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Inertial projection and contraction methods for pseudomonotone variational inequalities with non-Lipschitz operators and applications

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ABSTRACT
In this paper, some new accelerated iterative schemes are proposed to solve the variational inequality problem with a pseudomonotone and uniformly continuous operator in real Hilbert spaces. Strong convergence theorems of the suggested algorithms are obtained without the prior knowledge of the Lipschitz constant of the operator. Some numerical experiments and applications are performed to illustrate the advantages of the proposed methods with respect to several related ones.

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1. Introduction

Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $M:\mathcal{H}\to\mathcal{H}$ be a nonlinear mapping. Recall that the variational inequality problem is expressed as follows:

$$\text{find } x^*\in C \text{ such that } \langle Mx^*, x-x^*\rangle \geq 0 \quad \forall\, x\in C. \tag{VIP}$$

The solution set of (VIP) is denoted by $\mathrm{VI}(C,M)$. Many issues in science and society can be unified under the framework of the variational inequality model. The VIP plays an essential role in optimization theory and practical applications, see, e.g.[1–5]. Therefore, the VIP attracted considerable attention from many researchers and became an attractive field. Many scholars are interested not only in obtaining theoretical results but also in numerical methods to solve such problems. A large number of iterative algorithms have been proposed in the last decades to solve (VIP), see, e.g.[6–9] and the references therein.

Recently, the extragradient method (for short, EGM) proposed by Korpelevich [6] has been extensively studied by many scholars, and there are a large number of iterative schemes for finding numerical solutions to variational inequality problems by the EGM; see, e.g.[10–13]. It is important

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to emphasize that the EGM needs to compute the projection on the feasible set twice in each iteration, which may affect its computational efficiency when the feasible set has a complex structure. The projection and contraction method (PCM) introduced by He [7], Tseng's EGM offered by Tseng [8], and the subgradient extragradient method (SEGM) suggested by Censor et al. [9] can overcome this drawback. A common feature of the three methods mentioned above is that the projection on the feasible set needs to be evaluated only once in each iteration. Recently, variant forms based on these methods have been further investigated by researchers, see, e.g.[14–17] and the references therein.

Based on the SEGM and the PCM, Dong et al. [18] provided a modified subgradient extragradient method (MSEGM), which forms an iterative sequence that weakly converges to the solution of (VIP). Their basic examples demonstrate the numerical efficiency and advantages of the MSEGM compared to the SEGM and the PCM. Some applications appearing in medical imaging and machine learning tell us that the strong convergence is preferable to the weak convergence in an infinite-dimensional space. Recently, Thong and Gibali [19] and Gibali et al. [20] obtained some strongly convergent methods to solve the (VIP) with a monotone operator by combining the MSEGM, the Mann method and the viscosity method. On the other hand, the inertial idea has been studied by many researchers as a technique to accelerate the convergence speed of algorithms. They have constructed a large number of numerical methods to solve optimal control problems, signal processing, image recovery and other optimization problems; see, for instance, [21–25] and the references therein. In 2018, an inertial projection and contraction method (IPCM) by combining the PCM and the inertial method was introduced by Dong et al. [26] to solve the monotone (VIP). They showed the advantages of the IPCM compared with other algorithms through some computational tests and established the weak convergence theorem of the IPCM in real Hilbert spaces under appropriate assumptions. By associating the IPCM with the Mann method and the viscosity method, respectively, Thong et al. [27] and Cholamjiak et al. [28] established the strong convergence theorems of the proposed iterative schemes. The algorithms presented in [27, 28] use a fixed step size in each iteration, which indicates that the Lipschitz constant of the mapping must need to be received in advance. In practical large-scale nonlinear optimization problems, the Lipschitz constant is not easy to obtain or requires more calculation to estimate. It is known that there are some mappings that are not monotone, such as pseudomonotone mappings, and moreover the class of pseudomonotone mappings includes the class of monotone mappings. There are some numerical methods based on the SEGM and the PCM in the literature [28–31] that can solve pseudomonotone (VIP). Note that the approach stated in [29] achieves weak convergence, the algorithms offered in [28, 30] require the prior information of the Lipschitz constant of the mapping, and the methods presented in [31] uses the projection-type method to ensure strong convergence.

Motivated and stimulated by the results mentioned above, in this study, we introduce six inertial PCMs to solve variational inequalities in real Hilbert spaces. Our iterative schemes improve and extend some previously known results in the literature [14, 18–20, 26–28, 30]. More precisely, our contributions in this paper are as follows.

(i) The methods presented in this paper are designed to solve pseudomonotone (VIP). Note that the approaches stated in [19, 20, 27] will fail when the mapping is pseudomonotone rather than monotone because these methods are devised to solve monotone (VIP). Moreover, the variational inequality mapping associated with the suggested methods are uniformly continuous rather than Lipschitz continuous. Therefore, our iterative schemes are more useful and have a wider application.

(ii) Our algorithms use an Armijo-like criterion to automatically update the iteration step size, which makes them more intelligent in applications. It should be pointed out that the methods presented in [27, 28, 30] need to know the Lipschitz constant of the mapping, which limits the realization of such algorithms when the Lipschitz constant of the mapping associated with the problem is unknown.

(iii) We investigate and confirm the strong convergence of the suggested algorithms by applying the Mann-type method and the viscosity-type method, while the methods introduced in [18, 26, 29] only obtained weak convergence in Hilbert spaces. In addition, it should be noted that the methods proposed by Thong et al. [31] use a projection-type approach to ensure strong convergence, which affects their computational efficiency in infinite-dimensional spaces.

(iv) The iterative schemes devised in this paper combine inertial terms, which also accelerate the convergence speed of the algorithms without inertial terms proposed in [19, 20].

The rest of this paper is organized as follows. We recall some preliminaries that need to be used in Section 2. Section 3 presents the algorithms and analyzes their convergence. Some computational tests are presented to show the efficiency of the suggested approaches over several existing ones in Section 4. The proposed methods are investigated to solve optimal control problems in Section 5. Finally, the paper ends with a brief remark in Section 6, the last section.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y, z \in \mathcal{H}$, we have the following inequalities:

(1) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \alpha \in \mathbb{R}$;
(3) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2$, where $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

For every point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $P_C(x) := \operatorname{argmin}\{\|x-y\|, y \in C\}$. P_C is called the metric projection of $\mathcal{H}$ onto C. It is known that P_C has the following basic properties:

- $\langle x - P_C(x), y - P_C(x)\rangle \le 0 \ \forall y \in C$.
- $\|P_C(x) - P_C(y)\|^2 \le \langle P_C(x) - P_C(y), x - y\rangle \ \forall y \in \mathcal{H}$.

For any $x, y \in \mathcal{H}$, a mapping $M : \mathcal{H} \to \mathcal{H}$ is said to be:

(1) *η-strongly monotone* with $\eta > 0$ if $\langle Mx - My, x - y\rangle \ge \eta\|x-y\|^2$.
(2) *L-Lipschitz continuous* with $L > 0$ if $\|Mx - My\| \le L\|x-y\|$. If $L \in (0,1)$, then mapping M is called *contraction*.
(3) *Monotone* if $\langle Mx - My, x - y\rangle \ge 0$.
(4) *Pseudomonotone* if $\langle Mx, y - x\rangle \ge 0 \Longrightarrow \langle My, y - x\rangle \ge 0$.
(5) *Sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies that $\{Mx_n\}$ converges weakly to Mx.

According to the above definitions, it is easy to see that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4). But the inverse operation is usually not valid.

Lemma 2.1 ([32]): *Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\mu_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \mu_n = \infty$. Suppose that*

$$p_{n+1} \le \mu_n q_n + (1-\mu_n)p_n \quad \forall n \ge 1.$$

If $\limsup_{k\to\infty} q_{n_k} \le 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty} (p_{n_k+1} - p_{n_k}) \ge 0$, then $\lim_{n\to\infty} p_n = 0$.

3. Main results

In this section, we introduce six new iterative schemes based on the SEGM and the IPCM to solve pseudomonotone (VIP) in real Hilbert spaces. These algorithms guarantee the strong convergence with the aid of the Mann-type method and the viscosity-type method. The advantage of our approaches is that we do not need to know the Lipschitz constant of the mapping in advance. In fact, the variational inequality mapping associated only needs to satisfy the uniform continuity condition and not the Lipschitz continuity. To analyze the convergence of the algorithms, the mapping and parameters involved in our methods need to meet the following assumptions.

(C1) The feasible set C is a nonempty, closed and convex subset of $\mathcal{H}$.
(C2) The solution set of the (VIP) is nonempty, that is $\mathrm{VI}(C, M) \neq \emptyset$.
(C3) The mapping $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, uniformly continuous on $\mathcal{H}$, and sequentially weakly continuous on C.
(C4) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\mu_n} = 0$, where $\{\mu_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \mu_n = 0$ and $\sum_{n=1}^{\infty} \mu_n = \infty$. Let $\{\eta_n\} \subset (a, b) \subset (0, 1 - \mu_n)$ for some $a > 0$, $b > 0$.

3.1. The Mann-type inertial modified subgradient extragradient algorithm

The first iterative scheme is based on the IPCM, the SEGM and the Mann-type method, and its details are described in Algorithm 3.1.

Algorithm 3.1 The Mann-type inertial modified subgradient extragradient algorithm

Initialization: Take $\sigma > 0, \delta > 0, \zeta \in (0, 1), \phi \in (0, 1), \alpha \in (0, 2)$. Let $x_0, x_1 \in \mathcal{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set $u_n = x_n + \sigma_n(x_n - x_{n-1})$, where

$$\sigma_n = \begin{cases} \min\left\{\frac{\epsilon_n}{\|x_n - x_{n-1}\|}, \sigma\right\}, & \text{if } x_n \neq x_{n-1}; \\ \sigma, & \text{otherwise}. \end{cases} \tag{1}$$

Step 2. Compute $y_n = P_C(u_n - \gamma_n M u_n)$, where the step size γ_n is chosen to be the largest $\gamma \in \{\delta, \delta\zeta, \delta\zeta^2, \ldots\}$ satisfying

$$\gamma \|Mu_n - My_n\| \leq \phi \|u_n - y_n\| . \tag{2}$$

If $u_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $z_n = P_{T_n}(u_n - \alpha\gamma_n\chi_n My_n)$, where

$$T_n := \left\{x \in \mathcal{H} \mid \langle u_n - \gamma_n Mu_n - y_n, x - y_n\rangle \leq 0\right\} ,$$

and

$$\chi_n = (1 - \phi)\frac{\|u_n - y_n\|^2}{\|c_n\|^2}, \quad c_n = u_n - y_n - \gamma_n(Mu_n - My_n) . \tag{3}$$

Step 4. Compute $x_{n+1} = (1 - \mu_n - \eta_n)u_n + \eta_n z_n$. Set $n := n + 1$ and go to **Step 1**.

Remark 3.1: It follows from (1) and the assumptions on $\{\mu_n\}$ that

$$\lim_{n\to\infty} \frac{\sigma_n}{\mu_n}\|x_n - x_{n-1}\| = 0.$$

Indeed, we obtain $\sigma_n\|x_n - x_{n-1}\| \le \epsilon_n, \forall n \ge 1$, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\mu_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\sigma_n}{\mu_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\epsilon_n}{\mu_n} = 0.$$

The following lemmas are very helpful in analyzing the convergence of the algorithms.

Lemma 3.1 ([31, Lemma 3.1]): *Suppose that Assumptions (C1)–(C3) hold. The Armijo-like criteria* (2) *is well defined. Moreover, we obtain $\gamma_n \le \delta$.*

Lemma 3.2 ([31, Lemma 3.2]): *Suppose that Assumptions (C1)–(C3) hold. Let $\{u_n\}$ and $\{y_n\}$ be two sequences formulated by Algorithm* 3.1. *If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, M)$.*

Lemma 3.3: *Suppose that Assumptions (C1)–(C3) hold. Let $\{z_n\}$, $\{y_n\}$ and $\{u_n\}$ be three sequences created by Algorithm* 3.1. *Then, for all $x^* \in \mathrm{VI}(C, M)$,*

$$\|z_n - x^*\|^2 \le \|u_n - x^*\|^2 - \|u_n - z_n - \alpha\chi_n c_n\|^2 - \alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2.$$

Proof: From $x^* \in \mathrm{VI}(C, M) \subset C \subset T_n$ and the property of projection, we obtain

$$\begin{aligned}
2\|z_n - x^*\|^2 &= 2\|P_{T_n}(u_n - \alpha\gamma_n\chi_n My_n) - P_{T_n}(x^*)\|^2 \\
&\le 2\langle z_n - x^*, u_n - \alpha\gamma_n\chi_n My_n - x^*\rangle \\
&= \|z_n - x^*\|^2 + \|u_n - \alpha\gamma_n\chi_n My_n - x^*\|^2 - \|z_n - u_n + \alpha\gamma_n\chi_n My_n\|^2 \\
&= \|z_n - x^*\|^2 + \|u_n - x^*\|^2 + \alpha^2\gamma_n^2\chi_n^2\|My_n\|^2 - 2\langle u_n - x^*, \alpha\gamma_n\chi_n My_n\rangle \\
&\quad - \|z_n - u_n\|^2 - \alpha^2\gamma_n^2\chi_n^2\|My_n\|^2 - 2\langle z_n - u_n, \alpha\gamma_n\chi_n My_n\rangle \\
&= \|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|z_n - u_n\|^2 - 2\langle z_n - x^*, \alpha\gamma_n\chi_n My_n\rangle,
\end{aligned}$$

which implies that

$$\|z_n - x^*\|^2 \le \|u_n - x^*\|^2 - \|z_n - u_n\|^2 - 2\alpha\gamma_n\chi_n\langle z_n - x^*, My_n\rangle. \tag{4}$$

Combining the facts that M is pseudomonotone, $y_n \in C$, $x^* \in \mathrm{VI}(C, M)$ and (VIP), we can show that $\langle My_n, y_n - x^*\rangle \ge 0$, which means that $\langle My_n, z_n - x^*\rangle \ge \langle My_n, z_n - y_n\rangle$. Thus,

$$-2\alpha\gamma_n\chi_n\langle My_n, z_n - x^*\rangle \le -2\alpha\gamma_n\chi_n\langle My_n, z_n - y_n\rangle. \tag{5}$$

Since $z_n \in T_n$, we have $\langle u_n - \gamma_n Mu_n - y_n, z_n - y_n\rangle \le 0$. This shows that

$$\langle u_n - y_n - \gamma_n(Mu_n - My_n), z_n - y_n\rangle \le \gamma_n\langle My_n, z_n - y_n\rangle. \tag{6}$$

Using (5), (6) and the definition of c_n, we obtain

$$\begin{aligned}
-2\alpha\gamma_n\chi_n\langle My_n, z_n - x^*\rangle &\le -2\alpha\chi_n\langle c_n, z_n - y_n\rangle \\
&= -2\alpha\chi_n\langle c_n, u_n - y_n\rangle + 2\alpha\chi_n\langle c_n, u_n - z_n\rangle.
\end{aligned} \tag{7}$$

Now, we estimate $-2\alpha\chi_n\langle c_n, u_n - y_n\rangle$ and $2\alpha\chi_n\langle c_n, u_n - z_n\rangle$. From the definitions of χ_n and c_n, we have

$$\begin{aligned}\langle c_n, u_n - y_n\rangle &\geq \|u_n - y_n\|^2 - \gamma_n\|Mu_n - My_n\|\|u_n - y_n\| \\ &\geq \|u_n - y_n\|^2 - \phi\|u_n - y_n\|^2 \\ &= (1-\phi)\|u_n - y_n\|^2 = \chi_n\|c_n\|^2,\end{aligned}$$

which indicates that

$$-2\alpha\chi_n\langle c_n, u_n - y_n\rangle \leq -2\alpha\chi_n^2\|c_n\|^2. \tag{8}$$

According to the basic inequality $2ab = a^2 + b^2 - (a-b)^2$, we also have

$$2\alpha\chi_n\langle c_n, u_n - z_n\rangle = \|u_n - z_n\|^2 + \alpha^2\chi_n^2\|c_n\|^2 - \|u_n - z_n - \alpha\chi_n c_n\|^2. \tag{9}$$

It follows from the definition of c_n and (2) that

$$\begin{aligned}\|c_n\| &\leq \|u_n - y_n\| + \gamma_n\|Mu_n - My_n\| \\ &\leq \|u_n - y_n\| + \phi\|u_n - y_n\| \\ &= (1+\phi)\|u_n - y_n\|,\end{aligned}$$

which combining the definition of χ_n yields that

$$\chi_n^2\|c_n\|^2 = (1-\phi)^2\frac{\|u_n - y_n\|^4}{\|c_n\|^2} \geq \frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2. \tag{10}$$

Combining (4), (7)–(10), we conclude that

$$\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \|u_n - z_n - \alpha\chi_n c_n\|^2 - \alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2.$$

This completes the proof. ■

Theorem 3.1: *Suppose that Assumptions (C1)–(C4) hold. Then the sequence $\{x_n\}$ formed by Algorithm 3.1 converges to $x^* \in \mathrm{VI}(C, M)$ in norm, where $\|x^*\| = \min\{\|z\| : z \in \mathrm{VI}(C, M)\}$.*

Proof: First, we show that the sequence $\{x_n\}$ is bounded. Indeed, thanks to Lemma 3.3, one has

$$\|z_n - x^*\| \leq \|u_n - x^*\| \quad \forall n \geq 1. \tag{11}$$

From the definition of u_n, one sees that

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \mu_n \cdot \frac{\sigma_n}{\mu_n}\|x_n - x_{n-1}\|. \tag{12}$$

According to Remark 3.1, we have that $\frac{\sigma_n}{\mu_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Therefore, there exists a constant $Q_1 > 0$ such that

$$\frac{\sigma_n}{\mu_n}\|x_n - x_{n-1}\| \leq Q_1 \quad \forall n \geq 1,$$

which together with (11) and (12) implies

$$\|z_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\| + \mu_n Q_1. \tag{13}$$

By the definition of x_{n+1}, one obtains

$$\|x_{n+1} - x^*\| \leq \|(1-\mu_n-\eta_n)(u_n - x^*) + \eta_n(z_n - x^*)\| + \mu_n\|x^*\|. \tag{14}$$

It follows from (11) that

$$\begin{aligned}
&\|(1-\mu_n-\eta_n)(u_n - x^*) + \eta_n(z_n - x^*)\|^2 \\
&\quad \leq (1-\mu_n-\eta_n)^2\|u_n - x^*\|^2 + 2(1-\mu_n-\eta_n)\eta_n\|z_n - x^*\|\|u_n - x^*\| + \eta_n^2\|z_n - x^*\|^2 \\
&\quad \leq (1-\mu_n-\eta_n)^2\|u_n - x^*\|^2 + 2(1-\mu_n-\eta_n)\eta_n\|u_n - x^*\|^2 + \eta_n^2\|u_n - x^*\|^2 \\
&\quad = (1-\mu_n)^2\|u_n - x^*\|^2,
\end{aligned}$$

which yields that

$$\|(1-\mu_n-\eta_n)(u_n - x^*) + \eta_n(z_n - x^*)\| \leq (1-\mu_n)\|u_n - x^*\|. \tag{15}$$

Using (13), (14) and (15), we deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\mu_n)\|u_n - x^*\| + \mu_n\|x^*\| \\
&\leq (1-\mu_n)\|x_n - x^*\| + \mu_n(\|x^*\| + Q_1) \\
&\leq \max\{\|x_n - x^*\|, \|x^*\| + Q_1\} \\
&\leq \cdots \leq \max\{\|x_1 - x^*\|, \|x^*\| + Q_1\}.
\end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded. So the sequences $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded.

From (13), one sees that

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq (\|x_n - x^*\| + \mu_n Q_1)^2 \\
&= \|x_n - x^*\|^2 + \mu_n(2Q_1\|x_n - x^*\| + \mu_n Q_1^2) \\
&\leq \|x_n - x^*\|^2 + \mu_n Q_2
\end{aligned} \tag{16}$$

for some $Q_2 > 0$. By the definition of x_{n+1} and Assumption (C4), we find that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\mu_n-\eta_n)(u_n - x^*) + \eta_n(z_n - x^*) + \mu_n(-x^*)\|^2 \\
&= (1-\mu_n-\eta_n)\|u_n - x^*\|^2 + \eta_n\|z_n - x^*\|^2 + \mu_n\|x^*\|^2 \\
&\quad - \eta_n(1-\mu_n-\eta_n)\|u_n - z_n\|^2 - \mu_n(1-\mu_n-\eta_n)\|u_n\|^2 - \mu_n\eta_n\|z_n\|^2 \\
&\leq (1-\mu_n-\eta_n)\|u_n - x^*\|^2 + \eta_n\|z_n - x^*\|^2 + \mu_n\|x^*\|^2.
\end{aligned} \tag{17}$$

Combining Lemma 3.3, (16) and (17), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\mu_n-\eta_n)\|u_n - x^*\|^2 + \eta_n\|u_n - x^*\|^2 - \eta_n\|u_n - z_n - \alpha\chi_n c_n\|^2 \\
&\quad - \eta_n\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 + \mu_n\|x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \eta_n\|u_n - z_n - \alpha\chi_n c_n\|^2 \\
&\quad - \eta_n\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 + \mu_n(\|x^*\|^2 + Q_2).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\eta_n\|u_n - z_n - \alpha\chi_n c_n\|^2 + \eta_n\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 \\
&\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \mu_n(\|x^*\|^2 + Q_2).
\end{aligned} \tag{Eq1}$$

From the definition of u_n, we can write

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \|x_n - x^*\|^2 + 2\sigma_n\|x_n - x^*\|\|x_n - x_{n-1}\| + \sigma_n^2\|x_n - x_{n-1}\|^2 \\
&\le \|x_n - x^*\|^2 + 3Q\sigma_n\|x_n - x_{n-1}\|,
\end{aligned} \tag{18}$$

where $Q := \sup_{n\in\mathbb{N}}\{\|x_n - x^*\|, \sigma\|x_n - x_{n-1}\|\} > 0$. Take $t_n = (1-\eta_n)u_n + \eta_n z_n$. It follows from (11) that

$$\|t_n - x^*\| \le (1-\eta_n)\|u_n - x^*\| + \eta_n\|z_n - x^*\| \le \|u_n - x^*\|. \tag{19}$$

From (18) and (19), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\mu_n)(t_n - x^*) - \mu_n(u_n - t_n) - \mu_n x^*\|^2 \\
&\le (1-\mu_n)^2\|t_n - x^*\|^2 - 2\mu_n\langle u_n - t_n + x^*, x_{n+1} - x^*\rangle \\
&= (1-\mu_n)^2\|t_n - x^*\|^2 + 2\mu_n\langle u_n - t_n, x^* - x_{n+1}\rangle + 2\mu_n\langle x^*, x^* - x_{n+1}\rangle \\
&\le (1-\mu_n)\|t_n - x^*\|^2 + 2\mu_n\|u_n - t_n\|\|x_{n+1} - x^*\| + 2\mu_n\langle x^*, x^* - x_{n+1}\rangle \\
&\le (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\Bigg[2\eta_n\|u_n - z_n\|\|x_{n+1} - x^*\| \\
&\quad +2\langle x^*, x^* - x_{n+1}\rangle + \frac{3Q\sigma_n}{\mu_n}\|x_n - x_{n-1}\|\Bigg].
\end{aligned} \tag{Eq2}$$

Finally, we show that the sequence $\{\|x_n - x^*\|\}$ converges to zero. Throughout this paper, we always assume that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|) \ge 0$. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2) \\
&\quad = \liminf_{k\to\infty}[(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|)(\|x_{n_k+1} - x^*\| + \|x_{n_k} - x^*\|)] \ge 0.
\end{aligned}$$

By (Eq1) and Assumption (C4), we observe that

$$\begin{aligned}
&\eta_{n_k}\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_{n_k} - y_{n_k}\|^2 + \eta_{n_k}\|u_{n_k} - z_{n_k} - \alpha\chi_{n_k}c_{n_k}\|^2 \\
&\quad \le \limsup_{k\to\infty}[\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2] + \limsup_{k\to\infty}\mu_{n_k}(\|x^*\|^2 + Q_2) \\
&\quad = -\liminf_{k\to\infty}[\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2] \le 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|y_{n_k} - u_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|u_{n_k} - z_{n_k} - \alpha\chi_{n_k}c_{n_k}\| = 0.$$

From $\|c_{n_k}\| \geq (1-\phi)\|u_{n_k} - y_{n_k}\|$ and the definition of χ_{n_k}, we obtain

$$\begin{aligned}
\|u_{n_k} - z_{n_k}\| &\leq \|u_{n_k} - z_{n_k} - \alpha\chi_{n_k}c_{n_k}\| + \alpha\chi_{n_k}\|c_{n_k}\| \\
&= \|u_{n_k} - z_{n_k} - \alpha\chi_{n_k}c_{n_k}\| + \alpha(1-\phi)\frac{\|u_{n_k} - y_{n_k}\|^2}{\|c_{n_k}\|} \\
&\leq \|u_{n_k} - z_{n_k} - \alpha\chi_{n_k}c_{n_k}\| + \alpha\|u_{n_k} - y_{n_k}\|.
\end{aligned}$$

Hence, we get that $\lim_{k\to\infty}\|z_{n_k} - u_{n_k}\| = 0$. This together with the boundedness of $\{x_n\}$ further implies that

$$\lim_{k\to\infty} \eta_{n_k}\|u_{n_k} - z_{n_k}\|\|x_{n_k+1} - x^*\| = 0. \tag{20}$$

Moreover, using Remark 3.1 and Assumption (C4), we have

$$\|x_{n_k+1} - u_{n_k}\| = \mu_{n_k}\|u_{n_k}\| + \eta_{n_k}\|z_{n_k} - u_{n_k}\| \to 0, \quad \text{as } n \to \infty$$

and

$$\|x_{n_k} - u_{n_k}\| = \mu_{n_k} \cdot \frac{\sigma_{n_k}}{\mu_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0, \quad \text{as } n \to \infty.$$

From the above facts, we conclude that

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0, \quad \text{as } n \to \infty. \tag{21}$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$ when $j \to \infty$. Furthermore,

$$\limsup_{k\to\infty}\langle x^*, x^* - x_{n_k}\rangle = \lim_{j\to\infty}\langle x^*, x^* - x_{n_{k_j}}\rangle = \langle x^*, x^* - z\rangle. \tag{22}$$

We get that $u_{n_k} \rightharpoonup z$ since $\|x_{n_k} - u_{n_k}\| \to 0$. This together with $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$ and Lemma 3.2 yields that $z \in \mathrm{VI}(C, M)$. From the definition of x^* and (22), we obtain

$$\limsup_{k\to\infty}\langle x^*, x^* - x_{n_k}\rangle = \langle x^*, x^* - z\rangle \leq 0. \tag{23}$$

Combining (21) and (23), we find that

$$\limsup_{k\to\infty}\langle x^*, x^* - x_{n_k+1}\rangle \leq \limsup_{k\to\infty}\langle x^*, x^* - x_{n_k}\rangle \leq 0. \tag{24}$$

Thus, from Remark 3.1, (20), (24), (Eq2) and Lemma 2.1, we conclude that $x_n \to x^*$ as $n \to \infty$. That is the desired result. ■

3.2. First viscosity-type inertial modified subgradient extragradient algorithm

In this subsection, we introduce a viscosity-type inertial MSEGM for solving the (VIP). First, we use the following Assumption (C5) to replace the Assumption (C4) described in Section 3.

(C5) Let $f : \mathcal{H} \to \mathcal{H}$ be a κ-contraction mapping with $\kappa \in [0, 1)$. Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\epsilon_n}{\mu_n} = 0$, where $\{\mu_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty}\mu_n = 0$ and $\sum_{n=1}^{\infty}\mu_n = \infty$.

The Algorithm 3.2 is of the following form.

Algorithm 3.2 The first viscosity-type inertial modified subgradient extragradient algorithm.

Initialization: Take $\sigma > 0, \delta > 0, \zeta \in (0,1), \phi \in (0,1), \alpha \in (0,2)$. Let $x_0, x_1 \in \mathcal{H}$.

Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \sigma_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \gamma_n M u_n), \\ z_n = P_{T_n}(u_n - \alpha\gamma_n\chi_n M y_n), \\ T_n := \{x \in \mathcal{H} \mid \langle u_n - \gamma_n M u_n - y_n, x - y_n\rangle \le 0\}, \\ x_{n+1} = \mu_n f(u_n) + (1-\mu_n) z_n, \end{cases}$$

where $\{\sigma_n\}$, $\{\gamma_n\}$ and $\{\chi_n\}$ are defined in (1), (2) and (3), respectively.

Theorem 3.2: *Suppose that Assumptions (C1)–(C3) and (C5) hold. Then the sequence $\{x_n\}$ constructed by Algorithm* 3.2 *converges to $x^* \in \mathrm{VI}(C,M)$ in norm, where $x^* = P_{\mathrm{VI}(C,M)} \circ f(x^*)$.*

Proof: First, we show that the sequence $\{x_n\}$ is bounded. Using the definition of x_{n+1} and (13), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \mu_n\|f(u_n) - f(x^*)\| + \mu_n\|f(x^*) - x^*\| + (1-\mu_n)\|z_n - x^*\| \\ &\le \mu_n\kappa\|u_n - x^*\| + \mu_n\|f(x^*) - x^*\| + (1-\mu_n)\|u_n - x^*\| \\ &\le (1-(1-\kappa)\mu_n)\|x_n - x^*\| + (1-\kappa)\mu_n \frac{Q_1 + \|f(x^*) - x^*\|}{1-\kappa} \\ &\le \max\left\{\|x_n - x^*\|, \frac{Q_1 + \|f(x^*) - x^*\|}{1-\kappa}\right\} \\ &\le \cdots \le \max\left\{\|x_1 - x^*\|, \frac{Q_1 + \|f(x^*) - x^*\|}{1-\kappa}\right\}. \end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. We get that the sequences $\{u_n\}$, $\{z_n\}$ and $\{f(u_n)\}$ are also bounded. Combining Lemma 3.3, (13) and (16), we see that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \mu_n(\|f(u_n) - f(x^*)\| + \|f(x^*) - x^*\|)^2 + (1-\mu_n)\|z_n - x^*\|^2 \\ &\le \mu_n(\|u_n - x^*\| + \|f(x^*) - x^*\|)^2 + (1-\mu_n)\|z_n - x^*\|^2 \\ &= \mu_n\|u_n - x^*\|^2 + (1-\mu_n)\|z_n - x^*\|^2 \\ &\quad + \mu_n(2\|u_n - x^*\|\|f(x^*) - x^*\| + \|f(x^*) - x^*\|^2) \\ &\le \mu_n\|u_n - x^*\|^2 + (1-\mu_n)\|z_n - x^*\|^2 + \mu_n Q_3 \\ &\le \|x_n - x^*\|^2 - (1-\mu_n)\|u_n - z_n - \alpha\chi_n c_n\|^2 \\ &\quad - (1-\mu_n)\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 + \mu_n Q_4, \end{aligned}$$

where $Q_4 := Q_2 + Q_3$. Therefore, we obtain

$$\begin{aligned} &(1-\mu_n)\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 + (1-\mu_n)\|u_n - z_n - \alpha\chi_n c_n\|^2 \\ &\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \mu_n Q_4. \end{aligned} \tag{Eq3}$$

Using (11) and (18), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|\mu_n(f(u_n)-f(x^*))+(1-\mu_n)(z_n-x^*)\|^2+2\mu_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\le \mu_n\|f(u_n)-f(x^*)\|^2+(1-\mu_n)\|z_n-x^*\|^2+2\mu_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\le \mu_n\kappa\|u_n-x^*\|^2+(1-\mu_n)\|u_n-x^*\|^2+2\mu_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\ &\le (1-(1-\kappa)\mu_n)\|x_n-x^*\|^2+(1-\kappa)\mu_n\cdot\left[\frac{3Q}{1-\kappa}\cdot\frac{\sigma_n}{\mu_n}\|x_n-x_{n-1}\|\right.\\ &\quad\left.+\frac{2}{1-\kappa}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\right]. \end{aligned}\tag{Eq4}$$

Finally, we prove that the sequence $\{\|x_n-x^*\|\}$ converges to zero. By (Eq3) and Assumption (C5), we observe that

$$\begin{aligned}&(1-\mu_{n_k})\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_{n_k}-y_{n_k}\|^2+(1-\mu_{n_k})\|u_{n_k}-z_{n_k}-\alpha\chi_{n_k}c_{n_k}\|^2\\ &\le \limsup_{k\to\infty}[\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2+\mu_{n_k}Q_4]\le 0,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|y_{n_k}-u_{n_k}\|=0 \quad \text{and} \quad \lim_{k\to\infty}\|u_{n_k}-z_{n_k}-\alpha\chi_{n_k}c_{n_k}\|=0.$$

As stated in Theorem 3.1, it is easy to see that $\lim_{k\to\infty}\|z_{n_k}-u_{n_k}\|=0$. Moreover, using Remark 3.1 and Assumption (C5), we have

$$\|x_{n_k+1}-z_{n_k}\|=\mu_{n_k}\|z_{n_k}-f(x_{n_k})\|\to 0, \quad \text{as } n\to\infty$$

and

$$\|x_{n_k}-u_{n_k}\|=\mu_{n_k}\cdot\frac{\sigma_{n_k}}{\mu_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0, \quad \text{as } n\to\infty.$$

It follows that

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-z_{n_k}\|+\|z_{n_k}-u_{n_k}\|+\|u_{n_k}-x_{n_k}\|\to 0, \quad \text{as } n\to\infty. \tag{25}$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup z$ when $j\to\infty$. Furthermore,

$$\limsup_{k\to\infty}\langle f(x^*)-x^*,x_{n_k}-x^*\rangle=\lim_{j\to\infty}\langle f(x^*)-x^*,x_{n_{k_j}}-x^*\rangle=\langle f(x^*)-x^*,z-x^*\rangle. \tag{26}$$

We get that $u_{n_k}\rightharpoonup z$ since $\|x_{n_k}-u_{n_k}\|\to 0$, which together with $\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0$ and Lemma 3.2 implies that $z\in \mathrm{VI}(C,M)$. From the definition of x^* and (26), we obtain

$$\limsup_{k\to\infty}\langle f(x^*)-x^*,x_{n_k}-x^*\rangle=\langle f(x^*)-x^*,z-x^*\rangle\le 0. \tag{27}$$

Combining (25) and (27), we obtain

$$\limsup_{k\to\infty}\langle f(x^*)-x^*,x_{n_k+1}-x^*\rangle\le\limsup_{k\to\infty}\langle f(x^*)-x^*,x_{n_k}-x^*\rangle\le 0. \tag{28}$$

Thus, from Remark 3.1, (28), (Eq4) and Lemma 2.1, we conclude that $x_n\to x^*$ as $n\to\infty$. The proof of the Theorem 3.2 is now complete. ■

3.3. The second viscosity-type inertial modified subgradient extragradient algorithm

In this subsection, we introduce another viscosity-type iterative scheme that is different from Algorithm 3.2. The details of this scheme are described in Algorithm 3.3.

Algorithm 3.3 The second viscosity-type inertial modified subgradient extragradient algorithm.

Initialization: Take $\sigma > 0, \delta > 0, \zeta \in (0,1), \phi \in (0,1), \alpha \in (0,2)$. Let $x_0, x_1 \in \mathcal{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \sigma_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \gamma_n M u_n), \\ z_n = P_{T_n}(u_n - \alpha\gamma_n\chi_n M y_n), \\ T_n := \{x \in \mathcal{H} \mid \langle u_n - \gamma_n M u_n - y_n, x - y_n\rangle \le 0\}, \\ x_{n+1} = \mu_n f(z_n) + (1-\mu_n) z_n, \end{cases}$$

where $\{\sigma_n\}$, $\{\gamma_n\}$ and $\{\chi_n\}$ are defined in (1), (2) and (3), respectively.

Theorem 3.3: *Suppose that Assumptions (C1)–(C3) and (C5) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.3 converges to $x^* \in \mathrm{VI}(C,M)$ in norm, where $x^* = P_{\mathrm{VI}(C,M)} \circ f(x^*)$.*

Proof: The proof of this theorem is very similar to Theorem 3.2. First, we show that the sequence $\{x_n\}$ is bounded. Using the definition of x_{n+1} and (13), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \mu_n\|f(z_n) - f(x^*)\| + \mu_n\|f(x^*) - x^*\| + (1-\mu_n)\|z_n - x^*\| \\ &\le (1-(1-\kappa)\mu_n)\|x_n - x^*\| + (1-\kappa)\mu_n \frac{Q_1 + \|f(x^*) - x^*\|}{1-\kappa} \\ &\le \max\left\{\|x_1 - x^*\|, \frac{Q_1 + \|f(x^*) - x^*\|}{1-\kappa}\right\}. \end{aligned}$$

This indicates that the sequence $\{x_n\}$ is bounded. We also get that the sequences $\{u_n\}$, $\{z_n\}$ and $\{f(z_n)\}$ are bounded. Combining Lemma 3.3 and (16), we find that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \mu_n(\|z_n - x^*\| + \|f(x^*) - x^*\|)^2 + (1-\mu_n)\|z_n - x^*\|^2 \\ &= \mu_n\|z_n - x^*\|^2 + (1-\mu_n)\|z_n - x^*\|^2 \\ &\quad + \mu_n(2\|z_n - x^*\|\|f(x^*) - x^*\| + \|f(x^*) - x^*\|^2) \\ &\le \|z_n - x^*\|^2 + \mu_n Q_5 \\ &\le \|x_n - x^*\|^2 - \|u_n - z_n - \alpha\chi_n c_n\|^2 \\ &\quad - \alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 + \mu_n Q_6, \end{aligned}$$

where $Q_6 := Q_2 + Q_5$. Hence, we have

$$\begin{aligned} &\alpha(2-\alpha)\frac{(1-\phi)^2}{(1+\phi)^2}\|u_n - y_n\|^2 + \|u_n - z_n - \alpha\chi_n c_n\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \mu_n Q_6. \end{aligned}$$

Using (11) and (18), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\mu_n(f(z_n)-f(x^*))+(1-\mu_n)(z_n-x^*)+\mu_n(f(x^*)-x^*)\|^2\\&\le \mu_n\kappa\|z_n-x^*\|^2+(1-\mu_n)\|z_n-x^*\|^2+2\mu_n\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\\&\le (1-(1-\kappa)\mu_n)\|x_n-x^*\|^2+(1-\kappa)\mu_n\cdot\left[\frac{3Q}{1-\kappa}\cdot\frac{\sigma_n}{\mu_n}\|x_n-x_{n-1}\|\right.\\&\quad\left.+\frac{2}{1-\kappa}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\right].\end{aligned}$$

Finally, we show that the sequence $\{\|x_n-x^*\|\}$ converges to zero. It can be easily obtained by similar conclusions of Theorem 3.2. This completes the proof. ∎

In the next part, we will introduce three new simple numerical methods for solving the (VIP) that only need to calculate the projection once in each iteration.

3.4. The modified Mann-type inertial projection and contraction algorithm

Our first modified iterative process is stated in Algorithm 3.4. Compared with Algorithm 3.1, the calculation of the iterative sequence $\{z_n\}$ replaces the projection on the half-space with a display formula.

Algorithm 3.4 The modified Mann-type inertial projection and contraction algorithm.

Initialization: Take $\sigma>0,\delta>0,\zeta\in(0,1),\phi\in(0,1),\alpha\in(0,2)$. Let $x_0,x_1\in\mathcal{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases}u_n=x_n+\sigma_n(x_n-x_{n-1}),\\y_n=P_C(u_n-\gamma_nMu_n),\\z_n=u_n-\alpha\chi_nc_n,\\x_{n+1}=(1-\mu_n-\eta_n)u_n+\eta_nz_n,\end{cases}$$

where $\{\sigma_n\}$, $\{\gamma_n\}$ and $\{\chi_n\}$ are defined in (1), (2) and (3), respectively.

The following lemma plays an important role in studying the convergence of the algorithms.

Lemma 3.4: *Suppose that Assumptions (C1)–(C3) hold. Let $\{z_n\}$ and $\{u_n\}$ be two sequences produced by Algorithm 3.4. Then, for all $x^*\in \mathrm{VI}(C,M)$,*

$$\|z_n-x^*\|^2\le\|u_n-x^*\|^2-\frac{2-\alpha}{\alpha}\|u_n-z_n\|^2$$

and

$$\|u_n-y_n\|^2\le\left[\frac{1+\phi}{(1-\phi)\alpha}\right]^2\|u_n-z_n\|^2.$$

Proof: By the definition of z_n, one obtains

$$\begin{aligned}\|z_n - x^*\|^2 &= \|u_n - \alpha\chi_n c_n - x^*\|^2\\ &= \|u_n - x^*\|^2 - 2\alpha\chi_n\langle u_n - x^*, c_n\rangle + \alpha^2\chi_n^2\|c_n\|^2.\end{aligned}\tag{29}$$

Combining (2) and (3), we observe that

$$\begin{aligned}\langle u_n - x^*, c_n\rangle &= \langle u_n - y_n, u_n - y_n - \gamma_n(Mu_n - My_n)\rangle + \langle y_n - x^*, c_n\rangle\\ &\geq \|u_n - y_n\|^2 - \gamma_n\|u_n - y_n\|\|Mu_n - My_n\| + \langle y_n - x^*, c_n\rangle\\ &\geq (1-\phi)\|u_n - y_n\|^2 + \langle y_n - x^*, u_n - y_n - \gamma_n(Mu_n - My_n)\rangle.\end{aligned}\tag{30}$$

From $y_n = P_C(u_n - \gamma_n Mu_n)$ and the property of projection, we have

$$\langle u_n - y_n - \gamma_n Mu_n, y_n - x^*\rangle \geq 0.\tag{31}$$

Using $x^* \in \mathrm{VI}(C, M)$ and $y_n \in C$, we get that $\langle Mx^*, y_n - x^*\rangle \geq 0$. This together with the pseudomonotonicity of M yields that

$$\langle My_n, y_n - x^*\rangle \geq 0.\tag{32}$$

It follows from (3) that $(1-\phi)\|u_n - y_n\|^2 = \chi_n\|c_n\|^2$. This together with (30), (31) and (32) implies that

$$\langle u_n - x^*, c_n\rangle \geq (1-\phi)\|u_n - y_n\|^2 = \chi_n\|c_n\|^2.\tag{33}$$

Combining (29) and (33), we conclude that

$$\begin{aligned}\|z_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - 2\alpha\chi_n^2\|c_n\|^2 + \alpha^2\chi_n^2\|c_n\|^2\\ &= \|u_n - x^*\|^2 - \frac{2-\alpha}{\alpha}\|u_n - z_n\|^2.\end{aligned}$$

On the other hand, by the definition of z_n and (3), we have

$$\|u_n - y_n\|^2 = \frac{1}{\chi_n(1-\phi)}\|\chi_n c_n\|^2 = \frac{1}{\chi_n(1-\phi)\alpha^2}\|u_n - z_n\|^2.\tag{34}$$

Since $\|c_n\| \leq (1+\phi)\|u_n - y_n\|$, one has

$$\chi_n = (1-\phi)\frac{\|u_n - y_n\|^2}{\|c_n\|^2} \geq \frac{1-\phi}{(1+\phi)^2}.\tag{35}$$

It implies from (34) and (35) that

$$\|u_n - y_n\|^2 \leq \left[\frac{1+\phi}{(1-\phi)\alpha}\right]^2\|u_n - z_n\|^2.$$

The proof of the lemma is now complete. ■

Theorem 3.4: *Suppose that Assumptions (C1)–(C4) hold. Then the sequence $\{x_n\}$ generated by Algorithm* 3.4 *converges to $x^* \in \mathrm{VI}(C, M)$ in norm, where $\|x^*\| = \min\{\|z\| : z \in \mathrm{VI}(C, M)\}$.*

Proof: First, we show that the sequence $\{x_n\}$ is bounded. Indeed, thanks to Lemma 3.4, we have

$$\|z_n - x^*\| \le \|u_n - x^*\| \quad \forall\, n \ge 1. \tag{36}$$

Using the same facts as stated in Theorem 3.1, we get that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. Using Lemma 3.4, (16) and (17), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\mu_n-\eta_n)\|u_n - x^*\|^2 + \eta_n\|u_n - x^*\|^2 - \eta_n\frac{2-\alpha}{\alpha}\|u_n - z_n\|^2 + \mu_n\|x^*\|^2\\ &\le \|x_n - x^*\|^2 - \eta_n\frac{2-\alpha}{\alpha}\|u_n - z_n\|^2 + \mu_n(\|x^*\|^2 + Q_2).\end{aligned}$$

Thus, we have

$$\eta_n\frac{2-\alpha}{\alpha}\|u_n - z_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \mu_n(\|x^*\|^2 + Q_2). \tag{Eq5}$$

Moreover, we can get (Eq2) by using the same facts as declared in Theorem 3.1. Finally, we show that the sequence $\{\|x_n - x^*\|\}$ converges to zero. By (Eq5) and Assumption (C4), we have

$$\begin{aligned}\eta_{n_k}\frac{2-\alpha}{\alpha}\|u_{n_k} - z_{n_k}\|^2 &\le \limsup_{k\to\infty}[\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 + \mu_{n_k}(\|x^*\|^2 + Q_2)]\\ &\le 0,\end{aligned}$$

which implies that $\lim_{k\to\infty}\|z_{n_k} - u_{n_k}\| = 0$. In view of Lemma 3.4, we observe that $\lim_{k\to\infty}\|y_{n_k} - u_{n_k}\| = 0$. As asserted in Theorem 3.1, we can obtain the same result as (20)–(24). Therefore, we get that $x_n \to x^*$ as $n \to \infty$. This completes the proof. ■

3.5. First modified viscosity-type inertial projection and contraction algorithm

By replacing the calculation process of the iterative sequence $\{z_n\}$ in Algorithm 3.2, we get the following Algorithm 3.5.

Algorithm 3.5 The first modified viscosity-type inertial projection and contraction algorithm.

Initialization: Take $\sigma > 0$, $\delta > 0$, $\zeta \in (0,1)$, $\phi \in (0,1)$, $\alpha \in (0,2)$. Let $x_0, x_1 \in \mathcal{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases} u_n = x_n + \sigma_n(x_n - x_{n-1}),\\ y_n = P_C(u_n - \gamma_n M u_n),\\ z_n = u_n - \alpha\chi_n c_n,\\ x_{n+1} = \mu_n f(u_n) + (1-\mu_n)z_n,\end{cases}$$

where $\{\sigma_n\}$, $\{\gamma_n\}$ and $\{\chi_n\}$ are defined in (1), (2) and (3), respectively.

Theorem 3.5: *Suppose that Assumptions (C1)–(C3) and* (C5) *hold. Then the sequence $\{x_n\}$ designed by Algorithm* 3.5 *converges to $x^* \in \mathrm{VI}(C,M)$ in norm, where $x^* = P_{\mathrm{VI}(C,M)} \circ f(x^*)$.*

Proof: The proof of this theorem is very similar to Theorem 3.2. First, we show that the sequence $\{x_n\}$ is bounded. Using the same arguments as declared in Theorem 3.2, we get that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{f(u_n)\}$ are bounded. In view of Lemma 3.4 and (16), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \mu_n\|f(u_n)-x^*\|^2+(1-\mu_n)\|z_n-x^*\|^2\\
&\le \mu_n\|u_n-x^*\|^2+(1-\mu_n)\|z_n-x^*\|^2\\
&\quad +\mu_n(2\|u_n-x^*\|\cdot\|f(x^*)-x^*\|+\|f(x^*)-x^*\|^2)\\
&\le \mu_n\|u_n-x^*\|^2+(1-\mu_n)\|z_n-x^*\|^2+\mu_nQ_3\\
&\le \|x_n-x^*\|^2-(1-\mu_n)\frac{2-\alpha}{\alpha}\|u_n-z_n\|^2+\mu_nQ_4,
\end{aligned}$$

where $Q_4 := Q_2 + Q_3$. Thus, we obtain

$$(1-\mu_n)\frac{2-\alpha}{\alpha}\|u_n-z_n\|^2 \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\mu_nQ_4. \tag{Eq6}$$

Furthermore, we can get (Eq4) by using the same facts as stated in Theorem 3.2. Finally, we show that the sequence $\{\|x_n-x^*\|\}$ converges to zero. From (Eq6), one has

$$(1-\mu_{n_k})\frac{2-\alpha}{\alpha}\|u_{n_k}-z_{n_k}\|^2 \le 0,$$

which indicates that $\lim_{k\to\infty}\|z_{n_k}-u_{n_k}\|=0$. This together with Lemma 3.4 finds that $\lim_{k\to\infty}\|y_{n_k}-u_{n_k}\|=0$. As stated in Theorem 3.2, we can get the same facts as (25)–(28). Therefore, we obtain $x_n \to x^*$ as $n\to\infty$. The proof is completed. ■

3.6. The second modified viscosity-type inertial projection and contraction algorithm

Our last iterative scheme is stated in Algorithm 3.6.

Algorithm 3.6 The second modified viscosity-type inertial projection and contraction algorithm.

Initialization: Take $\sigma>0$, $\delta>0$, $\zeta\in(0,1)$, $\phi\in(0,1)$, $\alpha\in(0,2)$. Let $x_0, x_1\in\mathcal{H}$.
Iterative Steps: Calculate the next iteration point x_{n+1} as follows:

$$\begin{cases}
u_n = x_n+\sigma_n(x_n-x_{n-1}),\\
y_n = P_C(u_n-\gamma_nMu_n),\\
z_n = u_n-\alpha\chi_nc_n,\\
x_{n+1} = \mu_nf(z_n)+(1-\mu_n)z_n,
\end{cases}$$

where $\{\sigma_n\}$, $\{\gamma_n\}$ and $\{\chi_n\}$ are defined in (1), (2) and (3), respectively.

Theorem 3.6: *Suppose that Assumptions (C1)–(C3) and (C5) hold. Then the sequence $\{x_n\}$ determined by Algorithm 3.6 converges to $x^*\in \mathrm{VI}(C,M)$ in norm, where $x^* = P_{\mathrm{VI}(C,M)}\circ f(x^*)$.*

Proof: Combining the proofs of Theorems 3.3 and 3.5, we can easily get the desired conclusion. The proof is left to the readers to verify. ■

4. Numerical experiments

In this section, we provide some numerical experiments to demonstrate the advantages of the suggested methods and compare them with some known strongly convergent algorithms, which including the Algorithm 3.1 introduced by Shehu and Iyiola [14] (shortly, SI Alg. 3.1), Algorithms 3.1 and 3.2 presented by Thong and Gibali [19] (shortly, TG Alg. 3.1 and TG Alg. 3.2), Algorithms 3.1 and 3.2 proposed by Gibali et al. [20] (shortly, GTT Alg. 3.1 and GTT Alg. 3.2) and the Algorithm 4.3 suggested by Shehu et al. [30] (shortly, SDJ Alg. 4.3). All the programs are implemented in MATLAB 2018a on a personal computer.

Our parameters are set as follows. We set $\mu_n = 1/(n+1)$, $\eta_n = 0.5(1-\mu_n)$ and $f(x) = 0.1x$ for all the algorithms. Take $\delta = \zeta = 0.5$, $\phi = 0.4$, $\alpha = 1.5$ for the proposed algorithms, TG Alg. 3.1, TG Alg. 3.2, GTT Alg. 3.1 and GTT Alg. 3.2. Adopt inertial parameters $\sigma = 0.4$ and $\epsilon_n = 100/(n+1)^2$ in our algorithms. Choose $\zeta = 0.5$, $\phi = 0.4$ for SI Alg. 3.1. Pick fixed step size $\gamma_n = 0.5/L$ and $\alpha = 1.5$ in SDJ Alg. 4.3. We use $D_n = \|x_n - x^*\|$ to measure the nth iteration error of all algorithms, where x^* represents the solution to our problems.

Example 4.1: Consider the form of linear operator $M : \mathbb{R}^m \to \mathbb{R}^m$ ($m = 10, 30, 60, 100$) as follows: $M(x) = Gx + g$, where $g \in \mathbb{R}^m$ and $G = BB^{\mathsf{T}} + S + E$, matrix $B \in \mathbb{R}^{m\times m}$, matrix $S \in \mathbb{R}^{m\times m}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set C is a box constraint with the form $C = [-2, 5]^m$. It is easy to see that M is Lipschitz continuous monotone and its Lipschitz constant $L = \|G\|$. In this numerical example, all entries of B, E are generated randomly in $[0, 2]$ and S is generated randomly in $[-2, 2]$. Let $g = \mathbf{0}$. Then the solution set is $x^* = \{\mathbf{0}\}$. The maximum number of iterations 200 as a common stopping criterion and the initial values $x_0 = x_1$ are randomly generated by *5rand(m,1)* in MATLAB. The numerical results of all algorithms in different dimensions are shown in Figure 1.

Example 4.2: We consider an example in the Hilbert space $\mathcal{H} = L^2([0, 1])$ associated with the inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t$ and the induced norm $\|x\| := (\int_0^1 |x(t)|^2\,\mathrm{d}t)^{1/2}, \forall x, y \in \mathcal{H}$. Let the feasible set be the unit ball $C := \{x \in \mathcal{H} : \|x\| \le 1\}$. Define an operator $M : C \to \mathcal{H}$ by

$$(Mx)(t) = \int_0^1 (x(t) - G(t,s)g(x(s)))\,\mathrm{d}s + h(t), \quad t \in [0,1],\ x \in C,$$

where

$$G(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad g(x) = \cos x, \quad h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

It is known that operator M is monotone and L-Lipschitz continuous with $L = 2$, and $x^*(t) = \{\mathbf{0}\}$ is the solution of the corresponding variational inequality problem. Note that the projection on C is inherently explicit, that is,

$$P_C(x) = \begin{cases} \dfrac{x}{\|x\|}, & \text{if } \|x\| > 1; \\ x, & \text{if } \|x\| \le 1. \end{cases}$$

We choose the maximum number of iterations 50 as the common stopping criterion for all algorithms. Figure 2 shows the numerical behaviors of all algorithms with four starting points $x_0(t) = x_1(t)$.

Remark 4.1: From Examples 4.1 and 4.2, it is known that the proposed algorithms are efficient and robust. Furthermore, our algorithms are outperformance some existing known ones [14, 19, 20, 30], and these results are independent of the size of the dimension and the selection of the initial values. It should be pointed out that the SDJ Alg. 4.3 [30, Algorithms 4.3] needs to know the Lipschitz constant of the mapping M, while our suggested algorithms do not need it.

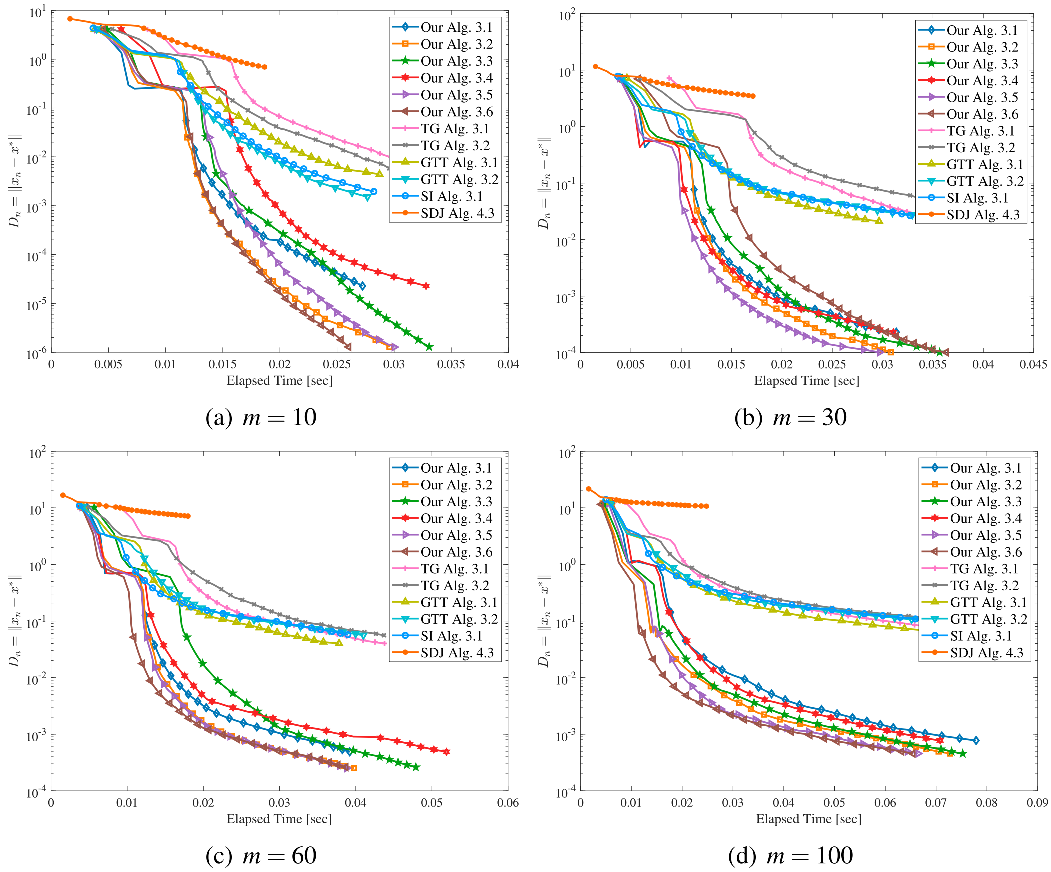


Figure 1. Numerical results for Example 4.1. (a) $m = 10$, (b) $m = 30$, (c) $m = 60$ and (d) $m = 100$.

5. Applications to optimal control problems

In this section, we use the proposed algorithms to solve the (VIP) that appears in optimal control problems. We recommend readers to refer to [33, 34] for detailed description of the problem. Our parameters are set as follows. For all algorithms, we set $\mu_n = 10^{-4}/(n+1)$, $\eta_n = 0.5(1-\mu_n)$, $f(x) = 0.1x$, $\delta = 1$, $\zeta = 0.5$, $\phi = 0.4$ and $\alpha = 1.5$. Take inertial parameters $\sigma = 0.01$ and $\epsilon_n = 10^{-4}/(n+1)^2$ in the proposed algorithms. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$, and the stopping criterion is either $\|p_{n+1} - p_n\| \leq 10^{-4}$ or reaching the maximum number of iterations 1000.

Example 5.1 (Control of a harmonic oscillator, see [35]):

$$
\begin{aligned}
\text{minimize} \quad & x_2(3\pi) \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = -x_1(t) + p(t) \quad \forall\, t \in [0, 3\pi], \\
& x(0) = 0, \\
& p(t) \in [-1, 1].
\end{aligned}
$$

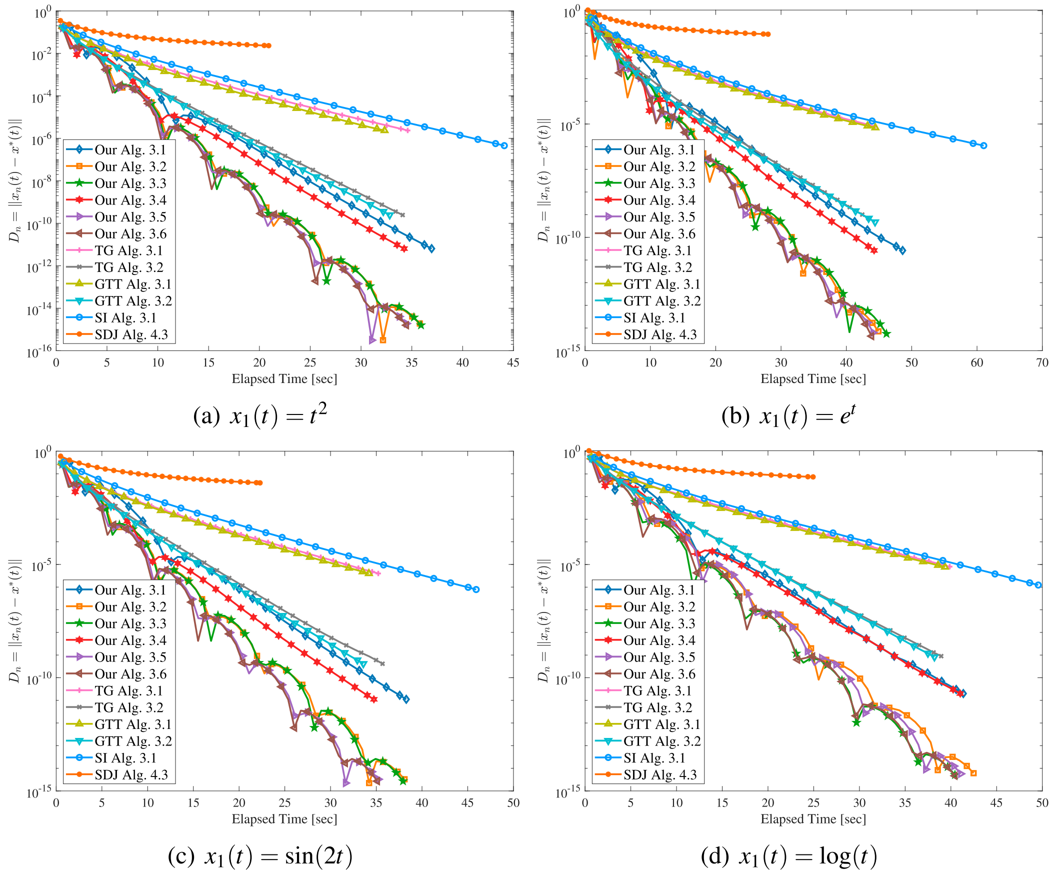


Figure 2. Numerical results for Example 4.2. (a) $x_1(t) = t^2$, (b) $x_1(t) = e^t$, (c) $x_1(t) = \sin(2t)$ and (d) $x_1(t) = \log(t)$.

The exact optimal control of Example 5.1 is known:

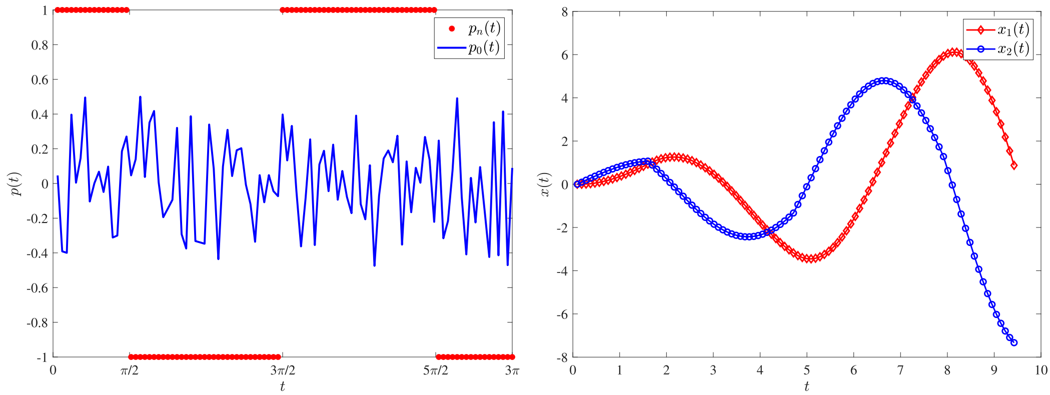
$$p^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2); \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

Figure 3 shows the approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.6.

We now consider two examples in which the terminal function is not linear.

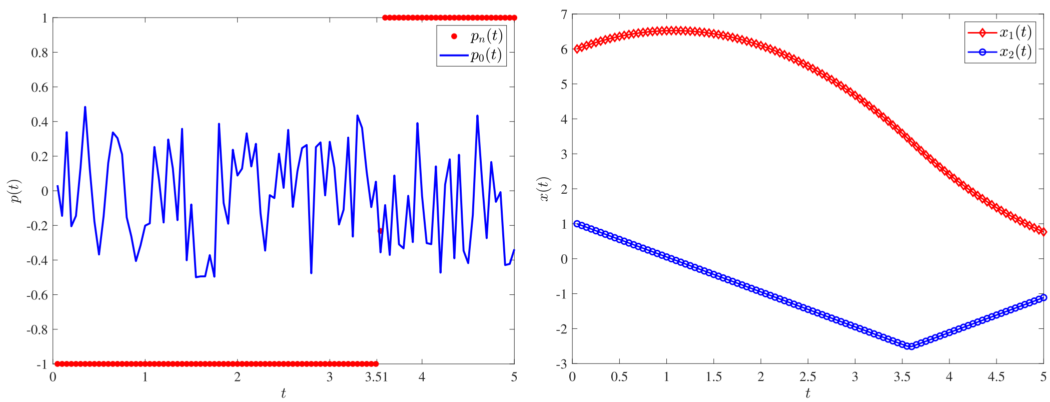
Example 5.2 (Rocket car [34]):

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}((x_1(5))^2 + (x_2(5))^2), \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = p(t) \quad \forall\, t \in [0, 5], \\ & x_1(0) = 6, \quad x_2(0) = 1, \\ & p(t) \in [-1, 1]. \end{aligned}$$



(a) Initial and optimal controls (b) Optimal trajectories

Figure 3. Numerical results for Example 5.1. (a) Initial and optimal controls. (b) Optimal trajectories.



(a) Initial and optimal controls (b) Optimal trajectories

Figure 4. Numerical results for Example 5.2. (a) Initial and optimal controls. (b) Optimal trajectories.

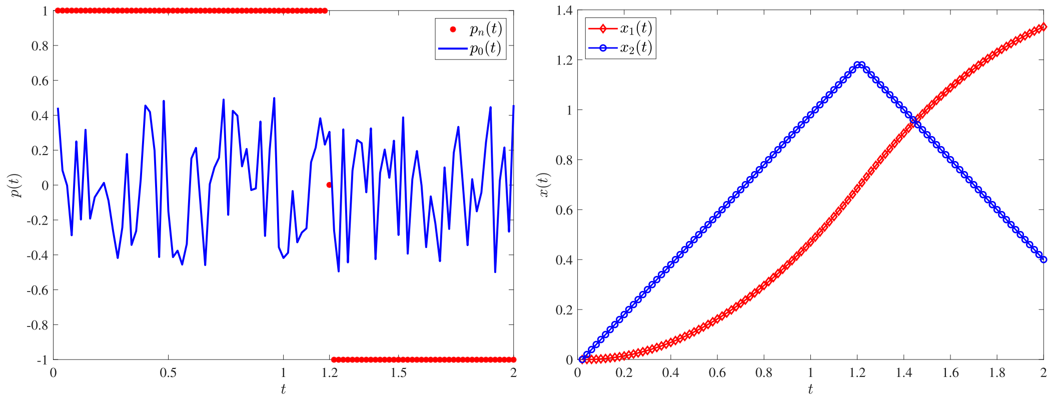
The exact optimal control of Example 5.2 is

$$p^* = \begin{cases} 1 & \text{if } t \in (3.517, 5]; \\ -1 & \text{if } t \in (0, 3.517]. \end{cases}$$

The approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.4 are plotted in Figure 4.

Example 5.3 (See [36]):

$$\begin{aligned} \text{minimize} \quad & -x_1(2) + (x_2(2))^2, \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = p(t) \quad \forall\, t \in [0, 2], \\ & x_1(0) = 0, \quad x_2(0) = 0, \\ & p(t) \in [-1, 1]. \end{aligned}$$



(a) Initial and optimal controls (b) Optimal trajectories

Figure 5. Numerical results for Example 5.3. (a) Initial and optimal controls. (b) Optimal trajectories.

Table 1. Comparison of the number of iterations and execution time of all algorithms in Examples 5.1–5.3.

	Example 5.1		Example 5.2		Example 5.3	
Algorithms	Iter.	CPU (s)	Iter.	CPU (s)	Iter.	CPU (s)
Our Alg. 3.1	201	0.1354	595	0.4340	417	0.1910
Our Alg. 3.2	90	0.0473	293	0.2501	207	0.1333
Our Alg. 3.3	90	0.0470	293	0.2278	207	0.1143
Our Alg. 3.4	224	0.1008	1000	1.0547	1000	0.6913
Our Alg. 3.5	101	0.0446	305	0.2857	350	0.2412
Our Alg. 3.6	101	0.0550	290	0.2712	339	0.2169
TG Alg. 3.1	202	0.0970	595	0.4884	417	0.1673
TG Alg. 3.2	91	0.0772	293	0.2204	207	0.1061
GTT Alg. 3.1	224	0.1147	1000	1.0601	1000	0.6316
GTT Alg. 3.2	101	0.0409	330	0.3138	346	0.2282
SI Alg. 3.1	91	0.0559	263	0.2043	181	0.1355

The exact optimal control of Example 5.3 is

$$p^*(t) = \begin{cases} 1 & \text{if } t \in [0, 1.2); \\ -1 & \text{if } t \in (1.2, 2]. \end{cases}$$

Figure 5 gives the approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.1.

Finally, the numerical performance of all algorithms in Examples 5.1–5.3 is shown in Table 1.

Remark 5.1: From Figures 3–5 and Table 1, we know that the suggested algorithms can work well when the terminal function is linear or nonlinear. Moreover, the step size of the Algorithm 4.3 suggested by Shehu et al. [30] requires the prior information of the Lipschitz constant of the mapping, and our algorithms can automatically update the iteration step size.

6. Conclusions

In this paper, we introduced several new iterative schemes to solve variational inequality problems in infinite-dimensional Hilbert spaces. Note that the variational inequality operator involved is pseudomonotone and uniformly continuous. These schemes are based on the MSEGM, the PCM, the inertial method, the viscosity-type method and the Mann-type method. Our methods use an Armijo-like step size criterion so that they do not need to know the Lipschitz constant of the mapping. Furthermore, they embedded inertial terms to accelerate the convergence speed of the algorithms. Strong convergence theorems of the suggested algorithms are obtained under reasonable assumptions on the parameters. The approaches established in this paper have competitive advantages over some known results in the literature and are more desirable in practical applications. In future work, we consider extending the results of this paper to Banach spaces with the help of the ideas in [37].

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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