On modified subgradient extragradient methods for pseudomonotone variational inequality problems with applications

Bing Tan[1] (ID) · **Songxiao Li**[1] (ID) · **Xiaolong Qin**[2]

Abstract

This paper presents several modified subgradient extragradient methods with inertial effects to approximate solutions of variational inequality problems in real Hilbert spaces. The operators involved are either pseudomonotone Lipschitz continuous or pseudomonotone non-Lipschitz continuous. The advantage of the suggested algorithms is that they can work adaptively without the prior information of the Lipschitz constant of the mapping involved. Strong convergence theorems of the proposed algorithms are established under some suitable conditions. Finally, some numerical experiments are given to verify the advantages and efficiency of the proposed iterative algorithms with respect to previously known ones.

1 Introduction and preliminaries

The goal of this paper is to provide several efficient and adaptive numerical methods to solve pseudomonotone variational inequality problems in real Hilbert spaces. Recall that the

---

✉ Songxiao Li
   jyulsx@163.com

   Bing Tan
   bingtan72@gmail.com

   Xiaolong Qin
   qxlxajh@163.com

1 Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China

2 Department of Mathematics, Zhejiang Normal University, Zhejiang, China

Springer /SBMAC

classical variational inequality problem (shortly, VIP) is given as follows:

$$\text{find } x^* \in C \text{ such that } \langle Mx^*, z - x^* \rangle \geq 0, \quad \forall z \in C, \tag{VIP}$$

where C is a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and $M : \mathcal{H} \rightarrow \mathcal{H}$ is an operator. Throughout the paper, the solution set of the (VIP) is denoted by VI(C, M), and is assumed to be nonempty. The variational inequality, as one of the fundamental problems in mathematics, provides a unified and useful framework for the study of many linear and nonlinear problems. It has become one of the effective mathematical methods and research tools in solving optimization problems, and has been widely used in many research fields (such as engineering, finance, mechanics, transportation modeling, operations management and optimal control), see, e.g., Mordukhovich (2018), Vuong and Shehu (2019), Bonacker et al. (2020), Cuong et al. (2020), Khan et al. (2015) and Sahu et al. (2021).

Let us first review the basic definitions of some mappings in nonlinear analysis, which will be used in the next sequel. Recall that a mapping $M : \mathcal{H} \rightarrow \mathcal{H}$ is said to be:

– *L-Lipschitz continuous* with $L > 0$ if $\|Mx - My\| \leq L\|x - y\|$, $\forall x, y \in \mathcal{H}$.
– *monotone* if $\langle Mx - My, x - y \rangle \geq 0$, $\forall x, y \in \mathcal{H}$.
– *pseudomonotone* if $\langle Mx, y - x \rangle \geq 0 \Rightarrow \langle My, y - x \rangle \geq 0$, $\forall x, y \in \mathcal{H}$.
– *sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies $\{Mx_n\}$ converges weakly to Mx.

In the past decades, researchers proposed a large number of iterative methods to solve variational inequality problems, among which the methods based on extragradient types are the focus of this paper. Recall that the classical extragradient method (shortly, EGM), which was introduced by Korpelevich (1976), requires computing the projection on the feasible set twice in each iteration. The computation of the projection is difficult if the feasible set is complex. In order to improve the computational efficiency of the iterative algorithm, scholars have proposed many improvements to the EGM, see, e.g., (He 1997; Solodov and Svaiter 1999; Tseng 2000; Censor et al. 2011; Malitsky 2015) and the references therein. It should be mentioned that the subgradient extragradient method (SEGM) proposed by Censor et al. (2011) replaces the projection on the feasible set in the second step of the EGM with a projection on the half-space in each iteration. It is known that the projection on a half-space can be computed explicitly (see, e.g., Cegielski 2012). Thus, the SEGM greatly improves the computational efficiency of the EGM. Note that the SEGM can only obtain the weak convergence in infinite-dimensional Hilbert spaces. From the viewpoint of the physically tangible property, the strong convergence, which is norm convergence, is often much more desirable than the weak convergence. This shows the theoretical value and potential applications of analyzing the strong convergence of iterative algorithms in infinite-dimensional Hilbert spaces. In the last decades, many techniques were developed to obtain strongly convergent numerical methods for solving variational inequality problems in infinite-dimensional Hilbert spaces; see, e.g., Mann-type methods (Kraikaew and Saejung 2014; Tan et al. 2021), viscosity-type methods (Shehu and Iyiola 2017; Thong et al. 2019), and projection-based methods (Censor et al. 2011; Cho 2020). In this paper, we consider several strongly convergent numerical algorithms for variational inequalities in the framework of real Hilbert spaces, which is motivated by the real applications of the (VIP) in infinite-dimensional spaces, such as machine learning and quantum mechanics.

On the other hand, it is noted that the Lipschitz constants of the operators corresponding to the problems studied in practical applications are difficult to obtain and estimate, which

will further affect the use of those algorithms whose step size is related to the prior information of the Lipschitz constant. Recently, some adaptive algorithms that do not require the prior information of the Lipschitz constant of the operator involved were proposed to solve variational inequality problems, see, e.g., Thong and Hieu (2018), Yang et al. (2018), Yang and Liu (2019), Liu and Yang (2020), Cai et al. (2021), Hieu et al. (2021) and the references therein. Among the step size selections of these methods, a non-monotonic step size criterion proposed by Liu and Yang (2020), and a new Armijo-type step size approach introduced by Cai et al. (2021) are desired to be mentioned. Their numerical experiments demonstrated the advantages and efficiency of the proposed algorithms over previously known ones.

It is known that the class of pseudomonotone mappings contains the class of monotone mappings. Recently, many algorithms were proposed in the literature to solve pseudomonotone variational inequalities in real Hilbert spaces; see, e.g., Hieu et al. (2021), Shehu et al. (2019), Jolaoso et al. (2020), Thong et al. (2020), Yang (2021), Grad and Lara (2021) and the references therein. However, a common feature enjoyed by these algorithms is the requirement that the operator satisfies the Lipschitz continuity, which may be difficult to be satisfied in practical applications. To overcome this drawback, scholars proposed many iterative schemes for solving non-Lipschitz continuous monotone (or pseudomonotone) variational inequality problems (see, e.g., Cai et al. 2021; Shehu et al. 2019; Cho 2020; Malitsky 2020; Reich et al. 2021; Tan and Cho 2021). On the other hand, the inertial extrapolation method based on discrete versions of a second-order dissipative dynamic system was widely studied as one of the acceleration techniques. Recently, inertial-type methods attracted a great deal of attention and interest from researchers in the optimization community, who proposed a large number of inertial-type numerical algorithms to solve image processing, signal recovery, variational inequality problems, equilibrium problems, split feasibility problems, fixed point problems, and variational inclusion problems, see, e.g., Hieu and Gibali (2020), Ceng and Shang (2021), Shehu and Yao (2020), Shehu and Gibali (2021) and the references therein. The main feature of inertial-type methods is the inclusion in each iteration of an inertial term, which is obtained from the combination of some previously known iteration points. This small change can improve the convergence speed of the original algorithm without the inertial term.

Inspired and motivated by the above work, this paper proposes several adaptive inertial subgradient extragradient methods to solve variational inequality problems in infinite-dimensional real Hilbert spaces. Our contributions in this paper are stated as follows: (1) the subgradient extragradient method introduced by Censor et al. (2011) is modified in two simple ways by using two different step sizes in each iteration; (2) two non-monotonic adaptive step size criteria are used to make the proposed algorithms work adaptively; (3) the variational inequality operators involved in the proposed methods are pseudomonotone Lipschitz continuous (or non-Lipschitz continuous); (4) the strong convergence of the iterative sequences generated by the proposed algorithms is established without the prior knowledge of the Lipschitz constant of the operator; (5) inertial extrapolation terms are added to the proposed algorithms to accelerate their convergence speed; and (6) some numerical experiments are given to verify the computational efficiency of the proposed iterative schemes compared to some known algorithms in the literature (Cai et al. 2021; Thong and Vuong 2019; Thong et al. 2020).

In the whole paper, we use the symbol $x_n \to x$ ($x_n \rightharpoonup x$) to represent the strong convergence (weak convergence) of the sequence $\{x_n\}$ to x, and use $P_C : \mathcal{H} \to C$ to denote the metric projection from $\mathcal{H}$ onto C, i.e., $P_C(x) := \arg\min\{\|x - y\|, \ y \in C\}$. We conclude the section by giving the following lemma that is crucial in the convergence analysis of the proposed algorithms.

Lemma 1.1 (Saejung and Yotkaew 2012) *Let* $\{p_n\}$ *be a positive sequence,* $\{s_n\}$ *be a sequence of real numbers, and* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Assume that*

$$p_{n+1} \leq (1 - \alpha_n)p_n + \alpha_n s_n, \quad \forall n \geq 1.$$

If $\limsup_{k \to \infty} s_{n_k} \leq 0$ *for every subsequence* $\{p_{n_k}\}$ *of* $\{p_n\}$ *satisfying* $\liminf_{k \to \infty} (p_{n_k+1} - p_{n_k}) \geq 0$, *then* $\lim_{n \to \infty} p_n = 0$.

2 Main results

In this section, we introduce several modified subgradient extragradient algorithms with inertial effects for solving pseudomonotone variational inequality problems in infinite-dimensional Hilbert spaces. The advantage of our algorithms is that they can work without the prior knowledge of the Lipschitz constant of the mapping and the strong convergence of the iterative sequence generated by the proposed algorithms can be guaranteed.

2.1 The first type of modified subgradient extragradient methods

In this subsection, two new iterative schemes are proposed for solving the (VIP) in real Hilbert spaces. We first introduce a new modified subgradient extragradient algorithm with a non-monotonic sequence of step sizes (see Algorithm 2.1 below) and assume that the proposed algorithm satisfies the following conditions.

(C1) The feasible set C is a nonempty, closed and convex subset of the real Hilbert space $\mathcal{H}$ and the solution set of the problem (VIP) is nonempty.

(C2) The operator $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathcal{H}$ and sequentially weakly continuous on C.

(C3) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n \to \infty} \frac{\epsilon_n}{\tau_n} = 0$, where $\{\tau_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$. Let $\{\sigma_n\} \subset (a, b) \subset (0, 1 - \tau_n)$ for some $a > 0, b > 0$.

We now state the first iterative scheme in Algorithm 2.1.

The following lemmas are important for the convergence analysis of our main results.

Lemma 2.1 *Suppose that Condition (C2) holds. Then the sequence* $\{\lambda_n\}$ *generated by* (2.3) *is well defined and* $\lim_{n \to \infty} \lambda_n = \lambda$ *and* $\lambda \in \left[\min\{\mu/L, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty} \xi_n\right]$.

Proof The proof is similar to Lemma 3.1 in Liu and Yang (2020) and thus we omit the details. □

Lemma 2.2 *Assume that Condition (C2) holds. Let* $\{q_n\}$ *be a sequence generated by Algorithm* 2.1. *Then, for all* $p \in \text{VI}(C, M)$,

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \beta^* \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right),$$

where $\beta^* = 2 - \beta - \frac{\beta \mu \lambda_n}{\lambda_{n+1}}$ *if* $\beta \in [1, 2/(1 + \mu))$ *and* $\beta^* = \beta - \frac{\beta \mu \lambda_n}{\lambda_{n+1}}$ *if* $\beta \in (0, 1)$.

Proof From the definition of q_n and the property of projection $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2$, $\forall x \in \mathcal{H}, y \in C$, we have

$$\|q_n - p\|^2 = \left\|P_{T_n}(u_n - \beta \lambda_n M v_n) - p\right\|^2$$

Algorithm 2.1

Initialization: Take $\phi > 0$, $\lambda_1 > 0$, $\beta \in (0, 2/(1 + \mu))$, $\mu \in (0, 1)$. Select $\{\epsilon_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ to satisfy Condition (C3). Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n ($n \geq 1$), calculate x_{n+1} as follows.

Step 1. Compute $u_n = x_n + \phi_n(x_n - x_{n-1})$, where

$$\phi_n = \begin{cases} \min\left\{ \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \phi \right\}, & \text{if } x_n \neq x_{n-1}; \\ \phi, & \text{otherwise.} \end{cases} \tag{2.1}$$

Step 2. Compute $v_n = P_C(u_n - \lambda_n M u_n)$. If $u_n = v_n$ or $M v_n = 0$, then stop and v_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $q_n = P_{T_n}(u_n - \beta\lambda_n M v_n)$, where

$$T_n := \{x \in \mathcal{H} \mid \langle u_n - \lambda_n M u_n - v_n, x - v_n \rangle \leq 0\}. \tag{2.2}$$

Step 4. Compute $x_{n+1} = (1 - \tau_n - \sigma_n)u_n + \sigma_n q_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\left\{ \mu \dfrac{\|u_n - v_n\|^2 + \|q_n - v_n\|^2}{2\langle M u_n - M v_n, q_n - v_n \rangle}, \lambda_n + \xi_n \right\}, & \text{if } \langle M u_n - M v_n, q_n - v_n \rangle > 0; \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases} \tag{2.3}$$

Set $n := n + 1$ and go to **Step 1**.

$$\begin{aligned} &\leq \|u_n - \beta\lambda_n M v_n - p\|^2 - \|u_n - \beta\lambda_n M v_n - q_n\|^2 \\ &= \|u_n - p\|^2 + (\beta\lambda_n)^2 \|M v_n\|^2 - 2\langle u_n - p, \beta\lambda_n M v_n \rangle - \|u_n - q_n\|^2 \\ &\quad - (\beta\lambda_n)^2 \|M v_n\|^2 + 2\langle u_n - q_n, \beta\lambda_n M v_n \rangle \\ &= \|u_n - p\|^2 - \|u_n - q_n\|^2 - 2\langle \beta\lambda_n M v_n, q_n - p \rangle \\ &= \|u_n - p\|^2 - \|u_n - q_n\|^2 - 2\langle \beta\lambda_n M v_n, q_n - v_n \rangle - 2\langle \beta\lambda_n M v_n, v_n - p \rangle. \end{aligned} \tag{2.4}$$

Since $p \in \text{VI}(C, M)$ and $v_n \in C$, we obtain $\langle Mp, v_n - p \rangle \geq 0$. By the pseudomonotonicity of mapping M, we have $\langle M v_n, v_n - p \rangle \geq 0$. Thus the inequality (2.4) reduces to

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - q_n\|^2 - 2\langle \beta\lambda_n M v_n, q_n - v_n \rangle. \tag{2.5}$$

Now we estimate $2\langle \beta\lambda_n M v_n, q_n - v_n \rangle$. Note that

$$-\|u_n - q_n\|^2 = -\|u_n - v_n\|^2 - \|v_n - q_n\|^2 + 2\langle u_n - v_n, q_n - v_n \rangle. \tag{2.6}$$

In addition,

$$\begin{aligned} &\langle u_n - v_n, q_n - v_n \rangle \\ &= \langle u_n - v_n - \lambda_n M u_n + \lambda_n M u_n - \lambda_n M v_n + \lambda_n M v_n, q_n - v_n \rangle \\ &= \langle u_n - \lambda_n M u_n - v_n, q_n - v_n \rangle + \lambda_n \langle M u_n - M v_n, q_n - v_n \rangle \\ &\quad + \langle \lambda_n M v_n, q_n - v_n \rangle. \end{aligned} \tag{2.7}$$

Since $q_n \in T_n$, one sees that

$$\langle u_n - \lambda_n M u_n - v_n, q_n - v_n \rangle \leq 0. \tag{2.8}$$

According to the definition of λ_{n+1}, it follows that

$$\langle M u_n - M v_n, q_n - v_n \rangle \leq \frac{\mu}{2\lambda_{n+1}} \|u_n - v_n\|^2 + \frac{\mu}{2\lambda_{n+1}} \|q_n - v_n\|^2. \tag{2.9}$$

Substituting (2.7), (2.8), and (2.9) into (2.6), we have

$$- \|u_n - q_n\|^2 \leq - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right) + 2 \langle \lambda_n M v_n, q_n - v_n \rangle,$$

which implies that

$$-2 \langle \beta \lambda_n M v_n, q_n - v_n \rangle \leq -\beta \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right)$$
$$+ \beta \|u_n - q_n\|^2. \qquad (2.10)$$

Combining (2.5) and (2.10), we conclude that

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \beta \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right)$$
$$- (1 - \beta) \|u_n - q_n\|^2. \qquad (2.11)$$

Note that

$$\|u_n - q_n\|^2 \leq 2 \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right),$$

which yields that

$$-(1 - \beta) \|u_n - q_n\|^2 \leq -2(1 - \beta) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right), \quad \forall \beta \geq 1.$$

This together with (2.11) implies

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \left(2 - \beta - \frac{\beta \mu \lambda_n}{\lambda_{n+1}}\right) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right), \quad \forall \beta \geq 1.$$

On the other hand, if $\beta \in (0, 1)$, then we obtain

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \beta \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right), \quad \forall \beta \in (0, 1).$$

This completes the proof of the lemma. $\square$

Remark 2.1 From Lemma 2.1 and the assumptions of the parameters μ and β (i.e., $\mu \in (0, 1)$ and $\beta \in (0, 2/(1 + \mu))$), we can obtain that $\beta^* > 0$ for all $n \geq n_0$ in Lemma 2.2 always holds.

Lemma 2.3 (Thong et al. 2020, Lemma 3.3) *Suppose that Conditions (C1)–(C3) hold. Let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences formulated by Algorithm* 2.1. *If there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that* $\{u_{n_k}\}$ *converges weakly to* $z \in \mathcal{H}$ *and* $\lim_{k \to \infty} \|u_{n_k} - v_{n_k}\| = 0$, *then* $z \in \mathrm{VI}(C, M)$.

We now in a position to prove our first main result of this section.

Theorem 2.1 *Suppose that Conditions (C1)–(C3) hold. Then the sequence* $\{x_n\}$ *generated by Algorithm* 2.1 *converges to* $p \in \mathrm{VI}(C, M)$ *in norm, where* $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, M)\}$.

Proof To begin with, our first goal is to show that the sequence $\{x_n\}$ is bounded. Indeed, thanks to Lemma 2.2 and Remark 2.1, one sees that

$$\|q_n - p\| \leq \|u_n - p\|, \quad \forall n \geq n_0. \qquad (2.12)$$

From the definition of u_n, one sees that

$$\|u_n - p\| \leq \|x_n - p\| + \tau_n \cdot \frac{\phi_n}{\tau_n} \|x_n - x_{n-1}\|. \tag{2.13}$$

According to Condition (C3), we have $\frac{\phi_n}{\tau_n} \|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Therefore, there exists a constant $Q_1 > 0$ such that

$$\frac{\phi_n}{\tau_n} \|x_n - x_{n-1}\| \leq Q_1, \quad \forall n \geq 1,$$

which together with (2.12) and (2.13) implies that

$$\|q_n - p\| \leq \|u_n - p\| \leq \|x_n - p\| + \tau_n Q_1, \quad \forall n \geq n_0. \tag{2.14}$$

Using the definition of x_{n+1} and (2.14), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \tau_n - \sigma_n)(u_n - p) + \sigma_n(q_n - p) - \tau_n p\| \\ &\leq (1 - \tau_n - \sigma_n)\|u_n - p\| + \sigma_n\|q_n - p\| + \tau_n\|p\| \\ &\leq (1 - \tau_n)\|u_n - p\| + \tau_n\|p\| \\ &\leq (1 - \tau_n)\|x_n - p\| + \tau_n(\|p\| + Q_1) \\ &\leq \max\{\|x_n - p\|, \|p\| + Q_1\}, \quad \forall n \geq n_0 \\ &\leq \cdots \leq \max\{\|x_{n_0} - p\|, \|p\| + Q_1\}. \end{aligned} \tag{2.15}$$

This implies that the sequence $\{x_n\}$ is bounded. We have that the sequences $\{u_n\}$ and $\{q_n\}$ are also bounded.

From (2.14), one sees that

$$\begin{aligned} \|u_n - p\|^2 &\leq (\|x_n - p\| + \tau_n Q_1)^2 \\ &= \|x_n - p\|^2 + \tau_n(2Q_1\|x_n - p\| + \tau_n Q_1^2) \\ &\leq \|x_n - p\|^2 + \tau_n Q_2 \end{aligned} \tag{2.16}$$

for some $Q_2 > 0$. Combining (2.16), Lemma 2.2 and the inequality $\|\tau x + \sigma y + \delta z\|^2 = \tau\|x\|^2 + \sigma\|y\|^2 + \delta\|z\|^2 - \tau\sigma\|x - y\|^2 - \tau\delta\|x - z\|^2 - \sigma\delta\|y - z\|^2$, where $\tau, \sigma, \delta \in [0, 1]$ and satisfies $\tau + \sigma + \delta = 1$, we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \tau_n - \sigma_n)(u_n - p) + \sigma_n(q_n - p) + \tau_n(-p)\|^2 \\ &\leq (1 - \tau_n - \sigma_n)\|u_n - p\|^2 + \sigma_n\|q_n - p\|^2 + \tau_n\|p\|^2 \\ &\leq (1 - \tau_n - \sigma_n)\|u_n - p\|^2 + \sigma_n\|u_n - p\|^2 + \tau_n\|p\|^2 \\ &\quad - \sigma_n\beta^*\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right) \\ &\leq \|x_n - p\|^2 - \sigma_n\beta^*\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right) \\ &\quad + \tau_n(\|p\|^2 + Q_2), \quad \forall n \geq n_0. \end{aligned} \tag{2.17}$$

It follows from (2.17) that

$$\begin{aligned} &\sigma_n\beta^*\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right) \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \tau_n(\|p\|^2 + Q_2), \quad \forall n \geq n_0. \end{aligned} \tag{2.18}$$

From the definition of u_n, we have

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - p\|^2 + 2\phi_n\|x_n - p\|\|x_n - x_{n-1}\| + \phi_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - p\|^2 + 3Q\phi_n\|x_n - x_{n-1}\|, \end{aligned} \tag{2.19}$$

where $Q := \sup_{n \in \mathbb{N}} \{\|x_n - p\|, \phi\|x_n - x_{n-1}\|\} > 0$. Setting $g_n = (1 - \sigma_n)u_n + \sigma_n q_n$, one has

$$\|g_n - u_n\| = \sigma_n \|u_n - q_n\|. \tag{2.20}$$

It follows from (2.12) that

$$
\begin{aligned}
\|g_n - p\| &= \|(1 - \sigma_n)(u_n - p) + \sigma_n(q_n - p)\| \\
&\leq (1 - \sigma_n)\|u_n - p\| + \sigma_n\|u_n - p\| \\
&= \|u_n - p\|, \quad \forall n \geq n_0.
\end{aligned} \tag{2.21}
$$

Combining (2.19), (2.20), (2.21), and the inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, $\forall x, y \in \mathcal{H}$, we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \sigma_n)u_n + \sigma_n q_n - \tau_n u_n - p\|^2 \\
&= \|(1 - \tau_n)(g_n - p) - \tau_n(u_n - g_n) - \tau_n p\|^2 \\
&\leq (1 - \tau_n)^2 \|g_n - p\|^2 - 2\tau_n \langle u_n - g_n + p, x_{n+1} - p \rangle \\
&= (1 - \tau_n)^2 \|g_n - p\|^2 + 2\tau_n \langle u_n - g_n, p - x_{n+1} \rangle + 2\tau_n \langle p, p - x_{n+1} \rangle \\
&\leq (1 - \tau_n)\|g_n - p\|^2 + 2\tau_n\|u_n - g_n\|\|x_{n+1} - p\| + 2\tau_n \langle p, p - x_{n+1} \rangle \\
&\leq (1 - \tau_n)\|x_n - p\|^2 + \tau_n \Big[2\sigma_n \|u_n - q_n\|\|x_{n+1} - p\| \\
&\quad + 2\langle p, p - x_{n+1} \rangle + \frac{3Q\phi_n}{\tau_n}\|x_n - x_{n-1}\| \Big], \quad \forall n \geq n_0.
\end{aligned} \tag{2.22}
$$

Finally, we need to show that the sequence $\{\|x_n - p\|\}$ converges to zero. We set

$$p_n = \|x_n - p\|^2, \quad s_n = \frac{3Q\phi_n}{\tau_n}\|x_n - x_{n-1}\| + 2\langle p, p - x_{n+1} \rangle + 2\sigma_n\|u_n - q_n\|\|x_{n+1} - p\|.$$

Then the last inequality in (2.22) can be written as $p_{n+1} \leq (1 - \tau_n)p_n + \tau_n s_n$ for all $n \geq n_0$. Note that the sequence $\{\tau_n\}$ is in $(0, 1)$ and $\sum_{n=1}^{\infty} \tau_n = \infty$. By Lemma 1.1, it remains to show that $\limsup_{k \to \infty} s_{n_k} \leq 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k \to \infty} (p_{n_k+1} - p_{n_k}) \geq 0$. For this purpose, we assume that $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$ such that $\liminf_{k \to \infty} (p_{n_k+1} - p_{n_k}) \geq 0$. From (2.18) and the assumption on $\{\tau_n\}$, one obtains

$$
\begin{aligned}
\sigma_{n_k}\beta^* &\left(\left\|u_{n_k} - v_{n_k}\right\|^2 + \left\|q_{n_k} - v_{n_k}\right\|^2 \right) \\
&\leq \limsup_{k \to \infty} \tau_{n_k}(\|p\|^2 + Q_2) + \limsup_{k \to \infty} (p_{n_k} - p_{n_k+1}) \\
&\leq -\liminf_{k \to \infty} (p_{n_k+1} - p_{n_k}) \leq 0,
\end{aligned}
$$

which together with Remark 2.1 yields

$$\lim_{k \to \infty} \|v_{n_k} - u_{n_k}\| = 0 \quad \text{and} \quad \lim_{k \to \infty} \|q_{n_k} - v_{n_k}\| = 0.$$

This implies that $\lim_{k \to \infty} \|q_{n_k} - u_{n_k}\| = 0$, which combining with the boundedness of $\{x_n\}$ yields that

$$\lim_{k \to \infty} \sigma_{n_k}\|u_{n_k} - q_{n_k}\|\|x_{n_k+1} - p\| = 0. \tag{2.23}$$

Moreover, we have

$$\left\|x_{n_k+1} - u_{n_k}\right\| \leq \tau_{n_k}\left\|u_{n_k}\right\| + \sigma_{n_k}\left\|u_{n_k} - q_{n_k}\right\| \to 0 \text{ as } k \to \infty,$$

and

$$\|x_{n_k} - u_{n_k}\| = \tau_{n_k} \cdot \frac{\phi_{n_k}}{\tau_{n_k}} \|x_{n_k} - x_{n_k-1}\| \to 0 \text{ as } k \to \infty.$$

It follows that

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{2.24}$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$. Furthermore,

$$\limsup_{k \to \infty} \langle p, p - x_{n_k} \rangle = \lim_{j \to \infty} \langle p, p - x_{n_{k_j}} \rangle = \langle p, p - z \rangle. \tag{2.25}$$

We obtain that $u_{n_k} \rightharpoonup z$ since $\|x_{n_k} - u_{n_k}\| \to 0$. This together with $\lim_{k \to \infty} \|u_{n_k} - v_{n_k}\| = 0$ and Lemma 2.3 yields $z \in \mathrm{VI}(C, M)$. From the definition of p, the property of projection $\langle x - P_C(x), y - P_C(x) \rangle \leq 0, \ \forall x \in \mathcal{H}, y \in C$ and (2.25), we have

$$\limsup_{k \to \infty} \langle p, p - x_{n_k} \rangle = \langle p, p - z \rangle \leq 0. \tag{2.26}$$

Combining (2.24) and (2.26), we obtain

$$\limsup_{k \to \infty} \langle p, p - x_{n_k+1} \rangle \leq 0. \tag{2.27}$$

This together with $\lim_{n \to \infty} \frac{\phi_n}{\tau_n} \|x_n - x_{n-1}\| = 0$ and (2.23) yields that $\limsup_{k \to \infty} s_{n_k} \leq 0$. Therefore, we conclude that $\lim_{n \to \infty} \|x_n - p\| = 0$. That is, $x_n \to p$ as $n \to \infty$. This completes the proof. □

Next, we provide a new Armijo-type iterative scheme (see Algorithm 2.2 below) for finding solutions to the non-Lipschitz continuous and pseudomonotone (VIP) in real Hilbert spaces. The following condition (C4) will replace the condition (C2) in Algorithm 2.1.

(C4) The mapping $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, uniformly continuous on $\mathcal{H}$ and sequentially weakly continuous on C.

The Algorithm 2.2 is stated as follows.

Algorithm 2.2

Initialization: Take $\phi > 0$, $\delta > 0$, $\ell \in (0, 1)$, $\mu \in (0, 1)$, $\beta \in (0, 2/(1 + \mu))$. Select $\{\epsilon_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n ($n \geq 1$), calculate x_{n+1} as follows.

Step 1. Compute $u_n = x_n + \phi_n(x_n - x_{n-1})$, where ϕ_n is defined in (2.1).

Step 2. Compute $v_n = P_C(u_n - \lambda_n M u_n)$. If $u_n = v_n$ or $M v_n = 0$, then stop and v_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $q_n = P_{T_n}(u_n - \beta \lambda_n M v_n)$, where T_n is defined in (2.2), $\lambda_n := \delta \ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\delta \ell^m \langle M v_n - M u_n, v_n - q_n \rangle \leq \frac{\mu}{2} \left[\|u_n - v_n\|^2 + \|v_n - q_n\|^2 \right]. \tag{2.28}$$

Step 4. Compute $x_{n+1} = (1 - \tau_n - \sigma_n)u_n + \sigma_n q_n$.
Set $n := n + 1$ and go to **Step 1**.

We need the following lemmas in order to analyze the convergence of Algorithm 2.2.

Lemma 2.4 *Suppose that Condition (C4) holds. Then the Armijo-like criteria (2.28) is well defined.*

Proof The proof is similar to Lemma 3.1 in Tan and Cho (2021). Therefore, we omit the details. □

Lemma 2.5 *Assume that Condition (C4) holds. Let $\{q_n\}$ be a sequence generated by Algorithm 2.2. Then, for all $p \in \mathrm{VI}(C, M)$,*

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \beta^{**} \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2 \right),$$

where $\beta^{**} = 2 - \beta - \beta\mu$ if $\beta \in [1, 2/(1 + \mu))$ and $\beta^{**} = \beta - \beta\mu$ if $\beta \in (0, 1)$.

Proof The proof is omitted since it follows the argument of Lemma 2.2. □

Remark 2.2 Note that $\beta^{**} > 0$ for all $n \geq 1$ in Lemma 2.5 always holds.

Lemma 2.6 *Suppose that Conditions (C1), (C3), and (C4) hold. Let $\{u_n\}$ and $\{v_n\}$ be two sequences generated by Algorithm 2.2. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k \to \infty} \|u_{n_k} - v_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, M)$.*

Proof A simple modification of Cai et al. (2021, Lemma 3.2) yields the desired conclusion and thus it is omitted. □

Theorem 2.2 *Suppose that Conditions (C1), (C3), and (C4) hold. Then the sequence $\{x_n\}$ created by Algorithm 2.2 converges to $p \in \mathrm{VI}(C, M)$ in norm, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, M)\}$.*

Proof The proof is similar to the proof of Theorem 2.1. Therefore, we omit some details of the proof. By Lemma 2.5 and similar statements in (2.12)–(2.14), we have

$$\|q_n - p\| \leq \|u_n - p\| \leq \|x_n - p\| + \tau_n Q_1, \quad \forall n \geq 1.$$

Applying a similar procedure as in (2.15), we can obtain that $\{x_n\}$, $\{u_n\}$, and $\{q_n\}$ are bounded. Combining (2.16), (2.17) and Lemma 2.5, we can show that

$$\sigma_n \beta^{**} \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2 \right)$$
$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \tau_n(\|p\|^2 + Q_2), \quad \forall n \geq 1.$$

Reviewing the statements from (2.19)–(2.22), we obtain

$$\|x_{n+1} - p\|^2 \leq (1 - \tau_n)\|x_n - p\|^2 + \tau_n \Big[2\sigma_n \|u_n - q_n\| \|x_{n+1} - p\|$$
$$+ 2\langle p, p - x_{n+1} \rangle + \frac{3Q\phi_n}{\tau_n} \|x_n - x_{n-1}\| \Big], \quad \forall n \geq 1.$$

The rest of the proof follows in the same way as that of Theorem 2.1 but we need apply Lemma 2.6 in place of Lemma 2.3. □

2.2 The second type of modified subgradient extragradient methods

In this subsection, two modified versions of the suggested Algorithms 2.1 and 2.2 are proposed to solve the pseudomonotone and Lipschitz continuous (or non-Lipschitz continuous) variational inequality problem in real Hilbert spaces. We first present a modified form of the

suggested Algorithm 2.1, see Algorithm 2.3 below for more details. Note that this method is different from the proposed Algorithm 2.1 in computing the values of the sequences $\{v_n\}$ and $\{q_n\}$.

Algorithm 2.3

Initialization: Take $\phi > 0$, $\lambda_1 > 0$, $\beta \in (1/(2 - \mu), 1/\mu)$, $\mu \in (0, 1)$. Select $\{\epsilon_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ to satisfy Condition (C3). Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n ($n \geq 1$), calculate x_{n+1} as follows.

$$\begin{cases} u_n = x_n + \phi_n(x_n - x_{n-1}), \\ v_n = P_C(u_n - \beta\lambda_n M u_n), \\ q_n = P_{H_n}(u_n - \lambda_n M v_n), \\ H_n = \{x \in \mathcal{H} \mid \langle u_n - \beta\lambda_n M u_n - v_n, x - v_n \rangle \leq 0\}, \\ x_{n+1} = (1 - \tau_n - \sigma_n)u_n + \sigma_n q_n, \end{cases}$$

where ϕ_n and λ_n are defined in (2.1) and (2.3), respectively.

The following lemma plays a crucial role in the convergence analysis of Algorithm 2.3.

Lemma 2.7 *Assume that Condition (C2) holds. Let $\{q_n\}$ be a sequence generated by Algorithm 2.3. Then, for all $p \in \mathrm{VI}(C, M)$,*

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \beta^\dagger \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2 \right),$$

where $\beta^\dagger = 2 - \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}$ if $\beta \in (0, 1]$ and $\beta^\dagger = \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}$ if $\beta > 1$.

Proof From (2.4) and (2.5), we obtain

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - q_n\|^2 - 2 \langle \lambda_n M v_n, q_n - v_n \rangle. \tag{2.29}$$

Now we estimate $2 \langle \lambda_n M v_n, q_n - v_n \rangle$. Note that

$$- \|u_n - q_n\|^2 = - \|u_n - v_n\|^2 - \|v_n - q_n\|^2 + 2 \langle u_n - v_n, q_n - v_n \rangle. \tag{2.30}$$

One can show that

$$\begin{aligned} &\langle u_n - v_n, q_n - v_n \rangle \\ &= \langle u_n - v_n - \beta\lambda_n M u_n + \beta\lambda_n M u_n - \beta\lambda_n M v_n + \beta\lambda_n M v_n, q_n - v_n \rangle \\ &= \langle u_n - \beta\lambda_n M u_n - v_n, q_n - v_n \rangle + \beta\lambda_n \langle M u_n - M v_n, q_n - v_n \rangle \\ &\quad + \langle \beta\lambda_n M v_n, q_n - v_n \rangle. \end{aligned} \tag{2.31}$$

Since $q_n \in H_n$, one has

$$\langle u_n - \beta\lambda_n M u_n - v_n, q_n - v_n \rangle \leq 0. \tag{2.32}$$

According to the definition of λ_{n+1}, we deduce that

$$\langle M u_n - M v_n, q_n - v_n \rangle \leq \frac{\mu}{2\lambda_{n+1}} \|u_n - v_n\|^2 + \frac{\mu}{2\lambda_{n+1}} \|q_n - v_n\|^2. \tag{2.33}$$

Substituting (2.31), (2.32), and (2.33) into (2.30), we have

$$- \|u_n - q_n\|^2 \leq - \left(1 - \frac{\beta\mu\lambda_n}{\lambda_{n+1}} \right) \left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2 \right) + 2\beta \langle \lambda_n M v_n, q_n - v_n \rangle,$$

which implies that

$$-2 \langle \lambda_n M v_n, q_n - v_n \rangle \leq -\left(\frac{1}{\beta} - \frac{\mu \lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right)$$
$$+ \frac{1}{\beta}\|u_n - q_n\|^2. \tag{2.34}$$

Combining (2.29) and (2.34), we conclude that

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \left(\frac{1}{\beta} - \frac{\mu \lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right)$$
$$- \left(1 - \frac{1}{\beta}\right)\|u_n - q_n\|^2. \tag{2.35}$$

Note that

$$\|u_n - q_n\|^2 \leq 2\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right),$$

which yields that

$$-\left(1 - \frac{1}{\beta}\right)\|u_n - q_n\|^2 \leq -2\left(1 - \frac{1}{\beta}\right)\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right), \quad \forall \beta \in (0, 1].$$

This together with (2.35) implies

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \left(2 - \frac{1}{\beta} - \frac{\mu \lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right), \quad \forall \beta \in (0, 1].$$

On the other hand, if $\beta > 1$, then we obtain

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \left(\frac{1}{\beta} - \frac{\mu \lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right), \quad \forall \beta > 1.$$

The proof is completed. □

Remark 2.3 From Lemma 2.1 and the assumptions of the parameters μ and β (i.e., $\mu \in (0, 1)$ and $\beta \in (1/(2 - \mu), 1/\mu)$), we can obtain that $\beta^\dagger > 0$ for all $n \geq n_1$ in Lemma 2.7 always holds.

Lemma 2.8 *Suppose that Conditions (C1)–(C3) hold. Let $\{u_n\}$ and $\{v_n\}$ be two sequences formulated by Algorithm 2.3. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k \to \infty} \|u_{n_k} - v_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, M)$.*

Proof The conclusion can be obtained by applying a similar statement in Thong et al. (2020, Lemma 3.3). □

Theorem 2.3 *Suppose that Conditions (C1)–(C3) hold. Then the sequence $\{x_n\}$ formed by Algorithm 2.3 converges to $p \in \mathrm{VI}(C, M)$ in norm, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, M)\}$.*

Proof It follows from Lemma 2.7 that $\beta^\dagger > 0$ for all $n \geq n_1$, which is similar to the conclusion of Remark 2.1. Thus, we can obtain the conclusion required by replacing Lemmas 2.2 and 2.3 in the proof of Theorem 2.1 with Lemmas 2.7 and 2.8, respectively. We omit the details of the proof to avoid repetition. □

Now, we state the last iterative scheme proposed in this paper in Algorithm 2.4 below. The difference between this scheme and the proposed Algorithm 2.3 is that it can solve the pseudomonotone and non-Lipschitz continuous (VIP) because it uses an Armijo-type step size (2.28) instead of the adaptive step size criterion (2.3).

The Algorithm 2.4 is described as follows.

Algorithm 2.4

Initialization: Take $\phi > 0$, $\delta > 0$, $\ell \in (0, 1)$, $\mu \in (0, 1)$, $\beta \in (1/(2 - \mu), 1/\mu)$. Select $\{\epsilon_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n ($n \geq 1$), calculate x_{n+1} as follows.

$$\begin{cases} u_n = x_n + \phi_n(x_n - x_{n-1}), \\ v_n = P_C(u_n - \beta\lambda_n M u_n), \\ q_n = P_{H_n}(u_n - \lambda_n M v_n), \\ H_n = \{x \in \mathcal{H} \mid \langle u_n - \beta\lambda_n M u_n - v_n, x - v_n\rangle \leq 0\}, \\ x_{n+1} = (1 - \tau_n - \sigma_n)u_n + \sigma_n q_n, \end{cases}$$

where ϕ_n and λ_n are defined in (2.1) and (2.28), respectively.

Lemma 2.9 *Assume that Condition (C4) holds. Let $\{q_n\}$ be a sequence generated by Algorithm 2.4. Then, for all $p \in \mathrm{VI}(C, M)$,*

$$\|q_n - p\|^2 \leq \|u_n - p\|^2 - \beta^{\ddagger}\left(\|u_n - v_n\|^2 + \|q_n - v_n\|^2\right),$$

where $\beta^{\ddagger} = 2 - \frac{1}{\beta} - \mu$ if $\beta \in (0, 1]$ and $\beta^{\ddagger} = \frac{1}{\beta} - \mu$ if $\beta > 1$.

Proof The proof follows the proof of Lemma 2.7 and so it is omitted. □

Remark 2.4 Note that $\beta^{\ddagger} > 0$ for all $n \geq 1$ in Lemma 2.9 always holds.

Lemma 2.10 *Suppose that Conditions (C1), (C3), and (C4) hold. Let $\{u_n\}$ and $\{v_n\}$ be two sequences generated by Algorithm 2.4. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k \to \infty} \|u_{n_k} - v_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, M)$.*

Proof We can obtain the conclusion by a simple modification of Cai et al. (2021, Lemma 3.2). □

Theorem 2.4 *Suppose that Conditions (C1), (C3), and (C4) hold. Then the sequence $\{x_n\}$ generated by Algorithm 2.4 converges to $p \in \mathrm{VI}(C, M)$ in norm, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, M)\}$.*

Proof The proof is similar to that of Theorem 2.2. However, we need to replace Lemmas 2.5 and 2.6 in Theorem 2.2 with Lemmas 2.9 and 2.10, respectively. Therefore, we omit the details of the proof. □

**Remark 2.5** We now explain the contribution of this paper in detail as follows.

1. We modify the subgradient extragradient method (SEGM) introduced by Censor et al. (2011) in two simple ways. Specifically, our algorithms use two different step sizes for computing the values of v_n and q_n in each iteration, while the SEGM by Censor et al. (2011)

employs the same step size for computing these two values in each iteration. Numerical experimental results will show that this modification improves the convergence speed and computational efficiency of the original method (see numerical results for our algorithms when $\beta = 1$ and $\beta \neq 1$ in Sect. 3).

2. The idea of step size selection (i.e., (2.1) and (2.28)) for the methods proposed in this paper comes from the recent work in Liu and Yang (2020) and Cai et al. (2021). The two step size criteria generate non-monotonic step size sequences, which improves the algorithms in the literature (see, e.g., Tan et al. 2021; Yang et al. 2018; Yang and Liu 2019; Hieu et al. 2021; Thong et al. 2020) that use non-increasing step size sequences. In addition, our Algorithms 2.2 and 2.4 can be used to solve non-Lipschitz continuous variational inequalities, which extends a large number of algorithms in the literature (see, e.g., Hieu et al. 2021; Shehu et al. 2019; Jolaoso et al. 2020; Thong et al. 2020; Yang 2021; Grad and Lara 2021) for solving Lipschitz continuous variational inequalities. On the other hand, the four iterative schemes suggested in this paper are designed to solve pseudomonotone variational inequalities. Therefore, our results extend many methods in the literature (see, e.g., Tan et al. 2021; Shehu and Iyiola 2017; Thong et al. 2019; Censor et al. 2011; Thong and Hieu 2018; Yang et al. 2018; Yang and Liu 2019) that can only solve monotone variational inequalities.

3. Our algorithms added inertial terms, which accelerates the convergence speed of our algorithms without inertial terms. In addition, the proposed schemes use the Mann-type method to obtain strong convergence. Thus, the results obtained in this paper are preferable to the weakly convergent algorithms in the literature (see, e.g., He 1997; Solodov and Svaiter 1999; Tseng 2000; Censor et al. 2011; Malitsky 2015; Hieu et al. 2021) in infinite-dimensional Hilbert spaces.

3 Numerical experiments

In this section, we provide several numerical examples to demonstrate the efficiency of our algorithms compared to some known ones in Cai et al. (2021), Thong and Vuong (2019) and Thong et al. (2020). All the programs are implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250S CPU @1.60 GHz computer with RAM 8.00 GB.

Example 3.1 Consider the linear operator $M : \mathbb{R}^m \to \mathbb{R}^m$ ($m = 20$) in the form $M(x) = Sx + q$, where $q \in \mathbb{R}^m$ and $S = NN^\mathsf{T} + Q + D$, N is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence S is positive symmetric definite). The feasible set C is given by $C = \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5, \ i = 1, \ldots, m\}$. It is clear that M is monotone and Lipschitz continuous with constant $L = \|S\|$. In this experiment, all entries of N, Q are generated randomly in $[-2, 2]$, D is generated randomly in $[0, 2]$ and $q = 0$. It can be checked that the solution of the (VIP) is $x^* = \{0\}$. We apply the proposed algorithms to solve this problem. Take $\phi = 0.6$, $\epsilon_n = 100/(n + 1)^2$, $\tau_n = 1/(n + 1)$ and $\sigma_n = 0.9(1 - \tau_n)$ for the suggested Algorithms 2.1–2.4. Choose $\lambda_1 = 1$, $\mu = 0.2$ and $\xi_n = 1/(n + 1)^{1.1}$ for the proposed Algorithms 2.1 and 2.3. Select $\delta = 2$, $\ell = 0.5$, $\mu = 0.2$ for the stated Algorithms 2.2 and 2.4. The maximum number of iterations 500 is used as a common stopping criterion. We use $D_n = \|x_n - x^*\|$ to measure the error of the nth iteration step. Figure 1 shows the numerical performance of the proposed algorithms for different parameter β.

(a) Our Algorithms 2.1 and 2.3   **(b)** Our Algorithms 2.2 and 2.4
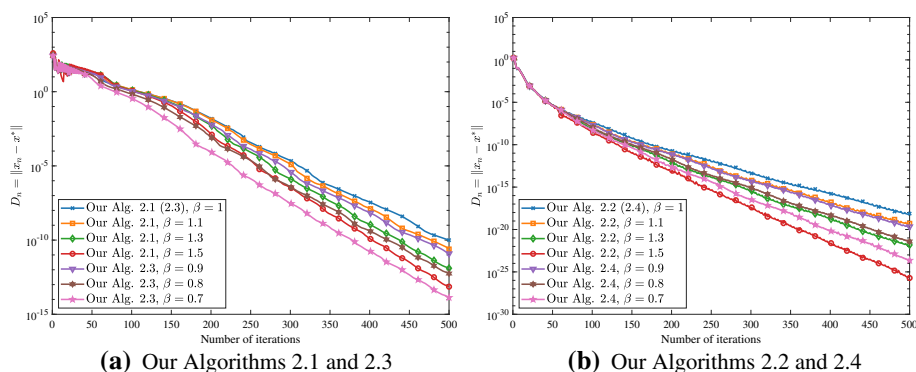
Fig. 1 Numerical results of our algorithms with different β in Example 3.1

Example 3.2 Let $\mathcal{H} = L^2([0, 1])$ be an infinite-dimensional Hilbert space with inner product

$$\langle x, y \rangle = \int_0^1 x(t)y(t)\,dt, \quad \forall x, y \in \mathcal{H},$$

and induced norm

$$\|x\| = \left(\int_0^1 |x(t)|^2\,dt \right)^{1/2}, \quad \forall x \in \mathcal{H}.$$

Let r, R be two positive real numbers such that $R/(k + 1) < r/k < r < R$ for some $k > 1$. Take the feasible set as $C = \{x \in \mathcal{H} : \|x\| \le r\}$. The operator $M : \mathcal{H} \to \mathcal{H}$ is given by

$$M(x) = (R - \|x\|)x, \quad \forall x \in \mathcal{H}.$$

It is not hard to check that operator M is pseudomonotone rather than monotone. For the experiment, we choose $R = 1.5$, $r = 1$, $k = 1.1$. The solution of the (VIP) with M and C given above is $x^*(t) = 0$. We compare the proposed Algorithms 2.1–2.4 with the Algorithm 2 introduced by Thong and Vuong (2019) (shortly, TV Alg. 2). Set $\tau_n = 1/(n + 1)$ and $\sigma_n = 0.9(1 - \tau_n)$ for all algorithms. Choose $\phi = 0.3$, $\epsilon_n = 100/(n + 1)^2$ for the suggested Algorithms 2.1–2.4. Take $\mu = 0.4$, $\lambda_1 = 1$ and $\xi_n = 1/(n + 1)^{1.1}$ for the suggested Algorithms 2.1 and 2.3. Select $\delta = 1$, $\ell = 0.5$ and $\mu = 0.4$ for the proposed Algorithm 2.2, Algorithm 2.4 and TV Alg. 2. The maximum number of iterations 50 is used as a common stopping criterion. The numerical behavior of the function $D_n = \|x_n(t) - x^*(t)\|$ of all algorithms with four starting points $x_0(t) = x_1(t)$ is shown in Fig. 2 and Table 1, where "CPU" in Table 1 indicates the execution time of all algorithms in seconds.

Example 3.3 Consider the Hilbert space $\mathcal{H} = l_2 := \{x = (x_1, x_2, \ldots, x_i, \ldots) \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ equipped with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i, \quad \forall x, y \in \mathcal{H},$$

and the induced norm

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad \forall x \in \mathcal{H}.$$

Table 1 Numerical results of all algorithms for Example 3.2

Algorithms	$x_1(t) = t^4$		$x_1(t) = \exp(t)$		$x_1(t) = \log(t)$		$x_1(t) = \sin(t)$	
	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 2.1, $\beta = 1.5$	6.71E−19	21.43	4.54E−19	21.48	8.06E−19	21.60	7.64E−19	21.42
Our Alg. 2.1 (2.3), $\beta = 1$	1.18E−13	20.46	9.75E−14	20.74	2.54E−13	20.87	1.48E−13	20.73
Our Alg. 2.2, $\beta = 1.5$	6.47E−19	43.86	3.59E−18	44.07	2.78E−18	44.30	1.02E−18	44.52
Our Alg. 2.2 (2.4), $\beta = 1$	7.50E−13	43.57	3.23E−12	43.49	2.89E−12	43.99	1.13E−12	44.03
Our Alg. 2.3, $\beta = 0.8$	4.20E−18	20.40	5.02E−20	20.60	1.18E−17	20.64	5.15E−18	20.71
Our Alg. 2.4, $\beta = 0.8$	5.11E−15	43.76	1.33E−14	43.29	1.30E−14	43.99	7.82E−15	44.12
TV Alg. 2	4.76E−08	34.74	2.95E−08	34.15	1.10E−07	34.62	6.71E−08	34.60
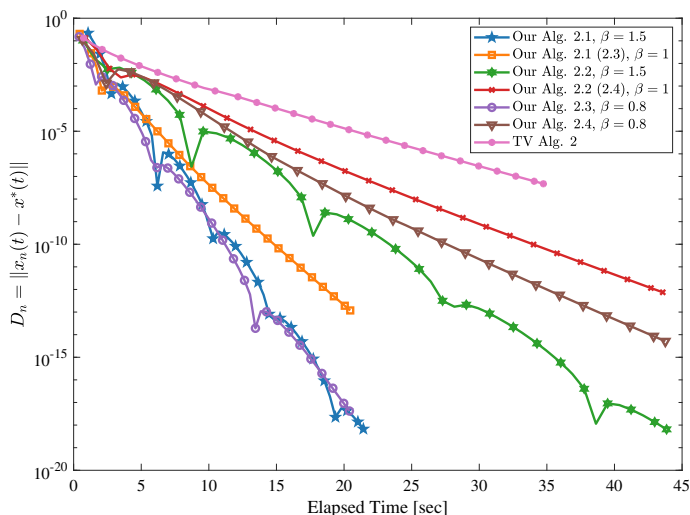
Fig. 2 Numerical behavior of all algorithms at $x_0(t) = x_1(t) = t^4$ for Example 3.2

Let $C := \{x \in \mathcal{H} : |x_i| \leq 1/i\}$. Define an operator $M : C \to \mathcal{H}$ by

$$Mx = \left(\|x\| + \frac{1}{(\|x\| + \alpha)} \right) x$$

for some $\alpha > 0$. It can be verified that mapping M is pseudomonotone on $\mathcal{H}$, uniformly continuous and sequentially weakly continuous on C, but not Lipschitz continuous on $\mathcal{H}$ (see Thong et al. 2020, Example 1). In the following cases, we take $\alpha = 0.5$, $\mathcal{H} = \mathbb{R}^m$ for different values of m. In these situations, the feasible set C is a box given by

$$C = \left\{ x \in \mathbb{R}^m : \frac{-1}{i} \leq x_i \leq \frac{1}{i}, \ i = 1, 2, \ldots, m \right\}.$$

We compare the proposed Algorithms 2.2 and 2.4 with several strongly convergent algorithms that can solve the (VIP) with uniformly continuous operators, including the Algorithm 3.1 introduced by Cai et al. (2021) (shortly, CDP Alg. 3.1) and the Algorithm 3 suggested by Thong et al. (2020) (shortly, TSI Alg. 3). Take $\tau_n = 1/(n+1)$, $\delta = 2$, $\ell = 0.5$, $\mu = 0.1$ for all the algorithms. Select $\sigma_n = 0.9(1 - \tau_n)$, $\phi = 0.4$ and $\epsilon_n = 100/(n+1)^2$ for the suggested Algorithms 2.2 and 2.4. Set $f(x) = 0.1x$ for CDP Alg. 3.1 and TSI Alg. 3. The initial values $x_0 = x_1 = 5rand(m,1)$ are randomly generated by MATLAB. The maximum number of iterations 200 is used as a common stopping criterion. The numerical performance of the function $D_n = \|x_n - x_{n-1}\|$ of all algorithms with four different dimensions is reported in Fig. 3 and Table 2, where "CPU" in Table 2 represents the execution time of all algorithms in seconds.

Next, we use the proposed algorithms to solve the (VIP) that appears in optimal control problems. Assume that $L_2([0, T], \mathbb{R}^m)$ represents the square-integrable Hilbert space with inner product $\langle p, q \rangle = \int_0^T \langle p(t), q(t) \rangle \, dt$ and norm $\|p\| = \sqrt{\langle p, p \rangle}$. The optimal control

Table 2 Numerical results of all algorithms for Example 3.3

Algorithms	$m = 10000$		$m = 50000$		$m = 100000$		$m = 200000$	
	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 2.2 (2.4), $\beta = 1$	4.48E−22	0.32	5.36E−22	0.91	8.41E−22	2.01	6.24E−22	11.72
Our Alg. 2.2, $\beta = 1.5$	1.58E−53	0.28	2.06E−53	0.78	3.94E−53	1.69	2.85E−53	10.14
Our Alg. 2.4, $\beta = 0.8$	7.73E−49	0.28	5.82E−49	0.80	1.07E−48	1.75	6.91E−49	10.82
CDP Alg. 3.1	2.00E−13	0.37	1.77E−13	0.95	2.59E−13	2.11	2.40E−13	12.53
TSI Alg. 3	2.65E−09	0.29	2.73E−09	0.79	2.97E−09	1.62	2.82E−09	10.34
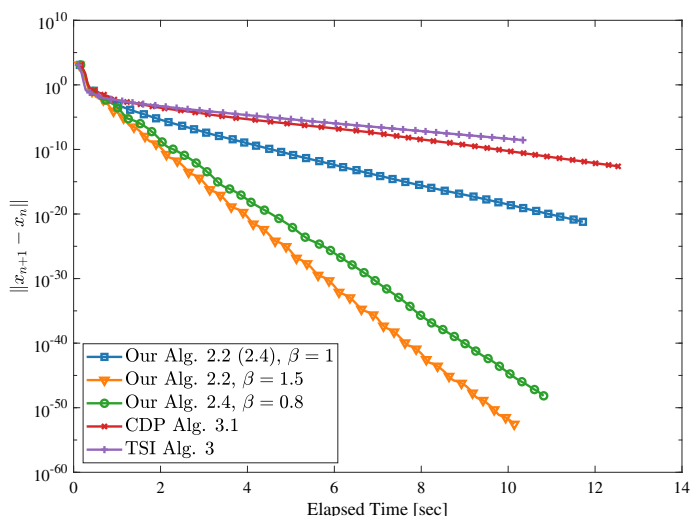
Fig. 3 Numerical behavior of all algorithms at $m = 200000$ for Example 3.3

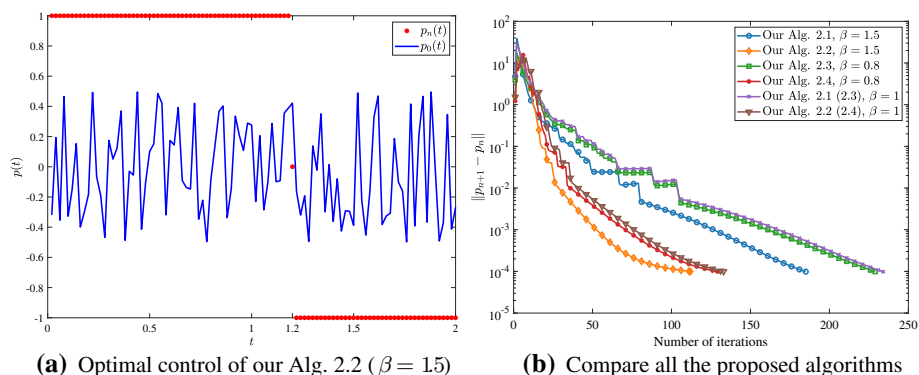problem is described as follows:

$$\begin{cases} p^*(t) \in \text{Argmin}\{g(p) \mid p \in V\}, \\ g(p) = \Phi(x(T)), \\ V = \left\{ p(t) \in L_2\left([0, T], \mathbb{R}^m\right) : p_i(t) \in \left[p_i^-, p_i^+\right], i = 1, 2, \dots, m \right\}, \\ \text{such that } \dot{x}(t) = Q(t)x(t) + W(t)p(t), \quad 0 \le t \le T, \quad x(0) = x_0, \end{cases} \quad (3.1)$$

where $g(p)$ means the terminal objective function, Φ is a convex and differentiable defined on the attainability set, $p(t)$ denotes the control function, V represents a set of feasible controls composed of m piecewise continuous functions, $x(t)$ stands for the trajectory, and $Q(t) \in \mathbb{R}^{n \times n}$ and $W(t) \in \mathbb{R}^{n \times m}$ are given continuous matrices for every $t \in [0, T]$. By the solution of problem (3.1), we mean a control $p^*(t)$ and a corresponding (optimal) trajectory $x^*(t)$ such that its terminal value $x^*(T)$ minimizes objective function $g(p)$. It is known that the optimal control problem (3.1) can be transformed into a variational inequality problem (see Preininger and Vuong 2018; Vuong and Shehu 2019). We first use the classical Euler discretization method to decompose the optimal control problem (3.1) and then apply the proposed algorithms to solve the variational inequality problem corresponding to the discretized version of the problem (see Preininger and Vuong 2018; Vuong and Shehu 2019 for more details). In the proposed Algorithms 2.1–2.4, we set $N = 100$, $\phi = 0.01$, $\epsilon_n = 10^{-4}/(n+1)^2$, $\tau_n = 10^{-4}/(n+1)$ and $\sigma_n = 0.9(1 - \tau_n)$. Pick $\lambda_1 = 0.4$, $\mu = 0.5$ and $\xi_n = 1/(n+1)^{1.1}$ for the proposed Algorithms 2.1 and 2.3. Select $\delta = 2$, $\ell = 0.5$ and $\mu = 0.5$ for the suggested Algorithms 2.2 and 2.4. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$ and the stopping criterion is $D_n = \|p_{n+1} - p_n\| \le 10^{-4}$.

Example 3.4 (Rocket car (Bressan and Piccoli 2007))

$$\begin{aligned} \text{minimize} \quad & -x_1(2) + (x_2(2))^2, \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = p(t), \quad \forall t \in [0, 2], \\ & x_1(0) = 0, \quad x_2(0) = 0, \quad p(t) \in [-1, 1]. \end{aligned}$$

(a) Optimal control of our Alg. 2.2 ($\beta = 1.5$)   **(b)** Compare all the proposed algorithms

Fig. 4 Numerical results for Example 3.4

The exact optimal control of Example 3.4 is $p^*(t) = 1$ if $t \in [0, 1.2)$ and $p^*(t) = -1$ if $t \in (1.2, 2]$. The approximate optimal control of the stated Algorithm 2.2 is plotted in Fig. 4a. Moreover, the numerical behavior of the proposed algorithms is shown in Fig. 4b.

Remark 3.1 We have the following observations for Examples 3.1–3.4: (i) as can be seen in Examples 3.2 and 3.3, the algorithms proposed in this paper converge faster than some known methods in the literature (Cai et al. 2021; Thong and Vuong 2019; Thong et al. 2020), and these results are independent of the choice of initial values and the size of the dimensions (see Figs. 2, 3, Tables 1, 2); (ii) our four algorithms can obtain a faster convergence speed when the appropriate value of β is chosen; in other words, our algorithms can achieve a higher error accuracy when the two step sizes used in each iteration are different (i.e., $\beta \neq 1$) than when they are the same (i.e., $\beta = 1$), see Figs. 1, 2, 3, 4 and Tables 1, 2; (iii) from Fig. 2 and Table 1, it can be seen that our Armijo-type Algorithms 2.2 and 2.4 take more computation time than the adaptive step size Algorithms 2.1 and 2.3, because the Armijo-type methods may take extra time to find a suitable step size in each iteration, while the adaptive step size type methods can use previously known information to automatically compute the next iteration step size; (iv) the proposed algorithms can work well in solving optimal control problems (see Fig. 4); and (v) notice that the operator M in Example 3.3 is non-Lipschitz continuous rather than Lipschitz continuous, which means that the algorithms proposed in the literature (see, e.g., Hieu et al. 2021; Shehu et al. 2019; Jolaoso et al. 2020; Thong et al. 2020; Yang 2021; Grad and Lara 2021) for solving Lipschitz continuous variational inequalities will not be available in this example. In summary, the iterative schemes suggested in this paper are useful, efficient and robust.

4 Conclusions

In this paper, we presented four new iterative schemes inspired by the inertial method, the subgradient extragradient method and the Mann-type method for solving the variational inequality problem with a pseudomonotone and Lipschitz continuous (or non-Lipschitz continuous) operator. Our algorithms use two non-monotonic step size criteria allowing them to work without the prior information about the Lipschitz constant of the operator. Strong convergence theorems of the proposed iterative algorithms are proved under some mild conditions in the framework of real Hilbert spaces. Finally, some numerical experimental results

show the computational efficiency of the suggested methods compared to previously known schemes.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

Bonacker E, Gibali A, Küfer KH (2020) Nesterov perturbations and projection methods applied to IMRT. J Nonlinear Var Anal 4:63–86

Bressan B, Piccoli B (2007) Introduction to the mathematical theory of control. American Institute of Mathematical Sciences, San Francisco

Cai G, Dong QL, Peng Y (2021) Strong convergence theorems for solving variational inequality problems with pseudo-monotone and non-Lipschitz operators. J Optim Theory Appl 188:447–472

Cegielski A (2012) Iterative methods for fixed point problems in Hilbert spaces, vol 2057. Lecture notes in mathematics. Springer, Heidelberg

Ceng LC, Shang M (2021) Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. Optimization 70:715–740

Censor Y, Gibali A, Reich S (2011) The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl 148:318–335

Censor Y, Gibali A, Reich S (2011) Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim Methods Softw 26:827–845

Cho SY (2020) A monotone Bregman projection algorithm for fixed point and equilibrium problems in a reflexive Banach space. Filomat 34:1487–1497

Cho SY (2020) A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings. J Nonlinear Convex Anal 21:1017–1026

Cuong TH, Yao JC, Yen ND (2020) Qualitative properties of the minimum sum-of-squares clustering problem. Optimization 69:2131–2154

Grad SM, Lara F (2021) Solving mixed variational inequalities beyond convexity. J Optim Theory Appl 190:565–580

He BS (1997) A class of projection and contraction methods for monotone variational inequalities. Appl Math Optim 35:69–76

Hieu DV, Gibali A (2020) Strong convergence of inertial algorithms for solving equilibrium problems. Optim Lett 14:1817–1843

Hieu DV, Muu LD, Quy PK, Duong HN (2021) New extragradient methods for solving equilibrium problems in Banach spaces. Banach J. Math. Anal. 15:Article ID 8

Jolaoso LO, Taiwo A, Alakoya TO, Mewomo OT (2020) A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods. J Optim Theory Appl 185:744–766

Khan AA, Tammer C, Zalinescu C (2015) Regularization of quasi-variational inequalities. Optimization 64:1703–1724

Korpelevich GM (1976) The extragradient method for finding saddle points and other problems. Èkonom Mat Metody 12:747–756

Kraikaew R, Saejung S (2014) Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J Optim Theory Appl 163:399–412

Liu H, Yang J (2020) Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput Optim Appl 77:491–508

Malitsky Y (2015) Projected reflected gradient methods for monotone variational inequalities. SIAM J Optim 25:502–520

Malitsky Y (2020) Golden ratio algorithms for variational inequalities. Math Program 184:383–410

Mordukhovich BS (2018) Variational analysis and applications. Springer monographs in mathematics. Springer, Berlin

Preininger J, Vuong PT (2018) On the convergence of the gradient projection method for convex optimal control problems with bang-bang solutions. Comput Optim Appl 70:221–238

Reich S, Thong DV, Dong QL, Li XH, Dung VT (2021) New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings. Numer Algorithms 87:527–549

Saejung S, Yotkaew P (2012) Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal 75:742–750

Sahu DR, Yao JC, Verma M, Shukla KK (2021) Convergence rate analysis of proximal gradient methods with applications to composite minimization problems. Optimization 70:75–100

Shehu Y, Gibali A (2021) New inertial relaxed method for solving split feasibilities. Optimi Lett 15:2109–2126

Shehu Y, Iyiola OS (2017) Strong convergence result for monotone variational inequalities. Numer Algorithms 76:259–282

Shehu Y, Yao JC (2020) Rate of convergence for inertial iterative method for countable family of certain quasi-nonexpansive mappings. J Nonlinear Convex Anal 21:533–541

Shehu Y, Dong QL, Jiang D (2019) Single projection method for pseudo-monotone variational inequality in Hilbert spaces. Optimization 68:385–409

Shehu Y, Iyiola OS, Li XH, Dong QL (2019) Convergence analysis of projection method for variational inequalities. Comput Appl Math 38:Article ID 161

Solodov MV, Svaiter BF (1999) A new projection method for variational inequality problems. SIAM J Control Optim 37:765–776

Tan B, Cho SY (2021) Inertial extragradient methods for solving pseudomonotone variational inequalities with non-Lipschitz mappings and their optimization applications. Appl Set-Valued Anal Optim 3:165–192

Tan B, Fan J, Li S (2021) Self-adaptive inertial extragradient algorithms for solving variational inequality problems. Comput Appl Math 40:Article ID 19

Thong DV, Hieu DV (2018) Weak and strong convergence theorems for variational inequality problems. Numer Algorithms 78:1045–1060

Thong DV, Vuong PT (2019) Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. Optimization 68:2207–2226

Thong DV, Vinh NT, Cho YJ (2019) Accelerated subgradient extragradient methods for variational inequality problems. J Sci Comput 80:1438–1462

Thong DV, Hieu DV, Rassias TM (2020) Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems. Optim Lett 14:115–144

Thong DV, Shehu Y, Iyiola OS (2020) A new iterative method for solving pseudomonotone variational inequalities with non-Lipschitz operators. Comput Appl Math 39:Article ID 108

Thong DV, Shehu Y, Iyiola OS (2020) Weak and strong convergence theorems for solving pseudo-monotone variational inequalities with non-Lipschitz mappings. Numer Algorithms 84:795–823

Tseng P (2000) A modified forward-backward splitting method for maximal monotone mappings. SIAM J Control Optim 38:431–446

Vuong PT, Shehu Y (2019) Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. Numer Algorithms 81:269–291

Yang J (2021) Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities. Appl Anal 100:1067–1078

Yang J, Liu H (2019) Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer Algorithms 80:741–752

Yang J, Liu H, Liu Z (2018) Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization 67:2247–2258