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ADAPTIVE INERTIAL SUBGRADIENT EXTRAGRADIENT METHODS FOR FINDING MINIMUM-NORM SOLUTIONS OF PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, four modified inertial subgradient extragradient methods with a new non-monotonic step size criterion are investigated for pseudomonotone variational inequality problems in real Hilbert spaces. Our algorithms employ two different step sizes in each iteration to update the values of iterative sequences, and they work well without the prior information about the Lipschitz constant of the operator. Strong convergence theorems of the proposed iterative schemes are established under some suitable and mild conditions. Some numerical examples are provided to demonstrate the computational efficiency and advantages of the proposed methods over other known ones.

1. **Introduction and preliminaries.** Let $\mathcal{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Throughout the paper, we use $Proj_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ to denote the metric projection from $\mathcal{H}$ onto $\mathcal{C}$, i.e., $Proj_{\mathcal{C}}(x) := \arg\min\{\|x-y\|, \, y \in \mathcal{C}\}$. It is known that the projection has the following property:

$$\langle x - Proj_{\mathcal{C}}(x), y - Proj_{\mathcal{C}}(x) \rangle \leq 0, \; \forall x \in \mathcal{H}, y \in \mathcal{C}.$$

Recall that a mapping $Q : \mathcal{H} \to \mathcal{H}$ is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if $\|Qx - Qy\| \leq L\|x-y\|, \; \forall x, y \in \mathcal{H}$;
(2) *monotone* if $\langle Qx - Qy, x - y \rangle \geq 0, \; \forall x, y \in \mathcal{H}$;
(3) *pseudomonotone* if $\langle Qx, y - x \rangle \geq 0 \Rightarrow \langle Qy, y - x \rangle \geq 0, \; \forall x, y \in \mathcal{H}$;

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(4) *sequentially weakly continuous* if for each sequence $\{x_n\}$ converging weakly to x, one has that $\{Qx_n\}$ converges weakly to Qx.

The purpose of this work is to present several efficient and adaptive numerical schemes for solving the variational inequality problem with a pseudomonotone operator in a real Hilbert space. Recall that the classical variational inequality problem is described as follows:

$$\text{find } x^\dagger \in \mathcal{C} \text{ such that } \langle Qx^\dagger, x - x^\dagger \rangle \geq 0,\ \forall x \in \mathcal{C}, \tag{VIP}$$

where $Q : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator. The solution set of the (VIP) is denoted by $\mathrm{VI}(\mathcal{C}, Q)$, and is assumed to be nonempty. Variational inequality theory is an important tool in transportation modeling, engineering mechanics, operations research and management, economics, optimal control, and others; see, e.g., [37, 1, 10, 9, 40]. In the last few decades, a large number of numerical methods were proposed to address variational inequality problems and related problems (see, e.g., [16, 11, 36, 5, 20, 3, 4]). Recall that the extragradient method introduced by Korpelevich [16] is one of the most popular methods for solving the (VIP). The method requires the computation of the projection on the feasible set twice in each iteration. It is well known that computing the projection is equivalent to solving a constrained minimum distance problem. The basic fact is that the projection is not easy to compute if the feasible set is complex. In the past decades, many efficient algorithms that only need to compute the projection on the feasible set once have been proposed to solve the (VIP); see, for example, [11, 36, 5, 20]. The computational efficiency of these methods is significantly improved by the fact that they evaluate the projection on the feasible set only once in each iteration.

Notice that the methods proposed in [16, 11, 36, 5, 20] are weakly convergent in infinite-dimensional Hilbert spaces under the condition that the mapping Q satisfies monotonicity and Lipschitz continuity. It is known that strong convergence is preferable to weak convergence in infinite-dimensional spaces (see [29] for more details). In the last decades, a large number of strongly convergent numerical algorithms were introduced for solving variational inequality problems in infinite-dimensional spaces; see, e.g., [6, 17, 12, 26, 23, 38] and the references therein. In addition, it is known that the class of pseudomonotone mappings contains the class of monotone mappings. Recently, numerical algorithms for pseudomonotone variational inequality problems in Hilbert spaces have received a lot of attention from scholars in the optimization community (see, e.g., [37, 10, 23, 35]). Notice that one of the drawbacks of the methods introduced in [16, 11, 36, 5, 20] is that the determination of step size requires prior information about the Lipschitz constant of mapping Q. This means that these methods will fail if the Lipschitz constant is unknown. However, the Lipschitz constant of the mapping is not always available in practical applications. Therefore, it is valuable to develop adaptive algorithms that do not require the Lipschitz constant of the mapping. Recently, Thong and Vuong [35] proposed a Mann-based Tseng extragradient method to solve pseudomonotone variational inequalities in Hilbert spaces. Their scheme is shown in Algorithm 1.1 below.

Under certain assumptions, the strong convergence of Algorithm 1.1 was established in real Hilbert spaces. Notice that Algorithm 1.1 employs line search step size rule (1.1) (also known as the Armijo type step criterion) making it possible to work without prior knowledge of the Lipschitz constant of the mapping. However, the use of criterion (1.1) may affect the computational efficiency of Algorithm 1.1 due to

Algorithm 1.1 The Algorithm 2 of Thong and Vuong [35]

Initialization: Give $\zeta > 0$, $\ell \in (0,1)$, and $\mu \in (0,1)$. Let $x_1 \in \mathcal{C}$ be arbitrary.
Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1. Compute $d_n = Proj_{\mathcal{C}}(x_n - \chi_n Q x_n)$, where $\chi_n := \zeta \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\zeta \ell^m \|Qx_n - Qd_n\| \leq \mu \|x_n - d_n\|. \tag{1.1}$$

Step 2. Compute $z_n = d_n - \chi_n (Qd_n - Qx_n)$.
Step 3. Compute $x_{n+1} = (1 - \theta_n - \alpha_n) x_n + \alpha_n z_n$.
Set $n := n + 1$ and go to **Step 1**.

the fact that the determination of step size χ_n may require multiple computations of the projection on the feasible set. To overcome this difficulty, Yang and Liu [38] proposed a new step size criterion that does not include any line search process (see Equation (1.2) below). By combining the Tseng extragradient method [36], the Moudafi viscosity method, and the adaptive step size criterion (1.2), Yang and Liu [38, Algorithm 1] introduced a new strongly convergent scheme to handle monotone variational inequalities in Hilbert spaces. Recently, this novel step size rule has attracted a great deal of interest and study from researchers, who inserted it into new algorithms to address variational inequalities (see, e.g., [32]). On the other hand, as one of the acceleration approaches, the inertial extrapolation method based on discrete versions of a second-order dissipative dynamic system has gained a lot of attention from researchers. The inertial-type methods are characterized by the fact that the next iteration is determined by the combination of the previous two (or more) iterations. It is worth noting that this minor modification can significantly improve the convergence speed of the original algorithm without inertial terms. Recently, many inertial-type algorithms were presented to handle variational inequalities, split feasibility problems, equilibrium problems, and others; see, e.g., [8, 13, 27, 25, 15] and the references therein.

Based on the inertial method, the subgradient extragradient method, the viscosity method, and the adaptive step size criterion in [38], Thong et al. [32] proposed a new iterative algorithm for solving pseudomonotone variational inequality problems in Hilbert spaces. Indeed, their method is illustrated in Algorithm 1.2 below.

The strong convergence of the iterative sequence generated by Algorithm 1.2 was proved under some mild conditions. It should be noted that the step size sequence generated by Algorithm 1.2 is non-increasing according to the definition of update method (1.2). To overcome this drawback, Liu and Yang [19] made a simple modification to (1.2) so that the algorithm used can produce a non-monotonic sequence of step sizes. Recently, this novel non-monotonic step size criterion suggested by Liu and Yang [19] has been used by many authors to address variational inequality problems (see, e.g., [21, 18]). Notice that the convergence of Algorithms 1.1 and 1.2 requires that the operator Q satisfies Lipschitz continuity, which may be difficult to satisfy (or hard to verify) in practical problems. Recently, scholars proposed a variety of algorithms that can handle monotone (or pseudomonotone) variational inequalities involving non-Lipschitz continuity to overcome this challenge; see, e.g., [7, 28, 33, 34, 2, 30]. We next state two approaches that are already existing in [34, 2], which help us to present the ideas in this paper. Recently, Thong et al. [34] offered a new iterative scheme for solving pseudomonotone variational inequalities

Algorithm 1.2 The Algorithm 3.2 of Thong et al. [32]

Initialization: Given $\rho > 0$, $\chi_1 > 0$, $\mu \in (0,1)$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and $x_n\,(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $v_n = x_n + \rho_n\left(x_n - x_{n-1}\right)$, where

$$\rho_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \rho\right\}, & \text{if } x_n \neq x_{n-1}; \\ \rho, & \text{otherwise.} \end{cases}$$

Step 2. Compute $d_n = Proj_{\mathcal{C}}\left(v_n - \chi_n Q v_n\right)$.
Step 3. Compute $z_n = Proj_{T_n}\left(v_n - \chi_n Q d_n\right)$, where the half space T_n is defined by

$$T_n = \left\{x \in \mathcal{H} : \langle v_n - \chi_n Q v_n - d_n, x - d_n\rangle \leq 0\right\}.$$

Step 4. Compute $x_{n+1} = \theta_n f\left(z_n\right) + \left(1 - \theta_n\right) z_n$ and update

$$\chi_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|v_n - d_n\|}{\|Q v_n - Q d_n\|}, \chi_n\right\}, & \text{if } Q v_n - Q d_n \neq 0; \\ \chi_n, & \text{otherwise.} \end{cases} \tag{1.2}$$

Set $n := n + 1$ and go to **Step 1**.

involving uniform continuity based on the subgradient extragradient method, the viscosity method, and the Armijo-type line search process. Now, their scheme is displayed in Algorithm 1.3 below.

Algorithm 1.3 The Algorithm 3 of Thong et al. [34]

Initialization: Given $\zeta > 0$, $\ell \in (0,1)$, and $\mu \in (0,1)$. Let $x_1 \in \mathcal{C}$ be arbitrary.
Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1. Compute $d_n = Proj_{\mathcal{C}}\left(x_n - \chi_n Q x_n\right)$, where $\chi_n := \zeta \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\zeta \ell^m \langle Q x_n - Q d_n, x_n - d_n\rangle \leq \mu \|x_n - d_n\|^2. \tag{1.3}$$

Step 2. Compute $z_n = Proj_{C_n}\left(x_n\right)$, where the half space C_n is defined by

$$C_n := \left\{x \in \mathcal{H} : \langle x_n - d_n - \chi_n\left(Q x_n - Q d_n\right), x - d_n\rangle \leq 0\right\}.$$

Step 3. Compute $x_{n+1} = \theta_n f\left(x_n\right) + \left(1 - \theta_n\right) z_n$.
Set $n := n + 1$ and go to **Step 1**.

The convergence of Algorithm 1.3 is established under some mild conditions in Hilbert spaces. Very recently, Cai et al. [2] introduced a strongly convergent iterative algorithm with a new Armijo-type step size criterion for finding solutions to pseudomonotone and non-Lipschitz continuous variational inequality problems. More precisely, the form of their iterative scheme is described in Algorithm 1.4 below.

It is worth noting that the criterion (1.4) used for the update step size of Algorithm 1.4 is different from the criterion (1.1) used in Algorithm 1.1 and the criterion (1.3) used in Algorithm 1.3. Specifically, the rule (1.4) uses the information of sequence z_n while the guidelines (1.1) and (1.3) do not use this information. Some numerical tests in [2] demonstrate the computational efficiency of Algorithm 1.4 in comparison to other known ones.

Algorithm 1.4 The Algorithm 3.1 of Cai et al. [2]

Initialization: Given $\zeta > 0$, $\ell \in (0,1)$, and $\mu \in (0,1)$. Let $x_1 \in \mathcal{C}$ be arbitrary.
Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1. Compute $d_n = Proj_{\mathcal{C}}\left(x_n - \chi_n Q x_n\right)$.
Step 2. Compute $z_n = Proj_{T_n}\left(x_n - \chi_n Q d_n\right)$, where the half space T_n is defined by

$$T_n = \left\{x \in \mathcal{H} : \langle x_n - \chi_n Q x_n - d_n, x - d_n\rangle \leq 0\right\},$$

and $\chi_n := \zeta \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\zeta \ell^{m} \langle Q d_n - Q x_n, d_n - z_n\rangle \leq \frac{\mu}{2}\left[\|x_n - d_n\|^2 + \|d_n - z_n\|^2\right]. \tag{1.4}$$

Step 3. Compute $x_{n+1} = \theta_n f\left(x_n\right) + \left(1 - \theta_n\right) z_n$.
Set $n := n + 1$ and go to **Step 1**.

Inspired and motivated by the above works, we introduce four strongly convergent modified subgradient extragradient methods to address pseudomonotone and Lipschitz continuous (or non-Lipschitz continuous) variational inequality problems in real Hilbert spaces. We conclude the section by giving the following lemma that is crucial in the convergence analysis of our algorithms.

Lemma 1.1 ([22]). *Let $\{p_n\}$ be a positive sequence, $\{s_n\}$ be a sequence of real numbers, and $\{\theta_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \theta_n = \infty$. Assume that*

$$p_{n+1} \leq (1 - \theta_n) p_n + \theta_n s_n, \ \forall n \geq 1.$$

If $\limsup_{k\to\infty} s_{n_k} \leq 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying

$$\liminf_{k\to\infty}(p_{n_k+1} - p_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} p_n = 0$.

2. **Main results.**

In this section, we introduce four new modified subgradient extragradient algorithms to address pseudomonotone and Lipschitz continuous (or non-Lipschitz continuous) variational inequalities in real Hilbert spaces. Our algorithms employ two novel non-monotonic step size criterion allowing them to work well without the Lipschitz constant of the operator. In the sequel, we use the symbol $x_n \rightharpoonup x$ ($x_n \to x$) to denote the weak convergence (strong convergence) of a sequence $\{x_n\}$ to x.

2.1. **Two methods for variational inequalities involving Lipschitz continuity.**

Two new iterative methods with non-monotonic step size criterion are proposed in this subsection to address the Lipschitz continuous and pseudomonotone variational inequality problems. We first assume that our algorithms satisfy the following conditions.

(C1) The feasible set $\mathcal{C}$ is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$, and the solution set of the problem (VIP) is nonempty.
(C2) The operator $Q : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathcal{H}$ and sequentially weakly continuous on $\mathcal{C}$.
(C3) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\theta_n} = 0$, where $\{\theta_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$.

Now, we are in a position to introduce our Algorithm 2.1.

The following lemmas are critical for analyzing the convergence of our algorithms.

Algorithm 2.1

Initialization: Take $\rho > 0$, $\chi_1 > 0$, $\phi \in (0, 2/(1+\mu))$, and $\mu \in (0,1)$. Select the sequences $\{\epsilon_n\}$ and $\{\theta_n\}$ to satisfy Condition (C3). Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

Step 1. Compute $v_n = (1-\theta_n)(x_n + \rho_n(x_n - x_{n-1}))$, where

$$\rho_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \rho\right\}, & \text{if } x_n \neq x_{n-1}; \\ \rho, & \text{otherwise.} \end{cases} \tag{2.1}$$

Step 2. Compute $d_n = Proj_{\mathcal{C}}(v_n - \chi_n Q v_n)$. If $v_n = d_n$ or $Qd_n = 0$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = Proj_{T_n}(v_n - \phi\chi_n Q d_n)$, where the half-space T_n is defined by

$$T_n = \{x \in \mathcal{H} : \langle v_n - \chi_n Q v_n - d_n, x - d_n\rangle \leq 0\}, \tag{2.2}$$

and update (set $b_n = \langle Qv_n - Qd_n, x_{n+1} - d_n\rangle$)

$$\chi_{n+1} = \begin{cases} \min\left\{\mu\dfrac{\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2}{2b_n}, \chi_n + \xi_n\right\}, & \text{if } b_n > 0; \\ \chi_n + \xi_n, & \text{otherwise.} \end{cases} \tag{2.3}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 2.1. *Assume that Condition (C2) holds. Then the sequence $\{\chi_n\}$ generated by (2.3) is well defined and $\lim_{n\to\infty} \chi_n$ exists.*

Proof. The proof is similar to Lemma 3.1 in [19] and thus we omit the details. □

Lemma 2.2. *Assume that Condition (C2) holds. Let $\{v_n\}$, $\{d_n\}$, and $\{x_{n+1}\}$ be three sequences formed by Algorithm 2.1. Then*

$$\|x_{n+1} - x^\dagger\|^2 \leq \|v_n - x^\dagger\|^2 - \phi^*\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \forall x^\dagger \in \mathrm{VI}(\mathcal{C}, Q),$$

where $\phi^* = 2 - \phi - \frac{\phi\mu\chi_n}{\chi_{n+1}}$ if $\phi \in [1, 2/(1+\mu))$ and $\phi^* = \phi - \frac{\phi\mu\chi_n}{\chi_{n+1}}$ if $\phi \in (0,1)$.

Proof. From the fact that

$$\|Proj_{\mathcal{C}}(x) - y\|^2 \leq \|x - y\|^2 - \|x - Proj_{\mathcal{C}}(x)\|^2, \ \forall x \in \mathcal{H}, y \in \mathcal{C}$$

and the definition of x_{n+1}, we obtain

$$\begin{aligned} &\|x_{n+1} - x^\dagger\|^2 \\ &= \|Proj_{T_n}(v_n - \phi\chi_n Q d_n) - x^\dagger\|^2 \\ &\leq \|v_n - \phi\chi_n Q d_n - x^\dagger\|^2 - \|v_n - \phi\chi_n Q d_n - x_{n+1}\|^2 \\ &= \|v_n - x^\dagger\|^2 + (\phi\chi_n)^2\|Qd_n\|^2 - 2\langle v_n - x^\dagger, \phi\chi_n Q d_n\rangle - \|v_n - x_{n+1}\|^2 \\ &\quad - (\phi\chi_n)^2\|Qd_n\|^2 + 2\langle v_n - x_{n+1}, \phi\chi_n Q d_n\rangle \\ &= \|v_n - x^\dagger\|^2 - \|v_n - x_{n+1}\|^2 - 2\langle \phi\chi_n Q d_n, x_{n+1} - x^\dagger\rangle \\ &= \|v_n - x^\dagger\|^2 - \|v_n - x_{n+1}\|^2 - 2\langle \phi\chi_n Q d_n, x_{n+1} - d_n\rangle \\ &\quad - 2\langle \phi\chi_n Q d_n, d_n - x^\dagger\rangle. \end{aligned} \tag{2.4}$$

It follows from $x^\dagger \in \mathrm{VI}(\mathcal{C}, Q)$ and $d_n \in \mathcal{C}$ that $\langle Qx^\dagger, d_n - x^\dagger\rangle \geq 0$, which together with the pseudomonotonicity of Q deduces that $\langle Qd_n, d_n - x^\dagger\rangle \geq 0$. Thus the inequality (2.4) reduces to

$$\|x_{n+1} - x^\dagger\|^2 \leq \|v_n - x^\dagger\|^2 - \|v_n - x_{n+1}\|^2 - 2\langle \phi\chi_n Qd_n, x_{n+1} - d_n\rangle. \tag{2.5}$$

Next, we estimate $2\langle \phi\chi_n Qd_n, x_{n+1} - d_n\rangle$. Notice that

$$-\|v_n - x_{n+1}\|^2 = -\|v_n - d_n\|^2 - \|d_n - x_{n+1}\|^2 + 2\langle v_n - d_n, x_{n+1} - d_n\rangle \tag{2.6}$$

and

$$\begin{aligned} &\langle v_n - d_n, x_{n+1} - d_n\rangle \\ &= \langle v_n - d_n - \chi_n Qv_n + \chi_n Qv_n - \chi_n Qd_n + \chi_n Qd_n, x_{n+1} - d_n\rangle \\ &= \langle v_n - \chi_n Qv_n - d_n, x_{n+1} - d_n\rangle + \chi_n\langle Qv_n - Qd_n, x_{n+1} - d_n\rangle \\ &\quad + \langle \chi_n Qd_n, x_{n+1} - d_n\rangle. \end{aligned} \tag{2.7}$$

In view of $x_{n+1} \in T_n$ and the definition of T_n, one obtains

$$\langle v_n - \chi_n Qv_n - d_n, x_{n+1} - d_n\rangle \leq 0. \tag{2.8}$$

From the definition of χ_{n+1}, we can show that

$$\langle Qv_n - Qd_n, x_{n+1} - d_n\rangle \leq \frac{\mu}{2\chi_{n+1}}\|v_n - d_n\|^2 + \frac{\mu}{2\chi_{n+1}}\|x_{n+1} - d_n\|^2. \tag{2.9}$$

Substituting (2.7), (2.8), and (2.9) into (2.6), we deduce

$$\begin{aligned} -\|v_n - x_{n+1}\|^2 \leq &-\left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) \\ &+ 2\langle \chi_n Qd_n, x_{n+1} - d_n\rangle. \end{aligned}$$

This means that

$$\begin{aligned} &-2\langle \phi\chi_n Qd_n, x_{n+1} - d_n\rangle \\ &\leq -\phi\left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) + \phi\|v_n - x_{n+1}\|^2. \end{aligned} \tag{2.10}$$

From (2.5) and (2.10), we obtain

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 \leq &\|v_n - x^\dagger\|^2 - \phi\left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) \\ &- (1 - \phi)\|v_n - x_{n+1}\|^2. \end{aligned} \tag{2.11}$$

Note that

$$-(1 - \phi)\|v_n - x_{n+1}\|^2 \leq -2(1 - \phi)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \ \forall \phi \geq 1.$$

This combining with (2.11) yields

$$\begin{aligned} &\|x_{n+1} - x^\dagger\|^2 \\ &\leq \|v_n - x^\dagger\|^2 - \left(2 - \phi - \frac{\phi\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \ \forall \phi \geq 1. \end{aligned}$$

On the other hand, if $\phi \in (0, 1)$, then

$$\begin{aligned} &\|x_{n+1} - x^\dagger\|^2 \\ &\leq \|v_n - x^\dagger\|^2 - \phi\left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \ \forall \phi \in (0, 1). \end{aligned}$$

This completes the proof. $\square$

Remark 2.3. We have $\phi^* > 0$ for all $n \geq n_0$ by using Lemma 2.1 and the assumptions on μ and ϕ.

Lemma 2.4. ([32, Lemma 3.3]) *Assume that Conditions (C1)–(C3) hold. Let $\{v_n\}$ and $\{d_n\}$ be two sequences formed by Algorithm 2.1. If there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|v_{n_k} - d_{n_k}\| = 0$, then $z \in \mathrm{VI}(\mathcal{C}, Q)$.*

We are now ready to analyze the convergence of Algorithm 2.1.

Theorem 2.5. *Assume that Conditions (C1)–(C3) hold. Then $\{x_n\}$ formed by Algorithm 2.1 converges to $x^\dagger \in \mathrm{VI}(\mathcal{C}, Q)$ in norm, where $\|x^\dagger\| = \min\{\|z\| : z \in \mathrm{VI}(\mathcal{C}, Q)\}$.*

Proof. In view of Lemma 2.2 and Remark 2.3, one obtains

$$\|x_{n+1} - x^\dagger\| \leq \|v_n - x^\dagger\|, \ \forall n \geq n_0. \tag{2.12}$$

It follows the definition of v_n that

$$\begin{aligned}
\|v_n - x^\dagger\| &= \|\left(1-\theta_n\right)\left(x_n + \rho_n\left(x_n - x_{n-1}\right)\right) - x^\dagger\| \\
&= \|\left(1-\theta_n\right)\left(x_n - x^\dagger\right) + \left(1-\theta_n\right)\rho_n\left(x_n - x_{n-1}\right) - \theta_n x^\dagger\| \\
&\leq \left(1-\theta_n\right)\|x_n - x^\dagger\| + \left(1-\theta_n\right)\rho_n\|x_n - x_{n-1}\| + \theta_n\|x^\dagger\| \\
&= \left(1-\theta_n\right)\|x_n - x^\dagger\| + \theta_n\left[\left(1-\theta_n\right)\frac{\rho_n}{\theta_n}\|x_n - x_{n-1}\| + \|x^\dagger\|\right].
\end{aligned} \tag{2.13}$$

From Condition (C3), we obtain $\frac{\rho_n}{\theta_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty}\left[\left(1-\theta_n\right)\frac{\rho_n}{\theta_n}\|x_n - x_{n-1}\| + \|x^\dagger\|\right] = \|x^\dagger\|.$$

Thus, there exists a constant $M_1 > 0$ such that

$$\left(1-\theta_n\right)\frac{\rho_n}{\theta_n}\|x_n - x_{n-1}\| + \|x^\dagger\| \leq M_1, \ \forall n \geq 1. \tag{2.14}$$

Combining (2.13) and (2.14), we deduce

$$\|v_n - x^\dagger\| \leq \left(1-\theta_n\right)\|x_n - x^\dagger\| + \theta_n M_1, \ \forall n \geq 1. \tag{2.15}$$

By using (2.12) and (2.15), we have

$$\begin{aligned}
\|x_{n+1} - x^\dagger\| &\leq (1-\theta_n)\|x_n - x^\dagger\| + \theta_n M_1 \\
&\leq \max\{\|x_n - x^\dagger\|, M_1\}, \ \forall n \geq n_0 \\
&\leq \cdots \leq \max\{\|x_{n_0} - x^\dagger\|, M_1\}.
\end{aligned}$$

This means that $\{x_n\}$ is bounded. Hence $\{v_n\}$ and $\{d_n\}$ are also bounded. In view of (2.15), one has

$$\begin{aligned}
\|v_n - x^\dagger\|^2 &\leq \left[\left(1-\theta_n\right)\|x_n - x^\dagger\| + \theta_n M_1\right]^2 \\
&= \left(1-\theta_n\right)^2\|x_n - x^\dagger\|^2 + \theta_n(2\left(1-\theta_n\right)M_1\|x_n - x^\dagger\| + \theta_n M_1^2) \\
&\leq \|x_n - x^\dagger\|^2 + \theta_n M_2, \ \forall n \geq 1,
\end{aligned} \tag{2.16}$$

where $M_2 := \sup_{n\in\mathbb{N}}\left\{2\left(1-\theta_n\right)M_1\|x_n - x^\dagger\| + \theta_n M_1^2\right\} > 0$. Combining Lemma 2.2 and (2.16), we obtain

$$\begin{aligned}
&\phi^*\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) \\
&\leq \|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 + \theta_n M_2, \ \forall n \geq n_0.
\end{aligned} \tag{2.17}$$

By using (2.12) and the definition of v_n, we have

$$\begin{aligned}\|x_{n+1}-x^\dagger\|^2 &\le \|(1-\theta_n)(x_n-x^\dagger)+(1-\theta_n)\rho_n(x_n-x_{n-1})-\theta_n x^\dagger\|^2\\ &\le \|(1-\theta_n)(x_n-x^\dagger)+(1-\theta_n)\rho_n(x_n-x_{n-1})\|^2+2\theta_n\langle -x^\dagger, v_n-x^\dagger\rangle\\ &\le (1-\theta_n)^2\|x_n-x^\dagger\|^2+2(1-\theta_n)\rho_n\|x_n-x^\dagger\|\|x_n-x_{n-1}\|\\ &\quad+\rho_n^2\|x_n-x_{n-1}\|^2+2\theta_n\langle -x^\dagger, v_n-x_{n+1}\rangle+2\theta_n\langle -x^\dagger, x_{n+1}-x^\dagger\rangle\end{aligned}$$

for all $n \ge n_0$. Thus

$$\begin{aligned}&\|x_{n+1}-x^\dagger\|^2\\ &\le (1-\theta_n)\|x_n-x^\dagger\|^2+\theta_n\Big[2(1-\theta_n)\|x_n-x^\dagger\|\frac{\rho_n}{\theta_n}\|x_n-x_{n-1}\|\\ &\quad+\rho_n\|x_n-x_{n-1}\|\frac{\rho_n}{\theta_n}\|x_n-x_{n-1}\|+2\|x^\dagger\|\|v_n-x_{n+1}\|\\ &\quad+2\langle x^\dagger, x^\dagger-x_{n+1}\rangle\Big],\ \forall n\ge n_0.\end{aligned}\tag{2.18}$$

Finally, we need to show that the sequence $\{\|x_n-x^\dagger\|\}$ converges to zero. We set

$$p_n=\|x_n-x^\dagger\|^2$$

and

$$\begin{aligned}s_n&=2(1-\theta_n)\|x_n-x^\dagger\|\frac{\rho_n}{\theta_n}\|x_n-x_{n-1}\|+\rho_n\|x_n-x_{n-1}\|\frac{\rho_n}{\theta_n}\|x_n-x_{n-1}\|\\ &\quad+2\|x^\dagger\|\|v_n-x_{n+1}\|+2\langle x^\dagger, x^\dagger-x_{n+1}\rangle.\end{aligned}$$

Then the inequality in (2.18) can be written as $p_{n+1}\le(1-\theta_n)p_n+\theta_n s_n$, $\forall n\ge n_0$. It is worth noting that $\{\theta_n\}\subset(0,1)$ and $\sum_{n=1}^\infty \theta_n=\infty$. According to Lemma 1.1, we assume that $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$ such that

$$\liminf_{k\to\infty}\left(p_{n_{k+1}}-p_{n_k}\right)\ge 0.$$

Combining (2.17), $\lim_{n\to\infty}\theta_n=0$, and Remark 2.3, we have

$$\begin{aligned}\phi^*\left(\|v_{n_k}-d_{n_k}\|^2+\|x_{n_k+1}-d_{n_k}\|^2\right) &\le \limsup_{k\to\infty}\theta_{n_k}M_2+\limsup_{k\to\infty}\left(p_{n_k}-p_{n_k+1}\right)\\ &\le -\liminf_{k\to\infty}\left(p_{n_k+1}-p_{n_k}\right)\\ &\le 0.\end{aligned}$$

This together with Remark 2.3 implies

$$\lim_{k\to\infty}\|d_{n_k}-v_{n_k}\|=0 \quad\text{and}\quad \lim_{k\to\infty}\|x_{n_k+1}-d_{n_k}\|=0.$$

Thus we have $\lim_{k\to\infty}\|x_{n_k+1}-v_{n_k}\|=0$, which combining with the boundedness of $\{x_n\}$ yields

$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k+1}\|\|x^\dagger\|=0.\tag{2.19}$$

It follows from the definition of v_n that

$$\begin{aligned}\|x_{n_k}-v_{n_k}\| &= \|(1-\theta_{n_k})\rho_{n_k}(x_{n_k}-x_{n_k-1})-\theta_{n_k}x_{n_k}\|\\ &\le \|(1-\theta_{n_k})\rho_{n_k}(x_{n_k}-x_{n_k-1})\|+\|\theta_{n_k}x_{n_k}\|\\ &= \theta_{n_k}\left[(1-\theta_{n_k})\frac{\rho_{n_k}}{\theta_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\|x_{n_k}\|\right].\end{aligned}$$

Thus we obtain $\lim_{k\to\infty}\|x_{n_k}-v_{n_k}\|=0$ by means of (2.14). Consequently, we conclude that

$$\|x_{n_k+1}-x_{n_k}\|\leq\|x_{n_k+1}-v_{n_k}\|+\|v_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{2.20}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to z as $j\to\infty$. Moreover

$$\limsup_{k\to\infty}\langle x^\dagger,x^\dagger-x_{n_k}\rangle=\lim_{j\to\infty}\langle x^\dagger,x^\dagger-x_{n_{k_j}}\rangle=\langle x^\dagger,x^\dagger-z\rangle. \tag{2.21}$$

We have that $v_{n_k}\rightharpoonup z$ due to the fact that $\lim_{k\to\infty}\|x_{n_k}-v_{n_k}\|=0$. This combining with $\lim_{k\to\infty}\|v_{n_k}-d_{n_k}\|=0$, in the light of Lemma 2.4, yields that $z\in \mathrm{VI}(\mathcal{C},Q)$. By using (2.21) and the definition of $x^\dagger$, we obtain

$$\limsup_{k\to\infty}\langle x^\dagger,x^\dagger-x_{n_k}\rangle=\langle x^\dagger,x^\dagger-z\rangle\leq 0. \tag{2.22}$$

From (2.20) and (2.22), we can show that

$$\limsup_{k\to\infty}\langle x^\dagger,x^\dagger-x_{n_k+1}\rangle\leq 0. \tag{2.23}$$

This combining with $\lim_{n\to\infty}\frac{\rho_n}{\theta_n}\|x_n-x_{n-1}\|=0$ and (2.19) infers that

$$\limsup_{k\to\infty} s_{n_k}\leq 0.$$

Therefore we conclude that $\lim_{n\to\infty}\|x_n-x^\dagger\|=0$. This completes the proof. □

Next, we present a modified version of Algorithm 2.1, which differs from Algorithm 2.1 in calculating the values of d_n and x_{n+1}. More precisely, the scheme is shown in Algorithm 2.2 below.

Algorithm 2.2

Initialization: Take $\rho>0$, $\chi_1>0$, $\phi\in(1/(2-\mu),1/\mu)$, and $\mu\in(0,1)$. Select the sequences $\{\epsilon_n\}$ and $\{\theta_n\}$ to satisfy Condition (C3). Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty}\xi_n<+\infty$. Let $x_0,x_1\in\mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n\geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $v_n=(1-\theta_n)(x_n+\rho_n(x_n-x_{n-1}))$, where ρ_n is defined in (2.1).
Step 2. Compute $d_n=Proj_{\mathcal{C}}(v_n-\phi\chi_n Qv_n)$. If $v_n=d_n$ or $Qd_n=0$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1}=Proj_{H_n}(v_n-\chi_n Qd_n)$, where the half-space H_n is defined by

$$H_n=\{x\in\mathcal{H}:\langle v_n-\phi\chi_n Qv_n-d_n,x-d_n\rangle\leq 0\}, \tag{2.24}$$

and update χ_{n+1} by (2.3).
Set $n:=n+1$ and go to **Step 1**.

Lemma 2.6. *Assume that Condition (C2) holds. Let $\{v_n\}$, $\{d_n\}$, and $\{x_{n+1}\}$ be three sequences created by Algorithm 2.2. Then*

$$\|x_{n+1}-x^\dagger\|^2\leq\|v_n-x^\dagger\|^2-\phi^\dagger\left(\|v_n-d_n\|^2+\|x_{n+1}-d_n\|^2\right),\ \forall x^\dagger\in\mathrm{VI}(\mathcal{C},Q),$$

where $\phi^\dagger=2-\frac{1}{\phi}-\frac{\mu\chi_n}{\chi_{n+1}}$ *if* $\phi\in(1/(2-\mu),1]$ *and* $\phi^\dagger=\frac{1}{\phi}-\frac{\mu\chi_n}{\chi_{n+1}}$ *if* $\phi\in(1,1/\mu)$.

Proof. It follows from (2.4) and (2.5) that

$$\|x_{n+1} - x^\dagger\|^2 \le \|v_n - x^\dagger\|^2 - \|v_n - x_{n+1}\|^2 - 2\langle \chi_n Q d_n, x_{n+1} - d_n\rangle. \tag{2.25}$$

Next, we estimate $2\langle \chi_n Q d_n, x_{n+1} - d_n\rangle$. We observe that

$$-\|v_n - x_{n+1}\|^2 = -\|v_n - d_n\|^2 - \|d_n - x_{n+1}\|^2 + 2\langle v_n - d_n, x_{n+1} - d_n\rangle \tag{2.26}$$

and

$$\begin{aligned} &\langle v_n - d_n, x_{n+1} - d_n\rangle \\ &= \langle v_n - d_n - \phi\chi_n Q v_n + \phi\chi_n Q v_n - \phi\chi_n Q d_n + \phi\chi_n Q d_n, x_{n+1} - d_n\rangle \\ &= \langle v_n - \phi\chi_n Q v_n - d_n, x_{n+1} - d_n\rangle + \phi\chi_n\langle Q v_n - Q d_n, x_{n+1} - d_n\rangle \\ &\quad + \langle \phi\chi_n Q d_n, x_{n+1} - d_n\rangle. \end{aligned} \tag{2.27}$$

From the definition of H_n and $x_{n+1} \in H_n$, one sees that

$$\langle v_n - \phi\chi_n Q v_n - d_n, x_{n+1} - d_n\rangle \le 0. \tag{2.28}$$

By using the definition of χ_{n+1}, we have

$$\langle Q v_n - Q d_n, x_{n+1} - d_n\rangle \le \frac{\mu}{2\chi_{n+1}}\|v_n - d_n\|^2 + \frac{\mu}{2\chi_{n+1}}\|x_{n+1} - d_n\|^2. \tag{2.29}$$

Substituting (2.27), (2.28), and (2.29) into (2.26), we deduce

$$\begin{aligned} -\|v_n - x_{n+1}\|^2 \le &-\left(1 - \frac{\phi\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) \\ &+ 2\phi\langle \chi_n Q d_n, x_{n+1} - d_n\rangle. \end{aligned}$$

This gives

$$\begin{aligned} &-2\langle \chi_n Q d_n, x_{n+1} - d_n\rangle \\ &\le -\left(\frac{1}{\phi} - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) + \frac{1}{\phi}\|v_n - x_{n+1}\|^2. \end{aligned} \tag{2.30}$$

From (2.25) and (2.30), we obtain

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 \le \|v_n - x^\dagger\|^2 &- \left(\frac{1}{\phi} - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right) \\ &- \left(1 - \frac{1}{\phi}\right)\|v_n - x_{n+1}\|^2. \end{aligned} \tag{2.31}$$

Note that

$$-\left(1 - \frac{1}{\phi}\right)\|v_n - x_{n+1}\|^2 \le -2\left(1 - \frac{1}{\phi}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \ \forall \phi \in (0, 1].$$

This together with (2.31) infers that

$$\begin{aligned} &\|x_{n+1} - x^\dagger\|^2 \\ &\le \|v_n - x^\dagger\|^2 - \left(2 - \frac{1}{\phi} - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \ \forall \phi \in (0, 1]. \end{aligned}$$

On the other hand, if $\phi > 1$, then we obtain

$$\begin{aligned} &\|x_{n+1} - x^\dagger\|^2 \\ &\le \|v_n - x^\dagger\|^2 - \left(\frac{1}{\phi} - \frac{\mu\chi_n}{\chi_{n+1}}\right)\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \ \forall \phi > 1. \end{aligned}$$

The proof is completed. □

Remark 2.7. Since $\phi \in (1/(2-\mu), 1/\mu)$ and $\mu \in (0,1)$, we can obtain that $\phi^\dagger > 0$ for all $n \geq n_0$.

Lemma 2.8. *Assume that Conditions (C1)–(C3) hold. Let $\{v_n\}$ and $\{d_n\}$ be two sequences formulated by Algorithm 2.2. If there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|v_{n_k} - d_{n_k}\| = 0$, then $z \in \mathrm{VI}(\mathcal{C}, Q)$.*

Proof. Following [32, Lemma 3.3], we can obtain the desired conclusion immediately. □

Theorem 2.9. *Assume that Conditions (C1)–(C3) hold. Then $\{x_n\}$ created by Algorithm 2.2 converges to $x^\dagger \in \mathrm{VI}(\mathcal{C}, Q)$ in norm, where $\|x^\dagger\| = \min\{\|z\| : z \in \mathrm{VI}(\mathcal{C}, Q)\}$.*

Proof. We can easily prove this theorem by replacing Lemma 2.2 and Lemma 2.4 in the proof of Theorem 2.5 with Lemma 2.6 and Lemma 2.8, respectively. Therefore the proof is omitted. □

2.2. **Two methods for variational inequalities involving non-Lipschitz continuity.** In this subsection, we provide two iterative algorithms with Armijo-type step size criterion for discovering the minimum-norm solutions to pseudomonotone and non-Lipschitz continuous variational inequality problems in real Hilbert spaces. We replace the condition (C2) in Section 2 with the following condition (C4).

(C4) The mapping $Q : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, uniformly continuous on $\mathcal{H}$, and sequentially weakly continuous on $\mathcal{C}$.

Now, our iterative schemes are stated in Algorithms 2.3 and 2.4 below.

Algorithm 2.3

Initialization: Take $\rho > 0$, $\zeta > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$, and $\phi \in (0, 2/(1+\mu))$. Select the sequences $\{\epsilon_n\}$ and $\{\theta_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

Step 1. Compute $v_n = (1-\theta_n)(x_n + \rho_n(x_n - x_{n-1}))$, where ρ_n is defined in (2.1).

Step 2. Compute $d_n = Proj_{\mathcal{C}}(v_n - \chi_n Q v_n)$. If $v_n = d_n$ or $Qd_n = 0$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = Proj_{T_n}(v_n - \phi\chi_n Q d_n)$, where T_n is defined in (2.2), $\chi_n := \zeta\ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\zeta\ell^m \langle Qd_n - Qv_n, d_n - x_{n+1}\rangle \leq \frac{\mu}{2}\left[\|v_n - d_n\|^2 + \|d_n - x_{n+1}\|^2\right]. \tag{2.32}$$

Set $n := n+1$ and go to **Step 1**.

The following lemmas are useful for the convergence analysis of our Algorithms 2.3 and 2.4.

Lemma 2.10. *Assume that Condition (C4) holds. Then the Armijo-like rule (2.32) is well defined.*

Proof. The proof is similar to Lemma 3.1 in [30]. Therefore we omit the details. □

Algorithm 2.4

Initialization: Take $\rho > 0$, $\zeta > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$, and $\phi \in (1/(2-\mu), 1/\mu)$. Select the sequences $\{\epsilon_n\}$ and $\{\theta_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

Step 1. Compute $v_n = (1-\theta_n)(x_n + \rho_n(x_n - x_{n-1}))$, where ρ_n is defined in (2.1).

Step 2. Compute $d_n = Proj_{\mathcal{C}}(v_n - \phi\chi_n Q v_n)$. If $v_n = d_n$ or $Qd_n = 0$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = Proj_{H_n}(v_n - \chi_n Q d_n)$, where H_n is defined in (2.24), $\chi_n := \zeta\ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying (2.32).

Set $n := n+1$ and go to **Step 1**.

Lemma 2.11. *Assume that Condition (C4) holds. Let $\{x_{n+1}\}$ be created by Algorithm 2.3. Then*

$$\|x_{n+1} - x^\dagger\|^2 \leq \|v_n - x^\dagger\|^2 - \phi^{**}\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \forall x^\dagger \in \mathrm{VI}(\mathcal{C}, Q),$$

where $\phi^{**} = 2 - \phi - \phi\mu$ if $\phi \in [1, 2/(1+\mu))$ and $\phi^{**} = \phi - \phi\mu$ if $\phi \in (0,1)$.

Proof. The proof is omitted since it follows the argument of Lemma 2.2. □

Lemma 2.12. *Assume that Condition (C4) holds. Let $\{x_{n+1}\}$ be formed by Algorithm 2.4. Then*

$$\|x_{n+1} - x^\dagger\|^2 \leq \|v_n - x^\dagger\|^2 - \phi^\ddagger\left(\|v_n - d_n\|^2 + \|x_{n+1} - d_n\|^2\right), \forall x^\dagger \in \mathrm{VI}(\mathcal{C}, Q),$$

where $\phi^\ddagger = 2 - \frac{1}{\phi} - \mu$ if $\phi \in (1/(2-\mu), 1]$ and $\phi^\ddagger = \frac{1}{\phi} - \mu$ if $\phi \in (1, 1/\mu)$.

Proof. The proof follows that of Lemma 2.6 and therefore it is omitted. □

Remark 2.13. We can easily see that $\phi^{**} > 0$ for all $n \geq 1$ in Lemma 2.11 and $\phi^\ddagger > 0$ for all $n \geq 1$ in Lemma 2.12 always hold.

Lemma 2.14. *Assume that Conditions (C1), (C3), and (C4) hold. Let $\{v_n\}$ and $\{d_n\}$ be created by Algorithm 2.3 (or Algorithm 2.4). If there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|v_{n_k} - d_{n_k}\| = 0$, then $z \in \mathrm{VI}(\mathcal{C}, Q)$.*

Proof. A simple modification of [2, Lemma 3.2] yields the conclusion and thus it is omitted. □

Theorem 2.15. *Assume that Conditions (C1), (C3), and (C4) hold. Then $\{x_n\}$ created by Algorithm 2.3 converges to $x^\dagger \in \mathrm{VI}(\mathcal{C}, Q)$ in norm, where $\|x^\dagger\| = \min\{\|z\| : z \in \mathrm{VI}(\mathcal{C}, Q)\}$.*

Proof. The proof is similar to that of Theorem 2.5, but we need to apply Lemma 2.11 and Lemma 2.14 in place of Lemma 2.2 and Lemma 2.4. We therefore omit the proof to avoid the redundancy. □

Theorem 2.16. *Assume that Conditions (C1), (C3), and (C4) hold. Then $\{x_n\}$ formed by Algorithm 2.4 converges to $x^\dagger \in \mathrm{VI}(\mathcal{C}, Q)$ in norm, where $\|x^\dagger\| = \min\{\|z\| : z \in \mathrm{VI}(\mathcal{C}, Q)\}$.*

Proof. By following the proof of Theorem 2.5 and replacing Lemma 2.2 and Lemma 2.4 with Lemma 2.12 and Lemma 2.14, respectively, we can easily obtain the desired conclusion. □

Remark 2.17. Note that the algorithms proposed in this paper can obtain strong convergence in real Hilbert spaces while the algorithms in [14, 24, 39] can only obtain weak convergence. On the other hand, our algorithms can be used to solve a wider range of pseudomonotone variational inequality problems, while the algorithms in the literature [24, 39] can only be applied to solve monotone variational inequalities. Therefore, the algorithms proposed in this paper have a broader range of applications.

The novelty of this paper is the introduction of a new parameter ϕ in our algorithms, which leads our methods to use different step size parameters in each iteration to compute the values of the sequences. The advantages of this change are illustrated in detail in the numerical experiments in Section 3. Moreover, when $S = I$ in Algorithm 3.1 of the literature [31], the degenerated algorithm is not the Algorithm 2.1 proposed in this paper. The Algorithm 2.1 suggested in this paper is different from the one in [31]. Specifically, we introduce a variable parameter ϕ in the calculation of x_{n+1}, which leads our algorithms to use two different step sizes in the calculation of d_n and x_{n+1} (when $\phi \neq 1$). Our numerical experiments in Section 3 show that the proposed algorithms achieve faster convergence speed and higher computational efficiency when a suitable value of ϕ is chosen. Thus, the algorithms proposed in this paper greatly improve the original subgradient extragradient algorithm introduced in [5] and enrich many results in the literature.

3. **Numerical experiments.** In this section, we implement two numerical examples occurring in finite and infinite-dimensional spaces to demonstrate the computational efficiency of the proposed algorithms compared to some related results. All the programs are implemented in MATLAB 2018a. In the following numerical experiments we use "CPU" to represent the execution time of all algorithms in seconds.

Example 3.1. Let $\mathcal{H} = L^2([0,1])$ be an infinite-dimensional Hilbert space with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\,\mathrm{d}t,\ \forall x, y \in \mathcal{H}$$

and induced norm

$$\|x\| = \left(\int_0^1 |x(t)|^2\,\mathrm{d}t\right)^{1/2},\ \forall x \in \mathcal{H}.$$

Assume that r and R are two positive real numbers such that

$$\frac{R}{k+1} < \frac{r}{k} < r < R$$

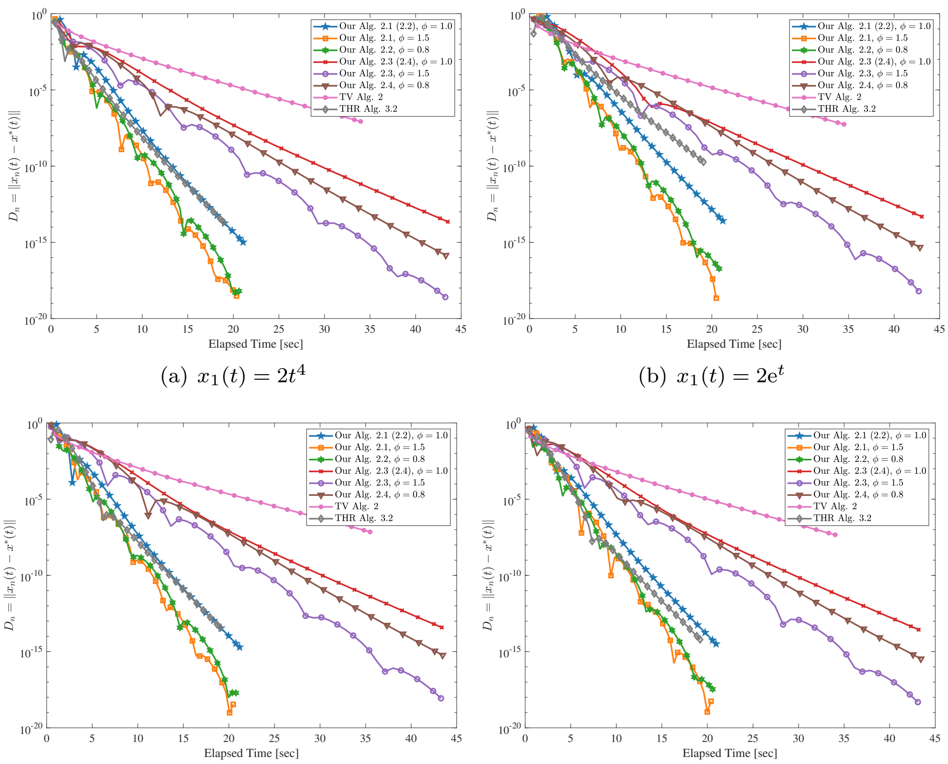
for some $k > 1$. Take the feasible set as $\mathcal{C} = \{x \in \mathcal{H} : \|x\| \leq r\}$. Let operator $Q : \mathcal{H} \to \mathcal{H}$ be given by

$$Q(x) = (R - \|x\|)x,\ \forall x \in \mathcal{H}.$$

It is not hard to check that operator Q is pseudomonotone rather than monotone. For the experiment, we select $R = 1.5$, $r = 1$, and $k = 1.1$. The solution of the (VIP) with Q and $\mathcal{C}$ given above is $x^*(t) = 0$. We compare the proposed Algorithms 2.1–2.4 with the Algorithm 2 introduced by Thong and Vuong [35] (shortly, TV Alg. 2) and the Algorithm 3.2 introduced by Thong et al. [32] (shortly, THR Alg. 3.2). The parameters of all algorithms are set as follows.

- Adopt $\theta_n = \frac{1}{n+1}$, $\rho = 0.3$, $\epsilon_n = \frac{100}{(n+1)^2}$ for the proposed Algorithms 2.1–2.4. Choose $\mu = 0.4$, $\chi_1 = 1$ and $\xi_n = \frac{1}{(n+1)^{1.1}}$ for Algorithms 2.1 and 2.2. Select $\zeta = 1$, $\ell = 0.5$ and $\mu = 0.4$ for Algorithms 2.3 and 2.4.
- Pick $\theta_n = \frac{1}{n+1}$, $\alpha_n = 0.9(1-\theta_n)$, $\zeta = 1$, $\ell = 0.5$ and $\mu = 0.4$ for TV Alg. 2.
- Take $\theta_n = \frac{1}{n+1}$, $\rho = 0.3$, $\epsilon_n = \frac{100}{(n+1)^2}$, $\mu = 0.4$, $\chi_1 = 1$ and $f(x) = 0.1x$ for THR Alg. 3.2.

The maximum number of iterations 50 is used as a common stopping criterion. The numerical results of $D_n = \|x_n(t) - x^*(t)\|$ of all algorithms with four initial points $x_0(t) = x_1(t)$ are reported in Figure 1 and Table 1.



(a) $x_1(t) = 2t^4$ (b) $x_1(t) = 2\mathrm{e}^t$

(c) $x_1(t) = 2\log(t)$ (d) $x_1(t) = 2\cos(t)$

FIGURE 1. Numerical behavior of all algorithms for Example 3.1

Example 3.2. Consider the Hilbert space $\mathcal{H} = \{x = (x_1, x_2, \ldots, x_i, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ equipped with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i d_i,\ \forall x, y \in \mathcal{H}$$

and induced norm

$$\|x\| = \sqrt{\langle x, x\rangle},\ \forall x \in \mathcal{H}.$$

TABLE 1. Numerical results of all algorithms for Example 3.1

Algorithms	$x_1(t)=2t^4$		$x_1(t)=2\mathrm{e}^t$		$x_1(t)=2\log(t)$		$x_1(t)=2\cos(t)$	
	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 2.1 (2.2), $\phi=1.0$	1.03E-15	21.07	2.59E-14	21.20	1.95E-15	21.17	3.19E-15	20.95
Our Alg. 2.1, $\phi=1.5$	3.01E-19	20.35	2.18E-19	20.49	3.44E-19	20.47	5.51E-19	20.40
Our Alg. 2.2, $\phi=0.8$	6.31E-19	20.62	1.88E-17	20.79	1.96E-18	20.77	3.41E-18	20.57
Our Alg. 2.3 (2.4), $\phi=1.0$	2.24E-14	43.54	4.88E-14	43.04	3.86E-14	43.36	2.75E-14	43.18
Our Alg. 2.3, $\phi=1.5$	2.56E-19	43.26	6.24E-19	42.70	8.66E-19	43.27	4.85E-19	43.09
Our Alg. 2.4, $\phi=0.8$	1.42E-16	43.31	4.74E-16	42.83	5.84E-16	43.45	3.37E-16	43.42
TV Alg. 2	8.50E-08	33.97	5.53E-08	34.51	7.13E-08	35.49	4.64E-08	34.02
THR Alg. 3.2	2.15E-14	18.92	1.88E-10	19.09	3.73E-14	19.05	6.25E-15	19.20

Let the feasible set be given by $\mathcal{C}=\left\{x\in\mathcal{H}:|x_i|\leq\frac{1}{i}\right\}$. Define an operator $Q:\mathcal{C}\to\mathcal{H}$ by

$$Qx=\left(\|x\|+\frac{1}{\|x\|+a}\right)x$$

for some $a>0$. It can be verified that mapping Q is pseudomonotone on $\mathcal{H}$, uniformly continuous and sequentially weakly continuous on $\mathcal{C}$, but not Lipschitz continuous on $\mathcal{H}$ (see [33, Example 1]). In the following cases, we set $a=0.5$, and $\mathcal{H}=\mathbb{R}^m$ for different values of m. We compare the proposed Algorithms 2.3 and 2.4 with the Algorithm 3 suggested by Thong et al. [34] (shortly, TSI Alg. 3) and the Algorithm 3.1 introduced by Cai et al. [2] (shortly, CDP Alg. 3.1). The parameters of all algorithms are choose as follows.

- Take $\theta_n=\frac{1}{n+1}$, $\rho=0.4$, $\epsilon_n=\frac{100}{(n+1)^2}$, $\zeta=2$, $\ell=0.5$ and $\mu=0.1$ for Algorithms 2.3 and 2.4.
- Pick $\theta_n=\frac{1}{n+1}$, $\zeta=2$, $\ell=0.5$, $\mu=0.1$ and $f(x)=0.1x$ for TSI Alg. 3 and CDP Alg. 3.1.

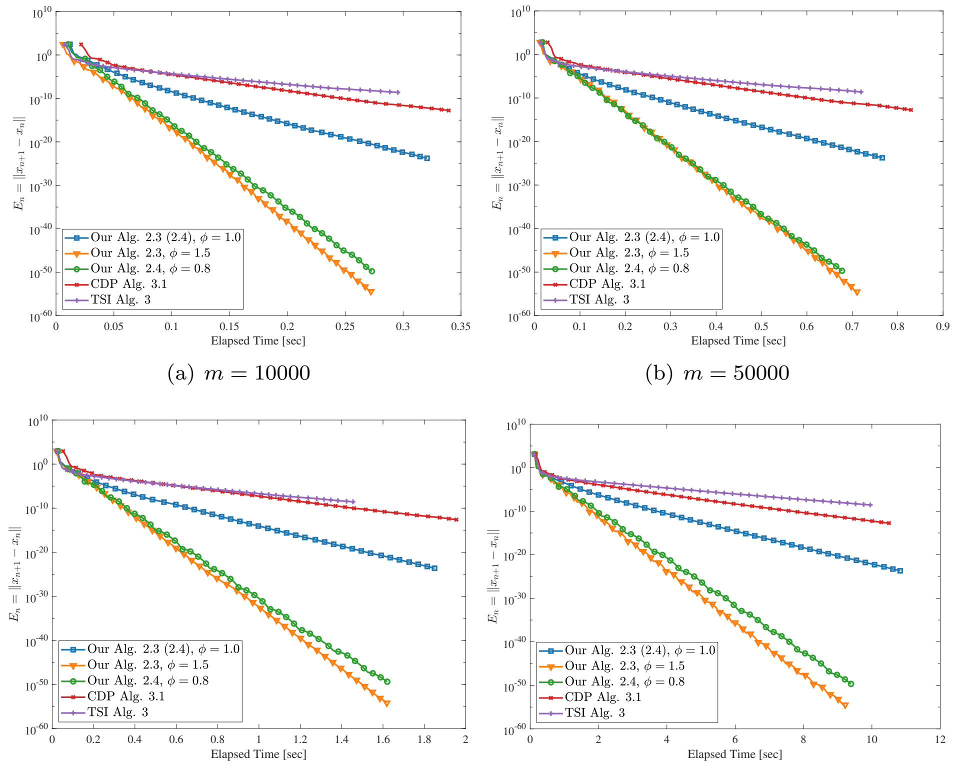
The maximum number of iterations 200 is used as a common stopping criterion and the initial values $x_0=x_1=5rand(m,1)$ are randomly generated by MATLAB. The numerical results of $E_n=\|x_n-x_{n-1}\|$ of all algorithms with four dimensions are given in Figure 2 and Table 2.

TABLE 2. Numerical results of all algorithms for Example 3.2

Algorithms	$m=10000$		$m=50000$		$m=100000$		$m=200000$	
	E_n	CPU	E_n	CPU	E_n	CPU	E_n	CPU
Our Alg. 2.3 (2.4), $\phi=1.0$	1.69E-24	0.32	1.83E-24	0.77	2.31E-24	1.85	2.08E-24	10.83
Our Alg. 2.3, $\phi=1.5$	3.93E-55	0.27	3.12E-55	0.71	6.46E-55	1.62	3.37E-55	9.22
Our Alg. 2.4, $\phi=0.8$	1.69E-50	0.27	1.94E-50	0.68	4.15E-50	1.62	2.11E-50	9.39
CDP Alg. 3.1	1.68E-13	0.34	1.79E-13	0.83	2.47E-13	1.95	1.86E-13	10.50
TSI Alg. 3	2.42E-09	0.30	2.46E-09	0.72	2.76E-09	1.45	2.75E-09	9.95

Remark 3.3. We have the following observations for Examples 3.1 and 3.2.

(1) As shown in Figures 1 and 2, Tables 1 and 2, our algorithms can obtain a smaller iteration error than the schemes in [35, 32, 34, 2] when they reach the same stopping criterion, and this result is independent of the choice of initial values and the size of dimensions. Moreover, notice that our Algorithms 2.1 and 2.2 with a new non-monotonic adaptive step criterion have a faster



(a) $m = 10000$ (b) $m = 50000$

(c) $m = 100000$ (d) $m = 200000$

FIGURE 2. Numerical behavior of all algorithms for Example 3.2

convergence speed than the Armijo-type methods (i.e., our Algorithms 2.3 and 2.4, and the Algorithm 2 introduced by Thong and Vuong [35]). It is also noticed that THR Alg 3.2 [32] in Table 1 requires less running time than the proposed algorithms, which is due to the fact that our proposed algorithms increase the computation time of the inertial step, while THR Alg 3.2 does not need to compute it.

(2) Our four algorithms with a new parameter ϕ have a higher accuracy when choosing the appropriate value of ϕ. Specifically, our Algorithm 2.1 with $\phi = 1.5$ and Algorithm 2.2 with $\phi = 0.8$ have a higher accuracy than when they are with $\phi = 1.0$ (see Figures 1 and 2). The same conclusion is reached for our Algorithms 2.3 and 2.4. Notice that this observation is also not related to the choice of initial values and the size of dimensions (see Tables 1 and 2).

(3) It should be noted that the operator Q in Example 3.1 is pseudomonotone rather than monotone, and many algorithms used in the literature (see, e.g., [20, 17, 26, 38, 8]) for solving monotone variational inequalities will not be available in Example 3.1. Furthermore, the operator Q in Example 3.2 is uniformly continuous rather than Lipschitz continuous, and many schemes in the literature (see, e.g., [10, 35, 32]) that require the operator to satisfy the Lipschitz continuity condition are not applicable in Example 3.2.

(4) Notice that Example 3.1 is an example that occurs in an infinite-dimensional Hilbert space. The four algorithms proposed in this paper obtain strong convergence theorems in real Hilbert spaces, which improves many weak convergence results in the literature (see, e.g., [10, 16, 11, 36, 5, 20, 8, 27, 19]).

Therefore, the iterative algorithms presented in this paper are useful, efficient, and robust.

4. **Conclusions.** To handle variational inequality problems (VIPs) in infinite-dimensional Hilbert spaces, we present four adaptive modified subgradient extragradient methods with inertial effects in this work. The first two approaches are intended to solve Lipschitz continuous and pseudomonotone VIPs. The last two schemes are designed to address non-Lipschitz continuous and pseudomonotone VIPs. Our contributions to this work are outlined below: (1) inertial terms are incorporated into our algorithms to enhance their convergence speed and accuracy; (2) the subgradient extragradient method proposed by Censor et al. [5] is modified by using two different step sizes in each iteration; (3) two novel non-monotonic step size criteria are employed so that the proposed algorithms can work adaptively without the Lipschitz constant of the operator; and (4) the strong convergence of the suggested algorithms is established under some appropriate conditions. Finally, the computational efficiency of the proposed algorithms compared to some known ones is verified by several numerical experiments occurring in finite and infinite-dimensional spaces.

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