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Inertial splitting algorithms for nonlinear operators of pseudocontractive and accretive types

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ABSTRACT
In this paper, inertial splitting algorithms for nonlinear operators of pseudocontractive and accretive types are proposed. Weak and strong convergence theorems are established in uniformly convex and q-uniformly smooth Banach spaces. Numerical examples are given to illustrate the effectiveness of our proposed algorithms.

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1. Introduction

Let E be a Banach space with dual space E^*. Recall that the generalized duality mapping J_q is defined by

$$J_q(x) := \{z \in E^* : \langle z, x\rangle = \|x\|^q, \|z\| = \|x\|^{q-1}\}, \quad \forall\, x \in E,$$

where $\langle \cdot, \cdot\rangle$ is the duality pairing between E and E^*. If $q = 2$, then J_2 is called the normalized duality mapping. Usually, we denote the normalized duality mapping J_2 by J. From now on, we use j_q to denote the single-valued generalized duality mapping. Let U be a nonempty subset of E, and let $T : U \to U$ be a mapping. We use Fix(T) to denote the fixed point set of T, that is, Fix(T) $= \{u \in E : Tu = u\}$. Recall that T is L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Tu - Tv\| \le L\|u-v\|, \quad \forall\, u, v \in U.$$

If $L \in (0, 1)$, then T is called a contractive mapping. If $L = 1$, then T is called a nonexpansive mapping. T is κ-strictly pseudocontractive if there exists a constant $\kappa \in (0, 1)$ such that

$$\langle Tu - Tv, j_q(u-v)\rangle \le \|u-v\|^q - \kappa\|(u-Tu) - (v - Tv)\|^q, \quad \forall\, u, v \in U,$$

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for some $j_q(u-v) \in J_q(u-v)$. T is pseudocontractive if

$$\langle Tu - Tv, j_q(u-v)\rangle \leq \|u-v\|^q, \quad \forall\, u, v \in U,$$

for some $j_q(u-v) \in J_q(u-v)$. T is strongly pseudocontractive if there exists a $\kappa \in (0,1)$ such that

$$\langle Tu - Tv, j_q(u-v)\rangle \leq \kappa\|u-v\|^q, \quad \forall\, u, v \in U,$$

for some $j_q(u-v) \in J_q(u-v)$. One knows that κ-strictly pseudocontractions are Lipschitz continuous with constant $L = \frac{\kappa+1}{\kappa}$ and the class of strongly pseudocontractive mappings is independent of the class of κ-strict pseudocontractions (see, e.g. [1,2]).

Approximating fixed points of nonexpansive mappings and their extensions, which is an important issue in nonlinear analysis and convex optimization, find wide applications in signal processing, medical imaging, economics, traffic networks and so on. Recently, many authors have done a lot of extensive research on nonexpansive mappings via iterative methods (see, e.g. [3–6]). Mann iterative process is efficient and attractive for dealing with fixed points of nonexpansive mappings

$$x_{n+1} = \lambda_n T x_n + (1-\lambda_n)x_n, \quad \forall\, n \geq 0,$$

where $\{\lambda_n\}$ is a real sequence in $(0,1)$ and T is a nonexpansive mapping. However, there is a flaw in this method that it is weakly convergent even in Hilbert spaces. In fact, in practical applications, we prefer strong convergence results to weak convergence results. A common way to obtain strong convergence results is to approximate a nonexpansive mapping with the aid of contractive mappings. In 2000, Moudafi [7] proposed a viscosity method for nonexpansive mappings and gave strong convergent results in Hilbert spaces. After that, many authors studied this method and extended it to Banach spaces (see, e.g. [8–10]). In particular, Qin et al. [11] introduced a splitting method and obtained its convergence analysis in the setting of real Banach spaces. However, from the viewpoint of the convergence speed, their method needed to be improved. In 1964, Polyak [12] introduced an inertial extrapolation for solving the smooth convex minimization problem. Later, this method was applied to accelerate the convergence speed of various iterative algorithms (see, e.g. [5,13–16]). Recently, various algorithms with inertial effects were studied in Banach spaces. In particular, Cholamjiak and Shehu [15] proposed a splitting algorithm with inertial extrapolation for inclusion problems in Banach spaces. In 2020, Shehu and Gibali [16] introduced an inertial Krasnoselskii-Mann algorithm for finding fixed points of nonexpansive mappings. A strong convergence was obtained for their inertial generalized forward-backward splitting method.

In this paper, we propose four viscosity-type splitting algorithms with inertial extrapolation for common solutions of the fixed-point problems of strictly pseudocontractive mappings and the inclusion problems of two accretive operators in uniformly convex and q-uniformly smooth Banach spaces. We give some applications and numerical examples to illustrate the convergence efficiency of our algorithms. It is worth noting that the mappings involved in our algorithms are κ-strictly pseudocontractive and accretive, which are more general than the above results. The main results presented in this paper extend and complement the recent results obtained in [11,15,16].

The structure of this paper is as follows. In Section 2, we give some lemmas, which will be used in our convergence analysis. In Section 3, we present two weak convergent splitting algorithms and two strong convergent splitting algorithms for our common solution problems in Banach spaces. In Section 4, some applications and numerical examples are proposed. Section 5, the last section, is the concluding remark of this paper.

2. Preliminaries

Let E be a real Banach space. The convex modulus of E is defined by the following function:

$$\delta_E(r) = \inf\left\{1 - \frac{\|u+v\|}{2} : u, v \in E, \|u\| = \|v\| = 1, \|u-v\| \geq r\right\},$$

which maps $(0,2]$ to $[0,1]$. If $\delta_E(r) > 0$ for any $r \in (0,2]$, then E is a uniformly convex Banach space. For any $p > 1$ and $r \in (0,2]$, if there exists a constant $\mu_p > 0$ such that $\delta_E(r) \geq \mu_p r^p$, then E is a p-uniformly convex Banach space. We see from [17] that E is a uniformly convex Banach space if there exists a convex, strictly increasing, continuous function $\psi : [0,+\infty) \to [0,+\infty)$ with $\psi(0) = 0$ and

$$\|tu + (1-t)v\|^p + (t^p(1-t) + (1-t)^p t)\psi(\|u-v\|) \leq t\|u\|^p + (1-t)\|v\|^p$$

for all $u, v \in B_r(0) := \{u \in E : \|u\| \leq r\}$ and $t \in [0,1]$, where $p > 1$ and $r > 0$ are two fixed real numbers. Particularly, we have

$$\|tu + (1-t)v\|^2 + t(1-t)\psi(\|u-v\|) \leq t\|u\|^2 + (1-t)\|v\|^2.$$

Let $\{x_n\}$ be a sequence in E. Recall that E is said to satisfy the Opial condition if whenever $\{x_n\}$ is a sequence in E which converges weakly to x as $n \to \infty$, then

$$\liminf_{n\to\infty} \|x_n \to x\| < \liminf_{n\to\infty} \|x_n \to y\|, \quad \forall\, y \in E, \quad y \neq x.$$

Let $B(0) = \{u \in E : \|u\| = 1\}$. The norm of E is Gâteaux differentiable if

$$\lim_{\tau\to 0} \frac{\|u+\tau v\| - \|u\|}{\tau}$$

exists for each $u, v \in B(0)$. In this case, E is a smooth Banach space. In smooth Banach space, J_q is single-valued and strongly weak* continuous. The smooth modulus of space E is defined by

$$\rho_E(t) = \left\{ \frac{\|u + tv\| + \|u - tv\|}{2} - 1 : u, v \in E, \|u\| = \|v\| = 1 \right\},$$

which maps $[0, +\infty)$ to $[0, +\infty)$. If $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$, then E is said to be uniformly smooth. For all $t > 0$ and $q > 1$, if there exists a $\nu_q > 0$ such that $\rho_E(t) \leq \nu_q t^q$, then E is a q-uniformly smooth Banach space. Xu [17] proved that there is no q-uniformly smooth Banach space with $q > 2$. It is obvious that every q-uniformly smooth Banach space is uniformly smooth. It is known that J_q is uniformly continuous on bounded sets of uniformly smooth Banach spaces.

Let $A : E \to 2^E$ be a set-valued operator and I be the identity operator on E. The domain of operator A is denoted by $D(A) = \{u \in E : Au \neq \emptyset\}$ and the range of operator A is denoted by $R(A) = \cup\{Au : u \in D(A)\}$. Recall that A is accretive if, for all $u, v \in D(A)$, there exists $j_q(u - v) \in J_q(u - v)$ such that

$$\langle \tilde{u} - \tilde{v}, j_q(u - v)\rangle \geq 0, \quad \forall\, \tilde{u} \in Au, \quad \tilde{v} \in Av.$$

If the range of $I + rA$ is precisely E for any $r > 0$, then A is an m-accretive operator. If A is an m-accretive operator, then the resolvent of A, which maps $R(I + rA)$ to $D(A)$, is a nonexpansive single-valued mapping and defined by $J_r^A = (I + rA)^{-1}$ for all $r > 0$. Recall that operator A is α-inverse strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle Au - Av, j_q(u - v)\rangle \geq \alpha \|Au - Av\|^q,$$

for all $u, v \in E$ and some $j_q(u - v) \in J_q(u - v)$. It is easy to see that each α-inverse strongly accretive operator is accretive and $\frac{1}{\alpha}$-Lipschitz continuous.

Let U be a closed and convex nonempty subset of E and let $Q : E \to U$ be a mapping. Q is called sunny if $Q(\tau u + (1 - \tau)Qu) = Qu$ for all $u \in E$ and $\tau \in (0, 1)$. Q is a retraction of E to U if $Q^2 = Q$ for all $u \in E$. Q is a sunny nonexpansive retraction if Q is sunny, nonexpansive and a retraction onto U. In Hilbert spaces, one knows that the sunny nonexpansive retraction Q coincides with the metric projection from E to U. If $Q : E \to U$ is a retraction, then the following statements are equivalent:

(i) Q is sunny and nonexpansive;
(ii) $\|Qu - Qv\|^q \leq \langle u - v, j_q(Qu - Qv)\rangle$ for all $u, v \in E$;
(iii) $\langle u - Qu, j_q(v - Qu)\rangle \leq 0$ for all $u \in E, v \in U$.

Next, we list some necessary lemmas, which play a significant role in the convergence analysis of our iterative algorithms.

Lemma 2.1 ([8,9]): *Let U be a nonempty convex closed subset of a uniformly smooth Banach space E. Suppose that $T : U \to U$ is a nonexpansive mapping* *with* $\mathrm{Fix(T)} \neq \emptyset$*, and* $f : U \to U$ *is a contraction. If* x_τ *is the unique solution of the equation* $x_\tau = (1-\tau)Tx_\tau + \tau f(x_\tau)$ *for each* $\tau \in (0,1)$*, then* $\{x_\tau\}$ *converges strongly to a fixed point* $x^* = Q^U_{\mathrm{Fix(T)}} f(x^*)$*, where* $Q^U_{\mathrm{Fix(T)}}$ *is the unique sunny nonexpansive retraction from U onto* $\mathrm{Fix(T)}$ *as* $\tau \to 0$*.*

Lemma 2.2 ([17]): *For each real q-uniformly smooth Banach space E, the following inequality holds:*

$$\|u+v\|^q \leq \|u\|^q + q\langle v, j_q(u)\rangle + \nu_q\|v\|^q, \quad \forall\, u, v \in E,$$

where ν_q *is some fixed positive constant.*

Lemma 2.3 ([11]): *Assume that E is a q-uniformly smooth Banach space, and U is a nonempty convex subset of E. Suppose that* $T : U \to E$ *is a k-strict pseudocontraction, and* $G : U \to E$ *is a mapping defined by* $G = \tau T + (1-\tau)I$*, where* $\tau \in (0, \min\{(\frac{qk}{\nu_q})^{q-1}, 1\})$*. Then G is a nonexpansive mapping, and* $\mathrm{Fix(G)} = \mathrm{Fix(T)}$*.*

Lemma 2.4 ([11]): *Suppose that E is a uniformly convex Banach space. Then there exists a strictly increasing continuous convex function* $\varphi : [0,\infty) \to [0,\infty)$ *with* $\varphi(0) = 0$ *such that*

$$\|\alpha u + \beta v + \eta z\|^p \leq \alpha\|u\|^p + \beta\|v\|^p + \eta\|z\|^p - \frac{\alpha^p\beta + \beta^p\alpha}{(\alpha+\beta)^p}\varphi(\|u-v\|),$$

where $u, v, z \in B_r(0) := \{x \in E, \|x\| \leq r\}$*,* $p > 1$ *is a real number and* $\alpha, \beta, \eta \in [0,1]$ *such that* $\alpha + \beta + \eta = 1$*.*

Lemma 2.5 ([11]): *Let U be a nonempty closed convex subset of a real uniformly convex Banach space E, and let* $T : U \to E$ *be a continuous pseudocontractive mapping. Then* $I-T$ *is demiclosed at zero.*

Lemma 2.6 ([18]): *Let E be a real Banach space and A be an m-accretive operator. For* $\alpha > 0$ *and* $\lambda > 0$*, the following equality holds:*

$$J^A_\lambda\left(\frac{\lambda}{\alpha}x + \left(1 - \frac{\lambda}{\alpha}\right)J^A_\alpha x\right) = J^A_\alpha x, \quad \forall\, x \in E.$$

Lemma 2.7 ([19]): *Suppose that* $\{\tau_n\}$ *is a sequence of nonnegative real numbers,* $\{\alpha_n\} \subset (0,1)$ *and* $\{\gamma_n\}$ *is a sequence of real numbers. Let* $\{s_n\}$ *be a sequence of nonnegative real numbers such that* $s_{n+1} \leq (1-\alpha_n)s_n + \gamma_n + \tau_n, \forall\, n \geq 1$*. If*

(i) $\limsup_{n\to\infty} \frac{\gamma_n}{\alpha_n} \leq 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} \tau_n < \infty$,

then $\lim_{n\to\infty} s_n = 0$*.*

Lemma 2.8 ([20]): *Let $q > 1$. Then, for any positive real numbers a and b, $(ab) \cdot q \leq (q-1) \cdot b^{\frac{q}{q-1}} + a^q$.*

Lemma 2.9 ([17]): *Let E be a real Banach space. Then, for any $u, v \in E$, the following inequality holds: $\|u+v\|^q \leq q\langle v, j_q(u+v)\rangle + \|u\|^q$, where $j_q(u+v) \in J_q(u+v)$ is the generalized duality mapping.*

Lemma 2.10 ([3]): *Let E be a real uniformly convex Banach space, and let U be a nonempty closed convex bounded subset of E. Then, for every nonexpansive mapping $T : U \to E$ and $t \in [0,1]$, there exists a strictly increasing and continuous convex function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that*

$$\|(tTu + (1-t)Tv) - T(tu + (1-t)v\| \leq \psi^{-1}(\|u - v\| - (Tu - Tv)),$$
$$\forall\, u, v \in U.$$

Lemma 2.11 ([21]): *Let $\{a_n\}$, $\{b_n\}$ and $\{\xi_n\}$ be sequences in $[0,+\infty)$ such that*

$$a_{n+1} \leq a_n + b_n(a_n - a_{n-1}) + \xi_n, \quad \forall\, n \geq 1,$$

where $\sum_{n=1}^{\infty} \xi_n < +\infty$. Suppose that there exists a real number b such that $0 \leq b_n \leq b < 1$ for all $n \in N$. Then the following statements hold.

(i) $\sum_{n=1}^{\infty} [a_n - a_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$;
(ii) *there exist* $a^* \in [0,+\infty)$ *such that* $\lim_{n\to\infty} a_n = a^*$.

3. Main results

In this section, the framework of Banach spaces is restricted to be uniformly convex and q-uniformly smooth. Weakly and strongly convergent inertial splitting algorithms are proposed and investigated for the common solution of the fixed point and inclusion problem. In order to obtain our main results, we always assume that the following conditions hold.

Condition 3.1: Let E be a uniformly convex and q-uniformly smooth Banach space with constant ν_q, and let U and V be two nonempty closed convex subsets of E. Suppose the following assumptions hold.

(i) $A : V \to 2^E$ is an m-accretive operator, and $B : U \to E$ is an α-inverse strongly accretive operator;
(ii) $T : U \to E$ is a κ-strict pseudocontraction with $\mathrm{Fix(T)} \neq \emptyset$;
(iii) $\mathrm{Fix(T)} \cap (A+B)^{-1}(0) \neq \emptyset$ and there exists a sunny nonexpansive retraction $Q_{U\cap V}^E$ from E onto $U \cap V$.

Condition 3.2: Suppose that $\{\epsilon_n\}$ $\{\delta_n\}$ and $\{\rho_n\}$ are real number sequences in $(0,1)$, and $\{\theta_n\}$ and $\{\gamma_n\}$ are two sequences in $(0,+\infty)$. Assume that the following conditions are satisfied.

(i) $0 < \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n < 1$ and $\sum_{n=0}^{\infty} \rho_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \epsilon_n \le \limsup_{n\to\infty} \epsilon_n < \min\{(\frac{\kappa q}{\nu_q})^{\frac{1}{q-1}}, 1\}$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < (\frac{\alpha q}{\nu_q})^{\frac{1}{q-1}}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Algorithm 1 The inertial splitting algorithm I.

Initialization: Let $\delta \in (0,1)$ and fix $x_0, x_1 \in U \cap V$ arbitrarily.
Iterative Steps: Given the current iterators x_n and x_{n-1}, calculate x_{n+1} as follows:
Step 1. Compute δ_n, such that $0 \le \delta_n \le \delta_n^*$, where

$$\delta_n^* = \begin{cases} \min\left\{\delta, \dfrac{\theta_n}{\|x_n - x_{n-1}\|}, \dfrac{\theta_n}{\|x_n - x_{n-1}\|^q}\right\}, & x_n - x_{n-1} \neq 0, \\ \delta, & otherwise. \end{cases} \tag{1}$$

Step 2. Compute

$$\begin{cases} p_n = x_n + \delta_n(x_n - x_{n-1}), \\ z_n = \epsilon_n T p_n + (1-\epsilon_n)p_n, \\ x_{n+1} = Q^E_{U\cap V}((1-\rho_n)J^A_{\gamma_n}(p_n - \gamma_n B p_n + e_n) + \rho_n z_n), \end{cases}$$

where $\{e_n\}$ is a sequence with $\sum_{n=0}^{\infty} \|e_n\| < \infty$.
Step 3. Set $n \leftarrow n+1$ and go to Step 1.

Remark 3.1: By Condition 3.2 and (1), one deduces that $\lim_{n\to\infty} \delta_n\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\delta_n}{\rho_n}\|x_n - x_{n-1}\| = 0$. In fact, it easy to find that

$$\lim_{n\to\infty} \delta_n\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \theta_n = 0.$$

Since $0 < \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n < 1$, one gets

$$\lim_{n\to\infty} \frac{\delta_n}{\rho_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\theta_n}{\rho_n} = 0.$$

Theorem 3.1: *Assume that E satisfies the Opial condition. Suppose that Condition 3.1 and Condition 3.2 satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to some points in* $\mathrm{Fix(T)} \cap (A+B)^{-1}(0)$.

Proof: *Step 1.* We show that $\{x_n\}$ is bounded. Take $x^* \in \mathrm{Fix(T)} \cap (A+B)^{-1}(0)$. Then

$$\|p_n - x^*\| \leq \|x_n - x^*\| + \delta_n \|x_n - x_{n-1}\|. \tag{2}$$

For all $x, y \in U$, we find from Lemma 2.2 and Condition 3.2 that

$$\begin{aligned}
&\|(I-\gamma_n B)x - (I-\gamma_n B)y\|^q \\
&\quad \leq \nu_q \gamma_n^q \|Bx - By\|^q - q\gamma_n \langle Bx - By, j_q(x-y)\rangle + \|x-y\|^q \\
&\quad \leq \nu_q \gamma_n^q \|Bx - By\|^q - q\gamma_n \alpha \|Bx - By\|^q + \|x-y\|^q \\
&\quad \leq \|x-y\|^q + (\nu_q \gamma_n^{(q-1)} - \alpha q)\gamma_n \|Bx - By\|^q \\
&\quad \leq \|x-y\|^q.
\end{aligned} \tag{3}$$

Hence, $I - \gamma_n B$ is nonexpansive. It is easy to see that $x^* = Tx^* = J^A_{\gamma_n}(x^* - \gamma_n x^* Bx^*) \in U \cap V$. Taking $T_n = \epsilon_n T + (1-\epsilon_n)I$. By Lemma 2.3, we get that T_n is a nonexpansive mapping and $\mathrm{Fix(T)} = \mathrm{Fix(T_n)}$. We yield from (3) that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|(1-\rho_n)J^A_{\gamma_n}(p_n - \gamma_n Bp_n + e_n) + \rho_n T_n p_n - x^*\| \\
&\leq (1-\rho_n)\|J^A_{\gamma_n}(p_n - \gamma_n Bp_n + e_n) - x^*\| + \rho_n \|x^* - T_n p_n\| \\
&\leq (1-\rho_n)\|p_n - x^*\| + \rho_n \|p_n - x^*\| + \|e_n\| \\
&\leq \|p_n - x^*\| + \|e_n\|.
\end{aligned} \tag{4}$$

Combining (2) and (4), we obtain that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \delta_n(\|x_n - x^*\| - \|x_{n-1} - x^*\|) + \|e_n\|. \tag{5}$$

This together with Remark 3.1 and Lemma 2.11 gives that $\{\|x_n - x^*\|\}$ is convergent. Hence, $\{x_n\}$ is bounded. Inequality (2) implies that $\{p_n\}$ is also bounded.

Step 2. We show that $\omega(x_n) \subset (A+B)^{-1}(0) \cap \mathrm{Fix(T)}$, where $\omega(x_n)$ denotes the weak accumulation set of $\{x_n\}$. From Lemma 2.9 and (3), we can deduce that

$$\begin{aligned}
&\|(p_n - \gamma_n Bp_n) - (x^* - \gamma_n Bx^*) + e_n\|^q \\
&\quad \leq \|(I-\gamma_n B)p_n - (I-\gamma_n B)x^*\|^q \\
&\qquad + q\langle e_n, j_q((I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n)\rangle \\
&\quad \leq q\|e_n\|\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n\|^{q-1} + \|p_n - x^*\|^q \\
&\qquad + \gamma_n(\nu_q \gamma_n^{q-1} - \alpha q)\|Bp_n - Bx^*\|^q.
\end{aligned} \tag{6}$$

Let $w_n = J^A_{\gamma_n}(p_n - \gamma_n Bp_n + e_n)$. Using the definition of the activeness of A and Lemma 2.4 yields

$$\|w_n - x^*\|^q \leq \|w_n - x^* + \frac{\gamma_n}{2}(\frac{p_n - \gamma_n Bp_n + e_n - w_n}{\gamma_n} - \frac{x^* - \gamma_n Bx^* - x^*}{\gamma_n})\|^q$$

$$
\begin{aligned}
&\leq \left\| \frac{(w_n - x^*) + ((I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n)}{2} \right\|^q \\
&\leq \frac{1}{2}\|w_n - x^*\|^q + \frac{1}{2}\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n\|^q \\
&\quad - \frac{1}{2^q}\varphi(\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n - (w_n - x^*)\|) \\
&\leq \|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n\|^q \\
&\quad - \frac{1}{2^q}\varphi(\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n - (w_n - x^*)\|), \qquad (7)
\end{aligned}
$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a strict increasing continuous convex function with $\varphi(0) = 0$. Substituting (6) into (7) gives

$$
\begin{aligned}
\|w_n - x^*\|^q &\leq q\|e_n\|\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n\|^{q-1} + \|p_n - x^*\|^q \\
&\quad - \gamma_n(\alpha q - \nu_q \gamma_n^{q-1})\|Bp_n - Bx^*\|^q \\
&\quad - \frac{1}{2^q}\varphi(\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n - (w_n - x^*)\|). \qquad (8)
\end{aligned}
$$

Using the convexity of $\|\cdot\|^q$, we get that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^q &\leq (1-\rho_n)\|w_n - x^*\|^q + \rho_n\|T_n p_n - x^*\|^q \\
&\leq \|p_n - x^*\|^q + q\|e_n\|\|(I-\gamma_n B)p_n - (I-\gamma_n B)x^* + e_n\|^{q-1} \\
&\quad - (1-\rho_n)\gamma_n(\alpha q - \nu_q \gamma_n^{q-1})\|Bp_n - Bx^*\|^q \\
&\quad - (1-\rho_n)\frac{1}{2^q}\varphi(\|(I-\gamma_n B)p_n \\
&\quad - (I-\gamma_n B)x^* + e_n - (w_n - x^*)\|). \qquad (9)
\end{aligned}
$$

Since

$$
\begin{aligned}
\|p_n - x^*\|^q &= \|x_n + \delta_n(x_n - x_{n-1}) - x^*\|^q \\
&\leq \|x_n - x^*\|^q + q\delta_n\langle x_n - x_{n-1}, j_q(x_n - x^*)\rangle + \nu_q\delta_n^q\|x_n - x_{n-1}\|^q
\end{aligned}
$$

and

$$
q\langle x_n - x_{n-1}, j_q(x_n - x^*)\rangle \leq \|x_n - x^*\|^q + \nu_q\|x_n - x_{n-1}\|^q - \|x_{n-1} - x^*\|^q,
$$

we have

$$
\begin{aligned}
\|p_n - x^*\|^q &\leq \|x_n - x^*\|^q + \nu_q(\delta_n + \delta_n^q)\|x_n - x_{n-1}\|^q \\
&\qquad + \delta_n(\|x_n - x^*\|^q - \|x_{n-1} - x^*\|^q) \\
&\leq \|x_n - x^*\|^q + 2\nu_q\delta_n\|x_n - x_{n-1}\|^q \\
&\qquad + \delta_n(\|x_n - x^*\|^q - \|x_{n-1} - x^*\|^q). \qquad (10)
\end{aligned}
$$

Substituting (10) back into (9) gives

$$\begin{aligned}\|x_{n+1}-x^*\|^q \leq{}& \|x_n-x^*\|^q+2\nu_q\delta_n\|x_n-x_{n-1}\|^q\\&+\delta_n(\|x_n-x^*\|^q-\|x_{n-1}-x^*\|^q)\\&+q\|e_n\|\|(I-\gamma_nB)p_n-(I-\gamma_nB)x^*+e_n\|^{q-1}\\&-(1-\rho_n)\gamma_n(\alpha q-\nu_q\gamma_n^{q-1})\|Bp_n-Bx^*\|^q\\&-(1-\rho_n)\frac{1}{2^q}\varphi(\|(I-\gamma_nB)p_n\\&-(I-\gamma_nB)x^*+e_n-(w_n-x^*)\|),\end{aligned}\tag{11}$$

which yields

$$\begin{aligned}&(1-\rho_n)\gamma_n(\alpha q-\nu_q\gamma_n^{q-1})\|Bp_n-Bx^*\|^q\\&\quad\leq(\|x_n-x^*\|^q-\|x_{n+1}-x^*\|^q)+2\nu_q\delta_n\|x_n-x_{n-1}\|^q\\&\qquad+\delta_n(\|x_n-x^*\|^q-\|x_{n-1}-x^*\|^q)\\&\qquad+q\|e_n\|\|(I-\gamma_nB)p_n-(I-\gamma_nB)x^*+e_n\|^{q-1}.\end{aligned}\tag{12}$$

We see from Step 1 that $\{\|x_n-x^*\|\}$ is convergent. Combining Condition 3.2, Remark 3.1 and (12), we get

$$\lim_{n\to\infty}\|Bp_n-Bx^*\|=0.\tag{13}$$

On the other hand, inequality (11) also implies that

$$\begin{aligned}&(1-\rho_n)\frac{1}{2^q}\varphi(\|(I-\gamma_nB)p_n-(I-\gamma_nB)x^*+e_n-(w_n-x^*)\|)\\&\quad\leq(\|x_n-x^*\|^q-\|x_{n+1}-x^*\|^q)+2\nu_q\delta_n\|x_n-x_{n-1}\|^q\\&\qquad+\delta_n(\|x_n-x^*\|^q-\|x_{n-1}-x^*\|^q)\\&\qquad+q\|e_n\|\|(I-\gamma_nB)p_n-(I-\gamma_nB)x^*+e_n\|^{q-1}.\end{aligned}\tag{14}$$

Similarly, we deduce from (14) that

$$\lim_{n\to\infty}\|p_n-\gamma_nBp_n+\gamma_nBx^*-w_n\|=0.\tag{15}$$

Since $w_n=J_{\gamma_n}^A(p_n-\gamma_nBp_n+e_n)$, by (13) and (15), we have

$$\lim_{n\to\infty}\|p_n-J_{\gamma_n}^A(p_n-\gamma_nBp_n)\|=0.\tag{16}$$

Using the uniformly convexity of E and Lemma 2.4, we get

$$\begin{aligned}&\|x_{n+1}-x^*\|^q\\&\quad\leq\rho_n\|T_np_n-x^*\|^q-\rho_n(1-\rho_n)\varphi(\|T_nx_n-w_n\|)+(1-\rho_n)\|w_n-x^*\|^q\end{aligned}$$

$$\begin{aligned}&\leq \rho_n\|p_n - x^*\|^q + (1-\rho_n)\|(p_n - \gamma_n Bp_n) - (x^* - \gamma_n Bx^*) + e_n\|^q \\ &\quad - \rho_n(1-\rho_n)\varphi(\|T_np_n - w_n\|).\end{aligned} \tag{17}$$

Combining (2), (6) and (17), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^q \leq (\|x_n - x^*\| + \delta_n\|x_n - x_{n-1}\|)^q &- \rho_n(1-\rho_n)\varphi(\|T_np_n - w_n\|) \\ &+ q\|e_n\|\|(I-\gamma_nB)p_n - (I-\gamma_nB)x^* + e_n\|^{q-1} \\ &+ \gamma_n(\nu_q\gamma_n^{q-1} - \alpha q)\|Bp_n - Bx^*\|^q.\end{aligned} \tag{18}$$

This implies that

$$\begin{aligned}\rho_n(1-\rho_n)\varphi(\|T_np_n - w_n\|) \leq (\|x_n - x^*\| + \delta_n\|x_n - x_{n-1}\|)^q - \|x_{n+1} - x^*\|^q \\ + q\|e_n\|\|(I-\gamma_nB)p_n - (I-\gamma_nB)x^* + e_n\|^{q-1}.\end{aligned} \tag{19}$$

We deduce from (19) that $\lim_{n\to\infty}\|T_np_n - w_n\| = 0$. This together with (16) yields $\lim_{n\to\infty}\|T_np_n - p_n\| = 0$. In addition, we find that

$$\begin{aligned}\|T_nx_n - x_n\| &= \|T_nx_n - T_np_n + T_np_n - p_n + p_n - x_n\| \\ &\leq 2\|p_n - x_n\| + \|T_np_n - p_n\| \\ &= 2\delta_n\|x_n - x_{n-1}\| + \|T_np_n - p_n\|.\end{aligned} \tag{20}$$

Combining Remark 3.1 and (20), we have $\lim_{n\to\infty}\|T_nx_n - x_n\| = 0$. Using the definition of T_n, we get that $\|Tx_n - x_n\| = \frac{1}{\epsilon_n}\|T_nx_n - x_n\|$. Hence, by Condition 3.2, we get that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. We see from Lemma 2.5 that $\omega(x_n) \subset \mathrm{Fix(T)}$. Without loss of generality, we assume that there exists a real number r such that $0 < r \leq \gamma_n$ for all $n \geq 0$. Since A is accretive, we have

$$\begin{aligned}0 \leq \Bigg\langle &\frac{x_n - J^A_{\gamma_n}(I-\gamma_nB)x_n}{\gamma_n} - \frac{x_n - J^A_r(I-rB)x_n}{r}, \\ &j_q(J^A_{\gamma_n}(I-\gamma_nB)x_n - J^A_r(I-rB)x_n)\Bigg\rangle.\end{aligned} \tag{21}$$

This implies that

$$\begin{aligned}&\|J^A_{\gamma_n}(I-\gamma_nB)x_n - J^A_r(I-rB)x_n\|^q \\ &\quad\leq \frac{|\gamma_n - r|}{\gamma_n}\langle J^A_{\gamma_n}(I-\gamma_nB)x_n - x_n, j_q(J^A_{\gamma_n}(I-\gamma_nB)x_n - J^A_r(I-rB)x_n)\rangle \\ &\quad\leq \frac{|\gamma_n - r|}{\gamma_n}\|J^A_{\gamma_n}(I-\gamma_nB)x_n - x_n\|\|J^A_{\gamma_n}(I-\gamma_nB)x_n - J^A_r(I-rB)x_n\|^{q-1}.\end{aligned} \tag{22}$$

Inequality (22) can be expressed as

$$\|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n\| \le \frac{|\gamma_n - r|}{\gamma_n}\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|. \tag{23}$$

Therefore, by the triangle inequality of norms and (23), we can obtain

$$\begin{aligned}
&\|J^A_r(I-rB)x_n - x_n\| \\
&\quad \le \|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\| + \|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n\| \\
&\quad \le \left(1+\frac{|\gamma_n - r|}{\gamma_n}\right)\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|.
\end{aligned} \tag{24}$$

On the other hand, we see that

$$\begin{aligned}
&\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\| \\
&\quad = \|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_{\gamma_n}(I-\gamma_n B)p_n + J^A_{\gamma_n}(I-\gamma_n B)p_n - p_n + p_n - x_n\| \\
&\quad \le \|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_{\gamma_n}(I-\gamma_n B)p_n\| \\
&\qquad + \|J^A_{\gamma_n}(I-\gamma_n B)p_n - p_n\| + \|p_n - x_n\| \\
&\quad \le 2\delta_n\|x_n - x_{n-1}\| + \|J^A_{\gamma_n}(I-\gamma_n B)p_n - p_n\|.
\end{aligned}$$

This together with (16) and (24) yields $\lim_{n\to\infty}\|J^A_r(I-rB)x_n - x_n\| = 0$. In view of Browder's demiclosedness principle, we obtain that $\omega(x_n) \subset \mathrm{Fix}(J^A_r(I-rB)) = (A+B)^{-1}(0)$.

Step 3. We show that $\omega(x_n)$ is a singleton. Suppose that $\{x_{n_i}\}$ converges weakly to x_1^* and $\{x_{n_j}\}$ converges weakly to x_2^*, respectively. By Step 2, we have $x_1^*, x_2^* \in \mathrm{Fix}(T) \cap (A+B)^{-1}(0)$. Let us show $x_1^* = x_2^*$. Assume $x_1^* \ne x_2^*$. Applying the Opial condition on the space E gives

$$\begin{aligned}
\liminf_{n\to\infty}\|x_n - x_1^*\| &= \liminf_{i\to\infty}\|x_{n_i} - x_1^*\| \\
&< \liminf_{i\to\infty}\|x_{n_i} - x_2^*\| = \liminf_{n\to\infty}\|x_n - x_2^*\| \\
&= \liminf_{j\to\infty}\|x_{n_j} - x_2^*\| < \liminf_{j\to\infty}\|x_{n_j} - x_1^*\| \\
&= \liminf_{n\to\infty}\|x_n - x_1^*\|.
\end{aligned}$$

This is a contradiction. So, we have $x_1^* = x_2^*$. Therefore, we conclude that $\{x_n\}$ converges weakly to an element of $\mathrm{Fix}(T) \cap (A+B)^{-1}(0)$. This completes the proof. ■

If mappings T, A and B are defined on space E, then the sunny nonexpansive retraction in Algorithm 1 can be removed.

Algorithm 2 The inertial splitting algorithm II.

Initialization: Let $\delta \in (0,1)$ and fix $x_0, x_1 \in E$ arbitrarily.

Iterative Steps: Given the current iterators x_n and x_{n-1}, calculate x_{n+1} as follows:

Step 1. Compute δ_n, such that $0 \le \delta_n \le \delta_n^*$, where δ_n^* is defined in (1).

Step 2. Compute

$$\begin{cases} p_n = x_n + \delta_n(x_n - x_{n-1}), \\ z_n = \epsilon_n T p_n + (1-\epsilon_n)p_n, \\ x_{n+1} = (1-\rho_n)J_{\gamma_n}^A(p_n - \gamma_n B p_n + e_n) + \rho_n z_n, \end{cases}$$

where $\{e_n\}$ is a sequence with $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Step 3. Set $n \leftarrow n+1$ and go to Step 1.

Corollary 3.1: *Assume that E satisfies the Opial condition. Suppose that Condition 3.1 and Condition 3.2 satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to some points in* $\mathrm{Fix(T)} \cap (A+B)^{-1}(0)$.

In the following, we introduce a strong convergence inertial algorithm for common solutions in Banach spaces. It is worth noting that the Opial condition is not necessary in our convergence analysis. Before that, we give some constraints on the parameters.

Condition 3.3: Assume that $\{\theta_n\}$ and $\{\gamma_n\}$ are real number sequences in $(0,+\infty)$ and $\{\epsilon_n\}, \{\eta_n\}, \{\sigma_n\}$ and $\{\rho_n\}$ are sequences in $(0,1)$ such that $\eta_n + \rho_n + \sigma_n = 1$, and

(i) $0 < \liminf_{n\to\infty} \epsilon_n \le \limsup_{n\to\infty} \epsilon_n < \min\{(\frac{\kappa q}{\nu_q})^{\frac{1}{q-1}}, 1\}, \sum_{n=0}^{\infty} |1 - \frac{\epsilon_{n+1}}{\epsilon_n}| < \infty$;

(ii) $\lim_{n\to\infty} \eta_n = 0, \sum_{n=0}^{\infty} \eta_n = \infty, \sum_{n=0}^{\infty} |\eta_n - \eta_{n+1}| < \infty$;

(iii) $\liminf_{\to\infty} \rho_n\sigma_n > 0, \sum_{n=0}^{\infty} |\rho_n - \rho_{n+1}| < \infty, \sum_{n=0}^{\infty} |\sigma_n - \sigma_{n+1}| < \infty$;

(iv) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < (\frac{\alpha q}{\nu_q})^{\frac{1}{q-1}}, \sum_{n=0}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$;

(v) $\theta_n = o(\eta_n)$, that is $\lim_{n\to\infty} \frac{\theta_n}{\eta_n} = 0$.

Remark 3.2: By Condition 3.2 and (25), one deduces that $\lim_{n\to\infty} \delta_n \|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\delta_n}{\eta_n}\|x_n - x_{n-1}\| = 0$. In fact, it is easy to find that

$$\lim_{n\to\infty} \frac{\delta_n}{\eta_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\theta_n}{\eta_n} = 0.$$

Algorithm 3 The inertial splitting algorithm III.

Initialization: Let $\delta \in (0,1)$ and fix $x_0, x_1 \in U \cap V$ arbitrarily.

Iterative Steps: Given the current iterators x_n and x_{n-1}, calculate x_{n+1} as follows:

Step 1. Compute δ_n such that $0 \le \delta_n \le \delta_n^*$, where

$$\delta_n^* = \begin{cases} \min\left\{\delta, \dfrac{\theta_n}{\|x_n - x_{n-1}\|}, \dfrac{\theta_n}{\|x_n - x_{n-1}\|^q}\right\}, & x_n - x_{n-1} \neq 0, \\ \delta, & \textit{otherwise}. \end{cases} \tag{25}$$

Step 2. Compute

$$\begin{cases} p_n = x_n + \delta_n(x_n - x_{n-1}), \\ z_n = \epsilon_n T p_n + (1-\epsilon_n)p_n, \\ x_{n+1} = Q^E_{U\cap V}(\eta_n f(x_n) + \sigma_n J^A_{\gamma_n}(p_n - \gamma_n B p_n + e_n) + \rho_n z_n), \end{cases}$$

where $\{e_n\}$ is a sequence with $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Step 3. Set $n \leftarrow n+1$ and go to Step 1.

This implies that

$$\lim_{n\to\infty} \delta_n \|x_n - x_{n-1}\| = 0.$$

Theorem 3.2: *Suppose that Condition 3.1 and Condition 3.3 satisfied. Let $f : U \to E$ be an h-contraction. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to some points to $x^* = Q^{U\cap V}_{\mathrm{Fix(T)}\cap(A+B)^{-1}(0)} f(x^*)$, where $Q^{U\cap V}_{\mathrm{Fix(T)}\cap(A+B)^{-1}(0)}$ is the unique sunny nonexpansive mapping from $U \cap V$ to $\mathrm{Fix(T)} \cap (A+B)^{-1}(0)$, that is, x^* is the unique solution of variational inequality $\langle f(x^*) - x^*, j_q(y - x^*)\rangle \le 0$ for all $y \in \mathrm{Fix(T)} \cap (A+B)^{-1}(0)$.*

Proof: We split the proof into three steps.

Step 1. We show that $\{x_n\}$ is bounded. For all $x, y \in U$, we find from (3) that $I - \gamma_n B$ is nonexpansive. For all $x^\star \in (A+B)^{-1}(0) \cap \mathrm{Fix(T)}$, we easily see that $x^\star = (I + \gamma_n A)(I - \gamma_n B)x^\star = Tx^\star$. Put $T_n = \epsilon_n T + (1-\epsilon_n)I$. By Lemma 2.3, we get that T_n is nonexpansive and $\mathrm{Fix(T)} = \mathrm{Fix(T_n)}$. Fixing $x^* \in (A+B)^{-1}(0) \cap \mathrm{Fix(T)}$, we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad \le \eta_n \|f(x_n) - x^*\| + \sigma_n \|J^A_{\gamma_n}(p_n - \gamma_n B p_n + e_n) - x^*\| + \rho_n \|T_n p_n - x^*\| \\ &\quad \le \eta_n \|f(x_n) - f(x^*)\| + \eta_n \|f(x^*) - x^*\| \end{aligned}$$

$$+ \sigma_n\|(p_n - \gamma_n Bp_n + e_n) - (I - \gamma_n B)x^*\| + \rho_n\|p_n - x^*\|$$
$$\leq h\eta_n\|x_n - x^*\| + (1-\eta_n)\|p_n - x^*\| + \eta_n\|f(x^*) - x^*\| + \sigma_n\|e_n\|. \quad (26)$$

Since

$$\|p_n - x^*\| \leq \|x_n - x^*\| + \delta_n\|x_n - x_{n-1}\|,$$

we have from (26) that

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1 - \eta_n(1-h))\|x_n - x^*\| + (1-\eta_n)\delta_n\|x_n - x_{n-1}\| \\ &\quad + \eta_n\|f(x^*) - x^*\| + \sigma_n\|e_n\| \\ &\leq (1 - \eta_n(1-h))\|x_n - x^*\| + \|e_n\| \\ &\quad + \eta_n(1-h)\left(\frac{\|f(x^*) - x^*\|}{1-h} + \frac{(1-\eta_n)\delta_n\|x_n - x_{n-1}\|}{\eta_n(1-h)}\right).\end{aligned} \quad (27)$$

From Remark 3.2, we conclude that

$$\lim_{n\to\infty} \frac{(1-\eta_n)\delta_n\|x_n - x_{n-1}\|}{\eta_n(1-h)} = 0.$$

Hence, there exists an $M_0 > 0$ such that $\frac{(1-\eta_n)\delta_n\|x_n - x_{n-1}\|}{\eta_n(1-h)} < M_0$ for all $n \geq 1$. By using the mathematical induction, we get from (27) that

$$\|x_{n+1} - x^*\| \leq \max\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-h} + M_0\} + \sum_{n=0}^{\infty}\|e_n\| < \infty.$$

This implies that $\{x_n\}$ is bounded. By (2), we see that $\{p_n\}$ is also bounded.

Step 2. We show that $\limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, j_q(x_n - \tilde{x})\rangle \leq 0$, where $\tilde{x} = Q^{U\cap V}_{\mathrm{Fix(T)}\cap(A+B)^{-1}(0)} f(\tilde{x})$. Since $\mathrm{Fix(T)} \cap (A+B)^{-1}(0)$ is convex and closed, and E is uniformly convex and q-uniformly smooth, we conclude that the sunny nonexpansive retraction onto it exists since this set can be viewed as the fixed point set of some nonexpansive mappings. Taking $\zeta_n = p_n - \gamma_n Bp_n + e_n$, we have

$$\begin{aligned}\|\zeta_n - \zeta_{n+1}\| &\leq \|(p_n - \gamma_n Bp_n + e_n) - (p_{n+1} - \gamma_{n+1}Bp_{n+1} + e_{n+1})\| \\ &\leq \|(p_n - \gamma_{n+1}Bp_n + e_{n+1}) - (p_{n+1} - \gamma_{n+1}Bp_{n+1} + e_{n+1})\| \\ &\quad + \|(p_n - \gamma_n Bp_n + e_n) - (p_n - \gamma_{n+1}Bp_n + e_{n+1})\| \\ &\leq \|p_{n+1} - p_n\| + |\gamma_{n+1} - \gamma_n|\|Bp_n\| + \|e_{n+1}\| + \|e_n\|.\end{aligned} \quad (28)$$

By the definition of z_n, we further have

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \|T_{n+1}p_{n+1} - T_{n+1}p_n\| + \|T_{n+1}p_n - T_n p_n\| \\ &\leq \|p_{n+1} - p_n\| + |1 - \frac{\epsilon_{n+1}}{\epsilon_n}|\|T_n p_n - p_n\|\end{aligned} \quad (29)$$

and

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| \le\ & \eta_{n+1}\|f(x_{n+1})-f(x_n)\|+|\eta_{n+1}-\eta_n|\|f(x_n)\| \\ & +\sigma_{n+1}\|J^A_{\gamma_n}\zeta_n-J^A_{\gamma_{n+1}}\zeta_{n+1}\|+|\sigma_{n+1}-\sigma_n|\|J^A_{\gamma_n}\zeta_n\| \\ & +\rho_{n+1}(\|p_{n+1}-p_n\|+|1-\frac{\epsilon_{n+1}}{\epsilon_n}|\|T_np_n-p_n\|) \\ & +|\rho_{n+1}-\rho_n|\|z_n\|. \end{aligned} \tag{30}$$

From Lemma 2.6 and (28), we obtain that

$$\begin{aligned}\|J^A_{\gamma_n}\zeta_n-J^A_{\gamma_n}\zeta_{n+1}\| &= \|J^A_{\gamma_n}\zeta_n-J^A_{\gamma_n}(\frac{\gamma_n}{\gamma_{n+1}}\zeta_{n+1}+(1-\frac{\gamma_n}{\gamma_{n+1}})J^A_{\gamma_{n+1}}\zeta_{n+1})\| \\ &\le \|\zeta_n-(\frac{\gamma_n}{\gamma_{n+1}}\zeta_{n+1}+(1-\frac{\gamma_n}{\gamma_{n+1}})J^A_{\gamma_{n+1}}\zeta_{n+1})\| \\ &\le \frac{|\gamma_n-\gamma_{n+1}|\|\zeta_{n+1}-J^A_{\gamma_{n+1}}\zeta_{n+1}\|}{\gamma_{n+1}}+\|\zeta_n-\zeta_{n+1}\| \\ &\le \frac{|\gamma_n-\gamma_{n+1}|\|\zeta_{n+1}-J^A_{\gamma_{n+1}}\zeta_{n+1}\|}{\gamma_{n+1}}+\|p_{n+1}-p_n\| \\ &\quad +|\gamma_{n+1}-\gamma_n|\|Bp_n\|+\|e_{n+1}\|+\|e_n\|. \end{aligned} \tag{31}$$

Substituting (31) into (30), we get

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\| \\ &\quad \le h\eta_{n+1}\|x_{n+1}-x_n\|+|\eta_{n+1}-\eta_n|\|f(x_n)\|+|\gamma_{n+1}-\gamma_n|\|Bp_n\| \\ &\qquad +\frac{|\gamma_n-\gamma_{n+1}|\|\zeta_{n+1}-J^A_{\gamma_{n+1}}\zeta_{n+1}\|}{\gamma_{n+1}}+\|e_{n+1}\|+\|e_n\| \\ &\qquad +|\sigma_{n+1}-\sigma_n|\|J^A_{\gamma_n}\zeta_n\|+(1-\eta_{n+1})\|p_{n+1}-p_n\| \\ &\qquad +|1-\frac{\epsilon_{n+1}}{\epsilon_n}|\|T_np_n-p_n\|+|\rho_{n+1}-\rho_n|\|T_np_n\|. \end{aligned} \tag{32}$$

By the definition of p_n, we have

$$\begin{aligned}\|p_{n+1}-p_n\| &\le \|(x_{n+1}+\delta_{n+1}(x_{n+1}-x_n))-(x_n+\delta_n(x_n-x_{n-1}))\| \\ &\le \|x_{n+1}-x_n\|+\delta_{n+1}\|x_{n+1}-x_n\|+\delta_n\|x_n-x_{n-1}\|. \end{aligned} \tag{33}$$

We deduce from (32) and (33) that

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\| \\ &\quad \le (1-\eta_{n+1}(1-h))\|x_{n+1}-x_n\|+|\eta_{n+1}-\eta_n|\|f(x_n)\| \\ &\qquad +\frac{|\gamma_n-\gamma_{n+1}|\|\zeta_{n+1}-J^A_{\gamma_{n+1}}\zeta_{n+1}\|}{\gamma_{n+1}}+|\gamma_{n+1}-\gamma_n|\|Bp_n\|+\|e_{n+1}\|+\|e_n\| \end{aligned}$$

$$+ |\sigma_{n+1} - \sigma_n| \|J_{\gamma_n}^A \zeta_n\| + |1 - \frac{\epsilon_{n+1}}{\epsilon_n}| \|T_n p_n - p_n\| + |\rho_{n+1} - \rho_n| \|T_n p_n\|$$
$$+ \delta_{n+1} \|x_{n+1} - x_n\| + \delta_n \|x_n - x_{n-1}\|. \tag{34}$$

Combining Condition 3.3, Remark 3.2 and Lemma 2.7, we get that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Setting $p = 2$ in Lemma 2.4, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le\ & \eta_n \|f(x_n) - x^*\|^2 + \sigma_n \|J_{\gamma_n}^A \zeta_n - x^*\|^2 \\
& + \rho_n \|T_n p_n - x^*\|^2 - \sigma_n \rho_n \varphi(\|J_{\gamma_n}^A \zeta_n - T_n p_n\|) \\
\le\ & \|x_n - x^*\|^2 + 2\eta_n \|x_n - x^*\| \|f(x^*) - x^*\| + \eta_n \|f(x^*) - x^*\|^2 \\
& + \sigma_n \|e_n\|^2 + 2\sigma_n \|p_n - x^*\| \|e_n\| - \sigma_n \rho_n \varphi(\|J_{\gamma_n}^A \zeta_n - T_n p_n\|) \\
& + 2(1 - \eta_n)\delta_n \|x_n - x_{n-1}\| \|x_n - x^*\| + (1 - \eta_n) \|x_n - x_{n-1}\|^2.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
& \sigma_n \rho_n \varphi(\|J_{\gamma_n}^A \zeta_n - T_n p_n\|) \\
& \quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\eta_n \|x_n - x^*\| \|f(x^*) - x^*\| \\
& \quad + \eta_n \|f(x^*) - x^*\|^2 + \sigma_n \|e_n\|^2 + 2\sigma_n \|p_n - x^*\| \|e_n\| \\
& \quad + 2(1 - \eta_n)\delta_n \|x_n - x_{n-1}\| \|x_n - x^*\| + (1 - \eta_n) \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{35}$$

Combining Condition 3.3, Lemma 2.4 and Remark 3.2, we conclude that

$$\lim_{n\to\infty} \|J_{\gamma_n}^A \zeta_n - T_n p_n\| = 0. \tag{36}$$

In addition, we also have

$$\begin{aligned}
\|J_{\gamma_n}^A \zeta_n - p_n\| & \le \|x_{n+1} - J_{\gamma_n}^A \zeta_n\| + \|x_{n+1} - p_n\| \\
& \le \eta_n \|f(x_n) - J_{\gamma_n}^A \zeta_n\| + \rho_n \|T_n p_n - J_{\gamma_n}^A \zeta_n\| \\
& \quad + \|x_{n+1} - x_n\| + \delta_n \|x_n - x_{n-1}\|.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|J_{\gamma_n}^A \zeta_n - p_n\| = 0, \tag{37}$$

due to the facts that $\lim_{n\to\infty} \eta_n = 0$, $\{x_n\}$ and $\{p_n\}$ are bounded, f is a contraction and $J_{\gamma_n}^A$ is nonexpansive. We can also get from (36) and (37) that

$$\lim_{n\to\infty} \|T_n p_n - p_n\| = 0. \tag{38}$$

Observe that

$$\begin{aligned}
\|T_n x_n - x_n\| & = \|T_n x_n - T_n p_n + T_n p_n - p_n + p_n - x_n\| \\
& \le 2\|p_n - x_n\| + \|T_n p_n - p_n\|
\end{aligned}$$

$$= 2\delta_n\|x_n - x_{n-1}\| + \|T_n p_n - p_n\|. \tag{39}$$

By Remark 3.2 and (39), we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{40}$$

By the definition of T_n, we see that $\|Tx_n - x_n\| = \frac{1}{\epsilon_n}\|T_n x_n - x_n\|$. Hence, it follows from Condition 3.3 that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{41}$$

Using Lemma 2.5, we have $\omega(x_n) \subset \mathrm{Fix(T)}$. Without loss of generality, we assume that there exists a real number r such that $0 < r \le \gamma_n$ for all $n \ge 0$. Since A is accretive, we have

$$\begin{aligned}&\langle \frac{x_n - J^A_{\gamma_n}(I-\gamma_n B)x_n}{\gamma_n} - \frac{x_n - J^A_r(I-rB)x_n}{r},\\ &j_q(J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n)\rangle \ge 0.\end{aligned} \tag{42}$$

This implies that

$$\begin{aligned}&\|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n\|^q\\ &\quad\le \frac{|\gamma_n - r|}{\gamma_n}\langle J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n, j_q(J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n)\rangle\\ &\quad\le \frac{|\gamma_n - r|}{\gamma_n}\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|\,\|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n\|^{q-1}.\end{aligned} \tag{43}$$

So, it follows from (43) that

$$\|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n\| \le \frac{|\gamma_n - r|}{\gamma_n}\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|. \tag{44}$$

Using the triangle inequality of norms and (44), we can obtain

$$\begin{aligned}\|J^A_r(I-rB)x_n - x_n\| &\le \|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|\\ &\quad + \|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_r(I-rB)x_n\|\\ &\le \left(1 + \frac{|\gamma_n - r|}{\gamma_n}\right)\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|.\end{aligned} \tag{45}$$

On the other hand,

$$\begin{aligned}&\|J^A_{\gamma_n}(I-\gamma_n B)x_n - x_n\|\\ &\quad= \|J^A_{\gamma_n}(I-\gamma_n B)x_n - J^A_{\gamma_n}(I-\gamma_n B)p_n + J^A_{\gamma_n}(I-\gamma_n B)p_n - p_n + p_n - x_n\|\end{aligned}$$

$$
\begin{aligned}
&\leq \|J_{\gamma_n}^A(I-\gamma_n B)x_n - J_{\gamma_n}^A(I-\gamma_n B)p_n\| \\
&\quad + \|J_{\gamma_n}^A(I-\gamma_n B)p_n - p_n\| + \|p_n - x_n\| \\
&\leq 2\delta_n\|x_n - x_{n-1}\| + \|J_{\gamma_n}^A(I-\gamma_n B)p_n - p_n\| \\
&\leq 2\delta_n\|x_n - x_{n-1}\| + \|J_{\gamma_n}^A\zeta_n - p_n\| + \|e_n\|.
\end{aligned}
$$

This together with (37) and (45) yields that

$$
\lim_{n\to\infty} \|J_r^A(I-rB)x_n - x_n\| = 0. \tag{46}
$$

Let $S = (1-\tau)T_n + \tau J_r^A(I-rB)$, where $\tau \in (0,1)$. In view of Lemma 2.3, we have that S is nonexpansive and $\mathrm{Fix}(S) = \mathrm{Fix}(T_n) \cap \mathrm{Fix}(J_r^A(I-rB)) = \mathrm{Fix}(T) \cap (A+B)^{-1}(0)$. Since

$$
\|Sx_n - x_n\| \leq (1-\tau)\|T_n x_n - x_n\| + \tau\|J_r^A(I-rB)x_n - x_n\|,
$$

we get from (40) and (46) that

$$
\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{47}
$$

Since f is a contraction and S is a nonexpansive mapping, we see that $(1-\lambda)S + \lambda f$ is contractive, where $\lambda \in (0,1)$. Thus, it has a unique fixed point, denoted by x^λ. That is,

$$
x^\lambda = (1-\lambda)Sx^\lambda + \lambda f(x^\lambda).
$$

Let $\tilde{x} = \lim_{\lambda\to 0} x^\lambda$. From Lemma 2.1, we get

$$
\tilde{x} = Q_{\mathrm{Fix(S)}}^{U\cap V} f(\tilde{x}) = Q_{(A+B)^{-1}(0)\cap \mathrm{Fix(T)}}^{U\cap V} f(\tilde{x}),
$$

where $Q_{(A+B)^{-1}(0)\cap \mathrm{Fix(T)}}^{U\cap V}$ is the unique sunny nonexpansive retraction of $U \cap V$ onto $(A+B)^{-1}(0) \cap \mathrm{Fix}(T)$. Hence, one obtains

$$
\begin{aligned}
\|x_n - x^\lambda\|^q &= \lambda\langle f(x^\lambda) - x_n, j_q(x^\lambda - x_n)\rangle + (1-\lambda)\langle Sx^\lambda - x_n, j_q(x^\lambda - x_n)\rangle \\
&= \lambda(\langle f(x^\lambda) - x^\lambda, j_q(x^\lambda - x_n)\rangle + \langle x^\lambda - x_n, j_q(x^\lambda - x_n)\rangle) \\
&\quad + (1-\lambda)(\langle Sx^\lambda - Sx_n, j_q(x^\lambda - x_n)\rangle + \langle Sx_n - x_n, j_q(x^\lambda - x_n)\rangle) \\
&\leq \lambda\langle f(x^\lambda) - x^\lambda, j_q(x^\lambda - x_n)\rangle + \lambda\|x^\lambda - x_n\|^q \\
&\quad + (1-\lambda)\|Sx^\lambda - Sx_n\|\|x^\lambda - x_n\|^{q-1} \\
&\quad + (1-\lambda)\|Sx_n - x_n\|\|x^\lambda - x_n\|^{q-1} \\
&\leq \lambda\langle f(x^\lambda) - x^\lambda, j_q(x^\lambda - x_n)\rangle + \|x^\lambda - x_n\|^q \\
&\quad + \|x^\lambda - x_n\|^{q-1}\|Sx_n - x_n\|.
\end{aligned} \tag{48}
$$

From (48), we see that

$$\langle x^\lambda - f(x^\lambda), j_q(x^\lambda - x_n)\rangle \le \frac{\|x^\lambda - x_n\|^{q-1}}{\lambda}\|Sx_n - x_n\|. \tag{49}$$

Fixing λ and letting $n \to \infty$ in (49), we conclude from (47) and (49) that

$$\limsup_{n\to\infty}\langle x^\lambda - f(x^\lambda), j_q(x^\lambda - x_n)\rangle \le 0. \tag{50}$$

Since E is q-uniformly smooth, one asserts that the limits $\limsup_{n\to\infty}$ and $\limsup_{\lambda\to 0}$ are interchangeable. Hence, it follows from (50) that

$$\limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, j_q(x_n - \tilde{x})\rangle \le 0. \tag{51}$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Since $Q^{U\cap V}_{\mathrm{Fix(T)}\cap(A+B)^{-1}(0)}$ is a sunny contraction, one gets

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^q &\le \langle \eta_n f(x_n) + \sigma_n J^A_{\gamma_n}(p_n - \gamma_n Bp_n + e_n) \\
&\quad + \rho_n T_n p_n - \tilde{x}, j_q(x_{n+1} - \tilde{x})\rangle \\
&\le \eta_n\langle f(x_n) - f(\tilde{x}), j_q(x_{n+1} - \tilde{x})\rangle + \eta_n\langle f(\tilde{x}) - \tilde{x}, j_q(x_{n+1} - \tilde{x})\rangle \\
&\quad + \sigma_n\|J^A_{\gamma_n}(p_n - \gamma_n Bp_n + e_n) - \tilde{x}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + \rho_n\|T_n p_n - \tilde{x}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\le \eta_n h\|x_n - \tilde{x}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} + \eta_n\langle f(\tilde{x}) - \tilde{x}, j_q(x_{n+1} - \tilde{x})\rangle \\
&\quad + \sigma_n\|p_n - \tilde{x}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} + \rho_n\|p_n - \tilde{x}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + \|e_n\|\,\|x_{n+1} - \tilde{x}\|^{q-1}.
\end{aligned} \tag{52}$$

From (2) and (52), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^q &\le (1 - \eta_n(1-h))\|x_n - \tilde{x}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + \eta_n\langle f(\tilde{x}) - \tilde{x}, j_q(x_{n+1} - \tilde{x})\rangle \\
&\quad + \delta_n\|x_n - x_{n-1}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + \delta_n\|x_n - x_{n-1}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + \|e_n\|\,\|x_{n+1} - \tilde{x}\|^{q-1}.
\end{aligned} \tag{53}$$

By Lemma 2.8, we get that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^q &\le (1 - \eta_n(1-h))\|x_n - \tilde{x}\|^q + \eta_n q\langle f(\tilde{x}) - \tilde{x}, j_q(x_{n+1} - \tilde{x})\rangle \\
&\quad + q\delta_n\|x_n - x_{n-1}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + q\delta_n\|x_n - x_{n-1}\|\,\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + q\|e_n\|\,\|x_{n+1} - \tilde{x}\|^{q-1}.
\end{aligned}$$

We see from Lemma 2.7 that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. This completes the proof. ■

If mappings T, A and B are defined on space E, then the sunny nonexpansive retraction in Algorithm 3 can be removed.

Algorithm 4 The inertial splitting algorithm IV.

Initialization: Let $\delta \in (0,1)$. Let $x_0, x_1 \in E$ be arbitrary.
Iterative Steps: Given the current iterators x_n and x_{n-1}, calculate x_{n+1} as follows:
Step 1. Compute δ_n, such that $0 \le \delta_n \le \delta_n^*$, where δ_n^* is defined in (25).
Step 2. Compute

$$\begin{cases} p_n = x_n + \delta_n(x_n - x_{n-1}), \\ z_n = \epsilon_n T p_n + (1-\epsilon_n)p_n, \\ x_{n+1} = \eta_n f(x_n) + \sigma_n J^A_{\gamma_n}(p_n - \gamma_n B p_n + e_n) + \rho_n z_n, \end{cases}$$

where $\{e_n\}$ is a sequence with $\sum_{n=0}^{\infty}\|e_n\| < \infty$.
Step 3. Set $n \leftarrow n+1$ and go to Step 1.

Corollary 3.2: *Suppose that Condition 3.1 and Condition 3.3 satisfied. Let $f: U \to E$ be a h-contraction. Then the sequence $\{x_n\}$ generated by Algorithm 4 converges strongly to some points to*

$$x^* = Q^E_{\mathrm{Fix(T)}\cap(A+B)^{-1}(0)} f(x^*),$$

where $Q^E_{\mathrm{Fix(T)}\cap(A+B)^{-1}(0)}$ is the unique sunny nonexpansive mapping from E to $\mathrm{Fix(T)}\cap(A+B)^{-1}(0)$, *that is, x^* is the unique solution of variational inequality* $\langle f(x^*) - x^*, j_q(y - x^*)\rangle \le 0$ *for all* $y \in \mathrm{Fix(T)}\cap(A+B)^{-1}(0)$.

4. Numerical results

In this section, we provide some numerical examples to demonstrate the computational performance of the suggested algorithms. In the following examples, since the related operators are defined in the whole spaces, we choose Algorithm 2 and Algorithm 4 for our numerical experiments. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU@1.60 GHz computer with RAM 8.00 GB.

Example 4.1: Let h and g be two convex, lower semi-continuous functions such that h is differentiable with L-Lipschitz continuous gradient, and the proximal

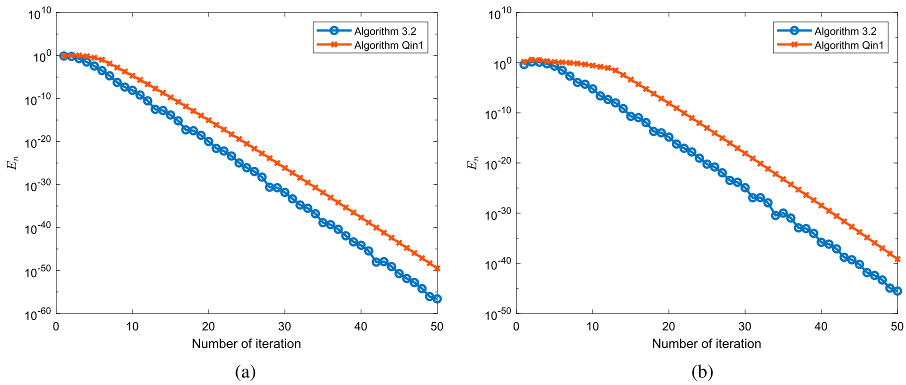

Figure 1. Compare the behaviour of Algorithm 2 and Qin et al.'s algorithm under different initial values. (a) Case I and (b) Case II.

mapping of g can be computed. The convex minimization problem is to find x^* such that

$$h(x^*) + g(x^*) \leq h(x) + g(x), \quad \forall\, x \in H.$$

Taking $A := \nabla h$ and $B := \partial g$, then the convex minimization problem can be reduced to the following inclusion problem: find $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*),$$

where ∇h is a gradient of h and ∂g is a subdifferential of g. Let $T : R^3 \to R^3$ be a mapping defined by $Tx = \frac{1}{2}x - \sin x$. We see that T is $\frac{1}{4}$-strict pseudo-contractive. Now we solve the following convex minimization problem:

$$\min_{x \in R^3} \|x\|_2^2 + \|x\|_1, \quad \text{such that } x^* \in \mathrm{Fix}(T), \tag{54}$$

where $x = (x_1, x_2, x_3) \in R^3$. Let $h(x) = \|x\|_2^2$ and $g(x) = \|x\|_1$. We obtain that $\nabla h(x) = 2x$. It is known that

$$(I + r\partial g)^{-1}x = (\max\{|x_1| - r, 0\}\mathrm{sign}(x_1), \max\{|x_2| - r, 0\}\mathrm{sign}(x_2), \max\{|x_3| - r, 0\}\mathrm{sign}(x_3)).$$

In case 1, we solve convex minimization problem (54) by Algorithm 2 and Theorem 2.1 in [11] (denoted by Algorithm Qin1). The iteration number $N = 50$, $E_n = \|x_n - x_{n-1}\|$ and the other parameters are chosen as follows. In Algorithm 2, we choose $e_n = 0$, $\theta_n = \frac{1}{n^{1.2}}$, $\delta = 0.2$, $\delta_n = 0.5\delta^*$, $\rho_n = 0.3$, $\gamma_n = 0.1$ and $\epsilon_n = 0.45$. The parameters selection in Algorithm Qin1 is consistent with Algorithm 2. We consider two different initial values (Case I: $x_0 = (-1, -3, 1)$ and $x_1 = (0.5, -1.5, 0.5)$, Case II: $x_0 = (10, 2.5, -8)$ and $x_1 = (-2.5, -1, 2.5)$) and the numerical results are shown in Figure 1.

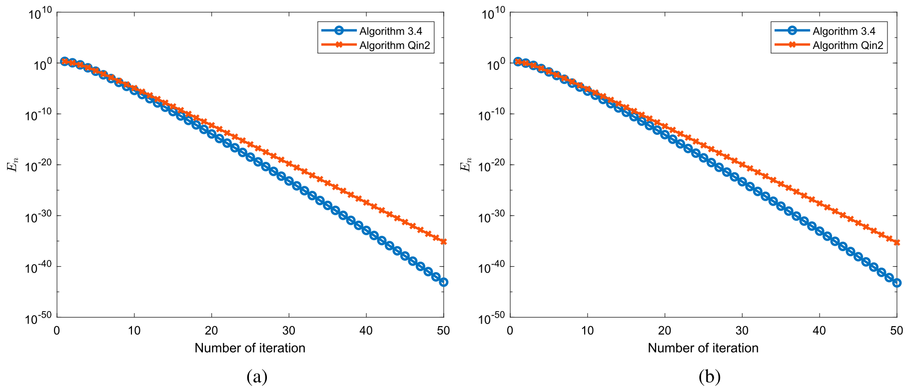

Figure 2. Compare the behaviour of Algorithm 4 and Qin et al.'s algorithm under different initial values. (a) Case III and (b) Case IV.

In case 2, we solve convex minimization problem (54) by Algorithm 4 and Theorem 2.2 in [11] (denoted by Algorithm Qin2). The iteration number $N = 50$, $E_n = \|x_n - x_{n-1}\|$ and the other parameters are chosen as follows. In Algorithm 4, we choose $f(x) = \frac{1}{2}x$, $\theta_n = \frac{1}{n^{1.2}}$, $\delta = 0.1$, $\delta_n = 0.5\delta^*$, $e_n = 0$, $\sigma_n = \rho_n = \frac{n}{2(n+1)}$, $\eta_n = \frac{1}{n+1}$, $\gamma_n = 0.1$ and $\epsilon_n = 0.45$. The parameters selection in Algorithm Qin2 is consistent with Algorithm 4. We consider two different initial values (Case III: $x_0 = (1, -4, 1)$ and $x_1 = (0.5, 3.5, 1.5)$, Case IV: $x_0 = (-3, 3, -1.5)$ and $x_1 = (1, -1, 3)$) and the numerical results are shown in Figure 2.

In the experiments, we choose the same iteration number and different initial values. It is clear from the experiments that Algorithm 2 and Algorithm 4 outperform Algorithm Qin1 and Algorithm Qin2 in the number of iterations, respectively.

Example 4.2: Suppose that H_1 and H_2 are two real Hilbert spaces and $S : H_1 \to H_2$ is a bounded linear operator with the adjoint S^*. Suppose that C and Q are two nonempty closed convex subsets of H_1 and H_2, respectively. In this example, we concern the following split convex feasibility problem:

$$\text{Find } x^* \in C \text{ such that } Sx^* \in Q. \tag{55}$$

Taking $Ax = \nabla(\frac{1}{2}\|Sx - P_Q Sx\|^2) = S^*(I - P_Q)Sx$ and $B = \partial i_C$, then (55) can be written in the form as follows:

$$\text{Find } x^* \text{ such that } 0 \in A(x^*) + B(x^*). \tag{56}$$

It is clear that A is 1-Lipschitz continuous and B is maximal monotone.

In this example, we set $Tx(t) = x(t)$, $H_1 = H_2 = L_2([0, 2\pi])$ with the inner product $\langle x, y\rangle := \int_0^{2\pi} x(t)y(t)\,\mathrm{d}t$ and the associated norm $\|x\|_2 :=$

$(\int_0^{2\pi} |x(t)|^2 \, dt)^{\frac{1}{2}}$. The half-spaces are defined as

$$C = \left\{ x \in L_2([0, 2\pi]) | \int_0^{2\pi} x(t) \, dt \leq 1 \right\},$$

and

$$Q = \left\{ x \in L_2([0, 2\pi]) | \int_0^{2\pi} |x(t) - \sin(t)|^2 \, dt \leq 16 \right\}.$$

Define a linear continuous operator $S : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Sx)(t) := x(t)$. Then $(S^* x)(t) = x(t)$ and $\|S\| = 1$. Now, we solve problem (55). Since $(Sx)(t) = x(t)$, problem (55) is actually a convex feasibility problem:

$$\text{find } x^* \in C \cap Q.$$

Moreover, it is easy to see that $x(t) = 0$ is a solution. Hence, the solution set of (55) is nonempty. For our numerical computation, we write the projections onto set C and the projections onto set Q as follows, respectively (see [22]):

$$J_{\gamma_n}^A(y) = P_C(y) = \begin{cases} \dfrac{1 - \int_0^{2\pi} y(t) \, dt}{4\pi^2} + y, & \displaystyle\int_0^{2\pi} y(t) \, dt > 1, \\ y, & \displaystyle\int_0^{2\pi} y(t) \, dt \leq 1; \end{cases}$$

$$P_Q(x) = \begin{cases} \sin(t) + \dfrac{4}{\sqrt{\int_0^{2\pi} |x(t) - \sin(t)|^2 dt}} & \\ \quad (x - \sin(t)), & \displaystyle\int_0^{2\pi} |x(t) - \sin(t)|^2 \, dt > 16, \\ x, & \displaystyle\int_0^{2\pi} |x(t) - \sin(t)|^2 \, dt \leq 16. \end{cases}$$

The error of the iterative algorithms is denoted by

$$E_n = \frac{1}{2} \|P_C(x_n) - x_n\|_2^2 + \frac{1}{2} \left\|P_Q(S(x_n)) - S(x_n)\right\|_2^2.$$

In this numerical experiment, we use the two algorithms mentioned in Example 4.1 to solve (55). The parameters of these algorithms are the same as those sets in Example 4.1. We consider four different initial values x_0 and x_1 (Case I: $x_0 = \frac{t^2}{5}$, $x_1 = \frac{2^t}{10}$; Case II: $x_0 = \frac{t^2}{5}$, $x_1 = \frac{t^3}{50}$; Case III: $x_0 = \frac{t^2}{15}$, $x_1 = \sin t + \frac{t^3}{30}$; Case IV: $x_0 = \frac{t^3}{10}$, $x_1 = \sin t + \frac{2^t}{15}$). The numerical results are reported in Figures 3 and 4.

It can be seen from Figure 3 and 4 that the two algorithms with inertial terms proposed in this paper converge faster than the algorithms without inertial terms suggested by Qin et al. [11].

Finally, we give an example that occurs in a Banach space. In this example, we compare the behaviour of the proposed algorithm (Algorithm 4) and Shehu

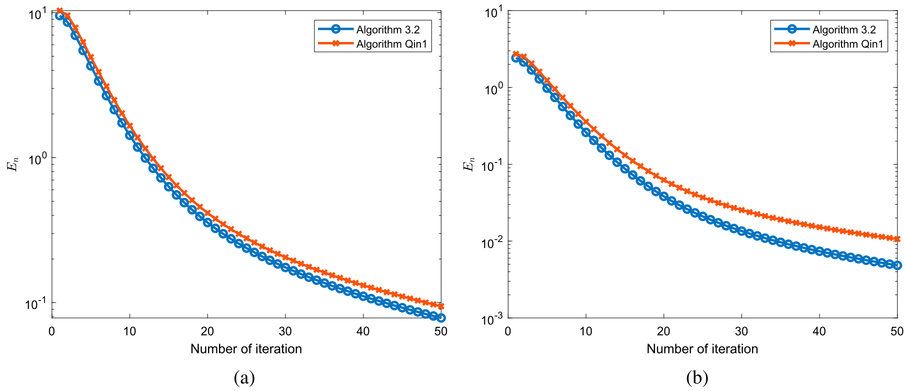

Figure 3. Compare the behaviour of Algorithm 2 and Qin et al.'s algorithm under different initial values. (a) Case I and (b) Case II.

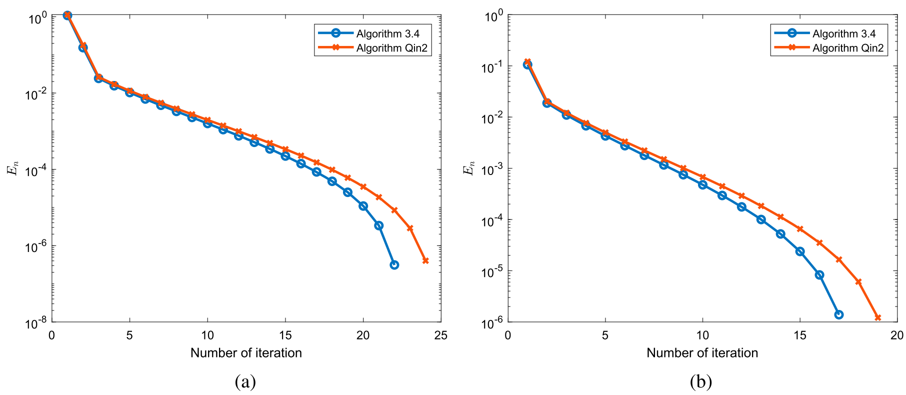

Figure 4. Compare the behaviour of Algorithm 4 and Qin et al.'s algorithm under different initial values. (a) Case III and (b) Case IV.

and Gibali's algorithm [16]. Since Algorithm 2 is the same as Shehu and Gibali's algorithm, in this case, we only compare the behaviour between Algorithm 4 and Shehu and Gibali's algorithm.

Example 4.3: Let $E = \ell^3(\mathbb{R})$ defined by $\ell^3(\mathbb{R}) := \{\bar{x} = (x_1, x_2, x_3, \ldots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^3 < \infty\}$, with norm $\|\cdot\|_{\ell^3} : \ell^3 \to [0, \infty)$ defined by $\|\bar{x}\|_{\ell^3} = (\sum_{i=1}^{\infty} |x_i|^3)^{\frac{1}{3}}$, for arbitrary $\bar{x} = (x_1, x_2, x_3, \ldots) \in \ell^3$. It is known that ℓ^3 is a uniformly convex and 2-uniformly smooth Banach space but not a Hilbert space. Let $A : \ell^3 \to \ell^3$ and $B : \ell^3 \to \ell^3$ be two mappings defined by

$$Ax = 5x$$

and

$$Bx = 2x + (1, 1, 1, 0, 0, 0, 0, \ldots),$$

respectively, where $x = (x_1, x_2, x_3, \ldots) \in \ell^3$. Now we show that A is an m-accretive operator and B is a $1/2$-inverse strongly accretive of order 2 operator. In fact, let $x, y \in \ell^3$, then we get

$$\langle Ax - Ay, j_2(x-y)\rangle = 5\langle x-y, j_2(x-y)\rangle = 5\|x-y\|_{\ell^3}^2$$

and $R(I + rA) = \ell^3$ for all $r > 0$. We also have

$$\begin{aligned}\langle Bx - By, j_2(x-y)\rangle &= \langle 2x - 2y, j_2(x-y)\rangle \\ &= 2\|x-y\|_{\ell^3}^2 = \frac{1}{2}\|Bx - By\|_{\ell^3}^2.\end{aligned}$$

On the other hand, for all $r > 0$, we have

$$\begin{aligned}J_r^A(x - rBx) &= (I + rA)^{-1}(x - rBx) \\ &= \frac{1-2r}{1+5r}x - \frac{r}{1+5r}(1,1,1,0,0,0,\ldots),\end{aligned}$$

where $x = (x_1, x_2, x_3, \ldots) \in \ell^3$. We compare the proposed algorithm (Algorithm 4) with the Algorithm 1 of Theorem 3 in [16]. Set $T(x) = x$ in our Algorithm 4 and keep the other parameters the same as in Example 4.1. Take $\theta = 0.1$, $\epsilon_n = 1/n^2$, $\theta_n = 0.5\bar{\theta}_n$, $r = 0.1$ and $\lambda_n = 1/n$ for Shehu and Gibali's Algorithm 1 [16]. The maximum number of iterations 200 is common stopping criterion and $D_n = \|x_n - x_{n-1}\|_{\ell^3}$ is measure of the error at the nth iteration step. In order to test the robustness of the proposed algorithm, we choose four different initial values as follows. The numerical behaviour of all algorithms for different initial values are shown in Figure 5:

(i) $x_0 = (0.6787, 0.7577, 0.7431, 0, 0, 0, \ldots)$,
$x_1 = (0.3922, 0.6554, 0.1711, 0, 0, 0, \ldots)$;

(ii) $x_0 = (7.6551, 7.9519, 1.8687, 0, 0, 0, \ldots)$,
$x_1 = (4.8976, 4.4558, 6.4631, 0, 0, 0, \ldots)$;

(iii) $x_0 = (37.5633, 12.7547, 25.2978, 0, 0, 0, \ldots)$,
$x_1 = (34.9538, 44.5451, 47.9645, 0, 0, 0, \ldots)$;

(iv) $x_0 = (61.6044, 47.3288, 35.1659, 0, 0, 0, \ldots)$,
$x_1 = (83.0828, 58.5264, 54.9723, 0, 0, 0, \ldots)$.

It is easy to check that the solution of the inclusion problem $0 \in (A + B)x^*$ is

$$x^* = (A+B)^{-1}(0) = \left\{\left(-\frac{1}{7}, -\frac{1}{7}, -\frac{1}{7}, 0, 0, 0, \ldots\right)\right\}.$$

The approximate solution obtained by our Algorithm 4 in (iv) is

$$x^* = (-0.14131868, -0.14131868, -0.14131868, 0, 0, 0, \ldots).$$

It can be seen from Figure 5 that the suggested algorithm (Algorithm 4) converges faster than Algorithm 1 presented by Shehu and Gibali [16], and the results are

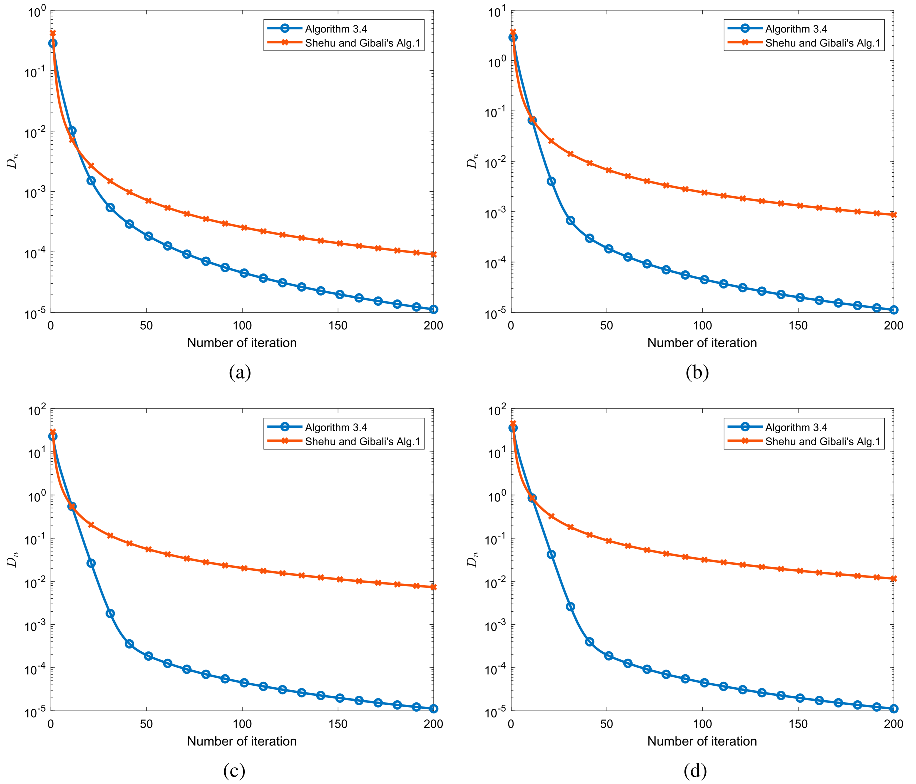

Figure 5. Compare the behaviour of Algorithm 4 and Shehu and Gibali's algorithm under different initial values. (a) Case (i), (b) Case (ii), (c) Case (iii), (d) Case (iv).

independent of the choice of initial values. Therefore, the algorithm proposed in this paper is efficient and robust.

5. Conclusions

In this paper, we proposed the inertial splitting algorithms for solving the common solution problem. Weak and strong convergence theorems are established in uniformly convex and q-uniformly smooth Banach spaces, for example, L_p with $1 < p < \infty$. One of the highlights is that our new algorithms converge faster than the associated ones from the viewpoint of numerical computation. The other highlight of this paper is that our new algorithms work for the class of κ-strictly pseudocontractive mappings, which include the class of nonexpansive mappings as a special case. The main results presented in this paper extend and complement the recent results obtained in [11,15,16].

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