Topics in Operator Algebras: Algebraic Conformal Field Theory

Bin Gui

Current page of writing

Contents

Notations

$\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$.

Unless otherwise stated, an **unbounded operator** $T : \mathcal{H} \to \mathcal{K}$ (where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces) denotes a linear map from a dense linear subspace $\mathscr{D}(T) \subset \mathcal{H}$ to $\mathcal{H}$. $\mathscr{D}(T)$ is called the **domain** of T. We let T^* be the adjoint of T. In practice, we are also interested in T^* defined on a dense subspace of its domain $\mathscr{D}(T^*)$. We call its restriction a **formal adjoint** of T and denote it by $T^\dagger$.

Given a Hilbert space $\mathcal{H}$, its inner product is denoted by $(\xi, \eta) \in \mathcal{H}^2 \mapsto \langle \xi | \eta \rangle$. We assume that it is linear on the first variable and antilinear on the second one. (Namely, we are following mathematician's convention.)

Whenever we write $\langle \xi, \eta \rangle$, we understand that it is linear on both variables. E.g. $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual space.

If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, we let

$$\mathrm{Hom}(\mathcal{H}, \mathcal{K}) = \{\text{Bounded linear maps } \mathcal{H} \to \mathcal{H}\} \qquad \mathrm{End}(\mathcal{H}) = \mathrm{Hom}(\mathcal{H}, \mathcal{H}) \tag{0.1}$$

If X is a set, the n-fold **configuration space** $\mathrm{Conf}^n(X)$ is

$$\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X : x_i \neq x_j \text{ if } i \neq j\} \tag{0.2}$$

Definition 0.1. A map of complex vector spaces $T : V \to V'$ is called **antilinear** or **conjugate linear** if $T(a\xi + b\eta) = \overline{a}T\xi + \overline{b}T\eta$ for all $\xi, \eta \in V$ and $a, b \in \mathbb{C}$. If V and V' are (complex) inner product spaces, we say that T is **antiunitary** if it is am antiliear surjective and satisfies $\|T\xi\| = \|\xi\|$ for all $\xi \in V$, equivalently,

$$\langle T\xi | T\eta \rangle = \overline{\langle \xi | \eta \rangle} \equiv \langle \eta | \xi \rangle \tag{0.3}$$

for all $\xi, \eta \in V$.

1 Introduction: PCT symmetry, Bisognano-Wichmann, Tomita-Takesaki

Algebraic quantum field theory (AQFT) is a mathematically rigorous approach to QFT using the language of functional analysis and operator algebras. The main subject of this course is 2d **algebraic conformal field theory (ACFT)**, namely, 2d CFT in the framework of AQFT.

1.1

Let $d \in \mathbb{Z}_+$. We first sketch general picture of an $(1 + d)$ dimensional Poincaré invariant QFT in the spirit of **Wightman axioms**. We consider Bosonic theory for simplicity.

We let $\mathbb{R}^{1,d}$ be the $(1+d)$-dimensional **Minkowski space**. So it is $\mathbb{R}^{1+d}$ but with metric tensor

$$ds^2 = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^d)^2 \tag{1.1}$$

Here x^0 denotes the time coordinate, and $x^1, \dots, x^d$ denote the spatial coordinates. The (restricted) **Poincaré group** is

$$\mathrm{P}^+(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}^+(1,d)$$

Here, $\mathbb{R}^{1,d}$ acts by translation on $\mathbb{R}^{1,d}$. $\mathrm{SO}^+(1,d)$ is the (restricted) **Lorentz group**, the identity component of the (full) Lorentz group $\mathrm{O}(1,d)$ whose elements are invertible linear maps on $\mathbb{R}^{1,d}$ preserving the Minkowski metric.

Remark 1.1. Any $g \in \mathrm{O}(1,d)$ must have determinent ± 1. One can show that $\mathrm{SO}^+(1,d)$ is precisely the elements $g \in \mathrm{O}(1,d)$ such that $\det g = 1$, and that g does not change the direction of time (i.e., if $\mathbf{v} = (v_0, \dots, v_d) \in \mathbb{R}^{1,d}$ satisfies $v_0 > 0$, then the first component of $g\mathbf{v}$ is > 0). See [Haag, Sec. I.2.1].

Definition 1.2. We say that $\mathbf{x} = (x_0, \dots, x_d), \mathbf{y} = (y_0, \dots, y_d) \in \mathbb{R}^{1,d}$ are **spacelike (separated)** if their Minkowski distance is negative, i.e.,

$$(x_0 - y_0)^2 < (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$$

1.2

A Poincaré invariant QFT consists of the following data:

(1) We have a Hilbert space $\mathcal{H}$.

(2) There is a (strongly continuous) projective unitary representation U of $\mathrm{P}^+(1,d)$ on $\mathcal{H}$. In particular, it restriction to the translation on the k-th component (where $k = 0,1,\ldots,d$) gives a one parameter unitary group $x^k \in \mathbb{R} \mapsto \exp(\mathbf{i}x^k P_k)$ where P_k is a self-adjoint operator on $\mathcal{H}$.

(3) (Positive energy) The following are positive operators:

$$P_0 \geqslant 0 \qquad (P_0)^2 - (P_1)^2 - \cdots - (P_d)^2 \geqslant 0$$

The operator P^0 is called the **Hamiltonian** or the **energy operator**. $P^1,\ldots,P^d$ are the momentum operators. $(P_0)^2 - (P_1)^2 - \cdots - (P_d)^2$ is the mass.

(4) We have a collection of **(quantum) fields** $\mathscr{Q}$, where each $\Phi \in \mathscr{Q}$ is an operator-valued function on $\mathbb{R}^{1,d}$. For each $\mathbf{x} \in \mathbb{R}^{1,d}$, $\Phi(x)$ is a "linear operator on $\mathcal{H}$".

(5) **(**Locality) If $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{1,d}$ are spacelike and $\Phi_1, \Phi_2 \in \mathscr{Q}$, then

$$[\Phi_1(x_1), \Phi_2(x_2)] = 0 \tag{1.2}$$

(6) (*-invariance) For each $\Phi \in \mathscr{Q}$, there exists $\Phi^\dagger \in \mathscr{Q}$ such that

$$\Phi(\mathbf{x})^\dagger = \Phi^\dagger(\mathbf{x}) \tag{1.3}$$

(7) (Poincaré invariance) There is a distinguished unit vector[1] Ω, called the **vacuum vector**, such that

$$U(g)\Omega = \Omega \qquad \forall g \in \mathrm{P}^+(1,d)$$

Moreover, for each $g \in \mathrm{P}^+(1,d)$ and $\Phi \in \mathscr{Q}$, we have

$$U(g)\Phi(\mathbf{x})U(g)^{-1} = \Phi(g\mathbf{x}) \tag{1.4}$$

(8) (Cyclicity) Vectors of the form

$$\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega \tag{1.5}$$

(where $n \in \mathbb{N}$, $\mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathbb{R}^{1,d}$ are mutually spacelike, and $\Phi_1,\ldots,\Phi_n \in \mathscr{Q}$) span a dense subspace of $\mathcal{H}$.

Remark 1.3. In some QFT, there is a factor (a function of $\mathbf{x}$) before $\Phi(g\mathbf{x})$ in the Poincaré invariance relation (1.4). Similarly, there is a factor before $\Phi^\dagger(\mathbf{x})$ in the $*$-invariance formula (1.3). We will encounter these more general covariance property later. In this section, we content ourselves with the simplest case that the factors are 1.

Remark 1.4. By the Poincaré invariance and the cyclicity, the action of $\mathrm{P}^+(1,d)$ is uniquely determined by $\mathscr{Q}$ by

$$U(g)\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(g\mathbf{x}_1)\cdots\Phi_n(g\mathbf{x}_n)\Omega \tag{1.6}$$

[1]A unit vector denotes a vector with length 1

1.3

Technically speaking, $\Phi(\mathbf{x})$ can not be viewed as a linear operator on $\mathcal{H}$. It cannot be defined even on a sufficiently large subspace of $\mathcal{H}$. One should think about **smeared fields**

$$\Phi(f) = \int_{\mathbb{R}^{1,d}} \Phi(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{1.7}$$

where $f \in C_c^\infty(\mathbb{R}^{1,d})$. (In contrast, we call $\Phi(\mathbf{x})$ a **pointed field**.) Then $\Phi(f)$ is usually a closable unbounded operator on $\mathcal{H}$ with dense domain $\mathscr{D}(\Phi(f))$. Moreover, $\mathscr{D}(\Phi(f))$ is preserved by any smeared operator $\Psi(g)$. Therefore, for any $f_1, \ldots, f_n \in C_c^\infty(\mathbb{R}^{1,d})$ the following vector can be defined in $\mathcal{H}$:

$$\Phi_1(f_1) \cdots \Phi_n(f_n)\Omega \tag{1.8}$$

The precise meaning of cyclicity in Subsec. 1.2 means that vectors of the form (1.8) span a dense subspace of $\mathcal{H}$. Locality means that for $f_1, f_2 \in C_c^\infty(\mathbb{R}^{1,d})$ compactly supported in spacelike regions, on a reasonable dense subspace of $\mathcal{H}$ (e.g., the subspace spanned by (1.8)) we have

$$[\Phi_1(f_1), \Phi_2(f_2)] = 0 \tag{1.9}$$

The $*$-invariance means that

$$\langle \Phi(f)\xi|\eta\rangle = \langle \xi|\Phi^\dagger(f)\eta\rangle \tag{1.10}$$

for each ξ, η in the this good subspace.

1.4

In the remaining part of this section, if possible, we also understand $\Phi(\mathbf{x})$ as a smeared operator $\Phi(f)$ where $f \in C_c^\infty(\mathbb{R}^{1,d})$ satisfies $\int f = 1$ and is supported in a small region containing $\mathbf{x}$. Thus, $\Phi(\mathbf{x})$ can almost be viewed as a closable operator. Hence the expression (1.5) makes sense in $\mathcal{H}$.

We now explore the consequences of positive energy. As we will see, it implies that $\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega$, a function of $\mathbf{x}_\bullet$, can be analytically continued.

The fact that $P_0 \geqslant 0$ implies that when $t \geqslant 0$, e^{tP_0} is a bounded linear operator with operator norm $\leqslant 1$. Therefore, if τ belongs to

$$\mathfrak{J} = \{\mathrm{Im}\tau \geqslant 0\}$$

then $e^{\mathbf{i}\tau P_0} = e^{\mathbf{i}\Re\tau} \cdot e^{-\mathrm{Im}\tau}$ is bounded. Indeed, $\tau \in \mathfrak{J} \mapsto e^{\mathbf{i}\tau P_0}$ is continuous, and is holomorphic on $\mathrm{Int}\mathfrak{J}$.

Let $\mathbf{e}_0 = (1, 0, \ldots, 0)$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$ be distinct. By the Poincaré covariance, the relation

$$e^{\mathbf{i}\tau P_0}\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(\mathbf{x}_1 + \tau e_0)\cdots\Phi_n(\mathbf{x}_n + \tau e_0)\Omega \tag{1.11}$$

holds for all real τ. Moreover, the LHS is continuous on $\mathfrak{I}$ and holomorphic on $\mathrm{Int}\mathfrak{I}$. This suggests that the RHS of (1.11) can also be defined as an element of $\mathcal{H}$ when $\tau \in \mathfrak{I}$.

1.5

We shall further explore the question: for which $\mathbf{x}_i$ is in $\mathbb{C}^d$ can $\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega$ be reasonably defined as an element of $\mathcal{H}$?

Remark 1.5. We expect that the smeared fields should be defined on any P_0-**smooth vectors**, i.e., vectors in $\bigcap_{k\geqslant 0}\mathscr{D}(P_0^k)$. For each $r > 0$, since one can find $C_{k,r} \geqslant 0$ such that $\lambda^{2k} \leqslant C_{k,r}e^{2r\lambda}$ for all $\lambda \geqslant 0$, we conclude that

$$\mathrm{Rng}(e^{-rP_0}) \equiv \mathscr{D}(e^{rP_0}) \subset \bigcap_{k\geqslant 0}\mathscr{D}(P_0^k) \tag{1.12}$$

The above remark shows that $\Phi_1(0)$, viewed as a smeared operator localized on a small neighborhood of 0, is definable on $e^{\mathbf{i}\zeta_2 P_0}\Phi_2(0)\Omega = \Phi_2(\zeta_2\mathbf{e}_0)\Omega$ whenever $\mathrm{Im}\zeta_2 > 0$. Thus, heuristically, $(\zeta_1, \zeta_2) \mapsto e^{\mathbf{i}\zeta_1 P_0}\Phi_1(\mathbf{x}_1)e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\Omega$ should also be holomorphic on

$$\{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \mathrm{Im}\zeta_1, \mathrm{Im}\zeta_2 > 0\}$$

More generally, the holomorphicity holds for

$$e^{\mathbf{i}\zeta_1 P_0}\Phi_1(\mathbf{x}_1)e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\cdots e^{\mathbf{i}\zeta_n P_0}\Phi_n(\mathbf{x}_n)\Omega$$

when $\mathrm{Im}\zeta_i > 0$. By Poincaré covariance, the above expression equals

$$\Phi_1(\mathbf{x}_1 + \zeta_1\mathbf{e}_0)\Phi_2(\mathbf{x}_2 + (\zeta_1 + \zeta_2)\mathbf{e}_0)\cdots\Phi_n(\mathbf{x}_n + (\zeta_1 + \cdots + \zeta_n)\mathbf{e}_0)\Omega$$

Therefore,

$$(\zeta_1, \ldots, \zeta_n) \mapsto \Phi_1(\mathbf{x}_1 + \zeta_1\mathbf{e}_0)\cdots\Phi_n(\mathbf{x}_n + \zeta_n\mathbf{e}_0) \in \mathcal{H} \tag{1.13}$$

should be holomorphic on $\{\zeta_\bullet \in \mathbb{C}^n : 0 < \mathrm{Im}\zeta_1 < \cdots < \mathrm{Im}\zeta_n\}$.

By the locality axiom, the order of products of quantum fields can be exchanged. Thus, our expectation for a reasonable QFT includes the following condition:

Conclusion 1.6. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$. Then (1.13) is holomorphic on

$$\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \mathrm{Im}\zeta_i > 0, \text{ and } \mathrm{Im}\zeta_i \neq \mathrm{Im}\zeta_j \text{ if } i \neq j\} \tag{1.14a}$$

Moreover, since (1.13) is also definable and continuous on

$$\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n : \mathbf{x}_1 + \zeta_1\mathbf{e}_1, \ldots, \mathbf{x}_n + \zeta_n\mathbf{e}_0 \text{ are mutually spacelike}\} \tag{1.14b}$$

we expect that the function (1.13) is continuous on the union of (1.14a) and (1.14b).

1.6

We have (informally) derived some consequences from the positivity of P_0. Note that since $P_0 \geqslant 0$, we have $U(g)P_0U(g)^{-1} \geqslant 0$ for each $g \in \mathrm{SO}^+(1,d)$. Since P_0 is the generator of the flow $t \in \mathbb{R} \mapsto t\mathbf{e}_0 \in \mathbb{R}^{1,d} \subset \mathrm{P}^+(1,d)$, $U(g)P_0U(g)^{-1}$ is the generator of the flow

$$t \in \mathbb{R} \mapsto g(t\mathbf{e}_0)g^{-1} = t \cdot g\mathbf{e}_0 \tag{1.15}$$

Therefore, if $g{=}(a_0,\dots,a_n)$, then

$$U(g)P_0U(g)^{-1} = a_0P_0 + \cdots + a_nP_n \tag{1.16}$$

Hence the RHS must be positive. But what are all the possible $g\mathbf{e}_0$?

Remark 1.7. One can show that the orbit of $\mathbf{e}_0 = (1,0,\dots,0)$ under $\mathrm{SO}^+(1,d)$ is the upper hyperbola with diameter 1, i.e., the set of all $(a_0,\dots,a_n) \in \mathbb{R}^{1,d}$ satisfying

$$a_0 > 0 \qquad (a_0)^2 - (a_1)^2 - \cdots - (a_n)^2 = 1 \tag{1.17}$$

Thus $\sum_i a_iP_i \geqslant 0$ for all such $a_\bullet$. What are the consequences of this positivity?

1.7

To simplify the following discussions, we set $d = 2$ and

$$t = x^0 \qquad x = x^1$$

We further set

$$u = t - x \qquad v = t + x \tag{1.18}$$

so that

$$t = \frac{u+v}{2} \qquad x = \frac{-u+v}{2} \tag{1.19}$$

The Minkowski metric becomes

$$\boxed{(dt)^2 - (dx)^2 = du \cdot dv} \tag{1.20}$$

Then

$$(u,v) \text{ is spacelike to } (u',v') \qquad \Longleftrightarrow \qquad (u-u')(v-v') < 0 \tag{1.21}$$

For each $\Phi \in \mathscr{Q}$, we write

$$\widetilde{\Phi}(u,v) := \Phi(t,x) = \Phi\Big(\frac{u+v}{2}, \frac{-u+v}{2}\Big) \tag{1.22}$$

We let H_0 and H_1 be the self-adjoint operators such that

$$H_0 = P_0 - P_1 \qquad H_1 = P_0 + P_1$$

so that they are the generators of the flow $t \mapsto (t,-t)$ and $t \mapsto (t,t)$.

Remark 1.8. Since $\mathbb{R}^{1,d}$ is an abelian group, we know that P_i commutes with P_j. Hence H_0 commutes with H_1.

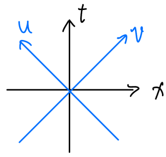

Figure 1.1. The coordinates u, v

1.8

The orbit of $\mathbf{e}_0$ under $\mathrm{SO}^+(1,1)$ is the unit upper hyperbola $(x^0)^2 - (x_1)^2 = 1, x^0 > 0$. Equivalently, it is $uv = 1, u > 0$. According to Subsec. 1.6, we conclude that $b_0 H_0 + b_1 H_1 \geqslant 0$ for each b_0, b_1 satisfying $b_0 b_1 = 1, b_0 > 0$ (equivalently, for each $b_0 > 0, b_1 > 0$). This implies

$$H_0 \geqslant 0 \qquad H_1 \geqslant 0 \tag{1.23}$$

Therefore, similar to the argument in Subsec. 1.5 (and specializing to the special case that $\mathbf{x}_1 = \cdots = \mathbf{x}_n = 0$), the holomorphicity of

$$(\zeta_\bullet, \gamma_\bullet) \mapsto e^{\mathbf{i}\zeta_1 H_0 + \mathbf{i}\gamma_1 H_1} \widetilde{\Phi}_1(0) e^{\mathbf{i}\zeta_2 H_0 + \mathbf{i}\gamma_2 H_1} \widetilde{\Phi}_2(0) \cdots e^{\mathbf{i}\zeta_n H_0 + \mathbf{i}\gamma_n H_1} \widetilde{\Phi}_n(0)\Omega$$

on the region $\mathrm{Im}\zeta_i > 0, \mathrm{Im}\gamma_i > 0$, together with locality, implies:

Conclusion 1.9. Let $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$. Then

$$(u_1, v_1, \ldots, u_n, v_n) \mapsto \widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}(u_n, v_n)\Omega \tag{1.24}$$

is holomorphic on

$$\{(u_\bullet, v_\bullet) \in \mathbb{C}^{2n} : \mathrm{Im}u_i > 0, \mathrm{Im}v_i > 0, \mathrm{Im}u_i \neq \mathrm{Im}u_j, \mathrm{Im}v_i \neq \mathrm{Im}v_j \text{ if } i \neq j\} \tag{1.25a}$$

and can be continuously extended to

$$\{(u_\bullet, v_\bullet) \in \mathbb{R}^{2n} : (u_i - u_j) \cdot (v_i - v_j) < 0 \text{ if } i \neq j\} \tag{1.25b}$$

Rigorously speaking, the above mentioned "continuity" of the extension should be understood in terms of distributions. Here, we ignore such subtlety and view pointed fields as smeared field in a small region.

1.9

We note that $\mathrm{diag}(-1, \pm 1)$ is not inside $\mathrm{SO}^+(1,1)$, since it reverses the time direction. Neither is $\mathrm{diag}(1, -1)$ in $\mathrm{SO}^+(1,1)$ because its determinant is negative. Consequently, the QFT is not necessarily symmetric under the following operations:

- **Time reversal** $t \mapsto -x$.
- **Parity transformation** $x \mapsto -x$.
- **PT transformation** $(t, x) \mapsto (-t, -x)$, the combination of time and parity inversions.

Mathematically, this means that the maps

$$\begin{aligned}
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega &\mapsto \Phi_1(-t_1, x_1) \cdots \Phi_n(-t_n, x_n)\Omega \\
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega &\mapsto \Phi_1(t_1, -x_1) \cdots \Phi_n(t_n, -x_n)\Omega \\
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega &\mapsto \Phi_1(-t_1, -x_1) \cdots \Phi_n(-t_n, -x_n)\Omega
\end{aligned}$$

(where $(t_1, x_1), \ldots, (t_n, x_n)$ are mutually spacelike) are not necessarily unitary. (Compare Rem. 1.4.) Simiarly, the QFT is not necessarily symmetric under **Charge conjugation** $\Phi \mapsto \Phi^\dagger$, which means that the map

$$\begin{aligned}
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega \mapsto\ & \Phi_n(t_n, x_n)^\dagger \cdots \Phi_1(t_1, x_1)^\dagger\Omega \\
=& \Phi_1^\dagger(t_1, x_1) \cdots \Phi_n^\dagger(t_n, x_n)\Omega
\end{aligned}$$

is not necessarily unitary. However, as we shall explain, the combination of PCT transformations is actually unitary, and hence is a symmetry of the QFT. This is called the PCT theorem.

1.10

To prove the PCT theorem, we shall first prove that the PT transformation, though not implemented by a unitary operator, is actually implemented by the analytic continuation of a one parameter unitary group.

Definition 1.10. The one parameter group $s \mapsto \Lambda(s) \in \mathrm{SO}^+(1,1)$ defined by

$$\Lambda(s)(u, v) = (e^{-s}u, e^s v) \tag{1.26}$$

is called the **Lorentz boost**. Equivalently,

$$\Lambda(s) \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \tag{1.27}$$

Define the (open) **right wedge** $\mathcal{W}$ and **left wedge** $-\mathcal{W}$ by

$$\mathcal{W} = \{(u, v) \in \mathbb{R}^2 : v > 0, u < 0\} = \{(t, x) \in \mathbb{R}^{1,1} : -x < t < x\} \tag{1.28}$$

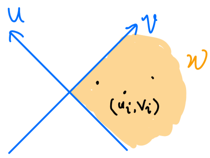

Figure 1.2.

Theorem 1.11 (PT theorem). *Let $(u_1, v_1), \dots, (u_n, v_n) \in \mathcal{W}$ be mutually spacelike (i.e. satisfying $(u_i - u_j)(v_i - v_j) < 0$ if $i \neq j$), cf. Fig. 1.2. Let $\Phi_1, \dots, \Phi_n \in \mathscr{Q}$. Let K be the self-adjoint generator of the Lorentz boost, i.e.,*

$$U(\Lambda(s)) = e^{\mathbf{i}sK}$$

Then $\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega$ belongs to the domain of $e^{-\pi K}$, and

$$e^{-\pi K}\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega = \Phi_1(-\mathbf{x}_1) \cdots \Phi_n(-\mathbf{x}_n)\Omega \tag{1.29}$$

Equivalently, $\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega$ belongs to the domain of $e^{-\pi K}$, and

$$e^{-\pi K}\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega = \widetilde{\Phi}_1(-u_1, -v_1) \cdots \widetilde{\Phi}_n(-u_n, -v_n)\Omega \tag{1.30}$$

Note that the requirement that $(u_1, v_1), \dots, (u_n, v_n) \in \mathcal{W}$ are spacelike means, after relabeling the subscripts, that

$$0 < v_1 < \cdots < v_n \qquad 0 < -u_1 < \cdots < -u_n$$

Proof. This theorem relies on the following fact that we shall prove rigorously in the future:

- ⋆ Let $T \geqslant 0$ be a self-adjoint operator on $\mathcal{H}$ with $\mathrm{Ker}(T) = 0$. Let $r > 0$. Then $\xi \in \mathcal{H}$ belongs to $\mathscr{D}(T^r)$ iff the function $s \in \mathbb{R} \mapsto T^{\mathbf{i}s}\xi \in \mathcal{H}$ can be extended to a continuous function F on

  $$\{z \in \mathbb{C} : -r \leqslant \mathrm{Im} z \leqslant 0\}$$

  and holomorphic on its interior. Moreover, for such ξ we have $F(-\mathbf{i}r) = T^r\xi$.

In fact, the function $F(z)$ is given by $z \mapsto T^z\xi$.

We shall apply this result to $T = e^{-K}$ and $r = \pi$. For that purpose, we must show that the $\mathcal{H}$-valued function of $s \in \mathbb{R}$ defined by

$$e^{\mathbf{i}\pi s}\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega = \widetilde{\Phi}_1(e^{-s}u_1, e^s v_1) \cdots \widetilde{\Phi}_n(e^{-s}u_n, e^s v_n)\Omega$$

can be extended to a continuous function on

$$\{z \in \mathbb{C} : 0 \leqslant \mathrm{Im} z \leqslant \pi\}$$

and holomorphic on its interior.

In fact, we can construct this $\mathcal{H}$-valued function, which is

$$z \mapsto \widetilde{\Phi}_1(e^{-z}u_1, e^z v_1)\cdots\widetilde{\Phi}_n(e^{-z}u_n, e^z v_n)\Omega$$

noting that the conditions in Conc. 1.9 are fulfilled. In particular, the condition $0 < \mathrm{Im} < \pi$ is used to ensure that, since $u_i < 0, v_i > 0$, we have $\mathrm{Im}(e^{-z}u_i) > 0$ and $\mathrm{Im}(e^z v_i) > 0$ as required by (1.25a). The value of this function at $z = \mathbf{i}\pi$ equals the RHS of (1.30). Therefore the theorem is proved. □

1.11

Theorem 1.12 (PCT theorem). *We have an antiunitary map $\Theta : \mathcal{H} \to \mathcal{H}$, called the **PCT operator**, such that*

$$\Theta \cdot \Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(-\mathbf{x}_1)^\dagger\cdots\Phi_n(-\mathbf{x}_n)^\dagger\Omega \tag{1.31}$$

for any $\Phi_1,\dots,\Phi_n \in \mathscr{Q}$ and mutually spacelike $\mathbf{x}_1,\dots,\mathbf{x}_n$.

Equivalently, Θ is defined by

$$\Theta \cdot \widetilde{\Phi}_1(u_1,v_1)\cdots\widetilde{\Phi}_n(u_n,v_n) = \widetilde{\Phi}_1(-u_1,-v_1)^\dagger\cdots\widetilde{\Phi}_n(-u_n,-v_n)^\dagger\Omega \tag{1.32}$$

Proof. The existence of an antilinear isometry Θ satisfying (1.32) is equivalent to showing that (cf. (0.3))

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_n(\mathbf{u}_n)\Omega|\widetilde{\Psi}_1(\mathbf{u}'_1)\cdots\widetilde{\Psi}_k(\mathbf{u}'_k)\Omega\rangle\\ =&\langle\widetilde{\Psi}_1(-\mathbf{u}'_1)^\dagger\cdots\widetilde{\Psi}_k(-\mathbf{u}'_k)^\dagger\Omega|\widetilde{\Phi}_1(-\mathbf{u}_1)^\dagger\cdots\widetilde{\Phi}_n(-\mathbf{u}_n)^\dagger\Omega\rangle\end{aligned}$$

if $\mathbf{u}_1,\dots\mathbf{u}_n$ are spacelike, and $\mathbf{u}'_1,\dots\mathbf{u}'_k$ are spacelike. (We do not assume that, say, $\mathbf{u}_1$ and $\mathbf{u}'_1$ are spacelike.)

It suffices to prove this in the special case that $\mathbf{u}_1,\dots,\mathbf{u}_n,\mathbf{u}'_1,\dots,\mathbf{u}'_k$ are mutually spacelike. Then the general case will follow that both sides of the above relation can be analytically continued to suitable regions as functions of $\mathbf{u}_1,\dots,\mathbf{u}_n$. For example, the fact that $H_0, H_1 \geqslant 0$ implies that

$$e^{\mathbf{i}\zeta H_0+\mathbf{i}\gamma H_1}\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_n(\mathbf{u}_n)\Omega = \widetilde{\Phi}_1(\mathbf{u}_1+(\zeta,\gamma))\cdots\widetilde{\Phi}_n(\mathbf{u}_n+(\zeta,\gamma))\Omega$$

is continuous on $\{(\zeta,\gamma)\in\mathbb{C}^2 : \mathrm{Im}\zeta \geqslant 0, \mathrm{Im}\gamma \geqslant 0\}$ and holomorphic on its interior.

Set $\Gamma_j = \Psi_k^\dagger$. Then the above relation is equivalent to

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\\ =&\langle\widetilde{\Phi}_1(-\mathbf{u}_1)\cdots\widetilde{\Phi}_1(-\mathbf{u}_n)\widetilde{\Gamma}_1(-\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(-\mathbf{u}'_k)\Omega|\Omega\rangle\end{aligned}$$

By the PT Thm. 1.11, this relation is equivalent to

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\\ =&\langle e^{-\pi K}\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\end{aligned}$$

But this of course holds since $e^{-\pi K}\Omega = \Omega$ by Poincaré invariance. □

1.12

Combining the PT Thm. 1.11 with the PCT Thm. 1.12, we conclude that $e^{-\pi K}$ is an injective positive operator, Θ is antinitary, and

$$\Theta e^{-\pi K} A\Omega = A^\dagger \Omega \tag{1.33a}$$

where A is a product of spacelike separated field in $\mathcal{W}$. The rigorous statement should be that

$$A = \Phi_1(f_1)\cdots\Phi_n(f_n)$$

where $\Phi_1,\ldots,\Phi_n \in \mathscr{Q}$, and $f_i \in C_c^\infty(O_i)$ where $O_1,\ldots,O_n \subset \mathcal{W}$ are open and mutually spacelike. If we let $\mathscr{A}(\mathcal{W})$ be the $*$-algebra generated by all such A, then by the Poincaré invariance, for each $g \in \mathrm{P}^+(1,d)$ we have

$$U(g)\mathscr{A}(\mathcal{W})U(g)^{-1} = \mathscr{A}(g\mathcal{W})$$

In particular, since for the Lorentz boost Λ we have $\Lambda(s)\mathcal{W} = \mathcal{W}$, we therefore have

$$e^{\mathbf{i}sK}\mathscr{A}(\mathcal{W})e^{-\mathbf{i}sK} = \mathscr{A}(\mathcal{W}) \tag{1.33b}$$

for all $s \in \mathbb{R}$. Since the PT transformation sends $\mathcal{W}$ to $-\mathcal{W}$, the definition of Θ clearly also implies (noting $\Theta^{-1} = \Theta^* = \Theta$)

$$\Theta\mathscr{A}(\mathcal{W})\Theta = \mathscr{A}(-\mathcal{W}) \tag{1.33c}$$

Note that since $\mathcal{W}$ is local to $-\mathcal{W}$, we have $[\mathscr{A}(\mathcal{W}),\mathscr{A}(-\mathcal{W})] = 0$. Therefore, $\Theta\mathscr{A}(\mathcal{W})\Theta$ is a subset of the (in some sense) commutant of $\mathscr{A}(\mathcal{W})$.

1.13

The set of formulas (1.33) is reminiscent of the Tomita-Takesaki theory, one of the deepest theories in the area of operator algebras. The setting is as follows.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Namely, $\mathcal{M}$ is a $*$-subalgebra of $\mathrm{End}(\mathcal{H})$ closed under the "strong operator topology". (We will formally introduce von Neumann algebras in a later section.) Let $\Omega \in \mathcal{H}$ be a unit vector. Assume that Ω is **cyclic** (i.e. $\mathcal{M}\Omega$ is dense) and **separating** (i.e., if $x \in \mathcal{M}$ and $x\Omega = 0$ then $x = 0$) under $\mathcal{M}$. Then the **Tomita-Takesaki theorem** says that the linear map

$$S : \mathcal{M}\Omega \to \mathcal{M}\Omega \qquad x\Omega \mapsto x^*\Omega$$

is antilinear and closable. Denote its closure also by S, and consider its polar decomposition $S = J\Delta^{\frac{1}{2}}$ where Δ is a positive closed operator, and J is an antiunitary map. Then Δ is injective, we have $J^{-1} = J^* = J$, and

$$\Delta^{\mathbf{i}s}\mathcal{M}\Delta^{-\mathbf{i}s} = \mathcal{M} \qquad J\mathcal{M}J = \mathcal{M}'$$

where $\mathcal{M}'$ is the commutant $\{y \in \mathrm{End}(\mathcal{H}) : xy = yx\ (\forall x \in \mathcal{M})\}$.We call Δ and J respectively the **modular operator** and the **modular conjugation**.

1.14

To relate the Tomita-Takesaki theory to QFT, one takes $\mathcal{M}$ to be $\mathfrak{A}(\mathcal{W})$, the von Neumann algebra generated by $\mathscr{A}(\mathcal{W})$. Note that the elements of $\mathscr{A}(\mathcal{W})$ are typically unbounded operators, whereas those of $\mathfrak{A}(\mathcal{W})$ are bounded. Thus, the meaning of "the von Neumann algebra generated by a set of closed/closable operators" should be clarified. This is an important notion, and we will study it in a later section.

To apply the setting of Tomita-Takesaki, one should first show that the vacuum vector is cyclic and separating under $\mathfrak{A}(\mathcal{W})$. This is not an easy task, although it is relatively easier to show that Ω is cyclic and separating under $\mathscr{A}(\mathcal{W})$. Moreover, we have

Theorem 1.13 (Bisognano-Wichmann). *Let Δ and J be the modular operator and the modular conjugation of $(\mathfrak{A}(\mathcal{W}), \Omega)$. Then $J = \Theta$ and $\Delta^{\frac{1}{2}} = e^{-\pi K}$.*

Since (1.33c) easily implies $\Theta\mathfrak{A}(\mathcal{W})\Theta = \mathfrak{A}(-\mathcal{W})$, together with $J\mathcal{M}J = \mathcal{M}'$ we obtain

$$\mathfrak{A}(\mathcal{W})' = \mathfrak{A}(-\mathcal{W}) \tag{1.34}$$

a version of **Haag duality**.

One of the main goals of this course is to give a rigorous and self-contained proof of the PCT theorem, the Bisognano-Wichmann theorem, and the Haag duality for 2d chiral conformal field theories.

1.15

For a general odd number $d > 0$, the above results should be modified as follows. Let K be the generator of the **Lorentz boost**

$$\Lambda(s) = \left(\begin{array}{cc|ccc} \cosh s & \sinh s & & 0 & \\ \sinh s & \cosh s & & & \\ \hline & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{array}\right)$$

Let $\Lambda(\mathbf{i}\pi) = \mathrm{diag}(-1,-1,1,\dots,1)$, which does not belong to $\mathrm{P}^+(1,d)$ since it reverses the time direction (although it has positive determinant). Then the PT Thm. 1.11 should be modified by replacing (1.29) with

$$e^{-\pi K}\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(\Lambda(\mathbf{i}\pi)\mathbf{x}_1)\cdots\Phi_n(\Lambda(\mathbf{i}\pi)\mathbf{x}_n)\Omega \tag{1.35}$$

Let $\rho = \mathrm{diag}(1,1,-1,\dots,-1)$, which has determinant 1 (since d is odd) and hence belongs to $\mathrm{SO}^+(1,d)$. Then the PCT Thm. 1.12 still holds verbatim. Let

$$\mathcal{W} = \{(a_0,\dots,a_n) \in \mathbb{R}^{1,d} : -a_1 < a_0 < a_1\} \tag{1.36}$$

Then the **Bisognano-Wichmann theorem** says that $e^{-\pi K}$ is the modular operator of $(\mathfrak{A}(\mathcal{W}), \Omega)$, and $\Theta U(\rho)$ is the modular conjugation.

We refer the readers to [Haag, Sec. V.4.1] and the reference therein for a detailed study.

Index

References

[Gui-S] Gui, B. (2021). Spectral Theory for Strongly Commuting Normal Closed Operators. See https://binguimath.github.io/

[Haag] Haag, G. Local quantum physics. Fields, particles, algebras. 2nd., rev. and enlarged ed. Berlin: Springer-Verlag (1996)

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA.
E-mail: binguimath@gmail.com