Pseudotraces on Almost Unital and Finite-Dimensional Algebras

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Abstract

We introduce the notion of almost unital and finite-dimensional (AUF) algebras, which are associative $\mathbb{C}$-algebras that may be non-unital or infinite-dimensional, but have sufficiently many idempotents. We show that the pseudotrace construction, originally introduced by Hattori and Stallings for unital finite-dimensional algebras, can be generalized to AUF algebras.

Let A be an AUF algebra. Suppose that G is a projective generator in the category $\mathrm{Coh}_{\mathrm{L}}(A)$ of finitely generated left A-modules that are quotients of free left A-modules, and let $B = \mathrm{End}_{A,-}(G)^{\mathrm{op}}$. We prove that the pseudotrace construction yields an isomorphism between the spaces of symmetric linear functionals $\mathrm{SLF}(A) \xrightarrow{\simeq} \mathrm{SLF}(B)$, and that the non-degeneracies on the two sides are equivalent.

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0 Introduction

In [Miy04], Miyamoto introduced the pseudo-q-trace construction for modules of vertex operator algebras (VOAs), generalizing the usual q-trace. His primary motivation was to address the failure of modular invariance for q-traces in the case of C_2-cofinite but irrational VOAs. While Zhu's theorem in [Zhu96] establishes modular invariance for q-traces in the rational setting, this result does not extend to the irrational case—unless q-traces are replaced with pseudo-q-traces.

Miyamoto's original approach is quite involved. Moreover, his dimension formula for the space of torus conformal blocks is expressed in terms of higher Zhu algebras. This presents two drawbacks: first, higher Zhu algebras are difficult to compute in practice; second, their connection to the VOA module category is not transparent.

Later, Arike [Ari10] and Arike-Nagatomo [AN13] introduced a simplified version of the pseudo-q-trace construction based on the idea of Hattori [Hat65] and Stallings [Sta65]. Below, we briefly outline this approach.

Let A be an algebra, and let B be a unital finite-dimensional algebra. Let M be a finite-dimensional A-B bimodule, projective as a right B-module. By the projectivity, there is a (finite) left coordinate system of M, namely, elements $\alpha_1,\dots,\alpha_n \in \mathrm{Hom}_B(B,M)$ and $\breve{\alpha}^1,\dots,\breve{\alpha}^n \in \mathrm{Hom}_M(M,B)$ satisfying $\sum_i \alpha_i \circ \breve{\alpha}^i = \mathrm{id}_M$. Then the linear map

$$A \to B \qquad x \mapsto \sum_i \breve{\alpha}^i \circ x \circ \alpha_i(1_B)$$

descends to a linear map $A/[A,A] \to B/[B,B]$ which is independent of the choice of the left coordinate system. Its pullback gives a linear map

$$\mathrm{SLF}(B) \to \mathrm{SLF}(A) \qquad \phi \mapsto \mathrm{Tr}^\phi \tag{0.1}$$

where $\mathrm{SLF}(A)$ is the space of symmetric linear functionals on A—that is, linear maps $\psi : A \to \mathbb{C}$ satisfying $\psi(xy) = \psi(yx)$ for all $x,y \in A$—and $\mathrm{SLF}(B)$ is the space of symmetric linear functionals on B. The above map is called the **pseudotrace construction**. Note that a typical choice of A is $\mathrm{End}_B(M)$.

The pseudotrace construction is applied to the VOA setting as follows. Let $\mathbb{V}$ be an $\mathbb{N}$-graded C_2-cofinite VOA with central charge c, and let $\mathbb{M}$ be a grading-restricted generalized $\mathbb{V}$-module. Then $\mathbb{M}$ admits a decomposition $\mathbb{M} = \bigoplus_{\lambda\in\mathbb{C}} \mathbb{M}_{[\lambda]}$ into generalized eigenspaces of $L(0)$, where each $\mathbb{M}_{[\lambda]}$ is finite-dimensional. Let $\mathrm{End}_{\mathbb{V}}(\mathbb{M})$ be the algebra of linear operators on $\mathbb{M}$ commuting with the action of $\mathbb{V}$, which is necessarily unital and finite-dimensional. Let B be a unital subalgebra of $\mathrm{End}_{\mathbb{V}}(\mathbb{M})^{\mathrm{op}}$. Assume that $\mathbb{M}$ is a projective right B-module, equivalently, each $\mathbb{M}_{[\lambda]}$ is B-projective. Let $\phi \in \mathrm{SLF}(B)$. Then for $v \in \mathbb{V}$, the expression

$$\mathrm{Tr}^\phi(Y_{\mathbb{M}}(v,z)q^{L(0)-\frac{c}{24}}) = \sum_{\lambda\in\mathbb{C}} \mathrm{Tr}^\phi\big(P(\lambda)Y_{\mathbb{M}}(v,z)q^{L(0)-\frac{c}{24}}P(\lambda)\big) \tag{0.2}$$

converges absolutely for $z \in \mathbb{C}$ and $0<|q|<1$, and defines a torus conformal block. Here, $P(\lambda)$ is the projection of $\overline{\mathbb{M}} := \prod_{\mu\in\mathbb{C}} \mathbb{M}_{[\mu]}$ onto $\mathbb{M}_{[\mu]}$. Then each $P(\lambda)Y_{\mathbb{M}}(v,z)q^{L(0)-\frac{c}{24}}P(\lambda)$

is a linear operator on $\mathbb{M}_{[\lambda]}$ commuting with the right action of B, and hence Tr^ϕ can be defined on it.

Based on this formulation, in [GR19, Conjecture 5.8], Gainutdinov and Runkel proposed a conjecture that directly relates the space of torus conformal blocks of a C_2-cofinite VOA $\mathbb{V}$ to the linear structure of the category $\mathrm{Mod}(\mathbb{V})$ of grading-restricted generalized $\mathbb{V}$-modules. Let $\mathbb{G}$ be a projective generator in $\mathrm{Mod}(\mathbb{V})$, and let $B = \mathrm{End}_{\mathbb{V}}(\mathbb{G})$. Then $\mathbb{G}$ is B-projective. The conjecture asserts that the linear map sending each $\phi \in \mathrm{SLF}(B)$ to (0.2) defines an isomorphism between $\mathrm{SLF}(B)$ and the space of torus conformal blocks of $\mathbb{V}$.

The purpose of this note is to establish results in the theory of associative algebras that are essential for proving the Gainutdinov-Runkel conjecture. The actual resolution of the conjecture will appear in the forthcoming paper [GZ25].

Our approach stems from recognizing a structural analogy between the Gainutdinov-Runkel conjecture and a classical result in associative algebra: If A is a unital finite-dimensional and M is a projective generator in the category of finite-dimensional left A-modules, then M is projective over $B := \mathrm{End}_A(M)^{\mathrm{op}}$, and the pseudotrace map (0.1) is a linear isomorphism. This result was suggested in [BBG21, Sec. 2] and was proved in [Ari10] in the special case that $M = Ae$ where e is a basic idempotent.

However, this classical result is not directly applicable to the Gainutdinov–Runkel conjecture. We need to generalize it to a larger class of associative algebras than unital finite-dimensional ones. In particular, we must consider infinite-dimensional algebras that can be approximated, in a certain sense, by finite-dimensional (and possibly unital) algebras. The need to consider infinite-dimensional associative algebras in the study of irrational VOAs has also been recognized in recent years from different perspectives, such as Huang's associative algebra $A^\infty(\mathbb{V})$ introduced in [Hua24], and the mode transition algebra introduced by Damiolini-Gibney-Krashen in [DGK25].

The infinite-dimensional algebra required for the proof of the Gainutdinov-Runkel conjecture is different from the above mentioned algebras. In [GZ25], we will show that the end

$$\mathbb{E} := \int_{\mathbb{M}\in\mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$$

a priori an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$, carries a structure of an associative $\mathbb{C}$-algebra that is compatible with its $\mathbb{V}^{\otimes 2}$-module structure. This algebra $\mathbb{E}$ is an example of an **almost unital and finite-dimensional algebra**[1] (abbreviated as **AUF algebra**), meaning that $\mathbb{E}$ has a collection of mutually orthogonal idempotents $(e_i)_{i\in\mathfrak{I}}$ such that $\mathbb{E} = \sum_{i,j\in\mathfrak{I}} e_i\mathbb{E}e_j$ where each summand $e_i\mathbb{E}e_j$ is finite-dimensional. (This sum is automatically direct.) In fact, $\mathbb{E}$ has only finitely many irreducibles. We call such algebra **strongly AUF**.

The main result of this note is a generalization of the aforementioned isomorphism between spaces of symmetric linear functionals to the setting of strongly AUF algebras. More precisely, we prove that the pseudotrace construction defines a linear isomorphism $\mathrm{SLF}(B) \simeq \mathrm{SLF}(A)$ where A is strongly AUF, M is a projective generator of the category $\mathrm{Coh}_{\mathrm{L}}(A)$ of **coherent left A-modules** (i.e., finitely generated left A-modules that are quotients of free ones), and $B = \mathrm{End}_A(M)^{\mathrm{op}}$. See Thm. 9.4. Moreover, we show that

[1] Here, "almost" modifies the entire phrase "unital and finite-dimensional", not just "unital".

the symmetric linear functional on B is non-degenerate if and only if the corresponding functional on A is non-degenerate. See Thm. 10.4.

Since the associative algebra structure on the end $\mathbb{E}$ will not be developed in this note, we present some alternative examples of AUF algebras for illustration. Let $U(\mathbb{V})$ be the universal algebra of $\mathbb{V}$ as defined in [FZ92]. Let

$$U(\mathbb{V})^{\mathrm{reg}} = \bigoplus_{\lambda,\mu\in\mathbb{C}} U(\mathbb{V})_{[\lambda,\mu]}$$

where $U(\mathbb{V})_{[\lambda,\mu]}$ is the subspace of joint generalized-eigenvector of the left and right actions of $L(0)$ corresponding to the eigenvalue λ and μ respectively. The following properties are shown in [MNT10]: Each $U(\mathbb{V})_{[\lambda,\mu]}$ is finite-dimensional. For each $\lambda,\mu,\nu\in\mathbb{C}$ one has

$$U(\mathbb{V})_{[\lambda,\mu]}U(\mathbb{V})_{[\mu,\nu]}\subset U(\mathbb{V})_{[\lambda,\nu]}$$

In particular, $U(\mathbb{V})^{\mathrm{reg}}$ is a subalgebra of $U(\mathbb{V})$. Moreover, there is an increasing sequence of idempotents $(1_n)_{n\in\mathbb{Z}_+}$ such that $U(\mathbb{V})^{\mathrm{reg}} = \bigcup_n 1_n U(\mathbb{V})^{\mathrm{reg}} 1_n$. (See [MNT10, Sec. 2.6].) Therefore, $U(\mathbb{V})^{\mathrm{reg}}$ is AUF, since the family of orthogonal idempotents in the definition of AUF algebras can be chosen to be $(1_{n+1}-1_n)_{n\in\mathbb{Z}_+}$.

For a more elementary and concrete example, consider the following. Let B be a unital finite-dimensional algebra. Let M be a right B-modules. Equip M with a grading

$$M=\bigoplus_{i\in\mathfrak{I}} M(i)$$

where each $M(i)$ is finite-dimensional and is preserved by the right action of B. Let A be

$$\begin{aligned}\mathrm{End}_B^0(M):=\{T\in\mathrm{End}(M):&(Tm)b=T(mb)\text{ for all }m\in M,b\in B,\\ &T|_{M(i)}=0\text{ for all by finitely many }i\in\mathfrak{I}\}\end{aligned}$$

Then A is clearly an AUF algebra, with the family of mutually orthogonal idempotents given by the projections e_i of M onto $M(i)$.

In fact, any strongly AUF algebra arises from such a construction. More precisely, an algebra is strongly AUF if and only if it is isomorphic to some $\mathrm{End}_B^0(M)$, where M and B satisfy the above conditions and, in addition, M is a projective generator in the category of right B-modules. See Thm. 11.9.

Note that the relationship between $\mathrm{End}_B^0(M)$ and C_2-cofinite VOAs is straightforward: If $\mathbb{M}\in\mathrm{Mod}(\mathbb{V})$ is equipped with the grading $\bigoplus_{\lambda\in\mathbb{C}}\mathbb{M}_{[\lambda]}$ given by the generalized eigenspaces of $L(0)$, and if B is a unital subalgebra of $\mathrm{End}_{\mathbb{V}}(\mathbb{M})^{\mathrm{op}}$ such that $\mathbb{M}$ is projective as a right B-module, then each $P(\lambda)Y_{\mathbb{M}}(v,z)q^{L(0)-\frac{c}{24}}P(\lambda)$ appearing in (0.2) lies $\mathrm{End}_B^0(\mathbb{M})$. Therefore, the main result of this note on pseudotraces (Thm. 10.4) can be applied to C_2-cofinite VOAs. Details of this application will be presented in [GZ25].

1 Preliminaries

Throughout this note, algebras are associative, not necessarily unital, and over $\mathbb{C}$. Let $\mathbb{N}=\{0,1,2,\dots\}$ and $\mathbb{Z}_+=\{1,2,\dots\}$. For any vector spaces V,W, we let $\mathrm{Hom}(V,W)=\mathrm{Hom}_{\mathbb{C}}(V,W)$ be the space of linear maps $V\to W$, and let $\mathrm{End}(V)=\mathrm{Hom}(V,V)$.

Let A be an algebra. Its opposite algebra is denoted by A^{op}. If M, N are left (resp. right) A-modules, we let $\mathrm{Hom}_{A,-}(M, N)$ (resp. $\mathrm{Hom}_{-,A}(M, N)$) be the space of linear maps $M \to N$ intertwining the left (resp. right) actions of A.

An **idempotent** $e \in A$ is an element satisfing $e^2 = e$. If $e, f \in A$ are idempotent, we write $e \leqslant f$ if $ef = fe = e$. Equivalently, $f = e + e'$ where $e' \in A$ is an idempotent **orthogonal** to e (i.e. $ee' = e'e = 0$). We say that a nonzero idempotent e is **primitive** if the only idempotent f satisfying $f \leqslant e$ is $f = 0$ and $f = e$.

In this section, we review some well-known facts about associative algebras. Since, unlike many references, our algebras are not assumed to be unital, we include proofs for the reader's convenience.

Definition 1.1. Let $u, v \in A$. We say that (u, v) is pair of **partial isometries in** A if the following are true:

(a) $p := vu$ and $q := uv$ are idempotents.

(b) $u \in qAp$ and $v \in pAq$.

In this case, we also say that u is a partial isometry from p to q, and that v is a partial isometry from q to p. We say that two idempotents are **equivalent** if there are partial isometries between them.

Proposition 1.2. *Let $e, f \in A$ be idempotents. Then an element of $\mathrm{Hom}_{A,-}(Ae, Af)$ is precisely the right multiplication of an element of eAf. In particular, we have an algebra isomorphism*

$$\mathrm{End}_{A,-}(Ae)^{\mathrm{op}} \simeq eAe$$

Proof. Clearly the right multiplication by some element of eAf yields an element of $\mathrm{Hom}_{A,-}(Ae, Af)$. Conversely, suppose that $T \in \mathrm{Hom}_{A,-}(Ae, Af)$. Let $x = T(e)$, which belongs to Af. Since $ex = eT(e) = T(ee) = T(e) = x$, we see that $x \in eAf$. For each $y \in A$, we have $T(ye) = yT(e) = yx = yex$, which shows that T is the right multiplication by x. □

Corollary 1.3. *Let e, f be idempotents in A. The following are equivalent:*

(1) $Ae \simeq Af$ as left A-modules.

(2) There is a partial isometry from e to f.

Proof. (1)⇒(2): Let $T \in \mathrm{Hom}_{A,-}(Ae, Af)$ be an isomorphism with inverse $T^{-1} \in \mathrm{Hom}_{A,e}(Af, Ae)$. By Prop. 1.2, T and T^{-1} are realized by the right multiplications of $u \in eAf$ and $v \in fAe$ respectively. Since $TT^{-1} = 1_{Af}$, we have $vu = f$. Since $T^{-1}T = 1_{Ae}$, we have $uv = e$.

(2)⇒(1): Let $u \in eAf$ and $v \in fAe$ such that $uv = e, vu = f$. Then the right multiplication of u on Ae has inverse being the right multiplication of v. So $Ae \simeq Af$. □

Corollary 1.4. *Let $e \in A$ be an idempotent. Let M be a left A-submodule of Ae. The following are equivalent.*

(1) M is a direct summand of Ae.

(2) $M = Af$ for some idempotent $f \leqslant e$ in A.

Proof. (2)⇒(1): $Ae = Af \oplus Af'$ where $f' = e - f$ is an idempotent.

(1)⇒(2): Let $Ae = M \oplus N$. Let $\varphi : Ae \to Ae$ be the projection on M vanishing on N. Then $\varphi \in \mathrm{End}(Ae)$. By Prop. 1.2, φ is the right multiplication by some $f \in eAe$. Since $\varphi \circ \varphi = \varphi$, clearly $f^2 = f$. Moreover, $M = \varphi(Ae) = (Ae)f = Af$. □

Corollary 1.5. *Let $e \in A$ be an idempotent. The following are equivalent.*

(1) Ae is an indecomposible left A-module.

(2) e is primitive.

Proof. This follows immediately from Cor. 1.4. □

Lemma 1.6. *Let M be a nonzero finitely-generated left A-module. Then M has a maximal proper left A-submodule N. Consequently, there is an epimorphism of M onto an irreducible module.*

Proof. Let $\xi_1, \ldots, \xi_n$ generate M. Without loss of generality, we assume that ξ_1 does not belong to the submodule N_0 generated by $\xi_2, \ldots, \xi_n$. By Zorn's lemma, there is a left submodule $N \leqslant M$ maximal with respect to the property that $N_0 \subset N$ and $\xi_1 \neq N$. Let us prove that N is a maximal proper submodule. Let $N < K \leqslant M$. Then by the maximality of N we must have $\xi_1 \in K$. So $\xi_1, \ldots, \xi_n \in M$, and hence $K = M$. So K is not proper. □

2 Almost unital algebras

In this section, we introduce the notion of almost unital algebras, which is weaker than being almost unital and finite-dimensional.

Definition 2.1. We say that an algebra A is **almost unital** if the following conditions are satisfied:

(a) For each $x \in A$, there is an idempotent $e \in A$ such that $x = exe$.

(b) For any finitely many idempotents $e_1, \ldots, e_n \in A$ there exists an idempotent $e \in A$ such that $e_i \leqslant e$ for all $1 \leqslant i \leqslant n$.

Throughout this section, unless otherwise stated, A is assumed to be almost unital.

Definition 2.2. We say that a left A-module M is **quasicoherent** if one of the following equivalent conditions hold:

(1) For each $\xi \in M$ we have $\xi \in A\xi$.

(2) For each $\xi \in M$ there exists an idempotent $e \in A$ such that $\xi = e\xi$.

(3) M is a quotient module of $\bigoplus_{i \in I} Ae_i$ where each $e_i \in A$ is an idempotent.

(4) M is a quotient module of a free left A-module $A^{\oplus I}$.

The category of quasicoherent left A-modules is denoted by $\mathbf{QCoh_L(A)}$.

Proof of equivalence. (1)⇒(2): For each $\xi \in M$, since $\xi \in A\xi$, we have $\xi = a\xi$ for some $a \in A$. Choose idempotent $e \in A$ such that $a \in eAe$. Then $e\xi = ea\xi = a\xi = \xi$.

(2)⇒(1): Obvious.

(2)⇒(3): For each $\xi \in M$, let $e_\xi \in A$ be an idempotent such that $e_\xi \xi = \xi$. Then we have a morphism $\bigoplus_{\xi\in M} Ae_\xi \to M$ whose restriction to Ae_ξ sends each $a \in Ae_\xi$ to $a\xi$. Then $\xi = e_\xi\xi$ implies that $\xi \in Ae_\xi\xi$, and hence ξ is in the range of this morphism. So this morphism is surjective.

(3)⇒(4): This is obvious, since we have an epimorphism $A \to Ae_i$ and hence an epimorphim $\bigoplus_{i\in I} A \to \bigoplus_{i\in I} Ae_i$.

(4)⇒(2): It suffices to show that $A^{\oplus I}$ satisfies the requirement of (2). Choose $\xi = (a_i)_{i\in I} \in A^{\oplus I}$. Then there are only finitely many $i \in I$ such that $a_i \neq 0$. Since A is almost unital, there exist idempotents $e_i \in A$ (where $i \in I$) such that $a_i = e_i a_i e_i$ for all $i \in I$. (If $a_i = 0$, then we choose $e_i = 0$). Choose idempotent $e \in A$ such that $e_i \leqslant e$ for all $i \in I$. Then $\xi = e\xi$. □

Definition 2.3. A left A-module M is called **coherent** if it is quasicoherent and finitely-generated. By the above proof of equivalence, it is clear that M is coherent iff M is a quotient of $\bigoplus_{i\in I} Ae_i$ where I is a *finite* index set and $e_i \in A$ is an idempotent. The category of coherent left A modules is denoted by $\mathbf{Coh_L(A)}$.

However, note that a coherent left A-module is not necessarily a quotient of $A^{\oplus n}$ where $n \in \mathbb{Z}_+$. Indeed, A is not necessarily finitely generated as a left A-module.

Remark 2.4. If $M \in \mathrm{QCoh_L}(A)$, then every submodule of M is quasicoherent, and every quotient module of M is quasicoherent M. However, if $M \in \mathrm{Coh_L}(A)$, then a submodule of M is not known to be coherent. Thus, $\mathrm{QCoh_L}(A)$ is an abelian category, while $\mathrm{Coh_L}(A)$ is not known to be abelian.

Proposition 2.5. *Let $M \in \mathrm{QCoh_L}(A)$. The following are equivalent.*

(1) M is projective in the category of left A-modules.

(2) M is projective in $\mathrm{QCoh_L}(A)$.

(3) M is a direct summand of $\bigoplus_{i\in I} Ae_i$ for some index set I and each $e_i \in A$ is an idempotent.

Proof. (3)⇒(1): It is well-known that a direct summand of a projective module is projective. Thus, it suffices to prove that $\bigoplus_{i\in I} Ae_i$ is projective. Let $\Phi : \bigoplus_{i\in I} Ae_i \to N$ be an epimorphism where N is a left A-module. Let $\Gamma : K \to N$ be an epimorphism. Let

$$\eta_i = \Phi(e_i)$$

Since Γ is surjective, there is $\xi_i \in K$ such that $\Gamma(\xi_i) = \eta_i$. Define $\Psi : \bigoplus_{i\in I} Ae_i \to K$ to be the morphism sending each $ae_i \in Ae_i$ to $ae_i\xi$. Then the following commute:

$$\begin{tikzcd} & \bigoplus_{i\in I} Ae_i \arrow[dl, "\Psi"'] \arrow[d, "\Phi"] \\ K \arrow[r, "\Gamma"'] & N \end{tikzcd} \qquad \begin{tikzcd} & ae_i \arrow[dl, mapsto] \arrow[d, mapsto] \\ ae_i\xi_i \arrow[r, mapsto] & ae_i\eta_i \end{tikzcd}$$

Note that $\mapsto$ holds since $\Gamma(ae_i\xi_i) = ae_i\Gamma(\xi_i) = ae_i\eta_i$, and $\downarrow$ holds since $\Phi(ae_i) = \Phi(ae_ie_i) = ae_i\Phi(e_i) = ae_i\eta_i$.

(1)⇒(2): Obvious.

(2)⇒(3): Choose an epimorphism $\bigoplus_{i\in I} Ae_i \to M$, which splits because M is projective. So M is a direct summand of $\bigoplus_{i\in I} Ae_i$. □

Proposition 2.6. *Let $M \in \mathrm{Coh}_{\mathrm{L}}(A)$. The following are equivalent.*

(1) M is projective in the category of left A-modules.

(2) M is projective in $\mathrm{QCoh}_{\mathrm{L}}(A)$.

(3) M is projective in $\mathrm{Coh}_{\mathrm{L}}(A)$.

(4) M is a direct summand of $\bigoplus_{i\in I} Ae_i$ for some finite index set I and each $e_i \in A$ is an idempotent.

Therefore, there is no ambiguity when talking about projective coherent left A-modules.

Proof. Clearly we have (1)⇒(2) and (2)⇒(3). By Prop. 2.5 we have (4)⇒(1). Assume (3). By Rem. 2.4, there is an epimorphism $\bigoplus_{i\in I} Ae_i \to M$ such that I is finite, and that it splits (because M is projective in $\mathrm{Coh}_{\mathrm{L}}(A)$). So (4) is true. □

Remark 2.7. If $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$, clearly M is irreducible in $\mathrm{QCoh}_{\mathrm{L}}(A)$ iff M is irreducible in the category of left A-modules; in this case we say that M is **irreducible**. Note that even if $M \in \mathrm{Coh}_{\mathrm{L}}(A)$, its irreducibility is understood as in $\mathrm{QCoh}_{\mathrm{L}}(A)$ but not as in $\mathrm{Coh}_{\mathrm{L}}(A)$.

Proposition 2.8. *Let M be a left A-module. The following are equivalent.*

(1) $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$ and M is irreducible.

(2) $M \simeq Ae/N$ where $e \in A$ is an idempotent and N is a maximal (proper) left ideal of Ae.

(3) $M \simeq A/N$ where N is a maximal proper left A-submodule of A.

Proof. (1)⇒(2): Let $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$ be irreducible. By Def. 2.2, M has an epimorphism Φ from some $\bigoplus_i Ae_i$ where $e_i \in A$ is an idempotent. The restriction of Φ to some Ae_i must be nonzero, and hence must be surjective (since M is irreducible). It follows that M has an epimorphism Ψ from Ae_i. Then $N = \mathrm{Ker}\Psi$ is a maximal proper left A-submodule of Ae_i, and $M \simeq Ae_i/N$.

(1)⇒(3): In the above proof, M also has an epimorphism from $\bigoplus_i A$ (since we have an epimorphism $A \to Ae_i$). Thus, replacing Ae_i with A_i in the above proof, we are done.

(2),(3)⇒(1): Clearly M is irreducible. That $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$ follows from Def. 2.2. □

3 Projective covers

Let A be an algebra, not necessarily almost unital. In this section, we recall some basic facts about projective covers. When A is unital, these results can be found in [AF92], for example. In the non-unital case, one can reduce to the unital setting by considering the unitalization of A. For the reader's convenience, we include complete proofs.

3.1 Basic facts

Definition 3.1. Let M be a left A-module. A left A-submodule $K \leqslant M$ is called **superfluous**, if for any left A-submodule $L \leqslant M$ satisfying $K + L = M$ we must have $L = M$.

Remark 3.2. Obviously, we have an equivalent description of superfluous submodules: Let $\pi : M \to M/K$ be the quotient map. Then $K \leqslant M$ is superfluous iff for any morphism of left A-modules $\varphi : N \to M$ such that $\pi \circ \varphi : N \to M/K$ is surjective, it must be true that φ is surjective.

Definition 3.3. Let M be a left A-module. A **projective cover** of M denotes a left A-module epimorphism $\varphi : P \twoheadrightarrow M$ where P is a projective left A-module, and $\mathrm{Ker}\varphi$ is superfluous in P.

The following property says that among the projective modules that have epimorphisms to M, the projective cover is the smallest one in the sense of direct summand.

Proposition 3.4. *Let $\varphi : P \to M$ be a projective cover of M. Let $\psi : Q \to M$ be an epimorphism where Q is projective. Then there is a morphism $\alpha : Q \to P$ such that the following diagram commutes.*

$$\begin{tikzcd} & P \arrow[d, two heads, "\varphi"] \\ Q \arrow[ur, "\alpha"] \arrow[r, two heads, "\psi"'] & M \end{tikzcd} \tag{3.1}$$

Moreover, for any such α, there is a left A-submodule $P' \leqslant Q$ such that $Q = \ker \alpha \oplus P'$ and that $\alpha|_{P'} : P' \xrightarrow{\simeq} P$ is an isomorphism.

By setting $L = \ker \alpha$, it follows that (3.1) is equivalent to

$$\begin{tikzcd} & P \arrow[d, two heads, "\varphi"] \\ L \oplus P \arrow[ur, "0\oplus \mathrm{id}_P"] \arrow[r, two heads, "0\oplus\varphi"'] & M \end{tikzcd} \tag{3.2}$$

Proof. The existence of α follows from that Q is projective and that φ is an epimorphism. Moreover, since $\ker \varphi$ is superfluous and $\varphi \circ \alpha$ is surjective, by Rem. 3.2, α is surjective. Therefore, since P is projective, the epimorophism α splits, i.e., there is a morphism $\beta : P \to Q$ such that $\alpha \circ \beta : P \to P$ equals id_P. One sees that $P' = \beta(P)$ fulfills the requirement. $\square$

It follows that projective covers are unique up to isomorphisms:

Corollary 3.5. *Let M be a left A-module with projective covers $\varphi : P \to M$ and $\psi : Q \to M$. Then there exists an isomorphism $\alpha : Q \to P$ such that (3.1) commutes.*

Proof. By Prop. 3.5, there exists α such that (3.1) commutes. It remains to show that α is an isomorphism. We assume that (3.1) equals (3.2). Since $0\oplus\varphi : L\oplus P \to M$ is a projective cover, $L + \ker(P) = \ker(0 \oplus \varphi)$ is superfluous, and hence L is superfluous. Thus, since $L + P$ equals $Q = L \oplus P$, we must have $Q = P$ and hence $L = 0$. So $\alpha = 0 \oplus \mathrm{id}_P$ is an isomorphism. $\square$

3.2 Projective covers of irreducibles

Proposition 3.6. *Suppose that $\varphi : P \to M$ is a projective cover of an irreducible left A-module M. Then P is indecomposible.*

Proof. Suppose that $P = P' \oplus P''$. Then one of $\varphi|_{P'}, \varphi|_{P''}$ (say $\varphi|_{P'}$) is nonzero. Since M is irreducible, $\varphi|_{P'} : P' \to M$ must be surjective. So the map $P' \hookrightarrow P' \oplus P'' \xrightarrow{\varphi} M$ is surjective. Since $\ker\varphi$ is superfluous, by Rem. 3.2, $P' \hookrightarrow P' \oplus P''$ is surjective, and hence $P'' = 0$. $\square$

Theorem 3.7. *Let $e \in A$ be a primitive idempotent satisfying*

$$\dim eAe < +\infty$$

Let K be any proper left A-submodule of Ae. Then K is superfluous. In other words, the quotient map $Ae \to Ae/K$ is the projective cover of Ae/K.

Proof. Step 1. Let $\varphi : Ae \to Ae/K$ be the quotient map. Let L be a submodule of Ae. Assume that $N + K = Ae$; in other words, if we let $\iota : N \hookrightarrow Ae$ be the inclusion, then $\varphi \circ \iota : N \to Ae/K$ is surjective. Our goal is to show that $N = Ae$.

Since Ae is projective and $\varphi \circ \iota$ is surjective, there is a morphism $\alpha : Ae \to N$ such that $\varphi = \varphi \circ \iota \circ \alpha$. Let $\beta = \iota \circ \alpha$. Then the following diagram commutes:

$$\begin{tikzcd}
 & & Ae \arrow[dll, "\alpha"'] \arrow[dl, "\beta"] \arrow[d, two heads, "\varphi"] \\
N \arrow[r, hook, "\iota"'] & Ae \arrow[r, two heads, "\varphi"'] & Ae/K
\end{tikzcd} \tag{3.3}$$

To prove that ι is surjective, it suffices to show that β is surjective.

Step 2. Suppose that β is not surjective. Let us find a contradiction. Since $\beta \in \mathrm{End}_{A,-}(Ae)$, by Prop. 1.2, β is the right multiplication by some $x \in eAe$. Let $R_x : eAe \to eAe$ be the right multiplication of x on eAe. Then R_x is not surjective. Otherwise, there exists $a \in A$ such that $R_x(eae) = e$, i.e., $eaex = e$. Then for each $b \in A$, we have $be = beaex = \beta(beae)$, contradicting the fact that β is not surjective.

It is well-known that if T is a linear operator on a finite-dimensional $\mathbb{C}$-vector space W, then W is the direct sum of generalized eigenspaces of T, and the projection operator of W onto each generalized eigenspace is a polynomial of T. Therefore, R_x has only one eigenvalue. Otherwise, there is a polynomial p such that $p(R_x) = R_{p(x)}$ is the projection of eAe onto a proper subspace, and hence $p(x)$ is an idempotent in eAe not equal to 0 or e. This is impossible, since e is assumed to be primitive.

Therefore, R_x has a unique eigenvalue, which must be 0 since R_x is not surjective. By linear algebra, R_x is nilpotent. Since $R_{x^n} = (R_x)^n$, it follows that x is nilpotent, and hence β is nilpotent. By (3.3), we have $\varphi = \varphi \circ \beta$, and hence $\varphi = \varphi \circ \beta = \varphi \circ \beta^2 = \varphi \circ \beta^3 = \cdots = 0$. This contradicts the fact that φ is a surjection onto a nonzero module, finishing the proof. $\square$

Corollary 3.8. *Let $e \in A$ be a primitive idempotent satisfying $\dim eAe < +\infty$. Then Ae has a unique proper maximal left A-submodule, denoted by* $\mathbf{rad}(\boldsymbol{Ae})$.

It follows from Thm. 3.7 that Ae is the projective cover of the irreducible $Ae/\mathrm{rad}(Ae)$.

Proof. By Lem. 1.6, Ae has at least one proper maximal left A-submodule. Suppose that $K \neq L$ are proper maximal left A-submodules of M. By the maximality, we have $K + L = M$. By Thm. 3.7, L is superfluous. So $K = M$, impossible. □

4 Left pseudotraces

Let A, B be algebras such that B is unital. Fix an A-B bimodule M. We do not assume that M_B is unital, i.e., $1_B \in B$ acts as the identity on M.

Definition 4.1. A **left coordinate system** of M denotes a collection of morphisms

$$\alpha_i \in \mathrm{Hom}_{-,B}(B, M) \qquad \check{\alpha}^i \in \mathrm{Hom}_{-,B}(M, B) \tag{4.1}$$

where i runs through an index set I such that the following conditions hold:

(a) For each $\xi \in M$, we have $\check{\alpha}^i(\xi) = 0$ for all but finitely many $i \in I$, and $\sum_{i \in I} \alpha_i \circ \check{\alpha}^i(\xi) = \xi$.

(b) For each $x \in A$ (viewed as an element of $\mathrm{End}_{-,B}(M)$), we have $x \circ \alpha_i = 0$ and $\check{\alpha}^i \circ x = 0$ for all but finitely many $i \in I$.

Remark 4.2. M is a projective right B-module iff there exists $(\alpha_i, \check{\alpha}^i)_{i \in I}$ of the form (4.1) satisfying condition (a).

Proof. Suppose that there exists $(\alpha_i, \check{\alpha}^i)_{i \in I}$ such that (a) holds. Define morphisms of right B-modules

$$\Phi : B^{\oplus I} \to M \qquad \oplus_i b_i \mapsto \sum_i \alpha_i(b_i)$$
$$\Psi : M \to B^{\oplus I} \qquad \xi \mapsto \oplus_i \check{\alpha}^i(\xi)$$

Then (a) implies that $\Phi \circ \Psi = \mathrm{id}_M$. Thus, M is a direct summand of $B^{\oplus I}$, and hence is projective as a right B-module.

Conversely, assume M is projective as a right B-module. Then we have an epimorphism $\Phi : B^{\oplus I} \to M$ and a morphism $\Psi : M \to B^{\oplus I}$ such that $\Phi \circ \Psi = \mathrm{id}_M$. For each $i \in I$, let $\iota_i : B \to B^{\oplus I}$ be the inclusion map of B into the i-th direct summand, and $\pi_i : B^{\oplus I} \to B$ be the projection map onto the i-th direct summand. Set

$$\alpha_i = \Phi \circ \iota_i \qquad \check{\alpha}^i = \pi_i \circ \Psi$$

Then $(\alpha_i, \check{\alpha}^i)_{i \in I}$ satisfies (a). □

Definition 4.3. Assume that M has a left coordinate system $(\alpha_i, \breve{\alpha}^i)_{i\in I}$. Define the ***B*-trace function**

$$\mathrm{Tr}^B : A \to B/[B,B] \qquad x \mapsto \sum_{i\in I} \breve{\alpha}^i \circ x \circ \alpha_i$$

where the RHS, originally an element of $\mathrm{End}_{-,B}(B) \simeq B$,[2] is descended to $B/[B,B]$.

Lemma 4.4. *The definition of* Tr^B *is independent of the choice of left coordinate systems.*

Proof. Suppose that $(\beta_j, \breve{\beta}^j)_{j\in J}$ is another left coordinate system of the A-B bimodule M. Let $I_x \subset I$ and $J_x \subset J$ be finite sets such that $\breve{\alpha}^i \circ x = 0, x\circ\alpha_i = 0$ for any $i \in I\backslash I_x$, and that $\breve{\beta}^j \circ x = 0, x \circ \beta_j = 0$ for any $j \in J\backslash J_x$. Then

$$\sum_{i\in I_x} \breve{\alpha}^i \circ x \circ \alpha_i = \sum_{i\in I_x, j\in J} \breve{\alpha}^i \circ x \circ \beta_j \circ \breve{\beta}^j \circ \alpha_i = \sum_{i\in I_x, j\in J_x} \breve{\alpha}^i \circ x \circ \beta_j \circ \breve{\beta}^j \circ \alpha_i$$

Since each $\breve{\alpha}^i \circ x \circ \beta_j$ and $\breve{\beta}^j \circ \alpha_i$ are in $\mathrm{End}_{-,B}(B) \simeq B$, the RHS above equals

$$\sum_{i\in I_x, j\in J_x} \breve{\beta}^j \circ \alpha_i \circ \breve{\alpha}^i \circ x \circ \beta_j = \sum_{j\in J_x} \breve{\beta}^j \circ x \circ \beta_j$$

in $B/[B,B]$. □

Proposition 4.5. Tr^B *is* ***symmetric****, i.e.,* $\mathrm{Tr}^B(xy) = \mathrm{Tr}^B(yx)$ *for any* $x,y \in A$. *Therefore,* Tr^B *descends to a linear map* $A/[A,A] \to B/[B,B]$.

Proof. Let $x, y \in A$. Let $I_0 \subset I$ be a finite set such that $\breve{\alpha}^i \circ x = \breve{\alpha}^i \circ y = 0$ and $x\circ\alpha_i = y \circ \alpha_i = 0$ for all $i \in I\backslash I_0$. Then

$$\mathrm{Tr}^B(xy) = \sum_{i\in I_0} \breve{\alpha}^i \circ x \circ y \circ \alpha_i = \sum_{i,j\in I_0} \breve{\alpha}^i \circ x \circ \alpha_j \circ \breve{\alpha}^j \circ y \circ \alpha_i$$

and similarly

$$\mathrm{Tr}^B(yx) = \sum_{i,j\in I_0} \breve{\alpha}^j \circ y \circ \alpha_i \circ \breve{\alpha}^i \circ x \circ \alpha_j$$

The two RHS's are equal in $B/[B,B]$, noting that $\breve{\alpha}^i \circ x \circ \alpha_j$ and $\breve{\alpha}^j \circ y \circ \alpha_i$ are both in $\mathrm{End}_{-,B}(B) \simeq B$. □

Definition 4.6. Let $\phi : B \to \mathbb{C}$ be a **symmetric linear functional (SLF)**, i.e., a linear map satisfying $\phi(ab) = \phi(ba)$ for all $a, b \in B$. The (left) **pseudotrace** associated to ϕ (and M), denoted by $\mathbf{Tr}^{\boldsymbol{\phi}}$, is defined to be

$$\mathrm{Tr}^\phi = \phi \circ \mathrm{Tr}^B : A \to \mathbb{C} \tag{4.2}$$

It is an SLF on A.

Thus, for each $x \in A$ we have

$$\mathrm{Tr}^\phi(x) = \sum_{i\in I} \phi(\breve{\alpha}^i \circ x \circ \alpha_i(1_B)) \tag{4.3}$$

[2]This isomorphism relies on the fact that B is unital.

5 AUF algebras and projective covers of irreducibles

Definition 5.1. An algebra A is called **almost unital and finite-dimensional (AUF)** if there is a family of mutually orthogonal idempotents $(e_i)_{i\in\mathfrak{I}}$ such that the following conditions hold:

(a) For each $i,j\in\mathfrak{I}$ we have $\dim e_iAe_j<+\infty$.

(b) $A=\sum_{i,j\in\mathfrak{I}}e_iAe_j$. (That is, for each $x\in A$ one can find a finite subset $I\subset\mathfrak{I}$ and a collection $(x_{i,j})_{i,j\in\mathfrak{I}}$ such that $x=\sum_{i,j\in I}e_ix_{i,j}e_j$.)

Note that (b) automatically impies $A=\bigoplus_{i,j\in\mathfrak{I}}e_iAe_j$.

It is illuminating to view an element $x\in A$ as an $\mathfrak{I}\times\mathfrak{I}$ matrix whose (i,j)-entry is e_ixe_j.

Remark 5.2. Each AUF algebra A is almost unital.

Proof. For each $x_1,\cdots,x_n\in A$, we can find a subset $I_0\subset\mathfrak{I}$ such that $x\in e'Ae'$, where $e'=\sum_{i\in I_0}e_i$. By choose $n=1$ and $x_1=x\in A$, we see $x=e'xe'$. By choosing idempotents $x_i=e_i\in A$, we see $e_i\leqslant e'$ for all $1\leqslant i\leqslant n$. □

Lemma 5.3. *In Def. 5.1, one can assume moreover that each e_i is primitive (in A).*

Proof. Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. For each $i\in\mathfrak{I}$, since e_iAe_i is a finite-dimensional left e_iAe_i-module, it is a finite direct sum of indecomposible left e_iAe_i-submodules. By Cor. 1.4 and 1.5, we have a finite direct sum $e_iAe_i=\bigoplus_{k\in\mathfrak{K}_i}e_iAf_{i,k}$ where $(f_{i,k})_{k\in\mathfrak{K}_i}$ is a finite family of mutually orthogonal idempotents in e_iAe_i, that $\sum_k f_{i,k}=e_i$, and that each $f_{i,k}$ is primitive in e_iAe_i. Clearly $f_{i,k}$ is also primitive in A. Replacing $(e_i)_{i\in\mathfrak{I}}$ by $(f_{i,k})_{i\in\mathfrak{I},k\in\mathfrak{K}_i}$ does the job. □

In the remaining part of this section, we always assume that A is AUF.

Remark 5.4. For each idempotents $e,f\in A$, we have

$$\dim eAf<+\infty$$

Indeed, one can find a finite set $I_0\subset\mathfrak{I}$ such that $e,f\in e'Ae'$ where $e'=\sum_{i\in I_0}e_i$. Then $\dim e'Ae'<+\infty$, and hence $\dim eAf<+\infty$.

It follows that each idempotent $e\in A$ has a (finite) orthogonal primitive decomposition $e=\varepsilon_1+\cdots+\varepsilon_n$. This follows from a decomposition of the finite-dimensional left eAe-module eAe into indecomposible submodules. □

Recall Rem. 2.7 about irreducibility.

Theorem 5.5. *The following are true.*

1. *For each primitive idempotent $e\in A$, let $\mathrm{rad}(Ae)$ be the unique proper maximal left submodule of Ae (cf. Cor. 3.8). Then $Ae\to Ae/\mathrm{rad}(Ae)$ gives a projective cover of the irreducible coherent module $Ae/\mathrm{rad}(Ae)$.*

2. *Any irreducible $M \in \mathrm{QCoh_L}(A)$ is isomorphic to $Ae/\mathrm{rad}(Ae)$ for some primitive idempotent $e \in A$.*

3. *Let e, f be primitive idempotents. Then the following are equivalent:*

   (1) $Ae \simeq Af$ as left A-modules.

   (2) $Ae/\mathrm{rad}(Ae) \simeq Af/\mathrm{rad}(Af)$ as left A-modules.

   (3) $e \simeq f$, i.e., there is a partial isometry (in A) from e to f.

Proof. Part 1 was already proved, cf. Thm. 3.7. (Note that Thm. 3.7 and its consequences are applicable since $\dim eAe < +\infty$ by Rem. 5.4.)

Part 2: By Prop. 2.8, M has an epimorphism Ψ from A. Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1 such that each e_i is primitive (Lem. 5.3). Then $A \simeq \bigoplus_i Ae_i$ as left A-modules. The restriction of Ψ to some Ae_i must be nonzero, and hence must be surjective. Therefore $M \simeq Ae_i/\mathrm{rad}(Ae_i)$.

Part 3: (1)⇒(2) is obvious. (2)⇒(1) follows from the uniqueness of projective covers (Cor. 3.5). (1)⇔(3) follows from Cor. 1.3. □

Corollary 5.6. *Let $P \in \mathrm{Coh_L}(A)$. The following are equivalent.*

(1) P is projective and indecomposible.

(2) P is the projective cover of an irreducible $M \in \mathrm{QCoh_L}(A)$, which (by Thm. 5.5) is isomorphic to Ae for some primitive idempotent $e \in A$.

Proof. (2)⇒(1): This follows from Prop. 3.6.

(1)⇒(2): By Lem. 1.6, P has an epimorphism to an irreducible, which (by Thm. 5.5) is of the form $Ae/\mathrm{rad}(Ae)$ where $e \in A$ is a primitive idempotent. We know that Ae is its projective cover. Since P is projective, by Prop. 3.4, Ae is a direct summand of P. Since P is indecomposible, we must have $P = Ae$. □

6 Pseudotraces and generating idempotents of strongly AUF algebras

Let A be AUF. In this section, we show that if $e \in A$ is a generating idempotent, any SLF ψ on A can be recovered from $\psi|_{eAe}$ via the pseudotrace construction.

Definition 6.1. An idempotent $e \in A$ is called **generating** if every irreducible $M \in \mathrm{QCoh_L}(A)$ has an epimorphism from Ae.

Proposition 6.2. *Let $e \in A$ be an idempotent. Let $e = \varepsilon_1 + \cdots + \varepsilon_n$ be an orthogonal primitive decomposition (cf. Rem. 5.4). The following are equivalent:*

(1) e is generating.

(2) Any primitive idempotent of A is isomorphic to ε_i for some i.

(3) Any irreducible $M \in \mathrm{QCoh_L}(A)$ is isomorphic to $A\varepsilon_i/\mathrm{rad}(A\varepsilon_i)$ for some i.

Proof. (1)⇒(3): Each irreducible $M \in \mathrm{QCoh_L}(A)$ has an epimorphism from $Ae = A\varepsilon_1 \oplus \cdots \oplus A\varepsilon_n$, and hence an epimorphism from some $A\varepsilon_i$. By Cor. 3.8, the kernel of this epimorophism is $\mathrm{rad}(A\varepsilon_i)$. Therefore, we have $A\varepsilon_i/\mathrm{rad}(A\varepsilon_i) \simeq M$.

(3)⇒(1): Obvious.

(2)⇔(3): Immediate from Thm. 5.5. □

Corollary 6.3. *Let $e, f \in A$ be idempotents such that $e \leqslant f$ and e is a generating idempotent of A. Then e is a generating idempotent of fAf.*

Proof. Let p be any primitive idempotent of fAf. Then p is a primitive idempotent of A. By Prop. 6.2, if we let $e = \varepsilon_1 + \cdots + \varepsilon_n$ be an orthogonal primitive decomposition, then there exist $1 \leqslant i \leqslant n$ and $u \in \varepsilon_i Ap, v \in pA\varepsilon_i$ such that $uv = \varepsilon_i$ and $vu = p$. So p is isomorphism in fAf to ε_i. By Prop. 6.2, we conclude that e is generating in fAf. □

Corollary 6.4. *The following are equivalent.*

(1) A has a generating idempotent.

(2) $\mathrm{QCoh_L}(A)$ has finitely many equivalence classes of irreducible objects.

(3) A has finitely many isomorphism classes of primitive idempotents.

If one of these conditions holds, we say that A is ***strongly AUF****.*

Proof. (1)⇒(2): Immediate from Prop. 6.2.

(2)⇔(3): Immediate from Thm. 5.5.

(2)⇒(1): Let $M_1, \ldots, M_n \in \mathrm{QCoh_L}(A)$ exhaust all equivalence classes of irreducibles. Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. For each $1 \leqslant k \leqslant n$, by Prop. 2.8, M_k has an epimorphism from A. Since $A = \bigoplus_{i\in\mathfrak{I}} Ae_i$, it follows that M_k has an epimorphism from Ae_{i_k} for some $i_k \in \mathfrak{I}$. If we assume at the beginning that $M_1, \ldots, M_n$ are mutually non-isomorphic, then $e_{i_1}, \ldots, e_{i_k}$ must be distinct, and hence mutually orthogonal. So $e = e_{i_1} + \cdots + e_{i_n}$ is a generating idempotent. □

Theorem 6.5. *Assume that A is strongly AUF, and let $e \in A$ be a generating idempotent. Then the A-(eAe) bimodule Ae has a left coordinate system. In particular, by Rem. 4.2, Ae is a projective right eAe-module.*

The following construction of left coordinate system is important and is motivated by [Ari10, Lem. 3.9].

Proof. Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. By Lem. 5.3, we can assume that each e_i is primitive. Let $e = \varepsilon_1 + \cdots + \varepsilon_n$ be an orthogonal primitive decomposition of e. By Prop. 6.2, there are partial isometries u_i, v_i such that

$$\begin{aligned} v_i u_i &= \varepsilon_{k_i} & \qquad u_i v_i &= e_i \\ u_i &\in e_i A \varepsilon_{k_i} & \qquad v_i &\in \varepsilon_{k_i} A e_i \end{aligned}$$

where $k_i \in \{1, \ldots, n\}$. In particular $u_i \in e_i Ae$ and $v_i \in eAe_i$. Let

$$\alpha_i \in \mathrm{End}_{-,eAe}(eAe, Ae) \qquad \check{\alpha}^i \in \mathrm{End}_{-,eAe}(Ae, eAe)$$

$$\alpha_i(exe) = u_i \cdot exe \qquad \breve{\alpha}^i(xe) = v_i \cdot xe$$

One checks easily that $(\alpha_i, \breve{\alpha}^i)_{i\in\mathfrak{I}}$ is a left coordinate system. □

The proof of [Ari10, Thm. 3.10] can be easily adapted to prove the following theorem.

Theorem 6.6. *Assume that A is strongly AUF, and let $e \in A$ be a generating idempotent. Then there is a linear isomorphism*

$$\mathrm{SLF}(A) \xrightarrow{\simeq} \mathrm{SLF}(eAe) \qquad \psi \mapsto \psi|_{eAe}$$

whose inverse is given by

$$\mathrm{SLF}(eAe) \xrightarrow{\simeq} \mathrm{SLF}(A) \qquad \phi \mapsto \mathrm{Tr}^\phi$$

Here, Tr^ϕ is the pseudotrace on A with respect to ϕ and the A-(eAe) bimodule Ae.

Proof. Let $u_i, v_i, \alpha_i, \breve{\alpha}^i$ be as in the proof of Thm. 6.5. For any $\phi \in \mathrm{SLF}(eAe)$, let us compute Tr^ϕ. Let $x \in A$, viewed as an element of $\mathrm{End}_{-,eAe}(Ae)$. Then $\breve{\alpha}^i \circ x \circ \alpha_i \in \mathrm{End}_{-,eAe}(eAe)$ equals (the left multiplication by) $v_i x u_i$. Then

$$\mathrm{Tr}^\phi(x) = \sum_{i\in\mathfrak{I}} \phi(v_i x u_i) \tag{6.1}$$

Note that the RHS is a finite sum since $u_i = e_i u_i$, and since and $xe_i = 0$ for all but finitely many i.

To show that $\mathrm{Tr}^\phi|_{eAe} = \phi$, we compute

$$\mathrm{Tr}^\phi(exe) = \sum_i \phi(v_i exe u_i) = \sum_i \phi(v_i exe \cdot eu_i)$$

Since $v_i exe, eu_i \in eAe$, and since ϕ is SLF, we have

$$\mathrm{Tr}^\phi(exe) = \sum_i \phi(eu_i \cdot v_i exe) = \sum_i \phi(ee_i exe) = \phi(exe)$$

Finally, let $\psi \in \mathrm{SLF}(A)$. Then for each $x \in A$,

$$\mathrm{Tr}^{\psi|_{eAe}}(x) = \sum_i \psi|_{eAe}(v_i x u_i) = \sum_i \psi(v_i x u_i) = \sum_i \psi(u_i v_i x) = \sum_i \psi(e_i x) = \psi(x)$$

This proves $\mathrm{Tr}^{\psi|_{eAe}} = \psi$. □

7 Projective generators of strongly AUF algebras

Let A be an AUF algebra.

Remark 7.1. A left A-module M is coherent if and only if M is a quotient module of $(Ae)^{\oplus n}$ where $n \in \mathbb{Z}_+$ and $e \in A$ is an idempotent.

Proof. "$\Leftarrow$" is obvious. Conversely, let $M \in \mathrm{Coh_L}(A)$. By Def. 2.3, M is a quotient module of $Ap_1 \oplus \cdots \oplus Ap_n$ where each p_i is an idempotent. By Rem. 5.2, one can find an idempotent $e \in A$ which is $\geqslant p_1, \dots, p_n$. Then M is a quotient module of $(Ae)^{\oplus n}$. □

Remark 7.2. By Rem. 7.1, if $M \in \mathrm{Coh_L}(A)$ and $x \in A$, then $\dim xM < +\infty$.

Proof. Suppose that M has an epimorphism from $N := (Ae)^{\oplus n}$ where $e \in A$ is an idempotent. Then $\dim xM \leqslant \dim xN$. Let $f \in A$ be an idempotent such that $x = fxf$. Then $xAe \subset fAe$, and hence

$$\dim xN = n \dim xAe \leqslant n \dim fAe < +\infty$$

□

7.1 Basic facts

Definition 7.3. Let $\mathscr{S}$ and $\mathscr{T}$ be classes of objects in $\mathrm{Coh_L}(A)$. We say that $\mathscr{S}$ **generates** $\mathscr{T}$ if each object of $\mathscr{T}$ is a quotient of a *finite* direct sum of objects in $\mathscr{S}$.

Definition 7.4. We say that $M \in \mathrm{Coh_L}(A)$ is a **generator** (of $\mathrm{Coh_L}(A)$) if it generates every object of $\mathrm{Coh_L}(A)$, i.e., every $N \in \mathrm{Coh_L}(A)$ is a quotient module of $M^{\oplus n}$ for some $n \in \mathbb{Z}_+$. A generator which is also projective is called a **projective generator**.

Example 7.5. Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. Then $\mathscr{S} := \{Ae_i : i \in \mathfrak{I}\}$ generates $\mathrm{Coh_L}(A)$.

Proof. By the proof of Rem. 5.2, for any idempotent $e \in A$ one can find a finite set $I_0 \subset I$ such that $e \leqslant \sum_{i\in I_0} e_i$. Therefore, $\mathscr{S}$ generates each Ae, and hence (by Rem. 7.1) generates $\mathrm{Coh_L}(A)$. □

Proposition 7.6. *Let $M \in \mathrm{Coh_L}(A)$ be projective. The following are equivalent.*

(1) M is a projective generator.

(2) Each irreducible $N \in \mathrm{Coh_L}(A)$ has an epimorphism from M.

Proof. (1)$\Rightarrow$(2): Obvious.

(2)$\Rightarrow$(1): Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. By Lem. 5.4, we assume that each e_i is primitive. By Exp. 7.5, it suffices to prove that M generates each Ae_i. By Thm. 5.5, Ae_i is the projective cover of the irreducible $N := Ae_i/\mathrm{rad}(Ae_i)$. By (2), M has an epimorphism to N. Since M is projective, by Prop. 3.4, Ae_i is isomorphic to a direct summand of M. □

Corollary 7.7. *Let $e \in A$ be an idempotent. Then the following are equivalent.*

(1) Ae is a (necessarily projective) generator.

(2) e is a generating idempotent.

Proof. (1)$\Rightarrow$(2): Clear from Def. 6.1. (2)$\Rightarrow$(1): Immediate from Prop. 7.6. □

Proposition 7.8. $\mathrm{Coh_L}(A)$ *has a projective generator if and only if A is strongly AUF.*

Proof. "$\Leftarrow$" follows from Cor. 6.4 and 7.7. Conversely, if $\mathrm{Coh_L}(A)$ has a projective generator M, by Rem. 7.1, an idempotent $e \in A$ can be found such that Ae generates M, and hence generates $\mathrm{Coh_L}(A)$. So e is a generating idempotent. Thus, by Cor. 6.4, $\mathrm{Coh_L}(A)$ has finitely many irreducibles. So A is strongly AUF. □

7.2 Projective generators and endomorphism algebras

Our next goal is to give criteria for projective generators in terms of the endomorphism algebras. We need the endomorphism algebras to be finite-dimensional:

Proposition 7.9. *Let $M, N \in \mathrm{Coh}_{\mathrm{L}}(A)$. Then*

$$\dim \mathrm{Hom}_{A,-}(M, N) < +\infty$$

Proof. By Def. 2.3, there is an epimorphism from a finite direct sum $\bigoplus_i Ae_i$ to M, where e_i is an idempotent. By taking composition with this epimorphism, we get

$$\mathrm{Hom}_{A,-}(M, N) \to \mathrm{Hom}_{A,-}\Big(\bigoplus_i Ae_i, N\Big) \simeq \bigoplus_i \mathrm{Hom}_{A,-}(Ae_i, N) \tag{7.1}$$

where the first map is injective. Thus, it suffices to prove that each $\mathrm{Hom}_{A,-}(Ae_i, N)$ is finite-dimensional.

Again, we can find an epimorphism $\Phi : \bigoplus_j Af_j \twoheadrightarrow N$ (where $\bigoplus_j$ is finite). Since Ae_i is projective, each $\alpha \in \mathrm{Hom}_{A,-}(Ae_i, N)$ can be lifted to some $\beta \in \mathrm{Hom}_{A,-}(Ae_i, \bigoplus_j Af_j)$ such that $\alpha = \Phi \circ \beta$. Thus

$$\dim \mathrm{Hom}_{A,-}(Ae_i, N) \leqslant \dim \mathrm{Hom}_{A,-}\Big(Ae_i, \bigoplus_j Af_j\Big) = \sum_j \dim \mathrm{Hom}_{A,-}(Ae_i, Af_j)$$

where $\dim \mathrm{Hom}_{A,-}(Ae_i, Af_j) = \dim e_i A f_j < +\infty$. □

Proposition 7.10. *Let M be a left A-module. Let $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$, and let $p, q \in B$ be idempotents. Then an element of $\mathrm{Hom}_{A,-}(Mp, Mq)$ is precisely the right multiplication of an element of pBq. In particular, we have a canonical isomorphism*

$$\mathrm{End}_{A,-}(Mp)^{\mathrm{op}} \simeq pBp$$

Consequently, the direct summands of the left A-module Mp correspond bijectively to the sub-idempotents of p in B.

Proof. This is similar to the proofs of Prop. 1.2 and Cor. 1.4. Any $y \in pBq$ defines a morphism $Mp \to Mq$ by right multiplication. Conversely, if $T \in \mathrm{Hom}_{A,-}(Mp, Mq)$, let $\widehat{T} : M \to M$ be $\widehat{T}(\xi) = T(\xi p)$. Then $\widehat{T} \in \mathrm{End}_{A,-}(M)$, and hence $\widehat{T}$ is the right multiplication by some $\widehat{y} \in B$. Note that $T = \widehat{T}|_{Mp}$, and hence $T(\xi p) = \xi p \widehat{y}$ for each $\xi \in M$. Since T has range in Mq, we have $T(\xi p) = \xi p \widehat{y} q$. So T is the right multiplication by $y := p\widehat{y}q \in pBq$. □

Theorem 7.11. *Let $M \in \mathrm{Coh}_{\mathrm{L}}(A)$. Let $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$ which is a finite-dimensional unital algebra (by Prop. 7.9). Let $p \in B$ be an idempotent. Consider the following statements:*

(1) As coherent left A-modules, Mp generates M.

(2) p is a generating idempotent of B.

Then (2)⇒(1). If M is projective, then (1)⇔(2).

Proof. (2)⇒(1): Since $\dim B < +\infty$, we have a primitive orthogonal decomposition $1_B = q_1 + \cdots + q_n$ where each $q_j \in B$ is a primitive idempotent. By Prop. 6.2, each q_j is isomorphic to a sub-idempotent of p. Thus Mq_j is isomorphic to a direct summand of the left A-module Mp. So Mp generates $\bigoplus_j Mq_j = M$.

(1)⇒(2): Let q be any primitive idempotent of B. Since Mp generates M and since M generates Mq, we have that Mp generates Mq. We claim that Mq is isomorphic to a direct summand of Mp. Then Prop. 7.10 will imply that q is isomorphic (in B) to a sub-idempotent of p. This implies (2), thanks to Prop. 6.2.

Let us prove the claim, assuming that M is projective. Since Mq is a direct summand of M, we see that Mq is projective. Since q is primitive in B, by Prop. 7.10, Mq is an indecomposible left A-module. Therefore, by Cor. 5.6, Mq is the projective cover of an irreducible $N \in \mathrm{Coh}_{\mathrm{L}}(A)$. Since Mp generates Mq, it generates N. Thus N has an epimorphism from a finite direct sum of Mp. Since N is irreducible, N has an epimorphism from Mp. Note that Mp is also projective. Therefore, by Prop. 3.4, Mq is isomorphic to a direct summand of Mp. □

Corollary 7.12. *Assume that $G \in \mathrm{Coh}_{\mathrm{L}}(A)$ is a projective generator. Let M be a left A-module. Then the following are equivalent.*

(1) $M \in \mathrm{Coh}_{\mathrm{L}}(A)$, and M is a projective generator (of $\mathrm{Coh}_{\mathrm{L}}(A)$).

(2) There exist $n \in \mathbb{Z}_+$ and a generating idempotent p of $B := \mathrm{End}_{A,-}(G^{\oplus n})^{\mathrm{op}}$ such that $M \simeq G^{\oplus n} \cdot p$.

In particular, if $e \in A$ is a generating idempotent, one can take $G = Ae$. Thus a projective generator of $\mathrm{Coh}_{\mathrm{L}}(A)$ is (up to isomorphisms) precisely of the form $(Ae)^{\oplus n}p$ where $n \in \mathbb{Z}_+$ and $p \in \mathrm{End}_{A,-}((Ae)^{\oplus n})^{\mathrm{op}}$ is a generating idempotent.

Proof. (2)⇒(1): By Thm. 7.11, M generates $G^{\oplus n}$. So M is a generator. Since $G^{\oplus n}p$ is a direct summand of the projective coherent module $G^{\oplus n}$, $G^{\oplus n}p$ is also projective and coherent.

(1)⇒(2): M has an epimorphism from $G^{\oplus n}$ for some $n \in \mathbb{Z}_+$. Since M is projective, this epimorphism splits. So M can be viewed as a direct summand of $G^{\oplus n}$. Let p be the projection of $G^{\oplus n}$ onto M, which can be viewed as an endomorphism of $G^{\oplus n}$. So p is an idempotent of B, and $M = G^{\oplus n}p$. Since M is a generator, it generates $G^{\oplus n}$. Since $G^{\oplus n}$ is projective, by Thm. 7.11, p is generating. □

8 Right pseudotraces

Let A be an AUF algebra. Let B be a unital algebra. Let M be an A-B bimodule, coherent as a left A-module.

For each $y \in B$ and $\xi \in M$, we write ξy as $y^{\mathrm{op}}\xi$. Namely, y^{op} is viewed as an element of $\mathrm{End}_{A,-}(M)$.

Definition 8.1. A **right coordinate system** of M denotes a collection of morphisms

$$\beta_j \in \mathrm{Hom}_{A,-}(Ae, M) \qquad \check{\beta}^j : \mathrm{Hom}_{A,-}(M, Ae)$$

where $e \in A$ is an idempotent (called the **domain idempotent**), and j runs through a *finite* index set J such that the $\sum_{j\in J} \beta_j \circ \breve{\beta}^j$ equals id_M.

Remark 8.2. M has a right coordinate system iff M is A-projective.

Proof. By Rem. 7.1, each $N \in \mathrm{Coh}_{\mathrm{L}}(A)$ has an epimorphism from $(Ae)^{\oplus n}$ where $e \in A$ is an idempotent and $n \in \mathbb{Z}_+$. This epimorphism splits iff N is projective in $\mathrm{Coh}_{\mathrm{L}}(A)$. Therefore, similar to Rem. 4.2, we see that M has a right coordinate system iff M is A-projective. □

Remark 8.3. In Def. 8.1, one can freely enlarge the domain idempotent e. More precisely, suppose that $f \in A$ is an idempotent such that $e \leqslant f$. One can define a new right coordinate system

$$\begin{gathered}\gamma_j \in \mathrm{Hom}_{A,-}(Af, M) \qquad \breve{\gamma}^j \in \mathrm{Hom}_{A,-}(M, Af)\\ \gamma_j(af) = \beta_j(ae) \qquad \breve{\gamma}^j(\xi) = \breve{\beta}^j(\xi)\end{gathered} \tag{8.1}$$

called the **canonical extension** of $(\beta_j, \breve{\beta}^j)_{j\in J}$.

Definition 8.4. Assume that M has a right coordinate system $(\beta_j, \breve{\beta}^j)_{j\in J}$. For each $\psi \in \mathrm{SLF}(A)$, define the (right) **pseudotrace** ${}^{\boldsymbol{\psi}}\mathbf{Tr}$ associated to ψ to be

$${}^{\psi}\mathrm{Tr} : B \to \mathbb{C} \qquad {}^{\psi}\mathrm{Tr}(y) = \sum_{j\in J} \psi\big((\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j)^{\mathrm{op}}\big)$$

noting that $\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j \in \mathrm{End}_{A,-}(Ae) \simeq (eAe)^{\mathrm{op}}$. In other words,

$${}^{\psi}\mathrm{Tr}(y) = \sum_{j\in J} \psi(\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e)) \tag{8.2}$$

Note that in (8.2) we have $\beta_j(e) \in M$, and hence $\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e) \in Ae$. So

$$\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e) = \breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e^2) = e\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e) \in eAe$$

Proposition 8.5. *Assume that M is A-projective. Let $\psi \in \mathrm{SLF}(A)$. Then ${}^{\psi}\mathrm{Tr} \in \mathrm{SLF}(B)$. Moreover, the definition of ${}^{\psi}\mathrm{Tr}$ is independent of the choice of right coordinate systems.*

Proof. From (8.1) and (8.2), it is clear that a canonical extension of the right coordinate system does not affect the value of ${}^{\psi}\mathrm{Tr}(y)$. Also, note that since A is AUF, for any idempotents $e_1, e_2 \in A$ there is an idempotent e_3 such that $e_1, e_2 \leqslant e_3$. Therefore, to compare ${}^{\psi}\mathrm{Tr}$ defined by two coordinate systems $(\alpha_\bullet, \breve{\alpha}^\bullet)$ and $(\beta_\star, \breve{\beta}^\star)$, by performing canonical extensions, it suffices to assume that their domain idempotents are equal. Then one can use the same argument as in Lem. 4.4 to show that $(\alpha_\bullet, \breve{\alpha}^\bullet)$ and $(\beta_\star, \breve{\beta}^\star)$ define the same ${}^{\psi}\mathrm{Tr}$. Finally, similar to the proof of Prop. 4.5, one shows that ${}^{\psi}\mathrm{Tr}$ is symmetric. □

Example 8.6. Let $M = Ae$ and $B = eAe$ where $e \in A$ is an idempotent. Then the identity map on Ae gives a right coordinate system. From this, one sees that if $\psi \in \mathrm{SLF}(A)$ then

$${}^{\psi}\mathrm{Tr} = \psi|_{eAe}$$

Example 8.7. More generally, let $M = (Ae)^{\oplus n}$ and $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$. So $B = eAe \otimes \mathbb{C}^{n\times n}$. Let

$$\mathrm{tr} : \mathbb{C}^{n\times n} \to \mathbb{C}$$

be the standard trace on $\mathbb{C}^{n\times n}$. A right coordinate system can be choosen to be the n canonical embeddings $Ae \to (Ae)^{\oplus n}$ and the n canonical projections $(Ae)^{\oplus n} \to Ae$. Then one easily sees that

$$^{\psi}\mathrm{Tr} = \psi|_{eAe} \otimes \mathrm{tr}$$

Proposition 8.8. *Assume that M is A-projective. Let $p \in B$ be an idempotent. Let $\psi \in \mathrm{SLF}(A)$. Let $^{\psi}\mathrm{Tr}_M : B \to \mathbb{C}$ be the pseudotrace associated to M. Then the pseudotrace $^{\psi}\mathrm{Tr}_{Mp} : pBp \to \mathbb{C}$ associated to the A-(pBp) bimodule Mp is equal to $^{\psi}\mathrm{Tr}_M\big|_{pBp}$, i.e.*

$$^{\psi}\mathrm{Tr}_{Mp} = {}^{\psi}\mathrm{Tr}_M\big|_{pBp}$$

Proof. Let $(\beta_\bullet, \check{\beta}^\bullet)$ be a right coordinate system (with domain idempotent $e \in A$) as in Def. 8.1. Then one has a right coordinate system

$$\begin{aligned} \gamma_j &\in \mathrm{Hom}_{A,-}(Ae, Mp) & \check{\gamma}^j &: \mathrm{Hom}_{A,-}(Mp, Ae) \\ \gamma_j(ae) &= \beta_j(ae)p & \check{\gamma}^j(\xi p) &= \check{\beta}^j(\xi p) \end{aligned}$$

noting that $Mp \leqslant M$, and hence $\check{\gamma}^j$ is simply the restriction of β^j to Mp. Using (8.2) one computes that for each $y \in B$,

$$\begin{aligned} {}^{\psi}\mathrm{Tr}_{Mp}(pyp) &= \sum_j \psi(\check{\gamma}^j \circ (pyp)^{\mathrm{op}} \circ \gamma_j(e)) = \sum_j \psi(\check{\beta}^j \circ (pyp)^{\mathrm{op}} \circ \beta_j(e)p) \\ = \sum_j \psi(\check{\beta}^j \circ (pyp)^{\mathrm{op}} \circ p^{\mathrm{op}} \circ \beta_j(e)) &= \sum_j \psi(\check{\beta}^j \circ (pyp)^{\mathrm{op}} \circ \beta_j(e)) = {}^{\psi}\mathrm{Tr}_M(pyp) \end{aligned}$$

□

9 Equivalence of left and right pseudotraces

Let A, B be algebras where B is unital.

9.1 Preliminary discussion

In this subsection, assume that A is AUF. We shall consider $M \in \mathrm{Coh}_{\mathrm{L}}(A)$ such that the left and the right pseudotrace constructions are both available to the A-$(\mathrm{End}_{A,-}(M)^{\mathrm{op}})$ bimodule M. By Rem. 8.2, M needs to be assumed A-projective. One also needs M to be $\mathrm{End}_{A,-}(M)^{\mathrm{op}}$-projective. In fact, these two conditions are precisely what ensure that both left and right pseudotraces can be defined.

Proposition 9.1. *Let M be an A-B bimodule. Assume that M is A-coherent. Then the following are equivalent.*

(1) M has a left coordinate system.

(2) M is B-projective.

Although this proposition will not be used in the current note, we include it here as it may be of use in the future.

Proof. (1)⇒(2): See Rem. 4.2.

(2)⇒(1): Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. By Rem. 7.2, each e_iM is finite-dimensional. Therefore, the right B-module e_iM has an epimorphism from $B^{\oplus n}$ which splits because M is B-projective (and hence e_iM is projective since $M = \bigoplus_{i\in\mathfrak{I}} e_iM$). Therefore, for each $i \in \mathfrak{I}$, there is a finite left coordinate system $\alpha_{i,\bullet} \in \mathrm{Hom}_{-,B}(B, e_iM)$ and $\check{\alpha}^{i,\bullet} \in \mathrm{Hom}_{-,B}(e_iM, B)$. Let

$$\begin{aligned} \gamma_{i,\bullet} &\in \mathrm{Hom}_{-,B}(B, M) &\qquad \check{\gamma}^{i,\bullet} &\in \mathrm{Hom}_{-,B}(M, B) \\ \gamma_{i,\bullet}(b) &= \alpha_{i,\bullet}(b) &\qquad \check{\gamma}^{i,\bullet}(\xi) &= \check{\alpha}^{i,\bullet}(e_i\xi) \end{aligned}$$

Then one checks easily that $(\gamma_{i,\bullet}, \check{\gamma}^{i,\bullet})_{i\in\mathfrak{I}}$ is a left coordinate system of M. □

9.2 Calculation of some left pseudotraces

In this subsection, A is not assumed to be AUF. Let M be an A-B bimodule.

The goal of this subsection is to prepare for the proof of the main Thm. 9.4. The following theorem is dual to Prop. 8.8.

Theorem 9.2. *Assume that M has a left coordinate system. Let $p \in B$ be a generating idempotent. Then the following are true.*

1. *The A-(pBp) bimodule Mp has a left coordinate system.*
2. *Let $\phi \in \mathrm{SLF}(B)$. Then on A, the pseudotrace associated to $\phi|_{pBp}$ and Mp is equal to the pseudotrace associated to ϕ and M. Namely,*

$$\mathrm{Tr}_{Mp}^{\phi|_{pBp}} = \mathrm{Tr}_{M}^{\phi} \tag{9.1}$$

In this theorem, we do not require that A is AUF.

Proof. Choose a left coordinate system for M:

$$\alpha_i \in \mathrm{Hom}_{-,B}(B, M) \qquad \check{\alpha}^i \in \mathrm{Hom}_{-,B}(M, B) \qquad i \in \mathfrak{I}$$

Since p is generating, similar to the proof of Thm. 6.5, we can find finitely many elements u_k, v_k in B such that

$$\begin{aligned} v_k u_k &= p_k &\qquad u_k v_k &= q_k \\ u_k &\in q_k B p_k &\qquad v_k &\in p_k B q_k \end{aligned}$$

where each $p_k, q_k \in B$ are idempotents, $1_B = \sum_k q_k$ is a primitive orthogonal decomposition of 1_B, and $p_k \leqslant p$ for each k. [3] Let

$$\theta_{i,k} \in \mathrm{Hom}_{-,pBp}(pBp, Mp) \qquad \breve{\theta}^{i,k} \in \mathrm{Hom}_{-,pBp}(Mp, pBp)$$
$$\theta_{i,k}(pyp) = \alpha_i(u_k \cdot pyp) \qquad \breve{\theta}^{i,k}(\xi p) = v_k \cdot \breve{\alpha}^i(\xi p)$$

noting that $\alpha_i(u_k \cdot pyp) = \alpha_i(u_k)pyp \in Mp$ and $v_k \cdot \breve{\alpha}^i(\xi p) = v_k \cdot \breve{\alpha}^i(\xi)p \in p_k Bp \subset pBp$.

For each $\xi \in M$, note that if $\breve{\alpha}^i(\xi) = 0$, then $\breve{\theta}^{i,k}(\xi p) = v_k \breve{\alpha}^i(\xi)p = 0$. Therefore, $\breve{\theta}^{i,k}(\xi p) = 0$ for all but finitely many i and k. Moreover, we compute

$$\begin{aligned}
&\sum_{i,k} \theta_{i,k} \circ \breve{\theta}^{i,k}(\xi p) = \sum_{i,k} \theta_{i,k}(v_k \breve{\alpha}^i(\xi p)) = \sum_{i,k} \alpha_i(u_k v_k \breve{\alpha}^i(\xi p)) \\
=&\sum_{i,k} \alpha_i(q_k \breve{\alpha}^i(\xi p)) = \sum_i \alpha_i \circ \breve{\alpha}^i(\xi p) = \xi p
\end{aligned}$$

where all the sums are finite. This proves that $(\theta, \breve{\theta})$ satisfies Def. 4.1-(a). It is easy to check Def. 4.1-(b). So we have proved that $(\theta, \breve{\theta})$ is a left coordinate system of Mp.

It remains to check (9.1). Choose any $x \in A$. By (4.3) and the fact that $1_{pBp} = p$,

$$\begin{aligned}
&\mathrm{Tr}_{Mp}^{\phi|_{pBp}}(x) = \sum_{i,k} \phi(\breve{\theta}^{i,k} \circ x \circ \theta_{i,k}(p)) = \sum_{i,k} \phi(\breve{\theta}^{i,k} \circ x \circ \alpha_i(u_k p)) \\
=&\sum_{i,k} \phi(\breve{\theta}^{i,k} \circ x \circ \alpha_i(u_k)) = \sum_{i,k} \phi(v_k \cdot \breve{\alpha}^i(x \circ \alpha_i(u_k)))
\end{aligned}$$

Since $\breve{\alpha}^i, x, \alpha_i$ commute with the right multiplication by v_k, and since ϕ is symmetric,

$$\begin{aligned}
&\mathrm{Tr}_{Mp}^{\phi|_{pBp}}(x) = \sum_{i,k} \phi(\breve{\alpha}^i(x \circ \alpha_i(u_k))v_k) = \sum_{i,k} \phi(\breve{\alpha}^i(x \circ \alpha_i(u_k v_k))) \\
=&\sum_i \phi(\breve{\alpha}^i(x \circ \alpha_i(1_B))) = \mathrm{Tr}_M^{\phi}(x)
\end{aligned}$$

This finishes the proof of (9.1). □

Corollary 9.3. *Assume that M has a left coordinate system. Let $n \in \mathbb{Z}_+$. Let $\widetilde{B} = B \otimes \mathbb{C}^{n \times n}$. Then the A-$\widetilde{B}$ bimodule $M^{\oplus n} \simeq M \otimes \mathbb{C}^{1,n}$ has a left coordinate system. Moreover, for each $\phi \in \mathrm{SLF}(B)$, we have*

$$\mathrm{Tr}_{M^{\oplus n}}^{\phi \otimes \mathrm{tr}} = \mathrm{Tr}_M^{\phi} \tag{9.2}$$

as pseudotraces on A associated to $\phi \otimes \mathrm{tr} \in \mathrm{SLF}(\widetilde{B})$ and $\phi \in \mathrm{SLF}(B)$, respectively.

Recall that $\mathrm{tr} \in \mathrm{SLF}(\mathbb{C}^{n \times n})$ is the standard trace on the $n \times n$ matrix algebra.

Proof. Choose a left coordinate system

$$\alpha_i \in \mathrm{Hom}_{-,B}(B, M) \qquad \breve{\alpha}^i \in \mathrm{Hom}_{-,B}(M, B)$$

[3] So p_k, q_k are similar to ε_{k_i}, e_i in the proof of Thm. 6.5.

of M. Define

$$\gamma_i \in \mathrm{Hom}_{-,\widetilde{B}}(\widetilde{B}, M^{\oplus n}) \qquad \check{\gamma}^i \in \mathrm{Hom}_{-,\widetilde{B}}(M^{\oplus n}, \widetilde{B})$$

such that

$$\gamma_i \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ & \vdots & \\ y_{n,1} & \cdots & y_{n,n} \end{bmatrix} = [\alpha_i(1_B), 0, \ldots, 0] \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ & \vdots & \\ y_{n,1} & \cdots & y_{n,n} \end{bmatrix} = [\alpha_i(y_{1,1}), \ldots, \alpha_i(y_{1,n})]$$

$$\check{\gamma}^i[\xi_1, \ldots, \xi_n] = \begin{bmatrix} \check{\alpha}^i(\xi_1) & \cdots & \check{\alpha}^i(\xi_n) \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

One checks easily that this is a left coordinate system of $M^{\oplus n}$. Now (9.2) follows by applying Thm. 9.2 to the A-$\widetilde{B}$ bimodule $M^{\oplus n}$ and the generating projection $p \in \widetilde{B}$, where p is the matrix whose $(1,1)$-entry is 1 and other entries are 0. $\square$

9.3 The main theorem

Assume that A is strongly AUF (cf. Cor. 6.4) so that A has a projective generator (cf. Prop. 7.8). The following generalization of Thm. 6.6 is the main theorem of this note.

Theorem 9.4. *Assume that $M \in \mathrm{Coh}_{\mathrm{L}}(A)$ is a projective generator. Assume that $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$ so that M is an A-B bimodule. Then M has left and right coordinate systems. Moreover, we have a linear isomorphism*

$$\mathrm{SLF}(A) \xrightarrow{\simeq} \mathrm{SLF}(B) \qquad \psi \mapsto {}^{\psi}\mathrm{Tr} \tag{9.3a}$$

whose inverse map is

$$\mathrm{SLF}(B) \xrightarrow{\simeq} \mathrm{SLF}(A) \qquad \phi \mapsto \mathrm{Tr}^{\phi} \tag{9.3b}$$

Of course, both pseudotraces are associated to M; we have suppressed the subscript M.

Proof. Note that $\dim B < +\infty$ by Prop. 7.9. So $\dim \mathrm{SLF}(B) < +\infty$. Since $M \in \mathrm{Coh}_{\mathrm{L}}(A)$ is A-projective, by Rem. 8.2, M has a right coordinate system. By Cor. 7.12, we may assume that $M = G \cdot p$ where

- $G = (Ae)^{\oplus n}$ for some $n \in \mathbb{Z}_+$ and generating idempotent $e \in A$.
- $M = Gp$ where p is a generating idempotent of $\widetilde{B} = \mathrm{End}_{A,-}(G)^{\mathrm{op}} = eAe \otimes \mathbb{C}^{n\times n}$.
- $B = p\widetilde{B}p$ (by Prop. 7.10).

By Thm. 6.5 and Cor. 9.3, G has a left coordinate system. Therefore, by Thm. 9.2, M has a left coordinate system.

By Thm. 6.6, we have $\dim \mathrm{SLF}(A) = \dim \mathrm{SLF}(eAe)$. Clearly we have a linear isomorphism

$$\mathrm{SLF}(eAe) \xrightarrow{\simeq} \mathrm{SLF}(eAe \otimes \mathbb{C}^{n\times n}) \qquad \omega \mapsto \omega \otimes \mathrm{tr}$$

So $\dim \mathrm{SLF}(eAe) = \dim \mathrm{SLF}(\widetilde{B})$. By Thm. 6.6, we have $\dim \mathrm{SLF}(\widetilde{B}) = \dim \mathrm{SLF}(B)$. This proves $\dim \mathrm{SLF}(A) = \dim \mathrm{SLF}(B) < +\infty$.

Choose any $\psi \in \mathrm{SLF}(A)$. By Exp. 8.7, ${}^{\psi}\mathrm{Tr}_G : \widetilde{B} \to \mathbb{C}$ equals $\psi|_{eAe} \otimes \mathrm{tr}$. By Prop. 8.8, on $B = p(eAe \otimes \mathbb{C}^{n\times n})p$ we have

$${}^{\psi}\mathrm{Tr}_M = (\psi|_{eAe} \otimes \mathrm{tr})\big|_B \qquad =: \phi$$

Now $\phi \in \mathrm{SLF}(B)$. By Thm. 9.2 and Cor. 9.3,

$$\mathrm{Tr}_M^{\phi} = \mathrm{Tr}_{Gp}^{(\psi|_{eAe}\otimes \mathrm{tr})|_B} = \mathrm{Tr}_G^{\psi|_{eAe}\otimes \mathrm{tr}} = \mathrm{Tr}_{Ae}^{\psi|_{eAe}}$$

By Thm. 6.6, $\mathrm{Tr}_{Ae}^{\psi|_{eAe}} = \psi$. So $\mathrm{Tr}_M^{\phi} = \psi$. We have thus proved that (9.3b) $\circ$ (9.3a) is the identity map on $\mathrm{SLF}(A)$. This finishes the proof. □

10 Equivalence of non-degeneracy of left and right pseudotraces

Definition 10.1. Let A be an algebra and $\psi \in \mathrm{SLF}(A)$. We say that ψ is **non-degenerate** if

$$\{x \in A : \psi(xA) = 0\} \equiv \{x \in A : \psi(xy) = 0, \forall y \in A\}$$

is zero.

In the following, A is always assumed to be AUF.

Lemma 10.2. *Let $e \in A$ be an idempotent, and let $\psi \in \mathrm{SLF}(A)$. If ψ is non-degenerate, then the restriction $\psi|_{eAe}$ is non-degenerate. Conversely, if $\psi|_{eAe}$ is non-degenerate and e is generating, then ψ is non-degenerate.*

Proof. Assume that ψ is non-degenerate. Choose $x \in eAe$ such that $\psi(xeAe) = 0$. Then

$$\psi(xA) = \psi(exeA) = \psi(xeAe) = 0$$

and hence $x = 0$. Therefore $\psi|_{eAe}$ is non-degenerate.

Conversely, assume that $\psi|_{eAe}$ is non-degenerate and e is generating. Choose $x \in A$ such that $\psi(xA) = 0$. Then for each $a, b \in A$,

$$\psi(eaxbe \cdot eAe) = \psi(eaxbeAe) = \psi(xbeAea) = 0$$

Therefore $eaxbe = 0$. Since b is arbitrary, we have $eaxAe = 0$. Since e is generating, it is not hard to show that the left A-module Ae is faithful. (See for example Lem. 11.6.) It follows from that $eax = 0$. Therefore $eAx = 0$. Similarly, eA is a faithful right A-module. Hence $x = 0$. This proves the non-degeneracy of ψ. □

Proposition 10.3. *Assume that $\psi \in \mathrm{SLF}(A)$ is non-degenerate. Let $M \in \mathrm{Coh}_{\mathrm{L}}(A)$ be projective, and let $B = \mathrm{End}^0_{A,-}(M)$. Then the right pseudotrace ${}^\psi\mathrm{Tr} \in \mathrm{SLF}(B)$ is non-degenerate.*

Proof. By Prop. 2.6, M can be viewed as a direct summand of $\bigoplus_{i=1}^n Ae_i$ where each $e_i \in A$ is an idempotent. Let $e \in A$ be an idempotent such that $e \geqslant e_i$ for all i. Then M is a direct summand of $(Ae)^{\oplus n}$. By Prop. 1.2, we have $\mathrm{End}^0_{A,-}(Ae)^{\mathrm{op}} = eAe$, and hence $\mathrm{End}^0_{A,-}((Ae)^{\oplus n}) = eAe \otimes \mathbb{C}^{n\times n}$. By Cor. 1.4, there is an idempotent $p \in eAe \otimes \mathbb{C}^{n\times n}$ such that $M = (Ae)^{\oplus n}p$. By Lem. 10.2, $\psi|_{eAe}$ is non-degenerate, and hence $\psi|_{eAe} \otimes \mathrm{tr} : eAe \otimes \mathbb{C}^{n\times n} \to \mathbb{C}$ is non-degenerate. By Lem. 10.2 again, the restriction of $\psi|_{eAe} \otimes \mathrm{tr}$ to $p(eAe \otimes \mathbb{C}^{n\times n})p$ (which is B due to Prop. 7.10) is non-degenerate. But this restriction is exactly ${}^\psi\mathrm{Tr}$ due to Exp. 8.7 and Prop. 8.8. $\square$

Theorem 10.4. *Assume that A is strongly AUF. Then in Thm. 9.4, for any $\psi \in \mathrm{SLF}(A)$, the non-degeneracy of ψ and of ${}^\psi\mathrm{Tr}$ are equivalent.*

Proof. We use the notation in the proof of Thm. 9.4. From that proof, we know ${}^\psi\mathrm{Tr} = (\psi|_{eAe} \otimes \mathrm{tr})|_B$. By Lem. 10.2, ψ is non-degenerate iff $\psi|_{eAe}$ is so, and $\psi|_{eAe} \otimes \mathrm{tr}$ is non-degenerate iff $(\psi|_{eAe} \otimes \mathrm{tr})|_B$ is so. The equivalence of the non-degeneracy of $\psi|_{eAe}$ and of $\psi|_{eAe} \otimes \mathrm{tr}$ is obvious. The proof is finished. $\square$

11 Classification of strongly AUF algebras

In this section, we fix an AUF algebra A.

Definition 11.1. For each left A-module M, let M^* be the space of linear functionals, which has a right A-module structure defined by

$$(\phi a)(m) = \phi(am) \qquad \text{for all } a \in A, m \in M$$

We define the **quasicoherent dual**

$$\begin{aligned} M^\vee =& \{\phi \in M^* : \phi \in \phi \cdot A\} \\ =& \{\phi \in M^* : \text{there exists an idempotent } e \in A \text{ such that } \phi = \phi e\} \end{aligned}$$

By Def. 2.2, $M^\vee$ is the largest right A-submodule of M that is quasicoherent.

Remark 11.2. Let $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$. Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. Then, as vector spaces, we clearly have

$$M = \bigoplus_{i\in\mathfrak{I}} e_i M \qquad M^* = \prod_{i\in\mathfrak{I}} (e_i M)^*$$

It follows easily that

$$M^\vee = \bigoplus_{i\in\mathfrak{I}} (e_i M)^*$$

Definition 11.3. For each $M \in \mathrm{QCoh_L}(A)$, we let

$$\mathrm{End}^0(M) = M \otimes_{\mathbb{C}} M^\vee$$

viewed as a subalgebra of $\mathrm{End}(M)$.[4] Suppose that B is an algebra, and M has a right B-module structure commuting with the left action of A, we let

$$\mathrm{End}^0_{-,B}(M) = \{T \in \mathrm{End}^0(M) : (T\xi)b = T(\xi b) \text{ for all } \xi \in M, b \in B\} \tag{11.1}$$

Remark 11.4. Let $M \in \mathrm{Coh_L}(A)$. By Rem. 7.2 we have $\dim e_i M < +\infty$. It follows from Rem. 11.2 that

$$\mathrm{End}^0(M) = \{T \in \mathrm{End}(M) : Te_i = 0 \text{ for all but finitely many } i \in \mathfrak{I}\}$$

Proposition 11.5. *Choose $M \in \mathrm{Coh_L}(A)$, and let $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$. Then for each generating idempotent $p \in B$, we have a linear isomorphism*

$$\mathrm{End}^0_{-,B}(M) \xrightarrow{\simeq} \mathrm{End}^0_{-,pBp}(Mp) \qquad S \mapsto S|_{Mp} \tag{11.2}$$

Proof. Step 1. Let $\widehat{B} = B^{\mathrm{op}} = \mathrm{End}_{A,-}(M)$, and let $\widehat{p} \in \widehat{B}$ be the opposite element of p. Then M has a left $\widehat{B}$-module structure commuting with the left action of A, and R_p is the left multiplication by $\widehat{p}$.

For each $S \in \mathrm{End}^0_{-,B}(M)$, note that $S|_{Mp} = S|_{\widehat{p}M}$ maps $\widehat{p}M$ into $\widehat{p}M$, because $S\widehat{p}\xi = \widehat{p}S\xi \in \widehat{p}M$ for each $\xi \in M$. It is clear that $S|_{Mp}$ commutes with the action of $\widehat{p}\widehat{B}\widehat{p}$. That $S|_{Mp}$ belongs to $\mathrm{End}^0(M)$ can be checked from Rem. 11.4. This proves that $S|_{Mp}$ belongs to $\mathrm{End}^0_{-,pBp}(Mp)$. We have thus proved that the linear map (11.2) is well-defined.

Step 2. Let us prove the surjectivity of (11.2). By Rem. 5.4, B is finite-dimensional. Therefore, we have an orthogonal primitive decomposition $1_{\widehat{B}} - \widehat{p} = f_1 + \cdots + f_n$ in $\widehat{B}$. In this case, we have

$$M = \widehat{p}M \oplus f_1 M \oplus \cdots \oplus f_n M$$

By Prop. 6.2, for each $1 \leqslant i \leqslant n$, f_i is isomorphic to a sub-idempotent q_i of $\widehat{p}$, i.e., there exist $u_i \in f_i \widehat{B} q_i$ and $v_i \in q_i \widehat{B} f_i$ such that $u_i v_i = f_i$ and $v_i u_i = q_i \leqslant \widehat{p}$ (where $q_i \in \widehat{B}$ is an idempotent).

Now, we choose $T \in \mathrm{End}^0_{-,pBp}(Mp) = \mathrm{End}^0_{-,pBp}(\widehat{p}M)$. Define a linear map

$$S : M \to M \qquad \xi \mapsto T(\widehat{p}\xi) + \sum_{i=1}^n u_i T(v_i \xi) \tag{11.3}$$

By Rem. 11.4, we have $S \in \mathrm{End}^0(M)$. We claim that S commutes with the action of $\widehat{B}$ (and hence $S \in \mathrm{End}^0_{-,B}(M)$). If this is proved, then since T clearly equals $S|_{Mp} = S|_{\widehat{p}M}$ (because $v_i \widehat{p} = 0$, see below), the proof of the surjectivity of (11.2) is complete.

Note that since $\widehat{p}, f_1, \dots, f_n$ are mutually orthogonal, we have

$$u_i u_j = 0 \qquad v_i v_j = 0 \qquad \text{for all } i, j$$

[4] That is, for each $\xi \in M, \phi \in M^\vee$, the operator $\xi \otimes \phi$ sends each $\eta \in M$ to $\phi(\eta) \cdot \xi$.

$$v_j u_i = 0 \qquad \text{for all } i \neq j$$
$$v_i \widehat{p} = 0 \qquad \widehat{p} u_i = 0 \qquad \text{for all } i$$

Using this observation and the fact that $T : \widehat{p}M \to \widehat{p}M$ commutes the left action of $\widehat{p}\widehat{B}\widehat{p}$, we compute that for each j and $\xi \in M$,

$$S(v_j\xi) = T(\widehat{p}v_j\xi) + 0 = T(v_j\xi)$$
$$v_jS(\xi) = v_jT(\widehat{p}\xi) + v_ju_jT(v_j\xi) \xlongequal{v_ju_j = q_j \in \widehat{p}\widehat{B}\widehat{p}} 0 + T(q_jv_j\xi) = T(v_j\xi)$$

and hence $S(v_j\xi) = v_jS(\xi)$; similarly,

$$S(u_j\xi) = T(\widehat{p}u_j\xi) + u_jT(v_ju_j\xi) \xlongequal{v_ju_j = q_j \in \widehat{p}\widehat{B}\widehat{p}} 0 + u_jq_jT(\widehat{p}\xi) = u_jT(\widehat{p}\xi)$$
$$u_jS(\xi) = u_jT(\widehat{p}\xi) + 0 = u_jT(\widehat{p}\xi)$$

and hence $S(u_j\xi) = u_jS(\xi)$. Moreover, for each $b \in \widehat{B}$ we have

$$S(\widehat{p}b\widehat{p}\xi) = T(\widehat{p}b\widehat{p}\xi) + 0 = \widehat{p}b\widehat{p}T(\widehat{p}\xi)$$
$$\widehat{p}b\widehat{p}S(\xi) = \widehat{p}b\widehat{p}T(\widehat{p}\xi) + 0 = \widehat{p}b\widehat{p}T(\widehat{p}\xi)$$

and hence $S(\widehat{p}b\widehat{p}\xi) = \widehat{p}b\widehat{p}S(\xi)$. This proves that S commutes with the left action of $\widehat{B}$, since $\widehat{B}$ is generated by $\{u_i, v_i : 1 \leqslant i \leqslant n\}$ and $\widehat{p}\widehat{B}\widehat{p}$—to see this, note that for each $b \in \widehat{B}$, by setting $f_0 = u_0 = v_0 = \widehat{p}$, we have

$$b = \sum_{i,j=0}^{n} f_ibf_j = \sum_{i,j=0}^{n} u_ib_{i,j}v_j$$

where each $b_{i,j} := v_ibu_j$ commutes with the left actions of A and satisfies $b_{i,j} = \widehat{p}b_{i,j}\widehat{p}$, and hence belongs to $\widehat{p}\widehat{B}\widehat{p}$.

Step 3. If $S \in \mathrm{End}^0_{-,B}(M)$ and $S|_{\widehat{p}M} = 0$, then for each $\xi \in M$, we have

$$S(\xi) = S(\widehat{p}\xi) + \sum_{i=1}^{n} S(f_i\xi) = S(\widehat{p}\xi) + \sum_{i=1}^{n} u_iS(v_i\xi)$$

where $\widehat{p}\xi, v_j\xi \in \widehat{p}M$. Therefore $S = 0$. This proves that (11.2) is injective. □

Lemma 11.6. *Suppose that $e \in A$ is a generating idempotent. Then we have a linear isomorphism*

$$A \xrightarrow{\simeq} \mathrm{End}^0_{-,eAe}(Ae) \tag{11.4}$$

sending each $a \in A$ to the left multiplication by a.

Proof. It is obvious that the left action on Ae by $a \in A$ belongs to $\mathrm{End}^0_{-,eAe}(Ae)$. Therefore, the map (11.4) is well-defined.

Suppose that the left multiplication of $a \in A$ on Ae is zero. Then $aAe = 0$. Since A is AUF and hence almost unital, there is an idempotent $p \in A$ such that $a = ap$. Since e is generating, by Cor. 7.7, Ae is a generator of $\mathrm{Coh_L}(A)$. Therefore, Ap is a quotient module of $(Ae)^{\oplus n}$ for some $n \in \mathbb{Z}_+$. Thus aAp is a quotient space of $(aAe)^{\oplus n}$, and hence $aAp = 0$. This proves $ap = 0$, and hence $a = 0$. We have thus proved that (11.4) is injective.

Choose $T \in \mathrm{End}^0_{-,eAe}(Ae)$. Since $T \in \mathrm{End}^0(Ae)$, by Rem. 11.2, there is an idempotent $f \in A$ such that $T = fTf$. It follows that $fTf|_{fAe}$ belongs to $\mathrm{End}_{-,eAe}(fAe)$. Since A is AUF, we may enlarge f so that $e \leqslant f$ also holds. We claim that $\mathrm{End}_{-,eAe}(fAe)$ consists of the left multiplications by elements of fAf. If this is true, then $T|_{fAe} = fTf|_{fAe}$ is the left multiplication by faf for some $a \in A$. It follows that for any $b \in A$, we have $Tbe = Tfbe = fafbe$, and hence T is the left multiplication by faf on Ae, finishing the proof that (11.4) is surjective.

By Cor. 6.3, the idempotent $e \in fAf$ is generating in fAf. Applying Prop. 11.5 to the finite-dimensional unital algebra fAf and its (finite-dimensional) coherent left module fAf, we see that $\mathrm{End}_{-,eAe}(fAe) = fAf|_{fAe}$. This proves the claim. □

Theorem 11.7. *Suppose that A is strongly AUF, and let G be a projective generator of $\mathrm{Coh_L}(A)$ (which exists due to Prop. 7.8). Set $B = \mathrm{End}_{A,-}(G)^{\mathrm{op}}$. Regard G as an A-B bimodule. Then we have a linear isomorphism*

$$A \xrightarrow{\simeq} \mathrm{End}^0_{-,B}(G) \tag{11.5}$$

sending each $a \in A$ to the left multiplication of a on G.

Proof. By Cor. 6.4 and Prop. 7.8, A has a generating idempotent e. If $G = Ae$, then $\mathrm{End}_{A,-}(G) = eAe$ due to Prop. 1.2. Therefore, by Lem. 11.6, the map (11.5) is bijective.

If $G = (Ae)^{\oplus n}$ where $n \in \mathbb{Z}_+$, one easily checks that $B = eAe \otimes \mathbb{C}^{n\times n}$ where $\mathbb{C}^{n\times n}$ is the matrix algebra of order n. The bijectivity of (11.5) then follows easily.

Finally, let G be any general projective generator. By Cor. 7.12, we may assume that $G = (Ae)^{\oplus n}p$ where $n \in \mathbb{Z}_+$, and p is a generating idempotent of $\widetilde{B} = \mathrm{End}_{A,-}((Ae)^{\oplus n})^{\mathrm{op}} \simeq eAe \otimes \mathbb{C}^{n\times n}$. By Prop. 7.10, we have $B = p\widetilde{B}p$. Therefore, by Prop. 11.5, the map

$$\mathrm{End}^0_{-,\widetilde{B}}((Ae)^{\oplus n}) \to \mathrm{End}^0_{-,B}(G)$$

sending each S to $S|_G$ is bijective. By the previous paragraph, the map

$$A \to \mathrm{End}^0_{-,\widetilde{B}}((Ae)^{\oplus n})$$

sending each a to the left multiplication by a is bijective. Therefore, their composition, namely (11.5), is bijective. □

Remark 11.8. In Thm. 11.7, the right B-module G is a **projective generator** in the category $\mathrm{Mod^R}(B)$ of right B-modules—that is, G is projective in $\mathrm{Mod^R}(B)$, and any object in $\mathrm{Mod^R}(B)$ has an epimorphism from a (possibly infinite) direct sum of G.

Proof. The projectivity of G in $\mathrm{Mod^R}(B)$ is due to Thm. 9.4 and Rem. 4.2. Using the notation in the proof of Thm. 11.7, we may assume $G = (Ae)^{\oplus n}p$ and $B = p(eAe \otimes \mathbb{C}^{n\times n})p$ where $e \in A$ and $p \in eAe \otimes \mathbb{C}^{n\times n}$ are generating idempotents. Since B is unital, B is generating in $\mathrm{Mod^R}(B)$. Therefore $(eAe \otimes \mathbb{C}^{n\times n})p$ is generating in $\mathrm{Mod^R}(B)$. Since $(eAe \otimes \mathbb{C}^{n\times n})p$ is a direct sum of $(eAe \otimes \mathbb{C}^{1\times n}) = (eAe)^{\oplus n}p = eG$, we conclude that eG is generating in $\mathrm{Mod^R}(B)$. Therefore G is generating in $\mathrm{Mod^R}(B)$. □

Theorem 11.9. *Let $\mathcal{A}$ be an algebra. The following are equivalent.*

(1) $\mathcal{A}$ is strongly AUF.

(2) $\mathcal{A}$ is isomorphic to $\mathrm{End}^0_{-,B}(M)$ where B is a unital finite-dimensional algebra, M is a projective generator in $\mathrm{Mod}^{\mathrm{R}}(B)$, the vector space M has a grading

$$M = \bigoplus_{i \in \mathfrak{I}} M(i)$$

where each $M(i)$ is finite-dimensional and is preserved by the right action of B, and $\mathrm{End}^0_{-,B}(M)$ is defined by

$$\begin{aligned}\mathrm{End}^0_{-,B}(M) := \{T \in \mathrm{End}(M) :& (Tm)b = T(mb) \text{ for all } m \in M, b \in B,\\ & T|_{M(i)} = 0 \text{ for all by finitely many } i \in \mathfrak{I}\}\end{aligned}$$

Proof. The direction (1)⇒(2) follows from Thm. 11.7 and Rem. 11.8. Let us prove the other direction.

Assume that $\mathcal{A} = \mathrm{End}^0_{-,B}(M)$ where $\mathrm{End}^0_{-,B}(M)$ is described as in (2). Let e_i be the projection of M onto $M(i)$. Then e_i clearly belongs to $\mathcal{A}$, and each $T \in \mathcal{A}$ can be written as $T = \sum_{i,j \in \mathfrak{I}} e_i T e_j$ where $e_i T e_j = 0$ for all but finitely many i, j. This proves that $\mathcal{A}$ is AUF.

Since M is a projective generator in $\mathrm{Mod}^{\mathrm{R}}(B)$, for each finite subset $I \subset \mathfrak{I}$, $M_I := \bigoplus_{i \in I} M(i)$ is projective in $\mathrm{Mod}^{\mathrm{R}}(B)$ (since it is a direct summand of M). Let $1_B = p_1 + \cdots + p_n$ be an orthogonal primitive decomposition of 1_B in B. By Thm. 5.5, irreducible finite-dimensional right B-modules are precisely those that are isomorphic to $p_k B/\mathrm{rad}(p_k B)$ for some k. Since M is generating in $\mathrm{Mod}^{\mathrm{R}}(B)$, it has an epimorphism to $p_k B/\mathrm{rad}(p_k B)$ for each k. This epimorphism must restrict to a nonzero morphism (and hence an epimorphism) $M(i_k) \to p_k B/\mathrm{rad}(p_k B)$. Let $I = \{i_1, \ldots, i_n\}$. Then M_I has an epimorphism to each irreducible right B-module. It follows from Prop. 7.6 that M_I is a projective generator in the category of finite-dimensional right B-modules.

Let $e_I = \sum_{i \in I} e_i$, which is an idempotent in $\mathcal{A}$. We claim that e_I is a generating idempotent in $\mathcal{A}$, which will complete the proof that $\mathcal{A}$ is strongly AUF.

Let ε be any primitive idempotent of $\mathcal{A}$. Then εM is a finite-dimensional right B-module, since any element of $\mathcal{A}$ has finite range when acting on M. Moreover, since ε is primitive in $\mathcal{A}$, the right B-module εM is indecomposible. Since εM is a direct summand of the projective right B-module M, it follows that εM is a finite-dimensional indecomposible projective right B-module. Therefore, since $M_I = e_I M$ is a projective generator, similar to the the end of the proof of Thm. 7.11, we conclude that the right B-module εM is isomorphic to a direct summand of $e_I M$. Thus, by Thm. 7.10, ε is isomorphic to a subidempotent of e_I in $\mathcal{A}$. This proves the claim that e_I is generating. □

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