A SELF-ADAPTIVE INERTIAL EXTRAGRADIENT ALGORITHM FOR SOLVING PSEUDO-MONOTONE VARIATIONAL INEQUALITY IN HILBERT SPACES

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Abstract. In this paper, we study the variational inequality involving pseudo-monotone and Lipschitz continuous mappings in real Hilbert spaces. We present a self-adaptive iterative method which combines the inertial technique and the Tseng's extragradient idea with a Armijo-like step size rule. The construction of our algorithm is without the prior knowledge of the Lipschitz constant of cost operators. Moreover, we also give some numerical experiments to demonstrate the efficiency of the our algorithm by comparing with existing ones.

1. Introduction

In this paper, we consider the following classical variational inequality problem in Hilbert spaces: find some points $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall \ \ x \in C, \tag{1.1}$$

where $A : H \to H$ is an operator and C is a nonempty closed convex subset in real Hilbert space H. This problem is denoted by VIP and the solution of this problem is denoted by Ω. It is known that the solution set Ω of variational inequality (1.1) is closed and convex. In recent decades, VIP has received a lot of attention by many authors and has became an important mathematical research subject. It is a useful mathematical model which have been widely applied to equilibrium problems in economics, optimization problems, complementarity problems and so on (see [1, 2, 3, 4, 5, 6, 7]). In order to solve the variational inequality problem and its variants, a number of iterative approaches have been proposed (see [8, 9, 10, 11, 12, 13, 14, 15]). Among these methods, the most notable and general directions are projection methods and regularized methods. In this paper, we focus on projection methods. Since the global convergence and simplicity of implementation, these methods have received much attention. Projection methods stem from Goldstein-Levitin-Polyak projection method (see [16, 17]) which consists of the following iteration: $x_{n+1} = P_C(x_n - \alpha Ax_n)$, where the step size α satisfies $\alpha \in (0, \frac{2\eta}{L^2})$, $A : H \to H$ is $\eta-$strongly monotone and $L-$Lipschitz and P_C denotes the Euclidean distance projection onto C. It is worth noting that the efficiency of this method depends heavily on the estimation of the Lipschitz constant L and the strongly monotone modulus η. As an improvement, in 1976, Korpelevich (see [18]) proposed an algorithm which called

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extragradient method:

$$\begin{cases} y_n = P_C(x_n - \alpha A x_n), \\ x_{n+1} = P_C(x_n - \alpha A y_n), \end{cases}$$

where $\alpha \in (0, \frac{1}{L})$ and P_C denotes the metric projection from H onto C. By comparison with the Goldstein-Levitin-Polyak projection method, an extra projection evaluation is required at each iteration in this method. But it is worth noting that the sequence converges if A is L-Lipschitz continuous and monotone, which avoids the strongly monotone assumption of A. In 2000, this method was expanded by Tseng (see [19]):

$$\begin{cases} y_n = P_C(x_n - \alpha A x_n), \\ x_{n+1} = y_n - \alpha(A y_n - A x_n), \end{cases}$$

where $\alpha \in (0, \frac{1}{L})$, A is a $L-$Lipschitz monotone continuous mapping and P_C denotes the metric projection from H onto C. It is worthwhile to point out that the Tseng's extragradient method only needs to calculate one projection. Recently, various modifications of Tseng's methods have been investigated in the framework of real infinite-dimensional spaces; see, e.g., [20, 21, 22, 23, 24, 25] and the references therein. In particular, Duong and Dang [26] introduced a new modification of Tseng's extragradient method (see Algorithm 1) for solving the VIP, where A is a monotone and Lipschitz continuous operator but the Lipschitz constant is not required to be known.

Algorithm 1

Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0 \in H$ be arbitrary.
Iterative Steps: Assume that $x_n \in H$ is known, calculate x_{n+1} as follows:
Step 1. Compute θ_n, such that $0 \leq \theta_n \leq \theta_n^*$, where $y_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying $\lambda \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|$.
Step 2. If $x_n = y_n$ then stop and x_n is the solution of VIP. Otherwise, compute

$$x_{n+1} = y_n - \lambda_n (A y_n - A x_n).$$

If $w_n = y_n$, then stop and y_n is a solution of inclusion problem (1.1). Otherwise,
Step 3. Set $n := n + 1$ and return to **Step 1**.

Inspired by the presented results, in this paper, we propose a self-adaptive iterative method which combines inertial technique and the Tseng's extragradient idea with a Armijo-like step size rule. In our algorithm the cost operator is pseudomonotone and Lipschitz continuous and the construction of our algorithm without the prior knowledge of the Lipschitz constant of cost operators. Under some suitable assumptions, we give a weak convergence theorem for solving pseudo-monotone variational inequality in framework of Hilbert spaces. Finally, we give some numerical experiments to demonstrate the efficiency of the our algorithm in comparisons with existing methods.

This paper is organized as follows. In Section 2, we recall some definitions and preliminary results, which will be used in our paper. In Section 3, we propose a self-adaptive inertial Tseng's extragradient algorithm for solving the variational inequality problem and analyze its convergence. Finally, numerical test examples and comparisons with other method are given in Section 4.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ and let C be a nonempty, closed and convex subset of H. In this paper, $x_n \rightharpoonup x$ indicates that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ indicates that the sequence $\{x_n\}$ converges strongly to x. One knows that for each $x \in H$, there exists a unique point in C, which is denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$, $\forall y \in C$. The mapping $P_C : H \to C$ is called the metric projection from H onto C. It is known that P_C has the following properties:

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H$;
(ii) $\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2, \forall x \in C, \forall y \in H$;
(iii) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \forall y \in C$.

For all $x, y \in H$ and $\lambda \in [0, 1]$, the following well-known results hold in a real Hilbert space: $\|\lambda x + (1-\lambda)y\|^2 + \lambda(1-\lambda)\|x-y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2$; $\|x\|^2 + \|y\|^2 = \|x+y\|^2 - 2\langle x, y\rangle$; $\|x+y\|^2 \leq \|x\|^2 + 2\langle x+y, y\rangle$.

We present some definitions and lemmas which will be used in our paper.

Definition 2.1. Let H be a real Hilbert space and let C be a closed convex subset of H. For all $x, y \in C$, an operator $A : H \to H$ is said to be:

(i) strongly monotone on C if there exists $\gamma > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \gamma\|x - y\|^2$;
(ii) monotone on C if $\langle Ax - Ay, x - y\rangle \geq 0$;
(iii) strongly pseudomonotone on C if there exists $\gamma > 0$ such that $\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, x - y\rangle \leq -\gamma\|x - y\|$;
(iv) pseudomonotone on C if $\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, x - y\rangle \leq 0$;
(v) L-Lipshhitz continuous on C if there exists $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$.

From the above definitions, it is easy to see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). In general, the converse implications are not true.

Lemma 2.2. [27] *Assume that $A : C \to H$ is a continuous and monotone operator. Then, x^* is a solution of (1.1) if and only if $\langle Ax, x - x^*\rangle \geq 0$, $\forall x \in C$.*

Lemma 2.3. [28] *Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H such that the following two conditions hold: (i) for each $x \in C$, $\lim_{n\to\infty}\|x_n - x\|$ exists, (ii) each sequential weak cluster point of $\{x_n\}$ is in C. Then, $\{x_n\}$ converges weakly to a point in C.*

Lemma 2.4. [29] *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences in $[0, +\infty)$ such that $a_{n+1} \leq a_n + c_n(a_n - a_{n-1}) + b_n$, $\forall n \geq 1$, $\sum_{n=1}^{\infty} b_n \leq \infty$, and suppose that there exists a real number α with $0 \leq c_n \leq \alpha < 1$ for all $n \in N$. Then, the following conclusions hold:*

(i) $\sum_{n=1}^{\infty}[a_n - a_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{t, 0\}$;
(ii) there exists $a^* \in [0, +\infty)$ such that $\lim_{n\to\infty} a_n = a^*$.

3. Main Results

In this section, we introduce a self-adaptive inertial Tseng's extragradient algorithm for solving the variational inequalities which combines the Tseng's extragradient method and inertial methods. Throughout this section, we assume that the operator $A : H \to H$ is a pseudo-monotone and Lipschitz mapping on H but the Lipschitz constant of A is not need to be known. We also assume that the solution set of VIP, which is denoted by Ω, is nonempty.

Algorithm 2

Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0, \frac{1}{3}]$, $\{\alpha_n\}$ is non-decreasing and $0 \leq \alpha_n \leq \alpha \leq \sqrt{5} - 2$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Assume that $x_n \in H$ is known, calculate x_{n+1} as follows:
Step 1. Compute $w_n = x_n + \alpha_n(x_n - x_{n-1})$.
Step 2. Compute $y_n = P_C(w_n - \lambda_n A w_n)$,
where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda \|Aw_n - Ay_n\| \leq \mu \|w_n - y_n\|. \tag{3.1}$$

Step 3. If $w_n = y_n$, then stop and w_n is the solution of VIP. Otherwise, compute $x_{n+1} = y_n - \lambda_n(Ay_n - Aw_n)$.
Step 4. Set $n := n + 1$ and return to **Step 1**.

Lemma 3.1. *In each iteration of Algorithm 2, the Armijo-like search rule (3.1) is well defined and* $\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma$.

Proof. Since A is $L - Lipschitz$ continuous on H, we have $\|Aw_n - A(P_C(w_n - \lambda_n A w_n))\| \leq L\|w_n - P_C(w_n - \lambda_n A w_n)\|$. This indicates that

$$\frac{\mu}{L}\|Aw_n - A(P_C(w_n - \lambda_n A w_n))\| \leq \mu\|w_n - P_C(w_n - \lambda_n A w_n)\|.$$

This implies that inequality (3.1) holds immediately provided $\lambda \leq \frac{\mu}{L}$. Hence, λ_n is well defined. It is easy to see that, $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$, then this lemma is proved. Suppose that $\lambda_n < \gamma$. Since λ_n is the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying (3.1) ($l \in (0,1)$), which implies that $\frac{\lambda_n}{l}$ must contradict with inequality (3.1), i.e., $\frac{\lambda_n}{l}\|Aw_n - A(P_C(w_n - \frac{\lambda_n}{l} w_n))\| > \mu\|w_n - P_C(w_n - \frac{\lambda_n}{l} w_n)\|$. Since A is $L - Lipschitz$ continuous on H, we have $\frac{\mu}{\frac{\lambda_n}{l}}\|w_n - P_C(w_n - \lambda_n A w_n)\| < L\|w_n - P_C(w_n - \frac{\lambda_n}{l} w_n)\|$, that is, $\lambda_n > \frac{\mu l}{L}$. This completes the proof. □

Lemma 3.2. *Suppose that* $\{x_n\}$ *is a sequence generated by Algorithm 2. Then,*

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - (1 - \mu^2)\|y_n - w_n\|^2, \quad \forall p \in \Omega. \tag{3.2}$$

Proof. Suppose $p \in \Omega$, by the definition of x_{n+1}, one has

(3.3)
$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|y_n-p\|^2+\lambda_n^2\|Ay_n-Aw_n\|^2-2\lambda_n\langle y_n-p, Ay_n-Aw_n\rangle\\ &= \|w_n-p\|^2+\|w_n-y_n\|^2-2\langle y_n-w_n, y_n-w_n\rangle+2\langle y_n-w_n, y_n-p\rangle\\ &\quad +\lambda_n^2\|Ay_n-Aw_n\|^2-2\lambda_n\langle y_n-p, Ay_n-Aw_n\rangle\\ &= \|w_n-p\|^2-\|w_n-y_n\|^2+2\langle y_n-w_n, y_n-p\rangle+\lambda_n^2\|Ay_n-Aw_n\|^2\\ &\quad -2\lambda_n\langle y_n-p, Ay_n-Aw_n\rangle.\end{aligned}$$

Since $y_n = P_C(w_n-\lambda_n Aw_n)$, one gets

$$\langle y_n-w_n+\lambda_n Aw_n, y_n-p\rangle \le 0. \tag{3.4}$$

This implies

$$\langle y_n-w_n, y_n-p\rangle \le -\langle \lambda_n Aw_n, y_n-p\rangle. \tag{3.5}$$

By the definition of λ_n, one sees from (3.3) and (3.5) that

(3.6)
$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|w_n-p\|^2-\|w_n-y_n\|^2-2\lambda_n\langle Aw_n, y_n-p\rangle+\lambda_n^2\|Ay_n-Aw_n\|^2\\ &\quad -2\lambda_n\langle y_n-p, Ay_n-Aw_n\rangle\\ &\le \|w_n-p\|^2-\|w_n-y_n\|^2+\mu^2\|y_n-w_n\|^2-2\lambda_n\langle y_n-p, Ay_n\rangle.\end{aligned}$$

Since $p \in VI(C,A)$, for all $x \in C$, one has $\langle Ap, x-p\rangle \ge 0$. By the pseudo-monotonicity of A, one has $\langle Ax, x-p\rangle \ge 0$, for all $x \in C$. Taking $x := y_n \in C$, one arrives at

$$\langle Ay_n, p-y_n\rangle \le 0. \tag{3.7}$$

Combining (3.6) and (3.7), one obtains

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|w_n-p\|^2-\|y_n-w_n\|^2+\mu^2\|y_n-w_n\|^2\\ &\le \|w_n-p\|^2-(1-\mu^2)\|y_n-w_n\|^2.\end{aligned} \tag{3.8}$$

This completes the proof. □

Lemma 3.3. *Let $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ are three sequences generated by Algorithm 2. Suppose that $\lim_{n\to\infty}\alpha_n = 0$. Then, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in \Omega$. Furthermore, $\|x_n-w_n\| \to 0$, $\|y_n-w_n\| \to 0$, and $\|y_n-x_n\| \to 0$ (as $n\to\infty$).*

Proof. By the definition of x_{n+1} and (3.1), one arrives at

$$\|x_{n+1}-y_n\| = \lambda_n\|Ay_n-Aw_n\| \le \mu\|y_n-w_n\|.$$

Therefore, $\|x_{n+1}-w_n\| \le \|x_{n+1}-y_n\|+\|y_n-w_n\| \le (1+\mu)\|y_n-w_n\|$. This implies that

$$\|y_n-w_n\| \ge \frac{1}{1+u}\|x_{n+1}-w_n\|. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\|x_{n+1}-p\|^2 \le \|w_n-p\|^2-\frac{1-\mu}{1+\mu}\|x_{n+1}-w_n\|^2. \tag{3.10}$$

Since $\mu \in (0, \frac{1}{3}]$, we have

$$\frac{1-\mu}{1+\mu} \geq \frac{1}{2}. \tag{3.11}$$

Combing (3.11) and (3.10), we get

$$\|x_{n+1}-p\|^2 \leq \|w_n - p\|^2 - \frac{1}{2}\|x_{n+1}-w_n\|^2. \tag{3.12}$$

From $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and (3.8), we get

$$\begin{aligned} \|w_n - p\|^2 &= \|(1+\alpha_n)(x_n - p) - \alpha_n(x_{n-1}-p)\|^2 \\ &\leq (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1}-p\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.13}$$

On the other hand, by the definition of w_n, we also have

$$\begin{aligned} \|x_{n+1}-w_n\|^2 &\geq \|x_{n+1}-x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1}-x_n\|\|x_n - x_{n-1}\| \\ &\geq (1-\alpha_n)\|x_{n+1}-x_n\|^2 + (\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.14}$$

Since $\{\alpha_n\}$ is non-decreasing, we obtain from (3.12), (3.13) and (3.14) that

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1}-p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\quad - \frac{1}{2}(1-\alpha_n)\|x_{n+1}-x_n\|^2 - \frac{1}{2}(\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2 \\ &= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1}-p\|^2 - \frac{1}{2}(1-\alpha_n)\|x_{n+1}-x_n\|^2 \\ &\quad + [\frac{1}{2}\alpha_n^2 + \frac{3}{2}\alpha_n]\|x_n - x_{n-1}\|^2 \\ &\leq (1+\alpha_{n+1})\|x_n - p\|^2 - \alpha_n\|x_{n-1}-p\|^2 - \gamma_n\|x_{n+1}-x_n\|^2 \\ &\quad + \mu_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.15}$$

where $\gamma_n := \frac{1}{2}(1-\alpha_n)$ and $\mu_n := \frac{1}{2}\alpha_n + \frac{3}{2}\alpha_n$. Let $T_n = \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \mu_n\|x_n - x_{n-1}\|^2$. It follows from (3.15) that

$$\begin{aligned} T_{n+1} - T_n &\leq \|x_{n+1}-p\|^2 - (1+\alpha_n)\|x_n - p\|^2 + \alpha_n\|x_{n-1}-p\|^2 \\ &\quad + \mu_{n+1}\|x_{n+1}-x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ &\leq -(\gamma_n - \mu_{n+1})\|x_{n+1}-x_n\|^2 \\ &\leq -(\frac{1}{2}(1-\alpha_{n+1}) - \frac{1}{2}\alpha_{n+1}^2 - \frac{3}{2}\alpha_{n+1})\|x_{n+1}-x_n\|^2 \\ &\leq -\frac{1-4\alpha-\alpha^2}{2}\|x_{n+1}-x_n\|^2. \end{aligned}$$

Hence, we get

$$T_{n+1} - T_n \leq -\delta\|x_{n+1}-x_n\|^2, \tag{3.16}$$

where $\delta := \frac{1-4\alpha-\alpha^2}{2}$. Since $0 \leq \alpha \leq \sqrt{5}-2$, we have $\delta \geq 0$. This implies that $T_{n+1} - T_n \leq 0$, $\forall n \geq 1$. Therefore, the sequence $\{T_n\}$ is nonincreasing. On the

other hand, we also have
(3.17)

$$T_{n+1} = \|x_{n+1} - p\|^2 - \alpha_{n+1}\|x_n - p\|^2 + \mu_{n+1}\|x_{n+1} - x_n\|^2 \geq -\alpha_{n+1}\|x_n - p\|^2,$$

and $T_n = \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \mu_n\|x_n - x_{n-1}\| \geq \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2$. For all $n \geq 1$, this implies that

$$\|x_n - p\|^2 \leq \alpha_n\|x_{n-1} - p\|^2 + T_n \leq \alpha\|x_{n-1} - p\|^2 + T_1. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned} -T_{n+1} &\leq \alpha_{n+1}\|x_n - p\|^2 \leq \alpha(\alpha\|x_{n-1} - p\|^2 + T_1) \\ &\leq \alpha^2\|x_{n-1} - p\|^2 + \alpha T_1 \leq \cdots \leq \alpha^n\|x_1 - p\|^2 + T_1(\alpha^n + \cdots + \alpha) \\ &\leq \|x_1 - p\|^2 + \frac{\alpha T_1}{1 - \alpha}. \end{aligned} \tag{3.19}$$

It follows from (3.16) and (3.19) that

$$\delta \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 \leq T_1 - T_{k+1} \leq \|x_1 - p\|^2 + \frac{T_1}{1 - \alpha}.$$

This implies

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 \leq +\infty. \tag{3.20}$$

Thus, we get

$$\|x_{n+1} - x_n\| \to 0. \tag{3.21}$$

Since $w_n = x_n + \alpha_n(x_n - x_{n-1})$, we have

$$\|x_{n+1} - w_n\|^2 = \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle.$$

Hence, from (3.21), we get

$$\|x_{n+1} - w_n\| \to 0. \tag{3.22}$$

Combining (3.13), (3.20) and Lemma 2.4, one sees that there exists an $l \in R$ such that

$$\lim_{n\to\infty} \|x_n - p\|^2 = l. \tag{3.23}$$

From (3.13), we get

$$\lim_{n\to\infty} \|w_n - p\|^2 = l. \tag{3.24}$$

Combining (3.21) and (3.22), we have

$$0 \leq \|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0. \tag{3.25}$$

It follows from (3.8) that

$$(1 - \mu^2)\|y_n - w_n\|^2 \leq \|w_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.26}$$

Since $\mu \in (1, \frac{1}{3}]$, we have

$$\lim_{n\to\infty} \|y_n - w_n\|^2 = 0. \tag{3.27}$$

From (3.25) and (3.27), we have

$$\lim_{n\to\infty}\|x_n-y_n\|^2 \le \lim_{n\to\infty}\|x_n-w_n\|^2+\lim_{n\to\infty}\|w_n-y_n\|^2=0. \tag{3.28}$$

We see from (3.23) that for each $p \in \Omega$, we have $\lim_{n\to\infty}\|x_n-p\|$ exists. Therefore, $\{x_n\}$ is bounded. This completes the proof. □

Lemma 3.4. *Suppose that $\{x_n\}$ is a sequence generated by Algorithm 2. Then each weak limit point of $\{x_n\}$ is a solution of VIP (1.1).*

Proof. Let z be a weak limit point of $\{x_n\}$. This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $z \in H$. Now, we show that $\liminf_{k\to\infty}\langle Aw_{n_k}, x-w_{n_k}\rangle \ge 0$. From Lemma 3.3, we have $\|y_{n_k}-w_{n_k}\| \to 0$ and $\|x_{n_k}-w_{n_k}\| \to 0$. Thus, $w_{n_k} \rightharpoonup z$ and $y_{n_k} \rightharpoonup z$. Since C is close, we get $z \in C$. Since $y_{n_k}=P_C(w_{n_k}-\lambda_{n_k}Aw_{n_k})$, we have $\langle w_{n_k}-\lambda_{n_k}Aw_{n_k}-y_{n_k}, x-y_{n_k}\rangle \le 0$, $\forall x \in C$. This equivalents to $\frac{1}{\lambda_{n_k}}\langle w_{n_k}-y_{n_k}, x-y_{n_k}\rangle \le \langle Aw_{n_k}, x-y_{n_k}\rangle, \forall x \in C$. This deduces that

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k}-y_{n_k}, x-y_{n_k}\rangle + \langle Aw_{n_k}, y_{n_k}-w_{n_k}\rangle \le \langle Aw_{n_k}, x-w_{n_k}\rangle, \forall x \in C. \tag{3.29}$$

From Lemma 3.3, we have $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-w_n\| \to 0$. Hence $\{w_n\}$ is also bounded. Since A is Lipschitz continuous on H, we have $\{Aw_{n_k}\}$ is bounded. We get from Lemma 3.1 that $\lambda_{n_k} \ge \frac{\mu l}{L}$. Let $k \to \infty$ in (3.29), we have

$$\liminf_{k\to\infty}\langle Aw_{n_k}, x-w_{n_k}\rangle \ge 0. \tag{3.30}$$

On the other hand, we see that

$$\langle Ay_{n_k}, x-y_{n_k}\rangle = \langle Ay_{n_k}-Aw_{n_k}, x-w_{n_k}\rangle + \langle Aw_{n_k}, x-w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k}-y_{n_k}\rangle. \tag{3.31}$$

From Lemma 3.3, we have $\|w_{n_k}-y_{n_k}\| \to 0 (k \to \infty)$, and by the Lipschiz continuous of A, we obtian

$$\lim_{k\to\infty}\|Ay_{n_k}-Aw_{n_k}\|=0 \tag{3.32}$$

Combining with (3.30) and (3.31), we have

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x-y_{n_k}\rangle \ge 0. \tag{3.33}$$

In the follows, we show that $z \in \Omega$. Let $\{\rho_k\}$ be a decreasing positive sequence and converges to zero. By (3.33), we can construct a strictly increasing positive integers sequence $\{N_k\}$ satisfying

$$\langle Ay_{n_j}, x-y_{n_j}\rangle + \rho_k \ge 0, \forall j \ge N_k. \tag{3.34}$$

For each k, set $p_{N_k}=\frac{Ay_{N_k}}{\|Ay_{N_k}\|^2}$. Thus, we have $\langle Ay_{N_k}, p_{N_k}\rangle = 1$. From (3.34), we have $\langle Ay_{N_k}, x+\rho_k p_{N_k}-y_{N_k}\rangle \ge 0$. Since A is pseudo-monotone mapping, we have $\langle A(x+\rho_k p_{N_k}), x+\rho_k p_{N_k}-y_{N_k}\rangle \ge 0$. This implies that

$$\langle Ax, x-y_{N_k}\rangle \ge \langle Ax - A(x+\rho_k p_{N_k}), x+\rho_k p_{N_k}-y_{N_k}\rangle - \rho_k\langle Ax, p_{N_k}\rangle. \tag{3.35}$$

Since $y_{n_k} \rightharpoonup z$ and A is Lipschitz continuous, we obtain $Ay_{N_k} \rightharpoonup Az$. Assume that $Az \ne 0$(otherwise, z is the solution). Since the normmapping is sequentially weakly lower semicontinuous, we have $0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|$. Since $\rho_k \to 0$ as $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty} \|\rho_k p_{N_k}\| = \limsup_{k\to\infty} (\frac{\rho_k}{\|Ay_{N_k}\|}) \leq \frac{\limsup_{k\to\infty} \rho_k}{\liminf_{k\to\infty} \|Ay_{N_k}\|} = 0.$$

Thus, $\lim_{k\to\infty} \rho_k p_{N_k} = 0$. Letting $k \to \infty$ in the right side of (3.35), we have $\liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0$. Therefore, for all $x \in C$, we have $\langle Ax, x - z\rangle = \lim_{k\to\infty}\langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0$. By Lemma 2.2, we obtain $z \in \Omega$. This completes the proof. □

Theorem 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then, $\{x_n\}$ converges weakly to an element of Ω.*

Proof. In Lemma 3.3 and Lemma 3.4, we obtain the following conclusions:

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists, for all $p \in \Omega$,
(ii) If $x_{n_k} \rightharpoonup z$ then $z \in \Omega$.

Therefore, it follows from Lemma 2.3 that the sequence $\{x_n\}$ converges weakly to an element of Ω. This completes the proof. □

4. Numerical experiments

In this section, we provide three numerical examples to illustrate the performance of our Algorithm 2. Furthermore, we also compare the convergence rate of our Algorithm 2 with Algorithm 1, which is defined without the inertial term (i.e., $\alpha_n = 0$). All programs are performed in MATLAB2018a on a PC Desktop Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.800 GHz, RAM 8.00 GB.

Example 4.1. Take $A : R^2 \to R^2$ by $A(x, y) = (x + y + \sin(x), -x + y + \sin(y))$, $\forall x, y \in R$. It is easy to see that A is $\sqrt{10}$-Lipschitz continuous and 1-strongly monotone. Therefore, the variational inequality (1.1) has a unique solution $x^* = (0, 0)$.

Our parameters are set as follows. Let $\gamma = 1$, $l = 0.5$, $\mu = 0.3$. Let x_0, x_1 are randomly generated by the MATLAB function $k\times$rand(m,1) (where, Case I: $k = 1$, Case II: $k = -1$, Case III: $k = 10$, Case IV: $k = -10$). rand(m,1) produces a random vector of m rows and 1 column, and each element is located at $(0, 1)$. Let $C = \{x \in R^2 | -5e \leq x \leq 5e\}$, where $e = (1, 1)$. $E_n = \|x_n - x^*\|_2$ denote the errors of the iterative Algorithms 2 and the stop criterion is $E_n < 10^{-8}$. We obtain the following numerical results shown in Table 1 and Figure 1.

Table 1. Compare the number of iterations for Example 4.1

Cases	$\alpha_n = 0.2$	$\alpha_n = 0.1$	$\alpha_n = 0$
I	42	50	56
II	50	58	65
III	59	68	77
IV	57	67	76

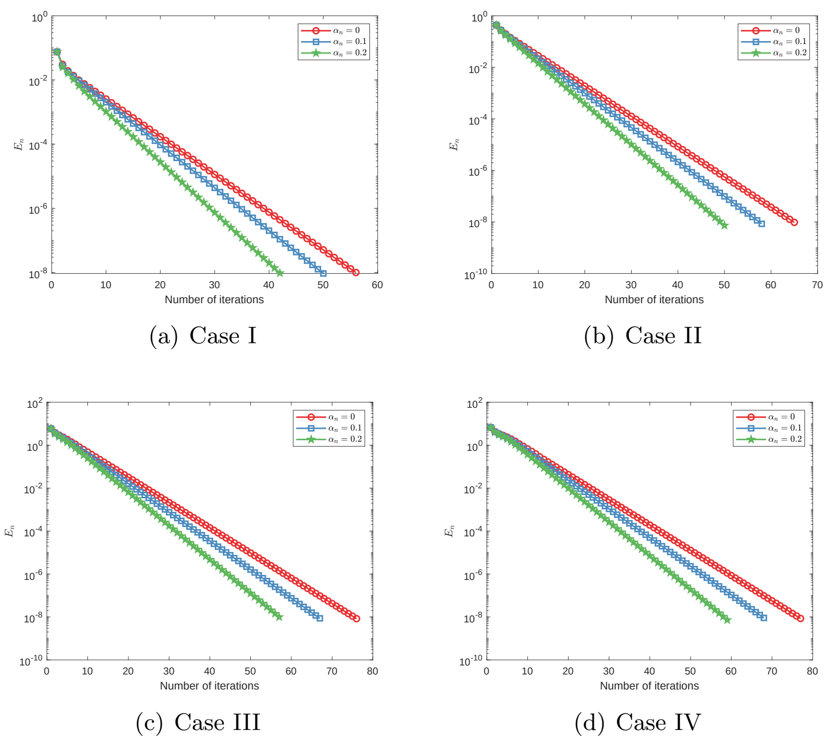
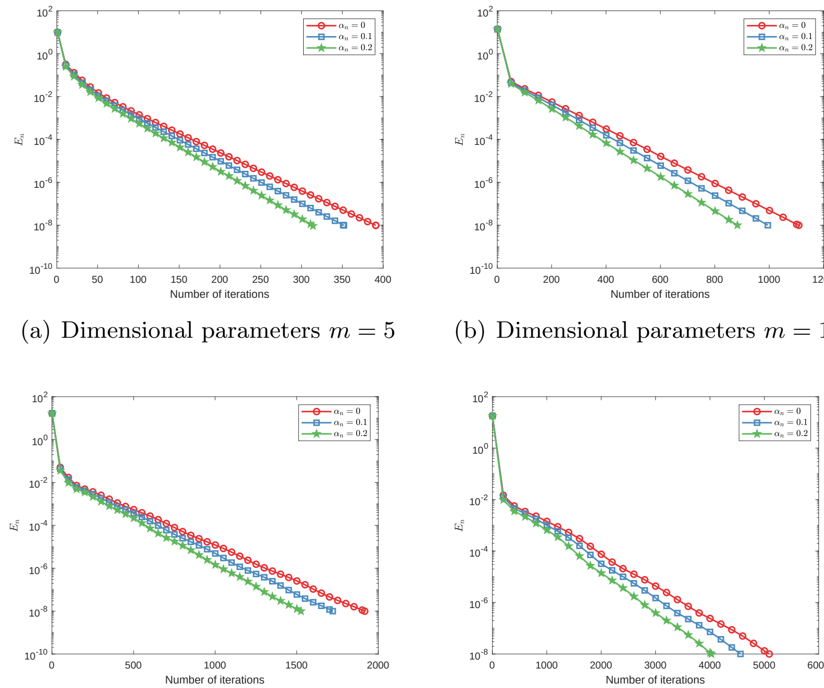


FIGURE 1. Convergence behavior of iteration error $\{E_n\}$ with different initial points for Example 4.1

Example 4.2. We consider the following linear operator: $A : R^m \to R^m$, where $A(x) = Mx + q$, $M = NN^T + S + D$, N is a $m \times m$ matrix, S is a $m \times m$ skew-symmetric matrix, D is a $m \times m$ diagonal matrix with the diagonal entries are nonnegative, all entries of q, N, S are generated randomly in $[-5, 5]$ and all entries of D are generated randomly in $[1, 5]$. We see from [8] that M is positive semi-definite. Therefore, A is pseudo-monotone and Lipschitz continuous with constant $L = \|M\|$. The feasible set C is given by $C = \{x \in \mathbb{R}^m : -5 \leq x_i \leq 5, \quad i = 1, \ldots, m\}$. In this example, the setting of our parameters are the same as in Example 4.1, and $x_0 = x_1$ =10*rand(m,1). The errors are denoted by $E_n = \|y_n - w_n\|$ and $E_n < 10^{-8}$ is our stopping criterion. We test the convergence of Algorithm 2 under different dimensional m and numerical results are shown in Table 2 and Figure 2.

TABLE 2. Compare the number of iterations for Example 4.2

Dimensional	$\alpha_n = 0.2$	$\alpha_n = 0.1$	$\alpha_n = 0$
5	314	352	391
10	883	995	1109
20	1526	1721	1917
50	4027	4561	5089



(a) Dimensional parameters $m = 5$ (b) Dimensional parameters $m = 10$

(c) Dimensional parameters $m = 20$ (d) Dimensional parameters $m = 50$

FIGURE 2. Convergence behavior of iteration error $\{E_n\}$ with different dimensional m for Example 4.2

Example 4.3. In this example, we consider the following nonlinear optimization problem:

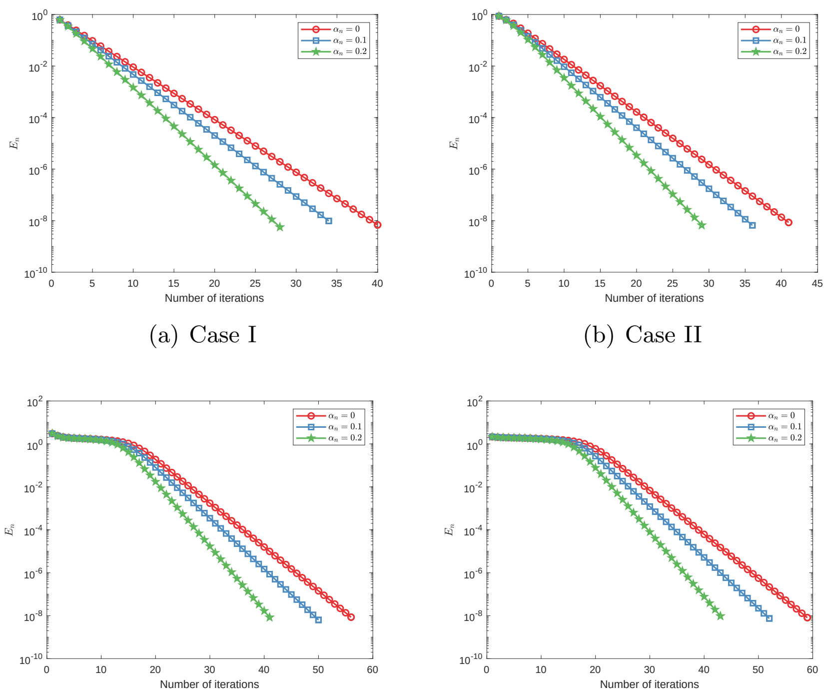
$$\begin{aligned} &\min f(x) = 1 + x_1^2 - e^{-x_2^2} \\ &\text{s.t.} \ -5e \leq x \leq 5e, \end{aligned} \tag{4.1}$$

where $x = (x_1, x_2)^T \in R^2$, $e = (1,1)^T$. One observes that $f(x)$ is a convex function with $\nabla f(x) = \left(2x_1, 2x_2 e^{-x_2^2}\right)^T$ and the unique solution is $x^* = (0,0)^T$. Taking $A(x) = \nabla f(x)$, it is easy to verify that $A(x)$ is pseudomonotone and Lipschizt continuous on the closed and convex subset $C = \left\{x \in \mathbb{R}^2 : -5e \leq x \leq 5e\right\}$. The selections of our parameters are the same as in Example 4.1, $E_n = \|x_n - x^*\|_2$ denote the errors of Algorithms 2 and the stop criterion is also $E_n < 10^{-8}$. We select four initial values and test our algorithm under different inertial parameters. We obtain the numerical results shown in Table 3 and Figure 3.

Remark 4.4. From results of numerical experiments, we see that sequences generated by our proposed Algorithm 2 involving the inertial technique converge more efficiently than by Algorithm 1, which is without the inertial terms.

TABLE 3. Compare the number of iterations for Example 4.3

Cases	$\alpha_n = 0.2$	$\alpha_n = 0.1$	$\alpha_n = 0$
I	28	34	40
II	29	36	41
III	41	50	56
IV	43	52	59



(a) Case I (b) Case II

(c) Case III (d) Case IV

FIGURE 3. Convergence behavior of iteration error $\{E_n\}$ with different initial points for Example 4.3

REFERENCES

[1] W. Takahashi, C.F. Wen, J.C. Yao, *The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space*, Fixed Point Theory **19** (2018), 407-419. 1

[2] X. Qin, N.T. An, *Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets*, Comput. Optim. Appl. (2019), doi: 10.1007/s10589-019-00124-7. 1

[3] Q.H. Ansari, F. Babu, J.C. Yao, *Regularization of proximal point algorithms in Hadamard manifolds*, J. Fixed Point Theory Appl. **21** (2019), Article ID 25. 1

[4] S.Y. Cho, W. Li, S.M. Kang, *Convergence analysis of an iterative algorithm for monotone operators*, J. Inequal. Appl. **2013** (2013), Article ID 199. 1

[5] N.T. An, N.M. Nam, *Solving k-center problems involving sets based on optimization techniques,* J. Global Optim. (2019), doi: 10.1007/s10898-019-00834-6. 1

[6] X. Zhao, *Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems*, Appl. Math. Optim. **78** (2018), 613-641. 1
[7] X. Qin, J.C. Yao, *A viscosity iterative method for a split feasibility problem*, J. Nonlinear Convex Anal. **20**, (2019), 1497-1506. 1
[8] P.T. Harker, J.S. Pang, *A damped-Newton method for the linear complementarity problem,* Lect. Appl. Math. 26 (1990), 265-284. 1, 4.2
[9] S.Y. Cho, *Strong convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space*, J. Appl. Anal. Comput. **8** (2018), 19-31. 1
[10] X. Qin, A. Petrusel, *CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces*, J. Nonlinear Convex Anal. **19** (2018), 157-165. 1
[11] W. Takahahsi, J.C. Yao, *The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces,* J. Nonlinear Convex Anal. **20**, (2019), 173-195. 1
[12] S.Y. Cho, *Generalized mixed equilibrium and fixed point problems in a Banach space*, J. Nonlinear Sci. Appl. **9**, (2016), 1083-1092. 1
[13] S.S. Chang, C.F. Wen, J.C. Yao, *Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces*, Optimization **67**, (2018), 1183-1196. 1
[14] S.S. Chang, C.F. Wen, J.C. Yao, *Zero point problem of accretive operators in Banach spaces*, Bull. Malaysian Math. Sci. Soc. **42**, (2019), 105-118. 1
[15] X. Qin, S.Y. Cho, L. Wang, *A regularization method for treating zero points of the sum of two monotone operators*, Fixed Point Theory Appl. **2014** (2014), Article ID 75. 1
[16] A.A. Goldstein, *Convex programming in hilbert space*, Bull. Amer. Math. Soc. **70** (1964), 709-710. 1
[17] E.S. Levitin, B.T. Polyak, *Constrained minimization problems*, USSR Comput. Math. Math. Phys. 6 (1966), 1-50. 1
[18] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematicheskie Metody **12** (1976), 747-756. 1
[19] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim. **38** (2000), 431-446. 1
[20] X. Qin, J.C. Yao, *Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators*, J. Inequal. Appl. **2016** (2016), Article ID 232. 1
[21] D.V. Thong, P.T. Vuong, *Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities*, Optimization, (2019), DOI: 10.1080/02331934.2019.1616191. 1
[22] D.V. Thong, D.V. Hieu, *Inertial extragradient algorithms for strongly pseudomonotone variational inequalities*, J. Comput. Appl. Math. **341** (2018), 80-98. 1
[23] G. Aviv, Y. Shehu, *An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces*, Optimization, **68** (2019), 13-32. 1
[24] S.Y. Cho, S.M. Kang, *Approximation of common solutions of variational inequalities via strict pseudocontractions*, Acta Math. Sci. **32**, (2012), 1607-1618. 1
[25] B.A. Bin Dehaish, *Weak and strong convergence of algorithms for the sum of two accretive operators with applications,* J. Nonlinear Convex Anal. **16** (2015), 1321-1336. 1
[26] D.V. Thong, D.V. Hieu, *Weak and strong convergence theorems for variational inequality problems*, Numer. Algor. **78** (2018), 1045-1060. 1
[27] W. Takahashi, *Nonlinear Functional Analysis-Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000. 2.2
[28] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597. 2.3
[29] F. Alvarez, H. Attouch. *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal. 9 (2001), 3-11. 2.4

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