# Pseudotraces on Almost Unital and Finite-Dimensional Algebras

BIN GUI, HAO ZHANG

**Abstract**

We introduce the notion of almost unital and finite-dimensional (AUF) algebras, which are associative $\mathbb{C}$-algebras that may be non-unital or infinite-dimensional, but have sufficiently many idempotents. We show that the pseudotrace construction, originally introduced by Hattori and Stallings for unital finite-dimensional algebras, can be generalized to AUF algebras.

Let $A$ be an AUF algebra. Suppose that $G$ is a projective generator in the category $\mathrm{Coh_L}(A)$ of finitely generated left $A$-modules that are quotients of free left $A$-modules, and let $B = \mathrm{End}_{A,-}(G)^{\mathrm{op}}$. We prove that the pseudotrace construction yields an isomorphism between the spaces of symmetric linear functionals $\mathrm{SLF}(A) \xrightarrow{\simeq} \mathrm{SLF}(B)$, and that the non-degeneracies on the two sides are equivalent.

# Contents

# 0 Introduction

In [Miy04], Miyamoto introduced the pseudo-$q$-trace construction for modules of vertex operator algebras (VOAs), generalizing the usual $q$-trace. His primary motivation was to address the failure of modular invariance for $q$-traces in the case of $C_2$-cofinite but irrational VOAs. While Zhu's theorem in [Zhu96] establishes modular invariance for $q$-traces in the rational setting, this result does not extend to the irrational case—unless $q$-traces are replaced with pseudo-$q$-traces.

Miyamoto's original approach is quite involved. Moreover, his dimension formula for the space of torus conformal blocks is expressed in terms of higher Zhu algebras. This presents two drawbacks: first, higher Zhu algebras are difficult to compute in practice; second, their connection to the VOA module category is not transparent.

Later, Arike [Ari10] and Arike-Nagatomo [AN13] introduced a simplified version of the pseudo-$q$-trace construction based on the idea of Hattori [Hat65] and Stallings [Sta65]. Below, we briefly outline this approach.

Let $A$ be an algebra, and let $B$ be a unital finite-dimensional algebra. Let $M$ be a finite-dimensional $A$-$B$ bimodule, projective as a right $B$-module. By the projectivity, there is a (finite) left coordinate system of $M$, namely, elements $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}_B(B, M)$ and $\check{\alpha}^1, \ldots, \check{\alpha}^n \in \mathrm{Hom}_M(M, B)$ satisfying $\sum_i \alpha_i \circ \check{\alpha}^i = \mathrm{id}_M$. Then the linear map

$$A \to B \qquad x \mapsto \sum_i \check{\alpha}^i \circ x \circ \alpha_i(1_B)$$

descends to a linear map $A/[A, A] \to B/[B, B]$ which is independent of the choice of the left coordinate system. Its pullback gives a linear map

$$\mathrm{SLF}(B) \to \mathrm{SLF}(A) \qquad \phi \mapsto \mathrm{Tr}^\phi \tag{0.1}$$

where $\mathrm{SLF}(A)$ is the space of symmetric linear functionals on $A$—that is, linear maps $\psi : A \to \mathbb{C}$ satisfying $\psi(xy) = \psi(yx)$ for all $x, y \in A$—and $\mathrm{SLF}(B)$ is the space of symmetric linear functionals on $B$. The above map is called the **pseudotrace construction**. Note that a typical choice of $A$ is $\mathrm{End}_B(M)$.

The pseudotrace construction is applied to the VOA setting as follows. Let $\mathbb{V}$ be an $\mathbb{N}$-graded $C_2$-cofinite VOA with central charge $c$, and let $\mathbb{M}$ be a grading-restricted generalized $\mathbb{V}$-module. Then $\mathbb{M}$ admits a decomposition $\mathbb{M} = \bigoplus_{\lambda \in \mathbb{C}} \mathbb{M}_{[\lambda]}$ into generalized eigenspaces of $L(0)$, where each $\mathbb{M}_{[\lambda]}$ is finite-dimensional. Let $\mathrm{End}_{\mathbb{V}}(\mathbb{M})$ be the algebra of linear operators on $\mathbb{M}$ commuting with the action of $\mathbb{V}$, which is necessarily unital and finite-dimensional. Let $B$ be a unital subalgebra of $\mathrm{End}_{\mathbb{V}}(\mathbb{M})^{\mathrm{op}}$. Assume that $\mathbb{M}$ is a projective right $B$-module, equivalently, each $\mathbb{M}_{[\lambda]}$ is $B$-projective. Let $\phi \in \mathrm{SLF}(B)$. Then for $v \in \mathbb{V}$, the expression

$$\mathrm{Tr}^\phi(Y_{\mathbb{M}}(v, z)q^{L(0) - \frac{c}{24}}) = \sum_{\lambda \in \mathbb{C}} \mathrm{Tr}^\phi\big(P(\lambda)Y_{\mathbb{M}}(v, z)q^{L(0) - \frac{c}{24}}P(\lambda)\big) \tag{0.2}$$

converges absolutely for $z \in \mathbb{C}$ and $0 < |q| < 1$, and defines a torus conformal block. Here, $P(\lambda)$ is the projection of $\overline{\mathbb{M}} := \prod_{\mu \in \mathbb{C}} \mathbb{M}_{[\mu]}$ onto $\mathbb{M}_{[\mu]}$. Then each $P(\lambda)Y_{\mathbb{M}}(v, z)q^{L(0) - \frac{c}{24}}P(\lambda)$

is a linear operator on $\mathbb{M}_{[\lambda]}$ commuting with the right action of $B$, and hence $\mathrm{Tr}^\phi$ can be defined on it.

Based on this formulation, in [GR19, Conjecture 5.8], Gainutdinov and Runkel proposed a conjecture that directly relates the space of torus conformal blocks of a $C_2$-cofinite VOA $\mathbb{V}$ to the linear structure of the category $\mathrm{Mod}(\mathbb{V})$ of grading-restricted generalized $\mathbb{V}$-modules. Let $\mathbb{G}$ be a projective generator in $\mathrm{Mod}(\mathbb{V})$, and let $B = \mathrm{End}_\mathbb{V}(\mathbb{G})$. Then $\mathbb{G}$ is $B$-projective. The conjecture asserts that the linear map sending each $\phi \in \mathrm{SLF}(B)$ to (0.2) defines an isomorphism between $\mathrm{SLF}(B)$ and the space of torus conformal blocks of $\mathbb{V}$.

The purpose of this note is to establish results in the theory of associative algebras that are essential for proving the Gainutdinov-Runkel conjecture. The actual resolution of the conjecture will appear in the forthcoming paper [GZ25].

Our approach stems from recognizing a structural analogy between the Gainutdinov-Runkel conjecture and a classical result in associative algebra: If $A$ is a unital finite-dimensional algebra and $M$ is a projective generator in the category of finite-dimensional left $A$-modules, then $M$ is projective over $B := \mathrm{End}_A(M)^{\mathrm{op}}$, and the pseudotrace map (0.1) is a linear isomorphism. This result was suggested in [BBG21, Sec. 2] and was proved in [Ari10] in the special case that $M = Ae$ where $e$ is a basic idempotent.

However, this classical result is not directly applicable to the Gainutdinov–Runkel conjecture. We need to generalize it to a larger class of associative algebras than unital finite-dimensional ones. In particular, we must consider infinite-dimensional algebras that can be approximated, in a certain sense, by finite-dimensional (and possibly unital) algebras. The need to consider infinite-dimensional associative algebras in the study of irrational VOAs has also been recognized in recent years from different perspectives, such as Huang's associative algebra $A^\infty(\mathbb{V})$ introduced in [Hua24], and the mode transition algebra introduced by Damiolini-Gibney-Krashen in [DGK25].

The infinite-dimensional algebra required for the proof of the Gainutdinov-Runkel conjecture is different from the above mentioned algebras. In [GZ25], we will show that the end

$$\mathbb{E} := \int_{\mathbb{M} \in \mathrm{Mod}(\mathbb{V})} \mathbb{M} \otimes_\mathbb{C} \mathbb{M}'$$

a priori an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 2})$, carries a structure of an associative $\mathbb{C}$-algebra that is compatible with its $\mathbb{V}^{\otimes 2}$-module structure. This algebra $\mathbb{E}$ is an example of an **almost unital and finite-dimensional algebra**[1] (abbreviated as **AUF algebra**), meaning that $\mathbb{E}$ has a collection of mutually orthogonal idempotents $(e_i)_{i \in \mathfrak{I}}$ such that $\mathbb{E} = \sum_{i,j \in \mathfrak{I}} e_i \mathbb{E} e_j$ where each summand $e_i \mathbb{E} e_j$ is finite-dimensional. (This sum is automatically direct.) In fact, $\mathbb{E}$ has only finitely many irreducibles. We call such algebra **strongly AUF**.

The main result of this note is a generalization of the aforementioned isomorphism between spaces of symmetric linear functionals to the setting of strongly AUF algebras. More precisely, we prove that the pseudotrace construction defines a linear isomorphism $\mathrm{SLF}(B) \simeq \mathrm{SLF}(A)$ where $A$ is strongly AUF, $M$ is a projective generator of the category $\mathrm{Coh}_\mathrm{L}(A)$ of **coherent left $A$-modules** (i.e., finitely generated left $A$-modules that are quotients of free ones), and $B = \mathrm{End}_A(M)^{\mathrm{op}}$. See Thm. 9.4. Moreover, we show that

---

[1]Here, "almost" modifies the entire phrase "unital and finite-dimensional", not just "unital".

the symmetric linear functional on $B$ is non-degenerate if and only if the corresponding functional on $A$ is non-degenerate. See Thm. 10.4.

Since the associative algebra structure on the end $\mathbb{E}$ will not be developed in this note, we present some alternative examples of AUF algebras for illustration. Let $U(\mathbb{V})$ be the universal algebra of $\mathbb{V}$ as defined in [FZ92]. Let

$$U(\mathbb{V})^{\mathrm{reg}} = \bigoplus_{\lambda,\mu \in \mathbb{C}} U(\mathbb{V})_{[\lambda,\mu]}$$

where $U(\mathbb{V})_{[\lambda,\mu]}$ is the subspace of joint generalized-eigenvectors of the left and right actions of $L(0)$ corresponding to the eigenvalues $\lambda$ and $\mu$ respectively. The following properties are shown in [MNT10]: Each $U(\mathbb{V})_{[\lambda,\mu]}$ is finite-dimensional. For each $\lambda, \mu, \nu \in \mathbb{C}$ one has

$$U(\mathbb{V})_{[\lambda,\mu]} U(\mathbb{V})_{[\mu,\nu]} \subset U(\mathbb{V})_{[\lambda,\nu]}$$

In particular, $U(\mathbb{V})^{\mathrm{reg}}$ is a subalgebra of $U(\mathbb{V})$. Moreover, there is an increasing sequence of idempotents $(1_n)_{n \in \mathbb{Z}_+}$ such that $U(\mathbb{V})^{\mathrm{reg}} = \bigcup_n 1_n U(\mathbb{V})^{\mathrm{reg}} 1_n$. (See [MNT10, Sec. 2.6].) Therefore, $U(\mathbb{V})^{\mathrm{reg}}$ is AUF, since the family of orthogonal idempotents in the definition of AUF algebras can be chosen to be $(1_{n+1} - 1_n)_{n \in \mathbb{Z}_+}$.

For a more elementary and concrete example, consider the following. Let $B$ be a unital finite-dimensional algebra. Let $M$ be a right $B$-modules. Equip $M$ with a grading

$$M = \bigoplus_{i \in \mathfrak{I}} M(i)$$

where each $M(i)$ is finite-dimensional and is preserved by the right action of $B$. Let $A$ be

$$\mathrm{End}_B^0(M) := \{T \in \mathrm{End}(M) : (Tm)b = T(mb) \text{ for all } m \in M, b \in B,$$
$$T|_{M(i)} = 0 \text{ for all by finitely many } i \in \mathfrak{I}\}$$

Then $A$ is clearly an AUF algebra, with the family of mutually orthogonal idempotents given by the projections $e_i$ of $M$ onto $M(i)$.

In fact, any strongly AUF algebra arises from such a construction. More precisely, an algebra is strongly AUF if and only if it is isomorphic to some $\mathrm{End}_B^0(M)$, where $M$ and $B$ satisfy the above conditions and, in addition, $M$ is a projective generator in the category of right $B$-modules. See Thm. 11.9.

Note that the relationship between $\mathrm{End}_B^0(M)$ and $C_2$-cofinite VOAs is straightforward: If $\mathbb{M} \in \mathrm{Mod}(\mathbb{V})$ is equipped with the grading $\bigoplus_{\lambda \in \mathbb{C}} \mathbb{M}_{[\lambda]}$ given by the generalized eigenspaces of $L(0)$, and if $B$ is a unital subalgebra of $\mathrm{End}_{\mathbb{V}}(\mathbb{M})^{\mathrm{op}}$ such that $\mathbb{M}$ is projective as a right $B$-module, then each $P(\lambda) Y_{\mathbb{M}}(v,z) q^{L(0) - \frac{c}{24}} P(\lambda)$ appearing in (0.2) lies $\mathrm{End}_B^0(\mathbb{M})$. Therefore, the main result of this note on pseudotraces (Thm. 10.4) can be applied to $C_2$-cofinite VOAs. Details of this application will be presented in [GZ25].

# 1 Preliminaries

Throughout this note, algebras are associative, not necessarily unital, and over $\mathbb{C}$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, \dots\}$. For any vector spaces $V, W$, we let $\mathrm{Hom}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)$ be the space of linear maps $V \to W$, and let $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

Let $A$ be an algebra. Its opposite algebra is denoted by $A^{\mathrm{op}}$. If $M, N$ are left (resp. right) $A$-modules, we let $\mathrm{Hom}_{A,-}(M, N)$ (resp. $\mathrm{Hom}_{-,A}(M, N)$) be the space of linear maps $M \to N$ intertwining the left (resp. right) actions of $A$.

An **idempotent** $e \in A$ is an element satisfing $e^2 = e$. If $e, f \in A$ are idempotent, we write $e \leqslant f$ if $ef = fe = e$. Equivalently, $f = e + e'$ where $e' \in A$ is an idempotent **orthogonal** to $e$ (i.e. $ee' = e'e = 0$). We say that a nonzero idempotent $e$ is **primitive** if the only idempotent $f$ satisfying $f \leqslant e$ is $f = 0$ and $f = e$.

In this section, we review some well-known facts about associative algebras. Since, unlike many references, our algebras are not assumed to be unital, we include proofs for the reader's convenience.

**Definition 1.1.** Let $u, v \in A$. We say that $(u, v)$ is pair of **partial isometries in** $A$ if the following are true:

(a) $p := vu$ and $q := uv$ are idempotents.

(b) $u \in qAp$ and $v \in pAq$.

In this case, we also say that $u$ is a partial isometry from $p$ to $q$, and that $v$ is a partial isometry from $q$ to $p$. We say that two idempotents are **equivalent** if there are partial isometries between them.

**Proposition 1.2.** *Let $e, f \in A$ be idempotents. Then an element of $\mathrm{Hom}_{A,-}(Ae, Af)$ is precisely the right multiplication of an element of $eAf$. In particular, we have an algebra isomorphism*

$$\mathrm{End}_{A,-}(Ae)^{\mathrm{op}} \simeq eAe$$

*Proof.* Clearly the right multiplication by some element of $eAf$ yields an element of $\mathrm{Hom}_{A,-}(Ae, Af)$. Conversely, suppose that $T \in \mathrm{Hom}_{A,-}(Ae, Af)$. Let $x = T(e)$, which belongs to $Af$. Since $ex = eT(e) = T(ee) = T(e) = x$, we see that $x \in eAf$. For each $y \in A$, we have $T(ye) = yT(e) = yx = yex$, which shows that $T$ is the right multiplication by $x$. $\square$

**Corollary 1.3.** *Let $e, f$ be idempotents in $A$. The following are equivalent:*

*(1) $Ae \simeq Af$ as left $A$-modules.*

*(2) There is a partial isometry from $e$ to $f$.*

*Proof.* (1)$\Rightarrow$(2): Let $T \in \mathrm{Hom}_{A,-}(Ae, Af)$ be an isomorphism with inverse $T^{-1} \in \mathrm{Hom}_{A,e}(Af, Ae)$. By Prop. 1.2, $T$ and $T^{-1}$ are realized by the right multiplications of $u \in eAf$ and $v \in fAe$ respectively. Since $TT^{-1} = 1_{Af}$, we have $vu = f$. Since $T^{-1}T = 1_{Ae}$, we have $uv = e$.

(2)$\Rightarrow$(1): Let $u \in eAf$ and $v \in fAe$ such that $uv = e, vu = f$. Then the right multiplication of $u$ on $Ae$ has inverse being the right multiplication of $v$. So $Ae \simeq Af$. $\square$

**Corollary 1.4.** *Let $e \in A$ be an idempotent. Let $M$ be a left $A$-submodule of $Ae$. The following are equivalent.*

*(1) $M$ is a direct summand of $Ae$.*

(2) $M = Af$ for some idempotent $f \leqslant e$ in $A$.

*Proof.* (2)$\Rightarrow$(1): $Ae = Af \oplus Af'$ where $f' = e - f$ is an idempotent.

(1)$\Rightarrow$(2): Let $Ae = M \oplus N$. Let $\varphi : Ae \to Ae$ be the projection on $M$ vanishing on $N$. Then $\varphi \in \operatorname{End}(Ae)$. By Prop. 1.2, $\varphi$ is the right multiplication by some $f \in eAe$. Since $\varphi \circ \varphi = \varphi$, clearly $f^2 = f$. Moreover, $M = \varphi(Ae) = (Ae)f = Af$. $\qquad\square$

**Corollary 1.5.** *Let $e \in A$ be an idempotent. The following are equivalent.*

(1) *$Ae$ is an indecomposible left $A$-module.*

(2) *$e$ is primitive.*

*Proof.* This follows immediately from Cor. 1.4. $\qquad\square$

**Lemma 1.6.** *Let $M$ be a nonzero finitely-generated left $A$-module. Then $M$ has a maximal proper left $A$-submodule $N$. Consequently, there is an epimorphism of $M$ onto an irreducible module.*

*Proof.* Let $\xi_1, \ldots, \xi_n$ generate $M$. Without loss of generality, we assume that $\xi_1$ does not belong to the submodule $N_0$ generated by $\xi_2, \ldots, \xi_n$. By Zorn's lemma, there is a left submodule $N \leqslant M$ maximal with respect to the property that $N_0 \subset N$ and $\xi_1 \neq N$. Let us prove that $N$ is a maximal proper submodule. Let $N < K \leqslant M$. Then by the maximality of $N$ we must have $\xi_1 \in K$. So $\xi_1, \ldots, \xi_n \in M$, and hence $K = M$. So $K$ is not proper. $\qquad\square$

# 2 Almost unital algebras

In this section, we introduce the notion of almost unital algebras, which is weaker than being almost unital and finite-dimensional.

**Definition 2.1.** We say that an algebra $A$ is **almost unital** if the following conditions are satisfied:

(a) For each $x \in A$, there is an idempotent $e \in A$ such that $x = exe$.

(b) For any finitely many idempotents $e_1, \ldots, e_n \in A$ there exists an idempotent $e \in A$ such that $e_i \leqslant e$ for all $1 \leqslant i \leqslant n$.

Throughout this section, unless otherwise stated, $A$ is assumed to be almost unital.

**Definition 2.2.** We say that a left $A$-module $M$ is **quasicoherent** if one of the following equivalent conditions hold:

(1) For each $\xi \in M$ we have $\xi \in A\xi$.

(2) For each $\xi \in M$ there exists an idempotent $e \in A$ such that $\xi = e\xi$.

(3) $M$ is a quotient module of $\bigoplus_{i \in I} Ae_i$ where each $e_i \in A$ is an idempotent.

(4) $M$ is a quotient module of a free left $A$-module $A^{\oplus I}$.

The category of quasicoherent left $A$-modules is denoted by $\mathbf{QCoh_L}(A)$.

*Proof of equivalence.* (1)⇒(2): For each $\xi \in M$, since $\xi \in A\xi$, we have $\xi = a\xi$ for some $a \in A$. Choose idempotent $e \in A$ such that $a \in eAe$. Then $e\xi = ea\xi = a\xi = \xi$.

(2)⇒(1): Obvious.

(2)⇒(3): For each $\xi \in M$, let $e_\xi \in A$ be an idempotent such that $e_\xi\xi = \xi$. Then we have a morphism $\bigoplus_{\xi \in M} Ae_\xi \to M$ whose restriction to $Ae_\xi$ sends each $a \in Ae_\xi$ to $a\xi$. Then $\xi = e_\xi\xi$ implies that $\xi \in Ae_\xi\xi$, and hence $\xi$ is in the range of this morphism. So this morphism is surjective.

(3)⇒(4): This is obvious, since we have an epimorphism $A \to Ae_i$ and hence an epimorphim $\bigoplus_{i \in I} A \to \bigoplus_{i \in I} Ae_i$.

(4)⇒(2): It suffices to show that $A^{\oplus I}$ satisfies the requirement of (2). Choose $\xi = (a_i)_{i \in I} \in A^{\oplus I}$. Then there are only finitely many $i \in I$ such that $a_i \neq 0$. Since $A$ is almost unital, there exist idempotents $e_i \in A$ (where $i \in I$) such that $a_i = e_i a_i e_i$ for all $i \in I$. (If $a_i = 0$, then we choose $e_i = 0$). Choose idempotent $e \in A$ such that $e_i \leqslant e$ for all $i \in I$. Then $\xi = e\xi$. $\qquad\square$

**Definition 2.3.** A left $A$-module $M$ is called **coherent** if it is quasicoherent and finitely-generated. By the above proof of equivalence, it is clear that $M$ is coherent iff $M$ is a quotient of $\bigoplus_{i \in I} Ae_i$ where $I$ is a *finite* index set and $e_i \in A$ is an idempotent. The category of coherent left $A$ modules is denoted by $\mathbf{Coh_L}(A)$.

However, note that a coherent left $A$-module is not necessarily a quotient of $A^{\oplus n}$ where $n \in \mathbb{Z}_+$. Indeed, $A$ is not necessarily finitely generated as a left $A$-module.

**Remark 2.4.** If $M \in \mathrm{QCoh_L}(A)$, then every submodule of $M$ is quasicoherent, and every quotient module of $M$ is quasicoherent $M$. However, if $M \in \mathrm{Coh_L}(A)$, then a submodule of $M$ is not known to be coherent. Thus, $\mathrm{QCoh_L}(A)$ is an abelian category, while $\mathrm{Coh_L}(A)$ is not known to be abelian.
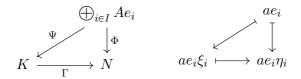
**Proposition 2.5.** *Let $M \in \mathrm{QCoh_L}(A)$. The following are equivalent.*

*(1) $M$ is projective in the category of left $A$-modules.*

*(2) $M$ is projective in $\mathrm{QCoh_L}(A)$.*

*(3) $M$ is a direct summand of $\bigoplus_{i \in I} Ae_i$ for some index set $I$ and each $e_i \in A$ is an idempotent.*

*Proof.* (3)⇒(1): It is well-known that a direct summand of a projective module is projective. Thus, it suffices to prove that $\bigoplus_{i \in I} Ae_i$ is projective. Let $\Phi : \bigoplus_{i \in I} Ae_i \to N$ be an epimorphism where $N$ is a left $A$-module. Let $\Gamma : K \to N$ be an epimorphism. Let

$$\eta_i = \Phi(e_i)$$

Since $\Gamma$ is surjective, there is $\xi_i \in K$ such that $\Gamma(\xi_i) = \eta_i$. Define $\Psi : \bigoplus_{i \in I} Ae_i \to K$ to be the morphism sending each $ae_i \in Ae_i$ to $ae_i\xi$. Then the following commute:

$$
\begin{array}{ccc}
 & \bigoplus_{i \in I} Ae_i & \\
{\scriptstyle \Psi} \swarrow & \downarrow {\scriptstyle \Phi} & \\
K \xrightarrow{\;\;\Gamma\;\;} & N &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & ae_i \\
 & \swarrow & \uparrow \\
ae_i\xi_i & \longmapsto & ae_i\eta_i
\end{array}
$$

Note that $\mapsto$ holds since $\Gamma(ae_i\xi_i) = ae_i\Gamma(\xi_i) = ae_i\eta_i$, and $\downarrow$ holds since $\Phi(ae_i) = \Phi(ae_ie_i) = ae_i\Phi(e_i) = ae_i\eta_i$.

(1)⇒(2): Obvious.

(2)⇒(3): Choose an epimorphism $\bigoplus_{i\in I} Ae_i \to M$, which splits because $M$ is projective. So $M$ is a direct summand of $\bigoplus_{i\in I} Ae_i$. $\qquad\square$

**Proposition 2.6.** *Let $M \in \mathrm{Coh}_L(A)$. The following are equivalent.*

*(1) $M$ is projective in the category of left $A$-modules.*

*(2) $M$ is projective in $\mathrm{QCoh}_L(A)$.*

*(3) $M$ is projective in $\mathrm{Coh}_L(A)$.*

*(4) $M$ is a direct summand of $\bigoplus_{i\in I} Ae_i$ for some finite index set $I$ and each $e_i \in A$ is an idempotent.*

Therefore, there is no ambiguity when talking about projective coherent left $A$-modules.

*Proof.* Clearly we have (1)⇒(2) and (2)⇒(3). By Prop. 2.5 we have (4)⇒(1). Assume (3). By Rem. 2.4, there is an epimorphism $\bigoplus_{i\in I} Ae_i \to M$ such that $I$ is finite, and that it splits (because $M$ is projective in $\mathrm{Coh}_L(A)$). So (4) is true. $\qquad\square$

**Remark 2.7.** If $M \in \mathrm{QCoh}_L(A)$, clearly $M$ is irreducible in $\mathrm{QCoh}_L(A)$ iff $M$ is irreducible in the category of left $A$-modules; in this case we say that $M$ is **irreducible**. Note that even if $M \in \mathrm{Coh}_L(A)$, its irreducibility is understood as in $\mathrm{QCoh}_L(A)$ but not as in $\mathrm{Coh}_L(A)$.

**Proposition 2.8.** *Let $M$ be a left $A$-module. The following are equivalent.*

*(1) $M \in \mathrm{QCoh}_L(A)$ and $M$ is irreducible.*

*(2) $M \simeq Ae/N$ where $e \in A$ is an idempotent and $N$ is a maximal (proper) left ideal of $Ae$.*

*(3) $M \simeq A/N$ where $N$ is a maximal proper left $A$-submodule of $A$.*

*Proof.* (1)⇒(2): Let $M \in \mathrm{QCoh}_L(A)$ be irreducible. By Def. 2.2, $M$ has an epimorphism $\Phi$ from some $\bigoplus_i Ae_i$ where $e_i \in A$ is an idempotent. The restriction of $\Phi$ to some $Ae_i$ must be nonzero, and hence must be surjective (since $M$ is irreducible). It follows that $M$ has an epimorphism $\Psi$ from $Ae_i$. Then $N = \mathrm{Ker}\Psi$ is a maximal proper left $A$-submodule of $Ae_i$, and $M \simeq Ae_i/N$.

(1)⇒(3): In the above proof, $M$ also has an epimorphism from $\bigoplus_i A$ (since we have an epimorphism $A \to Ae_i$). Thus, replacing $Ae_i$ with $A_i$ in the above proof, we are done.

(2),(3)⇒(1): Clearly $M$ is irreducible. That $M \in \mathrm{QCoh}_L(A)$ follows from Def. 2.2. $\qquad\square$

# 3 Projective covers

Let $A$ be an algebra, not necessarily almost unital. In this section, we recall some basic facts about projective covers. When $A$ is unital, these results can be found in [AF92], for example. In the non-unital case, one can reduce to the unital setting by considering the unitalization of $A$. For the reader's convenience, we include complete proofs.

## 3.1 Basic facts

**Definition 3.1.** Let $M$ be a left $A$-module. A left $A$-submodule $K \leqslant M$ is called **superfluous**, if for any left $A$-submodule $L \leqslant M$ satisfying $K + L = M$ we must have $L = M$.

**Remark 3.2.** Obviously, we have an equivalent description of superfluous submodules: Let $\pi : M \to M/K$ be the quotient map. Then $K \leqslant M$ is superfluous iff for any morphism of left $A$-modules $\varphi : N \to M$ such that $\pi \circ \varphi : N \to M/K$ is surjective, it must be true that $\varphi$ is surjective.

**Definition 3.3.** Let $M$ be a left $A$-module. A **projective cover** of $M$ denotes a left $A$-module epimorphism $\varphi : P \twoheadrightarrow M$ where $P$ is a projective left $A$-module, and $\mathrm{Ker}\,\varphi$ is superfluous in $P$.

The following property says that among the projective modules that have epimorphisms to $M$, the projective cover is the smallest one in the sense of direct summand.

**Proposition 3.4.** *Let $\varphi : P \to M$ be a projective cover of $M$. Let $\psi : Q \to M$ be an epimorphism where $Q$ is projective. Then there is a morphism $\alpha : Q \to P$ such that the following diagram commutes.*

$$
\begin{array}{ccc}
 & & P \\
 & {\scriptstyle\alpha}\nearrow & \downarrow{\scriptstyle\varphi} \\
Q & \xrightarrow{\ \psi\ } & M
\end{array}
\tag{3.1}
$$

*Moreover, for any such $\alpha$, there is a left $A$-submodule $P' \leqslant Q$ such that $Q = \ker \alpha \oplus P'$ and that $\alpha|_{P'} : P' \xrightarrow{\simeq} P$ is an isomorphism.*

By setting $L = \ker \alpha$, it follows that (3.1) is equivalent to

$$
\begin{array}{ccc}
 & & P \\
 & {\scriptstyle 0\oplus\mathrm{id}_P}\nearrow & \downarrow{\scriptstyle\varphi} \\
L \oplus P & \xrightarrow{\ 0\oplus\varphi\ } & M
\end{array}
\tag{3.2}
$$

*Proof.* The existence of $\alpha$ follows from that $Q$ is projective and that $\varphi$ is an epimorphism. Moreover, since $\ker \varphi$ is superfluous and $\varphi \circ \alpha$ is surjective, by Rem. 3.2, $\alpha$ is surjective. Therefore, since $P$ is projective, the epimorphism $\alpha$ splits, i.e., there is a morphism $\beta : P \to Q$ such that $\alpha \circ \beta : P \to P$ equals $\mathrm{id}_P$. One sees that $P' = \beta(P)$ fulfills the requirement. $\qquad\square$

It follows that projective covers are unique up to isomorphisms:

**Corollary 3.5.** *Let $M$ be a left $A$-module with projective covers $\varphi : P \to M$ and $\psi : Q \to M$. Then there exists an isomorphism $\alpha : Q \to P$ such that (3.1) commutes.*

*Proof.* By Prop. 3.5, there exists $\alpha$ such that (3.1) commutes. It remains to show that $\alpha$ is an isomorphism. We assume that (3.1) equals (3.2). Since $0\oplus\varphi : L\oplus P \to M$ is a projective cover, $L + \ker(P) = \ker(0 \oplus \varphi)$ is superfluous, and hence $L$ is superfluous. Thus, since $L + P$ equals $Q = L \oplus P$, we must have $Q = P$ and hence $L = 0$. So $\alpha = 0 \oplus \mathrm{id}_P$ is an isomorphism. $\qquad\square$

## 3.2 Projective covers of irreducibles

**Proposition 3.6.** *Suppose that $\varphi : P \to M$ is a projective cover of an irreducible left $A$-module $M$. Then $P$ is indecomposible.*

*Proof.* Suppose that $P = P' \oplus P''$. Then one of $\varphi|_{P'}, \varphi|_{P''}$ (say $\varphi|_{P'}$) is nonzero. Since $M$ is irreducible, $\varphi|_{P'} : P' \to M$ must be surjective. So the map $P' \hookrightarrow P' \oplus P'' \xrightarrow{\varphi} M$ is surjective. Since $\ker \varphi$ is superfluous, by Rem. 3.2, $P' \hookrightarrow P' \oplus P''$ is surjective, and hence $P'' = 0$. $\qquad\square$
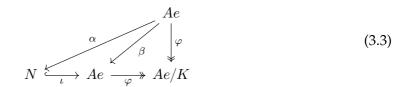
**Theorem 3.7.** *Let $e \in A$ be a primitive idempotent satisfying*

$$\dim eAe < +\infty$$

*Let $K$ be any proper left $A$-submodule of $Ae$. Then $K$ is superfluous. In other words, the quotient map $Ae \to Ae/K$ is the projective cover of $Ae/K$.*

*Proof.* Step 1. Let $\varphi : Ae \to Ae/K$ be the quotient map. Let $L$ be a submodule of $Ae$. Assume that $N + K = Ae$; in other words, if we let $\iota : N \hookrightarrow Ae$ be the inclusion, then $\varphi \circ \iota : N \to Ae/K$ is surjective. Our goal is to show that $N = Ae$.

Since $Ae$ is projective and $\varphi \circ \iota$ is surjective, there is a morphism $\alpha : Ae \to N$ such that $\varphi = \varphi \circ \iota \circ \alpha$. Let $\beta = \iota \circ \alpha$. Then the following diagram commutes:

$$
\begin{array}{c}
\\
\end{array}
\tag{3.3}
$$

To prove that $\iota$ is surjective, it suffices to show that $\beta$ is surjective.

Step 2. Suppose that $\beta$ is not surjective. Let us find a contradiction. Since $\beta \in \mathrm{End}_{A,-}(Ae)$, by Prop. 1.2, $\beta$ is the right multiplication by some $x \in eAe$. Let $R_x : eAe \to eAe$ be the right multiplication of $x$ on $eAe$. Then $R_x$ is not surjective. Otherwise, there exists $a \in A$ such that $R_x(eae) = e$, i.e., $eaex = e$. Then for each $b \in A$, we have $be = beaex = \beta(beae)$, contradicting the fact that $\beta$ is not surjective.

It is well-known that if $T$ is a linear operator on a finite-dimensional $\mathbb{C}$-vector space $W$, then $W$ is the direct sum of generalized eigenspaces of $T$, and the projection operator of $W$ onto each generalized eigenspace is a polynomial of $T$. Therefore, $R_x$ has only one eigenvalue. Otherwise, there is a polynomial $p$ such that $p(R_x) = R_{p(x)}$ is the projection of $eAe$ onto a proper subspace, and hence $p(x)$ is an idempotent in $eAe$ not equal to $0$ or $e$. This is impossible, since $e$ is assumed to be primitive.

Therefore, $R_x$ has a unique eigenvalue, which must be $0$ since $R_x$ is not surjective. By linear algebra, $R_x$ is nilpotent. Since $R_{x^n} = (R_x)^n$, it follows that $x$ is nilpotent, and hence $\beta$ is nilpotent. By (3.3), we have $\varphi = \varphi \circ \beta$, and hence $\varphi = \varphi \circ \beta = \varphi \circ \beta^2 = \varphi \circ \beta^3 = \cdots = 0$. This contradicts the fact that $\varphi$ is a surjection onto a nonzero module, finishing the proof. $\qquad\square$

**Corollary 3.8.** *Let $e \in A$ be a primitive idempotent satisfying $\dim eAe < +\infty$. Then $Ae$ has a unique proper maximal left $A$-submodule, denoted by* **rad($Ae$)**.

It follows from Thm. 3.7 that $Ae$ is the projective cover of the irreducible $Ae/\mathrm{rad}(Ae)$.

*Proof.* By Lem. 1.6, $Ae$ has at least one proper maximal left $A$-submodule. Suppose that $K \neq L$ are proper maximal left $A$-submodules of $M$. By the maximality, we have $K + L = M$. By Thm. 3.7, $L$ is superfluous. So $K = M$, impossible. □

## 4  Left pseudotraces

Let $A, B$ be algebras such that $B$ is unital. Fix an $A$-$B$ bimodule $M$. We do not assume that $M_B$ is unital, i.e., $1_B \in B$ acts as the identity on $M$.

**Definition 4.1.** A **left coordinate system** of $M$ denotes a collection of morphisms

$$\alpha_i \in \mathrm{Hom}_{-,B}(B, M) \qquad \breve{\alpha}^i \in \mathrm{Hom}_{-,B}(M, B) \tag{4.1}$$

where $i$ runs through an index set $I$ such that the following conditions hold:

(a) For each $\xi \in M$, we have $\breve{\alpha}^i(\xi) = 0$ for all but finitely many $i \in I$, and $\sum_{i \in I} \alpha_i \circ \breve{\alpha}^i(\xi) = \xi$.

(b) For each $x \in A$ (viewed as an element of $\mathrm{End}_{-,B}(M)$), we have $x \circ \alpha_i = 0$ and $\breve{\alpha}^i \circ x = 0$ for all but finitely many $i \in I$.

**Remark 4.2.** $M$ is a projective right $B$-module iff there exists $(\alpha_i, \breve{\alpha}^i)_{i \in I}$ of the form (4.1) satisfying condition (a).

*Proof.* Suppose that there exists $(\alpha_i, \breve{\alpha}^i)_{i \in I}$ such that (a) holds. Define morphisms of right $B$-modules

$$\Phi : B^{\oplus I} \to M \qquad \oplus_i b_i \mapsto \sum_i \alpha_i(b_i)$$

$$\Psi : M \to B^{\oplus I} \qquad \xi \mapsto \oplus_i \breve{\alpha}^i(\xi)$$

Then (a) implies that $\Phi \circ \Psi = \mathrm{id}_M$. Thus, $M$ is a direct summand of $B^{\oplus I}$, and hence is projective as a right $B$-module.

Conversely, assume $M$ is projective as a right $B$-module. Then we have an epimorphism $\Phi : B^{\oplus I} \to M$ and a morphism $\Psi : M \to B^{\oplus I}$ such that $\Phi \circ \Psi = \mathrm{id}_M$. For each $i \in I$, let $\iota_i : B \to B^{\oplus I}$ be the inclusion map of $B$ into the $i$-th direct summand, and $\pi_i : B^{\oplus I} \to B$ be the projection map onto the $i$-th direct summand. Set

$$\alpha_i = \Phi \circ \iota_i \qquad \breve{\alpha}^i = \pi_i \circ \Psi$$

Then $(\alpha_i, \breve{\alpha}^i)_{i \in I}$ satisfies (a). □

**Definition 4.3.** Assume that $M$ has a left coordinate system $(\alpha_i, \breve{\alpha}^i)_{i \in I}$. Define the **$B$-trace function**

$$\mathrm{Tr}^B : A \to B/[B,B] \qquad x \mapsto \sum_{i \in I} \breve{\alpha}^i \circ x \circ \alpha_i$$

where the RHS, originally an element of $\mathrm{End}_{-,B}(B) \simeq B,$[2] is descended to $B/[B,B]$.

**Lemma 4.4.** *The definition of $\mathrm{Tr}^B$ is independent of the choice of left coordinate systems.*

*Proof.* Suppose that $(\beta_j, \breve{\beta}^j)_{j \in J}$ is another left coordinate system of the $A$-$B$ bimodule $M$. Let $I_x \subset I$ and $J_x \subset J$ be finite sets such that $\breve{\alpha}^i \circ x = 0, x \circ \alpha_i = 0$ for any $i \in I \backslash I_x$, and that $\breve{\beta}^j \circ x = 0, x \circ \beta_j = 0$ for any $j \in J \backslash J_x$. Then

$$\sum_{i \in I_x} \breve{\alpha}^i \circ x \circ \alpha_i = \sum_{i \in I_x, j \in J} \breve{\alpha}^i \circ x \circ \beta_j \circ \breve{\beta}^j \circ \alpha_i = \sum_{i \in I_x, j \in J_x} \breve{\alpha}^i \circ x \circ \beta_j \circ \breve{\beta}^j \circ \alpha_i$$

Since each $\breve{\alpha}^i \circ x \circ \beta_j$ and $\breve{\beta}^j \circ \alpha_i$ are in $\mathrm{End}_{-,B}(B) \simeq B$, the RHS above equals

$$\sum_{i \in I_x, j \in J_x} \breve{\beta}^j \circ \alpha_i \circ \breve{\alpha}^i \circ x \circ \beta_j = \sum_{j \in J_x} \breve{\beta}^j \circ x \circ \beta_j$$

in $B/[B,B]$. $\qquad \square$

**Proposition 4.5.** $\mathrm{Tr}^B$ *is **symmetric**, i.e., $\mathrm{Tr}^B(xy) = \mathrm{Tr}^B(yx)$ for any $x, y \in A$. Therefore, $\mathrm{Tr}^B$ descends to a linear map $A/[A,A] \to B/[B,B]$.*

*Proof.* Let $x, y \in A$. Let $I_0 \subset I$ be a finite set such that $\breve{\alpha}^i \circ x = \breve{\alpha}^i \circ y = 0$ and $x \circ \alpha_i = y \circ \alpha_i = 0$ for all $i \in I \backslash I_0$. Then

$$\mathrm{Tr}^B(xy) = \sum_{i \in I_0} \breve{\alpha}^i \circ x \circ y \circ \alpha_i = \sum_{i,j \in I_0} \breve{\alpha}^i \circ x \circ \alpha_j \circ \breve{\alpha}^j \circ y \circ \alpha_i$$

and similarly

$$\mathrm{Tr}^B(yx) = \sum_{i,j \in I_0} \breve{\alpha}^j \circ y \circ \alpha_i \circ \breve{\alpha}^i \circ x \circ \alpha_j$$

The two RHS's are equal in $B/[B,B]$, noting that $\breve{\alpha}^i \circ x \circ \alpha_j$ and $\breve{\alpha}^j \circ y \circ \alpha_i$ are both in $\mathrm{End}_{-,B}(B) \simeq B$. $\qquad \square$

**Definition 4.6.** Let $\phi : B \to \mathbb{C}$ be a **symmetric linear functional (SLF)**, i.e., a linear map satisfying $\phi(ab) = \phi(ba)$ for all $a, b \in B$. The (left) **pseudotrace** associated to $\phi$ (and $M$), denoted by $\mathbf{Tr}^\phi$, is defined to be

$$\mathrm{Tr}^\phi = \phi \circ \mathrm{Tr}^B : A \to \mathbb{C} \tag{4.2}$$

It is an SLF on $A$.

Thus, for each $x \in A$ we have

$$\mathrm{Tr}^\phi(x) = \sum_{i \in I} \phi(\breve{\alpha}^i \circ x \circ \alpha_i(1_B)) \tag{4.3}$$

---

[2]This isomorphism relies on the fact that $B$ is unital.

# 5 AUF algebras and projective covers of irreducibles

**Definition 5.1.** An algebra $A$ is called **almost unital and finite-dimensional (AUF)** if there is a family of mutually orthogonal idempotents $(e_i)_{i\in\mathfrak{I}}$ such that the following conditions hold:

(a) For each $i, j \in \mathfrak{I}$ we have $\dim e_i A e_j < +\infty$.

(b) $A = \sum_{i,j\in\mathfrak{I}} e_i A e_j$. (That is, for each $x \in A$ one can find a finite subset $I \subset \mathfrak{I}$ and a collection $(x_{i,j})_{i,j\in\mathfrak{I}}$ such that $x = \sum_{i,j\in I} e_i x_{i,j} e_j$.)

Note that (b) automatically impies $A = \bigoplus_{i,j\in\mathfrak{I}} e_i A e_j$.

It is illuminating to view an element $x \in A$ as an $\mathfrak{I} \times \mathfrak{I}$ matrix whose $(i,j)$-entry is $e_i x e_j$.

**Remark 5.2.** Each AUF algebra $A$ is almost unital.

*Proof.* For each $x_1, \cdots, x_n \in A$, we can find a subset $I_0 \subset \mathfrak{I}$ such that $x \in e'Ae'$, where $e' = \sum_{i\in I_0} e_i$. By choose $n = 1$ and $x_1 = x \in A$, we see $x = e'xe'$. By choosing idempotents $x_i = e_i \in A$, we see $e_i \leqslant e'$ for all $1 \leqslant i \leqslant n$. $\square$

**Lemma 5.3.** *In Def. 5.1, one can assume moreover that each $e_i$ is primitive (in $A$).*

*Proof.* Let $(e_i)_{i\in\mathfrak{I}}$ be as in Def. 5.1. For each $i \in \mathfrak{I}$, since $e_i A e_i$ is a finite-dimensional left $e_i A e_i$-module, it is a finite direct sum of indecomposable left $e_i A e_i$-submodules. By Cor. 1.4 and 1.5, we have a finite direct sum $e_i A e_i = \bigoplus_{k\in\mathfrak{K}_i} e_i A f_{i,k}$ where $(f_{i,k})_{k\in\mathfrak{K}_i}$ is a finite family of mutually orthogonal idempotents in $e_i A e_i$, that $\sum_k f_{i,k} = e_i$, and that each $f_{i,k}$ is primitive in $e_i A e_i$. Clearly $f_{i,k}$ is also primitive in $A$. Replacing $(e_i)_{i\in\mathfrak{I}}$ by $(f_{i,k})_{i\in\mathfrak{I},k\in\mathfrak{K}_i}$ does the job. $\square$

In the remaining part of this section, we always assume that $A$ is AUF.

**Remark 5.4.** For each idempotents $e, f \in A$, we have

$$\dim eAf < +\infty$$

Indeed, one can find a finite set $I_0 \subset \mathfrak{I}$ such that $e, f \in e'Ae'$ where $e' = \sum_{i\in I_0} e_i$. Then $\dim e'Ae' < +\infty$, and hence $\dim eAf < +\infty$.

It follows that each idempotent $e \in A$ has a (finite) orthogonal primitive decomposition $e = \varepsilon_1 + \cdots + \varepsilon_n$. This follows from a decomposition of the finite-dimensional left $eAe$-module $eAe$ into indecomposable submodules. $\square$

Recall Rem. 2.7 about irreducibility.

**Theorem 5.5.** *The following are true.*

1. *For each primitive idempotent $e \in A$, let $\mathrm{rad}(Ae)$ be the unique proper maximal left submodule of $Ae$ (cf. Cor. 3.8). Then $Ae \to Ae/\mathrm{rad}(Ae)$ gives a projective cover of the irreducible coherent module $Ae/\mathrm{rad}(Ae)$.*

2. *Any irreducible $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$ is isomorphic to $Ae/\mathrm{rad}(Ae)$ for some primitive idempotent $e \in A$.*

3. *Let $e, f$ be primitive idempotents. Then the following are equivalent:*

   *(1) $Ae \simeq Af$ as left A-modules.*

   *(2) $Ae/\mathrm{rad}(Ae) \simeq Af/\mathrm{rad}(Af)$ as left A-modules.*

   *(3) $e \simeq f$, i.e., there is a partial isometry (in A) from $e$ to $f$.*

*Proof.* Part 1 was already proved, cf. Thm. 3.7. (Note that Thm. 3.7 and its consequences are applicable since $\dim eAe < +\infty$ by Rem. 5.4.)

Part 2: By Prop. 2.8, $M$ has an epimorphism $\Psi$ from $A$. Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1 such that each $e_i$ is primitive (Lem. 5.3). Then $A \simeq \bigoplus_i Ae_i$ as left $A$-modules. The restriction of $\Psi$ to some $Ae_i$ must be nonzero, and hence must be surjective. Therefore $M \simeq Ae_i/\mathrm{rad}(Ae_i)$.

Part 3: (1)$\Rightarrow$(2) is obvious. (2)$\Rightarrow$(1) follows from the uniqueness of projective covers (Cor. 3.5). (1)$\Leftrightarrow$(3) follows from Cor. 1.3. $\qquad\square$

**Corollary 5.6.** *Let $P \in \mathrm{Coh}_{\mathrm{L}}(A)$. The following are equivalent.*

*(1) P is projective and indecomposable.*

*(2) P is the projective cover of an irreducible $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$, which (by Thm. 5.5) is isomorphic to $Ae$ for some primitive idempotent $e \in A$.*

*Proof.* (2)$\Rightarrow$(1): This follows from Prop. 3.6.

(1)$\Rightarrow$(2): By Lem. 1.6, $P$ has an epimorphism to an irreducible, which (by Thm. 5.5) is of the form $Ae/\mathrm{rad}(Ae)$ where $e \in A$ is a primitive idempotent. We know that $Ae$ is its projective cover. Since $P$ is projective, by Prop. 3.4, $Ae$ is a direct summand of $P$. Since $P$ is indecomposable, we must have $P = Ae$. $\qquad\square$

# 6 Pseudotraces and generating idempotents of strongly AUF algebras

Let $A$ be AUF. In this section, we show that if $e \in A$ is a generating idempotent, any SLF $\psi$ on $A$ can be recovered from $\psi|_{eAe}$ via the pseudotrace construction.

**Definition 6.1.** An idempotent $e \in A$ is called **generating** if every irreducible $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$ has an epimorphism from $Ae$.

**Proposition 6.2.** *Let $e \in A$ be an idempotent. Let $e = \varepsilon_1 + \cdots + \varepsilon_n$ be an orthogonal primitive decomposition (cf. Rem. 5.4). The following are equivalent:*

*(1) e is generating.*

*(2) Any primitive idempotent of A is isomorphic to $\varepsilon_i$ for some i.*

*(3) Any irreducible $M \in \mathrm{QCoh}_{\mathrm{L}}(A)$ is isomorphic to $A\varepsilon_i/\mathrm{rad}(A\varepsilon_i)$ for some i.*

*Proof.* (1)⇒(3): Each irreducible $M \in \mathrm{QCoh}_\mathrm{L}(A)$ has an epimorphism from $Ae = A\varepsilon_1 \oplus \cdots \oplus A\varepsilon_n$, and hence an epimorphism from some $A\varepsilon_i$. By Cor. 3.8, the kernel of this epimorophism is $\mathrm{rad}(A\varepsilon_i)$. Therefore, we have $A\varepsilon_i/\mathrm{rad}(A\varepsilon_i) \simeq M$.

(3)⇒(1): Obvious.

(2)⇔(3): Immediate from Thm. 5.5. □

**Corollary 6.3.** *Let $e, f \in A$ be idempotents such that $e \leqslant f$ and $e$ is a generating idempotent of $A$. Then $e$ is a generating idempotent of $fAf$.*

*Proof.* Let $p$ be any primitive idempotent of $fAf$. Then $p$ is a primitive idempotent of $A$. By Prop. 6.2, if we let $e = \varepsilon_1 + \cdots + \varepsilon_n$ be an orthogonal primitive decomposition, then there exist $1 \leqslant i \leqslant n$ and $u \in \varepsilon_i Ap, v \in pA\varepsilon_i$ such that $uv = \varepsilon_i$ and $vu = p$. So $p$ is isomorphism in $fAf$ to $\varepsilon_i$. By Prop. 6.2, we conclude that $e$ is generating in $fAf$. □

**Corollary 6.4.** *The following are equivalent.*

*(1) $A$ has a generating idempotent.*

*(2) $\mathrm{QCoh}_\mathrm{L}(A)$ has finitely many equivalence classes of irreducible objects.*

*(3) $A$ has finitely many isomorphism classes of primitive idempotents.*

*If one of these conditions holds, we say that $A$ is **strongly AUF**.*

*Proof.* (1)⇒(2): Immediate from Prop. 6.2.

(2)⇔(3): Immediate from Thm. 5.5.

(2)⇒(1): Let $M_1, \ldots, M_n \in \mathrm{QCoh}_\mathrm{L}(A)$ exhaust all equivalence classes of irreducibles. Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1. For each $1 \leqslant k \leqslant n$, by Prop. 2.8, $M_k$ has an epimorphism from $A$. Since $A = \bigoplus_{i \in \mathfrak{I}} Ae_i$, it follows that $M_k$ has an epimorphism from $Ae_{i_k}$ for some $i_k \in \mathfrak{I}$. If we assume at the beginning that $M_1, \ldots, M_n$ are mutually non-isomorphic, then $e_{i_1}, \ldots, e_{i_k}$ must be distinct, and hence mutually orthogonal. So $e = e_{i_1} + \cdots + e_{i_n}$ is a generating idempotent. □

**Theorem 6.5.** *Assume that $A$ is strongly AUF, and let $e \in A$ be a generating idempotent. Then the $A$-$(eAe)$ bimodule $Ae$ has a left coordinate system. In particular, by Rem. 4.2, $Ae$ is a projective right $eAe$-module.*

The following construction of left coordinate system is important and is motivated by [Ari10, Lem. 3.9].

*Proof.* Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1. By Lem. 5.3, we can assume that each $e_i$ is primitive. Let $e = \varepsilon_1 + \cdots + \varepsilon_n$ be an orthogonal primitive decomposition of $e$. By Prop. 6.2, there are partial isometries $u_i, v_i$ such that

$$v_i u_i = \varepsilon_{k_i} \qquad u_i v_i = e_i$$
$$u_i \in e_i A\varepsilon_{k_i} \qquad v_i \in \varepsilon_{k_i} Ae_i$$

where $k_i \in \{1, \ldots, n\}$. In particular $u_i \in e_i Ae$ and $v_i \in eAe_i$. Let

$$\alpha_i \in \mathrm{End}_{-,eAe}(eAe, Ae) \qquad \breve{\alpha}^i \in \mathrm{End}_{-,eAe}(Ae, eAe)$$

$$\alpha_i(exe) = u_i \cdot exe \qquad \breve{\alpha}^i(xe) = v_i \cdot xe$$

One checks easily that $(\alpha_i, \breve{\alpha}^i)_{i \in \mathfrak{J}}$ is a left coordinate system. $\qquad\square$

The proof of [Ari10, Thm. 3.10] can be easily adapted to prove the following theorem.

**Theorem 6.6.** *Assume that $A$ is strongly AUF, and let $e \in A$ be a generating idempotent. Then there is a linear isomorphism*

$$\mathrm{SLF}(A) \xrightarrow{\simeq} \mathrm{SLF}(eAe) \qquad \psi \mapsto \psi|_{eAe}$$

*whose inverse is given by*

$$\mathrm{SLF}(eAe) \xrightarrow{\simeq} \mathrm{SLF}(A) \qquad \phi \mapsto \mathrm{Tr}^\phi$$

*Here, $\mathrm{Tr}^\phi$ is the pseudotrace on $A$ with respect to $\phi$ and the $A$-$(eAe)$ bimodule $Ae$.*

*Proof.* Let $u_i, v_i, \alpha_i, \breve{\alpha}^i$ be as in the proof of Thm. 6.5. For any $\phi \in \mathrm{SLF}(eAe)$, let us compute $\mathrm{Tr}^\phi$. Let $x \in A$, viewed as an element of $\mathrm{End}_{-,eAe}(Ae)$. Then $\breve{\alpha}^i \circ x \circ \alpha_i \in \mathrm{End}_{-,eAe}(eAe)$ equals (the left multiplication by) $v_i x u_i$. Then

$$\mathrm{Tr}^\phi(x) = \sum_{i \in \mathfrak{J}} \phi(v_i x u_i) \tag{6.1}$$

Note that the RHS is a finite sum since $u_i = e_i u_i$, and since and $x e_i = 0$ for all but finitely many $i$.

To show that $\mathrm{Tr}^\phi|_{eAe} = \phi$, we compute

$$\mathrm{Tr}^\phi(exe) = \sum_i \phi(v_i exe u_i) = \sum_i \phi(v_i exe \cdot eu_i)$$

Since $v_i exe, eu_i \in eAe$, and since $\phi$ is SLF, we have

$$\mathrm{Tr}^\phi(exe) = \sum_i \phi(eu_i \cdot v_i exe) = \sum_i \phi(ee_i exe) = \phi(exe)$$

Finally, let $\psi \in \mathrm{SLF}(A)$. Then for each $x \in A$,

$$\mathrm{Tr}^{\psi|_{eAe}}(x) = \sum_i \psi|_{eAe}(v_i x u_i) = \sum_i \psi(v_i x u_i) = \sum_i \psi(u_i v_i x) = \sum_i \psi(e_i x) = \psi(x)$$

This proves $\mathrm{Tr}^{\psi|_{eAe}} = \psi$. $\qquad\square$

# 7 Projective generators of strongly AUF algebras

Let $A$ be an AUF algebra.

**Remark 7.1.** A left $A$-module $M$ is coherent if and only if $M$ is a quotient module of $(Ae)^{\oplus n}$ where $n \in \mathbb{Z}_+$ and $e \in A$ is an idempotent.

*Proof.* "⇐" is obvious. Conversely, let $M \in \mathrm{Coh_L}(A)$. By Def. 2.3, $M$ is a quotient module of $Ap_1 \oplus \cdots \oplus Ap_n$ where each $p_i$ is an idempotent. By Rem. 5.2, one can find an idempotent $e \in A$ which is $\geqslant p_1, \ldots, p_n$. Then $M$ is a quotient module of $(Ae)^{\oplus n}$. □

**Remark 7.2.** By Rem. 7.1, if $M \in \mathrm{Coh_L}(A)$ and $x \in A$, then $\dim xM < +\infty$.

*Proof.* Suppose that $M$ has an epimorphism from $N := (Ae)^{\oplus n}$ where $e \in A$ is an idempotent. Then $\dim xM \leqslant \dim xN$. Let $f \in A$ be an idempotent such that $x = fxf$. Then $xAe \subset fAe$, and hence

$$\dim xN = n \dim xAe \leqslant n \dim fAe < +\infty$$

□

## 7.1 Basic facts

**Definition 7.3.** Let $\mathscr{S}$ and $\mathscr{T}$ be classes of objects in $\mathrm{Coh_L}(A)$. We say that $\mathscr{S}$ **generates** $\mathscr{T}$ if each object of $\mathscr{T}$ is a quotient of a *finite* direct sum of objects in $\mathscr{S}$.

**Definition 7.4.** We say that $M \in \mathrm{Coh_L}(A)$ is a **generator** (of $\mathrm{Coh_L}(A)$) if it generates every object of $\mathrm{Coh_L}(A)$, i.e., every $N \in \mathrm{Coh_L}(A)$ is a quotient module of $M^{\oplus n}$ for some $n \in \mathbb{Z}_+$. A generator which is also projective is called a **projective generator**.

**Example 7.5.** Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1. Then $\mathscr{S} := \{Ae_i : i \in \mathfrak{I}\}$ generates $\mathrm{Coh_L}(A)$.

*Proof.* By the proof of Rem. 5.2, for any idempotent $e \in A$ one can find a finite set $I_0 \subset I$ such that $e \leqslant \sum_{i \in I_0} e_i$. Therefore, $\mathscr{S}$ generates each $Ae$, and hence (by Rem. 7.1) generates $\mathrm{Coh_L}(A)$. □

**Proposition 7.6.** *Let $M \in \mathrm{Coh_L}(A)$ be projective. The following are equivalent.*

*(1) $M$ is a projective generator.*

*(2) Each irreducible $N \in \mathrm{Coh_L}(A)$ has an epimorphism from $M$.*

*Proof.* (1)⇒(2): Obvious.

(2)⇒(1): Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1. By Lem. 5.4, we assume that each $e_i$ is primitive. By Exp. 7.5, it suffices to prove that $M$ generates each $Ae_i$. By Thm. 5.5, $Ae_i$ is the projective cover of the irreducible $N := Ae_i / \mathrm{rad}(Ae_i)$. By (2), $M$ has an epimorphism to $N$. Since $M$ is projective, by Prop. 3.4, $Ae_i$ is isomorphic to a direct summand of $M$. □

**Corollary 7.7.** *Let $e \in A$ be an idempotent. Then the following are equivalent.*

*(1) $Ae$ is a (necessarily projective) generator.*

*(2) $e$ is a generating idempotent.*

*Proof.* (1)⇒(2): Clear from Def. 6.1. (2)⇒(1): Immediate from Prop. 7.6. □

**Proposition 7.8.** $\mathrm{Coh_L}(A)$ *has a projective generator if and only if $A$ is strongly AUF.*

*Proof.* "⇐" follows from Cor. 6.4 and 7.7. Conversely, if $\mathrm{Coh_L}(A)$ has a projective generator $M$, by Rem. 7.1, an idempotent $e \in A$ can be found such that $Ae$ generates $M$, and hence generates $\mathrm{Coh_L}(A)$. So $e$ is a generating idempotent. Thus, by Cor. 6.4, $\mathrm{Coh_L}(A)$ has finitely many irreducibles. So $A$ is strongly AUF. □

## 7.2 Projective generators and endomorphism algebras

Our next goal is to give criteria for projective generators in terms of the endomorphism algebras. We need the endomorphism algebras to be finite-dimensional:

**Proposition 7.9.** *Let $M, N \in \mathrm{Coh}_{\mathrm{L}}(A)$. Then*

$$\dim \mathrm{Hom}_{A,-}(M, N) < +\infty$$

*Proof.* By Def. 2.3, there is an epimorphism from a finite direct sum $\bigoplus_i Ae_i$ to $M$, where $e_i$ is an idempotent. By taking composition with this epimorphism, we get

$$\mathrm{Hom}_{A,-}(M, N) \to \mathrm{Hom}_{A,-}\Big(\bigoplus_i Ae_i, N\Big) \simeq \bigoplus_i \mathrm{Hom}_{A,-}(Ae_i, N) \tag{7.1}$$

where the first map is injective. Thus, it suffices to prove that each $\mathrm{Hom}_{A,-}(Ae_i, N)$ is finite-dimensional.

Again, we can find an epimorphism $\Phi : \bigoplus_j Af_j \twoheadrightarrow N$ (where $\bigoplus_j$ is finite). Since $Ae_i$ is projective, each $\alpha \in \mathrm{Hom}_{A,-}(Ae_i, N)$ can be lifted to some $\beta \in \mathrm{Hom}_{A,-}(Ae_i, \bigoplus_j Af_j)$ such that $\alpha = \Phi \circ \beta$. Thus

$$\dim \mathrm{Hom}_{A,-}(Ae_i, N) \leqslant \dim \mathrm{Hom}_{A,-}\Big(Ae_i, \bigoplus_j Af_j\Big) = \sum_j \dim \mathrm{Hom}_{A,-}(Ae_i, Af_j)$$

where $\dim \mathrm{Hom}_{A,-}(Ae_i, Af_j) = \dim e_i Af_j < +\infty$. $\qquad\square$

**Proposition 7.10.** *Let $M$ be a left $A$-module. Let $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$, and let $p, q \in B$ be idempotents. Then an element of $\mathrm{Hom}_{A,-}(Mp, Mq)$ is precisely the right multiplication of an element of $pBq$. In particular, we have a canonical isomorphism*

$$\mathrm{End}_{A,-}(Mp)^{\mathrm{op}} \simeq pBp$$

*Consequently, the direct summands of the left $A$-module $Mp$ correspond bijectively to the subidempotents of $p$ in $B$.*

*Proof.* This is similar to the proofs of Prop. 1.2 and Cor. 1.4. Any $y \in pBq$ defines a morphism $Mp \to Mq$ by right multiplication. Conversely, if $T \in \mathrm{Hom}_{A,-}(Mp, Mq)$, let $\widehat{T} : M \to M$ be $\widehat{T}(\xi) = T(\xi p)$. Then $\widehat{T} \in \mathrm{End}_{A,-}(M)$, and hence $\widehat{T}$ is the right multiplication by some $\widehat{y} \in B$. Note that $T = \widehat{T}|_{Mp}$, and hence $T(\xi p) = \xi p \widehat{y}$ for each $\xi \in M$. Since $T$ has range in $Mq$, we have $T(\xi p) = \xi p \widehat{y} q$. So $T$ is the right multiplication by $y := p\widehat{y}q \in pBq$. $\qquad\square$

**Theorem 7.11.** *Let $M \in \mathrm{Coh}_{\mathrm{L}}(A)$. Let $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$ which is a finite-dimensional unital algebra (by Prop. 7.9). Let $p \in B$ be an idempotent. Consider the following statements:*

*(1) As coherent left $A$-modules, $Mp$ generates $M$.*

*(2) $p$ is a generating idempotent of $B$.*

*Then (2)⇒(1). If $M$ is projective, then (1)⇔(2).*

*Proof.* (2)⇒(1): Since $\dim B < +\infty$, we have a primitive orthogonal decomposition $1_B = q_1 + \cdots + q_n$ where each $q_j \in B$ is a primitive idempotent. By Prop. 6.2, each $q_j$ is isomorphic to a sub-idempotent of $p$. Thus $Mq_j$ is isomorphic to a direct summand of the left $A$-module $Mp$. So $Mp$ generates $\bigoplus_j Mq_j = M$.

(1)⇒(2): Let $q$ be any primitive idempotent of $B$. Since $Mp$ generates $M$ and since $M$ generates $Mq$, we have that $Mp$ generates $Mq$. We claim that $Mq$ is isomorphic to a direct summand of $Mp$. Then Prop. 7.10 will imply that $q$ is isomorphic (in $B$) to a sub-idempotent of $p$. This implies (2), thanks to Prop. 6.2.

Let us prove the claim, assuming that $M$ is projective. Since $Mq$ is a direct summand of $M$, we see that $Mq$ is projective. Since $q$ is primitive in $B$, by Prop. 7.10, $Mq$ is an indecomposable left $A$-module. Therefore, by Cor. 5.6, $Mq$ is the projective cover of an irreducible $N \in \mathrm{Coh_L}(A)$. Since $Mp$ generates $Mq$, it generates $N$. Thus $N$ has an epimorphism from a finite direct sum of $Mp$. Since $N$ is irreducible, $N$ has an epimorphism from $Mp$. Note that $Mp$ is also projective. Therefore, by Prop. 3.4, $Mq$ is isomorphic to a direct summand of $Mp$. □

**Corollary 7.12.** *Assume that $G \in \mathrm{Coh_L}(A)$ is a projective generator. Let $M$ be a left $A$-module. Then the following are equivalent.*

(1) *$M \in \mathrm{Coh_L}(A)$, and $M$ is a projective generator (of $\mathrm{Coh_L}(A)$).*

(2) *There exist $n \in \mathbb{Z}_+$ and a generating idempotent $p$ of $B := \mathrm{End}_{A,-}(G^{\oplus n})^{\mathrm{op}}$ such that $M \simeq G^{\oplus n} \cdot p$.*

In particular, if $e \in A$ is a generating idempotent, one can take $G = Ae$. Thus a projective generator of $\mathrm{Coh_L}(A)$ is (up to isomorphisms) precisely of the form $(Ae)^{\oplus n}p$ where $n \in \mathbb{Z}_+$ and $p \in \mathrm{End}_{A,-}((Ae)^{\oplus n})^{\mathrm{op}}$ is a generating idempotent.

*Proof.* (2)⇒(1): By Thm. 7.11, $M$ generates $G^{\oplus n}$. So $M$ is a generator. Since $G^{\oplus n}p$ is a direct summand of the projective coherent module $G^{\oplus n}$, $G^{\oplus n}p$ is also projective and coherent.

(1)⇒(2): $M$ has an epimorphism from $G^{\oplus n}$ for some $n \in \mathbb{Z}_+$. Since $M$ is projective, this epimorphism splits. So $M$ can be viewed as a direct summand of $G^{\oplus n}$. Let $p$ be the projection of $G^{\oplus n}$ onto $M$, which can be viewed as an endomorphism of $G^{\oplus n}$. So $p$ is an idempotent of $B$, and $M = G^{\oplus n}p$. Since $M$ is a generator, it generates $G^{\oplus n}$. Since $G^{\oplus n}$ is projective, by Thm. 7.11, $p$ is generating. □

# 8 Right pseudotraces

Let $A$ be an AUF algebra. Let $B$ be a unital algebra. Let $M$ be an $A$-$B$ bimodule, coherent as a left $A$-module.

For each $y \in B$ and $\xi \in M$, we write $\xi y$ as $y^{\mathrm{op}}\xi$. Namely, $y^{\mathrm{op}}$ is viewed as an element of $\mathrm{End}_{A,-}(M)$.

**Definition 8.1.** A **right coordinate system** of $M$ denotes a collection of morphisms

$$\beta_j \in \mathrm{Hom}_{A,-}(Ae, M) \qquad \check{\beta}^j : \mathrm{Hom}_{A,-}(M, Ae)$$

where $e \in A$ is an idempotent (called the **domain idempotent**), and $j$ runs through a *finite* index set $J$ such that the $\sum_{j \in J} \beta_j \circ \breve{\beta}^j$ equals $\mathrm{id}_M$.

**Remark 8.2.** $M$ has a right coordinate system iff $M$ is $A$-projective.

*Proof.* By Rem. 7.1, each $N \in \mathrm{Coh}_\mathrm{L}(A)$ has an epimorphism from $(Ae)^{\oplus n}$ where $e \in A$ is an idempotent and $n \in \mathbb{Z}_+$. This epimorphism splits iff $N$ is projective in $\mathrm{Coh}_\mathrm{L}(A)$. Therefore, similar to Rem. 4.2, we see that $M$ has a right coordinate system iff $M$ is $A$-projective. $\qquad\square$

**Remark 8.3.** In Def. 8.1, one can freely enlarge the domain idempotent $e$. More precisely, suppose that $f \in A$ is an idempotent such that $e \leqslant f$. One can define a new right coordinate system

$$
\begin{aligned}
\gamma_j \in \mathrm{Hom}_{A,-}(Af, M) &\qquad \breve{\gamma}^j \in \mathrm{Hom}_{A,-}(M, Af) \\
\gamma_j(af) = \beta_j(ae) &\qquad \breve{\gamma}^j(\xi) = \breve{\beta}^j(\xi)
\end{aligned}
\tag{8.1}
$$

called the **canonical extension** of $(\beta_j, \breve{\beta}^j)_{j \in J}$.

**Definition 8.4.** Assume that $M$ has a right coordinate system $(\beta_j, \breve{\beta}^j)_{j \in J}$. For each $\psi \in \mathrm{SLF}(A)$, define the (right) **pseudotrace** ${}^\psi\mathrm{Tr}$ associated to $\psi$ to be

$$
{}^\psi\mathrm{Tr} : B \to \mathbb{C} \qquad {}^\psi\mathrm{Tr}(y) = \sum_{j \in J} \psi\big((\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j)^{\mathrm{op}}\big)
$$

noting that $\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j \in \mathrm{End}_{A,-}(Ae) \simeq (eAe)^{\mathrm{op}}$. In other words,

$$
{}^\psi\mathrm{Tr}(y) = \sum_{j \in J} \psi(\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e))
\tag{8.2}
$$

Note that in (8.2) we have $\beta_j(e) \in M$, and hence $\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e) \in Ae$. So

$$
\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e) = \breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e^2) = e\breve{\beta}^j \circ y^{\mathrm{op}} \circ \beta_j(e) \in eAe
$$

**Proposition 8.5.** *Assume that $M$ is $A$-projective. Let $\psi \in \mathrm{SLF}(A)$. Then ${}^\psi\mathrm{Tr} \in \mathrm{SLF}(B)$. Moreover, the definition of ${}^\psi\mathrm{Tr}$ is independent of the choice of right coordinate systems.*

*Proof.* From (8.1) and (8.2), it is clear that a canonical extension of the right coordinate system does not affect the value of ${}^\psi\mathrm{Tr}(y)$. Also, note that since $A$ is AUF, for any idempotents $e_1, e_2 \in A$ there is an idempotent $e_3$ such that $e_1, e_2 \leqslant e_3$. Therefore, to compare ${}^\psi\mathrm{Tr}$ defined by two coordinate systems $(\alpha_\bullet, \breve{\alpha}^\bullet)$ and $(\beta_\star, \breve{\beta}^\star)$, by performing canonical extensions, it suffices to assume that their domain idempotents are equal. Then one can use the same argument as in Lem. 4.4 to show that $(\alpha_\bullet, \breve{\alpha}^\bullet)$ and $(\beta_\star, \breve{\beta}^\star)$ define the same ${}^\psi\mathrm{Tr}$. Finally, similar to the proof of Prop. 4.5, one shows that ${}^\psi\mathrm{Tr}$ is symmetric. $\qquad\square$

**Example 8.6.** Let $M = Ae$ and $B = eAe$ where $e \in A$ is an idempotent. Then the identity map on $Ae$ gives a right coordinate system. From this, one sees that if $\psi \in \mathrm{SLF}(A)$ then

$$
{}^\psi\mathrm{Tr} = \psi|_{eAe}
$$

**Example 8.7.** More generally, let $M = (Ae)^{\oplus n}$ and $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$. So $B = eAe \otimes \mathbb{C}^{n \times n}$. Let

$$\mathrm{tr} : \mathbb{C}^{n \times n} \to \mathbb{C}$$

be the standard trace on $\mathbb{C}^{n \times n}$. A right coordinate system can be choosen to be the $n$ canonical embeddings $Ae \to (Ae)^{\oplus n}$ and the $n$ canonical projections $(Ae)^{\oplus n} \to Ae$. Then one easily sees that

$$^\psi \mathrm{Tr} = \psi|_{eAe} \otimes \mathrm{tr}$$

**Proposition 8.8.** *Assume that $M$ is $A$-projective. Let $p \in B$ be an idempotent. Let $\psi \in \mathrm{SLF}(A)$. Let $^\psi \mathrm{Tr}_M : B \to \mathbb{C}$ be the pseudotrace associated to $M$. Then the pseudotrace $^\psi \mathrm{Tr}_{Mp} : pBp \to \mathbb{C}$ associated to the $A$-$(pBp)$ bimodule $Mp$ is equal to $^\psi \mathrm{Tr}_M|_{pBp}$, i.e.*

$$^\psi \mathrm{Tr}_{Mp} = {}^\psi \mathrm{Tr}_M\big|_{pBp}$$

*Proof.* Let $(\beta_\bullet, \check{\beta}^\bullet)$ be a right coordinate system (with domain idempotent $e \in A$) as in Def. 8.1. Then one has a right coordinate system

$$\gamma_j \in \mathrm{Hom}_{A,-}(Ae, Mp) \qquad \check{\gamma}^j : \mathrm{Hom}_{A,-}(Mp, Ae)$$
$$\gamma_j(ae) = \beta_j(ae)p \qquad \check{\gamma}^j(\xi p) = \check{\beta}^j(\xi p)$$

noting that $Mp \leqslant M$, and hence $\check{\gamma}^j$ is simply the restriction of $\beta^j$ to $Mp$. Using (8.2) one computes that for each $y \in B$,

$$^\psi \mathrm{Tr}_{Mp}(pyp) = \sum_j \psi(\check{\gamma}^j \circ (pyp)^{\mathrm{op}} \circ \gamma_j(e)) = \sum_j \psi(\check{\beta}^j \circ (pyp)^{\mathrm{op}} \circ \beta_j(e)p)$$
$$= \sum_j \psi(\check{\beta}^j \circ (pyp)^{\mathrm{op}} \circ p^{\mathrm{op}} \circ \beta_j(e)) = \sum_j \psi(\check{\beta}^j \circ (pyp)^{\mathrm{op}} \circ \beta_j(e)) = {}^\psi \mathrm{Tr}_M(pyp)$$

$\square$

# 9 Equivalence of left and right pseudotraces

Let $A, B$ be algebras where $B$ is unital.

## 9.1 Preliminary discussion

In this subsection, assume that $A$ is AUF. We shall consider $M \in \mathrm{Coh}_L(A)$ such that the left and the right pseudotrace constructions are both available to the $A$-$(\mathrm{End}_{A,-}(M)^{\mathrm{op}})$ bimodule $M$. By Rem. 8.2, $M$ needs to be assumed $A$-projective. One also needs $M$ to be $\mathrm{End}_{A,-}(M)^{\mathrm{op}}$-projective. In fact, these two conditions are precisely what ensure that both left and right pseudotraces can be defined.

**Proposition 9.1.** *Let $M$ be an $A$-$B$ bimodule. Assume that $M$ is $A$-coherent. Then the following are equivalent.*

*(1) M has a left coordinate system.*

*(2) M is B-projective.*

Although this proposition will not be used in the current note, we include it here as it may be of use in the future.

*Proof.* (1)⇒(2): See Rem. 4.2.

(2)⇒(1): Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1. By Rem. 7.2, each $e_i M$ is finite-dimensional. Therefore, the right $B$-module $e_i M$ has an epimorphism from $B^{\oplus n}$ which splits because $M$ is $B$-projective (and hence $e_i M$ is projective since $M = \bigoplus_{i \in \mathfrak{I}} e_i M$). Therefore, for each $i \in \mathfrak{I}$, there is a finite left coordinate system $\alpha_{i,\bullet} \in \mathrm{Hom}_{-,B}(B, e_i M)$ and $\breve{\alpha}^{i,\bullet} \in \mathrm{Hom}_{-,B}(e_i M, B)$. Let

$$\gamma_{i,\bullet} \in \mathrm{Hom}_{-,B}(B, M) \qquad \breve{\gamma}^{i,\bullet} \in \mathrm{Hom}_{-,B}(M, B)$$
$$\gamma_{i,\bullet}(b) = \alpha_{i,\bullet}(b) \qquad \breve{\gamma}^{i,\bullet}(\xi) = \breve{\alpha}^{i,\bullet}(e_i \xi)$$

Then one checks easily that $(\gamma_{i,\bullet}, \breve{\gamma}^{i,\bullet})_{i \in \mathfrak{I}}$ is a left coordinate system of $M$. □

## 9.2 Calculation of some left pseudotraces

In this subsection, $A$ is not assumed to be AUF. Let $M$ be an $A$-$B$ bimodule.

The goal of this subsection is to prepare for the proof of the main Thm. 9.4. The following theorem is dual to Prop. 8.8.

**Theorem 9.2.** *Assume that $M$ has a left coordinate system. Let $p \in B$ be a generating idempotent. Then the following are true.*

1. *The $A$-$(pBp)$ bimodule $Mp$ has a left coordinate system.*

2. *Let $\phi \in \mathrm{SLF}(B)$. Then on $A$, the pseudotrace associated to $\phi|_{pBp}$ and $Mp$ is equal to the pseudotrace associated to $\phi$ and $M$. Namely,*

$$\mathrm{Tr}_{Mp}^{\phi|_{pBp}} = \mathrm{Tr}_M^{\phi} \tag{9.1}$$

*In this theorem, we do not require that $A$ is AUF.*

*Proof.* Choose a left coordinate system for $M$:

$$\alpha_i \in \mathrm{Hom}_{-,B}(B, M) \qquad \breve{\alpha}^i \in \mathrm{Hom}_{-,B}(M, B) \qquad i \in \mathfrak{I}$$

Since $p$ is generating, similar to the proof of Thm. 6.5, we can find finitely many elements $u_k, v_k$ in $B$ such that

$$v_k u_k = p_k \qquad u_k v_k = q_k$$
$$u_k \in q_k B p_k \qquad v_k \in p_k B q_k$$

where each $p_k, q_k \in B$ are idempotents, $1_B = \sum_k q_k$ is a primitive orthogonal decomposition of $1_B$, and $p_k \leqslant p$ for each $k$. [3] Let

$$\theta_{i,k} \in \operatorname{Hom}_{-,pBp}(pBp, Mp) \qquad \check{\theta}^{i,k} \in \operatorname{Hom}_{-,pBp}(Mp, pBp)$$

$$\theta_{i,k}(pyp) = \alpha_i(u_k \cdot pyp) \qquad \check{\theta}^{i,k}(\xi p) = v_k \cdot \check{\alpha}^i(\xi p)$$

noting that $\alpha_i(u_k \cdot pyp) = \alpha_i(u_k)pyp \in Mp$ and $v_k \cdot \check{\alpha}^i(\xi p) = v_k \cdot \check{\alpha}^i(\xi)p \in p_k Bp \subset pBp$.

For each $\xi \in M$, note that if $\check{\alpha}^i(\xi) = 0$, then $\check{\theta}^{i,k}(\xi p) = v_k \check{\alpha}^i(\xi)p = 0$. Therefore, $\check{\theta}^{i,k}(\xi p) = 0$ for all but finitely many $i$ and $k$. Moreover, we compute

$$\sum_{i,k} \theta_{i,k} \circ \check{\theta}^{i,k}(\xi p) = \sum_{i,k} \theta_{i,k}(v_k \check{\alpha}^i(\xi p)) = \sum_{i,k} \alpha_i(u_k v_k \check{\alpha}^i(\xi p))$$

$$= \sum_{i,k} \alpha_i(q_k \check{\alpha}^i(\xi p)) = \sum_i \alpha_i \circ \check{\alpha}^i(\xi p) = \xi p$$

where all the sums are finite. This proves that $(\theta, \check{\theta})$ satisfies Def. 4.1-(a). It is easy to check Def. 4.1-(b). So we have proved that $(\theta, \check{\theta})$ is a left coordinate system of $Mp$.

It remains to check (9.1). Choose any $x \in A$. By (4.3) and the fact that $1_{pBp} = p$,

$$\operatorname{Tr}_{Mp}^{\phi|_{pBp}}(x) = \sum_{i,k} \phi(\check{\theta}^{i,k} \circ x \circ \theta_{i,k}(p)) = \sum_{i,k} \phi(\check{\theta}^{i,k} \circ x \circ \alpha_i(u_k p))$$

$$= \sum_{i,k} \phi(\check{\theta}^{i,k} \circ x \circ \alpha_i(u_k)) = \sum_{i,k} \phi(v_k \cdot \check{\alpha}^i(x \circ \alpha_i(u_k)))$$

Since $\check{\alpha}^i, x, \alpha_i$ commute with the right multiplication by $v_k$, and since $\phi$ is symmetric,

$$\operatorname{Tr}_{Mp}^{\phi|_{pBp}}(x) = \sum_{i,k} \phi(\check{\alpha}^i(x \circ \alpha_i(u_k))v_k) = \sum_{i,k} \phi(\check{\alpha}^i(x \circ \alpha_i(u_k v_k)))$$

$$= \sum_i \phi(\check{\alpha}^i(x \circ \alpha_i(1_B))) = \operatorname{Tr}_M^{\phi}(x)$$

This finishes the proof of (9.1). $\qquad\square$

**Corollary 9.3.** *Assume that $M$ has a left coordinate system. Let $n \in \mathbb{Z}_+$. Let $\widetilde{B} = B \otimes \mathbb{C}^{n \times n}$. Then the $A$-$\widetilde{B}$ bimodule $M^{\oplus n} \simeq M \otimes \mathbb{C}^{1,n}$ has a left coordinate system. Moreover, for each $\phi \in \operatorname{SLF}(B)$, we have*

$$\operatorname{Tr}_{M^{\oplus n}}^{\phi \otimes \operatorname{tr}} = \operatorname{Tr}_M^{\phi} \tag{9.2}$$

*as pseudotraces on $A$ associated to $\phi \otimes \operatorname{tr} \in \operatorname{SLF}(\widetilde{B})$ and $\phi \in \operatorname{SLF}(B)$, respectively.*

Recall that $\operatorname{tr} \in \operatorname{SLF}(\mathbb{C}^{n \times n})$ is the standard trace on the $n \times n$ matrix algebra.

*Proof.* Choose a left coordinate system

$$\alpha_i \in \operatorname{Hom}_{-,B}(B, M) \qquad \check{\alpha}^i \in \operatorname{Hom}_{-,B}(M, B)$$

---

[3] So $p_k, q_k$ are similar to $\varepsilon_{k_i}, e_i$ in the proof of Thm. 6.5.

of $M$. Define

$$\gamma_i \in \operatorname{Hom}_{-,\widetilde{B}}(\widetilde{B}, M^{\oplus n}) \qquad \widecheck{\gamma}^i \in \operatorname{Hom}_{-,\widetilde{B}}(M^{\oplus n}, \widetilde{B})$$

such that

$$\gamma_i \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ & \vdots & \\ y_{n,1} & \cdots & y_{n,n} \end{bmatrix} = [\alpha_i(1_B), 0, \ldots, 0] \begin{bmatrix} y_{1,1} & \cdots & y_{1,n} \\ & \vdots & \\ y_{n,1} & \cdots & y_{n,n} \end{bmatrix} = [\alpha_i(y_{1,1}), \ldots, \alpha_i(y_{1,n})]$$

$$\widecheck{\gamma}^i[\xi_1, \ldots, \xi_n] = \begin{bmatrix} \widecheck{\alpha}^i(\xi_1) & \cdots & \widecheck{\alpha}^i(\xi_n) \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

One checks easily that this is a left coordinate system of $M^{\oplus n}$. Now (9.2) follows by applying Thm. 9.2 to the $A$-$\widetilde{B}$ bimodule $M^{\oplus n}$ and the generating projection $p \in \widetilde{B}$, where $p$ is the matrix whose $(1,1)$-entry is 1 and other entries are 0. $\qquad\square$

## 9.3 The main theorem

Assume that $A$ is strongly AUF (cf. Cor. 6.4) so that $A$ has a projective generator (cf. Prop. 7.8). The following generalization of Thm. 6.6 is the main theorem of this note.

**Theorem 9.4.** *Assume that $M \in \operatorname{Coh}_{\mathrm{L}}(A)$ is a projective generator. Assume that $B = \operatorname{End}_{A,-}(M)^{\mathrm{op}}$ so that $M$ is an $A$-$B$ bimodule. Then $M$ has left and right coordinate systems. Moreover, we have a linear isomorphism*

$$\operatorname{SLF}(A) \xrightarrow{\simeq} \operatorname{SLF}(B) \qquad \psi \mapsto {}^{\psi}\mathrm{Tr} \tag{9.3a}$$

*whose inverse map is*

$$\operatorname{SLF}(B) \xrightarrow{\simeq} \operatorname{SLF}(A) \qquad \phi \mapsto \mathrm{Tr}^{\phi} \tag{9.3b}$$

Of course, both pseudotraces are associated to $M$; we have suppressed the subscript $M$.

*Proof.* Note that $\dim B < +\infty$ by Prop. 7.9. So $\dim \operatorname{SLF}(B) < +\infty$. Since $M \in \operatorname{Coh}_{\mathrm{L}}(A)$ is $A$-projective, by Rem. 8.2, $M$ has a right coordinate system. By Cor. 7.12, we may assume that $M = G \cdot p$ where

- $G = (Ae)^{\oplus n}$ for some $n \in \mathbb{Z}_+$ and generating idempotent $e \in A$.

- $M = Gp$ where $p$ is a generating idempotent of $\widetilde{B} = \operatorname{End}_{A,-}(G)^{\mathrm{op}} = eAe \otimes \mathbb{C}^{n \times n}$.

- $B = p\widetilde{B}p$ (by Prop. 7.10).

By Thm. 6.5 and Cor. 9.3, $G$ has a left coordinate system. Therefore, by Thm. 9.2, $M$ has a left coordinate system.

By Thm. 6.6, we have $\dim \mathrm{SLF}(A) = \dim \mathrm{SLF}(eAe)$. Clearly we have a linear isomorphism

$$\mathrm{SLF}(eAe) \xrightarrow{\sim} \mathrm{SLF}(eAe \otimes \mathbb{C}^{n \times n}) \qquad \omega \mapsto \omega \otimes \mathrm{tr}$$

So $\dim \mathrm{SLF}(eAe) = \dim \mathrm{SLF}(\widetilde{B})$. By Thm. 6.6, we have $\dim \mathrm{SLF}(\widetilde{B}) = \dim \mathrm{SLF}(B)$. This proves $\dim \mathrm{SLF}(A) = \dim \mathrm{SLF}(B) < +\infty$.

Choose any $\psi \in \mathrm{SLF}(A)$. By Exp. 8.7, $^{\psi}\mathrm{Tr}_G : \widetilde{B} \to \mathbb{C}$ equals $\psi|_{eAe} \otimes \mathrm{tr}$. By Prop. 8.8, on $B = p(eAe \otimes \mathbb{C}^{n \times n})p$ we have

$$^{\psi}\mathrm{Tr}_M = \left. (\psi|_{eAe} \otimes \mathrm{tr}) \right|_B \qquad =: \phi$$

Now $\phi \in \mathrm{SLF}(B)$. By Thm. 9.2 and Cor. 9.3,

$$\mathrm{Tr}_M^{\phi} = \mathrm{Tr}_{Gp}^{(\psi|_{eAe} \otimes \mathrm{tr})|_B} = \mathrm{Tr}_G^{\psi|_{eAe} \otimes \mathrm{tr}} = \mathrm{Tr}_{Ae}^{\psi|_{eAe}}$$

By Thm. 6.6, $\mathrm{Tr}_{Ae}^{\psi|_{eAe}} = \psi$. So $\mathrm{Tr}_M^{\phi} = \psi$. We have thus proved that (9.3b) $\circ$ (9.3a) is the identity map on $\mathrm{SLF}(A)$. This finishes the proof. $\qquad\square$

## 10 Equivalence of non-degeneracy of left and right pseudotraces

**Definition 10.1.** Let $A$ be an algebra and $\psi \in \mathrm{SLF}(A)$. We say that $\psi$ is **non-degenerate** if

$$\{x \in A : \psi(xA) = 0\} \equiv \{x \in A : \psi(xy) = 0, \forall y \in A\}$$

is zero.

In the following, $A$ is always assumed to be AUF.

**Lemma 10.2.** *Let $e \in A$ be an idempotent, and let $\psi \in \mathrm{SLF}(A)$. If $\psi$ is non-degenerate, then the restriction $\psi|_{eAe}$ is non-degenerate. Conversely, if $\psi|_{eAe}$ is non-degenerate and $e$ is generating, then $\psi$ is non-degenerate.*

*Proof.* Assume that $\psi$ is non-degenerate. Choose $x \in eAe$ such that $\psi(xeAe) = 0$. Then

$$\psi(xA) = \psi(exeA) = \psi(xeAe) = 0$$

and hence $x = 0$. Therefore $\psi|_{eAe}$ is non-degenerate.

Conversely, assume that $\psi|_{eAe}$ is non-degenerate and $e$ is generating. Choose $x \in A$ such that $\psi(xA) = 0$. Then for each $a, b \in A$,

$$\psi(eaxbe \cdot eAe) = \psi(eaxbeAe) = \psi(xbeAea) = 0$$

Therefore $eaxbe = 0$. Since $b$ is arbitrary, we have $eaxAe = 0$. Since $e$ is generating, it is not hard to show that the left $A$-module $Ae$ is faithful. (See for example Lem. 11.6.) It follows from that $eax = 0$. Therefore $eAx = 0$. Similarly, $eA$ is a faithful right $A$-module. Hence $x = 0$. This proves the non-degeneracy of $\psi$. $\qquad\square$

**Proposition 10.3.** *Assume that $\psi \in \mathrm{SLF}(A)$ is non-degenerate. Let $M \in \mathrm{Coh}_L(A)$ be projective, and let $B = \mathrm{End}^0_{A,-}(M)$. Then the right pseudotrace ${}^\psi\mathrm{Tr} \in \mathrm{SLF}(B)$ is non-degenerate.*

*Proof.* By Prop. 2.6, $M$ can be viewed as a direct summand of $\bigoplus_{i=1}^n Ae_i$ where each $e_i \in A$ is an idempotent. Let $e \in A$ be an idempotent such that $e \geqslant e_i$ for all $i$. Then $M$ is a direct summand of $(Ae)^{\oplus n}$. By Prop. 1.2, we have $\mathrm{End}^0_{A,-}(Ae)^{\mathrm{op}} = eAe$, and hence $\mathrm{End}^0_{A,-}((Ae)^{\oplus n}) = eAe \otimes \mathbb{C}^{n \times n}$. By Cor. 1.4, there is an idempotent $p \in eAe \otimes \mathbb{C}^{n \times n}$ such that $M = (Ae)^{\oplus n}p$. By Lem. 10.2, $\psi|_{eAe}$ is non-degenerate, and hence $\psi|_{eAe} \otimes \mathrm{tr} : eAe \otimes \mathbb{C}^{n \times n} \to \mathbb{C}$ is non-degenerate. By Lem. 10.2 again, the restriction of $\psi|_{eAe} \otimes \mathrm{tr}$ to $p(eAe \otimes \mathbb{C}^{n \times n})p$ (which is $B$ due to Prop. 7.10) is non-degenerate. But this restriction is exactly ${}^\psi\mathrm{Tr}$ due to Exp. 8.7 and Prop. 8.8. $\qquad\square$

**Theorem 10.4.** *Assume that $A$ is strongly AUF. Then in Thm. 9.4, for any $\psi \in \mathrm{SLF}(A)$, the non-degeneracy of $\psi$ and of ${}^\psi\mathrm{Tr}$ are equivalent.*

*Proof.* We use the notation in the proof of Thm. 9.4. From that proof, we know ${}^\psi\mathrm{Tr} = (\psi|_{eAe} \otimes \mathrm{tr})|_B$. By Lem. 10.2, $\psi$ is non-degenerate iff $\psi|_{eAe}$ is so, and $\psi|_{eAe} \otimes \mathrm{tr}$ is non-degenerate iff $(\psi|_{eAe} \otimes \mathrm{tr})|_B$ is so. The equivalence of the non-degeneracy of $\psi|_{eAe}$ and of $\psi|_{eAe} \otimes \mathrm{tr}$ is obvious. The proof is finished. $\qquad\square$

# 11   Classification of strongly AUF algebras

In this section, we fix an AUF algebra $A$.

**Definition 11.1.** For each left $A$-module $M$, let $M^*$ be the space of linear functionals, which has a right $A$-module structure defined by

$$(\phi a)(m) = \phi(am) \qquad \text{for all } a \in A, m \in M$$

We define the **quasicoherent dual**

$$\begin{aligned} M^\vee &= \{\phi \in M^* : \phi \in \phi \cdot A\} \\ &= \{\phi \in M^* : \text{there exists an idempotent } e \in A \text{ such that } \phi = \phi e\} \end{aligned}$$

By Def. 2.2, $M^\vee$ is the largest right $A$-submodule of $M$ that is quasicoherent.

**Remark 11.2.** Let $M \in \mathrm{QCoh}_L(A)$. Let $(e_i)_{i \in \mathfrak{I}}$ be as in Def. 5.1. Then, as vector spaces, we clearly have

$$M = \bigoplus_{i \in \mathfrak{I}} e_i M \qquad M^* = \prod_{i \in \mathfrak{I}} (e_i M)^*$$

It follows easily that

$$M^\vee = \bigoplus_{i \in \mathfrak{I}} (e_i M)^*$$

**Definition 11.3.** For each $M \in \mathrm{QCoh}_L(A)$, we let

$$\mathrm{End}^0(M) = M \otimes_{\mathbb{C}} M^{\vee}$$

viewed as a subalgebra of $\mathrm{End}(M)$.[4] Suppose that $B$ is an algebra, and $M$ has a right $B$-module structure commuting with the left action of $A$, we let

$$\mathrm{End}^0_{-,B}(M) = \{T \in \mathrm{End}^0(M) : (T\xi)b = T(\xi b) \text{ for all } \xi \in M, b \in B\} \qquad (11.1)$$

**Remark 11.4.** Let $M \in \mathrm{Coh}_L(A)$. By Rem. 7.2 we have $\dim e_i M < +\infty$. It follows from Rem. 11.2 that

$$\mathrm{End}^0(M) = \{T \in \mathrm{End}(M) : Te_i = 0 \text{ for all but finitely many } i \in \mathfrak{I}\}$$

**Proposition 11.5.** *Choose* $M \in \mathrm{Coh}_L(A)$, *and let* $B = \mathrm{End}_{A,-}(M)^{\mathrm{op}}$. *Then for each generating idempotent* $p \in B$, *we have a linear isomorphism*

$$\mathrm{End}^0_{-,B}(M) \xrightarrow{\simeq} \mathrm{End}^0_{-,pBp}(Mp) \qquad S \mapsto S|_{Mp} \qquad (11.2)$$

*Proof.* Step 1. Let $\widehat{B} = B^{\mathrm{op}} = \mathrm{End}_{A,-}(M)$, and let $\widehat{p} \in \widehat{B}$ be the opposite element of $p$. Then $M$ has a left $\widehat{B}$-module structure commuting with the left action of $A$, and $R_p$ is the left multiplication by $\widehat{p}$.

For each $S \in \mathrm{End}^0_{-,B}(M)$, note that $S|_{Mp} = S|_{\widehat{p}M}$ maps $\widehat{p}M$ into $\widehat{p}M$, because $S\widehat{p}\xi = \widehat{p}S\xi \in \widehat{p}M$ for each $\xi \in M$. It is clear that $S|_{Mp}$ commutes with the action of $\widehat{p}\widehat{B}\widehat{p}$. That $S|_{Mp}$ belongs to $\mathrm{End}^0(M)$ can be checked from Rem. 11.4. This proves that $S|_{Mp}$ belongs to $\mathrm{End}^0_{-,pBp}(Mp)$. We have thus proved that the linear map (11.2) is well-defined.

Step 2. Let us prove the surjectivity of (11.2). By Rem. 5.4, $B$ is finite-dimensional. Therefore, we have an orthogonal primitive decomposition $1_{\widehat{B}} - \widehat{p} = f_1 + \cdots + f_n$ in $\widehat{B}$. In this case, we have

$$M = \widehat{p}M \oplus f_1 M \oplus \cdots \oplus f_n M$$

By Prop. 6.2, for each $1 \leqslant i \leqslant n$, $f_i$ is isomorphic to a sub-idempotent $q_i$ of $\widehat{p}$, i.e., there exist $u_i \in f_i \widehat{B} q_i$ and $v_i \in q_i \widehat{B} f_i$ such that $u_i v_i = f_i$ and $v_i u_i = q_i \leqslant \widehat{p}$ (where $q_i \in \widehat{B}$ is an idempotent).

Now, we choose $T \in \mathrm{End}^0_{-,pBp}(Mp) = \mathrm{End}^0_{-,pBp}(\widehat{p}M)$. Define a linear map

$$S : M \to M \qquad \xi \mapsto T(\widehat{p}\xi) + \sum_{i=1}^{n} u_i T(v_i \xi) \qquad (11.3)$$

By Rem. 11.4, we have $S \in \mathrm{End}^0(M)$. We claim that $S$ commutes with the action of $\widehat{B}$ (and hence $S \in \mathrm{End}^0_{-,B}(M)$). If this is proved, then since $T$ clearly equals $S|_{Mp} = S|_{\widehat{p}M}$ (because $v_i\widehat{p} = 0$, see below), the proof of the surjectivity of (11.2) is complete.

Note that since $\widehat{p}, f_1, \ldots, f_n$ are mutually orthogonal, we have

$$u_i u_j = 0 \qquad v_i v_j = 0 \qquad \text{for all } i, j$$

---

[4]That is, for each $\xi \in M, \phi \in M^{\vee}$, the operator $\xi \otimes \phi$ sends each $\eta \in M$ to $\phi(\eta) \cdot \xi$.

$$v_j u_i = 0 \qquad \text{for all } i \neq j$$
$$v_i \widehat{p} = 0 \qquad \widehat{p} u_i = 0 \qquad \text{for all } i$$

Using this observation and the fact that $T : \widehat{p}M \to \widehat{p}M$ commutes the left action of $\widehat{p}\widehat{B}\widehat{p}$, we compute that for each $j$ and $\xi \in M$,

$$S(v_j \xi) = T(\widehat{p} v_j \xi) + 0 = T(v_j \xi)$$

$$v_j S(\xi) = v_j T(\widehat{p}\xi) + v_j u_j T(v_j \xi) \xrightarrow{v_j u_j = q_j \in \widehat{p}\widehat{B}\widehat{p}} 0 + T(q_j v_j \xi) = T(v_j \xi)$$

and hence $S(v_j \xi) = v_j S(\xi)$; similarly,

$$S(u_j \xi) = T(\widehat{p} u_j \xi) + u_j T(v_j u_j \xi) \xrightarrow{v_j u_j = q_j \in \widehat{p}\widehat{B}\widehat{p}} 0 + u_j q_j T(\widehat{p}\xi) = u_j T(\widehat{p}\xi)$$

$$u_j S(\xi) = u_j T(\widehat{p}\xi) + 0 = u_j T(\widehat{p}\xi)$$

and hence $S(u_j \xi) = u_j S(\xi)$. Moreover, for each $b \in \widehat{B}$ we have

$$S(\widehat{p} b \widehat{p} \xi) = T(\widehat{p} b \widehat{p} \xi) + 0 = \widehat{p} b \widehat{p} T(\widehat{p}\xi)$$

$$\widehat{p} b \widehat{p} S(\xi) = \widehat{p} b \widehat{p} T(\widehat{p}\xi) + 0 = \widehat{p} b \widehat{p} T(\widehat{p}\xi)$$

and hence $S(\widehat{p} b \widehat{p} \xi) = \widehat{p} b \widehat{p} S(\xi)$. This proves that $S$ commutes with the left action of $\widehat{B}$, since $\widehat{B}$ is generated by $\{u_i, v_i : 1 \leqslant i \leqslant n\}$ and $\widehat{p}\widehat{B}\widehat{p}$—to see this, note that for each $b \in \widehat{B}$, by setting $f_0 = u_0 = v_0 = \widehat{p}$, we have

$$b = \sum_{i,j=0}^{n} f_i b f_j = \sum_{i,j=0}^{n} u_i b_{i,j} v_j$$

where each $b_{i,j} := v_i b u_j$ commutes with the left actions of $A$ and satisfies $b_{i,j} = \widehat{p} b_{i,j} \widehat{p}$, and hence belongs to $\widehat{p}\widehat{B}\widehat{p}$.

Step 3. If $S \in \operatorname{End}^0_{-,B}(M)$ and $S|_{\widehat{p}M} = 0$, then for each $\xi \in M$, we have

$$S(\xi) = S(\widehat{p}\xi) + \sum_{i=1}^{n} S(f_i \xi) = S(\widehat{p}\xi) + \sum_{i=1}^{n} u_i S(v_i \xi)$$

where $\widehat{p}\xi, v_j \xi \in \widehat{p}M$. Therefore $S = 0$. This proves that (11.2) is injective. $\qquad \square$

**Lemma 11.6.** *Suppose that $e \in A$ is a generating idempotent. Then we have a linear isomorphism*

$$A \xrightarrow{\simeq} \operatorname{End}^0_{-,eAe}(Ae) \tag{11.4}$$

*sending each $a \in A$ to the left multiplication by $a$.*

*Proof.* It is obvious that the left action on $Ae$ by $a \in A$ belongs to $\operatorname{End}^0_{-,eAe}(Ae)$. Therefore, the map (11.4) is well-defined.

Suppose that the left multiplication of $a \in A$ on $Ae$ is zero. Then $aAe = 0$. Since $A$ is AUF and hence almost unital, there is an idempotent $p \in A$ such that $a = ap$. Since $e$ is

generating, by Cor. 7.7, $Ae$ is a generator of $\mathrm{Coh}_{\mathrm{L}}(A)$. Therefore, $Ap$ is a quotient module of $(Ae)^{\oplus n}$ for some $n \in \mathbb{Z}_+$. Thus $aAp$ is a quotient space of $(aAe)^{\oplus n}$, and hence $aAp = 0$. This proves $ap = 0$, and hence $a = 0$. We have thus proved that (11.4) is injective.

Choose $T \in \mathrm{End}^0_{-,eAe}(Ae)$. Since $T \in \mathrm{End}^0(Ae)$, by Rem. 11.2, there is an idempotent $f \in A$ such that $T = fTf$. It follows that $fTf|_{fAe}$ belongs to $\mathrm{End}_{-,eAe}(fAe)$. Since $A$ is AUF, we may enlarge $f$ so that $e \leqslant f$ also holds. We claim that $\mathrm{End}_{-,eAe}(fAe)$ consists of the left multiplications by elements of $fAf$. If this is true, then $T|_{fAe} = fTf|_{fAe}$ is the left multiplication by $faf$ for some $a \in A$. It follows that for any $b \in A$, we have $Tbe = Tfbe = fafbe$, and hence $T$ is the left multiplication by $faf$ on $Ae$, finishing the proof that (11.4) is surjective.

By Cor. 6.3, the idempotent $e \in fAf$ is generating in $fAf$. Applying Prop. 11.5 to the finite-dimensional unital algebra $fAf$ and its (finite-dimensional) coherent left module $fAf$, we see that $\mathrm{End}_{-,eAe}(fAe) = fAf|_{fAe}$. This proves the claim. $\qquad\square$

**Theorem 11.7.** *Suppose that $A$ is strongly AUF, and let $G$ be a projective generator of $\mathrm{Coh}_{\mathrm{L}}(A)$ (which exists due to Prop. 7.8). Set $B = \mathrm{End}_{A,-}(G)^{\mathrm{op}}$. Regard $G$ as an $A$-$B$ bimodule. Then we have a linear isomorphism*

$$A \xrightarrow{\simeq} \mathrm{End}^0_{-,B}(G) \tag{11.5}$$

*sending each $a \in A$ to the left multiplication of $a$ on $G$.*

*Proof.* By Cor. 6.4 and Prop. 7.8, $A$ has a generating idempotent $e$. If $G = Ae$, then $\mathrm{End}_{A,-}(G) = eAe$ due to Prop. 1.2. Therefore, by Lem. 11.6, the map (11.5) is bijective.

If $G = (Ae)^{\oplus n}$ where $n \in \mathbb{Z}_+$, one easily checks that $B = eAe \otimes \mathbb{C}^{n \times n}$ where $\mathbb{C}^{n \times n}$ is the matrix algebra of order $n$. The bijectivity of (11.5) then follows easily.

Finally, let $G$ be any general projective generator. By Cor. 7.12, we may assume that $G = (Ae)^{\oplus n}p$ where $n \in \mathbb{Z}_+$, and $p$ is a generating idempotent of $\widetilde{B} = \mathrm{End}_{A,-}((Ae)^{\oplus n})^{\mathrm{op}} \simeq eAe \otimes \mathbb{C}^{n \times n}$. By Prop. 7.10, we have $B = p\widetilde{B}p$. Therefore, by Prop. 11.5, the map

$$\mathrm{End}^0_{-,\widetilde{B}}((Ae)^{\oplus n}) \to \mathrm{End}^0_{-,B}(G)$$

sending each $S$ to $S|_G$ is bijective. By the previous paragraph, the map

$$A \to \mathrm{End}^0_{-,\widetilde{B}}((Ae)^{\oplus n})$$

sending each $a$ to the left multiplication by $a$ is bijective. Therefore, their composition, namely (11.5), is bijective. $\qquad\square$

**Remark 11.8.** In Thm. 11.7, the right $B$-module $G$ is a **projective generator** in the category $\mathrm{Mod}^{\mathrm{R}}(B)$ of right $B$-modules—that is, $G$ is projective in $\mathrm{Mod}^{\mathrm{R}}(B)$, and any object in $\mathrm{Mod}^{\mathrm{R}}(B)$ has an epimorphism from a (possibly infinite) direct sum of $G$.

*Proof.* The projectivity of $G$ in $\mathrm{Mod}^{\mathrm{R}}(B)$ is due to Thm. 9.4 and Rem. 4.2. Using the notation in the proof of Thm. 11.7, we may assume $G = (Ae)^{\oplus n}p$ and $B = p(eAe \otimes \mathbb{C}^{n \times n})p$ where $e \in A$ and $p \in eAe \otimes \mathbb{C}^{n \times n}$ are generating idempotents. Since $B$ is unital, $B$ is generating in $\mathrm{Mod}^{\mathrm{R}}(B)$. Therefore $(eAe \otimes \mathbb{C}^{n \times n})p$ is generating in $\mathrm{Mod}^{\mathrm{R}}(B)$. Since $(eAe \otimes \mathbb{C}^{n \times n})p$ is a direct sum of $(eAe \otimes \mathbb{C}^{1 \times n}) = (eAe)^{\oplus n}p = eG$, we conclude that $eG$ is generating in $\mathrm{Mod}^{\mathrm{R}}(B)$. Therefore $G$ is generating in $\mathrm{Mod}^{\mathrm{R}}(B)$. $\qquad\square$

**Theorem 11.9.** *Let $\mathcal{A}$ be an algebra. The following are equivalent.*

*(1) $\mathcal{A}$ is strongly AUF.*

*(2) $\mathcal{A}$ is isomorphic to $\mathrm{End}^0_{-,B}(M)$ where $B$ is a unital finite-dimensional algebra, $M$ is a projective generator in $\mathrm{Mod}^{\mathrm{R}}(B)$, the vector space $M$ has a grading*

$$M = \bigoplus_{i \in \mathfrak{I}} M(i)$$

*where each $M(i)$ is finite-dimensional and is preserved by the right action of $B$, and $\mathrm{End}^0_{-,B}(M)$ is defined by*

$$\mathrm{End}^0_{-,B}(M) := \{T \in \mathrm{End}(M) : (Tm)b = T(mb) \text{ for all } m \in M, b \in B,$$
$$T|_{M(i)} = 0 \text{ for all by finitely many } i \in \mathfrak{I}\}$$

*Proof.* The direction (1)⇒(2) follows from Thm. 11.7 and Rem. 11.8. Let us prove the other direction.

Assume that $\mathcal{A} = \mathrm{End}^0_{-,B}(M)$ where $\mathrm{End}^0_{-,B}(M)$ is described as in (2). Let $e_i$ be the projection of $M$ onto $M(i)$. Then $e_i$ clearly belongs to $\mathcal{A}$, and each $T \in \mathcal{A}$ can be written as $T = \sum_{i,j \in \mathfrak{I}} e_i T e_j$ where $e_i T e_j = 0$ for all but finitely many $i, j$. This proves that $\mathcal{A}$ is AUF.

Since $M$ is a projective generator in $\mathrm{Mod}^{\mathrm{R}}(B)$, for each finite subset $I \subset \mathfrak{I}$, $M_I := \bigoplus_{i \in I} M(i)$ is projective in $\mathrm{Mod}^{\mathrm{R}}(B)$ (since it is a direct summand of $M$). Let $1_B = p_1 + \cdots + p_n$ be an orthogonal primitive decomposition of $1_B$ in $B$. By Thm. 5.5, irreducible finite-dimensional right $B$-modules are precisely those that are isomorphic to $p_k B / \mathrm{rad}(p_k B)$ for some $k$. Since $M$ is generating in $\mathrm{Mod}^{\mathrm{R}}(B)$, it has an epimorphism to $p_k B / \mathrm{rad}(p_k B)$ for each $k$. This epimorphism must restrict to a nonzero morphism (and hence an epimorphism) $M(i_k) \to p_k B / \mathrm{rad}(p_k B)$. Let $I = \{i_1, \ldots, i_n\}$. Then $M_I$ has an epimorphism to each irreducible right $B$-module. It follows from Prop. 7.6 that $M_I$ is a projective generator in the category of finite-dimensional right $B$-modules.

Let $e_I = \sum_{i \in I} e_i$, which is an idempotent in $\mathcal{A}$. We claim that $e_I$ is a generating idempotent in $\mathcal{A}$, which will complete the proof that $\mathcal{A}$ is strongly AUF.

Let $\varepsilon$ be any primitive idempotent of $\mathcal{A}$. Then $\varepsilon M$ is a finite-dimensional right $B$-module, since any element of $\mathcal{A}$ has finite range when acting on $M$. Moreover, since $\varepsilon$ is primitive in $\mathcal{A}$, the right $B$-module $\varepsilon M$ is indecomposable. Since $\varepsilon M$ is a direct summand of the projective right $B$-module $M$, it follows that $\varepsilon M$ is a finite-dimensional indecomposible projective right $B$-module. Therefore, since $M_I = e_I M$ is a projective generator, similar to the the end of the proof of Thm. 7.11, we conclude that the right $B$-module $\varepsilon M$ is isomorphic to a direct summand of $e_I M$. Thus, by Thm. 7.10, $\varepsilon$ is isomorphic to a subidempotent of $e_I$ in $\mathcal{A}$. This proves the claim that $e_I$ is generating. $\square$

# References

[AF92]  Frank W. Anderson and Kent R. Fuller. *Rings and categories of modules*, volume 13 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[AN13]  Yusuke Arike and Kiyokazu Nagatomo. Some remarks on pseudo-trace functions for orbifold models associated with symplectic fermions. *Int. J. Math.*, 24(2):1350008, 29, 2013.

[Ari10]    Yusuke Arike. Some remarks on symmetric linear functions and pseudotrace maps. *Proc. Japan Acad. Ser. A Math. Sci.*, 86(7):119–124, 2010.

[BBG21]    Anna Beliakova, Christian Blanchet, and Azat M. Gainutdinov. Modified trace is a symmetrised integral. *Selecta Math. (N.S.)*, 27(3):Paper No. 31, 51, 2021.

[DGK25]    Chiara Damiolini, Angela Gibney, and Daniel Krashen. Conformal blocks on smoothings via mode transition algebras. *Comm. Math. Phys.*, 406(6):Paper No. 131, 58, 2025.

[FZ92]     Igor B. Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, 66(1):123–168, 1992.

[GR19]     Azat M. Gainutdinov and Ingo Runkel. The non-semisimple Verlinde formula and pseudo-trace functions. *J. Pure Appl. Algebra*, 223(2):660–690, 2019.

[GZ25]     Bin Gui and Hao Zhang. How are pseudo-$q$-traces related to (co)ends? To appear, 2025.

[Hat65]    Akira Hattori. Rank element of a projective module. *Nagoya Math. J.*, 25:113–120, 1965.

[Hua24]    Yi-Zhi Huang. Associative algebras and the representation theory of grading-restricted vertex algebras. *Commun. Contemp. Math.*, 26(6):Paper No. 2350036, 46, 2024.

[Miy04]    Masahiko Miyamoto. Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness. *Duke Math. J.*, 122(1):51–91, 2004.

[MNT10]    Atsushi Matsuo, Kiyokazu Nagatomo, and Akihiro Tsuchiya. Quasi-finite algebras graded by Hamiltonian and vertex operator algebras. In *Moonshine: the first quarter century and beyond*, volume 372 of *London Math. Soc. Lecture Note Ser.*, pages 282–329. Cambridge Univ. Press, Cambridge, 2010.

[Sta65]    John Stallings. Centerless groups—an algebraic formulation of Gottlieb's theorem. *Topology*, 4:129–134, 1965.

[Zhu96]    Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA.
*E-mail*: binguimath@gmail.com        bingui@tsinghua.edu.cn

YAU MATHEMATICAL SCIENCES CENTER AND DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY, BEIJING, CHINA.
*E-mail*: zhanghao1999math@gmail.com        h-zhang21@mails.tsinghua.edu.cn