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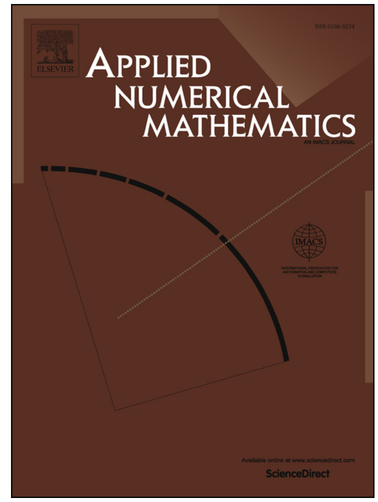

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SELF-ADAPTIVE INERTIAL SINGLE PROJECTION METHODS FOR VARIATIONAL INEQUALITIES INVOLVING NON-LIPSCHITZ AND LIPSCHITZ OPERATORS WITH THEIR APPLICATIONS TO OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, four accelerated subgradient extragradient methods are proposed to solve the variational inequality problem with a pseudo-monotone operator in real Hilbert spaces. These iterative schemes employ two new adaptive stepsize strategies that are significant when the Lipschitz constant of the mapping involved is unknown. Strong convergence theorems for the proposed algorithms are established under the condition that the operators are Lipschitz continuous and non-Lipschitz continuous. Numerical experiments on finite- and infinite-dimensional spaces and applications in optimal control problems are reported to demonstrate the advantages and efficiency of the proposed algorithms over some existing results.

Keywords. Variational inequality; Optimal control problem; Inertial subgradient extragradient method; Pseudomonotone mapping; Uniformly continuous mapping.

Mathematics Subject Classification. 47J20; 47J25; 47J30; 68W10; 65K15.

1. Introduction

The purpose of this paper is to construct several fast iterative algorithms to solve the following variational inequality problem (shortly, VIP) in real Hilbert spaces:

$$\text{find } x^* \in C \text{ such that } \langle Mx^*, z-x^*\rangle \geq 0, \quad \forall z \in C, \tag{VIP}$$

where C is a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and $M:\mathscr{H}\to\mathscr{H}$ is an operator. Throughout the paper, the solution set of the variational inequality problem (VIP) is denoted as $\mathrm{VI}(C,M)$ and is assumed to be non-empty. The theory of variational inequalities plays an important role in many fields and it can be used as a unifying framework for many problems; see, e.g., [1, 2, 3, 4, 5] and the references therein.

In recent years, many researchers have proposed various numerical algorithms to solve the (VIP) due to its non-existent explicit solution. In this paper, we are interested in the projection-based approaches. The earliest and simplest projection-based method for solving (VIP) is the projected gradient method, which generates an iterative sequence x_n starting from the initial point x_0 by means of the following way $x_{n+1} = P_C(x_n - \chi M x_n)$, where χ is a stepsize parameter that satisfies some restrictions and P_C denotes the metric projection from $\mathscr{H}$ onto C (see the definition in Sect. 2). However, the convergence of the method requires that the mapping M be Lipschitz continuous and strongly monotone, and these strong conditions greatly limit the

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applicability of the method. To overcome this difficulty, Korpelevich [6] proposed a new two-step iterative scheme to solve the monotone and Lipschitz continuous (VIP) in finite-dimensional spaces. This iterative scheme is now referred to as the extragradient method (abbreviated as EGM). Notice that the EGM needs to compute the projection onto the feasible set twice in each iteration, which will affect the execution efficiency of the algorithm when the feasible set is a general nonempty closed convex set. Recently, the EGM has attracted numerous interest and extensive research by researchers who have extended the method to infinite-dimensional spaces and accelerated the convergence speed of the scheme by various techniques; see, e.g., [7, 8, 9, 10, 11, 12] and the references therein. It is worth noting that the EGM may fail in cases where the Lipschitz constant of the mapping M is unknown or the mapping M is not Lipschitz continuous. In practical applications, the Lipschitz continuity condition may be difficult to satisfy and estimating a suitable Lipschitz constant may require more computations. To overcome these drawbacks, one research direction is to weaken the conditions of the operator M, and another is to automatically update the step size by adopting some adaptive step size criterion independent of the Lipschitz constant. In the last decades, scholars have made a lot of efforts and achieved some useful results, see, for example, [13, 14, 15, 16, 17, 18, 19, 20] and the references therein. It should be pointed out that the methods introduced in [14, 15, 17, 18, 20] can work without the prior knowledge of the Lipschitz constant of the mapping. We note here that the methods proposed in [14, 15] use an Armijo-type criterion (also known as the linesearch method) to automatically update the step size in each iteration. The disadvantage of the Armijo-type step size is that it may require a large amount of additional calculations because the value of the operator M and the evaluation of the projection may need to be computed many times in each iteration to find a suitable step size. To overcome this drawback, two new step size strategies without any linesearch process have been recently presented by Yang et al. in [17, 18]. These adaptive schemes automatically update the step size in each iteration by performing a simple calculation using some previously known information. However, the computational performance of the iterative algorithms proposed in [17, 18] may be affected due to the stepsize sequences generated by their two adaptive update schemes are non-increasing.

It is known that the condition that the operator M satisfies the Lipschitz continuity is relatively strong. Recently, combining the Armijo-type stepsize criterion and the extragradient method, scholars have proposed some new adaptive iterative methods to solve monotone and non-Lipschitz continuous variational inequality problems; see, e.g., [21, 22, 23, 24]. The weak convergence and strong convergence of these methods are established under some suitable conditions. On the other hand, the pseudo-monotone mappings, as a generalization of the monotone mappings, have been used in variational inequalities and other optimization problems. In recent years, many researchers proposed a large number of numerical methods to solve pseudomonotone variational inequalities; see, for instance, [16, 25, 26]. Moreover, several iterative schemes have been studied and developed to find solutions for variational inequalities with pseudomonotone and non-Lipschitz continuous operators, see, e.g., [27, 28, 29, 30, 31, 32]. Recently, the concept of inertial has received a lot of attention and research from researchers, who have combined the technique into numerical algorithms and proposed a large number of inertial methods. The basic idea of inertial methods, as an acceleration technique, is that the value of each iteration is determined by the combination of some previously known iterations. This small change can significantly improve the computational efficiency of the algorithms without inertial terms. In recent years, many inertial algorithms have been proposed to solve variational inequality problems, split feasibility problems, image processing problems and other optimization problems, see, for instance, [33, 34, 35, 36, 37, 38] and the references therein.

Inspired and motivated by the above work, four new adaptive iterative schemes are proposed in this paper to solve pseudomonotone variational inequalities in real Hilbert spaces. The advantages of these algorithms lie in four aspects: (1) two new step size criteria are used to make them work well without knowing the prior information of the Lipschitz constant of the mapping involved; (2) the operators M of the first two iterative methods only need to satisfy uniform continuity instead of Lipschitz continuity; (3) the strong convergence of the iterative sequences generated by the proposed iterative algorithms is established under some mild conditions; (4) our algorithms embed inertial terms making them converge faster than algorithms without inertial terms. Numerical experiments show that the iterative schemes proposed in this paper directly improve and generalize some of the results in [26, 29, 31, 32].

The paper is organized as follows. We first review some basic definitions and lemmas that need to be used in this paper in Sect. 2. The convergence analysis of the suggested adaptive iterative schemes is discussed and studied in detail in Sect. 3. Some numerical experiments are reported in Sect. 4 to verify the efficiency of the proposed algorithms over the existing ones. Applications of our methods to optimal control problem can be found in Sect. 5. Finally, we conclude the paper with a brief summary in Sect. 6, the last section.

2. Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y, z \in \mathscr{H}$, we have the following inequalities.

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle . \tag{2.1}$$

$$\|\varphi x + (1-\varphi)y\|^2 = \varphi\|x\|^2 + (1-\varphi)\|y\|^2 - \varphi(1-\varphi)\|x-y\|^2, \varphi \in \mathbb{R} . \tag{2.2}$$

$$\begin{aligned}\|\varphi x + \sigma y + \delta z\|^2 = \varphi\|x\|^2 + \sigma\|y\|^2 + \delta\|z\|^2 - \varphi\sigma\|x-y\|^2 - \varphi\delta\|x-z\|^2 \\ - \sigma\delta\|y-z\|^2, \text{where } \varphi, \sigma, \delta \in [0,1] \text{ with } \varphi+\sigma+\delta = 1 .\end{aligned} \tag{2.3}$$

For every point $x \in \mathscr{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $P_C(x) = \operatorname{argmin}\{\|x-y\|, y \in C\}$. P_C is called the metric projection of $\mathscr{H}$ onto C. It is known that P_C is nonexpansive and has the following basic properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall y \in C . \tag{2.4}$$

$$\|P_C(x) - y\|^2 \leq \|x-y\|^2 - \|x - P_C(x)\|^2, \quad \forall y \in C . \tag{2.5}$$

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x-y\rangle, \quad \forall y \in \mathscr{H} . \tag{2.6}$$

A mapping $M : \mathscr{H} \to \mathscr{H}$ is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if

$$\|Mx - My\| \leq L\|x-y\|, \quad \forall x, y \in \mathscr{H} .$$

If $L \in (0,1)$ then mapping M is called *contraction*. In particular, when $L = 1$, mapping M is called *nonexpansive*.

(2) α-*strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathscr{H}.$$

(3) *monotone* if

$$\langle Mx - My, x - y\rangle \geq 0, \quad \forall x, y \in \mathscr{H}.$$

(4) *pseudomonotone* if

$$\langle Mx, y - x\rangle \geq 0 \Longrightarrow \langle My, y - x\rangle \geq 0, \quad \forall x, y \in \mathscr{H}.$$

(5) *sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies $\{Mx_n\}$ converges weakly to Mx.

Remark 2.1. From the above definitions, it is easy to get that (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Notice that its opposite is generally not true. That is, there exist mappings that are pseudomonotone but not monotone; see, for example, [25, Example 6.10], [27, Example 5.4] and [30, Example 1].

We give some projection calculation formulas that need to be used in numerical experiments.

(1) The projection of x onto a half-space $H_{u,v} = \{x : \langle u, x\rangle \leq v\}$ is computed by

$$P_{H_{u,v}}(x) = x - \max\{[\langle u, x\rangle - v]/\|u\|^2, 0\}u.$$

(2) The projection of x onto a box $\mathrm{Box}[a,b] = \{x : a \leq x \leq b\}$ is computed by

$$P_{\mathrm{Box}[a,b]}(x)_i = \min\{b_i, \max\{x_i, a_i\}\}.$$

(3) The projection of x onto a ball $B[p,q] = \{x : \|x - p\| \leq q\}$ is computed by

$$P_{B[p,q]}(x) = p + \frac{q}{\max\{\|x - p\|, q\}}(x - p).$$

The following lemmas are important for the convergence analysis of our main results.

Lemma 2.1 ([39])**.** *Let $x \in \mathscr{H}$ and $\varphi \geq \sigma > 0$. The following inequality holds.*

$$\frac{\|x - P_C(x - \varphi Mx)\|}{\varphi} \leq \frac{\|x - P_C(x - \sigma Mx)\|}{\sigma}.$$

The following Lemma 2.2 was proved in [40, Prop. 2.1] (see also [41, Prop. 2.11]).

Lemma 2.2. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Suppose $M : \mathscr{H}_1 \to \mathscr{H}_2$ is uniformly continuous on bounded subsets of $\mathscr{H}_1$ and B is a bounded subset of $\mathscr{H}_1$. Then, $M(B)$ is bounded.*

Lemma 2.3 ([42])**.** *Assume that C is a closed and convex subset of a real Hilbert space $\mathscr{H}$. Let operator $M : C \to \mathscr{H}$ be continuous and pseudomonotone. Then, x^* is a solution of* (VIP) *if and only if $\langle Mx, x - x^*\rangle \geq 0$, $\forall x \in C$.*

Lemma 2.4 ([43])**.** *Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\varphi_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \varphi_n = \infty$. Assume that*

$$p_{n+1} \leq (1 - \varphi_n)p_n + \varphi_n q_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} q_{n_k} \leq 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty}(p_{n_k+1} - p_{n_k}) \geq 0$, then $\lim_{n\to\infty} p_n = 0$.

3. MAIN RESULTS

In this section, we introduce and investigate four new subgradient extragradient methods with inertial effects to solve pseudomonotone variational inequality problems. These iterative schemes use two new adaptive step size criteria making them work well without the priori information about the Lipschitz constant. The following conditions need to be satisfied in order to obtain the convergence theorems of the suggested algorithms.

(C1) The feasible set C is a nonempty, closed and convex subset of the real Hilbert space $\mathscr{H}$.

(C2) The operator $M:\mathscr{H}\to\mathscr{H}$ is pseudo-monotone, uniformly continuous on $\mathscr{H}$ and the operator $M:\mathscr{H}\to\mathscr{H}$ satisfies the following condition

$$\text{whenever } \{x_n\}\subset C,\ x_n\rightharpoonup z,\ \text{one has } \|Mz\|\le\liminf_{n\to\infty}\|Mx_n\|. \tag{C2-1}$$

(C3) The solution set of the problem (VIP) is nonempty, that is, $\mathrm{VI}(C,M)\neq\emptyset$.

(C4) The mapping $f:C\to C$ is ρ-contractive with constant $\rho\in[0,1)$.

(C5) Let $\{\varepsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\varepsilon_n}{\varphi_n}=0$, where $\{\varphi_n\}\subset(0,1)$ satisfies $\lim_{n\to\infty}\varphi_n=0$ and $\sum_{n=1}^{\infty}\varphi_n=\infty$.

3.1. Inertial viscosity-type subgradient extragradient algorithm.

Now we are ready to state the first adaptive iterative scheme with a new Armijo-type stepsize criterion, which is based on the inertial method, the subgradient extragradient method and the viscosity-type method. More precisely, this iterative scheme is formulated in detail in Algorithm 3.1.

Algorithm 3.1 Inertial viscosity-type subgradient extragradient algorithm

Initialization: Take $\theta>0$, $\delta>0$, $\ell\in(0,1)$, $\eta\in(0,1)$ and let $x_0,x_1\in\mathscr{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n\ge 1)$, calculate x_{n+1} as follows:

Step 1. Compute $q_n=x_n+\theta_n(x_n-x_{n-1})$, where

$$\theta_n=\begin{cases}\min\left\{\dfrac{\varepsilon_n}{\|x_n-x_{n-1}\|},\theta\right\}, & \text{if } x_n\neq x_{n-1},\\ \theta, & \text{otherwise.}\end{cases} \tag{3.1}$$

Step 2. Compute $y_n=P_C(q_n-\chi_n Mq_n)$. If $q_n=y_n$ or $My_n=0$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $z_n=P_{T_n}(q_n-\chi_n My_n)$, where

$$T_n:=\{x\in\mathscr{H}\mid\langle q_n-\chi_n Mq_n-y_n,x-y_n\rangle\le 0\}, \tag{3.2}$$

$\chi_n:=\delta\ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\delta\ell^m\langle My_n-Mq_n,y_n-z_n\rangle\le\frac{\eta}{2}\left[\|q_n-y_n\|^2+\|y_n-z_n\|^2\right]. \tag{3.3}$$

Step 4. Compute $x_{n+1}=\varphi_n f(z_n)+(1-\varphi_n)z_n$.
Set $n:=n+1$ and go to **Step 1**.

Remark 3.1. We have the following observations from Algorithm 3.1.

(i) We note here that inertial calculation criterion (3.1) is easy to implement since the term $\|x_n-x_{n-1}\|$ is known before calculating θ_n. Moreover, it follows from (3.1) and the assumptions on $\{\varphi_n\}$ that

$$\lim_{n\to\infty} \frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\| = 0\,.$$

Indeed, we obtain $\theta_n\|x_n - x_{n-1}\| \le \varepsilon_n, \forall n \ge 1$, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\varphi_n} = 0$ implies that $\lim_{n\to\infty} \frac{\theta_n}{\varphi_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\varepsilon_n}{\varphi_n} = 0$.

(ii) We prove that if $q_n = y_n$ or $My_n = 0$ then $y_n \in \mathrm{VI}(C,M)$. Indeed, from $0 < \chi_n \le \delta$ and Lemma 2.1, we have

$$0 = \frac{\|q_n - y_n\|}{\chi_n} = \frac{\|q_n - P_C(q_n - \chi_n M q_n)\|}{\chi_n} \ge \frac{\|q_n - P_C(q_n - \delta M q_n)\|}{\delta}\,,$$

which indicates that q_n is a solution of (VIP). Thus, $y_n \in \mathrm{VI}(C,M)$. On the other hand, since $y_n \in C$, one can sees that if $My_n = 0$ then $y_n = P_C(y_n - \delta My_n)$, that is $y_n \in \mathrm{VI}(C,M)$.

(iii) Note that the condition (C2-1) is used by many recent work on pseudomonotone variational inequalities, see, e.g., [26, 31, 44]. It is easy to check that Condition (C2-1) is weaker than the sequential weak continuity of the mapping M (see [26, Remark 3.2]).

Lemma 3.1. *Suppose that Conditions (C1)–(C3) hold. The Armijo-like criteria* (3.3) *is well defined. Moreover, we get that* $\chi_n \le \delta$.

Proof. If $q_n \in \mathrm{VI}(C,M)$ then $q_n = P_C(q_n - \delta M q_n)$, which implies that $q_n = y_n$. Thus, $m_n = 0$. If $q_n \notin \mathrm{VI}(C,M)$, we assume that the opposite of (3.3) holds, that is,

$$\begin{aligned}
&\delta\ell^m \left\langle MP_C(q_n - \delta\ell^m M q_n) - Mq_n, P_C(q_n - \delta\ell^m M q_n) - P_{T_n}(q_n - \delta\ell^m M y_n)\right\rangle \\
&> \frac{\eta}{2}\left[\|q_n - P_C(q_n - \delta\ell^m M q_n)\|^2 + \|P_C(q_n - \delta\ell^m M q_n) - P_{T_n}(q_n - \delta\ell^m M y_n)\|^2\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\delta\ell^m \|MP_C(q_n - \delta\ell^m M q_n) - Mq_n\| \cdot \|P_C(q_n - \delta\ell^m M q_n) - P_{T_n}(q_n - \delta\ell^m M y_n)\| \\
&> \eta\|q_n - P_C(q_n - \delta\ell^m M q_n)\| \cdot \|P_C(q_n - \delta\ell^m M q_n) - P_{T_n}(q_n - \delta\ell^m M y_n)\|\,.
\end{aligned}$$

Therefore, we get

$$\|Mq_n - MP_C(q_n - \delta\ell^m M q_n)\| > \eta \frac{\|q_n - P_C(q_n - \delta\ell^m M q_n)\|}{\delta\ell^m}\,. \tag{3.4}$$

We study two cases of q_n. First, suppose that $q_n \in C$, since M and P_C are continuous, we obtain

$$\lim_{m\to\infty} \|q_n - P_C(q_n - \delta\ell^m M q_n)\| = 0\,.$$

From the fact that M is uniformly continuous, one has

$$\lim_{m\to\infty} \|Mq_n - MP_C(q_n - \delta\ell^m M q_n)\| = 0\,,$$

which combining (3.4) yields

$$\lim_{m\to\infty} \frac{\|q_n - P_C(q_n - \delta\ell^m M q_n)\|}{\delta\ell^m} = 0\,. \tag{3.5}$$

Let $z_m = P_C(q_n - \delta\ell^m M q_n)$. According to the characteristics of projection (2.4), one obtains

$$\langle z_m - q_n + \delta\ell^m M q_n, x - z_m\rangle \ge 0, \quad \forall x \in C\,,$$

which means that

$$\langle (z_m - q_n)/\delta\ell^m, x - z_m\rangle + \langle Mq_n, x - z_m\rangle \geq 0, \quad \forall x \in C.$$

This together with (3.5) implies that $\langle Mq_n, x - q_n\rangle \geq 0, \forall x \in C$ when $m \to \infty$. This shows that $q_n \in \mathrm{VI}(C,M)$, which contradicts the hypothesis.

On the other hand, if $q_n \notin C$, then we obtain

$$\lim_{m\to\infty} \|q_n - P_C(q_n - \delta\ell^m Mq_n)\| = \|q_n - P_C(q_n)\| > 0,$$

and

$$\lim_{m\to\infty} \delta\ell^m \|Mq_n - MP_C(q_n - \delta\ell^m Mq_n)\| = 0.$$

Combining these with (3.4), we get an opposite. The proof is completed. □

Remark 3.2. It is worth noting that we did not use the pseudo-monotonicity of mapping M in the proof of Lemma 3.1.

The following two lemmas are very useful for proving the main results of this section.

Lemma 3.2. *Suppose that Conditions (C1)–(C3) hold. Let $\{q_n\}$ and $\{y_n\}$ be two sequences formulated by Algorithm 3.1. If there exists a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ such that $\{q_{n_k}\}$ converges weakly to $z \in \mathscr{H}$ and $\lim_{k\to\infty}\|q_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C,M)$.*

Proof. Since $\{y_n\} \subset C$, $q_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty}\|q_{n_k} - y_{n_k}\| = 0$, one gets $z \in C$. Using $y_{n_k} = P_C(q_{n_k} - \chi_{n_k} Mq_{n_k})$ and the property of projection (2.4), we have

$$\langle q_{n_k} - \chi_{n_k} Mq_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0, \quad \forall x \in C,$$

which can be written as follows

$$\chi_{n_k}^{-1}\langle q_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Mq_{n_k}, y_{n_k} - q_{n_k}\rangle \leq \langle Mq_{n_k}, x - q_{n_k}\rangle, \quad \forall x \in C. \tag{3.6}$$

Next, we prove that $\liminf_{k\to\infty}\langle Mq_{n_k}, x - q_{n_k}\rangle \geq 0$ by considering two possible situations of χ_{n_k}. First, we assume that $\liminf_{k\to\infty}\chi_{n_k} > 0$. Since the sequence $\{q_{n_k}\}$ is bounded and mapping M is uniformly continuous, in the light of Lemma 2.2, one gets that $\{Mq_{n_k}\}$ is bounded. Combining $\|q_{n_k} - y_{n_k}\| \to 0$ and (3.6), we have $\liminf_{k\to\infty}\langle Mq_{n_k}, x - q_{n_k}\rangle \geq 0$. Next, one supposes that $\liminf_{k\to\infty}\chi_{n_k} = 0$. Set $s_{n_k} = P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k})$. Since $\chi_{n_k}\ell^{-1} > \chi_{n_k}$, by means of Lemma 2.1, one obtains $\ell\|q_{n_k} - s_{n_k}\| \leq \|q_{n_k} - y_{n_k}\| \to 0$ as $k \to \infty$. Thus, $s_{n_k} \rightharpoonup z \in C$, which means that the sequence $\{s_{n_k}\}$ is bounded. This together with the uniform continuity of mapping M yields that

$$\lim_{k\to\infty}\|Mq_{n_k} - Ms_{n_k}\| \to 0. \tag{3.7}$$

Using (3.3), ones see that

$$\begin{aligned}&\chi_{n_k}\ell^{-1}\langle MP_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k}) - Mq_{n_k}, P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k}) - z_{n_k}\rangle\\ &> \frac{\eta}{2}\left[\|q_{n_k} - P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k})\|^2 + \|P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_n) - z_{n_k}\|^2\right].\end{aligned}$$

This combining with the Cauchy-Schwartz inequality ($\langle a,b\rangle \leq \|a\|\|b\|$) infers that

$$\begin{aligned}&\chi_{n_k}\ell^{-1}\|MP_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k}) - Mq_{n_k}\| \cdot \|P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k}) - z_{n_k}\|\\ &> \eta\|q_{n_k} - P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k})\| \cdot \|P_C(q_{n_k} - \chi_{n_k}\ell^{-1}Mq_{n_k}) - z_{n_k}\|,\end{aligned}$$

which implies

$$\chi_{n_k}\ell^{-1}\left\|Mq_{n_k}-MP_C\left(q_{n_k}-\chi_{n_k}\ell^{-1}Mq_{n_k}\right)\right\|>\eta\left\|q_{n_k}-P_C\left(q_{n_k}-\chi_{n_k}\ell^{-1}Mq_{n_k}\right)\right\|.$$

Therefore, we get $\frac{1}{\eta}\|Mq_{n_k}-Ms_{n_k}\|>\frac{\|q_{n_k}-s_{n_k}\|}{\chi_{n_k}\ell^{-1}}$. This combining with (3.7) yields that

$$\lim_{k\to\infty}\frac{\|q_{n_k}-s_{n_k}\|}{\chi_{n_k}\ell^{-1}}=0.$$

Moreover, according to the definition of s_{n_k} and the property of projection (2.4), we obtain

$$\left\langle q_{n_k}-\chi_{n_k}\ell^{-1}Mq_{n_k}-s_{n_k},x-s_{n_k}\right\rangle\leq 0,\quad\forall x\in C,$$

which yields

$$\frac{1}{\chi_{n_k}\ell^{-1}}\left\langle q_{n_k}-s_{n_k},x-s_{n_k}\right\rangle+\left\langle Mq_{n_k},s_{n_k}-q_{n_k}\right\rangle\leq\left\langle Mq_{n_k},x-q_{n_k}\right\rangle,\quad\forall x\in C.$$

Taking $k\to\infty$ on the left and right sides of the above inequality, one has

$$\liminf_{k\to\infty}\left\langle Mq_{n_k},x-q_{n_k}\right\rangle\geq 0. \tag{3.8}$$

Hence, we achieved the desired result.

Now, we show that $z\in\mathrm{VI}(C,M)$. Indeed, one sees that

$$\left\langle My_{n_k},x-y_{n_k}\right\rangle=\left\langle My_{n_k}-Mq_{n_k},x-q_{n_k}\right\rangle+\left\langle Mq_{n_k},x-q_{n_k}\right\rangle+\left\langle My_{n_k},q_{n_k}-y_{n_k}\right\rangle. \tag{3.9}$$

Since $\|q_{n_k}-y_{n_k}\|\to 0$ and mapping M is uniformly continuous, we have $\lim_{k\to\infty}\|Mq_{n_k}-My_{n_k}\|=0$. This together with (3.8) and (3.9) yields that $\liminf_{k\to\infty}\left\langle My_{n_k},x-y_{n_k}\right\rangle\geq 0$.

Next, we select a positive number decreasing sequence $\{\zeta_k\}$ such that $\zeta_k\to 0$ as $k\to\infty$. For any k, we represent the smallest positive integer with N_k such that

$$\left\langle My_{n_j},x-y_{n_j}\right\rangle+\zeta_k\geq 0,\quad\forall j\geq N_k. \tag{3.10}$$

It can be easily seen that the sequence $\{N_k\}$ is increasing because $\{\zeta_k\}$ is decreasing. Moreover, for any k, from $\{y_{N_k}\}\subset C$, we can assume $My_{N_k}\neq 0$ (otherwise, y_{N_k} is a solution) and set $u_{N_k}=My_{N_k}/\|My_{N_k}\|^2$. Then, we get $\langle My_{N_k},u_{N_k}\rangle=1,\forall k$. Now, we can deduce from (3.10) that $\langle My_{N_k},x+\zeta_k u_{N_k}-y_{N_k}\rangle\geq 0,\forall k$. According to the fact that M is pseudomonotone on $\mathscr{H}$, we can show that

$$\left\langle M\left(x+\zeta_k u_{N_k}\right),x+\zeta_k u_{N_k}-y_{N_k}\right\rangle\geq 0,$$

which further yields that

$$\left\langle Mx,x-y_{N_k}\right\rangle\geq\left\langle Mx-M\left(x+\zeta_k u_{N_k}\right),x+\zeta_k u_{N_k}-y_{N_k}\right\rangle-\zeta_k\left\langle Mx,u_{N_k}\right\rangle. \tag{3.11}$$

Now, we prove that $\lim_{k\to\infty}\zeta_k u_{N_k}=0$. We get that $y_{N_k}\rightharpoonup z$ since $q_{n_k}\rightharpoonup z$ and $\lim_{k\to\infty}\|q_{n_k}-y_{n_k}\|=0$. One assumes that $Mz\neq 0$ (otherwise, z is a solution). Since the mapping M satisfies the condition (C2-1), we obtain $0<\|Mz\|\leq\liminf_{k\to\infty}\|My_{n_k}\|$. Using $\{y_{N_k}\}\subset\{y_{n_k}\}$ and $\zeta_k\to 0$ as $k\to\infty$, we get

$$0\leq\limsup_{k\to\infty}\|\zeta_k u_{N_k}\|=\limsup_{k\to\infty}\left(\frac{\zeta_k}{\|My_{n_k}\|}\right)\leq\frac{\limsup_{k\to\infty}\zeta_k}{\liminf_{k\to\infty}\|My_{n_k}\|}=0.$$

That is, $\lim_{k\to\infty}\zeta_k u_{N_k}=0$. Thus, from the facts that M is uniformly continuous, sequences $\{y_{N_k}\}$ and $\{u_{N_k}\}$ are bounded and $\lim_{k\to\infty}\zeta_k u_{N_k}=0$, we can conclude from (3.11) that $\liminf_{k\to\infty}\langle Mx, x-y_{N_k}\rangle\geq 0$. Therefore,

$$\langle Mx, x-z\rangle=\lim_{k\to\infty}\langle Mx, x-y_{N_k}\rangle=\liminf_{k\to\infty}\langle Mx, x-y_{N_k}\rangle\geq 0,\quad \forall x\in C\,.$$

Consequently, we observe that $z\in \mathrm{VI}(C,M)$ by Lemma 2.3. This completes the proof. □

Remark 3.3. When M is monotone, it is not necessary to impose the sequential weak continuity (or Condition (C2-1)) of mapping M, see [39]. Notice that Lemma 3.2 clearly holds if the mapping M in Condition (C2) is Lipschitz continuous instead of uniformly continuous. Furthermore, if the step size in Algorithm 3.1 is a sequence of positive numbers, then Lemma 3.2 holds similarly.

Lemma 3.3. *Let $\{z_n\}$ be a sequence generated by Algorithm 3.1 and $p\in\mathrm{VI}(C,M)$. Then*

$$\|z_n-p\|^2\leq\|q_n-p\|^2-(1-\eta)\|q_n-y_n\|^2-(1-\eta)\|z_n-y_n\|^2.$$

Proof. It follows from the definition of z_n and (2.5) that

$$\begin{aligned}\|z_n-p\|^2&=\|P_{T_n}(q_n-\chi_n My_n)-p\|^2\\&\leq\|q_n-\chi_n My_n-p\|^2-\|q_n-\chi_n My_n-z_n\|^2\\&=\|q_n-p\|^2+\chi_n^2\|My_n\|^2-2\chi_n\langle q_n-p, My_n\rangle-\|q_n-z_n\|^2\\&\quad-\chi_n^2\|My_n\|^2+2\chi_n\langle q_n-z_n, My_n\rangle\\&=\|q_n-p\|^2-\|q_n-z_n\|^2+2\chi_n\langle p-z_n, My_n\rangle\\&=\|q_n-p\|^2-\|q_n-z_n\|^2-2\chi_n\langle My_n, y_n-p\rangle+2\chi_n\langle y_n-z_n, My_n\rangle\,.\end{aligned}$$

Since p is the solution of (VIP), we have $\langle Mp, x-p\rangle\geq 0$ for all $x\in C$. By the pseudomontonicity of mapping M, we get $\langle Mx, x-p\rangle\geq 0$ for all $x\in C$. Taking $x=y_n\in C$, one infers that

$$\langle My_n, p-y_n\rangle\leq 0\,.$$

Hence,

$$\begin{aligned}\|z_n-p\|^2&\leq\|q_n-p\|^2-\|q_n-z_n\|^2+2\chi_n\langle y_n-z_n, My_n\rangle\\&=\|q_n-p\|^2-\|q_n-y_n+y_n-z_n\|^2+2\chi_n\langle y_n-z_n, My_n\rangle\\&=\|q_n-p\|^2-\|q_n-y_n\|^2-\|y_n-z_n\|^2-2\langle q_n-y_n, y_n-z_n\rangle\\&\quad+2\chi_n\langle y_n-z_n, My_n\rangle\\&=\|q_n-p\|^2-\|q_n-y_n\|^2-\|y_n-z_n\|^2+2\langle y_n-q_n+\chi_n My_n, y_n-z_n\rangle\\&=\|q_n-p\|^2-\|q_n-y_n\|^2-\|y_n-z_n\|^2+2\langle q_n-\chi_n Mq_n-y_n, z_n-y_n\rangle\\&\quad+2\chi_n\langle My_n-Mq_n, y_n-z_n\rangle\,.\end{aligned}$$

According to $z_n\in T_n$ and the definition of T_n, one obtains

$$\langle q_n-\chi_n Mq_n-y_n, z_n-y_n\rangle\leq 0\,.$$

which infers that

$$\begin{aligned}\|z_n-p\|^2 &\le \|q_n-p\|^2-\|q_n-y_n\|^2-\|y_n-z_n\|^2+2\chi_n\langle My_n-Mq_n,y_n-z_n\rangle\\ &\le \|q_n-p\|^2-\|q_n-y_n\|^2-\|y_n-z_n\|^2+\eta\left[\|q_n-y_n\|^2+\|y_n-z_n\|^2\right]\\ &= \|q_n-p\|^2-(1-\eta)\|q_n-y_n\|^2-(1-\eta)\|y_n-z_n\|^2.\end{aligned}$$

This completes the proof. □

Theorem 3.1. *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to an element $p\in \mathrm{VI}(C,M)$, where $p=P_{\mathrm{VI}(C,M)}f(p)$.*

Proof. We divide the proof into four steps.
Claim 1. The sequence $\{x_n\}$ is bounded. It follows form Lemma 3.3 that

$$\|z_n-p\|\le\|q_n-p\|. \tag{3.12}$$

By the definition of q_n, one sees that

$$\|q_n-p\|\le\|x_n-p\|+\varphi_n\cdot\frac{\theta_n}{\varphi_n}\|x_n-x_{n-1}\|. \tag{3.13}$$

From Remark 3.1 (i), one gets $\frac{\theta_n}{\varphi_n}\|x_n-x_{n-1}\|\to 0$ as $n\to\infty$. Thus, there is a constant $Q_1>0$ that satisfies

$$\frac{\theta_n}{\varphi_n}\|x_n-x_{n-1}\|\le Q_1,\quad \forall n\ge 1. \tag{3.14}$$

Using (3.12), (3.13) and (3.14), we obtain

$$\|z_n-p\|\le\|q_n-p\|\le\|x_n-p\|+\varphi_nQ_1,\quad \forall n\ge 1. \tag{3.15}$$

Using the definition of x_{n+1} and (3.15), we have

$$\begin{aligned}\|x_{n+1}-p\| &\le \varphi_n\|f(z_n)-f(p)\|+\varphi_n\|f(p)-p\|+(1-\varphi_n)\|z_n-p\|\\ &\le (1-(1-\rho)\varphi_n)\|z_n-p\|+\varphi_n\|f(p)-p\|\\ &\le (1-(1-\rho)\varphi_n)\|x_n-p\|+(1-\rho)\varphi_n\frac{Q_1+\|f(p)-p\|}{1-\rho}\\ &\le \max\left\{\|x_n-p\|,\frac{Q_1+\|f(p)-p\|}{1-\rho}\right\}\\ &\le\cdots\le \max\left\{\|x_1-p\|,\frac{Q_1+\|f(p)-p\|}{1-\rho}\right\},\ \forall n\ge 1.\end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded. We get that the sequences $\{q_n\}$, $\{z_n\}$ and $\{f(z_n)\}$ are also bounded.
Claim 2.

$$(1-\eta)\left[\|q_n-y_n\|^2+\|y_n-z_n\|^2\right]\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\varphi_nQ_4$$

for some $Q_4>0$. Indeed, it follows from (3.15) that

$$\begin{aligned}\|q_n-p\|^2 &\le (\|x_n-p\|+\varphi_nQ_1)^2\\ &= \|x_n-p\|^2+\varphi_n(2Q_1\|x_n-p\|+\varphi_nQ_1^2)\\ &\le \|x_n-p\|^2+\varphi_nQ_2\end{aligned} \tag{3.16}$$

for some $Q_2 > 0$. Combining (2.2), (3.16) and Lemma 3.3, we see that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \varphi_n(\|f(z_n)-f(p)\|+\|f(p)-p\|)^2+(1-\varphi_n)\|z_n-p\|^2\\
&\le \varphi_n(\|z_n-p\|+\|f(p)-p\|)^2+(1-\varphi_n)\|z_n-p\|^2\\
&= \varphi_n\|z_n-p\|^2+(1-\varphi_n)\|z_n-p\|^2\\
&\quad +\varphi_n(\|f(p)-p\|^2+2\|z_n-p\|\cdot\|f(p)-p\|)\\
&\le \|z_n-p\|^2+\varphi_n Q_3\\
&\le \|x_n-p\|^2-(1-\eta)\left[\|q_n-y_n\|^2+\|y_n-z_n\|^2\right]+\varphi_n Q_4\,,
\end{aligned}$$

where $Q_3 := \sup_{n\in\mathbb{N}}\{\|z_n-p\|\cdot\|f(p)-p\|\}$ and $Q_4 := Q_2+Q_3$. Therefore, the desired result can be obtained through a simple deformation.

Claim 3.

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le (1-(1-\rho)\varphi_n)\|x_n-p\|^2+(1-\rho)\varphi_n\cdot\Big[&\frac{3Q}{1-\rho}\cdot\frac{\theta_n}{\varphi_n}\|x_n-x_{n-1}\|\\
&+\frac{2}{1-\rho}\langle f(p)-p,x_{n+1}-p\rangle\Big],\quad \forall n\ge n_0
\end{aligned}$$

for some $Q > 0$. Using the definition of q_n, we can show that

$$\begin{aligned}
\|q_n-p\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-p\|^2\\
&\le \|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2\\
&\le \|x_n-p\|^2+3Q\theta_n\|x_n-x_{n-1}\|\,,
\end{aligned}\tag{3.17}$$

where $Q := \sup_{n\in\mathbb{N}}\{\|x_n-p\|,\theta\|x_n-x_{n-1}\|\} > 0$. Using (2.1), (2.2), (3.12) and (3.17), we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\varphi_n(f(z_n)-f(p))+(1-\varphi_n)(z_n-p)+\varphi_n(f(p)-p)\|^2\\
&\le \|\varphi_n(f(z_n)-f(p))+(1-\varphi_n)(z_n-p)\|^2+2\varphi_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \varphi_n\|f(z_n)-f(p)\|^2+(1-\varphi_n)\|z_n-p\|^2+2\varphi_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le (1-(1-\rho)\varphi_n)\|z_n-p\|^2+2\varphi_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le (1-(1-\rho)\varphi_n)\|x_n-p\|^2+(1-\rho)\varphi_n\cdot\Big[\frac{3Q}{1-\rho}\cdot\frac{\theta_n}{\varphi_n}\|x_n-x_{n-1}\|\\
&\quad+\frac{2}{1-\rho}\langle f(p)-p,x_{n+1}-p\rangle\Big],\quad \forall n\ge 1\,.
\end{aligned}$$

Claim 4. The sequence $\{\|x_n-p\|\}$ converges to zero. From Lemma 2.4 and Remark 3.1 (i), it remains to show that $\limsup_{k\to\infty}\langle f(p)-p,x_{n_k+1}-p\rangle \le 0$ for any subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying $\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\ge 0$.

For this purpose, we assume that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\ge 0\,.$$

Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right)\\
&=\liminf_{k\to\infty}\left[(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)\right]\ge 0\,.
\end{aligned}$$

It follows from Claim 2 and the assumptions on $\{\varphi_n\}$ that

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\eta)\left[\|q_{n_k}-y_{n_k}\|^2+\|y_{n_k}-z_{n_k}\|^2\right]\\
&\leq \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right]+\limsup_{k\to\infty}\varphi_{n_k}Q_4\\
&=-\liminf_{k\to\infty}\left[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right]\\
&\leq 0\,,
\end{aligned}$$

which yields that $\lim_{k\to\infty}\|q_{n_k}-y_{n_k}\|=0$ and $\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|=0$. Hence, we get $\lim_{k\to\infty}\|z_{n_k}-q_{n_k}\|=0$. Moreover, using Remark 3.1 (i) and the assumptions on $\{\varphi_n\}$, we have

$$\|x_{n_k}-q_{n_k}\|=\varphi_{n_k}\cdot\frac{\theta_{n_k}}{\varphi_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0,\quad \text{as } k\to\infty\,,$$

and

$$\|x_{n_k+1}-z_{n_k}\|=\varphi_{n_k}\|z_{n_k}-f(z_{n_k})\|\to 0,\quad \text{as } k\to\infty\,.$$

Therefore, we conclude that

$$\|x_{n_k+1}-x_{n_k}\|\leq\|x_{n_k+1}-z_{n_k}\|+\|z_{n_k}-q_{n_k}\|+\|q_{n_k}-x_{n_k}\|\to 0,\ \text{ as } k\to\infty\,. \tag{3.18}$$

Since the sequence $\{x_{n_k}\}$ is bounded, there is a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that satisfies $x_{n_{k_j}}\rightharpoonup q$. Furthermore,

$$\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\lim_{j\to\infty}\langle f(p)-p,x_{n_{k_j}}-p\rangle=\langle f(p)-p,q-p\rangle\,. \tag{3.19}$$

We get that $q_{n_k}\rightharpoonup q$ since $\|x_{n_k}-q_{n_k}\|\to 0$. This together with $\lim_{k\to\infty}\|q_{n_k}-y_{n_k}\|=0$ and Lemma 3.2 yields that $q\in\mathrm{VI}(C,M)$. By the definition of $p=P_{\mathrm{VI}(C,M)}f(p)$, (2.4) and (3.19), we infer that

$$\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\langle f(p)-p,q-p\rangle\leq 0\,. \tag{3.20}$$

Combining (3.18) and (3.20), we see that

$$\limsup_{k\to\infty}\langle f(p)-p,x_{n_k+1}-p\rangle\leq\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle\leq 0\,. \tag{3.21}$$

Thus, from Remark 3.1 (i), (3.21), Claim 3 and Lemma 2.4, we conclude that $x_n\to p$ as $n\to\infty$. The proof of the Theorem 3.1 is now complete. □

3.2. **Inertial Mann-type subgradient extragradient algorithm.** In this subsection, we present a modified version of Algorithm 3.1 that uses the Mann-type approach to obtain strong convergence of the iterative sequence. Suppose that the following condition (D1) holds in order to study the convergence of the proposed algorithm.

(D1) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\epsilon_n}{\varphi_n}=0$, where ϵ_n is defined in (3.1) and $\{\varphi_n\}\subset(0,1)$ satisfies $\lim_{n\to\infty}\varphi_n=0$ and $\sum_{n=1}^{\infty}\varphi_n=\infty$. Let $\{\sigma_n\}\subset(a,b)\subset(0,1-\varphi_n)$ for some $a>0,b>0$.

The inertial Mann-type subgradient extragradient algorithm for solving (VIP) is stated in Algorithm 3.2.

Theorem 3.2. *Assume that Conditions (C1)–(C3) and (D1) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $p\in\mathrm{VI}(C,M)$, where $\|p\|=\min\{\|z\|:z\in\mathrm{VI}(C,M)\}$.*

Algorithm 3.2 Inertial Mann-type subgradient extragradient algorithm

Initialization: Take $\theta > 0$, $\delta > 0$, $\ell \in (0,1)$, $\eta \in (0,1)$ and let $x_0, x_1 \in \mathscr{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} q_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(q_n - \chi_n M q_n), \\ z_n = P_{T_n}(q_n - \chi_n M y_n), \\ x_{n+1} = (1 - \varphi_n - \sigma_n) q_n + \sigma_n z_n, \end{cases}$$

where θ_n, T_n and χ_n are defined in (3.1), (3.2) and (3.3), respectively.

Proof. We divide the proof into four steps.

Claim 1. The sequence $\{x_n\}$ is bounded. As stated in Claim 1 in Theorem 3.1, inequalities (3.12)–(3.15) also hold. By the definition of x_{n+1} and (3.15), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \varphi_n - \sigma_n)(q_n - p) + \sigma_n(z_n - p) - \varphi_n p\| \\ &\leq (1 - \varphi_n - \sigma_n)\|q_n - p\| + \sigma_n\|z_n - p\| + \varphi_n\|p\| \\ &\leq (1 - \varphi_n)\|q_n - p\| + \varphi_n\|p\| \\ &\leq (1 - \varphi_n)\|x_n - p\| + \varphi_n(\|p\| + Q_1) \\ &\leq \max\{\|x_n - p\|, \|p\| + Q_1\} \\ &\leq \cdots \leq \max\{\|x_1 - p\|, \|p\| + Q_1\}. \end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded. So the sequences $\{z_n\}$ and $\{q_n\}$ are also bounded.

Claim 2.

$$\begin{aligned} &\sigma_n(1-\eta)\left[\|q_n - y_n\|^2 + \|y_n - z_n\|^2\right] \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \varphi_n(\|p\|^2 + Q_2). \end{aligned}$$

From the definition of x_{n+1}, (2.3), (3.16) and Lemma 3.5, one obtains

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \varphi_n - \sigma_n)(q_n - p) + \sigma_n(z_n - p) + \varphi_n(-p)\|^2 \\ &\leq (1 - \varphi_n - \sigma_n)\|q_n - p\|^2 + \sigma_n\|z_n - p\|^2 + \varphi_n\|p\|^2 \\ &\leq (1 - \varphi_n - \sigma_n)\|q_n - p\|^2 + \sigma_n\|q_n - p\|^2 + \varphi_n\|p\|^2 \\ &\quad - \sigma_n(1-\eta)\left[\|q_n - y_n\|^2 + \|y_n - z_n\|^2\right] \\ &\leq \|x_n - p\|^2 - \sigma_n(1-\eta)\left[\|q_n - y_n\|^2 + \|y_n - z_n\|^2\right] + \varphi_n(\|p\|^2 + Q_2). \end{aligned}$$

The desired result can be obtained through a simple deformation.

Claim 3.

$$\begin{aligned} \|x_{n+1} - p\|^2 \leq (1 - \varphi_n)\|x_n - p\|^2 + \varphi_n \Big[& 2\sigma_n\|q_n - z_n\|\|x_{n+1} - p\| \\ & + 2\langle p, p - x_{n+1}\rangle + \frac{3Q\theta_n}{\varphi_n}\|x_n - x_{n-1}\| \Big], \ \forall n \geq 1. \end{aligned}$$

Setting $t_n = (1 - \sigma_n) q_n + \sigma_n z_n$, one has

$$\|t_n - q_n\| = \sigma_n\|q_n - z_n\|. \tag{3.22}$$

It follows from (3.12) that

$$\begin{aligned}\|t_n-p\| &= \|(1-\sigma_n)(q_n-p)+\sigma_n(z_n-p)\| \\ &\le (1-\sigma_n)\|q_n-p\|+\sigma_n\|q_n-p\| \\ &= \|q_n-p\|, \quad \forall n\ge 1.\end{aligned} \tag{3.23}$$

From (2.1), (3.17), (3.22) and (3.23), for all $n\ge 1$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\sigma_n)q_n+\sigma_n z_n-\varphi_n q_n-p\|^2 \\ &= \|(1-\varphi_n)(t_n-p)-\varphi_n(q_n-t_n)-\varphi_n p\|^2 \\ &\le (1-\varphi_n)^2\|t_n-p\|^2-2\varphi_n\langle q_n-t_n+p, x_{n+1}-p\rangle \\ &= (1-\varphi_n)^2\|t_n-p\|^2+2\varphi_n\langle q_n-t_n, p-x_{n+1}\rangle+2\varphi_n\langle p, p-x_{n+1}\rangle \\ &\le (1-\varphi_n)\|t_n-p\|^2+2\varphi_n\|q_n-t_n\|\|x_{n+1}-p\|+2\varphi_n\langle p, p-x_{n+1}\rangle \\ &\le (1-\varphi_n)\|x_n-p\|^2+\varphi_n\Big[2\sigma_n\|q_n-z_n\|\|x_{n+1}-p\| \\ &\quad +2\langle p, p-x_{n+1}\rangle+\frac{3Q\theta_n}{\varphi_n}\|x_n-x_{n-1}\|\Big], \ \forall n\ge 1.\end{aligned}$$

Claim 4. The sequence $\{\|x_n-p\|\}$ converges to zero. We assume that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that $\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\ge 0$. By Claim 2 and Condition (D1), we have

$$\begin{aligned}&\limsup_{k\to\infty}\sigma_{n_k}(1-\eta)\left[\|q_{n_k}-y_{n_k}\|^2+\|y_{n_k}-z_{n_k}\|^2\right] \\ &\le \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right]+\limsup_{k\to\infty}\varphi_{n_k}(\|p\|^2+Q_2) \\ &\le 0.\end{aligned}$$

This means that $\lim_{k\to\infty}\|q_{n_k}-y_{n_k}\|=0$ and $\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|=0$. Hence, we get $\lim_{k\to\infty}\|z_{n_k}-q_{n_k}\|=0$. This together with the boundedness of $\{x_n\}$ yields that

$$\lim_{k\to\infty}\sigma_{n_k}\|q_{n_k}-z_{n_k}\|\|x_{n_k+1}-p\|=0. \tag{3.24}$$

From Remark 3.1 (i) and Condition (D1), one gets $\lim_{k\to\infty}\|x_{n_k}-q_{n_k}\|=0$. Moreover, we have

$$\left\|x_{n_k+1}-q_{n_k}\right\|\le\varphi_{n_k}\|q_{n_k}\|+\sigma_{n_k}\|q_{n_k}-z_{n_k}\|\to 0, \quad \text{as } k\to\infty.$$

This combining with $\lim_{k\to\infty}\|x_{n_k}-q_{n_k}\|=0$ implies that $\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0$. Since the sequence $\{x_{n_k}\}$ is bounded, there is a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that satisfies $x_{n_{k_j}}\rightharpoonup q$. Furthermore,

$$\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle=\lim_{j\to\infty}\langle p, p-x_{n_{k_j}}\rangle=\langle p, p-q\rangle.$$

We get $q_{n_k}\rightharpoonup q$ since $\|x_{n_k}-q_{n_k}\|\to 0$. This together with $\lim_{k\to\infty}\|q_{n_k}-y_{n_k}\|=0$ and Lemma 3.2 yields that $q\in\mathrm{VI}(C,M)$. By the definition of $p=P_{\mathrm{VI}(C,M)}0$ and (2.4), we deduce that

$$\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle=\lim_{j\to\infty}\langle p, p-x_{n_{k_j}}\rangle=\langle p, p-q\rangle\le 0.$$

From $\|x_{n_k+1}-x_{n_k}\|\to 0$, we get

$$\limsup_{k\to\infty}\left\langle p, p-x_{n_k+1}\right\rangle\le 0. \tag{3.25}$$

Therefore, combining (3.24), (3.25), Remark 3.1 (i) and Claim 3, in the light of Lemma 2.4, we conclude that $x_n \to p$ as $n \to \infty$. This completes the proof of Theorem 3.2. $\square$

3.3. **Self-adaptive inertial viscosity-type subgradient extragradient algorithm.** In this subsection, a new adaptive iterative scheme without any linesearch process is introduced to solve the variational inequality problem with a pseudomonotone and Lipschitz continuous mapping. The strong convergence theorem of the suggested method is established without the prior knowledge of the Lipschitz constant of the mapping associated. Now, we replace the condition (C2) in Algorithms 3.1 and 3.2 with the following condition (E1) and then give the new adaptive Algorithm 3.3.

(E1) The mapping $M : \mathscr{H} \to \mathscr{H}$ is L-Lipschitz continuous, pseudomonotone on $\mathscr{H}$ and the mapping $M : \mathscr{H} \to \mathscr{H}$ satisfies the condition (C2-1).

The form of Algorithm 3.3 is shown below.

Algorithm 3.3 Self-adaptive inertial viscosity-type subgradient extragradient algorithm

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Initialization: Take $\theta > 0$, $\chi_1 > 0$, $\eta \in (0,1)$. Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathscr{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

Step 1. Compute $q_n = x_n + \theta_n(x_n - x_{n-1})$, where θ_n is defined in (3.1).

Step 2. Compute $y_n = P_C(q_n - \chi_n M q_n)$. If $q_n = y_n$ or $My_n = 0$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $z_n = P_{T_n}(q_n - \chi_n M y_n)$, where T_n is defined in (3.2).

Step 4. Compute $x_{n+1} = \varphi_n f(z_n) + (1 - \varphi_n) z_n$, and update

$$\chi_{n+1} = \begin{cases} \min\left\{\eta \dfrac{\|q_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle Mq_n - My_n, z_n - y_n\rangle}, \chi_n + \xi_n\right\}, & \text{if } \langle Mq_n - My_n, z_n - y_n\rangle > 0, \\ \chi_n + \xi_n, & \text{otherwise.} \end{cases} \tag{3.26}$$

Notice that the stepsize sequence generated in Algorithm 3.3 is non-monotonic due to the use of the new update method (3.26). Indeed, we have the following Lemma 3.4 which is crucial for the convergence analysis of the algorithm.

Lemma 3.4. *Suppose that Condition (E1) holds. Then the sequence $\{\chi_n\}$ generated by* (3.26) *is well defined and* $\lim_{n\to\infty} \chi_n = \chi$ *and* $\chi \in \left[\min\{\eta/L, \chi_1\}, \chi_1 + \Xi\right]$, *where* $\Xi = \sum_{n=1}^{\infty} \xi_n$.

Proof. Since mapping M is L-Lipschitz continuous, one gets $\|Mq_n - My_n\| \leq L\|q_n - y_n\|$. If $\langle Mq_n - My_n, z_n - y_n\rangle > 0$, then

$$\eta \frac{\|q_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle Mq_n - My_n, z_n - y_n\rangle} \geq \eta \frac{\|q_n - y_n\| \, \|z_n - y_n\|}{\|Mq_n - My_n\| \, \|z_n - y_n\|} = \eta \frac{\|q_n - y_n\|}{\|Mq_n - My_n\|} \geq \frac{\eta}{L}.$$

Thus, $\chi_n \geq \min\{\eta/L, \chi_1\}$. It follows from the definition of χ_{n+1} that $\chi_{n+1} \leq \chi_1 + \Xi$, where $\Xi = \sum_{n=1}^{\infty} \xi_n$. Thus, the sequence $\{\chi_n\}$ defined in (3.26) is bounded and $\chi_n \in [\min\{\eta/L, \chi_1\}, \chi_1 + \Xi]$. Let $(\chi_{n+1} - \chi_n)^+ = \max\{0, \chi_{n+1} - \chi_n\}$ and $(\chi_{n+1} - \chi_n)^- = \max\{0, -(\chi_{n+1} - \chi_n)\}$. By the definition of χ_n, one obtains $\sum_{n=1}^{\infty} (\chi_{n+1} - \chi_n)^+ \leq \sum_{n=1}^{\infty} \xi_n < +\infty$, which implies that the series $\sum_{n=1}^{\infty} (\chi_{n+1} - \chi_n)^+$ is convergent. Next we show the convergence of the series

$\sum_{n=1}^{\infty}(\chi_{n+1}-\chi_n)^-$. Assume that $\sum_{n=1}^{\infty}(\chi_{n+1}-\chi_n)^-=+\infty$. Note that $\chi_{n+1}-\chi_n=(\chi_{n+1}-\chi_n)^+-(\chi_{n+1}-\chi_n)^-$. Therefore,

$$\chi_{m+1}-\chi_1=\sum_{n=1}^{m}(\chi_{n+1}-\chi_n)=\sum_{n=1}^{m}(\chi_{n+1}-\chi_n)^+-\sum_{n=1}^{m}(\chi_{n+1}-\chi_n)^- .$$

Taking $m\to+\infty$ in the above equation, we get $\lim_{m\to+\infty}\chi_m\to-\infty$. That is a contradiction. Hence, we deduce that $\lim_{n\to\infty}\chi_n=\chi$ and $\chi\in[\min\{\eta/L,\chi_1\},\chi_1+\Xi]$. □

Remark 3.4. We remark here the following observations for Algorithm 3.3.

- The idea of the step size χ_n defined in (3.26) is derived from [45]. It is worth noting that the step size χ_n generated in Algorithm 3.3 is allowed to increase when the iteration increases. Therefore, the use of this type of step size reduces the dependence on the initial step size χ_1. On the other hand, because of $\sum_{n=1}^{\infty}\xi_n<+\infty$, which implies that $\lim_{n\to\infty}\xi_n=0$. Thus, χ_n may not increase when n is large enough. In fact, the stepsize sequence $\{\chi_n\}$ must eventually decrease to achieve convergence, due to the fact that the convergence is caused by $\frac{\chi_n}{\chi_{n+1}}\to1\,(n\to+\infty)$. If $\xi_n=0$, then the step size χ_n in Algorithm 3.3 is similar to the approaches in [17, 18, 26, 36, 38].
- Note that Lemma 3.2 in Algorithm 3.1 still holds when the Armijo-like criterion (3.3) is replaced by the adaptive stepsize (3.26).
- It should be noted that the stepsize update method and convergence conditions in Algorithm 3.1 and Algorithm 3.3 are different. Specifically, Algorithm 3.3 replaces the Arimijo-type criterion (3.3) and convergence condition (C2) in Algorithm 3.1 with update way (3.26) and convergence condition (E1), respectively. Each of these two iterative schemes has advantages and disadvantages, which will be discussed in detail in the following numerical experiments (cf. Section 4 and Section 5).

The following Lemma 3.5 plays an important role in the convergence analysis of Algorithm 3.3 and it can be easily obtained by using the same statement as Lemma 3.2 in [38].

Lemma 3.5 ([38]). *Assume that Conditions (C1), (C3) and (E1) hold. Let $\{z_n\}$ be a sequence generated by Algorithm 3.3. Then, for all $p\in\mathrm{VI}(C,M)$,*

$$\|z_n-p\|^2\le\|q_n-p\|^2-\Big(1-\eta\frac{\chi_n}{\chi_{n+1}}\Big)\|y_n-q_n\|^2-\Big(1-\eta\frac{\chi_n}{\chi_{n+1}}\Big)\|z_n-y_n\|^2 .$$

Theorem 3.3. *Assume that Conditions (C1), (C3), (C4), (C5) and (E1) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges strongly to an element $p\in\mathrm{VI}(C,M)$, where $p=P_{\mathrm{VI}(C,M)}f(p)$.*

Proof. According to Lemma 3.4, it follows that $\lim_{n\to\infty}\big(1-\eta\frac{\chi_n}{\chi_{n+1}}\big)=1-\eta>0$. Hence, there exists $n_0\in\mathbb{N}$ such that

$$1-\eta\frac{\chi_n}{\chi_{n+1}}>0,\quad \forall n\ge n_0 .$$

This combining with Lemma 3.5 yields that $\|z_n-p\|\le\|q_n-p\|$, $\forall n\ge n_0$. Thus, we get

$$\|z_n-p\|\le\|q_n-p\|\le\|x_n-p\|+\varphi_nQ_1,\quad \forall n\ge n_0 .$$

The conclusion of the theorem can be easily obtained by using some statements similar to Theorem 3.1. We leave it to the reader to verify. □

3.4. **Self-adaptive inertial Mann-type subgradient extragradient algorithm.** In this subsection, we replace the Armijo-type criterion (3.3) in Algorithm 3.2 with the new stepsize method (3.26) and then introduce a new numerical algorithm, the last iterative scheme stated in this paper. We now focus our attention to the description of Algorithm 3.4.

Algorithm 3.4 Self-adaptive inertial Mann-type subgradient extragradient algorithm

Initialization: Take $\theta > 0$, $\chi_1 > 0$, $\eta \in (0,1)$. Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathscr{H}$ be arbitrary.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

$$\begin{cases} q_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(q_n - \chi_n M q_n), \\ z_n = P_{T_n}(q_n - \chi_n M y_n), \\ x_{n+1} = (1 - \varphi_n - \sigma_n) q_n + \sigma_n z_n, \end{cases}$$

where θ_n, T_n and χ_n are defined in (3.1), (3.2) and (3.26), respectively.

Theorem 3.4. *Assume that Conditions (C1), (C3), (D1) and (E1) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.4 converges strongly to $p \in \mathrm{VI}(C,M)$, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C,M)\}$.*

Proof. The proof of the theorem is very similar to the proof of Theorem 3.2. We omit it here. □

Remark 3.5. We have the following observations for the offered Algorithms 3.1–3.4.

(i) Notice that the mapping M in Algorithm 3.1 and Algorithm 3.2 is pseudomonotone and uniformly continuous, while it is pseudomonotone and Lipschitz continuous in Algorithm 3.3 and Algorithm 3.4 (the Lipschitz constant does not need to be known). Moreover, the operator M in the presented algorithms only need to satisfy condition (C2-1) and not the sequence weak continuity. Therefore, the convergence conditions of the algorithms obtained in this paper are weaker than those in [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32], which makes them more widespread and useful in practical applications.

(ii) It should be highlighted that we use a new Armijo-type stepsize criterion in Algorithms 3.1 and 3.2 that exploits the information of z_n, which is actually influenced by the recent work of Cai, Dong and Peng [32]. In addition, Algorithms 3.3 and 3.4 embed a new non-monotonic stepsize criterion that overcomes the drawback of non-increasing stepsize sequences generated by the algorithms suggested in [17, 18, 26, 36, 38]. The use of these two new adaptive stepsize criteria allows the methods introduced in this paper to converge faster than some existing algorithms in the literature (see Sect. 4 and Sect. 5).

(iii) The proposed Algorithms 3.3 and 3.4 require only one evaluation of the projection on the feasible set in each iteration. However, the stated iterative schemes 3.1 and 3.2 need to compute the projection on the feasible set multiple times at each iteration because they use an Armijo-type criterion.

(iv) Our four iterative schemes are embedded with inertial effects, which allows them to accelerate the convergence speed of the algorithms. Furthermore, it is important to note that the inertial update approach (3.1) is easy to implement due to the fact that the term $\|x_n - x_{n-1}\|$ is known before updating θ_n.

(v) The algorithms offered in this paper obtain strong convergence theorems in real Hilbert spaces by applying the Mann-type method and the viscosity-type method. However, the strongly convergent methods presented in [19] are obtained by projection-type methods. It is known that projection-type methods are not easy to implement and converge slowly in infinite-dimensional spaces. Therefore, the iterative schemes provided in this paper are more useful.

4. NUMERICAL EXAMPLES

In this section, we perform some computational tests that occur in finite- and infinite-dimensional spaces, and compare the offered iterative schemes with several previously known strongly convergent algorithms, which including the Algorithm 3.1 introduced by Cai, Dong and Peng [32] (shortly, CDP Alg. 3.1), the Algorithm 3.1 presented by Thong et al. [26] (shortly, TYCR Alg. 3.1), the Algorithm 3 suggested by Thong, Shehu and Iyiola [29] (shortly, TSI Alg. 3) and the Algorithm 4 proposed by Reich et al. [31] (shortly, RTDLD Alg. 4). All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

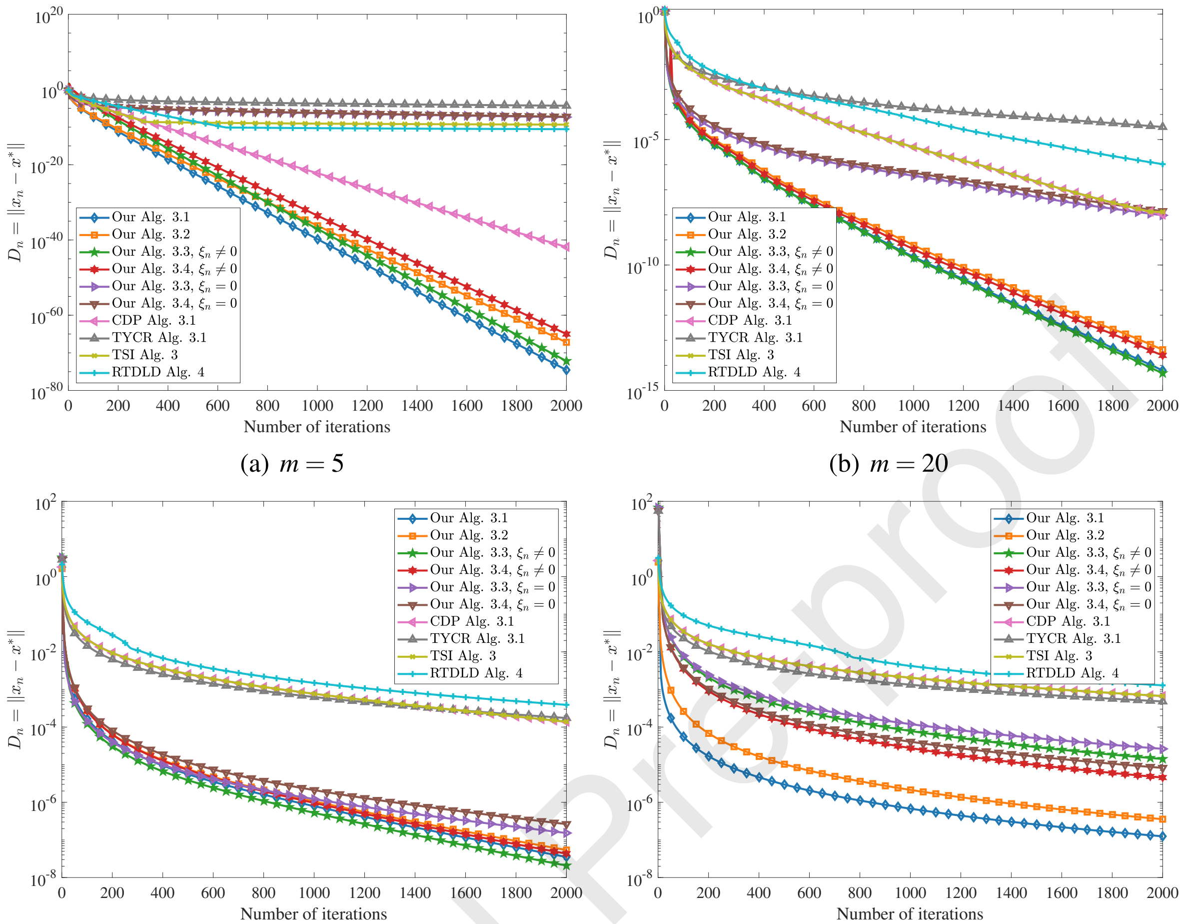
Example 4.1. Consider the form of linear operator $M:\mathbb{R}^m \to \mathbb{R}^m$ ($m=5,20,50,100$) as follows: $M(x)=Gx+g$, where $g\in\mathbb{R}^m$ and $G=BB^{\mathsf{T}}+S+E$, matrix $B\in\mathbb{R}^{m\times m}$, matrix $S\in\mathbb{R}^{m\times m}$ is skew-symmetric, and matrix $E\in\mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set C is a box constraint with the form $C=[-2,5]^m$. It is easy to see that M is Lipschitz continuous monotone and its Lipschitz constant $L=\|G\|$. In this numerical example, all entries of B,E are generated randomly in $[0,2]$, S is generated randomly in $[-2,2]$ and $g=\mathbf{0}$. It is easy to check that the solution set to this problem is $x^*=\{\mathbf{0}\}$. The parameters of all the algorithms are set as follows.

- Set $\varphi_n=1/(n+1)$, $\sigma_n=0.9(1-\varphi_n)$ and $f(x)=0.1x$ for all the algorithms.
- For the suggested Algorithm 3.1, Algorithm 3.2, CDP Alg. 3.1, TSI Alg. 3, RTDLD Alg. 4, we take Armijo parameters $\delta=2$, $\ell=0.5$ and $\eta=0.5$. Adopt inertial parameters $\theta=0.4$ and $\varepsilon_n=10/(n+1)^2$ for the proposed Algorithms 3.1–3.4.
- Pick $\eta=0.5$ and $\chi_1=0.0006$ for the suggested Algorithm 3.3, Algorithm 3.4 and TYCR Alg. 3.1. Take $\xi_n=1/(n+1)^{1.1}$ (or $\xi_n=0$) for the offered Algorithms 3.3 and 3.4. Choose $\chi=0.5/\eta$ for RTDLD Alg. 4.

The maximum number of iterations of 2000 as a common stopping criterion for all algorithms and the initial values $x_0=x_1$ are randomly generated by *rand(m,1)* in MATLAB. We use $D_n=\|x_n-x^*\|$ to measure the n-th iteration error of all algorithms. "Time" indicates the time in seconds required for all algorithms to reach the stopping criterion. The numerical results of all algorithms with four different dimensions are shown in Fig. 1, Fig. 2 and Table 1.

Example 4.2. Consider the Hilbert space $\mathscr{H}=l_2:=\{x=(x_1,x_2,\ldots,x_i,\ldots)\mid\sum_{i=1}^{\infty}|x_i|^2<+\infty\}$ equipped with inner product $\langle x,y\rangle=\sum_{i=1}^{\infty}x_iy_i$ and induced norm $\|x\|=\sqrt{\langle x,x\rangle}$ for any $x,y\in\mathscr{H}$. Let $C:=\{x=(x_1,x_2,\ldots,x_i,\ldots)\in\mathscr{H}:|x_i|\le 1/i,\, i=1,2,\ldots,n,\ldots\}$. Define an operator $M:C\to\mathscr{H}$ by

$$Mx=\left(\|x\|+\frac{1}{\|x\|+\varphi}\right)x,$$



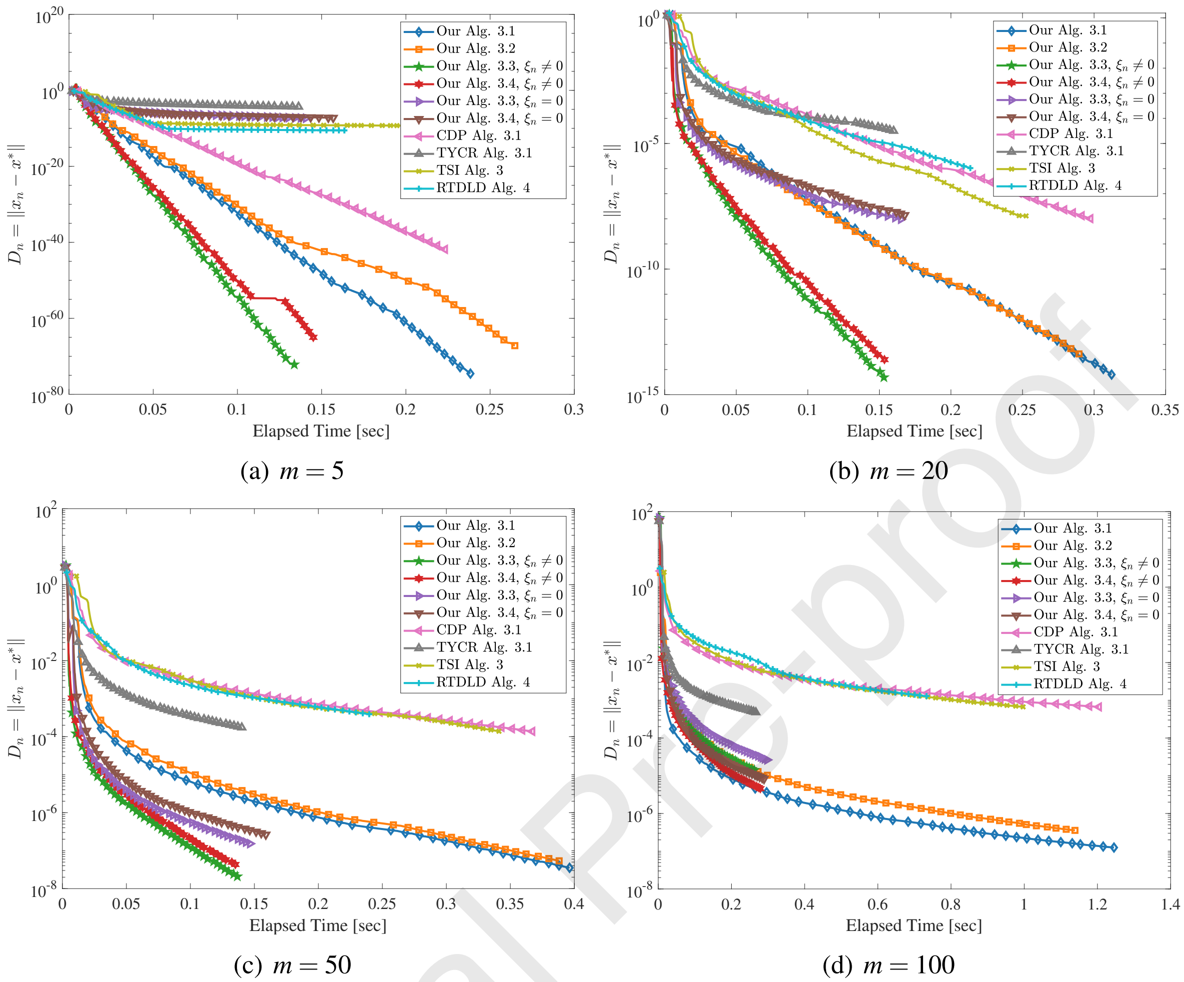
(a) $m = 5$ (b) $m = 20$

(c) $m = 50$ (d) $m = 100$

FIGURE 1. Compare the number of iterations of all algorithms in Example 4.1

TABLE 1. Numerical results of all algorithms for Example 4.1

Algorithms	$m = 5$		$m = 20$		$m = 50$		$m = 100$	
	D_n	Time	D_n	Time	D_n	Time	D_n	Time
Our Alg. 3.1	2.74E-75	0.2383	6.31E-15	0.3122	3.53E-08	0.3963	1.25E-07	1.2452
Our Alg. 3.2	6.32E-68	0.2647	4.21E-14	0.2895	5.39E-08	0.3881	3.56E-07	1.1386
Our Alg. 3.3 ($\xi_n \neq 0$)	6.60E-73	0.1338	4.87E-15	0.1532	2.08E-08	0.1367	1.42E-05	0.2626
Our Alg. 3.4 ($\xi_n \neq 0$)	9.77E-66	0.1450	2.49E-14	0.1535	4.31E-08	0.1351	4.55E-06	0.2790
Our Alg. 3.3 ($\xi_n = 0$)	4.60E-08	0.1406	9.57E-09	0.1651	1.53E-07	0.1466	2.64E-05	0.2977
Our Alg. 3.4 ($\xi_n = 0$)	5.80E-08	0.1568	1.41E-08	0.1677	2.64E-07	0.1589	8.22E-06	0.2890
CDP Alg. 3.1	1.41E-42	0.2232	1.02E-08	0.2974	0.000138	0.3672	0.000668	1.2041
TYCR Alg. 3.1	4.85E-05	0.1363	3.17E-05	0.1599	0.000174	0.1402	0.000481	0.2666
TSI Alg. 3	4.59E-10	0.2138	1.31E-08	0.2525	0.000139	0.3411	0.000669	0.9969
RTDLD Alg. 4	2.82E-11	0.1635	1.05E-06	0.2135	0.00039	0.2397	0.001289	0.7331



(a) $m = 5$ (b) $m = 20$

(c) $m = 50$ (d) $m = 100$

FIGURE 2. Compare the execution time of all algorithms in Example 4.1

for some $\varphi > 0$. It is easy to see that the solution $x^* = \{\mathbf{0}\}$, and moreover, M is pseudo-monotone on $\mathscr{H}$, uniformly continuous and sequentially weakly continuous on C but not Lipschitz continuous on $\mathscr{H}$ (see more details in [30]). In the following cases, we take $\varphi = 0.5$, $\mathscr{H} = \mathbb{R}^m$ for different values of m. In this case, the feasible set C is a box

$$C = \left\{x \in \mathbb{R}^m : \frac{-1}{i} \leq x_i \leq \frac{1}{i},\ i = 1, 2, \ldots, m\right\}.$$

We set $\chi_1 = 0.6$ for the suggested Algorithm 3.3, Algorithm 3.4 and TYCR Alg. 3.1, and keep the parameters of other algorithms the same as in Example 4.1. We use the maximum number of iterations of 200 as a common stopping criterion for all algorithms. The numerical performance of all algorithms with four different dimensions is reported in Table 2.

Example 4.3. Suppose that $\mathscr{H} = L^2([0,1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t$, $\forall x, y \in \mathscr{H}$ and norm $\|x\| := (\int_0^1 |x(t)|^2\, \mathrm{d}t)^{\frac{1}{2}}$. Let the feasible set be the unit ball $C := \{x \in \mathscr{H} : \|x\| \leq 1\}$. Define an operator $M : C \to \mathscr{H}$ by

$$Mx(t) = \int_0^1 \left[x(t) - \frac{4}{e^2 - 1} tse^{t+s} x(s)\right] \mathrm{d}s, \quad x \in C,\ t \in [0,1].$$

TABLE 2. Numerical results for Example 4.2

Algorithms	$m = 100$		$m = 1000$		$m = 10000$		$m = 100000$	
	D_n	Time	D_n	Time	D_n	Time	D_n	Time
Our Alg. 3.1	1.51E-53	0.0278	1.46E-53	0.0359	2.77E-53	0.2193	2.03E-53	1.3701
Our Alg. 3.2	4.20E-52	0.0298	3.67E-52	0.0373	3.76E-52	0.2476	1.79E-52	1.3908
CDP Alg. 3.1	2.09E-27	0.0446	2.13E-27	0.0472	4.08E-27	0.2256	3.34E-27	1.3900
TSI Alg. 3	5.51E-11	0.0433	5.53E-11	0.0558	5.00E-11	0.2130	4.07E-11	1.1559
RTDLD Alg. 4	4.33E-11	0.0267	4.89E-11	0.0316	3.20E-10	0.1428	0.030078	0.9524

It is known that the operator M is monotone and 2-Lipschitz continuous (see [32]). The parameters of all algorithms are the same as in Example 4.2. We choose the maximum number of iterations of 50 as the common stopping criterion and use $E_n = \|x_{n+1} - x_n\|$ to measure the error of the n-th step since we do not know the solution of the problem. The numerical results of all algorithms with four different initial values $x_0(t) = x_1(t)$ are shown in Table 3.

TABLE 3. Numerical results of all algorithms for Example 4.3

Algorithms	$x_0 = 3t^4 + 2$		$x_0 = e^t$		$x_0 = 3\cos(3t)$		$x_0 = 2\log(2t)$	
	E_n	Time	E_n	Time	E_n	Time	E_n	Time
Our Alg. 3.1	2.75E-04	42.70	1.36E-04	43.23	1.11E-04	42.64	4.77E-05	43.14
Our Alg. 3.2	1.69E-04	42.70	9.24E-05	43.25	7.03E-05	42.75	2.99E-05	43.09
Our Alg. 3.3 ($\xi_n \neq 0$)	2.90E-04	19.15	1.36E-04	19.38	1.10E-04	19.07	5.81E-06	19.27
Our Alg. 3.4 ($\xi_n \neq 0$)	1.36E-04	19.15	7.41E-05	19.34	4.50E-05	19.08	1.20E-05	19.36
Our Alg. 3.3 ($\xi_n = 0$)	2.90E-04	19.24	1.36E-04	19.69	1.10E-04	19.13	5.81E-06	19.24
Our Alg. 3.4 ($\xi_n = 0$)	1.36E-04	19.15	7.41E-05	19.28	4.50E-05	19.13	1.20E-05	19.34
CDP Alg. 3.1	6.24E-04	41.80	5.49E-04	41.23	0.000313	40.91	0.000123	41.13
TYCR Alg. 3.1	6.22E-04	17.23	4.90E-04	17.42	0.000357	17.18	0.000182	17.39
TSI Alg. 3	6.24E-04	38.81	5.35E-04	38.74	0.000327	37.43	0.000205	39.67
RTDLD Alg. 4	1.38E-03	31.63	9.10E-04	30.28	0.000615	31.41	0.000269	33.42

Example 4.4. We consider an example where the mapping M is not monotonic in an infinite-dimensional Hilbert space. Let $\mathscr{H} = L^2([0,1])$. Assume that the feasible set is a ball and its form is $C = \{x \in \mathscr{H} : \|x\| \leq 2\}$. Define a mapping $h : C \to \mathbb{R}$ by $h(m) = 1/(1 + \|m\|^2)$. It is easy to verify that the mappin h is bounded ($h(m) \in [0.2, 1]$) and L_h-Lipschitz continuous with $L_h = 16/25$. Recall that the Volterra integration operator $V : \mathscr{H} \to \mathscr{H}$ is given by

$$V(m)(t) = \int_0^t m(s)\,\mathrm{d}s, \forall t \in [0,1], m \in \mathscr{H}.$$

Then V is bounded linear monotone (see [46, Exercise 20.12]) and its operator norm is $\|V\| = \frac{2}{\pi}$. Now, we define the mapping $M : C \to \mathscr{H}$ as follows:

$$M(m)(t) = h(m)V(m)(t), \forall t \in [0,1], m \in C.$$

Note that M is not monotone. For example, take $n = 1$ and $m = 2$, then $\langle Mn - Mm, n - m\rangle = -\frac{1}{10} < 0$. In fact, M is pseudomonotone. Indeed, for all $m, n \in C$, assume that $\langle Mm, n - m\rangle \geq 0$, then we show that $\langle Mn, n-m\rangle \geq 0$. Note that $\langle Vm, n-m\rangle \geq 0$ (since $h(m) > 0.2$). Therefore, we obtain

$$\begin{aligned}\langle Mn, n-m\rangle &= h(n)\langle V(n), n-m\rangle \\ &\geq h(n)[\langle V(n), n-m\rangle - \langle V(m), n-m\rangle] \\ &= h(n)\langle V(n)-V(m), n-m\rangle \geq 0\,.\end{aligned}$$

Hence, M is pseudomonotone. Moreover, we have

$$\begin{aligned}\|Mm-Mn\| &= \|h(m)V(m)-h(n)V(n)\| \\ &\leq \|h(m)V(m)-h(m)V(n)\| + \|h(m)V(n)-h(n)V(n)\| \\ &\leq |h(m)|\|V(m)-V(n)\| + \|V(n)\|\|h(m)-h(n)\| \\ &\leq (|h(m)|\|V\| + \|V\|\|n\|L_h)\,\|m-n\| \\ &\leq \frac{114}{25\pi}\|m-n\|,\ \forall m,n \in C\,.\end{aligned}$$

Thus, mapping M is L-Lipschitz continuous with $L = 114/(25\pi)$.

The parameters of all algorithms are the same as in Example 4.3. The maximum number of iterations 50 is used as a common stopping criterion for all algorithms. Table 4 shows the numerical results of all algorithms with four starting points $x_0(t) = x_1(t)$.

TABLE 4. Numerical results of all algorithms for Example 4.4

Algorithms	$x_0 = t^2+1$		$x_0 = e^{2t}$		$x_0 = \cos(3t)$		$x_0 = \log(t)$	
	E_n	Time	E_n	Time	E_n	Time	E_n	Time
Our Alg. 3.1	7.42E-05	46.90	1.15E-04	62.32	7.43E-05	208.40	2.56E-04	267.81
Our Alg. 3.2	4.25E-05	48.77	5.69E-05	67.29	4.12E-05	224.38	1.42E-04	273.01
Our Alg. 3.3 ($\xi_n \neq 0$)	6.74E-05	45.99	1.45E-04	64.95	8.10E-05	228.36	2.37E-04	274.91
Our Alg. 3.4 ($\xi_n \neq 0$)	2.27E-05	47.18	3.19E-05	65.73	2.51E-05	255.71	8.21E-05	289.94
Our Alg. 3.3 ($\xi_n = 0$)	2.87E-05	41.79	6.03E-05	54.63	3.46E-05	162.80	7.91E-05	246.71
Our Alg. 3.4 ($\xi_n = 0$)	1.79E-05	43.65	2.57E-05	56.59	2.12E-05	179.93	5.51E-05	257.48
CDP Alg. 3.1	4.41E-04	41.54	2.95E-04	54.97	0.000433	141.55	0.001597	224.78
TYCR Alg. 3.1	2.22E-04	38.09	1.79E-04	50.39	0.000266	144.28	0.00073	234.08
TSI Alg. 3	4.08E-04	45.32	2.69E-04	56.87	0.00044	130.69	0.001575	259.97
RTDLD Alg. 4	3.74E-04	46.70	4.52E-04	60.02	0.000441	133.22	0.001158	242.93

Remark 4.1. We have the following observations from Examples 4.1–4.4.

(i) From Fig. 1, Fig. 2, Table 1–4, it can be seen that the algorithms proposed in this paper are easy to implement and efficient. Moreover, they have a faster convergence speed than some known algorithms in the literature [26, 29, 31, 32], and these results are not significantly related to the size of the dimension and the choice of initial values.

(ii) It is important to note that the variational inequality mapping M associated in Example 4.2 is uniformly continuous rather than Lipschitz continuous. The proposed Algorithm 3.3 and Algorithm 3.4 and the algorithms presented in [16, 25, 26] will not be available in this case because their convergence conditions require that the operator M be Lipschitz continuous. However, the stated Algorithm 3.1 and Algorithm 3.2 can work well due to the fact that they replace Lipschitz continuity with uniform continuity. Moreover, the operator M in Example 4.4 is pseudomonotone but not monotone. The algorithms proposed in [21, 22, 23, 24] for solving monotone variational inequalities will fail in the case that the operator M involved is pseudomonotone. Therefore, the four iterative schemes for solving variational inequalities with a pseudomonotone mapping presented in this paper have a broader scope of application.

(iii) Notice that Algorithm 3.1 and Algorithm 3.2 with an Armijo-type criterion take more execution time to reach the same stopping condition than Algorithm 3.3 and Algorithm 3.4 with a simple adaptive step size. This is due to the fact that the Armijo-type criterion may require multiple calculations of the values in operator M and multiple evaluations of the projections on the feasible set in each iteration. Furthermore, it should be highlighted that the proposed Algorithm 3.3 and Algorithm 3.4 apply a new non-monotonic step size rule. They have some advantages over the adaptive algorithms with a non-increasing step size sequence (i.e., the proposed Algorithm 3.3 ($\xi_n = 0$), Algorithm 3.4 ($\xi_n = 0$) and the Algorithm 3.1 offered in [26]).

5. Applications to optimal control problems

In this section, we use the proposed algorithms to solve the variational inequality that occurs in the optimal control problem. Assume that $L_2([0,T],\mathbb{R}^m)$ represents the square-integrable Hilbert space with inner product $\langle p,q\rangle = \int_0^T \langle p(t),q(t)\rangle \,\mathrm{d}t$ and norm $\|p\|_2 = \sqrt{\langle p,p\rangle}$. The optimal control problem is described as follows:

$$p^*(t) \in \operatorname{Argmin}\{g(p) \mid p \in V\}, \quad t \in [0,T], \tag{5.1}$$

where V represents a set of feasible controls composed of m piecewise continuous functions. Its form is expressed as follows:

$$V = \left\{p(t) \in L_2([0,T],\mathbb{R}^m) : p_i(t) \in \left[p_i^-, p_i^+\right], i = 1,2,\ldots,m\right\}. \tag{5.2}$$

In particular, the control $p(t)$ may be a piecewise constant function (bang-bang type). The terminal objective function has the form

$$g(p) = \Phi(x(T)), \tag{5.3}$$

where Φ is a convex and differentiable defined on the attainability set.

Assume that the trajectory $x(t) \in L_2([0,T]$ satisfies the constraints of the linear differential equation system:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Q(t)x(t) + W(t)p(t),\ 0 \le t \le T,\ x(0) = x_0, \tag{5.4}$$

where $Q(t) \in \mathbb{R}^{n\times n}$, $W(t) \in \mathbb{R}^{n\times m}$ are given continuous matrices for every $t \in [0,T]$. By the solution of problem (5.1)–(5.4), we mean a control $p^*(t)$ and a corresponding (optimal) trajectory $x^*(t)$ such that its terminal value $x^*(T)$ minimizes objective function (5.3). From the Pontryagin maximum principle, there exists a function $s^* \in L_2([0,T]$ such that the triple (x^*,s^*,p^*) solves for a.e. $t \in [0,T]$ the system

$$\frac{\mathrm{d}}{\mathrm{d}t}x^*(t) = Q(t)x^*(t) + W(t)p^*(t),\ \ x^*(0) = x_0\,, \tag{5.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s^*(t) = -Q(t)^{\mathsf{T}}s^*(t),\ \ s^*(T) = \nabla\Phi\left(x^*(T)\right), \tag{5.6}$$

$$0 \in W(t)^{\mathsf{T}}s^*(t) + N_V\left(p^*(t)\right), \tag{5.7}$$

where $N_V(p)$ is the normal cone to V at p defined by

$$N_V(p) := \begin{cases} \emptyset, & \text{if } p \notin V\,; \\ \{\iota \in \mathscr{H} : \langle \iota, q-p\rangle \le 0, \forall q \in V\}, & \text{if } p \in V\,. \end{cases}$$

Denoting $Gp(t) := W(t)^{\mathsf{T}}s(t)$, Khoroshilova [47] showed that Gp is the gradient of the objective function g. Therefore, system (5.5)–(5.7) is reduced to the variational inequality problem

$$\langle Gp^*, q-p^*\rangle \ge 0, \quad \forall q \in V\,. \tag{5.8}$$

Recently, there are many approaches to solve the optimal control problem, see, for example, [27, 47, 48, 49]. Note that our algorithms 3.1–3.4 guarantee strong convergence and do not require the Lipschitz constant. Furthermore, the addition of inertial terms makes them converge faster.

For the convenience of numerical computation, we discretize the continuous functions. Given the mesh size $h := T/N$ where N is a natural number. We identify any discretized control $p^N := (p_0, p_1, \ldots, p_{N-1})$ with its piece-wise constant extension:

$$p^N(t) = p_i,\ \ \forall t \in [t_i, t_{i+1}),\ \ t_i = ih,\ \ i = 0,1,\ldots,N\,.$$

Furthermore, we identify the discretized state $x^N := (x_0, x_1, \ldots, x_N)$ and co-state $s^N := (s_0, s_1, \ldots, s_N)$. They have the form of piecewise linear interpolation:

$$x^N(t) = x_i + \frac{t-t_i}{h}(x_{i+1}-x_i),\ \ \forall t \in [t_i, t_{i+1}),\ \ i = 0,1,\ldots,N-1\,,$$

and

$$s^N(t) = s_i + \frac{t_i-t}{h}(s_{i-1}-s_i),\ \ \forall t \in (t_{i-1}, t_i],\ \ i = N, N-1,\ldots,1\,.$$

We use the classical Euler discretization method to solve the systems of ODEs (5.5) and (5.6). The Euler discretization of the original system (5.1)–(5.4) is given by

$$\begin{aligned} \text{minimize}\ \ & \Phi_N\left(x^N, p^N\right) \\ \text{subject to}\ \ & x_{i+1}^N = x_i^N + h\left[Q(t_i)x_i^N + W(t_i)p_i^N\right],\ \ x^N(0) = x_0\,, \\ & s_i^N = s_{i+1}^N + hQ(t_i)^{\mathsf{T}}s_{i+1}^N,\ \ s(N) = \nabla\Phi(x_N)\,, \\ & p_i^N \in V\,. \end{aligned}$$

It is well known that the Euler discretization has the error estimate $O(h)$ [50]. This indicates that the difference between the discretized solution $p^N(t)$ and the original solution $p^*(t)$ is proportional to the mesh size h. That is, there exists a constant $K > 0$ such that $\left\|p^N - p^*\right\| \le Kh$.

Next, we present several mathematical examples to illustrate the computational performance of all the algorithms. Our parameters are set as follows. Set $\varphi_n = 10^{-4}/(n+1)$, $\sigma_n = 0.9(1-\varphi_n)$ and $f(x) = 0.1x$ for all algorithms. Take inertial parameters $\theta = 10^{-2}$ and $\varepsilon_n = 10^{-4}/(n+1)^2$ for the stated iterative schemes 3.1–3.4. Choose $\eta = 0.5$ and $\chi_1 = 0.4$ for the suggested

Algorithms 3.3, 3.4 and TYCR Alg. 3.1. The remaining parameters are the same as those set in Example 4.1. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1,1]$. The stopping criterion is either $D_n = \|p_{n+1} - p_n\| \leq 10^{-4}$, or maximum number of iterations which is set to 1000.

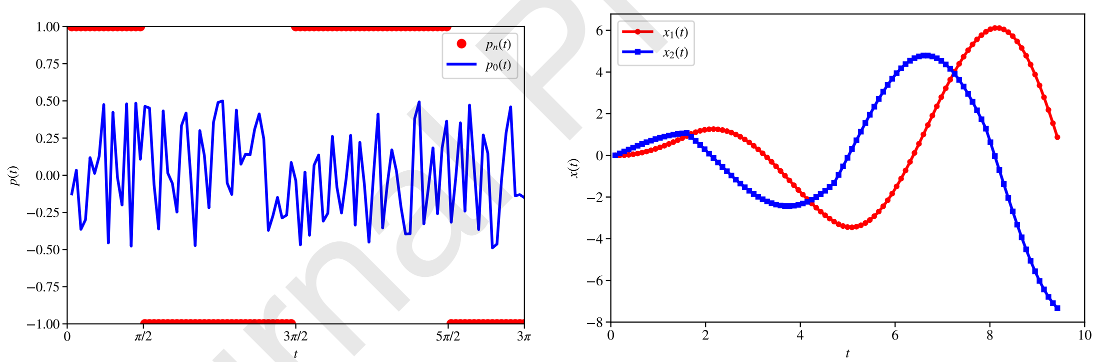
Example 5.1 (Control of a harmonic oscillator, see [51])**.**

$$\begin{aligned} \text{minimize} \quad & x_2(3\pi) \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t)\,, \\ & \dot{x}_2(t) = -x_1(t) + p(t), \;\; \forall t \in [0, 3\pi]\,, \\ & x(0) = 0\,, \\ & p(t) \in [-1,1]\,. \end{aligned}$$

The exact optimal control of Example 5.1 is known:

$$p^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2)\,; \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]\,. \end{cases}$$

Fig. 3 shows the approximate optimal control and the corresponding trajectories of Algorithm 3.3.



(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 3. Numerical results for Example 5.1

We now consider an example in which the terminal function is not linear.

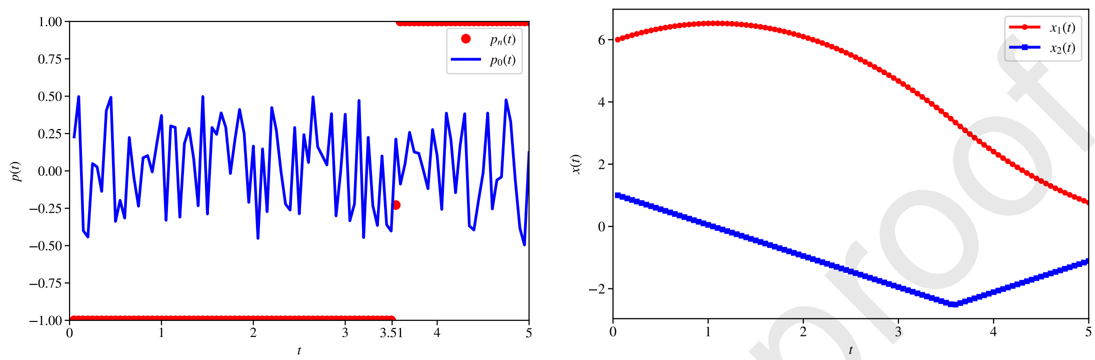
Example 5.2 (Rocket car [48])**.**

$$\begin{aligned} \text{minimize} \quad & 0.5\left((x_1(5))^2 + (x_2(5))^2\right)\,, \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t)\,, \\ & \dot{x}_2(t) = p(t), \;\; \forall t \in [0,5]\,, \\ & x_1(0) = 6, \;\; x_2(0) = 1\,, \\ & p(t) \in [-1,1]\,. \end{aligned}$$

The exact optimal control of Example 5.2 is

$$p^* = \begin{cases} 1 & \text{if } t \in (3.517, 5]; \\ -1 & \text{if } t \in (0, 3.517]. \end{cases}$$

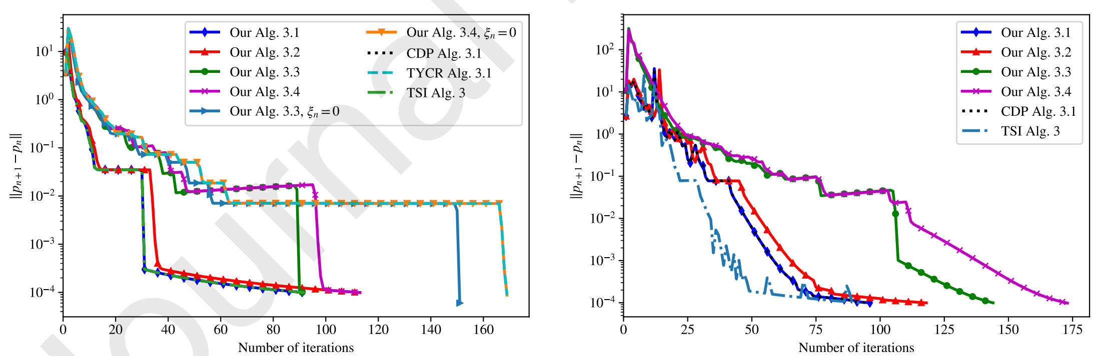
The approximate optimal control and the corresponding trajectories of Algorithm 3.1 are plotted in Fig. 4.



(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 4. Numerical results for Example 5.2

Finally, the numerical performance of all the algorithms in Examples 5.1 and 5.2 are shown in Fig. 5 and Table 5.



(a) Example 5.1 (b) Example 5.2

FIGURE 5. Numerical results for Examples 5.1 and 5.2

Remark 5.1. We draw the following observations from Examples 5.1 and 5.2.

(i) The offered Algorithms 3.1, 3.2, 3.3 and 3.4 can be applied to solve optimal control problems, and they perform well when the terminal function is linear or nonlinear (cf. Figs. 3 and 4).

(ii) As shown in Fig. 5 and Table 5, the algorithms proposed in this paper perform better when the terminal function is linear than when it is nonlinear, that is, they require less execution time and the number of termination iterations in the case where the terminal

TABLE 5. Comparison of the number of iterations and execution time of all algorithms in Examples 5.1 and 5.2

Algorithms	Example 5.1			Example 5.2		
	Iter.	Time (s)	D_n	Iter.	Time (s)	D_n
Our Alg. 3.1	90	0.052416	1.00E-04	95	0.091772	9.96E-05
Our Alg. 3.2	112	0.062005	9.93E-05	117	0.10633	9.96E-05
Our Alg. 3.3 ($\xi_n \neq 0$)	90	0.035136	1.00E-04	143	0.056123	9.96E-05
Our Alg. 3.4 ($\xi_n \neq 0$)	112	0.043454	9.93E-05	172	0.065834	9.84E-05
Our Alg. 3.3 ($\xi_n = 0$)	150	0.074606	6.00E-05	1000	0.37886	0.002684
Our Alg. 3.4 ($\xi_n = 0$)	168	0.081423	8.77E-05	1000	0.39503	0.002683
CDP Alg. 3.1	91	0.055566	9.89E-05	95	0.086908	9.96E-05
TYCR Alg. 3.1	168	0.07931	8.77E-05	1000	0.36277	0.002683
TSI Alg. 3	91	0.12166	9.89E-05	91	0.1255	9.98E-05
RTDLD Alg. 4	1000	0.47682	0.1625	1000	1.589	91.6813

function is linear. Moreover, the stated iterative schemes outperform the existing methods in the literature, in other words, the presented algorithms converge faster than the others for the same stopping criterion.

6. CONCLUSIONS

In this paper, we introduced and investigated four new iterative methods with adaptive stepsizes for solving variational inequalities in real Hilbert spaces which are based on the subgradient extragradient method, the Mann-type method, the viscosity-type method and an inertial extrapolation strategy. The first two methods with an Armijo-type stepsize are designed to solve the variational inequality problem with a pseudomonotone and non-Lipschitz continuous operator. The latter two adaptive iterative schemes are used to discover the solution of the variational inequality problem with a pseudomonotone and Lipschitz continuous operator (the Lipschitz constant does not need to be known). Strong convergence theorems of the proposed algorithms are established under some suitable conditions. The advantages of the suggested iterative algorithms over some related ones were confirmed by several numerical experiments.

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