From Segal's sewing to pseudo-q-traces and back

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Modular invariance

- A central theme in vertex operator algebras (VOAs) is modular invariance.
- An early breakthrough in this topic is Zhu's theorem (96): Assume that $\mathbb{V}$ is "C_2-cofinite and rational". Then the set of all $\mathrm{Tr}_{\mathbb{M}} q^{L_0 - \frac{c}{24}}$ span a $SL(2,\mathbb{Z})$-invariant space (where $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$).
- More generally: Let $Y(v,z)$ denote the vertex operators (where $v \in \mathbb{V}$). Then $(v,\tau) \in \mathbb{V} \times \mathbb{H} \mapsto \mathrm{Tr}_{\mathbb{M}} Y(v,1) q_\tau^{L_0 - \frac{c}{24}}$ (over all $\mathbb{M} \in \mathrm{Irr}$) span an $SL(2,\mathbb{Z})$-invariant space with dimension $\#\mathrm{Irr}(\mathbb{V})$. Here $q_\tau = e^{2\pi \mathbf{i} \tau}$.
- "C_2-cofinite" is a finiteness condition ensuring, e.g., that $\mathrm{Irr}(\mathbb{V})$ is finite. "Rational" means that every $\mathbb{V}$-mod is completely reducible.

Modular invariance beyond rationality

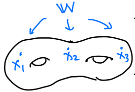
- However, this modular invariance does not hold when rationality is dropped: $\mathrm{Tr}_{\mathbb{M}} q_\tau^{L_0 - c/24}$ is a fractional power of $q_\tau = e^{2\mathbf{i}\pi\tau}$. However, without rationality (such as the $\mathcal{W}(p)$-algebra), an $SL(2,\mathbb{Z})$ action of $\mathrm{Tr}_{\mathbb{M}} q_\tau^{L_0 - c/24}$ will contain factors such as $\tau = \frac{1}{2\mathbf{i}\pi} \log q_\tau$.
- To rescue modular invariance, Miyamoto (04) introduced the **pseudo-q-trace** construction $(v,\tau) \in \mathbb{V} \times \mathbb{H} \mapsto \mathrm{Tr}^\omega_{\mathbb{M}} Y(v,1) q_\tau^{L_0 - \frac{c}{24}}$. For $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$, a pseudo-trace $\mathrm{Tr}^\omega_{\mathbb{M}}$ is a symmetric linear functional on a suitable subalgebra of $\mathrm{End}(\mathbb{M})$.
- Miyamoto showed that if $\mathbb{V}$ is C_2-cofinite, then the pseudo-q-traces form an $SL(2,\mathbb{Z})$-invariant space.
- Miyamoto's pseudo-q-traces were later simplified by Arike (10) and Arike-Nagatomo (11).

The goal of this talk

- The usual q-trace construction can be viewed as a special case of the **sewing construction** ($\approx$taking contraction) in Segal's functorial definition of CFT (88).
- However, pseudo-traces were not discussed in Segal's definition at all! Did Segal miss something?
- The goal of this talk is to explain our answer: No, Segal didn't miss anything! We will explain how the algebraic setting of pseudo-q-traces is compatible with Segal's geometric framework.
- Our approach is based on the theory of **conformal blocks** (CB). Traditional approaches to CB cannot recover pseudo-q-traces. We will present our formulation of CB, and point out its difference with the traditional ones.

Conformal blocks (CB)

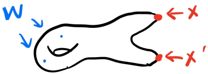
- We always assume for simplicity that $\mathbb{V}$ is C_2-cofinite.
- Fix a (possibly disconnected) N-pointed compact Riemann surface with local coordinates $\mathfrak{X} = (C; x_1, \cdots, x_N; \eta_1, \cdots, \eta_N)$. Associate $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ to $x_1, \cdots, x_N$. A **conformal block** (CB) is a linear map $\psi : \mathbb{W} \to \mathbb{C}$ invariant under the action defined by $\mathfrak{X}$ and $\mathbb{V}$ (Zhu 94, Frenkel&Ben-Zvi 04). The spaces of conformal blocks is denoted by $CB(\mathfrak{X}, \mathbb{W})$, or

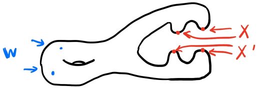
$CB\Big($  $\Big)$

- Traditional approaches take $\mathbb{W} = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ where $\mathbb{W}_i \in \mathrm{Mod}(\mathbb{V})$. But this is in sufficient for irrational VOAs.

Sewing compact Riemann surfaces

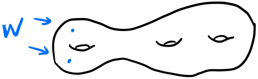
$\mathbb{X}'$ denotes the contragredient module of $\mathbb{X}$.

- Self sewing: $\mathfrak{X} =$ 

or 

,

  $\mathcal{S}\mathfrak{X} =$ 

or 

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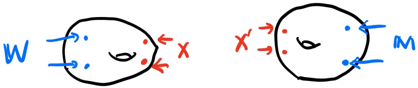
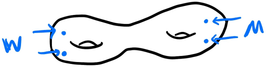
- Disjoint sewing: $\mathfrak{X} =$ 

or

, $\mathcal{S}\mathfrak{X} =$ 

  or 

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Convergence of sewing conformal blocks

Theorem (Convergence of sewing, G.-Zhang 24, arXiv: 2411.07707)

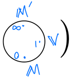
Suppose that $\psi : \mathbb{W} \otimes \mathbb{X} \otimes \mathbb{X}' \to \mathbb{C}$ *is a CB for* $\mathfrak{X}$. *Then* **Segal's sewing** $\mathcal{S}\psi : \mathbb{W} \to \mathbb{C}$ *defined by*

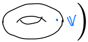
$$\mathcal{S}\psi(w) = \psi(w \otimes \underbrace{\cdot \otimes \cdot}_{\text{contraction}}).$$

is convergent and gives a CB for $\mathcal{S}\mathfrak{X}$.

- The problem of **factorization**: Does every conformal block for $\mathcal{S}\mathfrak{X}$ arise from a conformal block for $\mathfrak{X}$ via sewing?
- Answer: When $\mathbb{V}$ is rational: Yes (Damiolini-Gibney-Tarasca 19, G. 20). When $\mathbb{V}$ is irrational: Yes only for disjoint-sewing (G.-Zhang to appear).

Modular invariance as sewing-factorization

- The vertex operation for a $\mathbb{V}$-module $\mathbb{M}$ can be viewed as $\phi : \mathbb{V} \otimes \mathbb{M} \otimes \mathbb{M}' \to \mathbb{C}$ sending $v \otimes m \otimes m' \mapsto \langle Y(v,1)m, m' \rangle$. Then $\phi \in CB($ 

$)$.
- If we choose the local coordinate at $0, \infty$ to be $q^{-1}z$ and z^{-1} and sewing these two points, then $\mathcal{S}\psi : \mathbb{V} \to \mathbb{C}$ is $\mathcal{S}\psi(v) = \mathrm{Tr}_{\mathbb{M}} Y(v,1) q^{L_0}$ and belongs to $CB($ 

$)$.
- A key ingredient in Zhu's proof of modular invariance is the proof of genus-1 factorization for rational $\mathbb{V}$: any element of $CB($ 

$)$ can be written as $\mathcal{S}\psi(v) = \mathrm{Tr}_{\mathbb{M}} Y(v,1) q^{L_0}$ for some $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$.

The sewing-factorization (SF) theorem, preliminary version

- That Zhu's result fails for irrational $\mathbb{V}$ gives a typical example that factorization might not hold for self-sewing.

Theorem (The SF theorem (preliminary version), G.-Zhang, to appear)

If $\mathfrak{X} \mapsto \mathcal{S}\mathfrak{X}$ is a disjoint sewing, then any CB for $\mathcal{S}\mathfrak{X}$ can be written as $\mathcal{S}\phi$ for some CB ψ for $\mathfrak{X}$.

- In the following, I will explain how (Arike-Nagatomo's) pseudo-q-traces can be recovered from Segal's sewing.

Arike-Nagatomo's pseudo-q-traces

- Let $\mathbb{M}$ be a $\mathbb{V}$-module. We view $\mathbb{M}\otimes\mathbb{M}'$ as naturally a (non-unital) subalgebra of $\mathrm{End}(\mathbb{M})$, and write it as $\mathbb{M}\otimes\mathbb{M}' = \mathrm{End}^0(\mathbb{M})$. It has a subalgebra $\mathrm{End}^0_A(\mathbb{M}) := \mathrm{End}^0(\mathbb{M}) \cap \mathrm{End}_A(\mathbb{M})$.
- Suppose that A is a (necessarily finite dimensional) unital subalgebra of $\mathrm{End}_{\mathbb{V}}(\mathbb{M})^{\mathrm{op}}$ such that $\mathbb{M}$ is projective as a right A-module. Then the **pseudo-trace** construction (due to Hattori and Stallings 65) gives a linear map

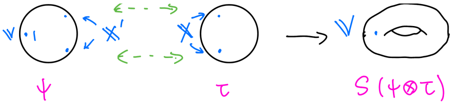
$$SLF(A) \to SLF(\mathrm{End}^0_A(\mathbb{M})) \qquad \omega \mapsto \mathrm{Tr}^\omega$$

  (where SLF =symmetric linear functionals).
- The **pseudo-q-trace** $v \in \mathbb{V} \mapsto \mathrm{Tr}^\omega Y(v,1)q^{L_0}$ belongs to $CB($ 

$)$.

From pseudo-q-traces to Segal's sewing

- Our sewing-factorization theorem shows that any element of $CB(\ \ \mathbb{V})$ can be written as Segal's sewing $\mathcal{S}(\psi \otimes \tau)$, where $\psi : \mathbb{V} \otimes \mathbb{X}' \to \mathbb{C}$ and $\tau : \mathbb{X} \to \mathbb{C}$ are CB and $\mathbb{X} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$:



How to interpret the pseudo-q-traces as Segal's sewing?

- Take $\mathbb{X} = \mathrm{End}^0_A(\mathbb{M})$ (as a $\mathbb{V}^{\otimes 2}$-submodule of $\mathrm{End}^0(\mathbb{M}) = \mathbb{M} \otimes \mathbb{M}'$), take $\tau = \mathrm{Tr}^\omega$, and notice that $v \otimes m' \otimes m \in \mathbb{V} \otimes \mathbb{M}' \otimes \mathbb{M} \mapsto \langle Y(v,1)m, m' \rangle$ descends to a conformal block $\psi : \mathbb{V} \otimes \mathrm{End}^0_A(\mathbb{M})' \to \mathbb{C}$. Then

$$\mathrm{Tr}^\omega Y(-,1)q^{L_0} = \mathcal{S}(\psi \otimes \tau)$$

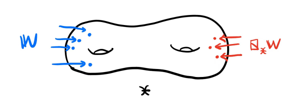
See arXiv: 2411.07707 for more discussions!

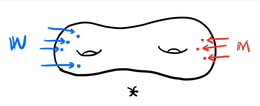
A more precise sewing-factorization theorem

We have shown the direction pseudo-q-traces $\longrightarrow$ Segal's sewing. To show the other direction in genus-1, we need a more precise version of sewing-factorization theorem that relates the dimensions of spaces of CB before sewing to those after sewing.

Dual fusion product $\boxbslash_{\mathfrak{X}}\mathbb{W}$ and fusion product $\boxtimes_{\mathfrak{X}}\mathbb{W}$

Theorem (G.-Zhang. 23, arXiv:2305.10180)

Let $\mathfrak{X}$ be $(N+L)$-pointed and $\mathbb{W}$ be a $\mathbb{V}^{\otimes N}$-module. Then there exists a $\mathbb{V}^{\otimes L}$-module $\boxbslash_{\mathfrak{X}}\mathbb{W}$ and $\gimel_{\mathfrak{X}} \in CB($ 

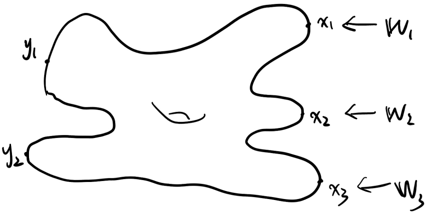
$)$ satisfying the universal property: for any $\mathbb{V}^{\otimes L}$-module $\mathbb{M}$ and any conformal block $\phi \in CB($ 

$)$, there exists a unique $T_\phi \in \mathrm{Hom}_{\mathbb{V}^{\otimes L}}(\mathbb{M}, \boxbslash_{\mathfrak{X}}\mathbb{W})$ such that $\phi = \gimel_{\mathfrak{X}} \circ (\mathbf{1} \otimes T_\phi)$.

$\boxbslash_{\mathfrak{X}}\mathbb{W}$ is called the **dual fusion product** of $\mathbb{W}$ along $\mathfrak{X}$ and $\gimel_{\mathfrak{X}}$ is called the **canonical conformal block**. The dual $\boxtimes_{\mathfrak{X}}\mathbb{W} = (\boxbslash_{\mathfrak{X}}\mathbb{W})'$ is called the **fusion product**.

Fusion products when $\mathbb{V}$ is rational

- If $\mathbb{V}$ is rational and $\mathfrak{Y}$ is



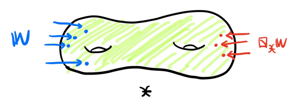
then

$$\boxbslash_{\mathfrak{Y}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3)$$

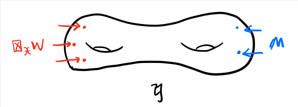
$$\simeq \bigoplus_{\mathbb{M}_1,\mathbb{M}_2 \in \mathrm{Irr}} \mathbb{M}_1 \otimes \mathbb{M}_2 \otimes CB(\mathfrak{Y}, \mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3 \otimes \mathbb{M}_1 \otimes \mathbb{M}_2)$$

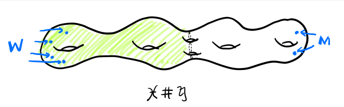
The sewing-factorization (SF) theorem

Theorem (SF theorem, G.-Zhang. to appear)

Recall that $\daleth_{\mathfrak{X}} \in CB($ 

$)$. *Then* $\phi \mapsto \mathcal{S}(\daleth_{\mathfrak{X}} \otimes \phi)$ *defines a linear isomorphism*

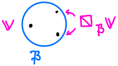
$CB($ 

$) \xrightarrow{\simeq} CB($ 

$)$

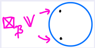
This isomorphism is called **SF isomorphism**.

The SF theorem for $CB($  $)$

- Let $\boxtimes_{\mathfrak{P}}\mathbb{V}$ be the fusion product associated to  (first discovered by Haisheng Li) which is a $\mathbb{V}^{\otimes 2}$-module. The SF Thm immediately implies:

Corollary

We have a canonical linear isomorphism

$CB($  $) \simeq CB($  $)$

- Example: If $\mathbb{V}$ is rational then $\boxtimes_{\mathfrak{P}}\mathbb{V} \simeq \bigoplus_{\mathbb{M}\in\mathrm{Irr}} \mathbb{M} \otimes_{\mathbb{C}} \mathbb{M}'$.

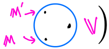
The non-unital associative algebra $\boxtimes_{\mathfrak{P}}\mathbb{V}$

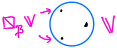
Theorem (G.-Zhang, to appear)

$\boxtimes_{\mathfrak{P}}\mathbb{V}$ has a natural (non-unital) associative $\mathbb{C}$-algebra structure that is compatible with its $\mathbb{V}^{\otimes 2}$-module structure.

- Since $\boxtimes_{\mathfrak{P}}\mathbb{V}$ is not unital, its left modules are not necessarily quotients of free modules. We say that a left $\boxtimes_{\mathfrak{P}}\mathbb{V}$-module if **quasicoherent** if it is the quotient of a free module. A quasicoherent left $\boxtimes_{\mathfrak{P}}\mathbb{V}$-module is called **coherent** if it is finitely-generated.

The functor $\mathfrak{F} : \mathrm{Mod}(\mathbb{V}) \to \mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{P}}\mathbb{V})$

- Let $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$. Recall that $\phi : \mathbb{V} \otimes \mathbb{M} \otimes \mathbb{M}' \to \mathbb{C}$ sending $v \otimes m \otimes m' \mapsto \langle Y(v,1)m, m'\rangle$ is an element of $\phi \in CB($ 

$)$.
- By the universal property for the dual fusion product, ψ is the composition of a $\mathbb{V}^{\otimes 2}$-module morphism $\mathbb{M}' \otimes \mathbb{M} \to \boxbslash_{\mathfrak{P}}\mathbb{V}$ and the canonical $\daleth_{\mathfrak{P}} \in CB($ 

$)$.
- The transpose of this $\mathbb{V}^{\otimes 2}$-module morphism is denoted by $\pi_{\mathbb{M}} : \boxtimes_{\mathfrak{P}}\mathbb{V} \to \mathbb{M} \otimes \mathbb{M}' = \mathrm{End}^0(\mathbb{M})$.

Proposition (G.-Zhang, to appear)

$\pi_{\mathbb{M}}$ is a $\mathbb{C}$-algebra homomorphism. Thus, $\mathfrak{F}(\mathbb{M}) := (\mathbb{M}, \pi_{\mathbb{M}})$ becomes a left $\boxtimes_{\mathfrak{P}}\mathbb{V}$-module which is indeed coherent. Moreover, if $\mathbb{M}$ is a projective generator of $\mathrm{Mod}(\mathbb{V})$, then $\pi_{\mathbb{M}}$ is faithful.

SLF on $\boxtimes_{\mathfrak{P}}\mathbb{V}$

Theorem (G.-Zhang, to appear)

Let $\mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{P}}\mathbb{V})$ *be the linear category of coherent left* $\boxtimes_{\mathfrak{P}}\mathbb{V}$*-modules. Then* $\mathfrak{F}:\mathrm{Mod}(\mathbb{V})\to\mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{P}}\mathbb{V})$ *is a linear equivalence.*

Theorem (G.-Zhang, to appear)

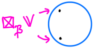
Let $\mathbb{M}$ *be a projective generator of* $\mathrm{Mod}(\mathbb{V})$*, equivalently, a projective generator of* $\mathrm{Coh}^{\mathrm{L}}(\boxtimes_{\mathfrak{P}}\mathbb{V})$*. Let* $A=\mathrm{End}_{\mathbb{V}}(\mathbb{M})^{\mathrm{op}}=\mathrm{End}_{\boxtimes_{\mathfrak{P}}\mathbb{V}}(\mathbb{M})^{\mathrm{op}}$*. Note* $\pi_{\mathbb{M}}:\boxtimes_{\mathfrak{P}}\mathbb{V}\to\mathrm{End}_A^0(\mathbb{M})$*. Then the pseudo-trace map*

$$SLF(A)\to SLF(\boxtimes_{\mathfrak{P}}\mathbb{V})\qquad \omega\mapsto\mathrm{Tr}^{\omega}\circ\pi_{\mathbb{M}}$$

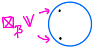
is a linear isomorphism, and its inverse is also given by the pseudo-trace construction.

The isomorphism $CB($  $) \simeq SLF\big(\mathrm{End}_{\mathbb{V}}(\mathbb{M})\big)$

- Due to the above theorem and the SF isomorphism

  $CB($  $) \simeq CB($  $)$

  and noting that

  $CB($  $) = SLF(\boxtimes_{\mathfrak{P}}\mathbb{V})$

  we obtain the following theorem conjectured by Gainutdinov-Runkel in 2016:

Theorem (G.-Zhang, to appear)

Let $\mathbb{M} \in \mathrm{Rep}(\mathbb{V})$ *be a projective generator. Then the combination of the pseudo-trace construction and the sewing-factorization isomorphism implements a linear isomorphism*

$CB($  $) \simeq SLF\big(\mathrm{End}_{\mathbb{V}}(\mathbb{M})\big)$

The isomorphism $CB\big($  $\big) \simeq SLF\big(\mathrm{End}_{\mathbb{V}}(\mathbb{M})\big)$

- Our isomorphism $CB\big($  $\big) \simeq SLF\big(\mathrm{End}_{\mathbb{V}}(\mathbb{M})\big)$ probably gives the first formula for $\dim CB\big($  $\big)$ that is both general and practical.
- For example, if $\mathbb{V} = \mathcal{W}(p)$, Adamović-Milas proved that its dimension is $3p-1$ using the modular differential equations and the analysis of the Zhu algebra of $\mathcal{W}(p)$.
- Now this result also follows from $CB\big($  $\big) \simeq SLF\big(\mathrm{End}_{\mathbb{V}}(\mathbb{M})\big)$: Nagatomo-Tsuchiya (09) showed that $\mathrm{Mod}(\mathcal{W}(p)) \simeq \mathrm{Rep}^{\mathrm{L}}(\overline{U}_q(sl_2))$ where $q = e^{\mathbf{i}\pi/p}$. Thus $\dim CB\big($  $\big) = \dim SLF(\overline{U}_q(sl_2)) = \dim Z(\overline{U}_q(sl_2))$. And Feigin-Gainutdinov-Semikhatov-Tipunin (06) computed that $\dim Z(\overline{U}_q(sl_2)) = 3p-1$.