_n+r_nBx^*\|$. This implies that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.\tag{3.5}$$

From Lemma 2.13 and (3.3), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|Tx_n-x^*\|^2-\alpha_n(1-\alpha_n)\varphi(\|Tx_n-z_n\|)+(1-\alpha_n)\|z_n-x^*\|^2\\ &\le \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\alpha_n(1-\alpha_n)\varphi(\|Tx_n-z_n\|)\\ &\le \|x_n-x^*\|^2-\alpha_n(1-\alpha_n)\varphi(\|Tx_n-z_n\|).\end{aligned}$$

Hence, $\alpha_n(1-\alpha_n)\varphi(\|Tx_n-z_n\|)\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2$. This implies that $\lim_{n\to\infty}\|Tx_n-z_n\|=0$. Since $\|Tx_n-x_n\|\le\|Tx_n-z_n\|+\|z_n-x_n\|$, combining with (3.5), we have $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. By Lemma 2.7 , we obtain $\omega_w(x_n)\subset \mathrm{Fix}(T)$. Since $\liminf_{n\to\infty} r_n>0$, there exists $\tau>0$ such that $r_n>\tau, \forall n\ge 1$. Then by Lemma 2.3, we have

$$\|J_\tau^A(I-\tau B)x_n-x_n\|\le 2\|J_{r_n}^A(I-r_nB)x_n-x_n\|.$$

Combining with (3.5), we have

$$\lim_{n\to\infty}\|J_\tau^A(I-\tau B)x_n-x_n\|=0.\tag{3.6}$$

By Browder's demiclosedness principle for nonexpansive mappings, we get $\omega_w(x_n)\subset \mathrm{Fix}(J_\tau^A(I-\tau B))=(A+B)^{-1}(0)$.

Step 3. Show that $\omega_w(x_n)$ is singleton.

Let $U_nx=\alpha_nTx+(1-\alpha_n)J_{r_n}^A(I-r_nB)x$, it is obvious that U_n is nonexpansive. Let $U_{n,m}=U_{n+m-1}U_{n+m-1}\dots U_n, \forall m,n\ge1$. Thus, $U_{n,m}$ is also nonexpansive and $x_{n+m}=U_{n,m}x_n$. For all $t\in[0,1]$, let $Q_{n,m}=\|(tx_{n+m}+(1-t)\xi_1)-U_{n,m}(tx_n+(1-t)\xi_1)\|$ and $w_n(t)=\|(1-t)\xi_1-\xi_2+tx_n\|$, where $\xi_1,\xi_2\in\mathrm{Fix}(T)\cap(A+B^{-1}(0))$. By Step 2, we get that for all $\xi\in\mathrm{Fix}(T)\cap(A+B^{-1}(0))$ and $\|U_n\xi-\xi\|\le e_n$, where $\lim_{n\to\infty}e_n=0$.

$$\begin{aligned}\|U_{n,m}\xi_1-\xi_1\| &\le \|U_{n+m-1}U_{n+m-2}\dots U_n\xi_1-\xi_1\|\\ &\le \|U_n\xi_1-\xi_1\|+\dots+\|U_{n+m-1}\xi_1-\xi_1\|\\ &\le \|e_n\|+\dots+\|e_{n+m-1}\|.\end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned} Q_{n,m}(t) &\leq \psi^{-1}(\|x_n - \xi_1\| - \|U_{n,m}x_n - U_{n,m}\xi_1\|) \\ &= \psi^{-1}(\|x_n - \xi_1\| - \|x_{n+m} - \xi_1 + \xi_1 - U_{n,m}\xi_1\|) \\ &\leq \psi^{-1}(\|x_n - \xi_1\| - \|x_{n+m} - \xi_1\| + \|e_n\| + \cdots + \|e_{n+m-1}\|). \end{aligned}$$

This implies that $\{Q_{n,m}(t)\}$ converges uniformly to zero as $n \to \infty$ for all $m \geq 1$. On the other hand, we have

$$\begin{aligned} w_{n+m}(t) &\leq \|U_{n,m}(tx_n + (1-t)\xi_1) - \xi_2\| + Q_{n,m}(t) \\ &\leq w_n(t) + +\|\xi_2 - U_{n,m}\xi_2\| + Q_{n,m}(t) \\ &\leq w_n(t) + Q_{n,m}(t) + \|e_n\| + \cdots + \|e_{n+m-1}\|. \end{aligned}$$

It is not hard to see that $\limsup_{n\to\infty} w_n(t) \leq \liminf_{n\to\infty} w_n(t)$. Thus, $\lim_{n\to\infty} w_n(t)$ exists for any $t \in [0,1]$. It follows from Lemma 2.9 that $\omega_w(x_n)$ is singleton. This completes the proof. □

3.2. **Strong convergence theorem.** In this subsection, we introduce a strong convergence theorem of a general type splitting method for solving inclusion problem (1.1) in the framework of uniformly convex and uniformly smooth Banach spaces.

Theorem 3.2. *Let E be a real uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^E$ be an m-accretive mapping and $B : E \to E$ be an α-inverse strongly accretive mapping. Assume that $(A+B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a sequence of positive real numbers and $\{\alpha_n\}, \{\lambda_n\}$ and $\{\delta_n\}$ be three sequences in $[0,1]$ with $\alpha_n + \lambda_n + \delta_n = 1$ and satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{\alpha}{c}$;
(iii) $\liminf_{n\to\infty} \delta_n > 0$.

Let the sequence $\{x_n\}$ generated by $u, x_1 \in E$ and Algorithm 3.2:

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}^A (x_n - r_n B x_n), n \leq 1. \tag{3.7}$$

where $J_{r_n}^A = (I + r_n A)^{-1}$. Then $\{x_n\}$ converges strongly to $z = Qu$, where Q is the unique sunny nonexpansive retraction of E onto $(A+B)^{-1}(0)$.

Proof. Let $T_n = J_{r_n}^A(I - r_n B)$ and $z_t = tu + (1-t)T_n z_t, t \in (0,1)$. Since $F(T_n) = (A+B)^{-1}(0)$, by Lemma 2.14 we have $z_t \to Q_{(A+B)^{-1}} u = z$ as $t \to 0$, where Q is the unique sunny nonexpansive retract form E to $(A+B)^{-1}$. It follows from Lemma 2.5 that T_n is nonexpansive. Then

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + \lambda_n \|x_n - z\| + \delta_n \|T_n x_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_n - z\|\} \\ &\ \ \vdots \\ &\leq \max\{\|u - z\|, \|x_1 - z\|\}. \end{aligned}$$

This implies that $\{x_n\}$ is bounded. From Lemma 2.1, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|\lambda_n(x_n-z)+\delta_n(T_nx_n-z)\|^2+2\alpha_n\langle u-z, j(x_{n+1}-z)\rangle \\ &\le \lambda_n\|x_n-z\|^2+\delta_n\|T_nx_n-z\|^2+2\alpha_n\langle u-z, j(x_{n+1}-z)\rangle.\end{aligned} \tag{3.8}$$

On the other hand, by Lemma 2.10, there exists a continuous, strictly increasing and convex function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\begin{aligned}\|T_nx_n-z\|^2 \le{}& \|x_n-z\|^2-r_n(2\alpha-r_nc)\|Bx_n-Bz\|^2 \\ &-\varphi(\|(I-J^A_{r_n})(I-r_nB)x_n-(I-J^A_{r_n})(I-r_nB)z\|).\end{aligned} \tag{3.9}$$

Replacing (3.8) into (3.9), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \le{}& (\lambda_n+\delta_n)\|x_n-z\|^2-\delta_nr_n(2\alpha-r_nc)\|Bx_n-Bz\|^2 \\ &-\delta_n\varphi(\|(x_n-r_nBx_n-T_nx_n+r_nBz\|) \\ &\quad+2\alpha_n\langle u-z, j(x_{n+1}-z)\rangle \\ \le{}& (1-\alpha_n)\|x_n-z\|^2-\delta_nr_n(2\alpha-r_nc)\|Bx_n-Bz\|^2 \\ &-\delta_n\varphi(\|(x_n-r_nBx_n-T_nx_n+r_nBz\|) \\ &\quad+2\alpha_n\langle u-z, j(x_{n+1}-z)\rangle.\end{aligned} \tag{3.10}$$

For each $n \ge 1$, we set

$$\begin{aligned}&s_n = \|x_n-z\|^2, \\ &\tau_n = 2\langle u-z, j(x_{n+1}-z)\rangle, \\ &\eta_n = \delta_nr_n(2\alpha-r_nc)\|Bx_n-Bz\|^2+\delta_n\varphi(\|(x_n-r_nBx_n-T_nx_n+r_nBz\|).\end{aligned}$$

From (3.10), we obtain

$$s_{n+1} \le (1-\alpha_n)s_n+\alpha_n\tau_n-\eta_n, \quad n \ge 1. \tag{3.11}$$

Now we show that $s_n \to 0 (n \to \infty)$ by consider the flowing tow possible cases.
Case 1 Suppose that there exists $N \ge 0$ such that $\{s_n\}$ is decreasing for $n \ge N$, that is, $\{s_n\}$ is convergent. By (3.11), we have

$$\eta_n \le s_n - s_{n+1} + \alpha_n(\tau_n - s_n). \tag{3.12}$$

Since $\{x_n\}$ is bounded, we get that $\{\tau_n\}$ is bounded. This implies that $\lim_{n\to\infty}\eta_n = 0$. Therefore,

$$\lim_{n\infty}\|Bx_n-Bz\|^2 = 0,$$

and

$$\lim_{n\to\infty}\|x_n-r_nBx_n-T_nx_n+r_nBz\| = 0.$$

Then

$$\lim_{n\to\infty}\|T_nx_n-x_n\| = 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, there exists $r > 0$ such that $r_n \ge r$. It is follows from Lemma 2.3 that

$$\lim_{n\to\infty}\|T_rx_n-x_n\| \le 2\lim_{n\to\infty}\|T_nx_n-x_n\| = 0.$$

By the definition of x_{n+1}, we have

$$\|x_{n+1}-x_n\| \le \alpha_n\|u-x_n\|+\delta_n\|T_nx_n-x_n\| \tag{3.13}$$

Since $z_t = tu + (1-t)T_r z_t, t \in (0,1)$ and $z_t \to Q_{(A+B)^{-1}}u = z$ as $t \to 0$, it follows from Lemma 2.14 that

$$\limsup_{n\to\infty} \langle u - z, j(x_n - z)\rangle \leq 0.$$

Since the duality mapping is norm-to-norm uniformly continuous on bounded sets, combining (3.13), we have $\limsup_{n\to\infty}\langle u - z, j(x_{n+1} - z)\rangle \leq 0$. That is, $\limsup_{n\to\infty} \tau_n \leq 0$. From (3.12), we obtain

$$s_n \leq \frac{1}{\alpha_n}(s_{n+1} - s_n) + \tau_n. \tag{3.14}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have $\liminf_{n\to\infty} \frac{s_{n+1}-s_n}{\alpha_n} = 0$. Setting $n \to \infty$ in (3.13), we obtain

$$\lim_{n\to\infty} s_n \leq \liminf_{n\to\infty}\left(\frac{1}{\alpha_n}(s_{n+1} - s_n) + \tau_n\right) \leq \liminf_{n\to\infty}\left(\frac{1}{\alpha_n}(s_{n+1} - s_n)\right) + \limsup_{n\to\infty}(\tau_n) \leq 0.$$

Thus $\{s_n\}$ converges to zero.

Case 2 Suppose that $\{s_n\}$ is not eventually decreasing. Then there exists $\{s_{n_k}\} \subset \{s_n\}$ such that $s_{n_k} \leq s_{n_k+1}, \forall k \geq 1$. Hence, by Lemma 2.12, we can define a subsequence $\{s_{\epsilon(n)}\}$ satisfies

$$\max\{s_{\epsilon(n)}, s_n\} \leq s_{\epsilon(n)+1}. \tag{3.15}$$

It follows from (3.12) that $0 \leq c_{\epsilon(n)} \leq \alpha_{\epsilon(n)}(\tau_{\epsilon(n)} - s_{\epsilon(n)}) \to 0$, $n \to \infty$. In a similar way to Case 1, we have

$$\limsup_{n\to\infty}\langle u - z, j(x_{\epsilon(n)+1} - z)\rangle \leq 0. \tag{3.16}$$

Therefore, $\limsup_{n\to\infty} \epsilon(n) \leq 0$. By (3.16), we have $\limsup_{\epsilon(n)} \leq 0$. By definition of x_{n+1}, we get

$$\begin{aligned}
\|x_{\epsilon(n)+1} - x_{\epsilon(n)}\| &= \|\alpha_{\epsilon(n)}(u - x_{\epsilon(n)}) + \delta_{\epsilon(n)}(T_{\epsilon(n)}x_{\epsilon(n)} - x_{\epsilon(n)})\| \\
&\leq \alpha_{\epsilon(n)}\|u - x_{\epsilon(n)}\| + \|T_{\epsilon(n)}x_{\epsilon(n)} - x_{\epsilon(n)}\| \\
&\leq \alpha_{\epsilon(n)}M_1 + \|T_{\epsilon(n)}x_{\epsilon(n)} - x_{\epsilon(n)}\|,
\end{aligned}$$

where $\limsup_{n\to\infty}\|u - x_{\epsilon(n)}\| \leq M_1 < \infty$. It follows from the definition of s_n that

$$\begin{aligned}
|s_{\epsilon(n)+1} - s_{\epsilon(n)}| &= |\|x_{\epsilon(n)+1} - z\|^2 - \|x_{\epsilon(n)} - z\|^2| \\
&\leq \|x_{\epsilon(n)+1} - x_{\epsilon(n)}\|(\|x_{\epsilon(n)+1} - z\| + \|x_{\epsilon(n)} - z\|) \\
&\leq (\alpha_{\epsilon(n)}M_1 + \|T_{\epsilon(n)}x_{\epsilon(n)} - x_{\epsilon(n)}\|)(\|x_{\epsilon(n)+1} - z\| + \|x_{\epsilon(n)} - z\|) \\
&\leq M(\alpha_{\epsilon(n)}M_1 + \|T_{\epsilon(n)}x_{\epsilon(n)} - x_{\epsilon(n)}\|)
\end{aligned} \tag{3.17}$$

Let $n \to \infty$. (3.17) implies that $|s_{\epsilon(n)+1} - s_{\epsilon(n)}| \to 0$. By (3.15), we have

$$0 \leq s_n \leq s_{\epsilon(n)+1} = s_{\epsilon(n)} + (s_{\epsilon(n)+1} - s_{\epsilon(n)}) \to 0, \quad n \to \infty.$$

Thus $s_n \to 0$, $n \to \infty$. Therefore, $x_n \to z, n \to \infty$. □

4. Numerical examples

In this section, we present numerical results to illustrate the convergence of Algorithm 3.1 and Algorithm 3.2.

Example 4.1 Let $A : R^N \to R^N$ be defined by $Ax = 2x + (1, 1, 1, \ldots, 1)$ and let $B : R^N \to R^N$ be defined by $Bx = 5x$ and $Tx = x + b$, where $x = (x_1, x_2, \ldots, x_N) \in R^N, b \in R^N$. It is easy to verify that A is $\frac{1}{2}$-inverse strongly accretive , B is m-accretive operator and T is nonexpansive mapping. Then we have

$$J_r^B(x - rAx) = (I + rB)^{-1}(x - rAx) \quad = \frac{1-2r}{1+5r}x - \frac{r}{1+5r}(1, 1, \ldots, 1).$$

In Figure 1, we choose $N = 10$, $r_n = 0.2$, and the number of iterations is 100. Others parameters are selected as follows. In Theorem 3.1, let $\alpha_n = \frac{1}{1000n+1}$, each component of b and x_1 is generated randomly in $(0, 1)$. For Theorem 3.2, we choose $\alpha_n = \frac{1}{1000n+1}, \lambda_n = \frac{1}{10n}, \delta_n = 1 - \alpha_n - \lambda_n$, is component of u and x_1 are generated randomly in $(0, 1)$. In Figure 2, we choose $N = 100$, $r_n = 0.1$, and the number of iterations is 500.

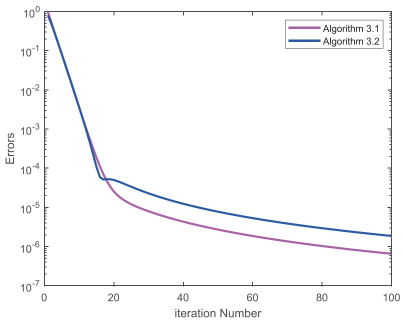

Figure 1. Example for $N = 10$, number of iterations is 100.

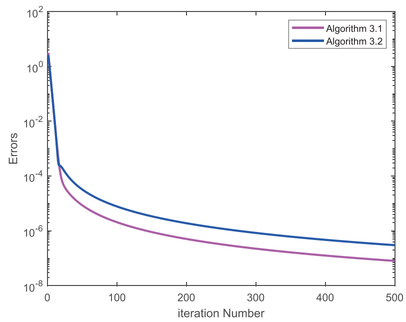

Figure 2. Example for $N = 100$, number of iterations is 500.

From the numerical results, we can see that the two algorithms converge to the unique solution. For example, let $N = 3$, number of iterations is 100. Then the solution is (-0.142853133653588, -0.142853133653588, -0.142853133653588). In Algorithm 3.1, the $Error = 3.583737462884978e - 07$, and in Algorithm 3.2, the $Error = 7.96000729545480e - 07$.

Example 4.2 Let E be a real Banach space. Let $F : E \to R$ is a convex smooth function and $G : E \to R$ be a convex, lower-semicontinuous and nonsmooth function. The convex minimization problem is: find $x^* \in E$ such that

$$F(x^*) + G(x^*) \le F(x) + G(x),$$

for all $x \in E$. This problem is equivalent to the problem of finding $x^* \in E$ such that

$$0 \in \nabla F(x^*) + \partial G(x^*).$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. Let $A = \nabla F$ and $B = \partial G$. We solve the following minimization problem:

$$\min_{x \in R^3} \|x\|_2^2 + (3,5,-1)x + 9 + \|x\|_1,$$

where $x = (y_1, y_2, y_3) \in R^3$. Let $F(x) = \|x\|_2^2 + (2,5,-1)x$ and $G(x) = \|x\|_1$. Thus, we get that $\nabla F(x) = 2x + (3,5,-1)$, and from [15], we have

$$(I+\partial G)^{-1}x = (\max\{|y_1|-r\}\text{sign}(y_1), \max\{|y_2|-r\}\text{sign}(y_2), \max\{|y_2|-r\}\text{sign}(y_2)).$$

We will solve this problem by Theorem 3.1 and Theorem 3.2. Let $r_n = 0.1$, and the number of iteration is 500. In Theorem 3.1, we set $Tx = x + u_0$, choose $\alpha_n = \frac{1}{100n+1}$, each component of u_0 and x_1 is generated randomly in $(0,1)$. In Theorem 3.2, we choose $\alpha_n = \frac{1}{1000n+1}, \lambda_n = \frac{1}{10n}, \delta_n = 1 - \alpha_n - \lambda_n$, each component of u and x_1 is generated randomly in $(0,1)$.

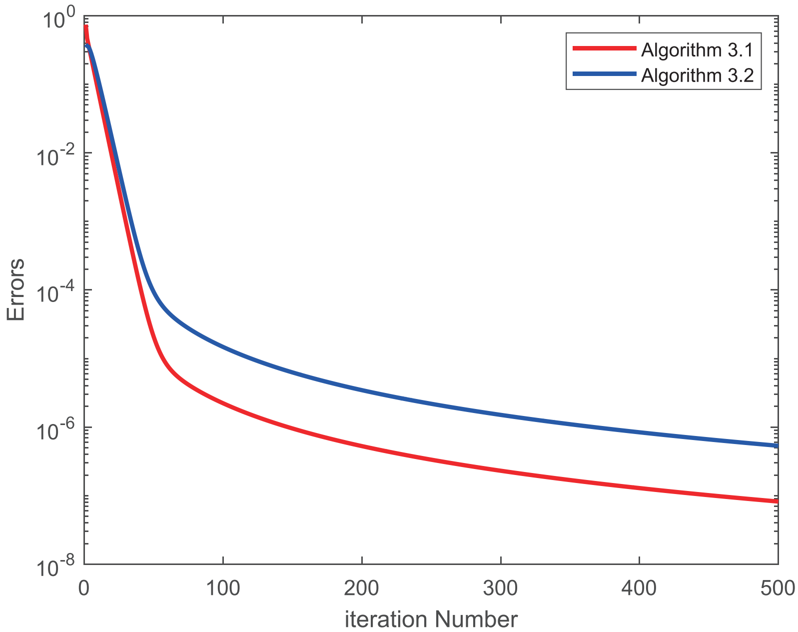

Figure 3. Number of iterations is 500.

From the numerical results, we can see that the two algorithms converge to the unique solution $(-1,-2,0)$. In Algorithm 3.1, the $Error = 2.02858226454799e-07$, and in Algorithm 3.2, the $Error = 6.38356864877978e - 07$.

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Yinglin Luo
Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China
E-mail address: `luoyinglink@163.com`