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Revisiting subgradient extragradient methods for solving variational inequalities

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Abstract

In this paper, several extragradient algorithms with inertial effects and adaptive non-monotonic step sizes are proposed to solve pseudomonotone variational inequalities in real Hilbert spaces. The strong convergence of the proposed methods is established without the prior knowledge of the Lipschitz constant of the mapping. Some numerical experiments are given to illustrate the advantages and efficiency of the proposed schemes over previously known ones.

Keywords Variational inequality · Inertial extragradient method · Armjio stepsize · Pseudomonotone mapping · Non-Lipschitz operator

Mathematics Subject Classification (2010) 47J20 · 47J25 · 47J30 · 68W10 · 65K15

1 Introduction

Our goal in this paper is to develop some efficient numerical algorithms to solve the following variational inequality problem (shortly, VIP):

$$\text{find } x^* \in C \text{ such that } \left\langle Mx^*, x - x^* \right\rangle \geq 0, \quad \forall x \in C, \tag{VIP}$$

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where C is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and $M:\mathcal{H}\to\mathcal{H}$ is an operator. Throughout the paper, the solution set of the (VIP) is denoted as $\mathrm{VI}(C,M)$ and is assumed to be nonempty. Variational inequalities have been widely applied to problems such as equilibrium problems in economics, operations research problems and urban transportation network modeling. They provide a general and useful framework for solving engineering problems, data science and other fields (see, e.g., [1–4]). Therefore, the study of numerical methods for solving variational inequalities has attracted the interest of many researchers.

Next, we introduce some algorithms in the literature for solving variational inequalities that will help us to develop several new efficient iterative schemes. One of the most popular methods for solving the (VIP) is the extragradient method (shortly, EGM) introduced by Korpelevich [5]. It should be noted that the EGM is an iterative scheme that requires computing the projection onto the feasible set twice in each iteration. If the projection onto the feasible set is difficult to compute, it will affect the execution efficiency of the method. To overcome this shortcoming, a large number of variants of the EGM, which only need to compute the projection on the feasible set once in each iteration, have been proposed to solve variational inequalities (see, e.g., [6–11]). The method to be emphasized is the subgradient extragradient method (shortly, SEGM) suggested by Censor, Gibali and Reich [8–10]. The SEGM replaces the projection on the feasible set in the second step of the EGM with the projection on a special half-space (noting that this projection can be computed explicitly). This modification significantly improves the computational efficiency of the EGM. In the last decade, a number of improved versions of the SEGM were proposed to solve variational inequalities, equilibrium problems, and other optimization problems (see, e.g., [12–15]). It is worth noting that the EGM and SEGM can only obtain weak convergence in infinite-dimensional Hilbert spaces. Some practical applications in quantum physics and machine learning show that strong convergence results are preferable to weak convergence results in infinite-dimensional spaces. Recently, scholars have proposed a large number of strongly convergent methods to solve variational inequalities (see, e.g., [9, 16–20]).

Another issue of concern in terms of the computational efficiency of iterative algorithms is the step size. The EGM and SEGM will fail if the Lipschitz constant of the mapping is unknown because the update of their step size requires a prior information of this constant. However, the Lipschitz constant of the mapping involved is not easily available in practical applications. Recently, some adaptive methods that do not require the prior knowledge of the Lipschitz constant were provided to solve variational inequalities (see, e.g., [17–21]). Note that the algorithms offered in [19–21] produce a non-increasing sequence of stepsizes, which will affect the execution efficiency of such algorithms. Recently, Liu and Yang [22] presented some iterative schemes with a non-monotonic sequence of stepsizes. Their numerical experiments illustrate the computational performance of the proposed algorithms. On the other hand, in practical applications of variational inequalities, the condition that the operator needs to satisfy Lipschitz continuity is strong, which will lead to the failure of those algorithms that require the operator to be Lipschitz continuous. To overcome this drawback, some methods with Armijo-type stepsizes were proposed for solving monotone (or pseudomonotone) uniformly continuous variational inequalities (see, e.g., [23–29] and the references therein). In recent years, inertial terms have attracted the attention of researchers as a technique to speed up the convergence speed of iterative algorithms. A common feature of inertial-type schemes is that the next iteration depends on the combination of the previous two iterations (see [30, 31] for more details). This small change greatly improves the computational performance of inertial-type algorithms. Recently, many researchers proposed a large number of inertia-type algorithms to solve variational inequalities, equilibrium problems, split feasibility problems, fixed point problems, and other optimization problems (see, e.g., [32–36] and the references therein).

Motivated by the above works and by the ongoing research in these directions, this paper proposes several modified subgradient extragradient methods for solving pseudomonotone variational inequalities in real Hilbert spaces. The operators involved in our algorithms are either Lipschitz continuous (the Lipschitz constant does not need to be known) or non-Lipschitz continuous. In addition, we use two new non-monotonic step size criteria that allow the proposed algorithms to work adaptively. The strong convergence theorems of the suggested methods are established under some mild conditions imposed on the parameters. Some numerical experiments and applications are given to verify the theoretical results.

The paper is organized as follows. In Section 2, we collect some definitions and lemmas that need to be used in the sequel. Section 3 presents the algorithms and analyzes their convergence. Some numerical examples are given in Section 4 to illustrate the efficiency of the proposed algorithms over some related ones. Finally, we conclude the paper with a brief summary in Section 5.

2 Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of the sequence $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y \in \mathcal{H}$, we have the following inequality:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.1}$$

$$\|\varphi x + (1-\varphi)y\|^2 = \varphi\|x\|^2 + (1-\varphi)\|y\|^2 - \varphi(1-\varphi)\|x-y\|^2, \quad \forall \varphi \in \mathbb{R}. \tag{2.2}$$

Let $P_C : \mathcal{H} \to C$ be the metric (nearest point) projection from $\mathcal{H}$ onto C, characterized by $P_C(x) := \arg\min\{\|x-y\|,\ y \in C\}$. It is known that P_C has the following basic properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall x \in \mathcal{H}, y \in C. \tag{2.3}$$

$$\|P_C(x) - y\|^2 \leq \|x-y\|^2 - \|x - P_C(x)\|^2, \quad \forall x \in \mathcal{H}, y \in C. \tag{2.4}$$

We give some projection calculation formulas that need to be used in numerical experiments.

(1) The projection of x onto a half-space $H_{u,v} = \{x : \langle u, x\rangle \leq v\}$ is computed by

$$P_{H_{u,v}}(x) = x - \max\left\{\frac{\langle u, x\rangle - v}{\|u\|^2}, 0\right\} u.$$

(2) The projection of x onto a box $\text{Box}[a,b] = \{x : a \le x \le b\}$ is computed by

$$P_{\text{Box}[a,b]}(x)_i = \min\{b_i, \max\{x_i, a_i\}\}.$$

(3) The projection of x onto a ball $B[p,q] = \{x : \|x-p\| \le q\}$ is computed by

$$P_{B[p,q]}(x) = p + \frac{q}{\max\{\|x-p\|, q\}}(x-p).$$

Recall that a mapping $M : \mathcal{H} \to \mathcal{H}$ is said to be

(i) *L-Lipschitz continuous* with $L > 0$ if $\|Mx - My\| \le L\|x - y\|, \ \forall x, y \in \mathcal{H}$;
(ii) *ρ-contractive* with $\rho \in [0, 1)$ if $\|Mx - My\| \le \rho\|x - y\|, \ \forall x, y \in \mathcal{H}$;
(iii) *monotone* if $\langle Mx - My, x - y\rangle \ge 0, \ \forall x, y \in \mathcal{H}$;
(iv) *pseudomonotone* if $\langle Mx, y - x\rangle \ge 0 \Rightarrow \langle My, y - x\rangle \ge 0, \ \forall x, y \in \mathcal{H}$;
(v) *sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies $\{Mx_n\}$ converges weakly to Mx.

Lemma 2.1 [37] *Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\alpha_n\}$ be a sequence in* $(0, 1)$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$*. Assume that*

$$p_{n+1} \le (1 - \alpha_n)p_n + \alpha_n q_n, \quad \forall n \ge 1.$$

If $\limsup_{k\to\infty} q_{n_k} \le 0$ *for every subsequence* $\{p_{n_k}\}$ *of* $\{p_n\}$ *satisfying* $\liminf_{k\to\infty} (p_{n_k+1} - p_{n_k}) \ge 0$*, then* $\lim_{n\to\infty} p_n = 0$.

3 Main results

In this section, we present four modified subgradient extragradient algorithms to approximate the solution of (VIP) in real Hilbert spaces and analyze their convergence. The advantage of the proposed algorithms is that they can work without the prior knowledge of the Lipschitz constant of the mapping.

3.1 The first type of modified subgradient extragradient methods

Our first iterative scheme is stated in Algorithm 3.1 below. To begin with, we assume that the Algorithm 3.1 satisfies the following conditions.

(C1) The feasible set C is a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$.
(C2) The operator $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathcal{H}$ and sequentially weakly continuous on C.
(C3) The mapping $f : C \to C$ is ρ-contractive with constant $\rho \in [0, 1)$.
(C4) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\varphi_n} = 0$, where $\{\varphi_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \varphi_n = 0$ and $\sum_{n=1}^{\infty} \varphi_n = \infty$.

We are now ready to introduce the Algorithm 3.1.

Algorithm 3.1

Initialization: Take $\theta > 0$, $\lambda_1 > 0$, $\beta \in (0, 2/(1+\eta))$ and $\eta \in (0, 1)$. Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n ($n \geq 1$), calculate x_{n+1} as follows:

Step 1. Compute $u_n = x_n + \theta_n(x_n - x_{n-1})$, where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1}; \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $y_n = P_C(u_n - \lambda_n M u_n)$. If $u_n = y_n$ or $My_n = 0$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $z_n = P_{T_n}(u_n - \beta\lambda_n M y_n)$, where

$$T_n := \{x \in \mathcal{H} \mid \langle u_n - \lambda_n M u_n - y_n, x - y_n\rangle \leq 0\}. \tag{3.2}$$

Step 4. Compute $x_{n+1} = \varphi_n f(x_n) + (1 - \varphi_n) z_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\eta \dfrac{\|u_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle Mu_n - My_n, z_n - y_n\rangle}, \lambda_n + \xi_n\right\}, & \text{if } \langle Mu_n - My_n, z_n - y_n\rangle > 0; \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go to **Step 1**.

The following lemmas are useful in the convergence analysis of Algorithm 3.1.

Lemma 3.1 *Suppose that Condition (C2) holds. Then the sequence $\{\lambda_n\}$ generated by (3.3) is well defined and $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lambda \in \left[\min\{\eta/L, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty} \xi_n\right]$.*

Proof The proof is similar to Lemma 3.1 in [22]. So we omit the details. □

Lemma 3.2 *Assume that Condition (C2) holds. Let $\{z_n\}$ be a sequence created by Algorithm 3.1. Then, for all $p \in \mathrm{VI}(C, M)$,*

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^* \left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

where $\beta^* = 2 - \beta - \frac{\beta\eta\lambda_n}{\lambda_{n+1}}$ if $\beta \in [1, 2/(1+\eta))$ and $\beta^* = \beta - \frac{\beta\eta\lambda_n}{\lambda_{n+1}}$ if $\beta \in (0, 1)$.

Proof From the definition of z_n and (2.4), we have

$$\begin{aligned} \|z_n - p\|^2 &= \left\|P_{T_n}(u_n - \beta\lambda_n M y_n) - p\right\|^2 \\ &\leq \|u_n - \beta\lambda_n M y_n - p\|^2 - \|u_n - \beta\lambda_n M y_n - z_n\|^2 \\ &= \|u_n - p\|^2 + (\beta\lambda_n)^2 \|My_n\|^2 - 2\langle u_n - p, \beta\lambda_n M y_n\rangle - \|u_n - z_n\|^2 \\ &\quad - (\beta\lambda_n)^2 \|My_n\|^2 + 2\langle u_n - z_n, \beta\lambda_n M y_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\langle \beta\lambda_n M y_n, z_n - p\rangle \\ &= \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\langle \beta\lambda_n M y_n, z_n - y_n\rangle - 2\langle \beta\lambda_n M y_n, y_n - p\rangle. \end{aligned} \tag{3.4}$$

Since $p \in \mathrm{VI}(C, M)$ and $y_n \in C$, we have $\langle Mp, y_n - p\rangle \geq 0$. By the pseudomonotonicity of mapping M, we obtain $\langle My_n, y_n - p\rangle \geq 0$. Thus the inequality (3.4) reduces to

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\left\langle \beta\lambda_n My_n, z_n - y_n\right\rangle. \tag{3.5}$$

Now we estimate $2\left\langle \beta\lambda_n My_n, z_n - y_n\right\rangle$. Note that

$$-\|u_n - z_n\|^2 = -\|u_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\left\langle u_n - y_n, z_n - y_n\right\rangle. \tag{3.6}$$

In addition,

$$\begin{aligned}
&\left\langle u_n - y_n, z_n - y_n\right\rangle \\
=\ &\left\langle u_n - y_n - \lambda_n Mu_n + \lambda_n Mu_n - \lambda_n My_n + \lambda_n My_n, z_n - y_n\right\rangle \\
=\ &\left\langle u_n - \lambda_n Mu_n - y_n, z_n - y_n\right\rangle \\
&+\lambda_n\left\langle Mu_n - My_n, z_n - y_n\right\rangle + \left\langle \lambda_n My_n, z_n - y_n\right\rangle.
\end{aligned} \tag{3.7}$$

Since $z_n \in T_n$, one has

$$\left\langle u_n - \lambda_n Mu_n - y_n, z_n - y_n\right\rangle \leq 0. \tag{3.8}$$

According to the definition of λ_{n+1}, it is easy to obtain

$$\left\langle Mu_n - My_n, z_n - y_n\right\rangle \leq \frac{\eta}{2\lambda_{n+1}}\|u_n - y_n\|^2 + \frac{\eta}{2\lambda_{n+1}}\|z_n - y_n\|^2. \tag{3.9}$$

Substituting (3.7), (3.8) and (3.9) into (3.6), we have

$$-\|u_n - z_n\|^2 \leq -\left(1 - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) + 2\left\langle \lambda_n My_n, z_n - y_n\right\rangle,$$

which implies that

$$\begin{aligned}
-2\left\langle \beta\lambda_n My_n, z_n - y_n\right\rangle \ \leq\ &-\beta\left(1 - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) \\
&+\beta\|u_n - z_n\|^2.
\end{aligned} \tag{3.10}$$

Combining (3.5) and (3.10), we conclude that

$$\begin{aligned}
\|z_n - p\|^2 \ \leq\ &\|u_n - p\|^2 - \beta\left(1 - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) \\
&-(1-\beta)\|u_n - z_n\|^2.
\end{aligned} \tag{3.11}$$

Note that

$$\|u_n - z_n\|^2 \leq 2\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

which yields that

$$-(1-\beta)\|u_n - z_n\|^2 \leq -2(1-\beta)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right), \quad \forall \beta \geq 1.$$

This together with (3.11) implies

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \left(2 - \beta - \frac{\beta\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right), \quad \forall \beta \geq 1.$$

On the other hand, if $\beta \in (0,1)$, then we obtain

$$\|z_n - p\|^2 \le \|u_n - p\|^2 - \beta\left(1 - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right), \quad \forall \beta \in (0,1).$$

This completes the proof of the lemma. □

Remark 3.1 From Lemma 3.1 and the assumptions of the parameters η and β (i.e., $\eta \in (0,1)$ and $\beta \in (0, 2/(1+\eta))$), we can obtain that $\beta^* > 0$ for all $n \ge n_0$ in Lemma 3.2 always holds.

Lemma 3.3 *([38, Lemma 3.3]) Suppose that Conditions (C1) and (C2) hold. Let $\{u_n\}$ and $\{y_n\}$ be two sequences formulated by Algorithm 3.1. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, M)$.*

Theorem 3.1 *Suppose that Conditions (C1)–(C4) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $x^\dagger \in \mathrm{VI}(C, M)$ in norm, where $x^\dagger = P_{\mathrm{VI}(C,M)}\left(f(x^\dagger)\right)$.*

Proof First, we show that the sequence $\{x_n\}$ is bounded. Indeed, thanks to Lemma 3.2 and Remark 3.1, one sees that

$$\|z_n - x^\dagger\| \le \|u_n - x^\dagger\|, \quad \forall n \ge n_0. \tag{3.12}$$

From the definition of u_n, one sees that

$$\|u_n - x^\dagger\| \le \|x_n - x^\dagger\| + \varphi_n \cdot \frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\|. \tag{3.13}$$

According to Condition (C4), we have $\frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Therefore, there exists a constant $Q_1 > 0$ such that

$$\frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\| \le Q_1, \quad \forall n \ge 1,$$

which together with (3.12) and (3.13) implies that

$$\|z_n - x^\dagger\| \le \|u_n - x^\dagger\| \le \|x_n - x^\dagger\| + \varphi_n Q_1, \quad \forall n \ge n_0. \tag{3.14}$$

Using the definition of x_{n+1} and (3.14), we have

$$\begin{aligned}
\|x_{n+1} - x^\dagger\| &= \|\varphi_n(f(x_n) - x^\dagger) + (1-\varphi_n)(z_n - x^\dagger)\| \\
&\le \varphi_n\|f(x_n) - f(x^\dagger)\| + \varphi_n\|f(x^\dagger) - x^\dagger\| + (1-\varphi_n)\|z_n - x^\dagger\| \\
&\le \varphi_n\kappa\|x_n - x^\dagger\| + \varphi_n\|f(x^\dagger) - x^\dagger\| + (1-\varphi_n)\|z_n - x^\dagger\| \\
&\le (1 - (1-\kappa)\varphi_n)\|x_n - x^\dagger\| + (1-\kappa)\varphi_n\frac{Q_1 + \|f(x^\dagger) - x^\dagger\|}{1-\kappa} \\
&\le \max\left\{\|x_n - x^\dagger\|, \frac{Q_1 + \|f(x^\dagger) - x^\dagger\|}{1-\kappa}\right\}, \quad \forall n \ge n_0 \\
&\le \cdots \le \max\left\{\|x_{n_0} - x^\dagger\|, \frac{Q_1 + \|f(x^\dagger) - x^\dagger\|}{1-\kappa}\right\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. We have that the sequences $\{u_n\}$, $\{z_n\}$ and $\{f(x_n)\}$ are also bounded.

From (3.14), one sees that

$$\begin{aligned} \|u_n - x^\dagger\|^2 &\leq (\|x_n - x^\dagger\| + \varphi_n Q_1)^2 \\ &= \|x_n - x^\dagger\|^2 + \varphi_n(2Q_1\|x_n - x^\dagger\| + \varphi_n Q_1^2) \\ &\leq \|x_n - x^\dagger\|^2 + \varphi_n Q_2 \end{aligned} \tag{3.15}$$

for some $Q_2 > 0$. By combining Lemma 3.2, (2.2), and (3.15), we obtain

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 &\leq \varphi_n \left(\|f(x_n) - f(x^\dagger)\| + \|f(x^\dagger) - x^\dagger\|\right)^2 + (1-\varphi_n)\|z_n - x^\dagger\|^2 \\ &\leq \varphi_n \left(\|x_n - x^\dagger\| + \|f(x^\dagger) - x^\dagger\|\right)^2 + (1-\varphi_n)\|z_n - x^\dagger\|^2 \\ &= \varphi_n\|x_n - x^\dagger\|^2 + (1-\varphi_n)\|z_n - x^\dagger\|^2 \\ &\quad +\varphi_n \left(2\|x_n - x^\dagger\| \cdot \|f(x^\dagger) - x^\dagger\| + \|f(x^\dagger) - x^\dagger\|^2\right) \\ &\leq \varphi_n\|x_n - x^\dagger\|^2 + (1-\varphi_n)\|z_n - x^\dagger\|^2 + \varphi_n Q_3 \\ &\leq \|x_n - x^\dagger\|^2 - (1-\varphi_n)\beta^* \left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) + \varphi_n Q_4, \ \forall n \geq n_0, \end{aligned}$$

where $Q_3 := \sup_{n\in\mathbb{N}} \left\{2\|x_n - x^\dagger\| \cdot \|f(x^\dagger) - x^\dagger\| + \|f(x^\dagger) - x^\dagger\|^2\right\}$, $Q_4 := Q_2 + Q_3$ and β^* is defined in Lemma 3.2. Thus we infer that

$$\begin{aligned} &(1-\varphi_n)\beta^* \left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) \\ &\leq \|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 + \varphi_n Q_4, \quad \forall n \geq n_0. \end{aligned} \tag{3.16}$$

By the definition of u_n, we have

$$\begin{aligned} \|u_n - x^\dagger\|^2 &\leq \|x_n - x^\dagger\|^2 + 2\theta_n\|x_n - x^\dagger\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - x^\dagger\|^2 + 3Q\theta_n\|x_n - x_{n-1}\|, \end{aligned} \tag{3.17}$$

where $Q := \sup_{n\in\mathbb{N}}\{\|x_n - x^\dagger\|, \theta\|x_n - x_{n-1}\|\} > 0$. Combining (2.1), (2.2), (3.12) and (3.17), we obtain

$$\begin{aligned} &\|x_{n+1} - x^\dagger\|^2 \\ &= \|\varphi_n(f(x_n) - f(x^\dagger)) + (1-\varphi_n)(z_n - x^\dagger) + \varphi_n(f(x^\dagger) - x^\dagger)\|^2 \\ &\leq \|\varphi_n(f(x_n) - f(x^\dagger)) + (1-\varphi_n)(z_n - x^\dagger)\|^2 + 2\varphi_n\langle f(x^\dagger) - x^\dagger, x_{n+1} - x^\dagger\rangle \\ &\leq \varphi_n\|f(x_n) - f(x^\dagger)\|^2 + (1-\varphi_n)\|z_n - x^\dagger\|^2 + 2\varphi_n\langle f(x^\dagger) - x^\dagger, x_{n+1} - x^\dagger\rangle \\ &\leq \varphi_n\kappa\|x_n - x^\dagger\|^2 + (1-\varphi_n)\|u_n - x^\dagger\|^2 + 2\varphi_n\langle f(x^\dagger) - x^\dagger, x_{n+1} - x^\dagger\rangle \\ &\leq (1-(1-\kappa)\varphi_n)\|x_n - x^\dagger\|^2 + (1-\kappa)\varphi_n \cdot \left[\frac{3Q}{1-\kappa} \cdot \frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\|\right. \\ &\quad \left. + \frac{2}{1-\kappa}\langle f(x^\dagger) - x^\dagger, x_{n+1} - x^\dagger\rangle\right], \quad \forall n \geq n_0. \end{aligned} \tag{3.18}$$

Finally, we need to show that the sequence $\{\|x_n - x^\dagger\|^2\}$ converges to zero. We set

$$p_n = \|x_n - x^\dagger\|^2, \ \ q_n = \frac{3Q\theta_n}{(1-\kappa)\varphi_n}\|x_n - x_{n-1}\| + \frac{2}{1-\kappa}\langle f(x^\dagger) - x^\dagger, x_{n+1} - x^\dagger\rangle.$$

Then the last inequality in (3.18) can be written as $p_{n+1} \leq (1-(1-\kappa)\varphi_n)p_n + (1-\kappa)\varphi_n q_n$, $\forall n \geq n_0$. Note that the sequence $\{(1-\kappa)\varphi_n\}$ is in $(0,1)$ and $\sum_{n=1}^{\infty}(1-\kappa)\varphi_n = \infty$. By Lemma 2.1, it remains to show that $\limsup_{k\to\infty} q_{n_k} \leq 0$ for every subsequence $\left\{p_{n_k}\right\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty}\left(p_{n_{k+1}} - p_{n_k}\right) \geq 0$. For this purpose, we assume that $\left\{p_{n_k}\right\}$ is a subsequence of $\{p_n\}$ such that $\liminf_{k\to\infty}\left(p_{n_{k+1}} - p_{n_k}\right) \geq 0$. From (3.16) and the assumption on $\{\varphi_n\}$, one obtains

$$
\begin{aligned}
&(1-\varphi_{n_k})\beta^*\left(\left\|u_{n_k} - y_{n_k}\right\|^2 + \left\|z_{n_k} - y_{n_k}\right\|^2\right) \\
\leq\ & \limsup_{k\to\infty} \varphi_{n_k} Q_4 + \limsup_{k\to\infty}\left(p_{n_k} - p_{n_k+1}\right) \\
\leq\ & -\liminf_{k\to\infty}\left(p_{n_k+1} - p_{n_k}\right) \leq 0.
\end{aligned}
$$

It follows from Remark 3.1 that

$$
\lim_{k\to\infty}\|y_{n_k} - u_{n_k}\| = 0 \ \text{ and } \lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0,
$$

which implies that $\lim_{k\to\infty}\|z_{n_k} - u_{n_k}\| = 0$. Moreover, we have

$$
\|x_{n_k+1} - z_{n_k}\| = \varphi_{n_k}\|z_{n_k} - f(x_{n_k})\| \to 0 \text{ as } n \to \infty,
$$

and

$$
\|x_{n_k} - u_{n_k}\| = \varphi_{n_k}\cdot\frac{\theta_{n_k}}{\varphi_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0 \text{ as } n \to \infty.
$$

It follows that

$$
\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{3.19}
$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$ and

$$
\limsup_{k\to\infty}\langle f(x^\dagger) - x^\dagger, x_{n_k} - x^\dagger\rangle = \lim_{j\to\infty}\langle f(x^\dagger) - x^\dagger, x_{n_{k_j}} - x^\dagger\rangle = \langle f(x^\dagger) - x^\dagger, z - x^\dagger\rangle. \tag{3.20}
$$

We obtain that $u_{n_k} \rightharpoonup z$ since $\|x_{n_k} - u_{n_k}\| \to 0$. This together with $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$ and Lemma 3.3 obtains $z \in \mathrm{VI}(C, M)$. From the definition of $x^\dagger$, (2.3) and (3.20), we have

$$
\limsup_{k\to\infty}\langle f(x^\dagger) - x^\dagger, x_{n_k} - x^\dagger\rangle = \langle f(x^\dagger) - x^\dagger, z - x^\dagger\rangle \leq 0. \tag{3.21}
$$

Combining (3.19) and (3.21), we obtain

$$
\limsup_{k\to\infty}\langle f(x^\dagger) - x^\dagger, x_{n_k+1} - x^\dagger\rangle \leq \limsup_{k\to\infty}\langle f(x^\dagger) - x^\dagger, x_{n_k} - x^\dagger\rangle \leq 0, \tag{3.22}
$$

which together with $\lim_{n\to\infty}\frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\| = 0$ yields that $\limsup_{k\to\infty} q_{n_k} \leq 0$. Therefore, we conclude that $\lim_{n\to\infty}\|x_n - x^\dagger\| = 0$. That is, $x_n \to x^\dagger$ as $n \to \infty$. This completes the proof. □

Next, we present an iterative scheme (see Algorithm 3.2) for solving (VIP) with a pseudomonotone and non-Lipschitz continuous operator. In our Algorithm 3.2, we replace the condition (C2) in Algorithm 3.1 with the following condition (C5).

(C5) The mapping $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, uniformly continuous on $\mathcal{H}$ and sequentially weakly continuous on C.

The form of Algorithm 2 is shown below.

Algorithm 3.2

Initialization: Take $\theta > 0$, $\delta > 0$, $\ell \in (0, 1)$, $\eta \in (0, 1)$ and $\beta \in (0, 2/(1+\eta))$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n ($n \geq 1$), calculate x_{n+1} as follows:
Step 1. Compute $u_n = x_n + \theta_n(x_n - x_{n-1})$, where θ_n is defined in (3.1).
Step 2. Compute $y_n = P_C(u_n - \lambda_n M u_n)$. If $u_n = y_n$ or $My_n = 0$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $z_n = P_{T_n}(u_n - \beta\lambda_n M y_n)$, where T_n is defined in (3.2), $\lambda_n := \delta\ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\delta\ell^m \langle My_n - Mu_n, y_n - z_n\rangle \leq \frac{\eta}{2}\left[\|u_n - y_n\|^2 + \|y_n - z_n\|^2\right]. \tag{3.23}$$

Step 4. Compute $x_{n+1} = \varphi_n f(x_n) + (1 - \varphi_n) z_n$.
Set $n := n + 1$ and go to **Step 1**.

Similar to Lemmas 3.1–3.3 in Algorithm 3.1, we have the following Lemmas 3.4–3.6 for Algorithm 3.2.

Lemma 3.4 *Suppose that Condition (C5) holds. Then the Armijo-like criteria* (3.23) *is well defined.*

Proof The proof is similar to the Lemma 3.1 in [29]. Therefore we omit the details. □

Lemma 3.5 *Assume that Condition (C5) holds. Let* $\{z_n\}$ *be a sequence generated by Algorithm 3.2. Then, for all* $p \in \mathrm{VI}(C, M)$,

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^{**}\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

where $\beta^{**} = 2 - \beta - \beta\eta$ *if* $\beta \in [1, 2/(1+\eta))$ *and* $\beta^{**} = \beta - \beta\eta$ *if* $\beta \in (0, 1)$.

Proof The proof follows the proof of Lemma 3.2 and thus it is omitted. □

Lemma 3.6 *Suppose that Conditions (C1) and (C5) hold. Let* $\{u_n\}$ *and* $\{y_n\}$ *be two sequences generated by Algorithm 3.2. If there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that* $\{u_{n_k}\}$ *converges weakly to* $z \in \mathcal{H}$ *and* $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, *then* $z \in \mathrm{VI}(C, M)$.

Proof The proof follows the proof of the Lemma 3.2 of [28]. So it is omitted. □

Theorem 3.2 *Suppose that Conditions (C1) and (C3)–(C5) hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges to* $x^\dagger \in \mathrm{VI}(C, M)$ *in norm, where* $x^\dagger = P_{\mathrm{VI}(C,M)}\left(f(x^\dagger)\right)$.

Proof The proof follows almost in the same way as that of Theorem 3.1, but we apply Lemma 3.5 and Lemma 3.6 in place of Lemmas 3.2 and 3.3, respectively. We leave it to the reader to verify it. □

3.2 The second type of modified subgradient extragradient methods

In this section, we introduce two new iterative schemes to solve the variational inequality problem (VIP). Our first scheme is shown in Algorithm 3.3 below.

Algorithm 3.3

Initialization: Take $\theta > 0$, $\lambda_1 > 0$, $\beta \in (1/(2-\eta), 1/\eta)$ and $\eta \in (0,1)$. Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \beta\lambda_n M u_n), \\ z_n = P_{H_n}(u_n - \lambda_n M y_n), \\ H_n = \{x \in \mathcal{H} \mid \langle u_n - \beta\lambda_n M u_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = \varphi_n f(x_n) + (1-\varphi_n) z_n, \end{cases}$$

where θ_n and λ_n are defined in (3.1) and (3.3), respectively.

Lemma 3.7 *Assume that Condition (C2) holds. Let $\{z_n\}$ be a sequence generated by Algorithm 3.3. Then, for all $p \in \mathrm{VI}(C, M)$,*

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^\dagger \left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

where $\beta^\dagger = 2 - \frac{1}{\beta} - \frac{\eta\lambda_n}{\lambda_{n+1}}$ if $\beta \in (0,1]$ and $\beta^\dagger = \frac{1}{\beta} - \frac{\eta\lambda_n}{\lambda_{n+1}}$ if $\beta > 1$.

Proof From (3.4) and (3.5), we obtain

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\langle \lambda_n M y_n, z_n - y_n\rangle. \tag{3.24}$$

Now we estimate $2\langle \lambda_n M y_n, z_n - y_n\rangle$. Note that

$$-\|u_n - z_n\|^2 = -\|u_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle u_n - y_n, z_n - y_n\rangle. \tag{3.25}$$

In addition,

$$\begin{aligned} &\langle u_n - y_n, z_n - y_n\rangle \\ = &\langle u_n - y_n - \beta\lambda_n M u_n + \beta\lambda_n M u_n - \beta\lambda_n M y_n + \beta\lambda_n M y_n, z_n - y_n\rangle \\ = &\langle u_n - \beta\lambda_n M u_n - y_n, z_n - y_n\rangle \\ &+\beta\lambda_n\langle M u_n - M y_n, z_n - y_n\rangle + \langle \beta\lambda_n M y_n, z_n - y_n\rangle. \end{aligned} \tag{3.26}$$

Since $z_n \in H_n$, one sees that

$$\langle u_n - \beta\lambda_n M u_n - y_n, z_n - y_n\rangle \leq 0. \tag{3.27}$$

According to the definition of λ_{n+1}, it is easy to obtain

$$\langle Mu_n - My_n, z_n - y_n\rangle \le \frac{\eta}{2\lambda_{n+1}} \|u_n - y_n\|^2 + \frac{\eta}{2\lambda_{n+1}} \|z_n - y_n\|^2 . \tag{3.28}$$

Substituting (3.26), (3.27) and (3.28) into (3.25), we have

$$-\|u_n - z_n\|^2 \le -\left(1 - \frac{\beta\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) + 2\beta\,\langle\lambda_n My_n, z_n - y_n\rangle\,,$$

which implies that

$$\begin{aligned} -2\,\langle\lambda_n My_n, z_n - y_n\rangle \;\le\; & -\left(\frac{1}{\beta} - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) \\ & + \frac{1}{\beta}\|u_n - z_n\|^2 . \end{aligned} \tag{3.29}$$

Combining (3.24) and (3.29), we conclude that

$$\begin{aligned} \|z_n - p\|^2 \;\le\; & \|u_n - p\|^2 - \left(\frac{1}{\beta} - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right) \\ & - \left(1 - \frac{1}{\beta}\right)\|u_n - z_n\|^2 . \end{aligned} \tag{3.30}$$

Note that

$$\|u_n - z_n\|^2 \le 2\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

which yields that

$$-\left(1 - \frac{1}{\beta}\right)\|u_n - z_n\|^2 \le -2\left(1 - \frac{1}{\beta}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right), \quad \forall\beta \in (0, 1].$$

This together with (3.30) implies

$$\|z_n - p\|^2 \le \|u_n - p\|^2 - \left(2 - \frac{1}{\beta} - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right), \quad \forall\beta \in (0, 1].$$

On the other hand, if $\beta > 1$, then we obtain

$$\|z_n - p\|^2 \le \|u_n - p\|^2 - \left(\frac{1}{\beta} - \frac{\eta\lambda_n}{\lambda_{n+1}}\right)\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right), \quad \forall\beta > 1.$$

This completes the proof of the lemma. □

Remark 3.2 From Lemma 3.1 and the assumptions of the parameters η and β (i.e., $\eta \in (0, 1)$ and $\beta \in (1/(2-\eta), 1/\eta)$), we can obtain that $\beta^\dagger > 0$ for all $n \ge n_1$ in Lemma 3.7 always holds.

Lemma 3.8 *Suppose that Conditions (C1) and (C2) hold. Let $\{u_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3.3. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, M)$.*

Proof The conclusion can be obtained by applying a similar statement in [38, Lemma 3.3]. □

Theorem 3.3 *Suppose that Conditions (C1)–(C4) hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.3 converges to* $x^\dagger \in \mathrm{VI}(C, M)$ *in norm, where* $x^\dagger = P_{\mathrm{VI}(C,M)}\left(f(x^\dagger)\right)$.

Proof The proof follows almost in the same way as that of Theorem 3.1, but we apply Lemmas 3.7 and 3.8 in place of Lemmas 3.2 and 3.3, respectively. We omit the details of the proof in order to avoid repetitive expressions. □

Now, we state the last scheme proposed in this paper in Algorithm 3.4. Notice that the Algorithm 3.4 can solve non-Lipschitz continuous variational inequalities.

Algorithm 3.4

Initialization: Take $\theta > 0$, $\delta > 0$, $\ell \in (0, 1)$, $\eta \in (0, 1)$ and $\beta \in (1/(2-\eta), 1/\eta)$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \beta\lambda_n M u_n), \\ z_n = P_{H_n}(u_n - \lambda_n M y_n), \\ H_n = \{x \in \mathcal{H} \mid \langle u_n - \beta\lambda_n M u_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = \varphi_n f(x_n) + (1-\varphi_n) z_n, \end{cases}$$

where θ_n and λ_n are defined in (3.1) and (3.23), respectively.

Lemma 3.9 *Assume that Condition (C5) holds. Let* $\{z_n\}$ *be a sequence generated by Algorithm 3.4. Then, for all* $p \in \mathrm{VI}(C, M)$,

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^\ddagger\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

where $\beta^\ddagger = 2 - \frac{1}{\beta} - \eta$ *if* $\beta \in (0, 1]$ *and* $\beta^\ddagger = \frac{1}{\beta} - \eta$ *if* $\beta > 1$.

Proof The proof of this lemma follows the proof of Lemma 3.7. So it is omitted. □

Lemma 3.10 *Suppose that Conditions (C1) and (C5) hold. Let* $\{u_n\}$ *and* $\{y_n\}$ *be two sequences generated by Algorithm 3.4. If there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that* $\{u_{n_k}\}$ *converges weakly to* $z \in \mathcal{H}$ *and* $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, *then* $z \in \mathrm{VI}(C, M)$.

Proof We can obtain the conclusion by a simple modification of [28, Lemma 3.2]. □

Theorem 3.4 *Suppose that Conditions (C1) and (C3)–(C5) hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.4 converges to* $x^\dagger \in \mathrm{VI}(C, M)$ *in norm, where* $x^\dagger = P_{\mathrm{VI}(C,M)}\left(f(x^\dagger)\right)$.

Proof The proof follows almost in the same way as that of Theorem 3.1, but we apply Lemmas 3.9 and 3.10 in place of Lemmas 3.2 and 3.3, respectively. □

Remark 3.3 We have the following observations for the proposed algorithms.

- Notice that if $\beta = 1$, then the proposed Algorithm 3.1 (respectively, Algorithm 3.2) and Algorithm 3.3 (respectively, Algorithm 3.4) are equivalent.
- The algorithms proposed in this paper obtain strong convergence in an infinite-dimensional Hilbert space, while the algorithms introduced in the literature [8, 10, 12, 21] can only obtain weak convergence. Therefore, our algorithms are preferable in infinite-dimensional Hilbert spaces.
- It should be noted that the proposed Algorithms 3.1 and 3.3 can solve pseudomonotone and Lipschitz continuous variational inequalities, while the suggested Algorithms 3.2 and 3.4 can solve pseudomonotone and non-Lipschitz continuous variational inequalities. The algorithms presented in this paper extend many results in the literature for solving monotone (or pseudomonotone) Lipschitz continuous variational inequalities (see, e.g., [16–21]) and monotone non-Lipschitz continuous variational inequalities (see, e.g., [23–25]).
- Our algorithms embed two adaptive step size criteria that allow them to work well without the prior information about the Lipschitz constant of the operator. The proposed Algorithms 3.1 and 3.3 apply a new non-monotonic step size criterion derived from Liu and Yang [22]. In addition, the suggested Algorithms 3.2 and 4 employ a new Armijo-type step size criterion, which comes from a recent paper by Cai et al. [28]. We embed these two latest step size criteria into the modified subgradient extragradient methods proposed in this paper. Numerical experimental results show that our algorithms are efficient and have a faster convergence speed than some previously known ones (see Section 4).

4 Numerical experiments

In this section, we provide several numerical examples to demonstrate the efficiency of our algorithms compared to some known ones. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250S CPU @1.60GHz computer with RAM 8.00 GB.

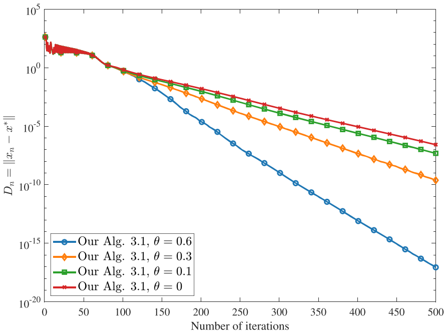
4.1 Theoretical examples

Example 4.1 Consider the linear operator $M : \mathbb{R}^m \to \mathbb{R}^m$ ($m = 20$) in the form $M(x) = Sx + q$, where $q \in \mathbb{R}^m$ and $S = NN^{\mathsf{T}} + Q + D$, N is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence S is positive symmetric definite). The feasible set C is given by $C = \{x \in \mathbb{R}^m : -2 \le x_i \le 5,\ i = 1, \ldots, m\}$. It is clear that M is monotone and Lipschitz continuous with constant $L = \|S\|$. In this experiment, all entries of N, Q are generated randomly in $[-2, 2]$, D is generated randomly in $[0, 2]$ and $q = \mathbf{0}$. It is easy to check that the solution of the variational inequality problem

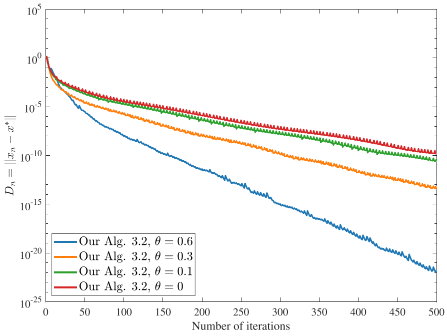
(VIP) is $x^* = \{\mathbf{0}\}$. We apply the proposed four algorithms to solve the variational inequality problem (VIP) with M and C given above. Take $\varphi_n = 1/(n+1)$, $\epsilon_n = 100/(n+1)^2$ and $f(x) = 0.1x$ for all algorithms. Choose $\lambda_1 = 1$ and $\eta = 0.2$ for Algorithms 3.1 and 3.3. Select $\delta = 2$, $\ell = 0.5$, $\eta = 0.2$ for Algorithms 3.2 and 4. The maximum number of iterations 500 is used as a stopping criterion. We use $D_n = \|x_n - x^*\|$ to measure the error of the nth iteration step. Next we test the effect of different parameters θ, β and ξ_n on the convergence behavior of the proposed algorithms. Figures 1, 2 and 3 show the numerical behavior of the proposed algorithms for different parameters θ, β and ξ_n, respectively.

Example 4.2 Let $\mathcal{H} = L^2([0,1])$ be an infinite-dimensional Hilbert space with inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)\,\mathrm{d}t$, $\forall x, y \in \mathcal{H}$ and induced norm $\|x\| = \left(\int_0^1 |x(t)|^2\,\mathrm{d}t\right)^{1/2}$, $\forall x \in \mathcal{H}$. Let r and R be two positive real numbers such that $R/(k+1) < r/k < r < R$ for some $k > 1$. Take the feasible set as $C = \{x \in \mathcal{H} : \|x\| \le r\}$. The operator $M : \mathcal{H} \to \mathcal{H}$ is given by

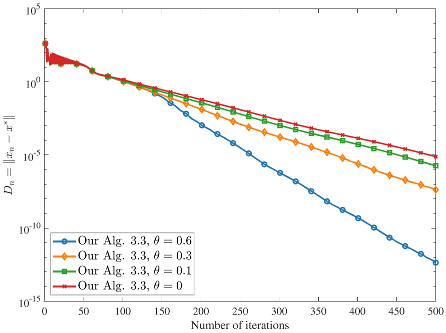
$$M(x) = (R - \|x\|)x, \quad \forall x \in \mathcal{H}.$$



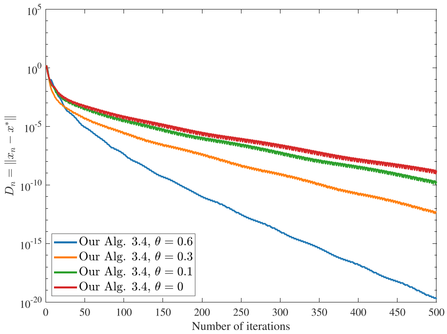
(a) Our Alg. 3.1, $\beta = 1.5$, $\xi_n = 1/(n+1)^{1.1}$



(b) Our Alg. 3.2, $\beta = 1.5$

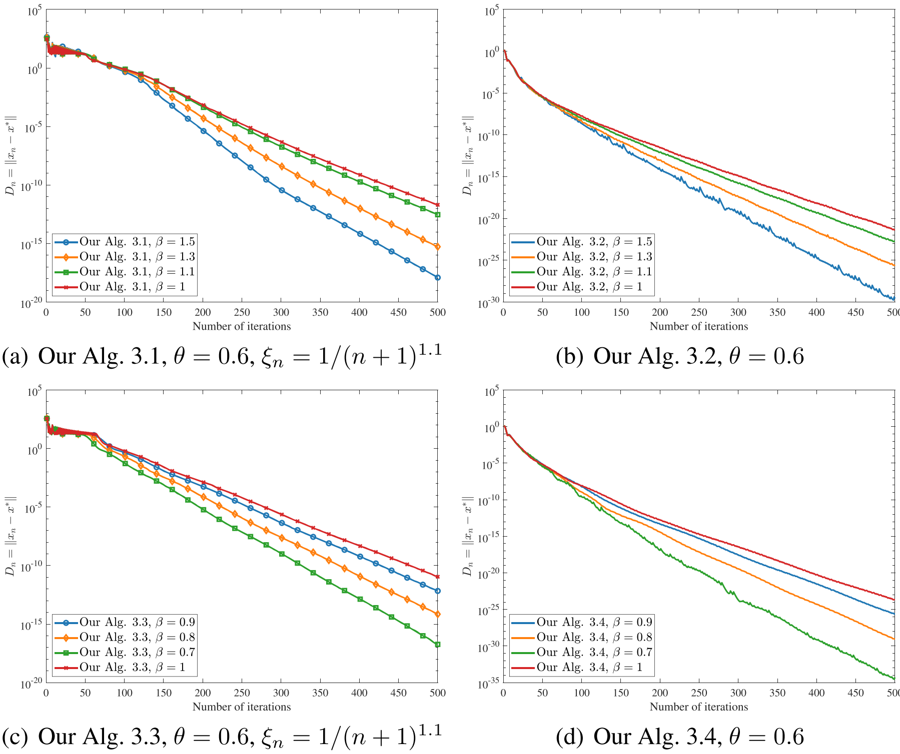


(c) Our Alg. 3.3, $\beta = 0.8$, $\xi_n = 1/(n+1)^{1.1}$



(d) Our Alg. 3.4, $\beta = 0.8$

Fig. 1 Example 4.1, compare θ



(a) Our Alg. 3.1, $\theta = 0.6$, $\xi_n = 1/(n+1)^{1.1}$

(b) Our Alg. 3.2, $\theta = 0.6$

(c) Our Alg. 3.3, $\theta = 0.6$, $\xi_n = 1/(n+1)^{1.1}$

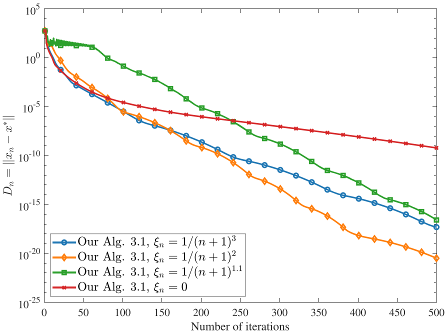
(d) Our Alg. 3.4, $\theta = 0.6$

Fig. 2 Example 4.1, compare β

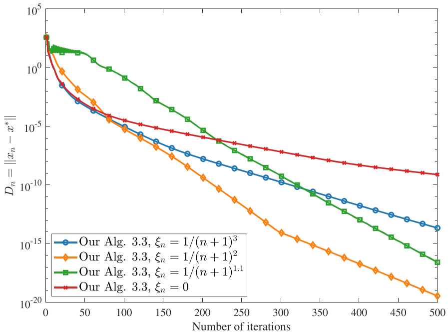
It is not hard to check that operator M is pseudomonotone rather than monotone. For the experiment, we choose $R = 1.5$, $r = 1$, $k = 1.1$. The solution of the problem (VIP) is $x^*(t) = 0$. We compare the proposed Algorithms 3.1–3.4 with the Algorithm 3.2 introduced by Thong, Hieu and Rassias [38] (shortly, THR Alg. 3.2). Set $\varphi_n = 1/(n+1)$, $\theta = 0.3$, $\epsilon_n = 100/(n+1)^2$ and $f(x) = 0.1x$ for all algorithms. Take $\eta = 0.4$ and $\lambda_1 = 1$ for the suggested Algorithm 3.1, Algorithm 3.3 and THR Alg. 3.2. Choose $\xi_n = 1/(n+1)^{1.1}$ for the suggested Algorithms 3.1 and 3.3. Select $\delta = 1$, $\ell = 0.5$ and $\eta = 0.4$ for the suggested Algorithms 3.2 and 3.4. The maximum number of iterations 50 is used as a common stopping criterion and $D_n = \|x_n(t) - x^*(t)\|$ is used to measure the error of the nth iteration step of all algorithms. The numerical behavior of all algorithms with four starting points $x_0(t) = x_1(t)$ is shown in Fig. 4.

Example 4.3 Consider the Hilbert space $\mathcal{H} = l_2 := \{x = (x_1, x_2, \ldots, x_i, \ldots) \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ equipped with inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, $\forall x, y \in \mathcal{H}$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$, $\forall x \in \mathcal{H}$. Let $C := \{x \in \mathcal{H} : |x_i| \le 1/i\}$. Define an operator $M : C \to \mathcal{H}$ by

$$Mx = \left(\|x\| + \frac{1}{(\|x\| + \varphi)}\right) x$$



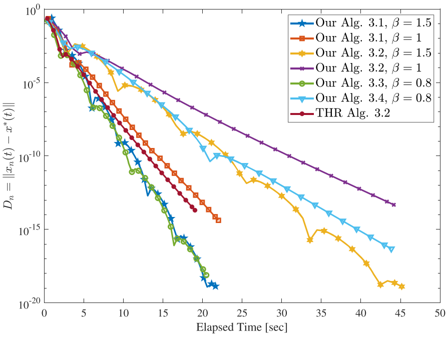
(a) Our Alg. 3.1, $\theta = 0.6$, $\beta = 1.5$



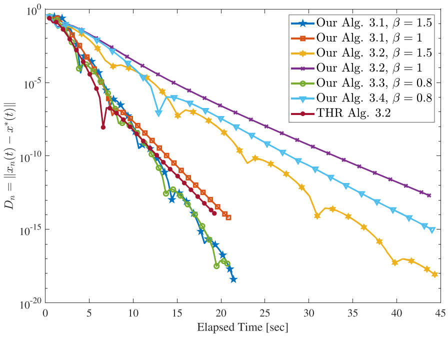
(b) Our Alg. 3.3, $\theta = 0.6$, $\beta = 0.8$

Fig. 3 Example 4.1, compare ξ_n

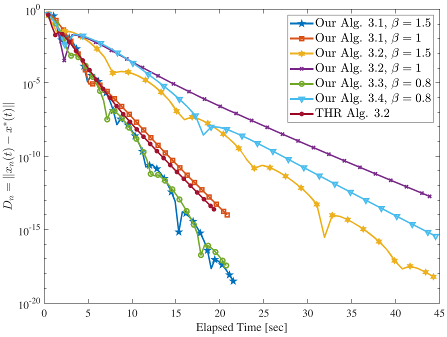
for some $\varphi > 0$. It can be verified that mapping M is pseudomonotone on $\mathcal{H}$, uniformly continuous and sequentially weakly continuous on C but not Lipschitz continuous on $\mathcal{H}$ (see [40]). In the following cases, we take $\varphi = 0.5$, $\mathcal{H} = \mathbb{R}^m$ for



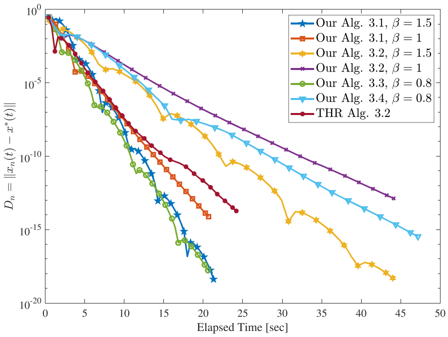
(a) $x_0(t) = x_1(t) = t^4$



(b) $x_0(t) = x_1(t) = \mathrm{e}^t$



(c) $x_0(t) = x_1(t) = \log(t)$



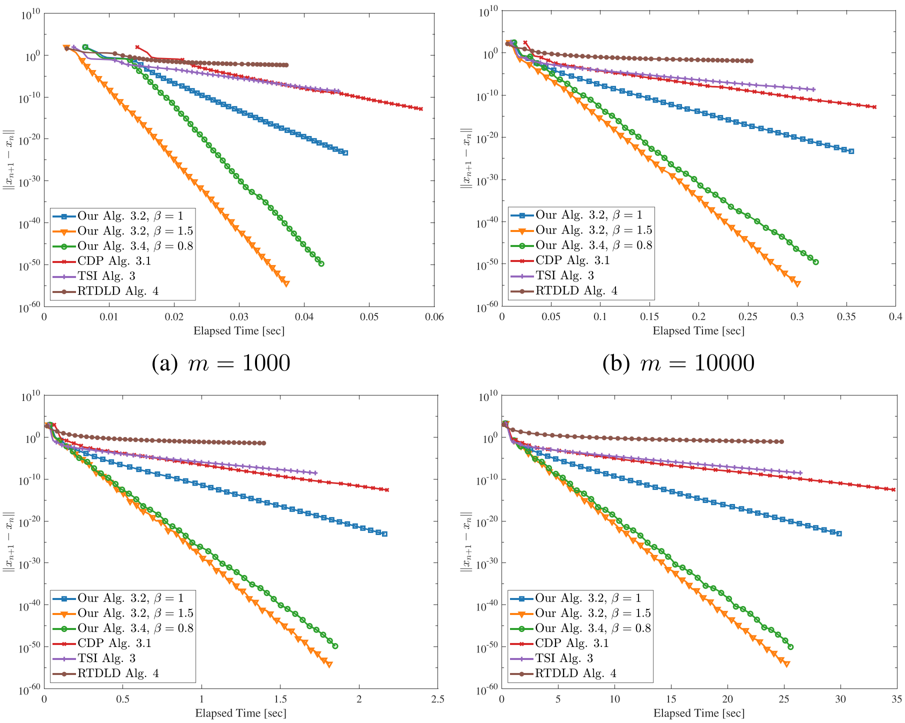
(d) $x_0(t) = x_1(t) = 2\sin(t)$

Fig. 4 Numerical results of all algorithms for Example 4.2

different values of m. In those cases, the feasible set C is a box $C = \{x \in \mathbb{R}^m : -1/i \leq x_i \leq 1/i, \ i = 1, 2, \ldots, m\}$. We compare the proposed Algorithms 3.2 and 4 with several strongly convergent algorithms that can solve the (VIP) with uniformly continuous operators, including the Algorithm 3.4 proposed by Reich et al. [27] (shortly, RTDLD Alg. 4), the Algorithm 3.1 introduced by Cai et al. [28] (shortly, CDP Alg. 3.1) and the Algorithm 3 suggested by Thong et al. [39] (shortly, TSI Alg. 3). Take $\varphi_n = 1/(n+1)$, $f(x) = 0.1x$, $\delta = 2$, $\ell = 0.5$, $\eta = 0.1$ for all algorithms. Choose $\lambda = 0.5/\eta$ for RTDLD Alg. 4. Select $\theta = 0.4$ and $\epsilon_n = 100/(n+1)^2$ for the suggested Algorithm 3.2 and Algorithm 3.4. The initial values $x_0 = x_1 =$ *5rand(m,1)* are randomly generated by MATLAB. The maximum number of iterations 200 is used as a common stopping criterion. The numerical performance of the sequence $\{\|x_n - x_{n-1}\|\}$ of all algorithms with four different dimensions is reported in Fig. 5.

Remark 4.1 We have the following observation for Examples 4.1–4.3.

- The following conclusions can be drawn from Example 4.1: (1) the proposed algorithms with inertial terms converge faster than those without inertial terms (see Fig. 1); (2) our algorithms can obtain a faster convergence speed when



(a) $m = 1000$ (b) $m = 10000$

(c) $m = 100000$ (d) $m = 500000$

Fig. 5 Numerical results of all algorithms for Example 4.3

choosing a suitable value of β (see Fig. 2), which indicates that the modified subgradient extragradient methods proposed in this paper are efficient; (3) the suggested Algorithms 3.1 and 3.3 converge faster when using a non-monotonic step size (i.e., $\xi_n \neq 0$) than when using a non-increasing step size (i.e., $\xi_n = 0$) (see Fig. 3).

- It can be seen from Figs. 4 and 5 that our algorithms converge faster than the schemes presented in [27, 28, 38, 39] and that these results are independent of the selection of initial values and the size of the dimensions. Therefore, the algorithms proposed in this paper are efficient and robust. Moreover, we can obtain that the proposed algorithms have a faster convergence speed when a suitable value of β is chosen. Specifically, the proposed Algorithm 3.1 (Algorithm 3.2) converges faster at $\beta = 1.5$ and the proposed Algorithm 3.3 (Algorithm 3.4) converges faster at $\beta = 0.8$ than when they are at $\beta = 1$.
- Note that our Algorithms 3.2 and 3.4 take more time to reach the common stopping criterion in infinite-dimensional spaces than the adaptive-type Algorithms 3.1 and 3.3, due to the fact that the proposed Algorithms 3.2 and 3.4 use an Armijo-type stepsize criterion which may require computing multiple projections on the feasible set in each iteration to find the appropriate step size and thus increases the computation time of the algorithms. On the other hand, note that the operator M in Example 4.3 is uniformly continuous rather than Lipschitz continuous. So we do not report numerical results for the proposed Algorithms 3.1 and 3.3 in this example because they are not available. At the same time, many algorithms used in the literature (see, e.g., [17–21]) for solving Lipschitz continuous variational inequalities will not be available. Therefore, the iterative schemes proposed in this paper improved and extended many known results in the literature for solving variational inequalities.

4.2 Application to optimal control problems

Next, we use the proposed algorithms to solve the (VIP) that appears in optimal control problems. Assume that $L_2\left([0, T], \mathbb{R}^m\right)$ represents the square-integrable Hilbert space with inner product $\langle p, q\rangle = \int_0^T \langle p(t), q(t)\rangle \, \mathrm{d}t$ and norm $\|p\| = \sqrt{\langle p, p\rangle}$. The optimal control problem is described as follows:

$$p^*(t) \in \operatorname{Argmin}\{g(p) \mid p \in V\}, \quad t \in [0, T], \tag{4.1}$$

where V represents a set of feasible controls composed of m piecewise continuous functions. Its form is expressed as follows:

$$V = \left\{p(t) \in L_2\left([0, T], \mathbb{R}^m\right) : p_i(t) \in \left[p_i^-, p_i^+\right], i = 1, 2, \ldots, m\right\}. \tag{4.2}$$

In particular, the control $p(t)$ may be a piecewise constant function (bang-bang type). The terminal objective function has the form

$$g(p) = \Phi(x(T)), \tag{4.3}$$

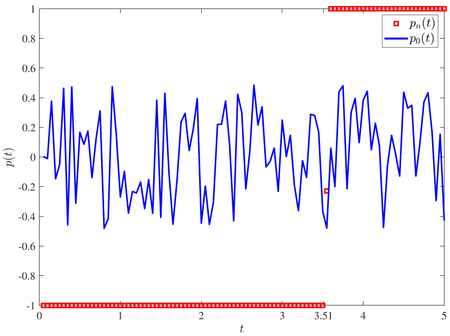
where Φ is a convex and differentiable defined on the attainability set. Assume that the trajectory $x(t) \in L_2([0, T]$ satisfies the constraints of the linear differential equation system:

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = Q(t)x(t) + W(t)p(t), \quad 0 \le t \le T, \quad x(0) = x_0, \tag{4.4}$$

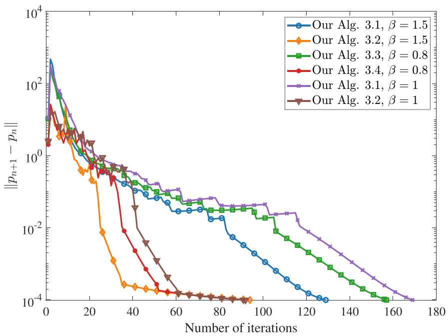
where $Q(t) \in \mathbb{R}^{n\times n}$, $W(t) \in \mathbb{R}^{n\times m}$ are given continuous matrices for every $t \in [0, T]$. By the solution of problem (4.1)–(4.4), we mean a control $p^*(t)$ and a corresponding (optimal) trajectory $x^*(t)$ such that its terminal value $x^*(T)$ minimizes objective function (4.3). It is known that the optimal control problem (4.1)–(4.4) can be transformed into a variational inequality problem (see [26, 41]). We first use the classical Euler discretization method to decompose the optimal control problem (4.1)–(4.4) and then apply the proposed algorithms to solve the variational inequality problem corresponding to the discretized version of the problem (see [26, 41] for more details). In the proposed Algorithms 3.1–3.4, we set $N = 100$, $\theta = 0.01$, $\epsilon_n = 10^{-4}/(n+1)^2$, $\varphi_n = 10^{-4}/(n+1)$ and $f(x) = 0.1x$. Pick $\lambda_1 = 0.4$, $\eta = 0.5$ and $\xi_n = 1/(n+1)^{1.1}$ for Algorithm 3.1 and Algorithm 3.3. Select $\delta = 2$, $\ell = 0.5$ and $\eta = 0.5$ for Algorithm 3.2 and Algorithm 3.4. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$ and the stopping criterion is $D_n = \|p_{n+1} - p_n\| \le 10^{-4}$.

Example 4.4 (Rocket car [41])

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\left((x_1(5))^2 + (x_2(5))^2\right), \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = p(t), \quad \forall t \in [0, 5], \\
& x_1(0) = 6, \quad x_2(0) = 1, \\
& p(t) \in [-1, 1].
\end{aligned}$$



(a) Optimal control of our Algorithm 3.1



(b) Compare all the proposed algorithms

Fig. 6 Numerical results for Example 4.4

The exact optimal control of Example 4.4 is

$$p^*(t) = \begin{cases} 1 & \text{if } t \in (3.517, 5]\,; \\ -1 & \text{if } t \in (0, 3.517]. \end{cases}$$

The approximate optimal control of the suggested Algorithm 3.1 is plotted in Fig. 6a. In addition, the numerical behavior of the stated algorithms is shown in Fig. 6b.

Remark 4.2 As it can be seen in Fig. 6, the algorithms proposed in this paper can solve the optimal control problem. Moreover, our algorithms can obtain a faster convergence speed when a suitable value of β is chosen, which is the same as the previous conclusion.

5 Conclusions

In this paper, we introduced four new efficient iterative schemes to solve pseudomonotone variational inequalities in the framework of infinite-dimensional Hilbert spaces. The proposed algorithms are motivated by the inertial method, the subgradient extragradient method, and the viscosity method. It is noted that our two schemes with Armijo-type step size criterion can solve non-Lipschitz continuous variational inequalities. The strong convergence of the iterative sequences generated by the proposed schemes is established without the prior knowledge of the Lipschitz constant of the involved mapping. Finally, some numerical experiments occurring on finite and infinite-dimensional spaces and applications in optimal control problems are given to demonstrate the computational efficiency and advantages of the suggested algorithms over some known schemes. The methods presented in this paper improved and extended some existing results in the literature for solving variational inequalities. In future work, we will consider extending the results of this paper to a reflexive Banach space with the help of the ideas in [42].

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Declarations

Conflict of interest The authors declare no competing interests.

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