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INERTIAL SHRINKING PROJECTION ALGORITHMS FOR SOLVING HIERARCHICAL VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we propose two inertial shrinking algorithms to approximate a solution of hierarchical variational inequality problems with nonexpansive mappings in Hilbert spaces. We prove strong convergence theorems under some mild conditions. Finally, we present some numerical examples to compare our algorithms with some existing algorithms, which illustrate the advantage of our proposed algorithms.

1. Introduction

Let C be a nonempty closed convex subset in a real Hilbert space H. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. The set of fixed points of a mapping $T : C \to C$ is defined by $\operatorname{Fix}(T) := \{x \in C : Tx = x\}$. Moudafi and Mainge [1] introduced the following hierarchical fixed point problem (in short, HFPP) for a nonexpansive mapping T with respect to a nonexpansive mapping S on C, namely,

$$\text{find } x^* \in \operatorname{Fix}(T) \quad \text{such that } \langle x^* - Sx^*, x^* - x\rangle \leq 0, \quad \forall x \in \operatorname{Fix}(T). \tag{1.1}$$

It is easy to check that solving HFPP (1.1) is equivalent to the fixed point problem:

$$\text{find } x^* \in C \quad \text{such that } x^* = P_{\operatorname{Fix}(T)} \circ Sx^*, \tag{1.2}$$

where $P_{\operatorname{Fix}(T)}$ stands for the metric projection of H onto $\operatorname{Fix}(T)$. The solution set of HFPP (1.1) is represented as $\Psi := \left\{x^* \in C : x^* = P_{\operatorname{Fix}(T)} \circ Sx^*\right\}$. We easily prove that HFPP (1.1) is equivalent to the variational inequality problem:

$$\text{find } x^* \in C \quad \text{such that } 0 \in (I - S)x^* + N_{\operatorname{Fix}(T)}x^*, \tag{1.3}$$

where I is the identity mapping on C and $N_{\operatorname{Fix}(T)}$ is the normal cone to $\operatorname{Fix}(T)$ defined by

$$N_{\operatorname{Fix}(T)} = \begin{cases} \{u \in H : \langle y - x, u\rangle \leq 0, \quad \forall y \in \operatorname{Fix}(T)\}, & \text{if } x \in \operatorname{Fix}(T), \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is worth mentioning that when $S = I$, the solution set of HFPP (1.1) is nothing but $\operatorname{Fix}(T)$. We note that HFPP (1.1) covers monotone variational inequalities and convex programming problems and acts as a useful framework for applied sciences,

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etc, see, for instance [2, 3, 4, 5, 6, 7]. By setting $S = I - \gamma A$, then HFPP (1.1) is reduced to a so-called hierarchical variational inequality problem: find $x^* \in \operatorname{Fix}(T)$ such that $\langle Ax^*, x - x^*\rangle \geq 0$, $\forall x \in \operatorname{Fix}(T)$. We note that HFPP (1.1) has the iteration algorithm $x_{n+1} = P_{\operatorname{Fix}(T)}(Sx_n)$ based upon relation (1.2). It will converge if a fixed point of the operator $P_{\operatorname{Fix}(T)} \circ S$ exists, and if S is averaged, not just nonexpansive. But calculating $P_{\operatorname{Fix}(T)} \circ S$ in this case is usually not easy. It would be nice if we could devise an algorithm that uses T itself, rather than $P_{\operatorname{Fix}(T)} \circ S$. For this purpose, Moudafi [2] introduced the following Mann iteration algorithm for solving HFPP (1.1):

$$x_{n+1} = (1 - \nu_n) x_n + \nu_n (\lambda_n S x_n + (1 - \lambda_n) T x_n), \quad \forall n \geq 0, \tag{1.4}$$

where $\{\nu_n\}$ and $\{\lambda_n\}$ are two sequences in $(0, 1)$. It should be noted that $\{x_n\}$ generated by (1.4) converges weakly to a solution of problem HFPP (1.1). It is worth mentioning that some algorithms in signal processing and image reconstruction may be written as the Mann iteration. The main feature of its corresponding convergence theorems provides a unified frame for analysing various specific algorithms. In practical applications, many problems, such as, quantum physics and image reconstruction, are in infinite dimensional spaces. To investigate these problems, norm convergence is usually preferable to the weak convergence. In 2008, Takahashi, Takeuchi and Kubota [8] established strong convergence of the Mann iteration with the aid of projections.

$$\begin{cases} y_n = \nu_n x_n + (1 - \nu_n) T x_n, \\ C_{n+1} = \{u \in C_n : \|y_n - u\| \leq \|x_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0. \end{cases} \tag{1.5}$$

They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of a nonexpansive mapping T. This method is now referred as the shrinking projection method. In recent years, many authors studied these projection-based methods in various spaces; see, e.g., [9, 10, 11, 12, 13].

In general, the convergence rate of Mann algorithm is slow. Fast convergence of algorithm is required in many practical applications. In particular, an inertial type extrapolation was first proposed by Polyak [14] as an acceleration process. In recent years, some authors have constructed different fast iterative algorithms by inertial extrapolation techniques, such as, inertial Mann algorithms [15], inertial forward-backward splitting algorithms [16], and fast iterative shrinkage-thresholding algorithm (FISTA) [17].

In 2008, Mainge [15] introduced the following inertial Mann algorithm by unifying the Mann algorithm and the inertial extrapolation:

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ x_{n+1} = (1 - \lambda_n) w_n + \lambda_n T(w_n), \quad n \geq 0. \end{cases} \tag{1.6}$$

Note that the iterative sequence $\{x_n\}$ generated by (1.6) converges weakly to a fixed point of T under some assumptions.

Recently, based on the projection method and the hybrid method, Malitsky and Semenov [19] introduced a new hybrid method without extrapolation step for solving variational inequality problems, and proved a strong convergence theorem. Their numerical experiments show that this method has a competitive performance. For related works, see [18, 20, 21, 22, 23].

Inspired and motivated by the above works. In this paper, by combining iterative methods (1.4), (1.5) and (1.6), we introduce two new inertial shrinking projection algorithms for solving HFPP (1.1). Two strong convergence theorems are established in the framework of real Hilbert spaces. We also give three numerical examples to illustrate the computational performance of our proposed algorithms over some previously known algorithms in [23, 24].

2. Preliminaries

Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let $\omega_w(x_n)$ denote the set of all weak limits of $\{x_n\}$. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$, where P_C is called the metric projection of H onto C. Moreover, $P_C x$ is characterized by the property: $P_C x \in C$ and $\langle P_C x - x, P_C x - y\rangle \leq 0$, $\forall y \in C$. This characterization implies the following inequality

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \text{ and } y \in C. \tag{2.1}$$

Lemma 2.1. [25] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$ and $a \in R$, $\left\{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u\rangle + a\right\}$ is convex and closed.*

Lemma 2.2. [26] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to +\infty$. Then $x \in \mathrm{Fix}(T)$.*

Lemma 2.3. [18] *Let $\{a_n\}$ and $\{\xi_n\}$ be nonnegative real sequences, assume that $\nu \in [0, 1)$, $\zeta \in R^+$ and for all $n \in N$ the following inequality holds: $a_{n+1} \leq \nu a_n + \zeta \xi_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} \xi_n < +\infty$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4. [27] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\} \subset H$, $u \in H$ and $q = P_C u$. If $\omega_w(x_n) \subset C$ and $\|x_n - u\| \leq \|u - q\|$, $\forall n \in N$, then $\{x_n\}$ converges strongly to q.*

3. The Inertial Shrinking projection algorithm

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S, T : C \to C$ be two nonexpansive mappings. Assume that $\Omega = \Psi \bigcap \mathrm{Fix(S)} \neq \emptyset$. Let*

$$\begin{aligned} &\{\delta_n\} \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \\ &\{\nu_n\} \in [\nu, 1), \nu \in (0, 1), \{\lambda_n\} \in [\lambda_1, \lambda_2] \subset (0, 1). \end{aligned} \tag{3.1}$$

Set $x_{-1}, x_0 \in C$ arbitrarily. Define a sequence $\{x_n\}$ by the following:

$$\left\{\begin{aligned} w_n &= x_n + \delta_n (x_n - x_{n-1}), \\ y_n &= (1-\nu_n) w_n + \nu_n (\lambda_n S w_n + (1-\lambda_n) T w_n), \\ C_{n+1} &= \left\{u \in C_n : \|y_n - u\|^2 \leq \|w_n - u\|^2\right\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 0. \end{aligned}\right. \tag{3.2}$$

Then the sequence $\{x_n\}$ defined by (3.2) converge strongly to $x^* = P_\Omega x_0$.

Proof. Obviously, Ψ is closed and convex, since $\Psi = \operatorname{Fix}\left(P_{\operatorname{Fix}(T)} \circ S\right) \neq \emptyset$ and Fix(S) is also closed and convex. Therefore, Ω is closed and convex and hence $P_\Omega x_0$ is well defined.

Step 1. We show that $\Omega \subset C_{n+1}$ for all n. From Lemma 2.1 we know that C_{n+1} is closed and convex. For all $z \in \Omega$ we have

$$\begin{aligned} \|y_n - z\|^2 &= \|(1-\nu_n)(w_n - z) + \nu_n (\lambda_n (S w_n - z) + (1-\lambda_n)(T w_n - z))\|^2 \\ &\leq (1-\nu_n)\|w_n - z\|^2 + \nu_n \Big(\lambda_n \|S w_n - z\|^2 \\ &\quad + (1-\lambda_n)\|T w_n - z\|^2 - \lambda_n (1-\lambda_n)\|S w_n - T w_n\|^2\Big) \\ &\leq \|w_n - z\|^2 - \nu_n \lambda_n (1-\lambda_n)\|S w_n - T w_n\|^2. \end{aligned} \tag{3.3}$$

So $z \in C_{n+1}$ for each $n \geq 0$ and hence $\Omega \subset C_{n+1} \subset C_n$, $\forall n \geq 0$.

Step 2. We show that $\{x_n\}$ converges weakly to $x^* \in \operatorname{Fix}(T)$. From $x_n = P_{C_n} x_0$, this together with the fact that $\Omega \subset C_n$ implies $\|x_n - x_0\| \leq \|z - x_0\|, \forall z \in \Omega$. In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|x^* - x_0\|, \quad \text{where } x^* = P_\Omega x_0. \tag{3.4}$$

The fact $x_{n+1} \in C_{n+1} \subset C_n$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, this implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Using (2.1), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \quad \forall n \geq 0. \tag{3.5}$$

Therefore, combining (3.4) and (3.5) we have

$$\sum_{n=1}^{N} \|x_{n+1} - x_n\|^2 \leq \sum_{n=1}^{N} \left(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2\right) \leq \|x^* - x_0\|^2 - \|x_1 - x_0\|^2,$$

which implies that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent and hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Next, by the definition of w_n in (3.2) and $\delta_1 \leq \delta_n \leq \delta_2$, $\forall n$, we have

$$\|w_n - x_n\| = \delta_n \|x_n - x_{n-1}\| \leq \max\{|\delta_1|, |\delta_2|\} \|x_n - x_{n-1}\| \to 0.$$

Therefore, $\|w_n - x_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| \to 0$. Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$, by the definiton of C_{n+1}, it follows that $\|y_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2 \to 0$. Furthermore, we have $\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0$. Again, since

$$\|w_n - y_n\| \leq \|w_n - x_n\| + \|x_n - y_n\| \to 0, \tag{3.7}$$

from (3.1), (3.3) and (3.7), we have

$$\nu_n\lambda_n(1-\lambda_n)\|Sw_n - Tw_n\|^2 \le \|w_n - z\|^2 - \|y_n - z\|^2 \le K\|w_n - y_n\|,$$

where $K := \sup_{n\ge 0}\{\|w_n - z\| + \|y_n - z\|\}$. From (3.7), we get

$$\lim_{n\to\infty}\|Sw_n - Tw_n\| = 0. \tag{3.8}$$

Further, from (3.2), (3.7) and (3.8), we have

$$\|Tw_n - w_n\| \le \frac{1}{\nu_n}\|y_n - w_n\| + \lambda_n\|Sw_n - Tw_n\| \to 0. \tag{3.9}$$

Therefore, $\|Sw_n - w_n\| \le \|Sw_n - Tw_n\| + \|Tw_n - w_n\| \to 0$. Since

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Tw_n\| + \|Tw_n - w_n\| + \|w_n - x_n\|\\ &\le 2\|w_n - x_n\| + \|Tw_n - w_n\| \to 0.\end{aligned} \tag{3.10}$$

Thus, it follows from (3.10) and Lemma 2.2 that every weak limit point of $\{x_n\}$ is a fixed point of the mapping T, i.e., $\omega_w(x_n) \subset \mathrm{Fix}(T)$. Therefore, $\{x_n\}$ converges weakly to $x^* \in \mathrm{Fix}(T)$. Further, $\{w_n\}$ converges weakly to $x^* \in \mathrm{Fix}(S)$.

Step 3. We show that $x^* \in \Psi$. From (3.2), we have $y_n - w_n = \nu_n(\lambda_n(Sw_n - w_n) + (1-\lambda_n)(Tw_n - w_n))$, Therefore,

$$\frac{1}{\nu_n\lambda_n}(w_n - y_n) = (I-S)w_n + \left(\frac{1-\lambda_n}{\lambda_n}\right)(I-T)w_n.$$

For all $u \in \mathrm{Fix}(T)$ and using monotonicity of $I - S$, we have

$$\begin{aligned}\left\langle \frac{w_n - y_n}{\nu_n\lambda_n}, w_n - u\right\rangle &= \langle (I-S)w_n - (I-S)u, w_n - u\rangle + \langle (I-S)u, w_n - u\rangle\\ &\quad + \frac{1-\lambda_n}{\lambda_n}\langle w_n - Tw_n, w_n - u\rangle\\ &\ge \langle (I-S)u, w_n - u\rangle + \frac{1-\lambda_n}{\lambda_n}\langle w_n - Tw_n, w_n - u\rangle.\end{aligned} \tag{3.11}$$

Using (3.1), (3.7) and (3.9) in (3.11), we have

$$\overline{\lim_{n\to\infty}}\langle u - Su, w_n - u\rangle \le 0, \quad \forall u \in \mathrm{Fix}(T).$$

By the fact that $\{w_n\}$ weakly converges to x^*, we have $\langle (I-S)u, x^* - u\rangle \le 0$, $\forall u \in \mathrm{Fix}(T)$. Since $\mathrm{Fix}(T)$ is convex, $\lambda u + (1-\lambda)x^* \in \mathrm{Fix}(T)$ for $\lambda \in (0,1)$,

$$\begin{aligned}&\langle (I-S)(\lambda u + (1-\lambda)x^*), x^* - (\lambda u + (1-\lambda)x^*)\rangle\\ &= \lambda\langle (I-S)(\lambda u + (1-\lambda)x^*), x^* - u\rangle,\end{aligned}$$

which implies $\langle (I-S)(\lambda u + (1-\lambda)x^*), x^* - u\rangle \le 0$, $\forall u \in \mathrm{Fix}(T)$. Taking limits $\lambda \to 0_+$, we have $\langle (I-S)x^*, x^* - u\rangle \le 0$, $\forall u \in \mathrm{Fix}(T)$, that is, $x^* \in \Psi$. Thus $x^* \in \Omega$.

Step 4. We show that $x_n \to x^*$, where $x^* = P_\Omega x_0$. Combining $\omega_w(x_n) \subset \Omega$, (3.4) and Lemma 2.4, we get that $\{x_n\}$ converge strongly to $x^* \in \Omega$, where $x^* = P_\Omega x_0$. ☐

Remark 3.2. (i) The term $(x_n - x_{n-1})$ in (3.2) introduces an inertial step that produces acceleration with proper choice of δ_n. It should be noted that in (3.2) that the term δ_n is a generalized term that was defined by the expression $\frac{n-1}{n+3}$ in [28, 29] and $\frac{t_n - 1}{t_{n+1}}$ (where $t_1 = 1, t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$) [17].

(ii) The conditions (3.1) on $\{\delta_n\}$, $\{\nu_n\}$ and $\{\lambda_n\}$ in the inertial shrinking projection algorithm (3.2) are obviously relaxed. Theorem 3.1 does not need the conditions $\sum_{n=0}^{\infty}\lambda_n<+\infty$ and $\lim_{n\to\infty}\frac{\|w_n-y_n\|}{\nu_n\lambda_n}=0$ in [2]. In fact, these two conditions are very strong, which prohibits the implementation of the related algorithms.

4. The Inertial shrinking projection algorithm without extrapolating step

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S,T:C\to C$ be two nonexpansive mappings. Assume that $\Omega=\Psi\bigcap\mathrm{Fix(S)}\neq\emptyset$. Let*

$$\begin{aligned}&\{\delta_n\}\subset[\delta_1,\delta_2],\delta_1\in(-\infty,0],\delta_2\in[0,\infty),\{\lambda_n\}\in[\lambda_1,\lambda_2]\subset(0,1)\\&\{\nu_n\}\subseteq(0,\nu],\nu\in\left(0,\frac{1}{1+\sigma}\right),\sigma\in(0,1),\lim_{n\to\infty}\inf\nu_n>0.\end{aligned}\tag{4.1}$$

Set $x_{-1},x_0,y_0\in C$ arbitrarily. Define two sequences $\{x_n\}$ and $\{y_n\}$ by the following:

$$\begin{cases}w_n=x_n+\delta_n\left(x_n-x_{n-1}\right),\\ y_{n+1}=\left(1-\nu_n\right)w_n+\nu_n\left(\lambda_nSy_n+\left(1-\lambda_n\right)Ty_n\right),\\ C_{n+1}=\left\{u\in C_n:\left\|y_{n+1}-u\right\|^2\leq\left(1-\nu_n\right)\left\|w_n-u\right\|^2+\nu_n\|y_n-u\|^2\right\},\\ x_{n+1}=P_{C_{n+1}}x_0,\quad n\geq0.\end{cases}\tag{4.2}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (4.2) converge strongly to $x^*\in\Omega$, where $x^*=P_\Omega x_0$.

Proof. As the same in Theorem 3.1, we have $P_\Omega x_0$ is well defined. From Lemma 2.1 we can easily observe that C_{n+1} is closed and convex. For all $z\in\Omega$ we have

$$\begin{aligned}\left\|y_{n+1}-z\right\|^2&=\left\|\left(1-\nu_n\right)w_n+\nu_n\left(\lambda_nSy_n+\left(1-\lambda_n\right)Ty_n\right)-z\right\|^2\\&\leq\left(1-\nu_n\right)\left\|w_n-z\right\|^2+\nu_n\Big(\lambda_n\left\|Sy_n-z\right\|^2+\left(1-\lambda_n\right)\left\|Ty_n-z\right\|^2\\&\quad-\lambda_n\left(1-\lambda_n\right)\left\|Sy_n-Ty_n\right\|^2\Big)\\&\leq\left(1-\nu_n\right)\left\|w_n-z\right\|^2+\nu_n\left\|y_n-z\right\|^2-\nu_n\lambda_n\left(1-\lambda_n\right)\left\|Sy_n-Ty_n\right\|^2\\&\leq\left(1-\nu_n\right)\left\|w_n-z\right\|^2+\nu_n\left\|y_n-z\right\|^2,\end{aligned}\tag{4.3}$$

which implies that $u\in C_{n+1}$ and hence $\Omega\subset C_{n+1}$ for all $n\geq0$. Using the same arguments in Theorem 3.1, we can get that the sequence $\{x_n\}$ is bounded. Further, we can prove that

$$\sum_{n=1}^{\infty}\left\|x_{n+1}-x_n\right\|^2<+\infty,\quad\text{and}\quad\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=0.\tag{4.4}$$

and

$$\lim_{n\to\infty}\left\|w_n-x_n\right\|=0,\quad\text{and}\quad\lim_{n\to\infty}\left\|w_n-x_{n+1}\right\|=0.\tag{4.5}$$

On the other hand, by the definition of w_n in (4.2), we have

(4.6)

$$\begin{aligned}\|w_n - x_{n+1}\|^2 \leq& \|x_n - x_{n+1}\|^2 + \delta_n^2\|x_n - x_{n-1}\| + \delta_n \left[\|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2\right]\\ \leq& (1+\delta_n)\|x_n - x_{n+1}\|^2 + \delta_n(1+\delta_n)\|x_n - x_{n-1}\|^2.\end{aligned}$$

From (4.6), the fact that $x_{n+1} \in C_{n+1}$ and (4.1), we obtain

(4.7)

$$\begin{aligned}\|y_{n+1} - x_{n+1}\|^2 \leq& (1-\nu_n)\left[(1+\delta_n)\|x_n - x_{n+1}\|^2 + \delta_n(1+\delta_n)\|x_n - x_{n-1}\|^2\right]\\ &+ \nu_n\left[\|y_n - x_n\|^2 + \|x_n - x_{n+1}\|^2 + 2\langle y_n - x_n, x_n - x_{n+1}\rangle\right]\\ \leq& (1-\nu_n)\left[(1+\delta_n)\|x_n - x_{n+1}\|^2 + \delta_n(1+\delta_n)\|x_n - x_{n-1}\|^2\right]\\ &+ \nu_n\left[(1+\sigma^2)\|y_n - x_n\|^2 + \left(1+\frac{1}{\sigma^2}\right)\|x_n - x_{n+1}\|^2\right]\\ \leq& \varphi^*\|y_n - x_n\|^2 + \xi_n,\end{aligned}$$

where $\varphi^* = \nu_n(1+\sigma) < 1$ and $\xi_n = \left(\delta_2 + \frac{1+2\sigma^2}{\sigma^2}\right)\|x_n - x_{n+1}\|^2 + \delta_2(1+\delta_2)\|x_n - x_{n-1}\|^2$. Since $\sum_{n=1}^{\infty}\|x_{n+1} - x_n\|^2 < +\infty$, we have $\sum_{n=1}^{\infty}\xi_n < \infty$. Therefore, applying Lemma 2.3 in (4.7), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{4.8}$$

From (4.4) and (4.8), we obtain

$$\|y_{n+1} - x_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{4.9}$$

Combining (4.5) and (4.8), we have

$$\|y_n - w_n\| \leq \|y_n - x_n\| + \|x_n - w_n\| \to 0. \tag{4.10}$$

It follows from (4.5) and (4.9), we obtain

$$\|y_{n+1} - w_n\| \leq \|y_{n+1} - x_n\| + \|x_n - w_n\| \to 0. \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\lim_{n\to\infty}\|w_n - y_{n+1} - \nu_n(w_n - y_n)\| = 0. \tag{4.12}$$

From (4.3), we have

$$\begin{aligned}\nu_n\lambda_n(1-\lambda_n)\|Sy_n - Ty_n\|^2 \leq& \|w_n - z\|^2 - \|y_{n+1} - z\|^2 + \nu_n\left(\|y_n - z\|^2 - \|w_n - z\|^2\right)\\ \leq& L\|w_n - y_{n+1}\| + M\|y_n - w_n\|,\end{aligned}$$

where $L := \sup_{n\geq 0}\{\|w_n - z\| + \|y_{n+1} - z\|\}$, $M := \sup_{n\geq 0}\{\|w_n - z\| + \|y_n - z\|\}$. Using (4.1), (4.10) and (4.11), we have

$$\lim_{n\to\infty}\|Sy_n - Ty_n\| = 0. \tag{4.13}$$

Further, from (4.2), we have

$$\nu_n\|Ty_n - y_n\| \leq \|y_{n+1} - w_n\| + \nu_n\|w_n - y_n\| + \nu_n\lambda_n\|Ty_n - Sy_n\|,$$

which implies

$$\|Ty_n - y_n\| \leq \frac{1}{\nu_n}\|y_{n+1} - w_n\| + \|w_n - y_n\| + \lambda_n \|Ty_n - Sy_n\|. \tag{4.14}$$

Hence, it follows from (4.1), (4.10), (4.11) and (4.13) that $\lim_{n\to\infty}\|Ty_n - y_n\| = 0$. From (4.13), we have $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$. The rest of the proof is similar to the proof of Theorem 3.1. □

Remark 4.2. It should be noted that Algorithm (4.2) is different from Algorithm (3.2). Note that the conditions (4.1) of our Algorithm (4.2) are different from Dong et al. Algorithm (5.3).

5. Numerical experiments

In this section, we do several computational experiments in support of the convergence of our proposed algorithms and compare with some existing algorithms in literatures. All the programs are performed in MATLAB2018a on a PC Desktop Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.800 GHz, RAM 8.00 GB. First, we introduce three algorithms, which solve our proposed problems.

In [24], Yao, Cho and Liou obtained the following theorem.

Theorem 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction with $\rho \in [0, 1)$, namely, $\|fx - fy\| \leq \rho\|x - y\|$ for all $x, y \in C$. Let $S, T : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Suppose that the following conditions are satisfied:*
(C1) $\lim_{n\to\infty}\nu_n = 0$ *and* $\sum_{n=1}^{\infty}\nu_n = \infty$*;*
(C2) $\lim_{n\to\infty}\frac{\zeta_n}{\nu_n} = 0$*;*
(C3) $\lim_{n\to\infty}\frac{|\nu_n - \nu_{n-1}|}{\nu_n} = 0$ *and* $\lim_{n\to\infty}\frac{|\zeta_n - \zeta_{n-1}|}{\zeta_n} = 0$ *or*
(C4) $\sum_{n=1}^{\infty}|\nu_n - \nu_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty}|\zeta_n - \zeta_{n-1}| < \infty$*.*
Set $x_{-1}, x_0 \in C$ arbitrarily. Define a sequence $\{x_n\}$ by the following:

$$\begin{cases} y_n = \zeta_n S x_n + (1 - \zeta_n) x_n, \\ x_{n+1} = \mathrm{P}_C[\nu_n f(x_n) + (1 - \nu_n) T y_n], \quad \forall n \geq 1. \end{cases} \tag{5.1}$$

Then the sequence $\{x_n\}$ defined by (5.1) converges strongly to a point $x^* \in \mathrm{F}(T)$.

Recently, Dong et al [23] obtained the following theorems.

Theorem 5.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S, T : C \to C$ be two nonexpansive mappings. Assume that $\Omega = \Psi \bigcap \mathrm{Fix(S)} \neq \emptyset$. Let $\{\delta_n\} \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \{\nu_n\} \in [\nu, 1), \nu \in (0, 1), \{\lambda_n\} \in [\lambda_1, \lambda_2] \subset (0, 1)$. Set $x_{-1}, x_0 \in C$ arbitrarily. Define a sequence $\{x_n\}$ by the following:*

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = (1 - \nu_n) w_n + \nu_n(\lambda_n S w_n + (1 - \lambda_n) T w_n), \\ C_n = \left\{u \in C : \|y_n - u\|^2 \leq \|w_n - u\|^2\right\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0. \end{cases} \tag{5.2}$$

Then the sequence $\{x_n\}$ *defined by* (5.2) *converge strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega x_0$.

Theorem 5.3. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $S, T : C \to C$ *be two nonexpansive mappings. Assume that* $\Omega = \Psi \bigcap \mathrm{Fix(S)} \neq \emptyset$. *Let* $\{\delta_n\} \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \nu_n \subset (0, \nu), \nu \in (0, \frac{1}{2}], \{\lambda_n\} \in [\lambda_1, \lambda_2] \subset (0, 1)$. *Set* $x_{-1}, x_0, y_0 \in C$ *arbitrarily. Define two sequences* $\{x_n\}$ *and* $\{y_n\}$ *by the following:*

$$\left\{\begin{aligned} w_n &= x_n + \delta_n (x_n - x_{n-1}), \\ y_{n+1} &= (1 - \nu_n) w_n + \nu_n (\lambda_n S y_n + (1 - \lambda_n) T y_n), \\ C_n &= \left\{u \in C : \|y_{n+1} - u\|^2 \leq (1 - \nu_n) \|w_n - u\|^2 + \nu_n \|y_n - u\|^2\right\}, \\ Q_n &= \{u \in C : \langle x_n - u, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n \geq 0. \end{aligned}\right. \tag{5.3}$$

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *defined by* (5.3) *converge strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega x_0$.

Let $C \subset H$ be a nonempty closed convex set of a real Hilbert space H. We consider the variational inequality problem:

$$\text{find } x^* \in C \quad \text{such that } \langle A(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{5.4}$$

where $A : H \to H$ is a mapping. Denote by $\mathrm{VI}(C, A)$ the solution of the variational inequality problem (5.4). Define $T : H \to H$ by $T := P_C$ and $S : H \to H$ by $S := I - \gamma A$, where $0 < \gamma < 2/L$ (L is the Lipschitz constant of the mapping A). From (1.1)–(1.3) we get $\mathrm{Fix}(P_C(I - \gamma A)) = \mathrm{VI}(C, A)$. Therefore, the variational inequality problem (5.4) is a special case of the hierarchical fixed point problem HFPP (1.1).

Example 5.4. Taking $A : R^2 \to R^2$ as follows:

$$A(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y)), \quad \forall x, y \in R.$$

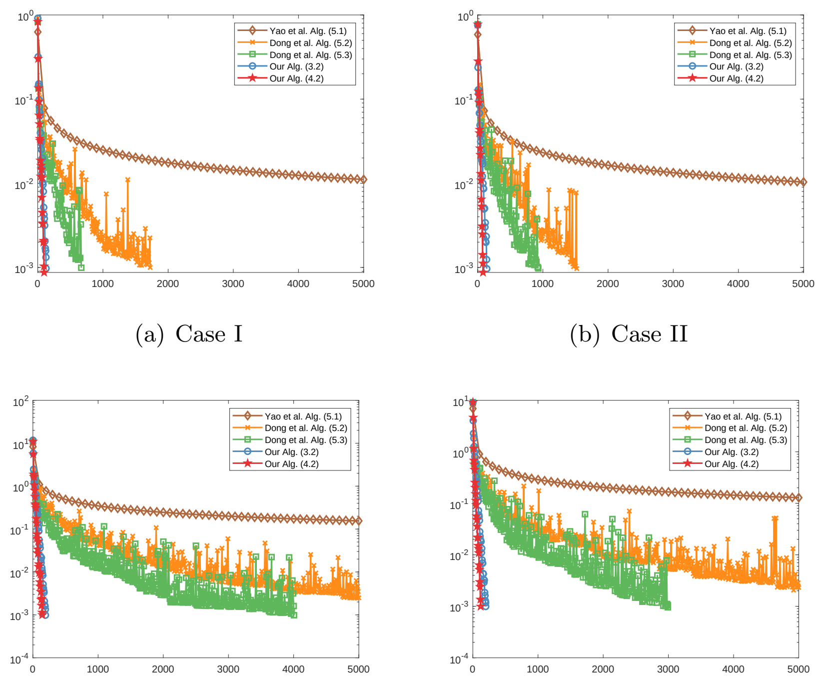
Let $C = \left\{x \in R^2 | -10e \leq x \leq 10e\right\}$, where $e = (1, 1)^\top$. It is not hard to check that A is Lipschitz continuous with constant $L = \sqrt{26}$ and 1-strongly monotone [30]. Therefore the variational inequality (5.4) has a unique solution $x^* = (0, 0)^\top$.

Our parameter settings are as follows. In Algorithm (5.1), set $\zeta_n = (n+1)^{-2}$, $\nu_n = (n+1)^{-1}$, $f(x) = 0.5x$ and $\gamma = 0.9/L$. In Algorithm (3.2), Algorithm (4.2), Algorithm (5.2) and Algorithm (5.3), set $\delta_n = 0.4$, $\lambda_n = 0.9$, $\nu_n = 0.4$ and $\gamma = 0.9/L$. Let $x_{-1} = x_0$, y_0 be randomly generated by the MATLAB function $k\times$rand(m,1) (where, Case I: $k = 1$, Case II: $k = -1$, Case III: $k = 10$, Case IV: $k = -10$). Denote by $E_n = \|x_n - x^*\|_2$ the error of the iterative algorithms. Maximum iteration 5000 or $E_n < 10^{-3}$ as a common stopping criterion. Our numerical results are shown in Table 1 and Figure 1.

Further, we performed a parameter analysis on the proposed Algorithm (3.2). Figure 2(a) shows the effect of inertial parameter on the convergence rate when $\lambda_n = 0.9, \nu_n = 0.4$, and Figure 2(b) shows the effect of choosing different λ_n and ν_n when $\delta_n = \frac{t_n - 1}{t_{n+1}}$.

TABLE 1. Compare the number of iterations for Example 5.4

Cases	Alg. (3.2)	Alg. (4.2)	Alg. (5.2)	Alg. (5.3)	Alg. (5.1)
I	98	127	668	1725	> 5000
II	82	139	930	1520	> 5000
III	144	192	4005	> 5000	> 5000
IV	120	197	2992	> 5000	> 5000

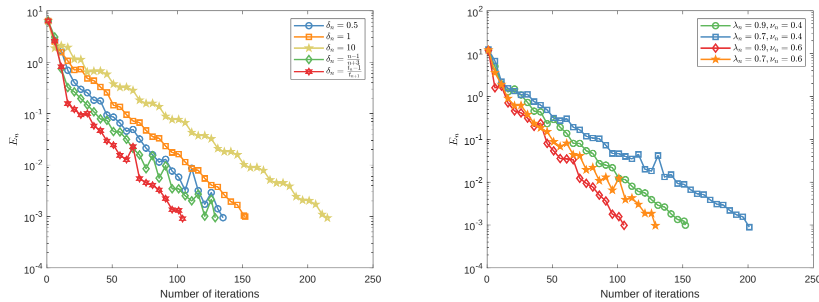


(a) Case I (b) Case II

(c) Case III (d) Case IV

FIGURE 1. Convergence behavior of $\{E_n\}$ for Example 5.4

Example 5.5. Consider the linear operator $A(x) = Mx + q$, where $q \in R^m$ and $M = NN^\top + U + D$, where N is a $m \times m$ matrix, U is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being non-negative (hence M is positive symmetric definite). The feasible set C is given by $C = \{x \in R^m : -5 \leq x_i \leq 5,\, i = 1, \ldots, m\}$. It is clear that A is monotone and Lipschitz continuous with constant $L = \|M\|$. In this experiment, all entries of N, U are generated randomly and uniformly in $[-5, 5]$ and D is generated randomly in $[1, 5]$. Let $q = 0$, then the solution set is $x^* = \{0\}$. Setting $m = 2$, our other parameters and stopping criterion are the same as in Example 5.4. Numerical results are reported in Table 2 and Figure 3.

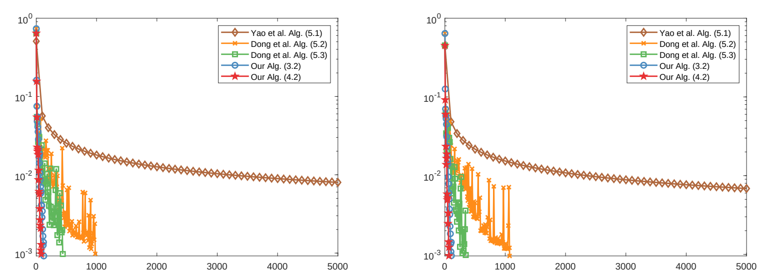


(a) Different δ_n with $\lambda_n = 0.9, \nu_n = 0.4$ (b) Different λ_n, ν_n with $\delta_n = \frac{t_n - 1}{t_{n+1}}$

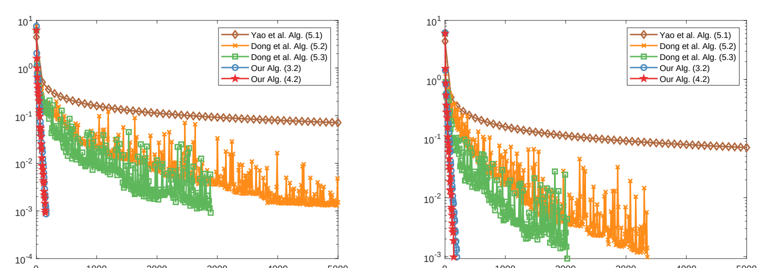
FIGURE 2. Parameter analysis for Example 5.4

TABLE 2. Compare the number of iterations for Example 5.5

Cases	Alg. (3.2)	Alg. (4.2)	Alg. (5.2)	Alg. (5.3)	Alg. (5.1)
I	83	125	440	978	> 5000
II	67	108	346	1073	> 5000
III	147	162	2890	> 5000	> 5000
IV	145	194	2025	3354	> 5000



(a) Case I (b) Case II



(c) Case III (d) Case IV

FIGURE 3. Convergence behavior of $\{E_n\}$ for Example 5.5

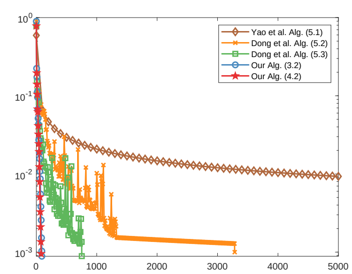
Example 5.6. Let us consider the following nonlinear optimization problem via

$$\begin{aligned} &\min F(x) = 1 + x_1^2 - e^{-x_2^2} \\ &\text{s.t.} \ \ -5e \leq x \leq 5e, \end{aligned} \tag{5.5}$$

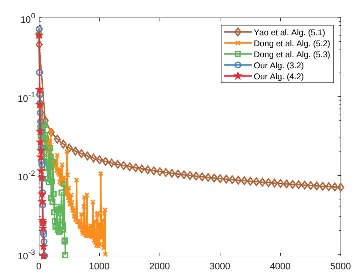
where $x = (x_1, x_2)^\top \in R^2$, $e = (1,1)^\top$. Observe that $\nabla F(x) = \left(2x_1, 2x_2 e^{-x_2^2}\right)^\top$ and the optimal solution for $F(x)$ is $x^* = (0,0)^\top$. Taking $A(x) = \nabla F(x)$, it is easy to check that $A(x)$ is monotone and Lipschizt continuous with constant $L = 2$ on the closed and convex subset $C = \left\{x \in R^2 : -5e \leq x \leq 5e\right\}$. Our parameters and stopping criterion are the same as in Example 5.4. Numerical results are reported in Table 3 and Figure 4.

Table 3. Compare the number of iterations for Example 5.6

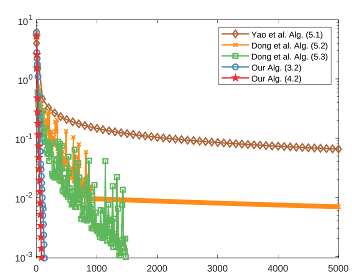
Cases	Alg. (3.2)	Alg. (4.2)	Alg. (5.2)	Alg. (5.3)	Alg. (5.1)
I	80	92	753	3284	> 5000
II	74	86	433	1098	> 5000
III	85	131	1479	> 5000	> 5000
IV	167	265	2998	> 5000	> 5000



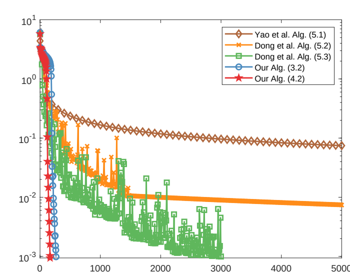
(a) Case I



(b) Case II



(c) Case III



(d) Case IV

Figure 4. Convergence behavior of $\{E_n\}$ for Example 5.6

Remark 5.7. Example 5.4–Example 5.6 show that our proposed Algorithm (3.2) and Algorithm (4.2) have better convergence behaviors than Algorithm (5.1), Algorithm (5.2) and Algorithm (5.3). In addition, Algorithm (3.2) and Algorithm (4.2) has the dual advantages of small oscillation and fast convergence. As shown in Figure 2, it should be noted that the Algorithm (3.2) has a faster convergence rate when $\delta_n = \frac{t_n - 1}{t_{n+1}}$ and $\lambda_n = 0.9, \nu_n = 0.6$.

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