Notes on Complex Analytic Geometry

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Forewords

This monograph is mainly intended to help my self-study of the theory of complex analytic spaces. I do not claim any originality for the results in this monograph. In writing this monograph, I drew on the following references: [GR-b] (influencing my writing of chapters 2,3,4,6), [Fis] (chapters 1,3,5), [BS] (chapter 6), [GPR] (chapters 3,4), [Vak17] (chapter 5), [Dem] (chapter 6).

The main goal of this monograph is to prove the semicontinuity theorem and base change theorems of Grauert. Namely: Thm. 6.6.2 and its immediate consequence Thm. 6.6.10, Thm. 6.7.4, and Thm. 6.7.10. Thus, Sec. 6.6 and 6.7 are the climax of this monograph. And all previous results can be viewed as paving the way for the proof and understanding of these theorems. I have tried to prove everything needed: the only exceptions are Stein theory (e.g. Cartan's Theorems) and Grauert direct image theorem, whose references are given in Chapter 6.

Base change theorems and semicontinuity theorem give satisfying answers to the following type of questions: Suppose that we have a complex manifold X and a (finite-rank) holomorphic vector bundle $\mathscr{E}$ on X (namely, a locally-free $\mathscr{O}_X$-module $\mathscr{E}$), and suppose we deform the complex structures of X and $\mathscr{E}$. Under what conditions can we extend an element of $\mathscr{E}(X)$ to global sections of the nearby complex manifolds? In mathematical physics, one also considers deformations of "(possibly) singular complex manifolds", a.k.a. *complex analytic spaces* ("complex spaces" for short). For instance, in 2-dimensional conformal field theory and string theory, one considers deformations of compact curves with possibly nodal singularities. Nodal curves are the "limits" of compact Riemann surfaces. ("Flat holomorphic maps" are a rigorous formulation of this limiting process.) So, even if one is primarily interested in smooth complex manifolds, general complex spaces are often inevitable.

Although, as mentioned at the beginning, this monograph was written to help myself learn about the subject, I would be more than happy if others interested in this subject could benefit from my writing.

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Chapter 1

Basic notions of complex spaces

1.1 Notations and conventions

The following notations and conventions are assumed throughout the monograph.

All rings and algebras are assumed to have a unity 1. Their morphisms are assumed to map 1 to 1. "Rings" and "$\mathbb{C}$-algebras" are always assumed to be commutative, unless otherwise stated. In general, an $\mathcal{B}$-algebra $\mathcal{A}$ means a morphism of rings $\mathcal{B} \to \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are already $\mathbb{C}$-algebras, we require the morphisms to be also $\mathbb{C}$-linear.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$.

$\mathbf{i} = \sqrt{-1}$.

$\{0\}, \mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \dots$ are called **(complex) number spaces**.

Unless otherwise stated, all vector spaces are over $\mathbb{C}$.

A neighborhood of a point x in a topological space means an *open* subset containing x.

A **precompact subset** U of a topological space X is a subset such that the closure U^{cl} in X is compact. A **nowhere dense subset** of X is a subset whose closure in X contains no non-empty open subsets of X.

$\mathbb{C}\{z_1, \dots, z_n\}$ denotes $\mathscr{O}_{\mathbb{C}^n,0}$, the algebra of convergent power series of $z_1, \dots, z_n$. It is clearly an integral domain. $\mathbb{C}[z_1, \dots, z_n]$ denotes the algebra of polynomials of $z_1, \dots, z_n$.

We assume the readers are familiar with the basic notions of sheaves and their maps (morphisms), sheafifications, image sheaves, kernels and cokernels of sheaves. For each presheaf $\mathscr{E}$ on a topological space X, we let $\mathscr{E}_x$ denote the stalk of $\mathscr{E}_x$ at x. The stalk of $s \in \mathscr{E}$ at x is denoted by s_x, or sometimes abbreviated to s when no confusion arises.

If $\varphi : X \to Y$ is a continuous map of topological spaces, then the **direct image** $\varphi_*\mathscr{E}$ denotes the sheaf on Y whose space of sections over any open $V \subset Y$ is

$\mathscr{E}(\varphi^{-1}(V))$, i.e.

$$(\varphi_*\mathscr{E})(V) = \mathscr{E}(\varphi^{-1}(V)).$$

If $\psi : Y \to Z$ is continuous, we clearly have

$$(\psi \circ \varphi)_*\mathscr{E} = \psi_*(\varphi_*\mathscr{E}).$$

If $f : \mathscr{E}_1 \to \mathscr{E}_2$ is an X-sheaf map, then we have a canonical $\varphi_* f : \varphi_*\mathscr{E}_1 \to \varphi_*\mathscr{E}_2$. φ_* is a **left exact functor** from the category of X-sheaves to that of Y-sheaves. (Cf. Rem. 1.9.6.)

If $\mathscr{F}$ is an $\mathscr{O}_Y$-module, the **inverse image** $\varphi^{-1}(\mathscr{F})$ is the sheafification of the presheaf on X associating to each open subsets of X:

$$U \mapsto \varinjlim_{V \supset \varphi(U)} \mathscr{F}(V)$$

where the direct limit is over all open subset $V \subset Y$ containing $\varphi(U)$. For each $x \in X$ there is a natural equivalence

$$(\varphi^{-1}\mathscr{F})_x \simeq \mathscr{F}_{\varphi(x)}. \tag{1.1.1}$$

$\mathscr{E}_U$, $\mathscr{E}|U$, $\mathscr{E}|_U$, $\mathscr{E} \restriction_U$ all denote the restriction of an X-sheaf $\mathscr{E}$ to the open subset U. If Y is a subset of X and $\iota : Y \hookrightarrow X$ is the inclusion map, we define the **set theoretic restriction**

$$\mathscr{E} \restriction_Y = \iota^{-1}(\mathscr{E}). \tag{1.1.2}$$

In particular, for each $y \in Y$, we have a canonical identification

$$(\mathscr{E} \restriction_Y)_y = \mathscr{E}_y. \tag{1.1.3}$$

Warning: in the future, we will define the restriction $\mathscr{E}|_Y = \mathscr{E}|Y$ when Y is a complex subspace of a complex space X and $\mathscr{E}$ is an $\mathscr{O}_X$-module. $\mathscr{E}|Y$ will be different from $\mathscr{E} \restriction_Y$. In particular, $(\mathscr{E}|_Y)_y$ is not $\mathscr{E}_y$.

We also write $\mathscr{E}(U)$ as $H^0(U, \mathscr{E})$.

Recall that the **support of an X-sheaf** $\mathscr{E}$, denoted by $\mathrm{Supp}(\mathscr{E})$, is the subset of all $x \in X$ such that $\mathscr{E}_x \neq 0$.

Remark 1.1.1. If Y is a closed subset of a topological space X, then there is a one-to-one correspondence between Y-sheaves $\mathscr{F}$ and X-sheaves $\mathscr{E}$ satisfying $\mathrm{Supp}(\mathscr{E}) \subset Y$: For any open $U \subset X$,

$$\mathscr{F}(U \cap Y) = \mathscr{E}(U). \tag{1.1.4}$$

Let $\iota : Y \hookrightarrow X$ be the inclusion. Then clearly $\iota_*\mathscr{F} = \mathscr{E}$ and $\mathscr{E} \restriction_Y = \mathscr{F}$. We often view $\mathscr{E}$ and $\mathscr{F}$ as the same thing.

If U is an open subset of $\mathbb{C}^N$, then a **holomorphic function** f on U is, by definition, a continuous function $f : U \to \mathbb{C}$ which is separately holomorphic on each variable (i.e., if $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N$ are fixed, then $f(z_\bullet) = f(z_1, \ldots, z_N)$ is holomorphic with respect to z_i).

Remark 1.1.2. The above definition agrees with our usual understanding of analytic functions, i.e., f has convergent power series expansions $f(z_\bullet) = \sum_{n_1,\ldots,n_N \in \mathbb{N}} a_{n_1,\ldots,n_N}(z_1 - w_1)^{n_1} \cdots (z_N - w_N)^{n_N}$ if $(w_\bullet) \in U$. To see this, choose a holomorphic f on U. Let us assume for simplicity $w_1 = \cdots = w_N = 0$, and U is the polydisc $\mathbb{D}_{R_\bullet} = \{(z_\bullet) \in \mathbb{C}^N : |z_1| < R_1, \ldots, |z_N| < R_N\}$ where $R_1, \ldots, R_N > 0$. Then for each $0 < r_i < R_i$ and $z_\bullet \in \mathbb{D}_{r_\bullet}$,

$$f(z_\bullet) = \oint\limits_{|\zeta_1|=r_1} \cdots \oint\limits_{|\zeta_N|=r_N} \frac{f(\zeta_\bullet)}{(\zeta_1 - z_1)\cdots(\zeta_N - z_N)} \cdot \frac{d\zeta_1 \cdots d\zeta_N}{(2\mathbf{i}\pi)^N}$$

by applying residue theorem successively to the variables $\zeta_1, \ldots, \zeta_N$. Write each $(\zeta_i - z_i)^{-1}$ as $\sum_{n_i=0}^{\infty} z_i^{n_i}/\zeta_i^{n_i+1}$ which converges absolutely and uniformly on $|\zeta_i| = r_i$ and $z_\bullet$ on any compact subset of $\mathbb{D}_{r_\bullet}$, and substitute them into the above integral, we get the desired series expansion which is absolutely and uniformly convergent on $|z_1| \leqslant r_1, \ldots, |z_N| \leqslant r_N$ for all $0 < r_i < R_i$. This proves one direction. For the other direction, namely absolutely convergent power series give holomorphic functions, one simply applies Morera's theorem to each complex variable.

Lemma 1.1.3 (Identitätssatz). *If X is a connected complex manifold, and if h is a non-zero (i.e. not constantly zero) holomorphic function on X, then h is non-zero when restricted to any open subset U of X.*

Proof. Consider the special case that X, U are open polydiscs in $\mathbb{C}^n$. We know the lemma is true when $n = 1$ (by e.g. taking power series). For a general n, if $h|_U = 0$, we may enlarge successively the disc-shape domains of each variable $z_1, \ldots, z_n$ on which h is constantly zero to get $h = 0$.

In general, we let Ω be the (clearly open) subset of all $x \in X$ such that h is constantly zero on a neighborhood of x (i.e. the germ of h at x is zero). If $x \in X\backslash\Omega$, then every neighborhood of $x \in X$ biholomorphic to an open polydisc must be disjoint from Ω, according to the previous paragraph. So $X\backslash\Omega$ is open. Since X is connected, Ω must be either $\varnothing$ or U. Thus $\Omega = \varnothing$ since $h \neq 0$. $\square$

1.2 $\mathbb{C}$-ringed spaces and sheaves of modules

1.2.1 $\mathbb{C}$-ringed spaces

Definition 1.2.1. A **$\mathbb{C}$-ringed space** is a topological space X together with a **sheaf of local $\mathbb{C}$-algebras** $\mathscr{O}_X$ on X (i.e., for each open $U \subset X$, $\mathscr{O}_X(U)$ is a $\mathbb{C}$-algebra with unity, and the additions and multiplications are compatible with the restriction to open subsets of U; each stalk $\mathscr{O}_{X,x}$ is a **local** $\mathbb{C}$**-algebra**).

By saying that $\mathscr{O}_{X,x}$ is a local $\mathbb{C}$-algebra, we mean that there is a unique maximal ideal $\mathfrak{m}_{X,x}$ of $\mathscr{O}_{X,x}$, and that we have an isomorphism of vector spaces

$$\mathbb{C} \xrightarrow{\simeq} \mathbb{C}_x := \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}, \qquad \lambda \mapsto \lambda 1.$$

We write $\mathfrak{m}_{X,x}$ as $\mathfrak{m}_x$ when no confusion arises. For each $f \in \mathscr{O}_{X,x}$, we let $f(x) \in \mathbb{C}$ denote the residue class of f in $\mathscr{O}_{X,x}/\mathfrak{m}_x$, called the **value** of f at x. In this way, any section of $\mathscr{O}_X$ can be viewed as a function.

$\mathscr{O}_X$ is called the **structure sheaf** of X. Each open subset $U \subset X$ is automatically a $\mathbb{C}$-ringed subspace of X with structure sheaf $\mathscr{O}_U := \mathscr{O}_X|_U$. □

For the sake of brevity, we write

$$\mathscr{O}(X) = \mathscr{O}_X(X) \tag{1.2.1}$$

The following important fact is obvious:

Proposition 1.2.2. *An element $f \in \mathscr{O}_{X,x}$ is a unit (i.e. invertible in the ring $\mathscr{O}_{X,x}$) iff $f(x) \neq 0$.*

Proof. $f(x) = 0$ iff $f \in \mathfrak{m}_{X,x}$ iff f is not a unit. □

Definition 1.2.3. A **morphism of** $\mathbb{C}$**-ringed spaces** $\varphi : X \to Y$ is a continuous map of topological spaces, together with a morphism of sheaves of $\mathbb{C}$-algebras $\varphi^\# : \mathscr{O}_Y \to \varphi_* \mathscr{O}_X$ (namely, $\varphi^\#$ is a sheaf map, and $\varphi^\# : \mathscr{O}_Y(V) \to \mathscr{O}_X(\varphi^{-1}(V))$ is a morphism of $\mathbb{C}$-algebras for each open $V \subset Y$), and for each $x \in X$ and $y = \varphi(x)$, the restriction $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a **morphism of local** $\mathbb{C}$**-algebras**, i.e. a morphism of $\mathbb{C}$-algebras such that

$$\varphi^\#(\mathfrak{m}_{Y,y}) \subset \mathfrak{m}_{X,x}. \tag{1.2.2}$$

The set of morphisms of $\mathbb{C}$-ringed spaces $X \to Y$ is denoted by $\mathrm{Mor}(X, Y)$. If $\varphi \in \mathrm{Mor}(X, Y)$ and $\psi \in \mathrm{Mor}(Y, Z)$, then their **composition** $\psi \circ \varphi \in \mathrm{Mor}(X, Z)$ is the usual composition of maps of sets, together with

$$(\psi \circ \varphi)^\# = \varphi^\# \circ \psi^\# : \mathscr{O}_{Z,\psi\circ\varphi(x)} \to \mathscr{O}_{X,x}$$

for all $x \in X$.

We leave it to the readers to define isomorphisms of $\mathbb{C}$-ringed spaces.

Proposition 1.2.4. *For each section $f \in \mathscr{O}_Y$ defined at $y = \varphi(x)$, we have*

$$(\varphi^\# f)(x) = f \circ \varphi(x). \tag{1.2.3}$$

Proof. This is true when $f = 1$ since $\varphi^\#$ preserves 1. It is also true when $f \in \mathfrak{m}_{Y,y}$. So it is true in general. $\square$

Thus, $\varphi^\#$ may be viewed as the transpose of φ. When studying morphisms between complex spaces, we may write $\varphi^\# f$ as $f \circ \varphi$ (cf. Rem. 1.4.2).

Example 1.2.5. A complex manifold is a $\mathbb{C}$-ringed space if we define the structure sheaf $\mathscr{O}_X$ to be the sheaf of (germs of) holomorphic functions. If X and Y are complex manifolds, then a holomorphic map from X to Y is a morphism of $\mathbb{C}$-ringed spaces.

1.2.2 Modules over $\mathbb{C}$-ringed spaces

We begin this section with the following general observation:

Remark 1.2.6. If $\mathscr{M}, \mathscr{N}$ are two subsheaves of an X-sheaf such that $\mathscr{M}_x = \mathscr{N}_x$ for all $x \in X$, then $\mathscr{M} = \mathscr{N}$. (For any $s \in \mathscr{M}$, $s_x \in \mathscr{M}_x = \mathscr{N}_x$ for all x on which s is defined. So $s \in \mathscr{N}$. So $\mathscr{M} \subset \mathscr{N}$, and vice versa.) Thus, we can talk about" the *unique* subsheaf of a given sheaf whose stalks are..." where the unique part is automatic.

Definition 1.2.7. A **presheaf of $\mathscr{O}_X$-modules** $\mathscr{E}$ on a $\mathbb{C}$-ringed space X is a sheaf such that for each open $U \subset X$, $\mathscr{E}(U)$ is an $\mathscr{O}(U)$-module, and that the linear combination and the action of $\mathscr{O}(U)$ on $\mathscr{E}(U)$ are compatible with the restriction to open subsets of U. If $\mathscr{E}$ is a sheaf, we call $\mathscr{E}$ an **$\mathscr{O}_X$-module**. We call the vector space

$$\mathscr{E}|x = \mathscr{E}_x/\mathfrak{m}_{X,x}\mathscr{E}_x = \mathscr{E}_x \otimes (\mathscr{O}_{X,x}/\mathfrak{m}_{X,x}) \tag{1.2.4}$$

the **fiber** of $\mathscr{E}$ at x. The right most expression of (1.2.4) will be explained in Rem. 1.9.3. The residue class of $s \in \mathscr{E}$ in $\mathscr{E}|x$ is denoted by $s(x)$ or $s|x$.

Definition 1.2.8. A **morphism of (presheaves of) $\mathscr{O}_X$-modules** $\varphi : \mathscr{E} \to \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are (presheaves of) $\mathscr{O}_X$-modules, is a sheaf map intertwining the actions of $\mathscr{O}_X$. More precisely, for each open $U \subset X$, $\varphi : s \in \mathscr{E}(U) \mapsto \varphi(s) \in \mathscr{F}(U)$ is a morphism of $\mathscr{O}(U)$-modules; if $V \subset U$ is open, then $\varphi(s|_U) = \varphi(s)|_U$.

φ is called **injective** resp. **surjective** if it is so as a sheaf map, namely $\varphi : \mathscr{E}_x \to \mathscr{F}_x$ is injective resp surjective for all $x \in X$. $\mathscr{E} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{G}$ is called **exact** if the corresponding sequence of stalk map $\mathscr{E}_x \xrightarrow{\varphi} \mathscr{F}_x \xrightarrow{\psi} \mathscr{G}_x$ is exact for all $x \in X$. φ is an **isomorphism** of $\mathscr{O}_X$-modules iff φ has an inverse iff φ is both injective and surjective. $\square$

Remark 1.2.9. In the following diagrams, assume that all objects are $\mathscr{O}_X$-modules, that all horizontal arrows are morphisms of $\mathscr{O}_X$-modules, and that the two horizontal lines are exact.

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{F} \arrow[r] \arrow[d, "\beta"'] & \mathscr{G} \arrow[d, "\gamma"'] \\
0 \arrow[r] & \mathscr{E}' \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{G}'
\end{tikzcd}
\tag{1.2.5}
$$

If there are morphisms β, γ such that the second square in (1.2.5) commutes, then β restricts to a (necessarily unique) morphism α such that the first square commutes.

$$
\begin{tikzcd}
\mathscr{E} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{F} \arrow[r] \arrow[d, "\beta"'] & \mathscr{G} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
\mathscr{E}' \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{G}' \arrow[r] & 0
\end{tikzcd}
\tag{1.2.6}
$$

If there are morphisms α, β such that the first square in (1.2.6) commutes, then β descends to a (necessarily unique) morphism γ such that the second square commutes.

Of course, the same observations hold for morphisms of modules of any commutative ring/algebra, and for general sheaf maps. $\square$

Remark 1.2.10 (Gluing construction of sheaves). Let $(V_\alpha)_{\alpha\in\mathfrak{A}}$ be an open cover of a topological space X. Suppose that for each $\alpha \in \mathfrak{A}$, we have a sheaf $\mathscr{E}^\alpha$, that for any $\alpha, \beta \in \mathfrak{A}$, we have a sheaf isomorphism $\phi_{\beta,\alpha} : \mathscr{E}^\alpha_{V_\alpha\cap V_\beta} \xrightarrow{\simeq} \mathscr{E}^\beta_{V_\alpha\cap V_\beta}$, that $\phi_{\alpha,\alpha} = \mathbf{1}$, and that $\phi_{\gamma,\alpha} = \phi_{\gamma,\beta}\phi_{\beta,\alpha}$ when restricted to $V_\alpha \cap V_\beta \cap V_\gamma$. Then we can define a sheaf $\mathscr{E}$ on X as follows. For any open $U \subset X$, $\mathscr{E}(U)$ is the set of all $(s_\alpha)_{\alpha\in\mathfrak{A}} \in \prod_{\alpha\in\mathfrak{A}} \mathscr{E}^\alpha(U \cap V_\alpha)$ (where each component s_α is in $\mathscr{E}^\alpha(U \cap V_\alpha)$) satisfying that $s_\beta|_{V_\alpha\cap V_\beta} = \phi_{\beta,\alpha}(s_\alpha|_{V_\alpha\cap V_\beta})$ for any $\alpha, \beta \in \mathfrak{A}$. If W is an open subset of U, then the restriction $\mathscr{E}(U) \to \mathscr{E}(W)$ is defined by that of $\mathscr{E}^\alpha(U\cap V_\alpha) \to \mathscr{E}^\alpha(W\cap V_\alpha)$. Then for each $\beta \in \mathfrak{A}$, we have a canonical isomorphism (trivialization) $\phi_\beta : \mathscr{E}_{V_\beta} \xrightarrow{\simeq} \mathscr{E}^\beta_{V_\beta}$ defined by $(s_\alpha)_{\alpha\in\mathfrak{A}} \mapsto s_\beta$. It is clear that for each $\alpha, \beta \in \mathfrak{A}$, we have $\phi_\beta = \phi_{\beta,\alpha}\phi_\alpha$ when restricted to $V_\alpha \cap V_\beta$.

In the case that X is a $\mathbb{C}$-ringed space, that each $\mathscr{E}^\alpha$ is an $\mathscr{O}_{V_\alpha}$-module, and that $\phi_{\beta,\alpha}$ is an equivalence of $\mathscr{O}_{V_\alpha\cap V_\beta}$-modules, then $\mathscr{E}$ is a sheaf of $\mathscr{O}_X$-modules. $\square$

Let X be a $\mathbb{C}$-ringed space.

Definition 1.2.11. A set of sections $\mathfrak{S} \subset \mathscr{O}_X(X)$ is said to **generate** the $\mathscr{O}_X$-module $\mathscr{E}$ if they generate each stalk $\mathscr{E}_x$, i.e., each element of $\mathscr{E}_x$ is an $\mathscr{O}_{X,x}$-linear combination of finitely many elements of $\mathfrak{S}$. Equivalently, this means that the $\mathscr{O}_X$-module morphism

$$\bigoplus_{s\in\mathfrak{S}} \mathscr{O}_X \to \mathscr{E}, \qquad \oplus_s f_s \mapsto \sum_s f_s \cdot s \tag{1.2.7}$$

(where $f_s \in \mathscr{O}_X$) is surjective. If it is also injective, we say $\mathfrak{S}$ **generates freely** $\mathscr{E}$.

Definition 1.2.12. We say an $\mathscr{O}_X$-module $\mathscr{E}$ is of **finite type** if each $x \in X$ is contained in a neighborhood U such that the restriction $\mathscr{E}|_U$ is generated by finitely many elements of $\mathscr{E}(U)$, or equivalently, there is a surjective $\mathscr{O}_U$-module morphism $\mathscr{O}_U^n \to \mathscr{E}|_U$.

Exercise 1.2.13. Show that if $\mathscr{E}$ is a finite type $\mathscr{O}_X$-module, then each stalk $\mathscr{E}_x$ is a finitely generated $\mathscr{O}_{X,x}$-module, and hence each fiber $\mathscr{E}|x$ is finite-dimensional.

Definition 1.2.14. If $\mathscr{E}_1, \mathscr{E}_2$ are $\mathscr{O}_X$-submodules of an $\mathscr{O}_X$-module $\mathscr{F}$. The sheafification of the presheaf

$$(\mathscr{E}_1 + \mathscr{E}_2)^{\mathrm{pre}}(U) = \mathscr{E}_1(U) + \mathscr{E}_2(U) \tag{1.2.8}$$

is denoted by $\mathscr{E}_1 + \mathscr{E}_2$. It is the unique subsheaf of $\mathscr{F}$ (cf. Rem. 1.2.6) whose stalks are $(\mathscr{E}_1 + \mathscr{E}_2)_x = \mathscr{E}_1 + \mathscr{E}_2$. It follows that if $\mathscr{E}_1$ is generated by $s_1, s_2, \dots \in \mathscr{E}_1(X)$ and $\mathscr{E}_2$ is generated by $t_1, t_2, \dots \in \mathscr{E}_2(X)$, then $\mathscr{E}_1 + \mathscr{E}_2$ is generated by $s_1, s_2, \dots, t_1, t_2, \dots$.

We recall the well-known

Theorem 1.2.15 (Nakayama's lemma). *Let $(\mathcal{A}, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$, and let $\mathcal{M}$ be a finitely generated $\mathcal{A}$-module. Choose a finite set of elements $s_1, \dots, s_n \in \mathcal{M}$. Then the following are equivalent.*

(1) $s_1, \dots, s_n$ generate the $\mathcal{A}$-module $\mathcal{M}$ (i.e. elements of $\mathcal{M}$ are $\mathcal{A}$-linear combinations of $s_1, \dots, s_n$).

(2) The residue classes of $s_1, \dots, s_n$ span the $(\mathcal{A}/\mathfrak{m})$-vector space $\mathcal{M}/(\mathfrak{m}\mathcal{M})$.

Proof. Clearly (1) implies (2). Let us prove (2)⇒(1). Assume $s_1, \dots, s_n$ span $\mathcal{M}/(\mathfrak{m}\mathcal{M})$. We extend the list $s_1, \dots, s_n$ to $s_1, \dots, s_N \in \mathcal{M}$ (where $N \geqslant n$) such that they generate $\mathcal{M}$. If $N = n$ then there is nothing to prove.

Assume $N > n$. Then every element of $\mathcal{M}$, and in particular s_N, can be written as

$$s_N = a_1 s_1 + \cdots + a_n s_n + \sigma$$

where $a_1, \dots, a_n \in \mathcal{A}$ and $\sigma \in \mathfrak{m}\mathcal{M}$. Since $s_1, \dots, s_N$ generate the $\mathcal{A}$-module $\mathcal{M}$, we have $\sigma = f_1 s_1 + \cdots + f_N s_N$ where $f_1, \dots, f_N \in \mathfrak{m}$. So

$$s_N = g_1 s_1 + \cdots + g_N s_N$$

where $g_{n+1}, g_{n+2}, \dots, g_N \in \mathfrak{m}$. Since $g_1 \in \mathfrak{m}$, $1 - g_1$ is invertible in $\mathcal{A}$. So

$$s_N = (1 - g_N)^{-1} \sum_{i=1}^{N-1} g_i s_i.$$

This proves that $s_1, \dots, g_{N-1}$ generate $\mathcal{M}$. By repeating this procedure several times, we conclude that $s_1, \dots, s_n$ generate $\mathcal{M}$. □

To apply Nakayama's lemma to sheaves of modules, we need the following observation:

Remark 1.2.16. Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Let $s_1, \dots, s_n$ be sections of $\mathscr{E}$ defined on a neighborhood of $x \in X$. Suppose (the germs of) $s_1, \dots, s_n$ generate the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$. Then there is a neighborhood U of x such that $s_1, \dots, s_n$ generate $\mathscr{E}|_U$. In particular, "$\mathscr{E}_x$ generates $\mathscr{E}|_U$".

Proof. Since $\mathscr{E}$ is finite-type, we may find U such that $\mathscr{E}|_U$ is generated by $t_1, \dots, t_m \in \mathscr{E}(U)$. Since $s_1, \dots, s_n$ generate $\mathscr{O}_x$, the germs of $t_1, \dots, t_m$ are $\mathscr{O}_{X,x}$-linear combinations of $s_1, \dots, s_n$. Thus, on a possibly smaller U, $t_1, \dots, t_m$ are $\mathscr{O}_X(U)$-linear combinations of $s_1, \dots, s_n$. So $s_1, \dots, s_n$ generate $\mathscr{E}|_U$. $\square$

Corollary 1.2.17. *Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Then $\mathrm{Supp}(\mathscr{E})$ is a closed subset of X.*

Proof. Assume the setting of Rem. 1.2.16. If $\mathscr{E}_x = 0$ then the stalks of $s_1, \dots, s_n$ are zero at x. So we may shrink U so that $s_1 = \cdots = s_n = 0$ in $\mathscr{E}(U)$. So $\mathscr{E}|_U = 0$. $\square$

Exercise 1.2.18. Use Nakayama's lemma and Rem. 1.2.16 to show that if $\mathscr{E}$ is a finite type $\mathscr{O}_X$-module, and if $s_1, \dots, s_n \in \mathscr{E}(U)$ (where U is a neighborhood of x) are such that $s_1(x), \dots, s_n(x)$ span the fiber $\mathscr{E}|x$, then they generate $\mathscr{E}|_V$ for a possibly smaller neighborhood V of x. (The opposite direction is obvious.) Nakayama's lemma is most often used in this form.

Corollary 1.2.19. *Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Then the* ***rank function*** *$x \in X \mapsto \dim(\mathscr{E}|x)$ is upper-semicontinuous.*

Definition 1.2.20. We say that an $\mathscr{O}_X$-module $\mathscr{E}$ is **free** if it is isomorphic to $\mathscr{O}_X^n$ for some $n \in \mathbb{N}$. We say $\mathscr{E}$ is **locally free** if each $x \in X$ is contained in a neighborhood U such that $\mathscr{E}|_U$ is free (or equivalently, that $\mathscr{E}|_U$ is generated freely by finitely many elements of $\mathscr{E}(U)$).

Exercise 1.2.21. Show that for a complex manifold X, locally free $\mathscr{O}_X$-modules $\mathscr{E}$ are the same as holomorphic vector bundles on X. Describe local trivializations and transition functions in terms of local free generators of $\mathscr{E}$.

Definition 1.2.22. An **ideal sheaf** $\mathcal{I}$ on a $\mathbb{C}$-ringed space X is an $\mathscr{O}_X$-submodule of $\mathscr{O}_X$. In particular, each stalk $\mathcal{I}_x$ is an ideal of $\mathscr{O}_{X,x}$. The **zero set** $N(\mathcal{I})$ is defined to be

$$\begin{aligned} N(\mathcal{I}) &:= \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}_x\} = \{x \in X : \mathcal{I}_x \subset \mathfrak{m}_{X,x}\} \\ &= \{x \in X : 1 \notin \mathcal{I}_x\} = \{x \in X : \mathcal{I}_x \neq \mathscr{O}_{X,x}\} = \mathrm{Supp}(\mathscr{O}_U/\mathcal{I}). \end{aligned} \tag{1.2.9}$$

Note that this is a closed subset of X by Cor. 1.2.17.

Proof. Note that $(\mathscr{O}_U/\mathcal{I})_x = \mathscr{O}_{U,x}/\mathcal{I}_x$. So $x \in \mathrm{Supp}(\mathscr{O}_U/\mathcal{I})$ iff $\mathscr{O}_{U,x}/\mathcal{I}_x \neq 0$ iff $\mathcal{I}_x \subsetneq \mathscr{O}_{U,x}$ iff $\mathcal{I}_x \subset \mathfrak{m}_x$ (as $\mathfrak{m}_x$ is the unique maximal ideal) iff $f(x) = 0$ for all $f \in \mathfrak{m}_x$. $\square$

Remark 1.2.23. If $\mathcal{I}$ is generated by $f_1, \dots, f_n \in \mathscr{O}(X)$, written as

$$\mathcal{I} = f_1\mathscr{O}_X + \cdots + f_n\mathscr{O}_X,$$

then clearly

$$N(\mathcal{I}) = \{\text{The common zeros of } f_1, \dots, f_n\}. \tag{1.2.10}$$

We also write $N(\mathcal{I})$ as $N(f_1, \dots, f_n)$.

1.3 Complex spaces and subspaces

Definition 1.3.1. A **(complex) model space** is

$$\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) := \big(N(\mathcal{I}), (\mathscr{O}_U/\mathcal{I})\restriction_{N(\mathcal{I})}\big) \tag{1.3.1}$$

where U is an open subset of a number space $\mathbb{C}^n$, $\mathscr{O}_U$ is the sheaf of holomorphic functions on U, $\mathcal{I}$ is a *finite-type* ideal of $\mathscr{O}_U$. $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is called the **analytic spectrum** of the sheaf $\mathscr{O}_U/\mathcal{I}$. Its underlying topological space is $\mathrm{Supp}(\mathscr{O}_U/\mathcal{I})$ as a subset of U, and its structure sheaf is $(\mathscr{O}_U/\mathcal{I})\restriction_{N(\mathcal{I})}$, whose stalk at any $x \in N(\mathcal{I})$ is $\mathscr{O}_{U,x}/\mathcal{I}_x$ (cf. (1.1.3)). With abuse of notations, one also writes for simplicity

$$\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) := \big(N(\mathcal{I}), \mathscr{O}_U/\mathcal{I}\big). \tag{1.3.2}$$

The stalk at $x \in N(\mathcal{I})$ of the structure sheaf is a local $\mathbb{C}$-algebra

$$\big(\mathscr{O}_{U,x}/\mathcal{I}_x, \mathfrak{m}_{U,x}/\mathcal{I}_x\big)$$

Definition 1.3.2. A $\mathbb{C}$-ringed Hausdorff space X is called a **complex space** if each point $x \in X$ is contained in a neighborhood V such that the $\mathbb{C}$-ringed space V (whose structure sheaf is defined by $\mathscr{O}_V := \mathscr{O}_X|_V$) is isomorphic to a model space. Sections of $\mathscr{O}_X(X)$ are called **holomorphic functions on** X. $\mathscr{O}_{X,x}$ is called an **analytic local $\mathbb{C}$-algebra**. Equivalently, an analytic local $\mathbb{C}$-algebra is $\mathbb{C}\{z_1, \dots, z_n\}/I$ for some finitely generated ideal I.[1]

If X, Y are complex spaces, a morphism $\varphi : X \to Y$ of $\mathbb{C}$-ringed spaces is called a **holomorphic map**. If φ has an inverse morphism $Y \to X$, we say that φ is a **biholomorphism**. Clearly, a holomorphic map φ is a biholomorphism iff it is a homeomorphism of topological spaces and induces isomorphisms $\varphi^\# : \mathscr{O}_{Y,\varphi(x)} \xrightarrow{\simeq} \mathscr{O}_{X,x}$ for each $x \in X$. $\square$

[1] As we shall see, $\mathbb{C}\{z_1, \dots, z_n\}$ is Noetherian. So the condition that I is finitely generated is redundant.

Definition 1.3.3. A **morphism of (analytic) local $\mathbb{C}$-algebras** $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a homomorphism of unital algebras sending $\mathfrak{m}_{Y,y}$ into $\mathfrak{m}_{X,x}$.

Definition 1.3.4. Let X be a complex space. An **open complex subspace** is $(U, \mathscr{O}_X|_U)$ where U is an open subset of X. A **closed complex subspace** is

$$\mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) := \big(N(\mathcal{I}), (\mathscr{O}_X/\mathcal{I}) \restriction_{N(\mathcal{I})}\big) \tag{1.3.3}$$

where $\mathcal{I}$ is a finite type ideal of $\mathscr{O}_X$. The stalk of the structure sheaf at $x \in N(\mathcal{I})$ is a local $\mathbb{C}$-algebra

$$\big(\mathscr{O}_{X,x}/\mathcal{I}_x, \mathfrak{m}_x/\mathcal{I}_x\big).$$

Remark 1.3.5. Let $X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. The construction of $\mathscr{O}_{X_0} = (\mathscr{O}_X/\mathcal{I}) \restriction_{N(\mathcal{I})}$ involves two sheafifications: one for quotient, and one for set-theoretic restriction. It would be convenient to combine these two into one: $\mathscr{O}_{X_0}$ is the sheafification of the presheaf $\mathscr{O}_{X_0}^{\mathrm{pre}}$ sending each open $U_0 \subset X_0$ (more precisely, $U_0 \subset N(\mathcal{I})$) to

$$\mathscr{O}_{X_0}^{\mathrm{pre}}(U_0) = \varinjlim_{U \supset U_0} \mathscr{O}_X(U)/\mathcal{I}(U) \tag{1.3.4}$$

where the direct limit is over all open $U \subset X$ containing U_0. Indeed, one can also take the direct limit over all open U satisfying $U \cap N(\mathcal{I}) = U_0$.

Remark 1.3.6. We have an obvious inclusion map which is holomorphic:

$$\iota : X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$$

such that $\iota^\# : \mathscr{O}_X \to \iota_*\mathscr{O}_{X_0} = \iota_*\iota^{-1}(\mathscr{O}_X/\mathcal{I})$ restricts to the quotient maps $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathcal{I}_x = (\iota_*\iota^{-1}(\mathscr{O}_X/\mathcal{I}))_x$ for all $x \in X$.

Proof. We explain the existence of such sheaf map $\iota^\#$. Choose any open $U \subset X$. Then by passing to direct limits (1.3.4), the quotient map $\mathscr{O}_X(U) \to \mathscr{O}_X(U)/\mathcal{I}(U)$ becomes a map $\mathscr{O}_X(U) \to \mathscr{O}_{X_0}^{\mathrm{pre}}(U \cap N(\mathcal{I}))$ whose composition with $\mathscr{O}_{X_0}^{\mathrm{pre}} \to \mathscr{O}_{X_0}$ gives $\mathscr{O}_X(U) \to \mathscr{O}_{X_0}(U \cap N(\mathcal{I})) = (\iota_*\mathscr{O}_{X_0})(U)$. □

Complex spaces arise from

Remark 1.3.7 (Gluing construction of complex spaces). Suppose X is a Hausdorff space with an open cover $\mathfrak{V} = (V_\alpha)$. Suppose that for each V_α there is a homoemorphism $\varphi_\alpha : V_\alpha \to U_\alpha$ where U_α is a complex space. Suppose also that for each α, β, the homeomorphism $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$ (where the source and the target are regarded as open subspaces of U_α and U_β respectively) can be extended to an isomorphism $\varphi_{\beta,\alpha}$ of $\mathbb{C}$-ringed spaces satisfying the **cocycle condition**: for all α, β, γ, we have $\varphi_{\alpha,\alpha} = \mathbf{1}$ and $\varphi_{\gamma,\alpha} = \varphi_{\gamma,\beta}\varphi_{\beta,\alpha}$ (from $\varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$ to $\varphi_\gamma(V_\alpha \cap V_\beta \cap V_\gamma)$). Then X is naturally a complex space such that $\varphi_\alpha : V_\alpha \to U_\alpha$ is extended to an isomorphism of $\mathbb{C}$-ringed spaces such that $\varphi_\beta = \varphi_{\beta,\alpha}\varphi_\alpha$ (from $V_\alpha \cap V_\beta$ to $\varphi_\beta(V_\alpha \cap V_\beta)$). Indeed, $\mathscr{O}_X$ is constructed by gluing all the V_α-sheaves $\varphi_\alpha^{-1}\mathscr{O}_{U_\alpha}$ (cf. Rem. 1.2.10).

Let us see some examples of complex spaces. We begin with an easier class of examples:

Definition 1.3.8. Let X be a complex space, and let $\mathscr{C}_X$ be the sheaf of complex valued continuous functions on X. Then there is a natural **morphism of sheaves of local $\mathbb{C}$-algebras** (i.e. a morphism of X-sheaves which preserve the linear structures and algebra multiplications when restricted to each open subset, and whose stalk maps send the maximal ideals into maximal ones)

$$\mathrm{red} : \mathscr{O}_X \to \mathscr{C}_X \tag{1.3.5}$$

sending each $f \in \mathscr{O}_X$ to f as a function (cf. Def. 1.2.1). red is called the **reduction map** of X. If $\mathrm{red} : \mathscr{O}_{X,x} \to \mathscr{C}_{X,x}$ is injective, we say that X is **reduced at** $x \in X$, or equivalently that x is a **reduced point** of X. If X is reduced everywhere, X is called a **reduced complex space**.

Thus, a holomorphic function on a reduced complex space can be viewed as a genuine continuous function without losing information. (Formally speaking: $\mathscr{O}_X$ is a subsheaf of $\mathscr{C}_X$.) For non-reduced spaces, holomorphic functions cannot be viewed as genuine functions.

Remark 1.3.9. In commutative algebra, there is a notion of reducedness: $\mathscr{O}_{X,x}$ is called reduced if it has no non-zero nilpotent element. We will see later that a complex space X is reduced at x iff $\mathscr{O}_{X,x}$ is a reduced ring. This is the famous Nullstellensatz.

Example 1.3.10. Let $U \subset \mathbb{C}^m \times \mathbb{C}^n$ be open, and let $\mathcal{I} = z_1\mathscr{O}_U + \cdots + z_m\mathscr{O}_U$. Then $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is naturally equivalent to the complex submanifold $U \cap (0 \times \mathbb{C}^n) \simeq U \cap \mathbb{C}^n$ (whose structure sheaf is the sheaf of holomorphic functions $f(\zeta_1, \dots, \zeta_n)$).

Proof. Clearly $N(\mathcal{I}) = U \cap \mathbb{C}^n$ (cf. Rem. 1.2.23). Consider the identity map $\varphi : U \cap \mathbb{C}^n \to X$ as a homeomorphism of topological spaces. In particular, we have an isomorphism $\mathrm{red}\varphi^\# : \mathscr{C}_X \to \mathscr{C}_{U\cap\mathbb{C}^n}$. We shall construct $\varphi^\# : \mathscr{O}_X = \mathscr{O}_U/\mathcal{I} \restriction_{N(\mathcal{I})} \to \mathscr{O}_{U\cap\mathbb{C}^n}$ such that φ is an isomorphism of $\mathbb{C}$-ringed spaces.

By (1.1.3), for each $x \in U \cap \mathbb{C}^n$,

$$\mathscr{O}_{X,x} = ((\mathscr{O}_U/\mathcal{I}) \restriction_{N(\mathcal{I})})_x \simeq \mathscr{O}_{\mathbb{C}^{m+n},x}/\mathcal{I}_x \simeq \mathscr{O}_{\mathbb{C}^n,x}$$

where the last isomorphism can be seen by taking power series expansions of $f(z_\bullet, \zeta_\bullet) = f(z_1, \dots, z_m, \zeta_1, \dots, \zeta_n)$ at n and throwing away every terms containing powers of $\zeta_\bullet$. Define a sheaf map

$$\varphi^\# : \mathscr{O}_X \xrightarrow{\mathrm{red}} \mathscr{C}_X \xrightarrow[\simeq]{\mathrm{red}\varphi^\#} \mathscr{C}_{U\cap\mathbb{C}^n}.$$

Its stalk map is $\mathscr{O}_{\mathbb{C}^n,x} \to \mathscr{C}_{U\cap\mathbb{C}^n,x}$ sending each f to the function f itself. From this we see that the stalk map is injective and has image $\mathscr{O}_{U\cap\mathbb{C}^n,x}$. This shows that $\varphi^\#$ is an injective sheaf map with image $\mathscr{O}_{U\cap\mathbb{C}^n}$. So $\varphi^\#$ restricts to an isomorphism of sheaves of local $\mathbb{C}$-algebras $\mathscr{O}_X \to \mathscr{O}_{U\cap\mathbb{C}^n}$. □

Remark 1.3.11. The proof of Exp. 1.3.10 suggests a way of understanding a *reduced* model space $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$: 1. Find the underlying topological space $N(\mathcal{I})$. 2. Understand each stalk $\mathscr{O}_{X,x} = \mathscr{O}_{U,x}/\mathcal{I}_x$ and show that $\mathrm{red} : \mathscr{O}_{X,x} \to \mathscr{C}_{X,x}$ is injective. 3. Find a familiar sheaf of local $\mathbb{C}$-subalgebras $\mathscr{A} \subset \mathscr{C}_X$ such that $\mathscr{A}_x = \mathrm{red}(\mathscr{O}_{X,x})$. Then $X \simeq (N(\mathcal{I}), \mathscr{A})$.

Exercise 1.3.12. Let U be a neighborhood of $0 \in \mathbb{C}^2$. Let z, w be the standard coordinates of $\mathbb{C}^2$. Let $\mathcal{I} = zw \cdot \mathscr{O}_U$, the ideal sheaf generated by the function zw. Show that $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is equivalent to the $\mathbb{C}$-ringed space whose underlying topological space is $N(\mathcal{I}) = \{(z,w) \in U : z = 0 \text{ or } w = 0\}$, and whose structure sheaf is the sheaf of continuous functions on open subsets of $N(\mathcal{I})$ that are holomorphic when restricted respectively to the z-axis and to the w-axis.

Example 1.3.13. Let $k \in \mathbb{Z}_+$. Let U be a neighborhood of $0 \in \mathbb{C}$. We call $\mathrm{Specan}(\mathscr{O}_U/z^k\mathscr{O}_U) = (0, \mathbb{C}\{z\}/z^k\mathbb{C}\{z\}) = (0, \mathbb{C}[z]/z^k\mathbb{C}[z])$ the **k-fold point**. It is not reduced when $k > 1$. A single reduced point is precisely a 1-fold point, which is the same as the connected 0-dimensional complex manifold $\mathbb{C}^0$.

We close this section by discussing a useful relationship between local-freeness and rank functions. A locally-free sheaf clearly has locally constant rank. The converse holds under some conditions which are often easy to verify:

Proposition 1.3.14. *Let X be a reduced complex space, and let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Then $\mathscr{E}$ is locally free if and only if the rank function $\mathbf{R} : x \in X \mapsto \dim(\mathscr{E}|x)$ is locally constant. Moreover, if $\mathbf{R}$ has constant value n, and if $s_1, \dots, s_n \in \mathscr{E}(X)$ generate $\mathscr{E}$, then $s_1, \dots, s_n$ generate $\mathscr{E}$ freely.*

Proof. Suppose $\mathbf{R}$ has constant value n and $s_1, \dots, s_n \in \mathscr{E}(X)$ generate $\mathscr{E}$. Then for each open $U \subset X$ and $f_1, \dots, f_n \in \mathscr{O}(U)$ satisfying $f_1 s_1 + \cdots + f_n s_n = 0$, we have for each $x \in U$ that $f_1(x)s_1(x) + \cdots + f_n(x)s_n(x) = 0$ where $s_i(x)$ is the restriction of s_i to the fiber $\mathscr{E}|x$. Clearly $s_1(x), \dots, s_n(x)$ span $\mathscr{E}|x$. Since $\dim(\mathscr{E}|x) = n$, $s_1(x), \dots, s_n(x)$ form a basis of $\mathscr{E}|x$. So $f_1(x) = \cdots = f_n(x) = 0$. As holomorphic functions on a reduced space are determined by their values, we have $f_1 = \cdots = f_n = 0$. This proves that $s_1, \dots, s_n$ are $\mathscr{O}_X$-free.

Assume in general that $\mathscr{E}$ is finite-type and $\mathbf{R}$ is locally constant. By shrinking X to a neighborhood of $x \in X$ we may assume $\mathbf{R}$ has constant value n. Choose $s_1, \dots, s_n \in \mathscr{E}_x$ whose values at x form a basis of $\mathscr{E}|x$. By Nakayam's lemma (Exe. 1.2.18), we may shrink X so that $s_1, \dots, s_n \in \mathscr{E}(X)$ generate $\mathscr{E}$. So by the first paragraph, $\mathscr{E}$ is locally-free. □

1.4 Holomorphic maps

In order to construct complex spaces by gluing model spaces (Rem. 1.3.7), and to understand holomorphic maps between complex spaces, we need to understand morphisms (i.e. holomorphic maps) between model spaces $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) \to \mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$ (where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open). This is a main goal of this section.

The first step is to understand the case that target is just V. As one may expect, holomorphic maps in this case are described by an n-tuple of holomorphic functions. Recall that $\mathrm{Mor}(X, Y)$ is the set of holomorphic maps from the complex space X to Y. Let $z_1, \dots, z_n$ be the standard coordinates of $\mathbb{C}^n$.

Theorem 1.4.1. *Let X be a complex space. Then the following map is bijective:*

$$\mathrm{Mor}(X, \mathbb{C}^n) \to \mathscr{O}(X)^n, \qquad \varphi \mapsto (\varphi^\# z_1, \dots, \varphi^\# z_n). \tag{1.4.1}$$

Remark 1.4.2. Due to this theorem, if $\psi : X \to Y$ is a holomorphic map and $f \in \mathscr{O}(Y)$, then we may write

$$f \circ \psi = \psi^\# f \tag{1.4.2}$$

by viewing f as a holomorphic map $Y \to \mathbb{C}$.

The proof of Thm. 1.4.1 relies on the Noetherian property of $\mathscr{O}_{X,x}$, whose proof is deferred to the next section.

Proof that (1.4.1) *is surjective assuming* (1.4.1) *is injective.* Assume (1.4.1) is injective for all complex spaces. Fix X and $F = (f_1, \dots, f_n) \in \mathscr{O}(X)^n$. We claim that each $x \in X$ is contained in a neighborhood U_x such that $F|_{U_x} \in \mathscr{O}(U_x)^n$ corresponds to some $\varphi_x \in \mathrm{Mor}(U_x, \mathbb{C}^n)$. By the injectivity, for every $x, y \in X$, φ_x and φ_y agree on $U_x \cap U_y$. Gluing all φ_x together gives the desired φ corresponding to F.

To prove the claim, we may assume U_x is a model space $\mathrm{Specan}(\mathscr{O}_V/\mathcal{I})$ where $V \subset \mathbb{C}^m$ is open and $\mathcal{I}$ is finite-type. Since the stalk $(\mathscr{O}_V/\mathcal{I})|_x$ equals $\mathscr{O}_{V,x}/\mathcal{I}_x$, we can further shrink U_x so that $F|_{U_x}$ can be lifted to $\widetilde{F}|_V \in \mathscr{O}(V)^n$. $\widetilde{F}$ can be viewed as a holomorphic map $V \to \mathbb{C}^n$. Its composition with the inclusion $\iota : \mathrm{Specan}(\mathscr{O}_V/\mathcal{I}) \hookrightarrow V$ gives the desired holomorphic map φ. □

Proof that (1.4.1) *is injective.* Let $\varphi_1, \varphi_2 \in \mathrm{Mor}(X, \mathbb{C}^n)$ correspond to the same element $(f_1, \dots, f_n)$ of $\mathscr{O}(X)^n$. By (1.2.3), $z_i \circ \varphi_\bullet(x) = (\varphi_\bullet^\# z_i)(x) = f_i(x)$. So φ_1 equals φ_2 as set maps, i.e. $\varphi_\bullet(x) = (f_1(x), \dots, f_n(x))$. Checking that they are equal as morphisms of $\mathbb{C}$-ringed spaces is equivalent to showing for any x that $\varphi_1^\# = \varphi_2^\#$ as maps from $\mathscr{O}_{\mathbb{C}^n, \varphi_\bullet(x)} = \mathscr{O}\{z_1 - f_1(x), \dots, z_n - f_n(x)\}$ to $\mathscr{O}_{X,x}$. We know that they both send each $z_i - f_i(x)$ to $f_i - f_i(x)$. So they are equal by the uniqueness part of the following proposition. □

The following proposition can be viewed as the infinitesimal version of Thm. 1.4.1. (This will become clear after the readers read Thm. 1.6.2.)

Proposition 1.4.3. *Let $\mathscr{O}_{X,x}$ be an analytic local $\mathbb{C}$-algebra. Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathscr{O}_{X,x}$. Then there is a unique morphism of local $\mathbb{C}$-algebras satisfying*

$$\Phi : \mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\} \to \mathscr{O}_{X,x}, \qquad z_i \mapsto f_i - f_i(x). \tag{1.4.3}$$

Note that as a morphism of *local* rings, Φ is assumed to send $\mathfrak{m}_{\mathbb{C}^n,0} = \sum_{j=1}^n z_j\mathbb{C}\{z_1, \dots, z_n\}$ into $\mathfrak{m}_{X,x}$.

Proof. Existence: By the second paragraph of the proof that (1.4.1) is surjective (which does not rely on the injectivity of (1.4.1)), by shrinking X, we may choose a holomorphic map $\phi : X \to \mathbb{C}^n$ corresponding to $(f_1 - f_1(x), \dots, f_n - f_n(x))$. Then the stalk map $\phi^\# : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,x}$ gives Φ.

Injectivity: Assume Φ_1, Φ_2 both satisfy the requirement. Then they clearly agree when restricted to the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. Now we choose $g \in \mathbb{C}\{z_\bullet\}$. For each $k \in \mathbb{N}$, we may write g as a polynomial of $z_\bullet$ plus $g_k \in \mathfrak{m}^k_{\mathbb{C}^n,0}$. So $\Phi_1(g) - \Phi_2(g)$ equals $\Phi_1(g_k) - \Phi_2(g_k)$, which belongs to $\mathfrak{m}^k_{X,x}$ since Φ_i sends $\mathfrak{m}_{\mathbb{C},0}$ into $\mathfrak{m}_{X,x}$. So $\Phi_1(g) - \Phi_2(g)$ belongs to $\bigcap_{k\in\mathbb{N}} \mathfrak{m}^k_{X,x}$, which is 0 due to the following theorem *and the fact that $\mathscr{O}_{X,x}$ is Noetherian.* □

Theorem 1.4.4 (Krull's intersection theorem). *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $\mathcal{M}$ be a finitely-generated A-module. Then $\bigcap_{k\in\mathbb{N}} \mathfrak{m}^k \cdot \mathcal{M} = 0$.*

Proof. The submodule $\mathcal{N} = \bigcap_{k\in\mathbb{N}} \mathfrak{m}^k \cdot \mathcal{M}$ is also finitely generated as A is Noetherian. Then $\mathcal{N} = 0$ will follow from $\mathfrak{m}\mathcal{N} = \mathcal{N}$ (equivalently, 0 spans the "fiber" $\mathcal{N}/\mathfrak{m}\mathcal{N}$) and Nakayama's lemma. That $\mathfrak{m}\mathcal{N} = \mathcal{N}$ is due to Artin-Rees lemma (applied to the $\mathfrak{m}$-stable filtration $(\mathfrak{m}^k\mathcal{M})_{k\in\mathbb{N}}$ to show that $(\mathcal{N} \cap \mathfrak{m}^k\mathcal{M})_{k\in\mathbb{N}} = (\mathcal{N})_{k\in\mathbb{N}}$ is $\mathfrak{m}$-stable). □

Recall that if I is an ideal of a ring A, an **I-filtration** $(\mathcal{M}_n)_{n\in\mathbb{N}}$ (of $\mathcal{M}_0$) is a descending chain of A-modules $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots$ satisfying $I\mathcal{M}_n \subset \mathcal{M}_{n+1}$ for all $n \in \mathbb{N}$. It is called **I-stable** if for some $N \in \mathbb{N}$ we have $I\mathcal{M}_n = \mathcal{M}_{n+1}$ for all $n \geqslant N$.

Theorem 1.4.5 (Artin-Rees lemma). *Let I be an ideal of a Noetherian ring A. Then for any I-stable filtration $(\mathcal{M}_n)_{n\in\mathbb{N}}$ inside a finitely-generated A-module $\mathcal{M}$, and for any submodule $\mathcal{N} \subset \mathcal{M}$, $(\mathcal{N} \cap \mathcal{M}_n)_{n\in\mathbb{N}}$ is I-stable.*

Proof. This follows from two ingredients: 1. The graded ring $A_\bullet = (A, I, I^2, \cdots)$ is a quotient of the Noetherian ring $A[z_1, \dots, z_m]$ if I is generated by m elements. So $A_\bullet$ is Noetherian. 2. An I-filtration $(\mathcal{M}_0)_{n\in\mathbb{N}}$ of finitely-generated A-modules is I-stable iff the graded $A_\bullet$-module $\mathcal{M}_\bullet = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \cdots)$ is finitely-generated. See [AM, Sec. 10.3] for details. □

The uniqueness part of Thm. 1.4.1 can be formulated in the following form.

Remark 1.4.6 (Substitution rule). Let X be a complex space, let $\mathcal{I}$ be a finite type ideal of $\mathscr{O}_X$ containing $f_1 - g_1, \dots, f_n - g_n$ where $f_\bullet, g_\bullet \in \mathscr{O}(X)$. Let $F = (f_1, \dots, f_n)$ and $G = (g_1, \dots, g_n)$. Let $h \in \mathscr{O}_{\mathbb{C}^n}$. Then $F^\# h$ and $G^\# h$ restrict to the same (locally defined) holomorphic function of $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$, i.e. they are equal as elements of $\mathscr{O}_X/\mathcal{I}$.

Proof. f_i and g_i are equal as holomorphic functions of Y. So by Thm. 1.4.1, F and G are the same holomorphic map $X \to \mathbb{C}^n$. So $F^\# h$ equals $G^\# h$ as elements of $\mathscr{O}_Y$. □

Example 1.4.7. Let $U \subset \mathbb{C}^2$ be open, let $f \in \mathscr{O}(U)$, and let $\mathcal{I}$ be the ideal sheaf of $\mathscr{O}_U$ generated by $z_2 - f(z_1, z_2)$. Then for each $h \in \mathscr{O}_{\mathbb{C}^2}$, $h(z_1, z_2)$ and $h(z_1, f(z_1, z_2))$ are equal as elements of $\mathscr{O}_U/\mathcal{I}$.

We have seen how a holomorphic map from a model space $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ to $V \subset \mathbb{C}^n$ looks like. The next question is when this map "has image in $\mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$"? This is answered by the following theorem whose proof does not rely on the Noetherian property.

Theorem 1.4.8. *Let $\varphi : X \to Y$ be a holomorphic map of complex spaces. Let $X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ and $Y_0 = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$ be closed complex subspaces of X and Y respectively. Then the following are equivalent:*

(a) There is a (necessarily unique) holomorphic map $\psi : X_0 \to Y_0$ such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "\psi"] \arrow[d, hook] & Y_0 \arrow[d, hook] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd} \tag{1.4.4}$$

(b) For each $x \in X$ and $y = \varphi(x)$, the stalk map $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ satisfies

$$\varphi^\#(\mathcal{J}_y) \subset \mathcal{I}_x$$

Proof. Assume (a). If $x \in X_0$, then each $f \in \mathcal{J}_y \subset \mathscr{O}_{Y,y}$ is sent by the transpose $\iota^\#_{Y_0,Y}$ to 0. Also f is sent by $\varphi^\#$ to $\varphi^\#(f) \in \mathscr{O}_{X,x}$, and then sent by $\iota^\#_{X_0,X}$ to $\varphi^\#(f) + \mathcal{I}_x$ in $\mathscr{O}_{X_0,x} = \mathscr{O}_{X,x}/\mathcal{I}_x$, which must be 0 since (1.4.4) commutes. So $\varphi^\#(f) \in \mathcal{I}_x$.

If $x \in X \backslash X_0$, then $x \neq N(\mathcal{I})$. So $\mathcal{I}_x = \mathscr{O}_{X,x_0}$. Then clearly $\varphi^\#(\mathcal{J}_y) \subset \mathcal{I}_x$. (b) is proved.

Now assume (b). If $y \notin N(\mathcal{J})$, then $\mathcal{J}_y = \mathscr{O}_{Y,y}$. So $1 \in \mathcal{J}_y$, and so $1 = \varphi^\#(1)$ belongs to $\mathcal{I}_x$. Therefore $x \notin N(\mathcal{I})$. This proves $\varphi(N(\mathcal{I})) \subset N(\mathcal{J})$. So ψ exists as a continuous map of topological spaces, and such a map is clearly unique.

Choose $x \in X_0$ i.e. $x \in N(\mathcal{I})$. By (b), we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X_0,x} = \mathscr{O}_{X,x}/\mathcal{I}_x & \mathscr{O}_{Y_0,y} = \mathscr{O}_{Y,y}/\mathcal{J}_y \arrow[l, "\psi^\#"'] \\
\mathscr{O}_{X,x} \arrow[u, two heads] & \mathscr{O}_{Y,y} \arrow[l, "\varphi^\#"] \arrow[u, two heads]
\end{tikzcd}$$

for a unique stalk map $\psi^\# : \mathscr{O}_{Y_0,y} \to \mathscr{O}_{X_0,x}$, which is clearly a morphism of local $\mathbb{C}$-algebras. It remains to show that these stalk maps can be assembled into a sheaf map.

Recall the presheaves in Rem. 1.3.5. For each open $V \subset Y$, (b) implies $\varphi^\#(\mathcal{J}(V)) \subset \mathcal{I}(\varphi^{-1}(V))$. So the map $\varphi^\# : \mathscr{O}_Y(V) \to (\varphi_*\mathscr{O}_X)(V) = \mathscr{O}_X(\varphi^{-1}(V))$ descends to

$$\mathscr{O}_Y(V)/\mathcal{J}(V) \to \mathscr{O}_X(\varphi^{-1}(V))/\mathcal{I}(\varphi^{-1}(V)).$$

By taking direct limit over all V containing a fixed open $V_0 \subset Y_0$, we obtain

$$\mathscr{O}_{Y_0}^{\mathrm{pre}}(V_0) \to \mathscr{O}_{X_0}^{\mathrm{pre}}(\psi^{-1}(V_0))$$

Its composition with

$$\mathscr{O}_{X_0}^{\mathrm{pre}}(\psi^{-1}(V_0)) \to \mathscr{O}_{X_0}(\psi^{-1}(V_0)) = (\psi_*\mathscr{O}_{X_0})(V_0)$$

gives a presheaf map $\mathscr{O}_{Y_0}^{\mathrm{pre}} \to \psi_*\mathscr{O}_{X_0}$ whose sheafification is the desired $\psi^\# : \mathscr{O}_{Y_0} \to \psi_*\mathscr{O}_{X_0}$. □

1.5 Weierstrass division theorem and Noetherian property of $\mathscr{O}_{X,x}$

1.5.1 Main results

Now that we have seen the importance of the Noetherian property, we prove this in this section. Since $\mathscr{O}_{X,x}$ is a quotient of $\mathscr{O}_{\mathbb{C}^n,0}$, it suffices to prove that $\mathscr{O}_{\mathbb{C}^n,0}$ is Noetherian. The proof relies on Weierstrass division theorem, which we state below.

Definition 1.5.1. We say that $f(z) \in \mathbb{C}\{z\}$ has **order** $k \in \mathbb{N} \cup \{\infty\}$ if $f(z) = z^k(a_k + a_{k+1}z + a_{k+2}z^2 + \cdots)$ and $a_k \neq 0$; f has order ∞ iff $f = 0$. More generally, for $m \in \mathbb{N}$, we say that $f(w_\bullet, z) = f(w_1, \dots, w_m, z) \in \mathbb{C}\{w_\bullet, z\}$ has **order** k **(in** z**)** if $f(0, z) \in \mathbb{C}\{z\}$ has order k. Equivalently, $f(w_\bullet, z) = \sum_{i=0}^\infty a_k(w_\bullet)z^k$ where

$$a_0(0) = \cdots = a_{k-1}(0) = 0, \qquad a_k(0) \neq 0. \tag{1.5.1}$$

That f has order ∞ in z means $a_i(0) = 0$ for all i.

Recall that the **degree** of a polynomial $p(w_\bullet, z) \in \mathbb{C}\{w_\bullet\}[z]$ is the smallest power of z whose coefficient is a non-zero element of $\mathbb{C}\{w_\bullet\}$. The degree of zero polynomial is set to be $-\infty$. $\square$

Remark 1.5.2. Let $f(w_\bullet, z)$ have order $k < \infty$ in z, defined on a neighborhood of 0. Then inside this neighborhood we can find a smaller one $U \times V \subset \mathbb{C}^m \times \mathbb{C}$ such that $f(0, z)$ has one zero in V^{cl} (namely $z = 0$) with multiplicity k. By Rouché's theorem, we may shrink U such that for each fixed $w_\bullet \in U$, the holomorphic function $f(w_\bullet, z)$ of z has k zeros in V counting multiplicities; see Fig. 1.5.1.

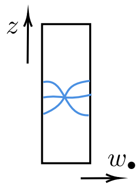

Figure 1.5.1

In the following, we suppress the variable $w_\bullet$ when necessary.

Theorem 1.5.3 (Weierstrass division theorem (WDT)). *Suppose $g \in \mathbb{C}\{w_\bullet, z\}$ has order $k < \infty$ in z. Then for each $f \in \mathbb{C}\{w_\bullet, z\}$, there exist unique $q \in \mathbb{C}\{w_\bullet, z\}$ and $r \in \mathbb{C}\{w_\bullet\}[z]$ with degree $< k$ such that $f = gq + r$.*

We shall prove the Noetherian property using the following (almost) equivalent form of WDT.

Theorem 1.5.4 (Weierstrass division theorem (WDT)). *Suppose $g \in \mathbb{C}\{w_\bullet, z\} = \mathscr{O}_{\mathbb{C}^{m+1}}$ has order $k < \infty$ in z. Then $\mathscr{O}_{\mathbb{C}^{m+1},0}/g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is a rank-k free $\mathscr{O}_{\mathbb{C}^m}$-module. $1, z, \dots, z^{k-1}$ are a set of free generators.*

Theorem 1.5.5. *Every analytic local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$ is Noetherian.*

Proof. It suffices to discuss $\mathscr{O}_{\mathbb{C}^n,0}$. We prove this by induction on n. The case $n = 0$ is trivial. Suppose the case $m = n - 1$ is known. We prove the case $m + 1$. Choose any ideal non-zero $I \subset \mathscr{O}_{\mathbb{C}^{m+1},0}$. Choose $0 \neq g \in I$. Then on a complex line passing through 0, 0 must be an isolated zero of h. (Otherwise, on each line, g vanishes on a neighborhood of 0. So g vanishes on each line (and hence each domain containing 0) by complex analysis.) By choosing new coordinates, we may assume the last coordinate axis is that line. Namely, writing $g = g(w_1, \dots, w_m, z)$, g has finite order in z.

By WDT, $\mathscr{O}_{\mathbb{C}^{m+1},0}/g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is a finitely-generated $\mathscr{O}_{\mathbb{C}^m,0}$-module. Its submodule $I/I \cap g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is generated by finitely many elements $f_1, \dots, f_N \in I$, thanks to the assumption that $\mathscr{O}_{\mathbb{C}^m,0}$ is Noetherian. So elements of I are $\mathscr{O}_{\mathbb{C}^{m+1},0}$-linear combinations of $f_1, \dots, f_N, g$. $\square$

1.5.2 Proof of WDT

We prove the first version of WDT following [GR-b].

Proof of the uniqueness. Let $f = gq_1 + r_1 = gq_2 + r_2$. Then $g(q_1 - q_2) = r_2 - r_1$. Choose a small enough neighborhood $U \times V \subset \mathbb{C}^m \times \mathbb{C}$ as in Rem. 1.5.2 such that for all fixed $w_\bullet \in U$, $g(z)$ has k zeros in V (counting multiplicities). So $g(q_1 - q_2)$ has $\geqslant k$ zeros in z. Since $r_2 - r_1$ has degree $< k$ in z, for the fixed $w_\bullet$, the number of zeros of $r_2 - r_1$ is either $< k$ (which is impossible), or is ∞. Since the latter is the only possible case, we conclude $(r_1 - r_2)(z) = 0$ for all $w_\bullet$. And $(q_1 - q_2)(z) = 0$ since it is so outside the (finitely many) zeros of g. (One can also deduce $q_1 = q_2$ from the fact that $\mathscr{O}_{\mathbb{C}^{m+1},0}$ is an integral domain.) □

Discussion. We now discuss the proof of the existence part. Let $\widehat{f}, \widehat{g}$ be the first k terms in their power series expansions of z. So

$$g(w_\bullet, z) = \underbrace{a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}}_{\widehat{g}} + z^k (a_k + a_{k+1} z + a_{k+2} z^2 + \cdots)$$

where all $a_i = a_i(w_\bullet) \in \mathbb{C}\{w_\bullet\}$ and $a_0(0) = \cdots = a_{k-1}(0) = 0$, $a_k(0) \neq 0$. So $(g - \widehat{g})z^{-k}$ and similarly $(f - \widehat{f})z^{-k}$ are naturally elements of $\mathbb{C}\{w_\bullet, z\}$. Moreover, $(g - \widehat{g})z^{-k}$ is a unit.

A naïve attempt to find the decomposition $f = gq + r$ is to write

$$f = g \cdot \frac{f - \widehat{f}}{g} + \widehat{f}$$

since clearly $\widehat{f} \in \mathbb{C}\{w_\bullet\}[z]$ has degree $< k$ in z. This certainly works for single-variable functions. However, when $m > 0$, the expression $(f - \widehat{f})/g$ might not be continuous at the origin. (Take for instance the quotient to be $z^2/(wz + z^2)$.) We can only divide $f - \widehat{f}$ by $g - \widehat{g}$, which gives an element of $\mathbb{C}\{w_\bullet, z\}$. So we write

$$f = (g - \widehat{g}) \cdot \frac{f - \widehat{f}}{g - \widehat{g}} + \widehat{f} = g \cdot \frac{f - \widehat{f}}{g - \widehat{g}} + \widehat{f} + \underbrace{\left(-\widehat{g} \cdot \frac{f - \widehat{f}}{g - \widehat{g}} \right)}_{f_1}$$

We then decompose f_1, find f_2, and then repeat this procedure again and again to produce an infinite series, which we hope would converge to the expected decomposition. Namely, we let $f_0 = f$. So the above defines f_1 in terms of f_0. We define in a similar way f_{n+1} in terms of f_n:

$$f_n = g \cdot \frac{f_n - \widehat{f}_n}{g - \widehat{g}} + \widehat{f}_n + f_{n+1}. \tag{1.5.2}$$

Substituting $f_0, f_1, \ldots, f_n$ into f, we get

$$\begin{aligned} f &= \left(g \cdot \frac{f_0 - \widehat{f}_0}{g - \widehat{g}} + \widehat{f}_0\right) + f_1 \\ &= \left(g \cdot \frac{f_0 - \widehat{f}_0}{g - \widehat{g}} + \widehat{f}_0\right) + \left(g \cdot \frac{f_1 - \widehat{f}_1}{g - \widehat{g}} + \widehat{f}_1\right) + f_2 = \cdots \\ &= g \cdot \sum_{i=0}^{n} \frac{f_i - \widehat{f}_i}{g - \widehat{g}} + \sum_{i=0}^{n} \widehat{f}_i + f_{n+1}. \end{aligned} \tag{1.5.3}$$

In the following formal proof, we give careful analysis when $n \to \infty$. $\square$

Finishing the proof of WDT. For each $(r_\bullet, \rho) = (r_1, \ldots, r_m, \rho) \in \mathbb{R}_{>0}^m \times \mathbb{R}_{>0}$, define a norm $\|\cdot\|_{r_\bullet,\rho}$ on $\mathbb{C}\{w_\bullet, z\}$ as follows: if $h = \sum_{i_1,\ldots,i_m,j\in\mathbb{N}} b_{i_\bullet,j} w_1^{i_1} \cdots w_m^{i_m} z^j$ then

$$\|h\|_{r_\bullet,\rho} = \sum_{i_1,\ldots,i_m,j\in\mathbb{N}} |b_{i_\bullet,j}| r_1^{i_1} \cdots r_m^{i_m} \rho^j,$$

which might take value ∞. We have

$$\|h_1 h_2\|_{r_\bullet,\rho} \leqslant \|h_1\|_{r_\bullet,\rho} \cdot \|h_2\|_{r_\bullet,\rho} \qquad \|h - \widehat{h}\|_{r_\bullet,\rho} \leqslant \|h\|_{r_\bullet,\rho}. \tag{1.5.4}$$

We write (1.5.2) as

$$\begin{aligned} -f_{n+1} &= \frac{\widehat{g}}{(g - \widehat{g})} \cdot (f_n - \widehat{f}_n) \\ &= \frac{\widehat{g}}{z^{-k}(g - \widehat{g})} \cdot z^{-k}(f_n - \widehat{f}_n) =: \beta \cdot \alpha_n. \end{aligned} \tag{1.5.5}$$

By the first paragraph in the previous *Discussion*, we have $\beta, \alpha_n \in \mathbb{C}\{w_\bullet, z\}$. Choose $r_\bullet, \rho$ such that f, g are defined (and holomorphic) and $g - \widehat{g}$ has no zeros in the polydisc D with multiradii $r_\bullet, \rho$ except at the origin. Then (1.5.5) shows that all f_n are defined in this domain.

Slightly shrink ρ so that $C := \|f\|_{r_\bullet,\rho} < \infty$. *Now we use the condition that g has order k in z in full power*: it tells us that $\beta(0, z) = 0$. So we may shrink $r_\bullet$ such that $\|\beta\|_{r_\bullet,\rho} < \frac{1}{2}\rho^k$. Clearly $\|f_n - \widehat{f}_n\|_{r_\bullet,\rho} = \rho^k \|\alpha_n\|_{r_\bullet,\rho}$. So by (1.5.4),

$$\|f_{n+1}\|_{r_\bullet,\rho} < \frac{1}{2}\|f_n - \widehat{f}_n\|_{r_\bullet,\rho} \leqslant \frac{1}{2}\|f_n\|_{r_\bullet,\rho}.$$

Thus $\|f_n\|_{r_\bullet,\rho} < 2^{-n}C$. So $\|z^{-k}(f_n - \widehat{f}_n)\|_{r_\bullet,\rho} < 2^{-n}\rho^{-k}C$ and $\|\widehat{f}_n\|_{r_\bullet,\rho} < 2^{-n}C$.

The uniform norm on the polydisc with multi-radii $(r_\bullet, \rho)$ is clearly $\leqslant \|\cdot\|_{r_\bullet,\rho}$. So $f_n \to 0$ uniformly on the polydisc D. The infinite series $\sum_{i=0}^{\infty} \frac{z^{-k}(f_i - \widehat{f}_i)}{z^{-k}(g - \widehat{g})}$ converges uniformly to a continuous function q on any compact subset of D. q

is holomorphic, since it is so on each variable by Morera's theorem. Similarly, $\sum_{i=0}^{\infty} \widehat{f}_i$ converges uniformly to a holomorphic r. Residue theorem and the fact that contour integrals commute with (uniformly convergent) infinite sum show that r does not have $\geqslant k$ powers of z (since each $\widehat{f}_n$ does not). Thus, we obtain the decomposition $f = gq + r$ by letting $n \to \infty$ in (1.5.3). □

1.6 Germs of complex spaces

Definition 1.6.1. The **category of germs of complex spaces** denotes the one whose objects are (X, x) where X is a complex space and x is a marked point. If $U \subset X$ is a neighborhood of x then (X, x) is identified with (U, x). A **morphism of germs** from (X, x) to (Y, y) is a holomorphic map $\varphi : U \to Y$ where $U \subset X$ is a neighborhood of x such that $\varphi(x) = y$. Two morphisms $\varphi_1, \varphi_2 : (X, x) \to (Y, y)$ are regarded equal if there is a neighborhood U of x such that $\varphi_1|_U$ equals $\varphi_2|_U$ as holomorphic maps $U \to Y$. Composition of morphisms are the usual one for holomorphic functions (i.e. for $\mathbb{C}$-ringed spaces).

An **isomorphism of germs of complex spaces** $\varphi : (X, x) \to (Y, y)$ is a morphism of germs with inverses, namely, there is a morphism $\psi : (Y, y) \to (X, x)$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are $\mathbf{1}$ on neighborhoods of x and y respectively. Equivalently, there are neighborhoods $U \ni x$ and $V \ni y$ such that $\varphi : U \to V$ is a biholomorphism, and that $\varphi(x) = y$. □

The category of analytic local $\mathbb{C}$-algebras is understood in the obvious way: the morphisms are defined by Def. 1.3.3.

Theorem 1.6.2. *The contravariant functor $\mathfrak{F}$ from the category of germs of complex spaces to the category of analytic local $\mathbb{C}$-algebras, sending (X, x) to $\mathscr{O}_{X,x}$ and sending $\varphi : (X, y) \to (Y, y)$ to $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$, is an **antiequivalence of categories**. Namely:*

(1) For each (X, x) and (Y, y), the following map is bijective

$$\mathfrak{F} : \mathrm{Mor}\big((X, x), (Y, y)\big) \to \mathrm{Mor}\big(\mathscr{O}_{Y,y}, \mathscr{O}_{X,x}\big), \qquad \varphi \mapsto \varphi^\#. \tag{1.6.1}$$

(2) Each analytic local $\mathbb{C}$-algebra is isomorphic to $\mathfrak{F}((X, x))$ for some germ of complex space (X, x).

Part (2) is obvious. Let us prove part (1).

Proof. Assume without loss of generality that Y is a model space $\mathrm{Specan}(\mathscr{O}_V/\mathscr{J})$ where $V \subset \mathbb{C}^n$ is open and $y = 0$.

Suppose $\varphi_1^\#, \varphi_2^\# : \mathscr{O}_{Y,y} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$ are equal. Then for each $j = 1, \dots, n$, $\varphi_1^\# z_j$ equals $\varphi_2^\# z_j$ as elements of $\mathscr{O}_{X,x}$. So they are equal on X if we shrink

X to a smaller neighborhood of x. By Thm. 1.4.1, φ_1 and φ_2 are equal as holomorphic maps $X \to V$, and hence are equal as $X \to Y$. So the map $\mathfrak{F}$ in (1.6.1) is injective.

Next, we choose a morphism $\Phi : \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$. Let $f_1 = \Phi(z_1), \dots, f_n = \Phi(z_n)$, which are elements of $\mathscr{O}(X)$ if we shrink X to a smaller neighborhood of x. View $F = (f_1, \dots, f_n) \in \mathscr{O}(X)^n$ as a holomorphic map $\varphi : X \to \mathbb{C}^n$. Replace X by $\varphi^{-1}(V)$ such that $\varphi : X \to V$. Note that $\varphi(x) = 0$. So $h \in \mathscr{O}_{\mathbb{C}^n,0} \mapsto h \circ \varphi = \varphi^\# h \in \mathscr{O}_{X,x}$ is a morphism of local $\mathbb{C}$-algebras. It agrees with $\mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \xrightarrow{\Phi} \mathscr{O}_{X,x}$ on $z_1, \dots, z_n$ by the very definition of F. So they agree on any element of $\mathscr{O}_{\mathbb{C}^n,0}$ due to Prop. 1.4.3. We conclude $\varphi^\#(h) = \Phi([h])$ for all $h \in \mathscr{O}_{\mathbb{C}^n,0}$ (where $[h]$ denotes the residue class of h in $\mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0$). In particular, we have $\varphi^\# \mathscr{J}_0 = 0$ in $\mathscr{O}_{X,x}$.

Shrink V and $X \subset \varphi^{-1}(V)$, and choose $g_1, \dots, g_k \in \mathscr{O}_{\mathbb{C}^n}(V)$ generating the ideal $\mathscr{J}_0$ and sent by $\varphi^\#$ to $0 \in \mathscr{O}(X)$. Since $\mathscr{J}$ is finite-type, by Rem. 1.2.16, we can shrink V such that $g_1, \dots, g_k$ generate $\mathscr{J}$. Thus $\varphi^\# \mathscr{J} = 0$ in $\varphi_* \mathscr{O}_X$. By Thm. 1.4.8, φ restricts to a holomorphic map $\widetilde{\varphi} : X \to Y$. $\widetilde{\varphi}^\# : \mathscr{O}_{Y,y} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$ equals Φ since $\varphi^\# : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,x}$ factors as $\mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \xrightarrow{\widetilde{\varphi}^\#} \mathscr{O}_{X,x}$. This proves that $\mathfrak{F}$ is surjective. $\square$

Corollary 1.6.3. *Let X, Y be complex spaces, $x \in Y, y \in Y$, and $\Phi : \mathscr{O}_{Y,y} \xrightarrow{\simeq} \mathscr{O}_{X,x}$ be an isomorphism of local $\mathbb{C}$-algebras. Then there are neighborhoods $U \ni x, V \ni y$ and a biholomorphism $\varphi : U \xrightarrow{\simeq} V$ whose transpose $\varphi^\# : \mathscr{O}_{V,y} \to \mathscr{O}_{U,x}$ equals Φ.*

Definition 1.6.4. An analytic local $\mathbb{C}$-algebra is called **regular** if it is isomorphic to $\mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\}$ for some n.

Corollary 1.6.5. *Let X be a complex space and $x \in X$. If $\mathscr{O}_{X,x}$ is regular, then there is a neighborhood U of x biholomorphic to an open subset of $\mathbb{C}^n$ for some n.*

Definition 1.6.6. We say that X is **smooth at** x (equivalently, x is a **smooth point** of X) if $\mathscr{O}_{X,x}$ is regular. We say that X is **smooth** (equivalently, X is a complex manifold) if it is smooth everywhere.

1.7 Immersions and closed embeddings; generating $\mathscr{O}_{X,x}$ analytically

Definition 1.7.1. A holomorphic map $\varphi : X \to Y$ is called an **immersion at** $x \in X$ if $\varphi^\# : \mathscr{O}_{Y,\varphi(y)} \to \mathscr{O}_{X,x}$ is surjective. φ is called an **immersion** if it is an immersion at every $x \in X$. φ is called a **closed (resp. open) embedding** if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\simeq"'] & & Y \\ & Y_0 \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{1.7.1}$$

where Y_0 is a closed (resp. open) complex subspace of Y and $X \xrightarrow{\simeq} Y_0$ is a biholomorphic map.

A closed embedding is clearly an immersion. Moreover, an immersion is locally a closed embedding:

Proposition 1.7.2. *Let $\varphi : X \to Y$ be an immersion at x. Then there are neighborhoods V of $y = \varphi(x)$ and $U \subset \varphi^{-1}(V)$ of x such that $\varphi : U \to V$ is a closed embedding. In particular, φ is an immersion on U.*

Proof. By assumption, $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is surjective. Let J be its kernel, and choose generating elements $g_1, \dots, g_k \in J$. By shrinking Y to a neighborhood of y (and shrink X accordingly), we assume $g_1, \dots, g_k \in \mathscr{O}_Y(Y)$. Let $\mathscr{J} = g_1\mathscr{O}_Y + \cdots + g_k\mathscr{O}_Y$. Then $\mathscr{J}_x = J$. Define a closed subspace $Z = \mathrm{Specan}(\mathscr{O}_Y/\mathscr{J})$ of Y. Then $\varphi^\#$ factors as

$$\varphi^\# : \mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{Y,y}/J = \mathscr{O}_{Z,y} \xrightarrow[\simeq]{\Psi} \mathscr{O}_{X,x}.$$

By Cor. 1.6.3, we may shrink X so that there is an open embedding $\widetilde{\varphi} : X \to Z$, $\widetilde{\varphi}(x) = y$, such that $\widetilde{\varphi}^\# : \mathscr{O}_{Z,y} \to \mathscr{O}_{X,x}$ equals Ψ. Let $\iota : Z \to Y$ be the inclusion. Then $(\iota\widetilde{\varphi})^\# = \widetilde{\varphi}^\#\iota^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ equals $\varphi^\#$. By Thm. 1.6.2, we may find open $U \ni x$ such that $\varphi = \iota\widetilde{\varphi}$ on U. Since $\widetilde{\varphi}(U)$ is an open subset of Z, we may find open $V \subset Y$ such that $\widetilde{\varphi}(U) = V \cap Z = V \cap N(\mathscr{J})$. So φ restricts to the biholomorphism $\widetilde{\varphi} : U \to \widetilde{\varphi}(U)$ where $\widetilde{\varphi}(U)$ is a closed subspace of V. □

We now discuss when an immersion is a closed embedding and give some examples.

Proposition 1.7.3. *Let X be complex spaces and $\varphi : X \to Y$ a holomorphic immersion. Assume that φ is an injective and closed map[2] of topological spaces. Suppose we have a finite type ideal $\mathscr{J}$ of $\mathscr{O}_Y$ such that $N(\mathscr{J})$ equals the image of φ, and that*

$$\mathscr{J}_y = \mathrm{Ker}(\mathscr{O}_{Y,y} \xrightarrow{\varphi^\#} \mathscr{O}_{X,x}) \tag{1.7.2}$$

for all $x \in X$ and $y = \varphi(x)$. Then φ is a closed embedding. More precisely, φ restricts to a biholomorphism

$$\widetilde{\varphi} : X \xrightarrow{\simeq} \mathrm{Specan}(\mathscr{O}_Y/\mathscr{J}). \tag{1.7.3}$$

[2] φ is called closed if it maps closed subsets to closed subsets.

We will see in Cor. 2.7.8 that the assumption on the existence of $\mathcal{J}$ is redundant.

Proof. Let $Y_0 := \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$. By Thm. 1.4.8, the restriction (1.7.3) as a holomorphic map exists, i.e., we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & Y_0 \arrow[d, hook] \\ & Y \end{tikzcd}$$

The underlying topological space of $Y_0 := \mathrm{Specan}(\mathscr{O}_X/\mathcal{J})$ is $N(\mathcal{J})$. So $\widetilde{\varphi}$ is a continuous closed bijection from X to $N(\mathcal{J})$, which is therefore a homeomorphism. For each $x \in X, y = \varphi(x)$, the stalk map $\widetilde{\varphi}^{\#} : \mathscr{O}_{Y_0,y} = \mathscr{O}_{Y,y}/\mathcal{J}_y \to \mathscr{O}_{X,x}$ is surjective since φ is an immersion, and is injective by (1.7.2). So $\widetilde{\varphi}$ is a biholomorphism. $\square$

Example 1.7.4. The holomorphic map $\iota : 0 \times \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^n$ is an immersion and a closed injective map, and the kernels of $\iota^{\#}$ at the level of stalks are the stalks of the ideal $\mathcal{I} = z_1\mathscr{O}_{\mathbb{C}^{m+n}} + \cdots + z_m\mathscr{O}_{\mathbb{C}^{m+n}}$. Thus, by Prop. 1.7.3, ι restricts to a biholomorphism $0 \times \mathbb{C}^n \xrightarrow{\simeq} \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^{m+n}}/\mathcal{I})$. This reproves Exp. 1.3.10.

Example 1.7.5. Let X be a complex space, and let $\mathcal{I}, \mathcal{J}$ be finite-type ideals of $\mathscr{O}_X$. Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. So $\mathscr{O}_Y = (\mathscr{O}_X/\mathcal{I})|_{N(\mathcal{I})}$. Then

$$\widetilde{\mathcal{J}} = \big((\mathcal{I} + \mathcal{J})/\mathcal{I}\big) \restriction_{N(\mathcal{I})}$$

is a finite-type ideal of $\mathscr{O}_Y$, and is the unique ideal whose stalk at each $x \in N(\mathcal{I})$ equals $(\mathcal{I}_x + \mathcal{J}_x)/\mathcal{I}_x$. Then there is a biholomorphism

$$\mathrm{Specan}(\mathscr{O}_X/(\mathcal{I} + \mathcal{J})) \xrightarrow[\simeq]{\varphi} \mathrm{Specan}(\mathscr{O}_Y/\widetilde{\mathcal{J}}). \tag{1.7.4}$$

which equals $N(\mathcal{I}+\mathcal{J}) \xrightarrow{=} N(\mathcal{I}) \cap N(\mathcal{J})$ as maps of topological spaces, and whose stalk maps are

$$\mathscr{O}_{Y,x}/\widetilde{\mathcal{J}}_x = \frac{\mathscr{O}_{X,x}/\mathcal{I}_x}{(\mathcal{I}_x + \mathcal{J}_x)/\mathcal{I}_x} \quad \xrightarrow{\simeq} \quad \mathscr{O}_{X,x}/(\mathcal{I}_x + \mathcal{J}_x).$$

Proof. The key point is to show that the above stalk isomorphisms can be assembled into a sheaf isomorphism. Consider the diagram

$$\begin{tikzcd} & \mathrm{Specan}(\mathscr{O}_Y/\widetilde{\mathcal{J}}) \arrow[d, hook] \\ \mathrm{Specan}(\mathscr{O}_X/(\mathcal{I} + \mathcal{J})) \arrow[ur, "\varphi"] \arrow[r, "\alpha"] \arrow[dr, hook] & Y \arrow[d, hook] \\ & X \end{tikzcd} \tag{1.7.5}$$

By Thm. 1.4.8, there is a holomorphic map α such that the lower triangle commutes. The stalk maps are $\alpha^\# : \mathscr{O}_{X,x}/\mathcal{I}_x \to \mathscr{O}_{X,x}/(\mathcal{I}_x + \mathcal{J}_x)$, with kernel $(\mathcal{I}_x + \mathcal{J}_x/\mathcal{I}_x)$. These kernels can be assembled into the ideal sheaf $\widetilde{\mathcal{J}}$ on $N(\mathcal{I})$. Thus, Prop. 1.7.3 guarantees that there is a biholomorphism making the upper triangle in (1.7.5) commutes. $\square$

Exp. 1.7.5 shows that a closed complex subspace of a closed subspace is again a closed subspace of the original space. Thus, we have more generally:

Corollary 1.7.6. *If $\alpha : X \to Y$ and $\beta : Y \to Z$ are closed embeddings, then so is the composition $\beta \circ \alpha : X \to Z$.*

Let us consider the special case $\varphi : X \to \mathbb{C}^n$, where φ is represented by $(f_1, \dots, f_n) \in \mathscr{O}_X^n$ (cf. Thm. 1.4.1). Then φ is an immersion at x iff the morphism of analytic local $\mathbb{C}$-algebras defined in Prop. 1.4.3, namely $\mathbb{C}\{z_\bullet\} \to \mathscr{O}_{X,x}$ sending z_j to $f_j - f_j(x)$, is surjective. This actually mean that $f_1, \dots, f_n$ *generate (analytically) the analytic local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$*. (They certainly do not generate the ring $\mathscr{O}_{X,x}$ algebraically. But one can imagine that the subalgebra generated algebraically by $f_\bullet$ is "dense" in $\mathscr{O}_{X,x}$, where the density means approximation by power series of $f_1, \dots, f_n$.) The situation is similar to the case of a surjective morphism of $\mathbb{C}$-algebras $\mathbb{C}[z_\bullet] \to A$, whose algebro-geometric meaning is that the affine scheme $\mathrm{Spec}(A)$ is embedded into the affine plane $\mathbb{C}^n$.

We must find a criterion on whether $f_1, \dots, f_n$ generate $\mathscr{O}_{X,x}$ (analytically). At first sight, this problem seems not easy even if X is smooth. (For instance, take $f_1, \dots, f_n$ to be some arbitrary holomorphic functions and deduce whether they generate $\mathscr{O}_{X,x}$.) There is indeed a simple criterion, which is proved using the (holomorphic version of) inverse function theorem. To begin with, we define:

Definition 1.7.7. If X is a complex space and $x \in X$, the vector space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is called the **cotangent space** of X at x, and its dual space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the **tangent space**. Since $\mathscr{O}_{X,x}$ is Noetherian, $\mathfrak{m}_{X,x}$ is finitely-generated, and hence $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is finite-dimensional.

It is inspiring to write the residue class of $f - f(x)$ (where $f \in \mathscr{O}(X)$) in the cotangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ as $d_x f$.

Theorem 1.7.8. *Let X be a complex space and $x \in X$. Let $f_1, \dots, f_n \in \mathscr{O}(X)$. Consider $(f_1, \dots, f_n)$ as a holomorphic map $\varphi : X \to \mathbb{C}^n$ (cf. Thm. 1.4.1). The following are equivalent.*

(1) φ is an immersion at x.

(2) The morphism of analytic local $\mathbb{C}$-algebras $\Phi : \mathscr{O}_{\mathbb{C}^n,\varphi(x)} \to \mathscr{O}_{X,x}$ sending each z_i to f_i (cf. Prop. 1.4.3) is surjective.

(3) *(The residue classes of)* $f_1 - f_1(x), \dots, f_n - f_n(x)$ *span* $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$.

(4) *(The germs of)* $f_1 - f_1(x), \dots, f_n - f_n(x)$ *generate the ideal* $\mathfrak{m}_{X,x}$.

If any of these conditions holds, we say that $f_1, \dots, f_n$ ***generate (the algebra)*** $\mathscr{O}_{X,x}$ ***analytically***.

Proof. Assume for simplicity that $\varphi(x) = 0$. Clearly (1)⇔(2) and (3)⇔(4). (Note that (3)⇒(4) follows from Nakayama's lemma.) It remains to prove (2)⇔(3).

Assume (2). Choose any $g \in \mathfrak{m}_{X,x}$. Then there is $h(z_\bullet) \in \mathscr{O}_{\mathbb{C}^n,0}$ sent by Φ to g. We may write $h(z_\bullet) = \sum_i a_i z_i +$ an element of $\mathfrak{m}_{\mathbb{C}^n,0}^2$ where $a_i \in \mathbb{C}$. Since $\Phi(z_i) = f_i$ and $\Phi(\mathfrak{m}_{\mathbb{C}}^2) \subset \mathfrak{m}_{X,x}^2$, we have $g \in \sum_i a_i f_i + \mathfrak{m}_{X,x}^2$. This proves (3).

Asume (3). By discarding some elements, we may assume that $f_1, \dots, f_n$ form a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. Assume X is a model space $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ where $U \subset \mathbb{C}^N$ is open and $x = 0$. So $\mathscr{O}_{X,x} = \mathscr{O}_{\mathbb{C}^N,0}/\mathcal{I}_0$, $\mathfrak{m}_{X,x} = \mathfrak{m}_{\mathbb{C}^N,0}/\mathcal{I}_0$, and hence

$$\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 = \mathfrak{m}_{\mathbb{C}^N,0}/(\mathfrak{m}_{\mathbb{C}^N,0}^2 + \mathcal{I}_0). \tag{1.7.6}$$

Lift $f_\bullet$ to elements of $\mathscr{O}_{\mathbb{C}^N,0}$, still denoted by $f_\bullet$. Then we can extend $f_1, \dots, f_n$ to a list $f_1, \dots, f_N$ whose residue classes form a basis of $\mathfrak{m}_{\mathbb{C}^N,0}/\mathfrak{m}_{\mathbb{C}^N,0}^2$ such that $f_{n+1}, \dots, f_N \in \mathcal{I}_0$. By the inverse function theorem, we may assume $x = 0$ and $f_1, \dots, f_N$ are the standard coordinates $z_1, \dots, z_N$ of $\mathbb{C}^N$. By shrinking U, we may assume $z_{n+1}, \dots, z_N \in \mathcal{I}(U)$.

Assume for simplicity that $\mathcal{I}$ is generated by $z_{n+1}, \dots, z_N$ together with $g_1, \dots, g_k \in \mathcal{I}(U)$. Let $\mathcal{I}_1 = z_{n+1}\mathscr{O}_U + \cdots + z_N\mathscr{O}_U$. Then by Exp. 1.7.5, $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is naturally a closed subspace of $X_1 = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I}_1)$ (defined by $g_1, \dots, g_k$). By Exp. 1.7.4, X_1 is naturally equivalent to $U \cap (\mathbb{C}^n \times 0)$. So the map $(z_1, \dots, z_n) : X_1 \to \mathbb{C}^n$ is an open embedding. φ is its restriction to X, which is therefore an immersion at 0. This proves (1) and hence (2). □

Remark 1.7.9. Assume that X, Y are complex manifolds and $\varphi : X \to Y$ is a closed embedding of complex spaces. Let $x \in X$. Then by Thm. 1.7.8, φ is an immersion at x in the sense of complex differential manifolds, namely, it induces an injective map of tangent spaces (since its transpose is a surjective map of cotangent spaces). Therefore, since φ is also a homeomorphism from X to its image in Y, as in the case of real differential manifolds, one can find a neighborhood V of $y = \varphi(x)$, a biholomorphic map β from V to a neighborhood $\widetilde{V}$ of $0 \in \mathbb{C}^n$, and biholomorphism α from $U = \varphi^{-1}(V)$ to a neighborhood $\widetilde{U}$ of $0 \in \mathbb{C}^m$ (where $m \leqslant n$), such that $\beta \circ \varphi \circ \alpha^{-1} : \widetilde{U} \to \widetilde{V}$ is the restriction of the closed embedding $\mathbb{C}^m \simeq \mathbb{C}^m \times \{0\} \hookrightarrow \mathbb{C}^m \times \mathbb{C}^{n-m}$.

We give an application of analytically generating elements.

Proposition 1.7.10.

Let $\Phi, \Psi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ be morphisms of analytic local $\mathbb{C}$-algebras. Assume $f_1, \dots, f_n \in \mathscr{O}_{Y,y}$ generate the algebra $\mathscr{O}_{Y,y}$ analytically.

(1) If $\Phi(f_i) = \Psi(f_i)$ for all $i = 1, \dots, n$, then $\Phi = \Psi$.

(2) Let I be the ideal of $\mathscr{O}_{X,x}$ generated by $\Phi(f_i) - \Psi(f_i)$ for all i. Then I contains $\Phi(h) - \Psi(h)$ for every $h \in \mathscr{O}_{Y,y}$.

Proof. (1): By Prop. 1.4.3, we have a (unique) morphism $\Upsilon : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{Y,y}$ sending z_i to $f_i - f_i(x)$. So $\Phi \circ \Upsilon$ and $\Psi \circ \Upsilon$ agree at $z_1, \dots, z_n$. So $\Phi \circ \Upsilon = \Psi \circ \Upsilon$ by Prop. 1.4.3. By assumption, Υ is surjective. So $\Phi = \Psi$.

(2): Apply (1) to the restriction $\Phi, \Psi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}/I$. □

Prop. 1.7.10-(2) is the stalk version of a geometric construction called equalizer.

1.8 Equalizers of $X \rightrightarrows Y$

Definition 1.8.1. Let $\varphi, \psi : X \to Y$ be holomorphic maps of complex spaces. A **kernel** or an **equalizer of the double arrow** $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$ is a complex space E and a holomorphic map $\iota : E \to X$ such that $\varphi \circ \iota = \psi \circ \iota$, and that for every complex space S and holomorphic map $\mu : S \to X$ satisfying $\varphi \circ \mu = \psi \circ \mu$ there is a unique holomorphic $\widetilde{\mu} : S \to E$ such that $\mu = \iota \circ \widetilde{\mu}$.

$$\begin{tikzcd} S \arrow[d, dashed, "\widetilde{\mu}"'] \arrow[dr, "\mu"] & & \\ E \arrow[r, "\iota"'] & X \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & Y \end{tikzcd} \tag{1.8.1}$$

It is easy to see that equalizers are unique up to isomorphisms.

The main result of this section is:

Theorem 1.8.2. *Every double arrow $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$ of holomorphic maps has an equalizer which is the inclusion map of a closed subspace $\iota : E = \mathrm{Specan}(\mathscr{O}_X/\mathscr{I}) \hookrightarrow X$. This is called the* ***canonical equalizer****. The finite-type ideal $\mathscr{I}$ is uniquely determined by the fact that for all $x \in X$:*

(a) If $\varphi(x) \neq \psi(x)$, then $\mathscr{I}_x = \mathscr{O}_{X,x}$.

(b) If $\varphi(x) = \psi(x)$, then by considering $\varphi^\#, \psi^\#$ as stalk maps $\mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$, $\mathscr{I}_x$ is the ideal of $\mathscr{O}_{X,x}$ generated by all $\varphi^\#(f) - \psi^\#(f)$ (where $f \in \mathscr{O}_{Y,\varphi(x)}$).

Moreover, $N(\mathcal{I})$, the underlying set of E, is $\Delta = \{x \in X : \varphi(x) = \psi(x)\}$.

From Prop. 1.7.10, it is clear that $\mathcal{I}_x$ is generated by $\varphi^{\#}(f_i) - \psi^{\#}(f_i)$ if $f_1, \dots, f_n \in \mathscr{O}_{Y,y}$ generate the algebra $\mathscr{O}_{Y,y}$ analytically, e.g. $z_1, \dots, z_n$ if Y is a model space in $\mathbb{C}^n$.

Remark 1.8.3. From Thm. 1.8.2, it is clear that if $E_0 \to X$ is an equalizer of $X \rightrightarrows Y$, then it is a closed embedding, and equals the composition of a unique biholomorphism $E_0 \xrightarrow{\simeq} E$ and the inclusion map $E \hookrightarrow X$ where E is the canonical equalizer.

Construction of E. We define a finite-type ideal $\mathcal{I}$ satisfying (a) and (b). We shall first define it locally and then glue the pieces. Then $\mathcal{I}$ gives E.

Let $\Omega = X \backslash \Delta$ which is open. We set $\mathcal{I}_\Omega = \mathscr{O}_X|_\Omega$. For each $x \in \Delta$, we choose a neighborhood $V_y \subset Y$ of $y = \varphi(x)$ biholomorphic to a model space. So we can choose finitely many $f_1, \dots f_n \in \mathscr{O}_Y(V_y)$ embedding V_y onto a closed subspace of an open subset of $\mathbb{C}^n$. $U_x = \varphi^{-1}(V_y) \cap \psi^{-1}(V_y)$ is a neighborhood of x, and we set $\mathcal{I}_{U_x}$ to be the ideal of $\mathscr{O}_{U_x}$ generated by $\varphi^{\#}(f_1) - \psi^{\#}(f_1), \dots, \varphi^{\#}(f_n) - \psi^{\#}(f_n)$ (defined on U_x).

We claim that these locally defined finitely-generated ideals are compatible. If $p \in U_x \cap \Delta$ then, as $\varphi(p) = \psi(p)$, by Prop. 1.7.10 or by substitution rule (Rem. 1.4.6), the stalk $(\mathcal{I}_{U_x})_p$ is the ideal generated by all $\varphi^{\#}(f) - \psi^{\#}(f) \in \mathscr{O}_{X,p}$ where $f \in \mathscr{O}_{Y,\varphi(p)}$. If $p \in U_x \cap \Omega$, then as $\varphi(p) \neq \psi(p)$ and $(f_1, \dots, f_n)$ is an embedding, there is some f_i among $f_1, \dots, f_n$ such that $\varphi^{\#}(f_i) - \psi^{\#}(f_i)$ has non-zero value at p, and hence its germ at p is not in $\mathfrak{m}_{X,p}$. This proves $(\mathcal{I}_{U_x})_p = \mathscr{O}_{X,p}$. Combining these two cases together, we see that $\mathcal{I}_\Omega$ and $\mathcal{I}_{U_x}$ (for all $x \in \Delta$) are compatible. This defines $\mathcal{I}$.

If $\varphi(x) \neq \psi(x)$, then $\mathcal{I}_x = \mathscr{O}_{X,x}$ shows $x \notin N(\mathcal{I})$. If $\varphi(x) = \psi(x)$, then $\varphi^{\#}(f) - \psi^{\#}(f)$ vanishes at x by (1.2.3). So $\mathcal{I}_x$ vanishes at x. So $x \in N(\mathcal{I})$. This proves $\Delta = N(\mathcal{I})$. □

Proof that E is an equalizer. It is easy to check $\varphi \circ \iota = \psi \circ \iota$. Choose any holomorphic $\mu : S \to X$ such that $\varphi \circ \mu = \psi \circ \mu$. For any $s \in S$, let $x = \mu(s)$. Then $\varphi(x) = \psi(x)$. Choose any $f \in \mathscr{O}_{Y,\varphi(x)}$. Then $\varphi \circ \mu = \psi \circ \mu$ shows that $\mu^{\#}$ sends $\varphi^{\#}(f) - \psi^{\#}(f)$ to $0 \in \mathscr{O}_{S,s}$. Thus $\mu^{\#} : \mathscr{O}_{X,x} \to \mathscr{O}_{S,s}$ vanishes on $\mathcal{I}_x$. Thus, by Thm. 1.4.8, there is a unique holomorphic $\widetilde{\mu} : S \to E$ such that the triangle in (1.8.1) commutes. □

The proof of Thm. 1.8.2 is finished. From the proof, we know:

Remark 1.8.4. Assume the setting of Thm. 1.8.2. Assume $\varphi(x) = \psi(x) =: y$. Let V_y be a neighborhood of y biholomorphic to a model space. More precisely, we choose $(f_1, \dots, f_n) \in \mathscr{O}_Y(V_y)^n$ which, considered as a holomorphic map $V_y \to \mathbb{C}^n$, is a closed embedding of V_y into an open subset of $\mathbb{C}^n$. Let $U_x = \varphi^{-1}(V_y) \cap \psi^{-1}(V_y)$. Then the ideal sheaf $\mathcal{I}|_{U_x}$ is generated by $\varphi^{\#}(f_1) - \psi^{\#}(f_1), \dots, \varphi^{\#}(f_n) - \psi^{\#}(f_n) \in \mathscr{O}(U_x)$.

1.9 $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$, $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$, and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$

We fix a $\mathbb{C}$-ringed space X.

1.9.1 Tensor product

Definition 1.9.1. Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{O}_X$-modules. Consider the presheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules defined by $\mathscr{G}(U) = \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$. The tensor product of restriction maps $\mathscr{E}(U) \to \mathscr{E}(V)$ and $\mathscr{F}(U) \to \mathscr{F}(V)$ gives the restriction map $\mathscr{G}(U) \to \mathscr{G}(V)$. The sheafification of $\mathscr{G}$ is denoted by $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ or simply $\mathscr{E} \otimes \mathscr{F}$ and called the **tensor product** of $\mathscr{E}$ and $\mathscr{F}$.

Remark 1.9.2. Let A be a commutative ring, and fix an A-module $\mathcal{N}$. Recall the following basic facts:

1. **Tensor products commute with direct limits**. More precisely, let $(\mathcal{M}_\alpha)$ be a direct system of A-modules. Then the canonical map $\mathcal{M}_\beta \otimes_A \mathcal{N} \to (\varinjlim_\alpha \mathcal{M}_\alpha) \otimes_A \mathcal{N}$ (for each fixed β) defines, by passing to the direct limit, an isomorphism
$$\varinjlim_\alpha (\mathcal{M}_\alpha \otimes_A \mathcal{N}) \xrightarrow{\simeq} (\varinjlim_\alpha \mathcal{M}_\alpha) \otimes_A \mathcal{N}. \tag{1.9.1}$$
(Proof: Construct the inverse map explicitly.)

2. **The tensor product functor $-\otimes \mathcal{N}$ is right exact**. Namely, if
$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \to 0$$
is an exact sequence of A-modules, then so is
$$\mathcal{M}_1 \otimes \mathcal{N} \xrightarrow{f\otimes \mathbf{1}} \mathcal{M}_2 \otimes \mathcal{N} \xrightarrow{g\otimes \mathbf{1}} \mathcal{M}_3 \otimes \mathcal{N} \to 0.$$
Identify $\mathcal{M}_3$ with $\mathrm{Coker} f = \mathcal{M}_2/f(\mathcal{M}_1)$. Then the right exactness of tensor product is equivalent to that **tensor products commute with cokernels**: we have an equivalence of A-modules
$$\mathrm{Coker}\big(\mathcal{M}_1 \otimes_A \mathcal{N} \xrightarrow{f\otimes \mathbf{1}} \mathcal{M}_2 \otimes_A \mathcal{N}\big) \quad \xrightarrow{\simeq} \quad \mathrm{Coker}\big(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2\big) \otimes_A \mathcal{N} \tag{1.9.2}$$
descended from the canonical morphism
$$\mathcal{M}_2 \otimes_A \mathcal{N} \longrightarrow \frac{\mathcal{M}_2}{f(\mathcal{M}_1)} \otimes_A \mathcal{N}. \tag{1.9.3}$$

$\square$

Proof. We have a well-defined map sending $\frac{\mathcal{M}_2}{f(\mathcal{M}_1)} \times \mathcal{N}$ to $\frac{\mathcal{M}_2 \otimes_A \mathcal{N}}{(f\otimes \mathbf{1})(\mathcal{M}_1 \otimes_A \mathcal{N})}$ (i.e. the LHS of (1.9.2)) sending $[\xi] \times \eta$ to $[\xi \otimes_A \eta]$, where $[\cdots]$ stands for the residue classes, and $\xi \in \mathcal{M}_2, \eta \in \mathcal{N}$. This map is clearly A-biinvariant. So it gives an A-module morphism from the RHS to the LHS of (1.9.2), which is clearly the inverse of the map in (1.9.2) from LHS to RHS. So (1.9.2) is an isomorphism. $\square$

Remark 1.9.3. We can now use (1.9.2) to explain the last equality of (1.2.4):

$$\begin{aligned} &\mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{O}_{X,x}/\mathfrak{m}_x) = \mathscr{E}_x \otimes \mathrm{Coker}(\mathfrak{m}_x \hookrightarrow \mathscr{O}_{X,x}) \\ \simeq &\mathrm{Coker}(\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x \otimes \mathscr{O}_{X,x}) \simeq \mathrm{Coker}(\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x) = \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x \end{aligned}$$

since the image of the multiplication map $\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x$ is $\mathfrak{m}_x\mathscr{E}_x$.

Proposition 1.9.4. *The canonical morphism of $\mathscr{O}(U)$-modules*

$$\mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \to \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$$

(where $U \ni x$ is open and the map is the tensor product of $\mathscr{E}(U) \to \mathscr{E}_x$ and $\mathscr{F}(U) \to \mathscr{F}_x$) induces an isomorphism

$$(\mathscr{E} \otimes \mathscr{F})_x = \varinjlim_{U \ni x} \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \quad \xrightarrow{\simeq} \quad \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x. \tag{1.9.4}$$

Proof. Define a canonical map from $\mathscr{E}_x \times \mathscr{F}_x$ to $\varinjlim_{U\ni x} \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$ and show that it is $\mathscr{O}_{X,x}$-biinvariant. This descends to the inverse map of (1.9.4). $\square$

Corollary 1.9.5. *For each $\mathscr{O}_X$-module $\mathscr{F}$, the functor $-\otimes \mathscr{F}$ on the abelian category of $\mathscr{O}_X$-modules is right exact: if*

$$\mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$$

is exact, then so is

$$\mathscr{E}_1 \otimes \mathscr{F} \to \mathscr{E}_2 \otimes \mathscr{F} \to \mathscr{E}_3 \otimes \mathscr{F} \to 0.$$

Proof. Exactness of sheaves can be checked at the level of stalks. Then this follows from the isomorphism (1.9.4) and the right exactness of $-\otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$. $\square$

1.9.2 Hom

We leave it to the readers to check the following easy facts:

Remark 1.9.6. Let A be a commutative ring, and fix an A-module $\mathcal{N}$:

1. $\mathrm{Hom}_A(\mathcal{N}, -)$ **is a left exact functor**. Namely, for any exact sequence of A-modules

$$0 \to \mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3, \tag{1.9.5}$$

we have an exact sequence

$$0 \to \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_1) \xrightarrow{f_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2) \xrightarrow{g_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_3)$$

where f_* sends T to $f \circ T$ and g_* is defined similarly. Equivalently, $\mathrm{Hom}_A(\mathcal{N}, -)$ **commutes with kernels**: there is a equivalence

$$\mathrm{Hom}_A\big(\mathcal{N}, \mathrm{Ker}(\mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3)\big) \simeq \mathrm{Ker}\big(\mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2) \xrightarrow{g_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_3)\big) \tag{1.9.6}$$

induced by the obvious inclusion

$$\mathrm{Hom}_A\big(\mathcal{N}, \mathrm{Ker}(\mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3)\big) \hookrightarrow \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2).$$

2. $\mathrm{Hom}_A(-, \mathcal{N})$ **is a left exact contravariant functor**. for any exact sequence of A-modules

$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \to 0 \tag{1.9.7}$$

we have an exact sequence

$$0 \to \mathrm{Hom}_A(\mathcal{M}_3, \mathcal{N}) \xrightarrow{g^*} \mathrm{Hom}_A(\mathcal{M}_2, \mathcal{N}) \xrightarrow{f^*} \mathrm{Hom}_A(\mathcal{M}_1, \mathcal{N})$$

where f^* sends T to $T \circ f$ and g^* is defined similarly. Equivalently, $\mathrm{Hom}_A(-, \mathcal{N})$ **turns cokernels into kernels**: there is a canonical equivalence

$$\mathrm{Hom}_A\big(\mathrm{Coker}(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2), \mathcal{N}\big) \simeq \mathrm{Ker}\big(\mathrm{Hom}_A(\mathcal{M}_2, \mathcal{N}) \xrightarrow{f^*} \mathrm{Hom}_A(\mathcal{M}_1, \mathcal{N})\big) \tag{1.9.8}$$

induced by the obvious inclusion

$$\mathrm{Hom}_A\big(\mathrm{Coker}(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2), \mathcal{N}\big) \hookrightarrow \mathrm{Hom}_A(\mathcal{M}_2, \mathcal{N}).$$

Definition 1.9.7. Let $\mathscr{E}, \mathscr{F}$ be $\mathscr{O}_X$-modules. The **hom space** $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$ is defined to be the space of all $\mathscr{O}_X$-module morphims from $\mathscr{E}$ to $\mathscr{F}$.

The presheaf of $\mathscr{O}_X$-modules sending each open $U \subset X$ to the $\mathscr{O}(U)$-module $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)$, and whose restriction map is the obvious restriction of sheaf morphisms, is automatically a sheaf of $\mathscr{O}_X$-modules. It is called the **hom sheaf** and denoted by $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$.

The dual and the double dual of $\mathscr{E}$ is defined by

$$\mathscr{E}^\vee = \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X), \qquad \mathscr{E}^{\vee\vee} = (\mathscr{E}^\vee)^\vee. \tag{1.9.9}$$

$\square$

Exercise 1.9.8. Describe canonical equivalences

$$\mathscr{E} \simeq \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{E} \simeq \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{E}). \tag{1.9.10}$$

In general, the stalks of $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$ cannot be identified with $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)$. But good things happen when $\mathscr{E}$ is coherent, as we will see in Cor. 2.2.4.

1.10 $(\mathscr{O}_X\text{−mod}) \otimes_{\mathscr{O}_S} (\mathscr{O}_S\text{−mod})$; pullback sheaves

Definition 1.10.1. Let $\varphi : X \to S$ be a holomorphic map of complex spaces. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module and $\mathscr{M}$ an $\mathscr{O}_S$-module. Then $\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M} = \mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{E}$ denotes the sheafification of the presheaf of $\mathscr{O}_X$-modules sending each open $U \subset X$ to

$$(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})^{\mathrm{pre}}(U) = \varinjlim_{V \supset \varphi(U)} \mathscr{E}(U) \otimes_{\mathscr{O}_S(V)} \mathscr{M}(V) \tag{1.10.1}$$

where the direct limit is over all open subset $V \subset S$ containing $\varphi(U)$, and $g \in \mathscr{O}_S(V)$ acts on $\varsigma \in \mathscr{E}(U)$ as

$$g \cdot \varsigma := \varphi^{\#}(g) \cdot \varsigma. \tag{1.10.2}$$

For each $x \in X$, we have a canonical equivalence

$$(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_x \simeq \mathscr{E}_x \otimes_{\mathscr{O}_{S,\varphi(x)}} \mathscr{M}_{\varphi(x)}. \tag{1.10.3}$$

Thus $\mathscr{M} \mapsto \mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}$ is a right exact functor.

Definition 1.10.2. The **pullback sheaf** of $\mathscr{M}$ along φ is the $\mathscr{O}_X$-module defined by

$$\varphi^* \mathscr{M} := \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{M} \tag{1.10.4}$$

whose stalk at x is $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,\varphi(x)}} \mathscr{M}_x$. It can be viewed as the induced representation of $\mathscr{M}$. Thus we may write

$$\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M} = \mathscr{E} \otimes_{\mathscr{O}_X} \varphi^* \mathscr{M}. \tag{1.10.5}$$

If $V \subset S$ is open and $\sigma \in \mathscr{M}(V)$, then the **pullback section** $\varphi^*(\sigma) \in \varphi^* \mathscr{M}(\varphi^{-1}(V))$ is the image of

$$1 \otimes \sigma \in \mathscr{O}(\varphi^{-1}(V)) \otimes_{\mathscr{O}(V)} \mathscr{M}(V) \to (\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{M})(\varphi^{-1}(V)) = (\varphi_* \varphi^* \mathscr{M})(V). \tag{1.10.6}$$

This gives a canonical morphism of $\mathscr{O}_S$-modules

$$\mathscr{M} \to \varphi_* \varphi^* \mathscr{M}. \tag{1.10.7}$$

If $g : \mathscr{M}_1 \to \mathscr{M}_2$ is a morphism of $\mathscr{O}_S$-modules, we define an $\mathscr{O}_X$-module morphism

$$\varphi^* g := \mathbf{1} \otimes g : \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_1 \to \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_2, \tag{1.10.8}$$

called the **pullback of** g.

The notation $\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}$ is a generalization of $\mathscr{E} \otimes_{\mathbb{C}} W$ for a ($\mathbb{C}$-)vector space W by viewing $\mathbb{C}$ as the structure sheaf of the single reduced point $\{0\}$, and by viewing the holomorphic map as the obvious one $X \to \{0\}$.

Proposition 1.10.3. *(φ^*, φ_*) is a pair of **adjoint functors** between the categories of $\mathscr{O}_X$-modules and $\mathscr{O}_S$-modules (with φ^* the left adjoint and φ_* the right one). Namely, there is a natural isomphism*

$$\mathrm{Hom}_{\mathscr{O}_X}(\varphi^* \mathscr{M}, \mathscr{E}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}, \varphi_* \mathscr{E}). \tag{1.10.9}$$

*The word **natural** means that for any morphisms $g : \mathscr{M}_2 \to \mathscr{M}_1$ of $\mathscr{O}_S$-modules and $f : \mathscr{E}_1 \to \mathscr{E}_2$ of $\mathscr{O}_X$-modules, $\varphi^* g$ and $\varphi_* f$ induce a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{O}_X}(\varphi^* \mathscr{M}_1, \mathscr{E}_1) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}_1, \varphi_* \mathscr{E}_1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{O}_X}(\varphi^* \mathscr{M}_2, \mathscr{E}_2) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}_2, \varphi_* \mathscr{E}_2) \end{array} \tag{1.10.10}$$

Proof. Given a morphism $F : \varphi^* \mathscr{M} \to \mathscr{E}$, the composition of $\mathscr{M} \to \varphi_* \varphi^* \mathscr{M}$ with $\varphi_* F : \varphi_* \varphi^* \mathscr{M} \to \varphi_* \mathscr{E}$ gives a morphism $G : \mathscr{M} \to \varphi_* \mathscr{E}$. Informally,

$$G(\sigma) = F(1 \otimes \sigma). \tag{1.10.11}$$

We leave it to the readers to check that $F \mapsto G$ is natural.

Conversely, given $G : \mathscr{M} \to \varphi_* \mathscr{E}$. The $\mathscr{O}(U)$-module morphisms

$$\mathscr{O}(U) \otimes_{\mathscr{O}(V)} \mathscr{M}(V) \to \mathscr{E}(U), \qquad f \otimes \sigma \mapsto f \cdot G(\sigma)|_U$$

for all open $U \subset X$ and $V \supset \varphi(U)$ pass to $F : \varphi^* \mathscr{M} \to \mathscr{E}$. This gives the inverse of the above construction. $\square$

Definition 1.10.4. Let $\iota : Y = \mathrm{Specan}(\mathscr{O}_X / \mathcal{I}) \hookrightarrow X$ be a closed subspace of X. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the **(sheaf theoretic) restriction of $\mathscr{E}$ to** Y, denoted by $\mathscr{E}|_Y$ or $\mathscr{E}|Y$ is

$$\mathscr{E}|_Y = \iota^* \mathscr{E} = (\mathscr{O}_X / \mathcal{I}) \restriction_{N(\mathcal{I})} \otimes_{\mathscr{O}_X} \mathscr{E}. \tag{1.10.12}$$

Remark 1.10.5. If $\iota : Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \to X$ is an embedding of closed complex subspace, one may view an $\mathscr{O}_Y$-module $\mathscr{F}$ as the corresponding $\mathscr{O}_X$-module $\iota_*\mathscr{F}$. A more precise statement is that the functor ι_* from the category of $\mathscr{O}_Y$-modules to the category of $\mathscr{O}_X$-modules annihilated by the multiplication of $\mathcal{I}$, sending each morphism φ to $\iota_*\varphi$, is an equivalence of categories. (Cf. Thm. 1.6.2 or Thm. 2.2.2 for the precise meaning.) An inverse functor can be chosen to be ι^*. In particular, we have a canonical equivalence $\mathscr{F} \simeq \iota^*\iota_*\mathscr{F}$ for any $\mathscr{O}_Y$-module $\mathscr{F}$ and $\mathscr{E} \simeq \iota_*\iota^*\mathscr{E}$ for any $\mathscr{O}_X$-module $\mathscr{E}$ annihilated by $\mathcal{I}$ (so that $\mathscr{E} = \mathscr{E}/\mathcal{I}\mathscr{E} \simeq \mathscr{E}\otimes_{\mathscr{O}_X}(\mathscr{O}_X/\mathcal{I})$). These equivalences are the identity maps at the level of stalks.

Moreover, the functor ι_* is an equivalence of tensor categories. Namely, we have natural isomorphisms

$$\iota_*(\mathscr{F}_1 \otimes_{\mathscr{O}_Y} \mathscr{F}_2) \simeq (\iota_*\mathscr{F}_1) \otimes_{\mathscr{O}_X} (\iota_*\mathscr{F}_2).$$

Note that since $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$ is surjective (if $y \in Y$), we have

$$\mathscr{F}_{1,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{F}_{2,y} \simeq \mathscr{F}_{1,y} \otimes_{\mathscr{O}_{X,y}} \mathscr{F}_{2,y}. \tag{1.10.13}$$

If $\mathscr{E}$ is an $\mathscr{O}_X$-module, we also have a natural isomorphism

$$\iota_*(\mathscr{E}|_Y) \simeq (\mathscr{O}_X/\mathcal{I}) \otimes_{\mathscr{O}_X} \mathscr{E}. \tag{1.10.14}$$

Thus, the study of the restriction $\mathscr{E}|_Y$ can be turned into the study of an $\mathscr{O}_X$-module. $\square$

1.11 Fiber products

Definition 1.11.1. Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. A **fiber product** of these two maps is a complex space $X \times_S Y$ together with holomorphic maps $\mathrm{pr}_X : X \times_S Y \to X$ and $\mathrm{pr}_Y : X \times_S Y \to Y$ satisfying:

(1) $\varphi \circ \mathrm{pr}_X = \psi \circ \mathrm{pr}_Y$.

(2) For each complex space Z and holomorphic maps $\alpha : Z \to X$ and $\beta : Z \to Y$ satisfying $\varphi\circ\alpha = \psi\circ\beta$ there is a unique holomorphic map $\alpha\vee\beta : Z \to X\times_S Y$ such that $\alpha = \mathrm{pr}_X \circ (\alpha \vee \beta)$ and that $\beta = \mathrm{pr}_Y \circ (\alpha \vee \beta)$.

$$\begin{tikzcd}
& & Z \arrow[dll, bend right, "\alpha"'] \arrow[dl, dashed, "\alpha\vee\beta"] \arrow[ddl, bend left, "\beta"] \\
X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & \\
S & Y \arrow[l, "\psi"] &
\end{tikzcd} \tag{1.11.1}$$

The commutative square diagram above involving $S, X, Y, X \times_S Y$ is called a **Cartesian square**. $\mathrm{pr}_Y : X \times_S Y \to Y$ is called the **pullback/base change** of $\varphi : X \to S$ along $\psi : Y \to S$. $\square$

The following is easy to check:

Proposition 1.11.2. *In Def. 1.11.1, let $\gamma : Z' \to Z$ be a holomorphic map. Then*

$$(\alpha \vee \beta) \circ \gamma = (\alpha \circ \gamma) \vee (\beta \circ \gamma) : Z' \to X \times_S Y. \tag{1.11.2}$$

Fiber products are clearly unique up to isomorphisms. The following is easy to check.

Remark 1.11.3. Suppose that the following two small commuting square diagrams are both Cartesian, then the largest rectangular square is also Cartesian.

$$\begin{tikzcd}
X \arrow[d] & X \times_S Y \arrow[l] \arrow[d] & (X \times_S Y) \times_Y Z \arrow[l] \arrow[d] \\
S & Y \arrow[l] & Z \arrow[l]
\end{tikzcd}$$

Namely, $(X \times_S Y) \times_Y Z$, together with its maps to X and Z, is a pullback of $X \to S$ along $Z \to S$. This can be generalized to more complicated situations. For instance, if the following 4 small cells are Cartesian squares, then so is the largest square diagram.

$$\begin{tikzcd}
X_1 \arrow[d] & Z_1 \arrow[l] \arrow[d] & Z_3 \arrow[l] \arrow[d] \\
X \arrow[d] & Z \arrow[l] \arrow[d] & Z_2 \arrow[l] \arrow[d] \\
S & Y \arrow[l] & Y_1 \arrow[l]
\end{tikzcd}$$

Example 1.11.4. Let U, V be open subsets of a complex space X. Then $U \cap V$ is a fiber product $U \times_X V$: we have Cartesian square

$$\begin{tikzcd}
U \arrow[d, hook] & U \cap V \arrow[l, hook'] \arrow[d, hook] \\
X & V \arrow[l, hook']
\end{tikzcd}$$

Definition 1.11.5. Let $\varphi : X \to S$, $\psi : Y \to S$, $\alpha : X' \to X$, $\beta : Y' \to Y$ be holomorphic maps of complex spaces. Assume $X \times_S Y$ exists. Assume we have a fiber product $X' \times_S Y'$ of $\varphi \circ \alpha : X' \to S$ and $\psi \circ \beta : Y' \to S$. Then

$$\alpha \times \beta : X' \times_S Y' \to X \times_S Y \tag{1.11.3}$$

denotes $(\alpha\circ \mathrm{pr}_{X'})\vee(\beta\circ\mathrm{pr}_{Y'})$, the unique holomorphic map making the following diagram commute.

$$
\begin{tikzcd}
 & X' \arrow[dl, "\alpha"'] & X'\times_S Y' \arrow[l, "\mathrm{pr}_{X'}"'] \arrow[dl, dashed, "\alpha\times\beta"'] \arrow[d, "\mathrm{pr}_{Y'}"] \\
X \arrow[d, "\varphi"'] & X\times_S Y \arrow[l, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & Y' \arrow[dl, "\beta"] \\
S & Y \arrow[l, "\psi"] &
\end{tikzcd}
\tag{1.11.4}
$$

The following is easy to check:

Proposition 1.11.6. *In Def. 1.11.5, let $\mu: Z\to X'$, $\nu: Z\to Y'$ be holomorphic maps of complex spaces such that $\varphi\circ\alpha\circ\mu=\psi\circ\beta\circ\nu$. Then we have equality*

$$
(\alpha\times\beta)\circ(\mu\vee\nu)=(\alpha\circ\mu)\vee(\beta\circ\nu): Z\to X\times_S Y. \tag{1.11.5}
$$

Let $\widetilde{\alpha}: X''\to X'$, $\widetilde{\beta}: Y''\to Y'$ be holomorphic maps of complex spaces, and assume that a fiber product $X''\times_S Y''$ exists for $\varphi\circ\alpha\circ\widetilde{\alpha}: X''\to S$ and $\psi\circ\beta\circ\widetilde{\beta}: Y''\to S$. Then

$$
(\alpha\times\beta)\circ(\widetilde{\alpha}\times\widetilde{\beta})=(\alpha\circ\widetilde{\alpha})\times(\beta\circ\widetilde{\beta}): X''\times_S Y''\to X\times_S Y. \tag{1.11.6}
$$

Remark 1.11.7. There are no canonical fiber products of give holomorphic $\varphi: X\to S$, $\psi: Y\to S$. But suppose that a fiber product $X\times_S Y$ exists and is fixed. Then for each open $U\subset X$ and $X\subset Y$, there is a unique (open) **fiber product** $U\times_S V$ **inside** $X\times_S Y$. which is the open complex subspace

$$
U\times_S V:=\mathrm{pr}_X^{-1}(U)\cap\mathrm{pr}_Y^{-1}(V)
$$

of $X\times_S Y$. The projections $\mathrm{pr}_U: U\times_S V\to U$ and $\mathrm{pr}_V: U\times_S V\to V$ are defined respectively by the restrictions of $\mathrm{pr}_X,\mathrm{pr}_Y$.

Moreover, assume that $\alpha: X'\to X$, $\beta: Y'\to Y$ are holomorphic, and a fiber product $X'\times_S Y'$ is fixed. Let $U'\subset X'$ and $V'\subset Y'$ be open such that $\alpha(U')\subset U$, $\beta(V')\subset V$. Let $U'\times_S V'$ be the fiber product inside $X'\times_S Y'$. The we have a commutative diagram

$$
\begin{tikzcd}
X'\times_S Y' \arrow[r, "\alpha\times\beta"] & X\times_S Y \\
U'\times_S V' \arrow[u, hook] \arrow[r, "\alpha|_{U'}\times\beta|_{V'}"] & U\times_S V \arrow[u, hook]
\end{tikzcd}
\tag{1.11.7}
$$

□

Proof. Show that the inclusion $U \times_S V \hookrightarrow X \times_S Y$ is the product of $U \hookrightarrow X$ and $V \hookrightarrow Y$ and $U' \times_S V' \hookrightarrow X' \times_S Y'$ similarly. Then apply Prop. 1.11.6. $\square$

With the help of the above observation, we can prove:

Lemma 1.11.8 (Gluing fiber products). *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. Let $(U_i)_{i\in\mathfrak{I}}$ and $(V_t)_{t\in\mathfrak{T}}$ be open covers of X and Y respectively. Suppose that for each $i \in \mathfrak{I}$ and $t \in \mathfrak{T}$ there exists a fiber product $U_i \times_S V_t$. Then a fiber product $X \times_S Y$ exists.*

Proof. It suffices to assume (V_t) has only one member, which is Y. So each $U_i \times_S Y$ exists. To simplify notations, for each $i, j, k \in \mathfrak{I}$ we set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. We let $U_{ij} \times_i Y$ and $U_{ijk} \times_i Y$ denote the corresponding open fiber products inside $U_i \times_S Y$. So $U_{ij} \times_i Y$ and $U_{ij} \times_j Y$ are isomorphism but not identical.

We now apply the gluing construction Rem. 1.3.7 to construct $X \times Y$ by gluing all $U_i \times Y$ together. As gluing of topological spaces the process is trivial. To glue the structures of complex spaces, we must assign an isomorphism $\pi_{j,i} : U_{ij} \times_i Y \xrightarrow{\simeq} U_{ij} \times_j Y$ for all i, j. This is chosen to be $\mathbf{1}_{U_{ij}} \times_{j,i} \mathbf{1}_Y$ defined by Def. 1.11.5. (Note that this is not an identity map since the source does not equal the target. The symbol $\times_{j,i}$ reflects the fact that this product relies on both i and j.)

Clearly $\pi_{i,i}$ is the identity. To finish checking the cocycle condition, we must show that the holomorphic maps $\pi_{k,i}$ and $\pi_{k,j} \circ \pi_{j,i}$ are equal when restricted to open subsets $U_{ijk} \times_i Y \to U_{ijk} \times_k Y$. By Rem. 1.11.7, $\pi_{k,i}$ restricts to $\mathbf{1}_{U_{ijk}} \times_{k,i} \mathbf{1}_Y$, and $\pi_{k,j} \circ \pi_{j,i}$ restricts to $(\mathbf{1}_{U_{ijk}} \times_{k,j} \mathbf{1}_Y) \circ (\mathbf{1}_{U_{ijk}} \times_{j,i} \mathbf{1}_Y)$, which equals $\mathbf{1}_{U_{ijk}} \times_{k,i} \mathbf{1}_Y$ by Prop. 1.11.6.

Thus the complex space $X \times_S Y$ is constructed. We leave it to the readers to define pr_X and pr_Y. $\square$

1.12 Fiber products and inverse images of subspaces

Proposition 1.12.1. *Let $\varphi : X \to S$ be a holomorphic map of complex spaces, and let $\mathscr{J}$ be a finite type ideal of $\mathscr{O}_S$. Then we have a Cartesian square*

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & \varphi^{-1}(S_0) := \mathrm{Specan}(\mathscr{O}_X/\mathscr{J}\mathscr{O}_X) \arrow[l, hook'] \arrow[d, "\tilde{\varphi}"'] \\ S & S_0 := \mathrm{Specan}(\mathscr{O}_S/\mathscr{J}) \arrow[l, hook'] \end{tikzcd} \tag{1.12.1}$$

*where $\mathscr{J}\mathscr{O}_X$ is the (necessarily unique) finite-type ideal of $\mathscr{O}_X$ whose stalks $(\mathscr{J}\mathscr{O}_X)_x$ are generated by $\mathscr{J}_{\varphi(x)}$ (more precisely, by $\varphi^\#(\mathscr{J}_{\varphi(x)})$, cf. (1.10.2)). $\varphi^{-1}(S_0) :=$ $\mathrm{Specan}(\mathscr{O}_X/\mathscr{J}\mathscr{O}_X)$ is called the **inverse image of** S_0 along φ.*

Proof. If $V \subset S$ is open and $\mathcal{J}|_V$ is generated by finitely many $g_1, g_2, \cdots \in \mathcal{J}(V)$, then $(\mathcal{J}\mathscr{O}_X)|_{\varphi^{-1}(V)}$ is defined to be the ideal of $\mathscr{O}_X|_{\varphi^{-1}(V)}$ generated by $\varphi^{\#}(g_1), \varphi^{\#}(g_2), \ldots$. Clearly the stalks of $(\mathcal{J}\mathscr{O}_X)|_{\varphi^{-1}(V)}$ satisfy the requirement. Thus, these ideals are compatible for different V, and can be glued together and form the desired ideal $\mathcal{J}\mathscr{O}_X$. To check that (1.12.1) is Cartesian one uses Thm. 1.4.8. □

Remark 1.12.2. Using the explicit construction of $\mathcal{J}$ in the proof of Prop. 1.12.1, one sees that the underlying set of $\varphi^{-1}(S_0)$ is the usual preimage of S_0, i.e., $\{x \in X : \varphi(x) \in S_0\}$.

Remark 1.12.3. As an $\mathscr{O}_X$-module, $\mathscr{O}_{\varphi^{-1}(S_0)}$ has a natural equivalence

$$\mathscr{O}_{\varphi^{-1}(S_0)} = \mathscr{O}_X/\mathcal{J}\mathscr{O}_X \simeq \mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathcal{J}) = \varphi^*(\mathscr{O}_{S_0}) \tag{1.12.2}$$

Thus, for any $\mathscr{O}_X$-module $\mathscr{E}$, we have canonical equivalences of $\mathscr{O}_X$-modules

$$\mathscr{E}|_{\varphi^{-1}(S_0)} = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_{\varphi^{-1}(S_0)} \simeq \mathscr{E} \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathcal{J}) \simeq \mathscr{E}/\mathcal{J}\mathscr{E} \tag{1.12.3}$$

Proof. Using the right exactness of $\mathscr{O}_X \otimes_{\mathscr{O}_S} -$, we have

$$\begin{aligned} &\mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathcal{J}) = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathrm{Coker}(\mathcal{J} \hookrightarrow \mathscr{O}_S) \\ \simeq &\mathrm{Coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathcal{J} \to \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S) \simeq \mathrm{Coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathcal{J} \to \mathscr{O}_X) \end{aligned}$$

which equals $\mathscr{O}_X/\mathcal{J}\mathscr{O}_X$ since the term insider the last Coker is the multiplication map. (Compare Rem. 1.9.3.) This proves (1.12.2). (1.12.3) follows from a similar argument. □

Example 1.12.4. Let $\mathcal{I}, \mathcal{J}$ be finite-type ideals of $\mathscr{O}_S$. Using Thm. 1.4.8 again, one easily checks that there is a Cartesian square that breaks into two commuting triangles.

$$\begin{tikzcd}
X = \mathrm{Specan}(\mathscr{O}_S/\mathcal{I}) \arrow[d, hook] & X \cap Y := \mathrm{Specan}(\mathscr{O}_S/(\mathcal{I}+\mathcal{J})) \arrow[l] \arrow[d] \arrow[dl, hook, dashed] \\
S & Y = \mathrm{Specan}(\mathscr{O}_S/\mathcal{J}) \arrow[l, hook']
\end{tikzcd} \tag{1.12.4}$$

Thus, the inverse image of Y along X is naturally equivalent to the closed subspace $X \cap Y := \mathrm{Specan}(\mathscr{O}_S/(\mathcal{I}+\mathcal{J}))$ of S, called the **intersection of** X **and** Y. (Compare this with Exp. 1.7.5.) In view of this equivalence, we shall view $X \cap Y$ as closed subspaces of X and Y in the future.

Proposition 1.12.5. *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic, and let X_0 and Y_0 be complex subspaces of X, Y respectively. Assume that $X \times_S Y$ is a fiber product of φ and ψ. Recall $\mathrm{pr}_X : X \times_S Y \to X$ and $\mathrm{pr}_Y : X \times_S Y \to Y$. Then the intersection*

$$X_0 \times_S Y_0 := \mathrm{pr}_X^{-1}(X_0) \cap \mathrm{pr}_Y^{-1}(Y_0)$$

is a fiber product of $X_0 \hookrightarrow X \xrightarrow{\varphi} S$ and $Y_0 \hookrightarrow Y \xrightarrow{\psi} S$, called the ***(closed) fiber product inside*** *$X \times_S Y$. The projections of $\mathrm{pr}_X^{-1}(X_0) \cap \mathrm{pr}_Y^{-1}(Y_0)$ to X_0 and Y_0 are respectively the restrictions of pr_X and pr_Y. Moreover, the inclusion $X_0 \times_S Y_0 \hookrightarrow X \times_S Y$ equals the product of $X_0 \hookrightarrow X$ and $Y_0 \hookrightarrow Y$.*

Proof. The four cells are Cartesian squares. So is the largest one (Rem. 1.11.3).

$$\begin{tikzcd}
X_0 & \mathrm{pr}_X^{-1}(X_0) \arrow[l] & X_0 \times_S Y_0 \arrow[l, hook'] \\
X \arrow[d, "\varphi"'] \arrow[from=u, hook] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"'] \arrow[from=u, hook] & \mathrm{pr}_Y^{-1}(Y_0) \arrow[l, hook'] \arrow[d] \arrow[from=u, hook] \\
S & Y \arrow[l, "\psi"'] & Y_0 \arrow[l, hook']
\end{tikzcd} \tag{1.12.5}$$

The claim about inclusions is obvious. □

Remark 1.12.6. The closed fiber product $X_0 \times_S Y_0 \subset X \times_S Y$ can be written more explicitly. Choose finite-type ideals $\mathcal{I} \subset \mathscr{O}_X$ and $\mathcal{J} \subset \mathscr{O}_Y$ defining X_0, Y_0 respectively. Then $X_0 \times_S Y_0$ is defined by the ideal $\mathcal{K} \subset \mathscr{O}_{X \times_S Y}$ generated by $\mathrm{pr}_X^{\#}(\mathcal{I})$ and $\mathrm{pr}_Y^{\#}(\mathcal{J})$. More precisely: each stalk $\mathcal{K}_{x \times y}$ is generated by $\mathrm{pr}_X^{\#}(\mathcal{I}_x)$ and $\mathrm{pr}_Y^{\#}(\mathcal{J}_y)$.

In practice, we may assume X and Y are small enough such that $\mathcal{I}$ is generated by $f_1, \dots, f_m \in \mathscr{O}(X)$ and $\mathcal{J}$ is generated by $g_1, \dots, g_n \in \mathscr{O}(Y)$. Then all $\mathrm{pr}_X^{\#}(f_i)$ and $\mathrm{pr}_Y^{\#}(g_j)$ generate $\mathcal{K}$. □

Remark 1.12.7. Similar to Rem. 1.11.7, suppose we have holomorphic $\alpha : X' \to X$, $\beta : Y' \to Y$, $\varphi : X \to S$, $\psi : Y \to S$. Let $X_0 \subset X, Y_0 \subset Y, X_0' \subset X', Y_0' \subset Y'$ be closed subspaces such that α restricts to $\alpha : X_0' \to X_0$ and β restricts to $\beta : Y_0' \to Y_0$ (in the sense of Thm. 1.4.8). Then for the closed fiber products $X_0 \times_S Y_0 \subset X \times_S Y$ and X_0', the following diagram commutes.

$$\begin{tikzcd}
X' \times_S Y' \arrow[r, "\alpha \times \beta"] & X \times_S Y \\
X_0' \times_S Y_0' \arrow[u, hook] \arrow[r, "\alpha|_{X_0'} \times \beta|_{Y_0'}"] & X_0 \times_S Y_0 \arrow[u, hook]
\end{tikzcd} \tag{1.12.6}$$

1.13 Fiber products, direct products, and equalizers

Definition 1.13.1. Let X, Y be complex spaces. A **direct product** of X, Y, or simply a **product** of X, Y, is a fiber product of $X \to 0$ and $Y \to 0$ and denoted by $X \times Y$ (together with the projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$).

To spell out the definition: For each complex space Z and holomorphic $\alpha : Z \to X, \beta : Z \to Y$, there is a unique holomorphic map $\alpha \vee \beta : Z \to X \times Y$ such that the following diagram commute.

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[d, "\alpha\vee\beta"'] \arrow[dr, "\beta"] & \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y \end{tikzcd}$$

If $f \in \mathscr{O}_X$ and $g \in \mathscr{O}_Y$, we write

$$f \otimes 1 := \mathrm{pr}_X^\#(f), \qquad 1 \otimes g := \mathrm{pr}_Y^\#(g), \qquad f \otimes g := \mathrm{pr}_X^\#(f)\mathrm{pr}_Y^\#(g). \tag{1.13.1}$$

If $x \in X$ and $y \in Y$, we define the **completed tensor product**

$$\mathscr{O}_{X,x}\widehat{\otimes}\mathscr{O}_{Y,y} := \mathscr{O}_{X\times Y,x\times y}$$

which is well-defined up to isomorphisms by Cor. 1.6.3. □

Remark 1.13.2. One can also view $\mathscr{O}_{X\times_S Y,x\times y}$ as $\mathscr{O}_{X,x}\widehat{\otimes}_{\mathscr{O}_{S,s}}\mathscr{O}_{Y,y}$ (if $s = \varphi(x) = \psi(y)$), a completed tensor product over $\mathscr{O}_{S,s}$. In the case that either φ or ψ is "finite", the stalk $\mathscr{O}_{X\times_S Y,x\times y}$ is actually equal to the usual tensor product $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$. This will be studied in the next chapter.

Example 1.13.3. $\mathbb{C}^{m+n}$ is naturally a product of $\mathbb{C}^m$ and $\mathbb{C}^n$.

Proof. Apply Thm. 1.4.1. □

Lemma 1.13.4. *For every complex spaces X, Y there is a product $X \times Y$.*

Proof. We know this is true when X, Y are number spaces, and hence when X, Y are open subspaces of number spaces (cf. Exp. 1.11.7), and hence if X, Y are model spaces (due to Prop. 1.12.5), and hence for all complex spaces (by Lemma 1.11.8). □

Remark 1.13.5. If X and Y are model spaces $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ and $\mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$ where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open, $\mathcal{I}$ is generated by $f_1, f_2, \dots \in \mathcal{I}(U)$, and $\mathcal{J}$ is generated by $g_1, g_2, \dots \in \mathcal{J}(V)$, then $X \times Y$ as a closed direct product inside $U \times V$ can be written down explicitly with the help of Rem. 1.12.6: it is the model space $\mathrm{Specan}(\mathscr{O}_{U\times V}/\mathcal{K})$ where $\mathcal{K}$ is the ideal generated by all $f_i \otimes 1$ and $1 \otimes g_j$.

In the following, we give two proofs that fiber products always exist. We need the following notion:

Proposition 1.13.6. *Let $\varphi : X \to Y$ be a holomorphic map. Then $\mathbf{1}_X \vee \varphi : X \to X \times Y$ is an equalizer:*

$$X \xrightarrow{\mathbf{1}\vee\varphi} X \times Y \overset{\varphi\circ\mathrm{pr}_X}{\underset{\mathrm{pr}_Y}{\rightrightarrows}} Y \tag{1.13.2}$$

The canonical equalizer $\mathfrak{G}(\varphi)$ of $X \times Y \rightrightarrows Y$ (which is a closed subspace of $X \times Y$) is called the ***graph of*** *φ.*

Proof. Let Z be a complex space. Any holomorphic map $Z \to X \times Y$ is $\alpha \vee \beta$ for some $\alpha : Z \to X$ and $\beta : Z \to Y$. Suppose that the compositions of $\alpha \vee \beta$ with $\varphi \circ \mathrm{pr}_X$ and with pr_Y are equal. Then $\varphi \circ \alpha = \beta$. Then we may find a holomorphic map $Z \to X$ such that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[d] \arrow[dr, "\alpha\vee\beta"] & \\ X \arrow[r, "\mathbf{1}\vee\varphi"] & X \times Y \end{tikzcd}$$

Indeed, we can choose this map to be α. Then by Prop. 1.11.2, $(\mathbf{1} \vee \varphi) \circ \alpha = \alpha \vee (\varphi \circ \alpha) = \alpha \vee \beta$. On the other hand, if we have another such holomorphic map $\psi : Z \to X$. Composing the above triangle with $\mathrm{pr}_X : X \times Y \to X$ shows that $\psi = \mathrm{pr}_X \circ (\mathbf{1} \vee \varphi) \circ \psi$ equals $\mathrm{pr}_X \circ (\alpha \vee \beta) = \alpha$. This proves the uniqueness of such ψ. $\square$

Remark 1.13.7. Using Thm. 1.8.2, one can give a more explicit description of the graph of $\varphi : X \to Y$. We write it as $\mathrm{Specan}(\mathscr{O}_{X\times Y}/\mathscr{J})$ for a finite-type ideal $\mathscr{J}$. Let $x \in X, y \in Y$. If $y \neq \varphi(x)$ then $\mathscr{J}_{x\times y} = \mathscr{O}_{X\times Y,x\times y}$. If $y = \varphi(x)$ then $\mathscr{J}_{x\times y}$ is the ideal of $\mathscr{O}_{X\times Y,x\times y}$ generated by

$$(f \circ \varphi) \otimes 1 - 1 \otimes f \tag{1.13.3}$$

for all $f \in \mathscr{O}_{Y,y}$ (equivalently, for a set of f generating the algebra $\mathscr{O}_{Y,y}$ analytically). The underlying topological space of the graph is $\{x \times y \in X \times Y : y = \varphi(x)\}$.

Remark 1.13.8. The graph construction shows that every holomorphic map $\varphi : X \to Y$ is the composition of a closed embedding $X \xrightarrow{\mathbf{1}\vee\varphi} X \times Y$ (cf. Rem. 1.8.3) and a projection of direct product $X \times Y \xrightarrow{\mathrm{pr}_Y} Y$. Thus, very often, the study of general holomorphic maps reduces to the studies of these two special types of maps. As an application of this observation, we prove:

Theorem 1.13.9. *For any holomorphic maps of complex spaces $\varphi : X \to S, \psi : Y \to S$, there exists a fiber product $X \times_S Y$.*

Proof. We want to show that the pullback of φ along ψ exists. We know it exists when ψ is a closed embedding due to Prop. 1.12.1. It also exists when ψ is a projection $S \times Y_1 \to S$: in that case $X \times_S Y$ is given by the Cartesian square

$$\begin{tikzcd} X & X \times Y_1 \arrow[l] \arrow[d, "\varphi\times\mathbf{1}"] \\ S \arrow[from=u, "\varphi"'] & S \times Y_1 \arrow[l] \end{tikzcd} \tag{1.13.4}$$

(We leave it to the readers to check that this commutative diagram is indeed Cartesian.) The general case follows from Rem. 1.13.8 and the fact that the pullback of a pullback is a pullback (Rem. 1.11.3). □

We now give another way of constructing fiber products. This construction is very explicit when X and Y are model spaces.

Proposition 1.13.10. *Let $\varphi : X \to S, \varphi : Y \to S$ be holomorphic maps of complex spaces. Let $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ be the projections of $X \times Y$. Then the canonical equalizer E of the following double arrow is a fiber product $X \times_S Y$:*

$$E \xhookrightarrow{\iota} X \times Y \underset{\psi\circ\mathrm{pr}_Y}{\overset{\varphi\circ\mathrm{pr}_X}{\rightrightarrows}} S \tag{1.13.5}$$

The projections of E to X, Y are $\mathrm{pr}_X \circ \iota$ and $\mathrm{pr}_Y \circ \iota$ respectively. We call E the (closed) ***fiber product of X, Y inside the direct product*** *$X \times Y$.*

Proof. That E is an equalizer means that $\varphi \circ (\mathrm{pr}_X \circ \iota) = \psi \circ (\mathrm{pr}_Y \circ \iota)$, and that for every holomorphic $\alpha \vee \beta : Z \to X \times Y$ whose compositions with $\varphi \circ \mathrm{pr}_X$ and with $\psi \circ \mathrm{pr}_Y$ are the same (namely, $\varphi \circ \alpha = \psi \circ \beta$) there is a unique holomorphic $\gamma : Z \to E$ such that $\iota \circ \gamma = \alpha \vee \beta$ (namely, $(\mathrm{pr}_X \circ \iota) \circ \gamma = \alpha$ and $(\mathrm{pr}_Y \circ \iota) \circ \gamma = \beta$). This means precisely that E equipped with $\mathrm{pr}_X \circ \iota$ and $\mathrm{pr}_Y \circ \iota$ is a fiber product. □

Remark 1.13.11. Using Thm. 1.8.2, we can describe the fiber product $X \times_S Y$ inside a given $X \times Y$ easily: It is $\mathrm{Specan}(\mathscr{O}_{X\times Y}/\mathscr{J})$ where $\mathscr{J}$ is a finite-type ideal. Let $x \in X, y \in Y$. If $\varphi(x) \neq \psi(y)$ then $\mathscr{J}_{x\times y} = \mathscr{O}_{X\times Y,x\times y}$. If $\varphi(x) = \psi(y)$ then $\mathscr{J}_{x\times y}$ is the ideal of $\mathscr{O}_{X\times Y,x\times y}$ generated by

$$(f \circ \varphi) \otimes 1 - 1 \otimes (f \circ \psi) \tag{1.13.6}$$

for all $f \in \mathscr{O}_{S,\varphi(x)}$ (equivalently, for a set of f generating the algebra $\mathscr{O}_{S,\varphi(x)}$ analytically). The underlying topological space of $X \times_S Y$ is $\{x \times y \in X \times Y : \varphi(x) = \psi(y)\}$.

From this, it is clear that given a fiber product $X \times_S Y$, if $x \in X, y \in Y$ and $\varphi(x) = \psi(y)$, then there is a unique point of $X \times_S Y$, denoted by (x, y) or $x \times y$, whose projections to X, Y are x, y respectively. Moreover, all points of $X \times_S Y$ are in this form. □

Exercise 1.13.12. Show that the pullback of $\varphi \times \psi : X \times Y \to S \times S$ along the **diagonal map** Δ_S defined by $\mathbf{1}_S \vee \mathbf{1}_S : S \to S \times S$ is a fiber product $X \times_S Y$.

We have seen that fiber products can be constructed from equalizers. Conversely, equalizers can also be viewed as special cases of fiber products:

Proposition 1.13.13. *Let $\varphi, \psi : X \to Y$ be holomorphic maps, and let $\Delta_Y : Y \to Y \times Y$ be the diagonal map of Y with image $\widetilde{Y}$ being a closed subspace of $Y \times Y$, called the* ***diagonal of*** *$Y \times Y$. Then the inverse image E of $\widetilde{Y}$ along $\varphi \vee \psi : X \to Y \times Y$ is the canonical equalizer of $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$.*

Proof. Write $\widetilde{Y}$ as $\mathrm{Specan}(\mathscr{O}_{Y \times Y}, \mathcal{J})$. Then by Rem. 1.13.7, $\mathcal{J}_{y,y'} = \mathscr{O}_{Y \times Y, y \times y'}$ if $y \neq y'$, and $\mathcal{J}_{y,y'}$ is generated by all $f \otimes 1 - 1 \otimes f$ where $f \in \mathscr{O}_{Y,y}$.

Write E as $\mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then by Prop. 1.12.1, if $\varphi(x) \neq \psi(x)$ then $\mathcal{I}_x$ equals $\mathscr{O}_{X,x}$ (since $\mathcal{J}_{\varphi(x),\psi(x)} = \mathscr{O}_{Y \times Y, \varphi(x) \times \psi(x)}$); if $\varphi(x) = \psi(x)$ then $\mathcal{I}_x$ is generated by $(f \otimes 1 - 1 \otimes f) \circ (\varphi \vee \psi)$ (i.e. by $f \circ \varphi - f \circ \psi$) for all $f \in \mathscr{O}_{Y,\varphi(x)}$. Comparing this description with Thm. 1.8.2, we see that E is the canonical equalizer. $\square$

Chapter 2

Finite holomorphic maps and coherence

2.1 Coherent sheaves

We fix a $\mathbb{C}$-ringed space X.

Definition 2.1.1. An $\mathscr{O}_X$-module $\mathscr{E}$ is called **coherent** if the following conditions are satisfied:

1. $\mathscr{E}$ is of finite-type.
2. For every open set $U \subset X$, any $n \in \mathbb{N}$, and any $\mathscr{O}_U$-module morphism $\varphi : \mathscr{O}_U^n \to \mathscr{E}|_U$, the kernel $\mathrm{Ker}\varphi$ is a finite-type $\mathscr{O}_U$-module.

Set $s_1 = \varphi(1, 0, \cdots, 0), \ldots, s_n = \varphi(0, 0, \ldots, 1)$. Then $\mathrm{Ker}\varphi$ is called the **sheaf of relations of** $s_1, \ldots, s_n$ and denoted by $\mathscr{Rel}(s_\bullet) = \mathscr{Rel}(s_1, \ldots, s_n)$.

In other words, $\mathscr{Rel}(s_\bullet)$ is the sheaf of all $(f_1, \ldots, f_n) \in \mathscr{O}_U^n$ such that $f_1 s_1 + \cdots + f_n s_n = 0$. A coherent $\mathscr{O}_X$-module is a finite-type $\mathscr{O}_X$-module such that any sheaf of relations is finite-type.

Remark 2.1.2. It is clear that a finite type submodule of a coherent $\mathscr{O}_X$-module is coherent.

Theorem 2.1.3. *Let $0 \to \mathscr{E} \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \to 0$ be an exact sequence of $\mathscr{O}_X$-modules. If two of the three sheaves are coherent, then the remaining one is also coherent.*

We view $\mathscr{E}$ as a subsheaf of $\mathscr{F}$.

Proof of $\mathscr{E}, \mathscr{F}$ coherent $\Rightarrow \mathscr{G}$ coherent. Since $\mathscr{F}$ is finite-type and φ is surjective, $\mathscr{G}$ is finite-type. Choose any $x \in X$, any neighborhood $U \ni x$, and any $t_1, \ldots, t_n \in \mathscr{G}(U)$.

We shall show that $\mathscr{Rel}(t_\bullet)$ is generated by finitely many global sections after shrinking U to a smaller neighborhood of x.

Shrink U so that we can find $s_1, \dots, s_n \in \mathscr{F}(U)$ sent to $t_1, \dots, t_n$ by φ, and that $\mathscr{E}|_U$ is generated by some elements $e_1, \dots, e_k \in \mathscr{E}(U)$. As $\mathscr{F}$ is coherent, $\mathscr{Rel}(e_\bullet, s_\bullet)$ is finite-type. So we can further shrink U so that $\mathscr{Rel}(e_\bullet, s_\bullet)$ is generated by $(f_1^l, \dots, f_k^l, g_1^l, \dots, g_n^l) \in \mathscr{O}(U)^{k+n}$ for finitely many l.

Clearly $(g_1^l, \dots, g_n^l) \in \mathscr{O}(U)^n$ are in $\mathscr{Rel}(t_\bullet)$. We claim that they generate $\mathscr{Rel}(t_\bullet)$. Choose any $y \in U$ and $h_1, \dots, h_n \in \mathscr{O}_{X,y}$ such that $h_1t_1 + \cdots + h_nt_n = 0$ in $\mathscr{G}_x$. So $h_1s_1 + \cdots + h_ns_n \in \mathscr{E}_y$. So $\mu_1e_1 + \cdots + \mu_ke_k + h_1s_1 + \cdots + h_ns_n = 0$ in $\mathscr{F}_y$ for some $\mu_1, \dots, \mu_k \in \mathscr{O}_{X,y}$. So $(\mu_\bullet, h_\bullet) \in \mathscr{Rel}(e_\bullet, s_\bullet)_y$. So $(\mu_\bullet, h_\bullet)$ is an $\mathscr{O}_{X,y}$-linear combination of $(f_\bullet^l, g_\bullet^l)$. Hence $(h_\bullet)$ is an $\mathscr{O}_{X,y}$-linear combination of $(g_\bullet^l)$. □

Proof of $\mathscr{F}, \mathscr{G}$ coherent $\Rightarrow$ $\mathscr{E}$ coherent. As $\mathscr{E}$ is a subsheaf of $\mathscr{F}$ and $\mathscr{F}$ is coherent, the sheaves of relations of $\mathscr{E}$ are clearly finite-type. Let us prove that $\mathscr{E}$ is finite-type. Choose $x \in X$ and a neighborhood $U \ni x$ such that $\mathscr{F}|_U$ is generated by $s_1, \dots, s_n \in \mathscr{F}(U)$. Then each $t_i = \varphi(s_i)$ is in $\mathscr{G}(U)$. Since $\mathscr{G}$ is coherent, $\mathscr{Rel}(t_\bullet)$ is finite-type. Thus, after shrinking U to a smaller neighborhood, $\mathscr{Rel}(t_\bullet)$ is generated by $(f_1^l, \dots, f_n^l) \in \mathscr{O}(U)^n$ for finitely many l.

Let $e^l = f_1^l s_1 + \cdots + f_n^l s_n$. Then $\varphi(e^l) = 0$, and hence $e^l \in \mathscr{E}(U)$. We claim that $e^1, e^2, \dots$ generate $\mathscr{E}|_U$. Choose any $y \in U$ and $\sigma \in \mathscr{E}_y$. Then $\varphi(\sigma) = 0$ and $\sigma = g_1s_1 + \cdots + g_ns_n$ for some $g_1, \dots, g_n \in \mathscr{O}_{X,y}$. So $(g_\bullet) \in \mathscr{Rel}(t_\bullet)_y$. Hence $(g_\bullet)$ is an $\mathscr{O}_{X,y}$-linear combination of $(f_\bullet^1), (f_\bullet^2), \dots$. So σ is the same $\mathscr{O}_{X,y}$-linear combination of $e^1, e^2, \dots$. □

Proof of $\mathscr{E}, \mathscr{G}$ coherent $\Rightarrow$ $\mathscr{F}$ coherent. Step 1. We prove that $\mathscr{F}$ is finite-type. Choose $x \in X$ and a neighborhood $U \ni x$. Shrink U so that we can find $s_1, \dots, s_n \in \mathscr{F}(U)$ such that $t_1 = \varphi(s_1), \dots, t_n = \varphi(s_n)$ generate $\mathscr{G}|_U$, and that there are $e_1, \dots, e_k \in \mathscr{E}(U)$ generating $\mathscr{E}|_U$. Then for each $y \in U$ and $\sigma \in \mathscr{E}_y$, $\varphi(\sigma) = f_1t_1 + \cdots + f_nt_n$ for some $f_1, \dots, f_n \in \mathscr{O}_{X,y}$. So $\sigma - f_1s_1 - \cdots - f_ns_n$ belongs to $\mathscr{E}_y$, which is an $\mathscr{O}_{X,y}$-linear combination of $e_1, \dots, e_k$. This shows that $s_1, \dots, s_n, e_1, \dots, e_k$ generate $\mathscr{F}|_U$.

Step 2. We prove that all sheaves of relations of $\mathscr{F}$ are finite-type. Again we choose $x \in X$ and a neighborhood $U \ni x$. Choose any $s_1, \dots, s_n \in \mathscr{F}(U)$, and let $t_\bullet = \varphi(s_\bullet)$. Since $\mathscr{Rel}(t_\bullet)$ is finite-type, we may shrink U to a smaller neighborhood such that we can find $G \in \mathscr{O}(U)^{n \times k}$ (i.e. an $\mathscr{O}(U)$-valued $n \times k$ matrix) such that the columns $G_{\bullet,1}, \dots, G_{\bullet,k} \in \mathscr{O}(U)^n$ generate $\mathscr{Rel}(t_\bullet)$. Set

$$(e_1, \dots, e_k) = (s_1, \dots, s_n)G \qquad \in \mathscr{F}(U)^k,$$

namely, $e_j = \sum_{i=1}^n s_iG_{i,j}$. Then $e_1, \dots, e_n$ are killed by φ, i.e. they are in $\mathscr{E}(U)$. As $\mathscr{Rel}(e_\bullet)$ is finite-type, we may shrink U and find a $k \times m$ matrix $E \in \mathscr{O}(U)^{k \times m}$ whose columns generate $\mathscr{Rel}(e_\bullet)$. Let $F = GE$ (which is in $\mathscr{O}(U)^{n \times m}$). We claim that the columns of F generate $\mathscr{Rel}(s_\bullet)$.

Choose any $y \in U$ and an element of $\mathscr{R}el(s_\bullet)_y$, written as an $n \times 1$ matrix $A \in \mathscr{O}_{X,x}^{n\times 1}$. So $(s_1, \dots, s_n)A = 0$. Hence $(t_1, \dots, t_n)A = 0$. So $A \in \mathscr{R}el(t_\bullet)_y$. Since $G_{\bullet,1}, \dots, G_{\bullet,k}$ generate $\mathscr{R}el(t_\bullet)_y$, we may write $A = GB$ for some $B \in \mathscr{O}_{X,y}^{k\times 1}$. So $(e_1, \dots, e_k)B = 0$. Thus, as $E_{\bullet,1}, \dots, E_{\bullet,m}$ generate $\mathscr{R}el(e_\bullet)_y$, we may write $B = EC$ for some $C \in \mathscr{O}_{X,y}^{m\times 1}$. Thus $A = FC$. So A is an $\mathscr{O}_{X,y}$-linear combination of columns of F. □

Corollary 2.1.4. *$\mathscr{E}_1, \mathscr{E}_2$ are coherent $\mathscr{O}_X$-modules if and only if $\mathscr{E}_1 \oplus \mathscr{E}_2$ is coherent.*

Proof. The exactness of $0 \to \mathscr{E}_1 \to \mathscr{E}_1 \oplus \mathscr{E}_2 \to \mathscr{E}_2 \to 0$ shows that "$\mathscr{E}_1, \mathscr{E}_2$ coherent" $\Rightarrow$ "$\mathscr{E}_1 \oplus \mathscr{E}_2$ coherent", and that if $\mathscr{E}_1 \oplus \mathscr{E}_2$ is coherent then $\mathscr{E}_2$ is finite type and the sheaves of relations of $\mathscr{E}_1$ are finite-type. Exchanging the roles of $\mathscr{E}_1, \mathscr{E}_2$ shows that "$\mathscr{E}_1 \oplus \mathscr{E}_2$ coherent" $\Rightarrow$ "$\mathscr{E}_1, \mathscr{E}_2$ coherent". □

Corollary 2.1.5. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of coherent $\mathscr{O}_X$-modules. Then* $\mathrm{Im}\varphi, \mathrm{Ker}\varphi, \mathrm{Coker}\varphi$ *are coherent.*

Proof. $\mathrm{Im}\varphi$ is finite-type since $\mathscr{F} \to \mathrm{Im}\varphi$ is surjective and $\mathscr{F}$ is finite-type. The sheaves of relations of $\mathrm{Im}\varphi$ are finite-type because $\mathscr{G}$ is coherent and $\mathrm{Im}\varphi$ is its $\mathscr{O}_X$-submodule. So $\mathrm{Im}\varphi$ is coherent. That $\mathrm{Ker}\varphi$ and $\mathrm{Coker}\varphi$ are coherent follows from Thm. 2.1.3 and the exact sequences $0 \to \mathrm{Ker}\varphi \to \mathscr{F} \to \mathrm{Im}\varphi \to 0$ and $0 \to \mathrm{Im}\varphi \to \mathscr{G} \to \mathrm{Coker}\varphi \to 0$. □

Corollary 2.1.6. *If $\mathscr{E}, \mathscr{F}$ are coherent $\mathscr{O}_X$-submodules of a coherent $\mathscr{O}_X$-module $\mathscr{G}$, then $\mathscr{E} + \mathscr{F}$ and $\mathscr{E} \cap \mathscr{F}$ are coherent.*

Note that the **intersection sheaf** $\mathscr{E} \cap \mathscr{F}$ is defined to be the sheaf of all sections of $\mathscr{G}$ whose germ at each $x \in X$ belongs to $\mathscr{E}_x \cap \mathscr{F}_x$. It is easy to check that $(\mathscr{E} \cap \mathscr{F})_x$ is canonically equivalent to $\mathscr{E}_x \cap \mathscr{F}_x$.

Proof. Clearly $\mathscr{E} + \mathscr{F}$ is finite-type and hence coherent. So by Cor. 2.1.5, $\mathscr{E}/(\mathscr{E} \cap \mathscr{F}) \simeq (\mathscr{E} + \mathscr{F})/\mathscr{F}$ is coherent, and hence $\mathscr{E} \cap \mathscr{F}$ is coherent. □

Definition 2.1.7. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-module. If $\mathscr{L}$ is an $\mathscr{O}_X$-submodule of $\mathscr{G}$, we define $\varphi^{-1}(\mathscr{L})$ to be the $\mathscr{O}_X$-module such that for each open $U \subset X$,

$$\varphi^{-1}(\mathscr{L})(U) = \{s \in \mathscr{F}(U) : \varphi(s)_x \in \mathscr{L}_x \text{ for all } x \in U\} \tag{2.1.1}$$

where $\varphi(s)_x$ is the germ of $\varphi(s)$ at x.

We have an obvious canonical equivalence

$$\varphi^{-1}(\mathscr{L})_x \simeq \varphi^{-1}(\mathscr{L}_x). \tag{2.1.2}$$

Therefore, by checking at the level of stalks, we see that the sequence

$$0 \to \mathrm{Ker}(\varphi) \to \varphi^{-1}(\mathscr{L}) \to \mathscr{L} \to 0 \tag{2.1.3}$$

is exact. Thus, by Thm. 2.1.3 and Cor. 2.1.5, we have:

Corollary 2.1.8. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-module. If $\mathscr{L}$ is a coherent $\mathscr{O}_X$-submodule of $\mathscr{G}$, then $\varphi^{-1}(\mathscr{L})$ is $\mathscr{O}_X$-coherent.*

Theorem 2.1.9. *Assume that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module. Then an $\mathscr{O}_X$-module $\mathscr{E}$ is coherent if and only if for each $x \in X$ there is a neighborhood $U \ni x$ such that $\mathscr{E}|_U$ is isomorphic to $\mathrm{Coker}\varphi$ for some morphism of free $\mathscr{O}_U$-modules $\varphi : \mathscr{O}_U^m \to \mathscr{O}_U^n$ (where $m, n \in \mathbb{N}$).*

Indeed, the "only if" part does not need $\mathscr{O}_X$ to be coherent.

Proof. "If": Since $\mathscr{O}_U$ is coherent, $\mathscr{O}_U^m$ and $\mathscr{O}_U^n$ are coherent. So $\mathrm{Coker}\varphi$ is coherent by Cor. 2.1.5.

"Only if": Let $\mathscr{E}$ be coherent. Choose $x \in X$. Since $\mathscr{E}$ is finite-type, we may find a neighborhood U such that there is a surjective $\psi : \mathscr{O}_U^n \to \mathscr{E}|_U$. Since $\mathscr{E}$ is coherent, $\mathrm{Ker}\psi$ is finite-type. Thus, after shrinking U, we may find a surjective $\pi : \mathscr{O}_U^m \to \mathrm{Ker}\psi$. Then $\mathscr{E}|_U \simeq \mathrm{Coker}(\iota \circ \pi)$ where $\iota : \mathrm{Ker}\psi \to \mathscr{O}_U^n$ is the inclusion. □

Corollary 2.1.10. *For any coherent $\mathscr{O}_X$-modules $\mathscr{E}, \mathscr{F}$, the tensor product $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ is coherent.*

Proof. Choose any $x \in X$. By Thm. 2.1.9, we may shrink X to a neighborhood of x such that $\mathscr{E} \simeq \mathrm{Coker}\varphi$ where $\varphi : \mathscr{O}_X^m \to \mathscr{O}_X^n$ is a morphism. By the right exactness of $- \otimes \mathscr{F}$ (cf. Prop. 1.9.5), $\mathscr{E} \otimes \mathscr{F}$ is equivalent to $\mathrm{Coker}(\mathscr{O}_X^m \otimes \mathscr{F} \to \mathscr{O}_X^n \otimes \mathscr{F})$, which is $\mathrm{Coker}(\mathscr{F}^m \to \mathscr{F}^n)$. By Cor. 2.1.4, $\mathscr{F}^m, \mathscr{F}^n$ are coherent. So the cokernel is coherent by Cor. 2.1.5. □

We end this section with some more criteria on coherence.

Proposition 2.1.11. *Let $\varphi : X \to S$ be a morphism of $\mathbb{C}$-ringed spaces, and let $\mathscr{E}$ be a finite-type $\mathscr{O}_S$-module. Then $\varphi^*\mathscr{E}$ is a finite type $\mathscr{O}_X$-module. If moreover $\mathscr{E}$ is $\mathscr{O}_S$-coherent and $\mathscr{O}_X$ is $\mathscr{O}_X$-coherent, then $\varphi^*\mathscr{E}$ is a coherent $\mathscr{O}_X$-module.*

Proof. If $\mathscr{E}$ is finite-type, then for each $x \in X$, we may shrink X to a neighborhood of x such that $\mathscr{E}$ is generated by finitely many $s_1, s_2, \dots \in \mathscr{E}(X)$. So $\varphi^*\mathscr{E} = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{E}$ is generated by all $\varphi^* s_i = 1 \otimes s_i$. So $\varphi^*\mathscr{E}$ is finite-type.

Now assume $\mathscr{E}$ is $\mathscr{O}_S$-coherent and $\mathscr{O}_X$ is $\mathscr{O}_X$-coherent. By Thm. 2.1.9, we may shrink X so that $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{O}_S^m \to \mathscr{O}_S^n)$. Then

$$\begin{aligned}\varphi^*\mathscr{E} &\simeq \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathrm{Coker}(\mathscr{O}_S^m \to \mathscr{O}_S^n) \simeq \mathrm{Coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S^m \to \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S^n)\\ &\simeq \mathrm{Coker}(\mathscr{O}_X^m \to \mathscr{O}_X^n)\end{aligned}$$

which is $\mathscr{O}_X$-coherent by Thm. 2.1.9 □

Proposition 2.1.12 (Extension principle). *Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ be a closed complex subspace of a complex space X where $\mathcal{I}$ is finite-type. Let $\iota : Y \to X$ be the inclusion, and let $\mathscr{E}$ be an $\mathscr{O}_Y$-module. Assume that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module. Then $\mathscr{E}$ is a coherent $\mathscr{O}_Y$-module if and only if $\iota_*\mathscr{E}$ is a coherent $\mathscr{O}_X$-module.*

Extension principle is an important special case of Finite mapping Thm. 2.7.1 which we will prove later.

Proof. We identify $\mathscr{E}$ with $\iota_*\mathscr{E}$ and $\mathscr{O}_Y$ with $\iota_*\mathscr{O}_Y = \mathscr{O}_X/\mathcal{I}$. (Cf. Rem. 1.10.5.) Clearly $\mathcal{I}$ is $\mathscr{O}_X$-coherent. So $\mathscr{O}_Y$ is $\mathscr{O}_X$-coherent by Cor. 2.1.5.

Assume $\iota_*\mathscr{E}$ is $\mathscr{O}_X$-coherent. Then by Prop. 2.1.11, $\mathscr{E} \simeq \iota^*\iota_*\mathscr{E}$ is a finite-type $\mathscr{O}_Y$-module. Suppose that after shrinking X we have a morphism $\alpha : \mathscr{O}_Y^n \to \mathscr{E}$. Since $\mathscr{O}_Y^n$ is $\mathscr{O}_X$-coherent, $\mathrm{Ker}\alpha$ is $\mathscr{O}_X$-coherent by Cor. 2.1.5. So $\mathrm{Ker}\alpha$ (or more precisely, $\iota_*(\mathrm{Ker}\alpha)$) is a finite-type $\mathscr{O}_X$-module. So by Prop. 2.1.11, it is a finite-type $\mathscr{O}_Y$-module.

Assume $\mathscr{E}$ is $\mathscr{O}_Y$-coherent. Then by Thm. 2.1.9, $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{O}_Y^m \to \mathscr{O}_Y^n)$ after shrinking X to a neighborhood of $x \in Y \subset X$. Since $\mathscr{O}_Y$ is $\mathscr{O}_X$-coherent, by Cor. 2.1.4, $\mathscr{O}_Y^m, \mathscr{O}_Y^n$ are $\mathscr{O}_X$-coherent. So $\mathscr{E}$ is $\mathscr{O}_X$-coherent by Cor. 2.1.5. □

Corollary 2.1.13. *Let Y be a closed complex subspace of X. Assume $\mathscr{O}_X$ is $\mathscr{O}_X$-coherent. Then $\mathscr{O}_Y$ is $\mathscr{O}_Y$-coherent.*

Proof. Write $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ where $\mathcal{I}$ is a finite-type ideal of $\mathscr{O}_X$. So $\mathcal{I}$ is $\mathscr{O}_X$-coherent. Hence $\mathscr{O}_Y = \mathscr{O}_X/\mathcal{I}$ is $\mathscr{O}_X$-coherent, and hence $\mathscr{O}_Y$-coherent by Extension principle. □

Thus, if we can show that $\mathscr{O}_{\mathbb{C}^n}$ is coherent for any n, then all model spaces, and hence all complex spaces have coherent structure sheaves.

2.2 Germs of coherent sheaves; coherence of hom sheaves

Let X be a $\mathbb{C}$-ringed space.

An important reason for studying coherent sheaves is that germs of coherent sheaves are equivalent to finitely-generated modules of local analytic $\mathbb{C}$-algebras, just as germs of complex spaces are equivalent to local analytic $\mathbb{C}$-algebras (Thm. 1.6.2). Let us be more precise.

Definition 2.2.1. Let X be a $\mathbb{C}$-ringed space and $x \in X$. The **category of germs of coherent modules at** x is the category whose objects are coherent $\mathscr{O}_U$-modules $\mathscr{E}_U$ where $U \ni x$ is open. If $V \subset U$ is a neighborhood of x, then $\mathscr{E}_U$ and $\mathscr{E}_V := \mathscr{E}_U|_V$ are viewed as the same object.

A **morphism** between two objects $\mathscr{E}_U, \mathscr{F}_U$ is an element $\varphi \in \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{E}_V, \mathscr{F}_V)$ for a possibly smaller neighborhood $V \ni x$. Two morphisms are regarded as equal if then agree when restricted to a possibly smaller neighborhood of x on which both are defined. Compositions of morphisms are defined in the obvious way. Thus, in this category the set of morphisms from $\mathscr{E}_U$ to $\mathscr{F}_U$ is precisely the stalk $\mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)_x$ of $\mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)$. □

Theorem 2.2.2. *Let X be a $\mathbb{C}$-ringed space and $x \in X$. Assume that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module, and $\mathscr{O}_{X,x}$ is Noetherian. Then the functor $\mathfrak{F}$ from the category of germs of coherent modules at x to the category of finitely-generated $\mathscr{O}_{X,x}$-modules, sending $\mathscr{E}_U$ to the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$ and sending each $\varphi \in \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)_x$ (namely, each $\varphi \in \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{E}_V, \mathscr{F}_V)$ for a possibly smaller neighborhood $V \ni x$) to the corresponding stalk map $\mathscr{E}_x \to \mathscr{F}_x$, is an* ***equivalence of categories****. Namely, the following two statements hold:*

(1) For each objects $\mathscr{E}_U, \mathscr{F}_U$, the following $\mathscr{O}_{X,x}$-module morphism is bijective:

$$\mathfrak{F} : \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)_x \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \tag{2.2.1}$$

(2) Each finitely-generated $\mathscr{O}_{X,x}$-module is isomorphic to $\mathfrak{F}(\mathscr{E}_U)$ for some object $\mathscr{E}_U$. Namely, it is isomorphic to $\mathscr{E}_{U,x}$.

Remark 2.2.3. If only (1) resp. (2) is satisfied, we say $\mathfrak{F}$ is **fully-faithful** resp. **essentially surjective**. These names also apply to contravariant functors.

From the proof, we shall see that the $\mathfrak{F}$ in (2.2.1) is an isomorphism even without assuming that $\mathscr{O}_X, \mathscr{F}_U$ are coherent or $\mathscr{O}_{X,x}$ is Noetherian.

Proof of (2). Choose any finitely generated $\mathscr{O}_{X,x}$-module $\mathcal{M}$. Then we have a surjective morphism $\alpha : \mathscr{O}_{X,x}^n \to \mathcal{M}$. $\mathrm{Ker}\alpha$ is an $\mathscr{O}_{X,x}$-submodule of $\mathscr{O}_{X,x}^n$, which is finitely-generated since $\mathscr{O}_{X,x}$ is Noetherian. Thus we have a surjective $\beta : \mathscr{O}_{X,x}^m \to \mathrm{Ker}\alpha$. Let $\gamma : \mathscr{O}_{X,x}^m \to \mathscr{O}_{X,x}^n$ be the composition of β and the inclusion $\iota : \mathrm{Ker}\alpha \to \mathscr{O}_{X,x}^n$. Then $\mathcal{M} \simeq \mathrm{Coker}\gamma$.

We can extend γ to an $\mathscr{O}_U$-module morphism $\varphi : \mathscr{O}_U^m \to \mathscr{O}_U^n$ for some neighborhood $U \ni x$. Namely, the stalk map of φ at x is γ. (For instance, choose U such that $s_1 = \gamma(1, 0, \dots, 0), \dots, s_n = \gamma(0, 0, \dots, 1) \in \mathscr{O}_{X,x}^n$ can be defined on U. Then φ is defined to be the $\mathscr{O}_U$-module morphism sending $(1, 0, \dots, 0) \in \mathscr{O}(U)^m$ to $s_1 \in \mathscr{O}(U)^n$, etc., and $(0, 0, \dots, 1)$ to s_n.) Then $\mathrm{Coker}\varphi$ is a coherent $\mathscr{O}_U$-module (Cor. 2.1.4 and 2.1.5) whose stalk at x is $\mathrm{Coker}\gamma \simeq \mathcal{M}$. □

Proof of (1). Choose an $\mathscr{O}_U$-module morphism $\varphi : \mathscr{E}_U \to \mathscr{F}_U$ such that $\mathfrak{F}(\varphi) = 0$. So the stalk map $\varphi : \mathscr{E}_{U,x} \to \mathscr{F}_{U,x}$ is zero. Since $\mathscr{E}_U$ is finite-type, $\mathscr{E}_{U,x}$ is finitely-generated. So we may choose $s_1, \dots, s_n \in \mathscr{E}_{U,x}$ generating $\mathscr{E}_{U,x}$. We may find a neighborhood $V \ni x$ in U such that $s_1, \dots, s_n \in \mathscr{E}(V)$, that $\varphi(s_1) = \cdots = \varphi(s_n) = 0$ in $\mathscr{F}(V)$, and that (by Rem. 1.2.16 and that $\mathscr{E}_U$ is finite-type) $s_1, \dots, s_n$ generate $\mathscr{E}_V$. So φ sends all sections of $\mathscr{E}_V$ to 0. This proves that $\mathfrak{F}$ is injective and uses only the condition that $\mathscr{E}_U$ is finite-type.

We now prove that $\mathfrak{F}$ is surjective. Choose any $\eta \in \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)$. By Thm. 2.1.9, there is a neighborhood $V \ni x$ inside U and an $\mathscr{O}_V$-module morphism $\alpha : \mathscr{O}_V^m \to \mathscr{O}_V^n$ such that $\mathscr{E}_V = \mathrm{Coker}(\alpha)$. Let $\pi_x : \mathscr{O}_{V,x}^n \to \mathscr{E}_x = \mathrm{Coker}(\alpha_x : \mathscr{O}_{V,x}^m \to \mathscr{O}_{V,x}^n)$ be the quotient map. Let η' be $\mathscr{O}_{V,x}^n \xrightarrow{\pi_x} \mathscr{E}_x \xrightarrow{\eta} \mathscr{F}_x$. Then as argued in the proof of part (2), the stalk map η' can be extended to an $\mathscr{O}_V$-module morphism $\widetilde{\eta}' : \mathscr{O}_V^n \to \mathscr{F}_V$ after shrinking V. $\widetilde{\eta}' \circ \alpha : \mathscr{O}_V^m \to \mathscr{F}_V$ has stalk map $\eta \circ \pi_x \circ \alpha_x$ at x, which is 0. So by the injectivity of $\mathfrak{F}$, we may shrink V so that $\widetilde{\eta}' \circ \alpha = 0$. So $\widetilde{\eta}'$ equals $\mathscr{O}_V^n \xrightarrow{\pi} \mathscr{E}_V = \mathrm{Coker}(\alpha) \xrightarrow{\widetilde{\eta}} \mathscr{F}_V$ for some $\mathscr{O}_V$-module morphism $\widetilde{\eta}$. Then $\widetilde{\eta}_x = \eta$, i.e. $\mathfrak{F}(\widetilde{\eta}) = \eta$. $\square$

Let us emphasize the following crucial special case of Thm. 2.2.2:

Corollary 2.2.4. *Let X be a $\mathbb{C}$-ringed space and $x \in X$. Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{O}_X$-modules. Then the canonical $\mathscr{O}_{X,x}$-module morphism*

$$\mathfrak{F} : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})_x \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \tag{2.2.2}$$

is injective if $\mathscr{E}$ is finite-type, and is bijective if $\mathscr{E}$ is coherent.

Corollary 2.2.5. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module.*

1. *The contravariant functor $\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{F})$ on the category of coherent $\mathscr{O}_X$-modules is left exact, where the contravariant functor sends each $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{E}_2)$ to $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}_2, \mathscr{F}) \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{F}), \psi \mapsto \psi \circ \varphi$.*

2. *Assume that $\mathscr{F}$ is coherent. Then the functor $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, -)$ on the category of $\mathscr{O}_X$-modules is left exact, where the functor sends each $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{E}_2)$ to $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}_1) \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}_2), \psi \mapsto \varphi \circ \psi$.*

Note that these two exactness is equivalent to saying that we have equivalences

$$\mathscr{H}om_{\mathscr{O}_X}\big(\mathrm{Coker}(\mathscr{E}_1 \to \mathscr{E}_2), \mathscr{F}\big) \simeq \mathrm{Ker}\big(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}_2, \mathscr{F}) \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}_1, \mathscr{F})\big) \tag{2.2.3a}$$

$$\mathscr{H}om_{\mathscr{O}_X}\big(\mathscr{F}, \mathrm{Ker}(\mathscr{E}_1 \to \mathscr{E}_2)\big) \simeq \mathrm{Ker}\big(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}_1) \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}_2)\big) \tag{2.2.3b}$$

induced by the obvious inclusions

$$\mathscr{H}om_{\mathscr{O}_X}\big(\mathrm{Coker}(\mathscr{E}_1 \to \mathscr{E}_2), \mathscr{F}\big) \hookrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}_2, \mathscr{F}),$$
$$\mathscr{H}om_{\mathscr{O}_X}\big(\mathscr{F}, \mathrm{Ker}(\mathscr{E}_1 \to \mathscr{E}_2)\big) \hookrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}_1).$$

Proof. Let $\mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$ be an exact sequence of coherent $\mathscr{O}_X$-modules. Then we have $0 \to \mathscr{H}om(\mathscr{F}, \mathscr{E}_3) \to \mathscr{H}om(\mathscr{F}, \mathscr{E}_2) \to \mathscr{H}om(\mathscr{F}, \mathscr{E}_1)$ which, thanks to Cor. 2.2.4, gives stalk maps $0 \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{E}_{3,x}) \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{E}_{2,x}) \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{E}_{1,x})$ at each $x \in X$ which is exact by Rem. 1.9.6. This proves part 1. Part 2 is proved in a similar way. $\square$

Corollary 2.2.6. *Assume that $\mathscr{E}, \mathscr{F}$ are coherent $\mathscr{O}_X$-modules. Then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$ is coherent. So $\mathscr{E}^\vee$ is coherent if $\mathscr{E}, \mathscr{O}_X$ are coherent.*

Proof. If $\mathscr{E} = \mathscr{O}_X^n$ then $\mathscr{H}om(\mathscr{E}, \mathscr{F}) \simeq \mathscr{F}^n$ is coherent by Cor. 2.1.4. In the general case, choose $x \in X$. Then by Thm. 2.1.9 we may shrink X to a neighborhood of x such that $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{E}_1 \to \mathscr{E}_2)$ where $\mathscr{E}_1, \mathscr{E}_2$ are free $\mathscr{O}_X$-modules. The coherence of $\mathscr{H}om(\mathscr{E}, \mathscr{F})$ follows from (2.2.3a) and Cor. 2.1.5. $\square$

2.3 Supports and annihilators of coherent sheaves; image spaces

In this section, we assume X, Y are complex spaces.

From Rem. 1.10.5, we know that if $\mathcal{I}$ is a finite-type ideal of $\mathscr{O}_X$ annihilating an $\mathscr{O}_X$-module $\mathscr{E}$, then the study of $\mathscr{E}$ is equivalent to the study of the $\mathscr{O}_Y$-module $\mathscr{E}|_Y$ where $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. A natural question is whether we can find a largest such $\mathcal{I}$, i.e., a smallest such Y. To study this problem, we introduce:

Definition 2.3.1. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the **annihilator sheaf** of $\mathscr{E}$, written as $\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})$ or simply $\mathscr{Ann}(\mathscr{E})$, is the ideal sheaf of $\mathscr{O}_X$ defined to be the kernel of the $\mathscr{O}_X$-module morphism $\mathscr{O}_X \to \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) =: \mathscr{End}_{\mathscr{O}_X}(\mathscr{E})$ sending each $f \in \mathscr{O}_X$ to the multiplication of f on $\mathscr{E}$. So we have an exact sequence

$$0 \to \mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{O}_X \to \mathscr{End}_{\mathscr{O}_X}(\mathscr{E}). \tag{2.3.1}$$

If $\mathscr{E}$ and $\mathscr{O}_X$ are coherent then so is $\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})$ (due to Cor. 2.1.5 and 2.2.6).

Similarly, if A is a commutative ring and $\mathcal{M}$ an A-module, then the **annihilator** $\mathrm{Ann}_A(\mathcal{M})$ is defined to be the kernel of $A \to \mathrm{End}_A(\mathcal{M})$. □

Remark 2.3.2. (2.3.1) gives an exact sequence of stalk maps at each x. Assume that $\mathscr{E}$ is $\mathscr{O}_X$-coherent. Then by Prop. 2.2.4, $\mathscr{End}_{\mathscr{O}_X}(\mathscr{E})_x \simeq \mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{E}_x)$. This shows that we have a canonical equivalence of $\mathscr{O}_{X,x}$-modules

$$\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})_x \simeq \mathrm{Ann}_{\mathscr{O}_{X,s}}(\mathscr{E}_x) \tag{2.3.2}$$

if $\mathscr{E}$ is coherent.

Definition 2.3.3. Assume $\mathscr{O}_X$ is coherent. Given a coherent $\mathscr{O}_X$-module $\mathscr{E}$, we define the **support of** $\mathscr{E}$, written as $\mathrm{Supp}(\mathscr{E})$, to be the complex space

$$\mathrm{Supp}(\mathscr{E}) = \mathrm{Specan}\big(\mathscr{O}_X/\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})\big). \tag{2.3.3}$$

Remark 2.3.4. $\mathrm{Ann}(\mathscr{E}_x) = \mathscr{O}_{X,x}$ iff $1 \in \mathrm{Ann}(\mathscr{E}_x)$ iff 1 annihilates $\mathscr{E}_x$ iff $\mathscr{E}_{X,x} = 0$. This shows that the underlying topological space of $\mathrm{Supp}(\mathscr{E})$ defined above (i.e. the set of all x such that $\mathscr{O}_{X,x}/\mathscr{Ann}(\mathscr{E})_x \neq 0$) agrees with the usual one (i.e. the set of all x such that $\mathscr{E}_x \neq 0$) when $\mathscr{E}$ is coherent.

Remark 2.3.5. We know that the support (as a set) of a finite-type $\mathscr{O}_X$-module is a closed subset of X (Cor. 1.2.17). Now we know that if $\mathscr{E}, \mathscr{O}_X$ are coherent, then $\mathrm{Supp}(\mathscr{E})$ as a set is an **analytic subset** of X, which means that it is $N(\mathcal{I})$ for a finite-type ideal $\mathcal{I}$.

Our definition of analytic subsets seems stronger than the usual one, which says that a subset $A \subset X$ is analytic if each $x \in X$ is contained in a neighborhood U such that $A \cap U$ is the zero set of finitely many elements of $\mathscr{O}(U)$. These two definitions are indeed equivalent, which follows from the coherence of the radicals of coherent ideals. See Cor. 3.2.8.

Convention 2.3.6. If $\mathscr{E}, \mathscr{O}_X$ are coherent, we understand $\mathrm{Supp}(\mathscr{E})$ as a complex subspace of X. Otherwise we understand it as only a subset of X.

Exercise 2.3.7. Show that if $\mathcal{I}$ is a finite-type (and hence coherent) ideal of $\mathscr{O}_X$, then

$$\mathrm{Supp}(\mathscr{O}_X/\mathcal{I}) = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}). \tag{2.3.4}$$

Definition 2.3.8. Let $\varphi : X \to Y$ be a holomorphic map of complex spaces. Assume that $\mathscr{O}_Y, \varphi_*\mathscr{O}_X$ are coherent $\mathscr{O}_Y$-modules and $\mathrm{Im}(\varphi) = \{\varphi(x) : x \in X\}$ is a closed subset of Y. We define the **image space** $\varphi(X)$ of φ to be

$$\varphi(X) = \mathrm{Supp}(\varphi_*\mathscr{O}_X) = \mathrm{Specan}\big(\mathscr{O}_Y/\mathscr{A}nn_{\mathscr{O}_Y}(\varphi_*\mathscr{O}_X)\big). \tag{2.3.5}$$

Then $\varphi^\# : \mathscr{O}_{\varphi(X)} \to \varphi_*\mathscr{O}_X$ is clearly injective.

The notation $\varphi(X)$ and the name "image space" is justified by the following lemma.

Lemma 2.3.9. *The underlying topological space of $\varphi(X)$ is the usual one $\mathrm{Im}(\varphi) = \{\varphi(x) : x \in X\}$. In particular, $\mathrm{Im}(\varphi)$ is an analytic subset of Y.*

Proof. Choose $y \in Y$. We show that $(\varphi_*\mathscr{O}_X)_y = 0$ iff $y \notin \mathrm{Im}(\varphi)$. First assume $(\varphi_*\mathscr{O}_X)_y = 0$. Choose a neighborhood V of y. The non-zero element $1 \in (\varphi_*\mathscr{O}_X)(V) = \mathscr{O}_X(\varphi^{-1}(V))$ becomes 0 in $(\varphi_*\mathscr{O}_X)_y$, which means that we may shrink V so that $1 = 0$ in $\mathscr{O}_X(\varphi^{-1}(V))$. So $\varphi^{-1}(V) = \emptyset$. Hence $y \notin \mathrm{Im}(\varphi)$. Conversely, suppose $y \notin \mathrm{Im}(\varphi)$. Since $\mathrm{Im}(\varphi)$ is closed, we may find a small enough neighborhood $V \ni y$ such that $\varphi^{-1}(V) = \emptyset$. So $(\varphi_*\mathscr{O}_X)_y = 0$. $\square$

Remark 2.3.10. In the setting of Def. 2.3.8, using (2.3.2), it is easy to see that we have a canonical equivalence of $\mathscr{O}_{Y,y}$-modules

$$\mathscr{A}nn_{\mathscr{O}_Y}(\varphi_*\mathscr{O}_X)_y \simeq \mathrm{Ker}\big(\varphi^\# : \mathscr{O}_{Y,y} \to (\varphi_*\mathscr{O}_X)_y\big). \tag{2.3.6}$$

Exercise 2.3.11. Assume that $\varphi_*\mathscr{O}_X$ and $\mathscr{O}_Y$ are $\mathscr{O}_Y$-coherent and φ is a closed map. Show that if X is reduced then the complex space $\varphi(X)$ is reduced. (Recall Def. 1.3.8.) Show that if A is an analytic subset of X, then the set $\varphi(A)$ is analytic in Y.

To study a coherent sheaf $\mathscr{E}$ one can restrict the underlying complex space to $\mathrm{Supp}(\mathscr{E})$. Likewise, to study φ when $\varphi_*\mathscr{O}_X$ and $\mathscr{O}_Y$ are coherent and $\mathrm{Im}(\varphi)$ is closed, one can restrict the codomain of φ to $\varphi(X)$:

Proposition 2.3.12. *Let $\varphi : X \to Y$ be holomorphic. Assume that $\mathscr{O}_Y, \varphi_*\mathscr{O}_X$ are coherent $\mathscr{O}_Y$-modules and $\mathrm{Im}(\varphi)$ is closed in Y. Then there is a unique holomorphic map $\widetilde{\varphi} : X \to \varphi(Y)$ (the restriction of φ) such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi}"'] & & Y \\ & \varphi(X) \arrow[ur, hook] & \end{tikzcd}$$

Proof. This follows immediately from Thm. 1.4.8. □

Let us give another application of supports of coherent sheaves. Recall that if A is a commutative ring and $\mathcal{M}$ is an A-module, an element $a \in A$ is called a **zero divisor of** $\mathcal{M}$ if $a\xi = 0$ for a non-zero $\xi \in \mathcal{M}$. Equivalently a is a zero divisor iff $\mathrm{Ker}(\mathcal{M} \xrightarrow{\times a} \mathcal{M})$ is non-zero. If a is not a zero divisor of $\mathcal{M}$, we call it a **non zero-divisor of** $\mathcal{M}$, not to be confused with a **non-zero zero divisor**, which is by definition a zero divisor which itself is not zero. Finally, a zero divisor means a zero divisor of A.

In the following we assume $\mathscr{O}_X$ is coherent, which is redundant after Oka's coherence theorem is proved.

Proposition 2.3.13. *Let X be a complex space, $\mathscr{E}$ a coherent $\mathscr{O}_X$-module, and choose $f \in \mathscr{O}(X)$. Then*

$$Z = \{x \in X : \textit{The germ of } f \textit{ at } x \textit{ is a zero divisor of } \mathscr{E}_x\}$$

is an analytic subset of X. In particular, the set of $x \in X$ such that f is a non zero-divisor of $\mathscr{E}_x$ is open in X.

Proof. Let $\mathscr{K} = \mathrm{Ker}(\mathscr{E} \xrightarrow{\times f} \mathscr{E})$, which is coherent by Cor. 2.1.5. Then $\mathrm{Supp}(\mathscr{K})$ is a complex subspace of X. A point $x \in X$ belongs to $\mathrm{Supp}(\mathscr{K})$ iff $\mathscr{K}_x = \mathrm{Ker}(\mathscr{E}_x \xrightarrow{\times f} \mathscr{E}_x)$ is non-zero iff f is a zero divisor of $\mathscr{E}_x$. This shows that Z equals $\mathrm{Supp}(\mathscr{K})$ as sets. □

2.4 Finite maps and proper maps

The proof of coherence of the structure sheaves of complex spaces is closely related to the study of finite holomorphic maps $\varphi : X \to Y$ and the coherence of $\varphi_*\mathscr{O}_X$. In this section, we discuss finite maps in the purely topological setting.

We assume X, Y are topological spaces. Recall that a continuous map $\varphi : X \to Y$ is called **closed** if φ sends closed subsets of X to closed subsets of Y.

Proposition 2.4.1. *Let $\varphi : X \to Y$ be a continuous map. Then the following are equivalent.*

(1) φ is a closed map.

(2) For each $y \in Y$,

$$\{\varphi^{-1}(V) : V \subset Y \text{ is a neighborhood of } y\}$$

is a ***basis of neighborhoods of*** *$\varphi^{-1}(y)$, which means that for each open $U \subset X$ containing $\varphi^{-1}(y)$ there is a neighborhood $V \ni y$ such that $\varphi^{-1}(V) \subset U$.*

Proof. Assume (1). For each open $U \subset X$ containing $\varphi^{-1}(y)$, let $V \subset Y$ be defined by $Y\backslash V = \varphi(X\backslash U)$ where $\varphi(X\backslash U)$ is closed because φ is closed. So V is open and clearly contains y. Since $V \cap \varphi(X\backslash U) = \emptyset$, $\varphi^{-1}(V) \cap (X\backslash U) = \emptyset$. So $\varphi^{-1}(V) \subset U$. This proves (2).

Assume (2). Choose any closed subset $E \subset X$. We shall show that $\varphi(E)$ is closed in Y. Choose any $y \in Y\backslash\varphi(E)$. Then $X\backslash E$ is a neighborhood of $\varphi^{-1}(y)$. So we can choose a neighborhood $V \subset Y$ of y such that $\varphi^{-1}(V) \subset X\backslash E$. So $\varphi^{-1}(V) \cap E = \emptyset$, and hence $V \cap \varphi(E) = \emptyset$. This proves that y is an interior point of $Y\backslash\varphi(E)$. So $Y\backslash\varphi(E)$ is open, and (1) is proved. □

Remark 2.4.2. The above proposition shows that closedness is a local property (with respect to the base Y): If Y has an open cover $(V_\alpha)_\alpha$ such that for each α, the restriction $\varphi : \varphi^{-1}(V_\alpha) \to V_\alpha$ is closed. Then $\varphi : X \to Y$ is closed.

Definition 2.4.3. A continuous map $\varphi : X \to Y$ is called **finite** if it is a closed map and if $\varphi^{-1}(y)$ is a finite set for all $y \in Y$. The composition of two finite maps is clearly finite. If $\varphi : X \to Y$ is a holomorphic map of complex spaces which is finite as a continuous map of topological spaces, we say φ is a **finite holomorphic map**.

Remark 2.4.4. A main reason that we require finite maps to be closed is the following: Suppose φ is finite. Given $y \in Y$, choose mutually disjoint neighborhoods $U_x \subset X$ for all $x \in \varphi^{-1}(y)$. Then by Prop. 2.4.1, there is a sufficiently small neighborhood $V \subset Y$ of y such that

$$\varphi^{-1}(V) = \coprod_{x \in \varphi^{-1}(y)} \varphi^{-1}(V) \cap U_x. \tag{2.4.1}$$

In other words, we can shrink each U_x to a smaller neighborhood of x such that

$$\varphi^{-1}(V) = \coprod_{x \in \varphi^{-1}(y)} U_x. \tag{2.4.2}$$

From this it is clear that the restriction $\varphi|_{U_x} : U_x \to Y$ is finite.

As applications of this observation, we prove several important facts about direct images.

Proposition 2.4.5. *Let $\varphi : X \to Y$ be a finite continuous map, and let $\mathscr{E}$ be an X-sheaf. Then for each $y \in Y$, we have an isomorphism of abelian groups*

$$\Phi : (\varphi_*\mathscr{E})_y \xrightarrow{\simeq} \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{E}_x \tag{2.4.3}$$

defined componentwisely by the obvious restriction maps.

If φ is a morphism of $\mathbb{C}$-ringed spaces and $\mathscr{E}$ is an $\mathscr{O}_Y$-module, then Φ is clearly an isomorphism of $\mathscr{O}_{Y,y}$-modules. Moreover, Φ *is an isomorphism of* $(\varphi_*\mathscr{O}_X)_y$*-modules* if we let $(\varphi_*\mathscr{O}_X)_y \simeq \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{O}_{X,x}$ act on the codomain of Φ componentwisely.

Proof. Ψ is defined by passing to the direct limit of the map

$$\Phi_V : \mathscr{E}(\varphi^{-1}(V)) \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{E}_x \tag{2.4.4}$$

over all open $V \ni y$. If $s \in \mathscr{E}(\varphi^{-1}(V))$ and $\Phi_V(s) = 0$, then we may find disjoint neighborhoods $U_x \ni x$ such that $s|_{U_x} = 0$. After shrinking V such that (2.4.1) holds, we have $s = 0$. So Φ is injective.

On the other hand, choose $s_x \in \mathscr{E}_x$ for each $x \in \varphi^{-1}(y)$. Then we may choose small enough neighborhoods $U_x \ni x$ and $V \ni y$ such that $s_x \in \mathscr{E}(U_x)$ and (2.4.2) holds. Let $s \in \mathscr{E}(\varphi^{-1}(V))$ be s_x when restricted to U_x. Then $\Phi_V(s) = s_x$. So Φ is surjective. □

Recall that for an arbitrary continuous map φ, the functor φ_* is left exact.

Corollary 2.4.6. *Let $\varphi : X \to Y$ be a finite continuous map. Then φ_* is an* ***exact functor*** *(i.e. a left and right exact functor) from the category of X-sheaves to that of Y-sheaves. Namely: if a sequence of maps of X-sheaves*

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0, \tag{2.4.5}$$

is exact, then the following is exact:

$$0 \to (\varphi_*\mathscr{E})_y \to (\varphi_*\mathscr{F})_y \to (\varphi_*\mathscr{G})_y \to 0. \tag{2.4.6}$$

Indeed, (2.4.5) *is exact if and only if* (2.4.6) *is exact.*

Proof. By Prop. 2.4.5, (2.4.6) is the same as

$$0 \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{E}_x \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{F}_x \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{G}_x \to 0.$$

The equivalence of the exactness of (2.4.5) and (2.4.6) follows. □

Proposition 2.4.7 (Base change proposition). *Let $\pi : X \to S$ be a finite continuous map. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module and $\mathscr{M}$ an $\mathscr{O}_S$-module. Then we have a (clearly natural) $\mathscr{O}_S$-module isomorphism*

$$\begin{aligned} \Upsilon : (\pi_*\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{M} &\xrightarrow{\simeq} \pi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}) \\ \sigma \otimes \mu \in \mathscr{E}(\pi^{-1}(W)) \otimes_{\mathscr{O}_S(W)} \mathscr{M}(W) &\mapsto \sigma \otimes \mu \in (\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})(\pi^{-1}(W)) \end{aligned} \tag{2.4.7}$$

for all open $W \subset S$.

Note that the stalk map of Φ at any $t \in S$ is the canonical morphism

$$\Upsilon : (\pi_*\mathscr{E})_t \otimes_{\mathscr{O}_{S,t}} \mathscr{M}_t \longrightarrow \pi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_t \tag{2.4.8}$$

Proof. By Prop. 2.4.5, the stalk map (2.4.8) can be extended to a commutative diagram

$$\begin{tikzcd} (\pi_*\mathscr{E})_t \otimes_{\mathscr{O}_{S,t}} \mathscr{M}_t \arrow[r, "\Upsilon"] \arrow[d, "\simeq"'] & \pi(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_t \arrow[d, "\simeq"] \\ \left(\bigoplus_{x\in\pi^{-1}(t)} \mathscr{E}_x\right) \otimes_{\mathscr{O}_{S,t}} \mathscr{M}_t \arrow[r, "\simeq"] & \bigoplus_{x\in\pi^{-1}(t)} (\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_x \end{tikzcd} \tag{2.4.9}$$

where the other three morphisms of $\mathscr{O}_{S,t}$-modules are isomorphisms. So (2.4.8) is an isomorphism. □

Lemma 2.4.8. *Let $\varphi : X \to Y$ be a finite continuous map. Assume that $\mathscr{E}$ is a coherent $\mathscr{O}_X$-module. Then each $y \in Y$ is contained in neighborhood $V \subset Y$ such that $\mathscr{E}|_{\pi^{-1}(V)}$ is the cokernel of a morphism of free $\mathscr{O}_{\pi^{-1}(V)}$-modules.*

Proof. Choose V such that (2.4.2) holds and U_x is a small enough neighborhood of $x \in \varphi^{-1}(y)$ such that $\mathscr{E}|_{U_x}$ is equivalent to $\mathrm{Coker}(\mathscr{O}_{U_x}^m \to \mathscr{O}_{U_x}^n)$. The natural numbers m, n might initially depend on x, but we can enlarge m, n so that they do not. Then $\mathscr{E}|_{\pi^{-1}(V)}$ is clearly the cokernel of a morphism $\mathscr{O}_{\pi^{-1}(V)}^m \to \mathscr{O}_{\pi^{-1}(V)}^n$. □

Definition 2.4.9. A continuous map $\varphi : X \to Y$ is called **proper** if for each compact subset $K \subset Y$, $\varphi^{-1}(K)$ is compact.

Finite maps are special cases of proper maps as shown by the following proposition. Indeed, a deep theorem by Grauert says that if φ is a proper holomorphic map then $\varphi_*\mathscr{E}$ is $\mathscr{O}_Y$-coherent whenever $\mathscr{E}$ is $\mathscr{O}_X$-coherent. In particular, $\varphi_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent. So we can study $f(X)$. In the special case that φ is finite, the study of the coherence of $\varphi_*\mathscr{O}_X$ is crucial to the proof of coherence of all structure sheaves of complex spaces.

Proposition 2.4.10. *Let $\varphi : X \to Y$ be a continuous map of topological spaces. If X and Y are locally compact and Y is Hausdorff, then the following are equivalent.*

(1) φ is proper.

(2) φ is closed, and $\varphi^{-1}(y)$ is compact for each $y \in Y$.

Thus, a finite map is precisely a proper map whose fibers $\varphi^{-1}(y)$ are all discrete sets.

Note: To prove (1)⇒(2) we don't need X to be locally compact. To prove (2)⇒(1) we don't need Y to be Hausdorff.

Proof. Assume (1). Let us prove that φ is closed by proving part (2) of Prop. 2.4.1. Choose $y \in Y$ and any neighborhood $U \supset \varphi^{-1}(y)$. Since Y is locally compact, we can fix a precompact neighborhood $V_0 \subset Y$ of y. Then $E := (X \backslash U) \cap \varphi^{-1}(V_0^{\mathrm{cl}})$ is compact by the properness of φ. Let $\mathfrak{V}$ be the set of all precompact open subsets of V_0 containing y. Then $\bigcap_{V \in \mathfrak{V}} V^{\mathrm{cl}} = \{y\}$ since Y is Hausdorff, and hence $E \cap \bigcap_{V \in \mathfrak{V}} \varphi^{-1}(V^{\mathrm{cl}}) = \varnothing$. So by the compactness of E, there is $V \in \mathfrak{V}$ such that $E \cap \varphi^{-1}(V^{\mathrm{cl}}) = 0$. So $\varphi^{-1}(V^{\mathrm{cl}}) \subset U$.

Assume (2). For each $y \in Y$, since $\varphi^{-1}(y)$ is compact and X is locally compact, we may find a precompact neighborhood $U \subset X$ of $\varphi^{-1}(y)$. By Prop. 2.4.1, we can find a neighborhood V of y such that $\varphi^{-1}(V) \subset U$. So $\varphi^{-1}(V)^{\mathrm{cl}}$ is compact since it is closed in U^{cl}. From this we conclude that any compact $K \subset Y$ can be covered by finitely many open sets $V_1, V_2, \dots$ such that $\varphi^{-1}(V_j)^{\mathrm{cl}}$ is compact. Then $\varphi^{-1}(K)$ as a closed subset of $\bigcup_j \varphi^{-1}(V_j)^{\mathrm{cl}}$ is compact. □

The following important fact says that properness and finiteness are preserved by base changes.

Proposition 2.4.11. *Let $\pi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. If π is proper resp. finite, then $\mathrm{pr}_Y : X \times_S Y \to Y$ is proper resp. finite.*

Proof. As a topological space, $X \times_S Y$ is the closed subset of all $x \times y \in X \times Y$ such that $\pi(x) = \psi(y)$ (Rem. 1.13.11). The relation $\mathrm{pr}_Y^{-1}(y) = \pi^{-1}(\psi(y)) \times y$ shows that the fibers of pr_Y are finite sets if those of π are finite. It also shows that if $K \subset Y$ is compact then $\mathrm{pr}_Y^{-1}(K)$ is a (clearly closed) subset of $\pi^{-1}(\psi(K)) \times K$ which is compact if π is proper. So pr_Y is proper if π is so. □

2.5 Weierstrass maps and Weierstrass preparation theorem

The goal of this section is to study an important class of finite holomorphic maps called Weierstrass maps.

Definition 2.5.1. Let S be a complex space. Let $k \in \mathbb{N}$. For each $i = 1, \dots, k$, we choose a polynomial of degree n_i

$$p_i(z_i) = 1 \otimes a_{i,0} + (1 \otimes a_{i,1})z_i + \cdots + (1 \otimes a_{i,n_i})z_i^{n_i} \qquad \in \mathscr{O}(\mathbb{C}^k \times S)[z_i]$$

where $a_{i,j} \in \mathscr{O}(S)$, $n_i \in \mathbb{Z}_+$, and $a_{i,n_i}(t) \neq 0$ for all $t \in S$. Consider p_i as an element of $\mathscr{O}(\mathbb{C}^k \times S)$. Let

$$X = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k \times S}/\mathcal{I}) \qquad \mathcal{I} = p_1 \mathscr{O}_{\mathbb{C}^k \times S} + \cdots + p_k \mathscr{O}_{\mathbb{C}^k \times S}. \tag{2.5.1}$$

Then the holomorphic map $\pi : X \to S$ defined by restricting the projection $\mathrm{pr}_S : \mathbb{C}^k \times S \to S$ is called a **Weierstrass map**.

Recall that by our notations, $1 \otimes a_{i,j}$ means $\mathrm{pr}_S^\# a_{i,j} = a_{i,j} \circ \mathrm{pr}_S$. We shall write $1 \otimes a_{i,j}$ as $a_{i,j}$ if no confusion arises.

Proposition 2.5.2. *Weierstrass maps are finite.*

Proof. Clearly each fiber of $\pi : X \to S$ is a finite set. To check that π is closed, by Rem. 2.4.2, it suffices to check it locally with respect to the base.

By Rem. 1.5.2 we can shrink S and find an open disc $D \subset \mathbb{C}$ such that for each $t \in S$ and each i, the polynomial $p_i(z_i, t)$ of z_i has n_i zeros in D counting multiplicities. So X (as a topological space, namely $N(\mathcal{I})$) is a closed subset of $(D^{\mathrm{cl}})^k \times S$. Therefore $\pi : X \to S$ is the restriction of the projection $(D^{\mathrm{cl}})^k \times S \to S$ to a closed subset, which is closed because the projection $(D^{\mathrm{cl}})^k \times S \to S$ is proper and hence closed (Prop. 2.4.10). $\square$

The next proposition says that a canonical pullback $Y \to T$ of a Weierstrass map $X \to S$ along a holomorphic map $\psi : T \to S$ is Weierstrass.

Proposition 2.5.3. *Assume the setting of Def. 2.5.1. Let $\psi : T \to S$ be a holomorphic map of complex spaces. Set*

$$\widetilde{a}_{i,j} = a_{i,j} \circ \psi \in \mathscr{O}(T)$$
$$\widetilde{p}_i(z_i) = 1 \otimes \widetilde{a}_{i,0} + (1 \otimes \widetilde{a}_{i,1})z_i + \cdots + (1 \otimes \widetilde{a}_{i,n_i})z_i^{n_i} \qquad \in \mathscr{O}(\mathbb{C}^k \times T)[z_i]$$

and set

$$Y = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k \times T}/\mathcal{J}) \qquad \mathcal{J} = \widetilde{p}_1 \mathscr{O}_{\mathbb{C}^k \times T} + \cdots + \widetilde{p}_k \mathscr{O}_{\mathbb{C}^k \times T}. \tag{2.5.2}$$

Then the Cartesian square

$$\begin{tikzcd} \mathbb{C}^k \times S \arrow[d] & \mathbb{C}^k \times T \arrow[l, "\mathbf{1} \times \psi"'] \arrow[d] \\ S & T \arrow[l, "\psi"'] \end{tikzcd}$$

restricts to a Cartesian square

$$\begin{tikzcd} X & Y \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\pi}"] \\ S & T \arrow[l, "\psi"] \arrow[from=1-1, to=2-1, "\pi"'] \end{tikzcd}$$

Proof. By Prop. 1.12.1 we have a Cartesian square

$$\begin{tikzcd} X \arrow[d, hook] & Y \arrow[l] \arrow[d, hook] \\ \mathbb{C}^k \times S & \mathbb{C}^k \times T \arrow[l] \end{tikzcd}$$

which, together with Rem. 1.11.3, proves our proposition. □

The following theorem is the first major result of this chapter. Many subsequent major results in this chapter are proved using this theorem.

Theorem 2.5.4 (Fundamental theorem of Weierstrass maps). *Assume the setting of Def. 2.5.1. Then*

$$\{z_1^{\nu_1}\cdots z_k^{\nu_k} : 0 \leqslant \nu_i \leqslant n_i - 1 \textit{ for all } 1 \leqslant i \leqslant k\} \tag{2.5.3}$$

(or more precisely, these elements acted on by $\mathrm{pr}^{\#}_{\mathbb{C}^k\times S\to\mathbb{C}^k}$) is a set of free generators of the $\mathscr{O}_S$-module $\pi_*\mathscr{O}_X$.

In particular, $\pi_*\mathscr{O}_X$ is a free $\mathscr{O}_S$-module of rank $n_1n_2\cdots n_k$.

Lemma 2.5.5. *If Thm. 2.5.4 holds when S is smooth, then Thm. 2.5.4 holds when S is any complex space.*

Proof. Note that Thm. 2.5.4 is local by nature since it can be checked at the level of stalks. So we may assume S is so small that it is a closed subspace of an open subset $\Omega \subset \mathbb{C}^m$, and that each $a_{i,j}$ is the restriction of an element of $\mathscr{O}(\Omega)$. Therefore, by Prop. 2.5.3, we have a Weierstrass map $Y \hookrightarrow \mathbb{C}^k \times \Omega \to \Omega$ (which we also denote by π) such that the following two squares are Cartesian.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{C}^k \times S \arrow[r, hook] \arrow[d] & \mathbb{C}^k \times \Omega \arrow[d] \\ S \arrow[r, hook] & \Omega \end{tikzcd}$$

In particular, $\pi : X \to S$ is the base change of $\pi : Y \to \Omega$ to S.

Write $S = \mathrm{Specan}(\mathscr{O}_\Omega/\mathcal{I})$. Then by Rem. 1.12.3, $\mathscr{O}_X$ is $\mathscr{O}_Y \otimes_{\mathscr{O}_\Omega} \mathscr{O}_S$ (if we regard $\mathscr{O}_X$ as an $\mathscr{O}_Y$-module and $\mathscr{O}_S$ as an $\mathscr{O}_\Omega$-module). By Base change Prop. 2.4.7, we have canonical isomorphisms of $\mathscr{O}_\Omega$-modules

$$\pi_*\mathscr{O}_X \simeq \pi_*(\mathscr{O}_Y \otimes_{\mathscr{O}_\Omega} \mathscr{O}_S) \simeq \pi_*\mathscr{O}_Y \otimes_{\mathscr{O}_\Omega} \mathscr{O}_S.$$

Equivalently, $\pi_*\mathscr{O}_X \simeq \pi_*\mathscr{O}_Y|_S$ as $\mathscr{O}_S$-modules. Since we assume that Thm. 2.5.4 holds for $\pi : Y \to \Omega$, we know that $\pi_*\mathscr{O}_Y$ is generated freely by (2.5.3). So $\pi_*\mathscr{O}_X$ is generated freely by (the restrictions to S of) (2.5.3). $\square$

Due to Lemma 2.5.5, we can assume that:

Convention 2.5.6. In the remaining part of this section, S is an open subset of $\mathbb{C}^m$. Let $t_\bullet = (t_1, \ldots, t_m)$ be the variables of S.

To prepare for the proof, we let $N(p_i) \subset \mathbb{C} \times S$ be the subset of all $(z_i, t_\bullet)$ such that $p_i(z_i, t_\bullet) = 0$. For each $(t_\bullet) \in S$, define a subset of $\mathbb{C}$

$$N(p_i)_{t_\bullet} = N(p_i) \cap \mathrm{pr}^{-1}_{\mathbb{C}\times S \to S}(t_\bullet),$$

namely, the set of all z_i satisfying $p_i(z_i, t_\bullet) = 0$. Then by Prop. 2.4.5, we have an obvious isomorphism of $\mathscr{O}_{S,t_\bullet}$-modules

$$(\pi_*\mathscr{O}_X)_{t_\bullet} \simeq \bigoplus_{\substack{w_i \in N(p_i)_{t_\bullet} \\ 1 \leqslant i \leqslant k}} \mathscr{O}_{\mathbb{C}^k \times S, (w_\bullet, t_\bullet)}/\mathcal{I}_{(w_\bullet, t_\bullet)} \tag{2.5.4}$$

where

$$\mathcal{I}_{(w_\bullet, t_\bullet)} = p_1 \mathscr{O}_{\mathbb{C}^k \times S, (w_\bullet, t_\bullet)} + \cdots + p_k \mathscr{O}_{\mathbb{C}^k \times S, (w_\bullet, t_\bullet)}.$$

Our goal is to show that (2.5.3) generates (2.5.4) freely.

2.5.1 Proof of Thm. 2.5.4, I

In this subsection, we assume $(t_\bullet) = 0 \in S \subset \mathbb{C}^m$ for simplicity, and show that (2.5.3) generate $(\pi_*\mathscr{O}_X)_0$. We let $(\tau_\bullet)$ denote a set of general variables of S. (2.5.4) reads

$$(\pi_*\mathscr{O}_X)_0 \simeq \bigoplus_{\substack{w_i \in N(p_i)_0 \\ 1 \leqslant i \leqslant k}} \mathscr{O}_{\mathbb{C}^k \times S, (w_\bullet, 0)}/\mathcal{I}_{(w_\bullet, 0)}. \tag{2.5.5}$$

Lemma 2.5.7. *(2.5.3) generates $(\pi_*\mathscr{O}_X)_0$.*

Proof-special case. We consider the special case that for each i, $N(p_i)_0$ is the single point 0. In this case, $p_i(z_i, \tau_\bullet)$ has order n_i in z_i (recall Def. 1.5.1). (Namely, p_i is, up to multiplication by a nowhere zero element of $\mathscr{O}(S)$, a *Weierstrass polynomial* of z_i.) Now (2.5.5) reads

$$(\pi_* \mathscr{O}_X)_0 \simeq \mathscr{O}_{\mathbb{C}^k \times S,(0,0)} / \mathcal{I}_{(0,0)}.$$

We explain the proof when $k = 2$. The general case follows from exactly the same argument.

Choose $f(z_1, z_2, \tau_\bullet) \in \mathscr{O}_{\mathbb{C}^2 \times S,(0,0)}$. Then by WDT (Weierstrass division theorem),

$$f(z_1, z_2, \tau_\bullet) = \sum_{j=0}^{n_1 - 1} z_1^j \cdot g_j(z_2, \tau_\bullet) \qquad \mod \ p_1 \mathscr{O}_{\mathbb{C}^2 \times S,(0,0)}$$

where $g_j \in \mathscr{O}_{\mathbb{C} \times S,(0,0)}$. Apply WDT again, we have

$$g_j(z_2, \tau_\bullet) = \sum_{l=0}^{n_2 - 1} z_2^l \cdot h_l(\tau_\bullet) \qquad \mod \ p_2 \mathscr{O}_{\mathbb{C} \times S,(0,0)}$$

where $h_l \in \mathscr{O}_{S,0}$. This finishes the proof. □

To prove the general case, for each $w_i \in N(p_i)_0$ we define integer

$$\mu_i(w_i) = \{\text{The multiplicity of the root } z_i = w_i \text{ of } p_i(z_i, 0)\}$$

So $0 < \mu_i(w_i) \leqslant n_i$.

Lemma 2.5.8. *For each i, choose $w_i \in N(p_i)_0$. Then there is an $\mathscr{O}_{S,0}$-coefficients polynomial $q_1(z_\bullet, \tau_\bullet)$ of $z_1, \dots, z_n$ with multi-degree $\leqslant \big(n_1 - \mu_1(w_1), \dots, n_k - \mu_k(w_k)\big)$ satisfying the following conditions.*

1. *Its germ at $(w_\bullet, 0)$ is an invertible element of the ring $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)} / \mathcal{I}_{(w_\bullet,0)}$.*
2. *Its germ at $(\widetilde{w}_\bullet, 0)$ is 0 in the ring $\mathscr{O}_{\mathbb{C}^k \times S,(\widetilde{w}_\bullet,0)} / \mathcal{I}_{(\widetilde{w}_\bullet,0)}$ for any $(\widetilde{w}_\bullet) = (\widetilde{w}_1, \dots, \widetilde{w}_k) \in \mathbb{C}^k$ such that $\widetilde{w}_i \in N(p_i)_0$ (for all i) and that $(\widetilde{w}_\bullet) \neq (w_\bullet)$.*

This lemma can be viewed as a partition of unity of $(\pi_* \mathscr{O}_X)_0$. We postpone the proof of this lemma until after proving Lemma 2.5.7.

Proof of Lemma 2.5.7-general case. In view of (2.5.5), it suffices to show prove the following claim:

- Choose any $(w_\bullet) \in \mathbb{C}^k$ such that $w_i \in N(p_i)_0$, and choose any $f(z_\bullet, \tau_\bullet) \in (\pi_* \mathscr{O}_X)_0$ which is zero in $\mathscr{O}_{\mathbb{C}^k \times S,(\widetilde{w}_\bullet,0)} / \mathcal{I}_{(\widetilde{w}_\bullet,0)}$ whenever $(\widetilde{w}_\bullet) \neq (w_\bullet)$. Then f belongs to the $\mathscr{O}_{S,0}$-submodule of $(\pi_* \mathscr{O}_X)_0$ generated by (2.5.3).

- Namely, there is an $\mathscr{O}_{S,0}$-coefficients polynomial $q(z_\bullet, \tau_\bullet)$ of $z_\bullet$ with multi-degree $\leqslant (n_1 - 1, \ldots, n_k - 1)$ whose germ at $(w_\bullet, 0)$ is equal to the germ of f mod $\mathcal{I}_{(w_\bullet,0)}$, and whose germ at $(\widetilde{w}_\bullet, 0)$ (where $(\widetilde{w}_\bullet) \neq (w_\bullet)$) is in $\mathcal{I}_{(\widetilde{w}_\bullet,0)}$.

Let q_1 be as in Lemma 2.5.8, whose germ at $(w_\bullet, 0)$ is an invertible element of $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}$. Note that f/q_1 is in $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}$ (but not in $(\pi_* \mathscr{O}_X)_0$). We now apply the proof of the special case to f/q_1. Then by WDT (noting that $p_i(z_i, \tau_\bullet)$ has order $\mu_i(w_i)$ in $z_i - w_i$), there is an $\mathscr{O}_{S,0}$-coefficients polynomial $q_2(z_\bullet, \tau_\bullet)$ of $z_\bullet$ with multi-degree $\leqslant (\mu_1(n_1) - 1, \ldots, \mu_k(n_k) - 1)$ which equals f/q_1 in $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}/\mathcal{I}_{(w_\bullet,0)}$. Then f and $q := q_1 q_2$ are clearly equal in $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}/\mathcal{I}_{(w_\bullet,0)}$. They are also equal in $\mathscr{O}_{\mathbb{C}^k \times S,(\widetilde{w}_\bullet,0)}/\mathcal{I}_{(\widetilde{w}_\bullet,0)}$ since both are 0. □

We are done with the proof of Lemma 2.5.7.

2.5.2 Proof of Lemma 2.5.8

Definition 2.5.9. A polynomial $q(z, \tau_\bullet) \in \mathbb{C}\{\tau_\bullet\}[z]$ is called a **Weierstrass polynomial of z** if it is monic and the degree equals the order in z. Namely,

$$q(z, \tau_\bullet) = a_0(\tau_\bullet) + a_1(\tau_\bullet) z + \cdots + a_{n-1}(\tau_\bullet) z^{n-1} + z^n \tag{2.5.6}$$

where $a_0, \ldots, a_{n-1} \in \mathbb{C}\{\tau_\bullet\}$, and

$$a_0(0) = a_1(0) = \cdots = a_{n-1}(0) = 0.$$

Theorem 2.5.10 (Weierstrass preparation theorem (WPT)). *Choose $f(z, \tau_\bullet) \in \mathbb{C}\{z, \tau_\bullet\}$ with finite order n in z. Then there exist a unique invertible $u \in \mathbb{C}\{z, \tau_\bullet\}$ and a Weiertrass polynomial $q \in \mathbb{C}\{\tau_\bullet\}[z]$ of z such that in $\mathbb{C}\{z, \tau_\bullet\}$ we have*

$$f = uq.$$

Proof. Uniqueness: $f = uq$ can be written as $q = u^{-1} f$. Write $q(z, \tau_\bullet) = z^n - r$ where the polynomial $r \in \mathbb{C}\{\tau_\bullet\}[z]$ of z has degree $< n$. Then $z^n = u^{-1} f + r$ gives the unique Weierstrass division of z^n by f. So u, q are unique.

Existence: By WDT, we have $z^n = \alpha f + r$ where $\alpha \in \mathbb{C}\{z, \tau_\bullet\}$ and $r \in \mathbb{C}\{\tau_\bullet\}[z]$ has degree $< n$. Now, $z^n = \alpha(z, 0) f(z, 0) + r(z, 0)$ gives the unique Weierstrass division of z^n by $f(z, 0)$. Since f has order n in z, we may write $f(z, 0) = z^n h(z)$ where $h \in \mathbb{C}\{z\}$ is invertible. So $z^n = h(z)^{-1} \cdot f(z, 0)$ also gives a Weierstrass division. Therefore $r(z, 0) = 0$ and $\alpha(z, 0) = h(z)^{-1}$. So $\alpha(0, 0) \neq 0$, i.e. α is invertible in $\mathbb{C}\{z, \tau_\bullet\}$. We have $f = \alpha^{-1} q$ where $q = z^n - r$. □

We are ready to prove Lemma 2.5.8.

Proof of Lemma 2.5.8. Recall the polynomials p_i in Def. 2.5.1. By WPT, for each $w_i \in N(p_i)_0$, in the ring $\mathbb{C}\{z_i - w_i, \tau_\bullet\}$, $p_i(z_i, \tau_\bullet)$ equals a unit times a Weierstrass polynomial $r_{i,w_i}(z_i, \tau_\bullet)$ of $z_i - w_i$. So $r_{i,w_i}(z_i, \tau_\bullet) \in \mathscr{O}_{S,0}[z_i]$ has degree $\mu_i(w_i)$ in z_i, and $r_{i,w_i}(z_i, 0) = (z_i - w_i)^{\mu_i(w_i)}$. So in the ring $\mathscr{O}_{\mathbb{C}^k \times S, (\widetilde{w}_\bullet, 0)}/\mathcal{I}_{(\widetilde{w}_\bullet, 0)}$, r_{i,w_i} is invertible when $\widetilde{w}_i \neq w_i$ (since $r_{i,w_i}(\widetilde{w}_i, 0) \neq 0$), and is 0 when $\widetilde{w}_i = w_i$. Thus

$$R_i := \prod_{\substack{\widetilde{w}_i \in N(p_i)_0 \\ \widetilde{w}_i \neq w_i}} r_{i,w_i}$$

is invertible in $\mathscr{O}_{\mathbb{C}^k \times S, (\widetilde{w}_\bullet, 0)/\mathcal{I}_{(\widetilde{w}_\bullet, 0)}}$ when $\widetilde{w}_i = w_i$ and is zero when $\widetilde{w}_i \neq w_i$. $R_i \in \mathscr{O}_{S,0}[z_i]$ has degree $n - \mu_i(w_i)$ in z_i. So $p_1 = \prod_{i=1}^k R_i$ gives the desired polynomial. $\square$

2.5.3 Proof of Thm. 2.5.4, II

Finishing the proof of Thm. 2.5.4. We have already shown that the set (2.5.3) (which has $n_1 \cdots n_k$ elements) generate $\pi_* \mathscr{O}_X$. In particular, $\pi_* \mathscr{O}_X$ is a finite-type $\mathscr{O}_S$-module. To show that (2.5.3) generates $\pi_* \mathscr{O}_X$ freely, by Prop. 1.3.14, it suffices to show that the fiber $(\pi_* \mathscr{O}_X)|y = (\pi_* \mathscr{O}_X) \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathfrak{m}_{S,y})$ has dimension $n_1 \cdots n_k$ for each $y \in S$.

By Base change Prop. 2.4.7, $(\pi_* \mathscr{O}_X)|y$ is canonically equivalent to

$$\pi_*(\mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathfrak{m}_{S,y})),$$

which equals $\pi_* \mathscr{O}_{X_y} = \mathscr{O}(X_y)$ (where $X_y = \pi^{-1}(y)$ is the inverse image of y and is a closed subspace of X) by Rem. 1.12.3. By Prop. 2.5.3, $\pi : \pi^{-1}(y) \to \{y\}$ is a Weierstrass map. It is the restriction of $\mathbb{C}^k \to \{y\}$ to the complex subspace of $\mathbb{C}^k$ defined by the ideal sheaf generated by $p_i(z_i, y) = a_{i,0}(y) + a_{i,1}(y) z_i + \cdots + a_{i,n_i}(y) z_i^{n_i}$ for all $1 \leqslant i \leqslant k$. Thus, it suffices to prove the following lemma. $\square$

Lemma 2.5.11. *Let $X = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k}/\mathcal{I})$ where $\mathcal{I}$ is the ideal sheaf generated by $p_1, \dots, p_k$ where each $p_i(z_i) \in \mathbb{C}[z_i]$ has degree n_i. Then $\mathscr{O}(X)$ has dimension $n_1 \cdots n_k$.*

Proof. We are still in the setting of Def. 2.5.1, but assuming that S is a single point 0. So $N(p_i)_0 = N(p_i)$. By (2.5.5),

$$\mathscr{O}(X) \simeq \bigoplus_{\substack{w_i \in N(p_i) \\ 1 \leqslant i \leqslant k}} \mathscr{O}_{\mathbb{C}^k, w_\bullet}/\mathcal{I}_{w_\bullet}.$$

Clearly $\mathcal{I}_{w_\bullet}$ is the ideal generated by $(z_i - w_i)^{\mu_i(w_i)}$ for all $1 \leqslant i \leqslant k$. So

$$\left\{ \prod_{i=1}^k (z_i - w_i)^{\nu_i} : 0 \leqslant \nu_i \leqslant \mu_i(w_i) - 1 \right\}$$

is a basis of $\mathscr{O}_{\mathbb{C}^k, w_\bullet}/\mathcal{I}_{w_\bullet}$. This calculates the dimension of $\mathscr{O}(X)$. $\square$

2.6 Coherence of $\mathscr{O}_X$

The goal of this section is to prove that $\mathscr{O}_X$ is coherent for every complex space X. By Cor. 2.1.13, it suffices to prove that $\mathscr{O}_{\mathbb{C}^n}$ is coherent. The role that Thm. 2.5.4 plays in the proof of coherence of $\mathscr{O}_{\mathbb{C}^n}$ is similar to the role that WDT plays in the proof that $\mathscr{O}_{\mathbb{C}^n,0}$ is Noetherian.

Lemma 2.6.1. *Assume that X is an open subset of $\mathbb{C}^n$. Assume that for each open connected $U \subset X$ and each non-zero $h \in \mathscr{O}(U)$, $\mathscr{O}_U/h\mathscr{O}_U$ is a coherent $\mathscr{O}_U/h\mathscr{O}_U$-module. Then $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module.*

More precisely, our assumption is that the structure sheaf of $\mathrm{Specan}(\mathscr{O}_U/h\mathscr{O}_U)$ is coherent.

Proof. Choose any open connected $U \subset X$ and $h_1, \dots, h_N \in \mathscr{O}(U)$. We want to show that $\mathscr{R}el(h_1, \dots, h_N)$ is a finite-type $\mathscr{O}_U$-submodule of $\mathscr{O}_U^N$. We assume one of $h_1, \dots, h_N$ is non-zero, say $h_1 \neq 0$; otherwise the proof is obvious. For each $f \in \mathscr{O}_U$, we let $[f]$ denotes its residue class in $\mathscr{O}_Y = (\mathscr{O}_U/h_1\mathscr{O}_U) \restriction_{N(h_1\mathscr{O}_U)}$ where $Y = \mathrm{Specan}(\mathscr{O}_U/h_1\mathscr{O}_U)$.

Choose any $x \in U$. By assumption, $\mathscr{O}_Y$ is coherent. So $\mathscr{R}el([h_2], \dots, [h_N])$ is a finite type $\mathscr{O}_Y$-submodule of $\mathscr{O}_Y^{N-1}$. Thus, after shrinking U to a smaller neighborhood of x, we may find $(s_2^i, \dots, s_N^i) \in \mathscr{O}(U)$ (for finitely many i) such that $([s_2^i], \dots, [s_N^i])$ generate $\mathscr{R}el([h_2], \dots [h_N])$. This means:

(a) For each i, $s_2^i h_2 + \cdots + s_N^i h_N \in h_1\mathscr{O}_U$. (This can be checked at the level of stalks.)

(b) For each $y \in U$ and $(\sigma_2, \dots, \sigma_N) \in \mathscr{O}_{U,y}^{N-1}$ such that

$$\sigma_2 h_2 + \cdots + \sigma_N h_N \in h_1\mathscr{O}_{U,y},$$

there exist $f_i \in \mathscr{O}_{U,y}$ for all i and $g_2, \dots, g_N \in \mathscr{O}_{U,y}$ such that

$$(\sigma_2, \dots, \sigma_N) = \sum_i f_i(s_2^i, \dots, s_N^i) + h_1(g_2, \dots, g_N). \tag{2.6.1}$$

By (a), we may shrink U further so that for each i, we may find $s_1^i \in \mathscr{O}(U)$ such that $s_1^i h_1 + s_2^i h_2 + \cdots + s_N^i h_N = 0$. We claim that

$$(s_1^i, \dots, s_N^i)$$

for all i and

$$(-h_2, h_1, 0, \dots, 0), \quad (-h_3, 0, h_1, \dots, 0), \quad \dots, \quad (-h_N, 0, 0, \dots, h_1)$$

(which are clearly in $\mathscr{Rel}(h_1, h_2, \dots, h_N)$) generate $\mathscr{Rel}(h_1, h_2, \dots, h_N)$.

Choose any $y \in U$ and $(\sigma_1, \dots, \sigma_N) \in \mathscr{O}_{U,y}^N$ in $\mathscr{Rel}(h_1, \dots, h_N)_y$, namely $\sigma_1 h_1 + \cdots + \sigma_N h_N = 0$. Then by (b), we can find $f_i, g_2, \dots, g_N \in \mathscr{O}_{U,y}$ such that the relation

$$
\begin{aligned}
(\sigma_1, \sigma_2, \dots, \sigma_N) = & \sum_i f_i(s_1^i, s_2^i, \dots, s_N^i) \\
& + \sum_{j=2}^N g_j(-h_j, 0, \dots, \underset{\substack{\uparrow \\ i\text{-th component}}}{h_1}, \dots, 0)
\end{aligned}
\tag{2.6.2}
$$

holds for the last $N-1$ components. To show that it holds also for the first component, we write the RHS of (2.6.2) as $(\widetilde{\sigma}_1, \sigma_2, \dots, \sigma_N)$, which is an element of $\mathscr{Rel}(h_1, \dots, h_N)_y$. So

$$
\sigma_1 h_1 + \sigma_2 h_2 \cdots + \sigma_N h_N = \widetilde{\sigma}_1 h_1 + \sigma_2 h_2 \cdots + \sigma_N h_N = 0,
$$

which shows $(\sigma_1 - \widetilde{\sigma}_1)h_1 = 0$. Since h_1 is a non-zero element of $\mathscr{O}(U)$, by the Identitätssatz 1.1.3, the germ of h_1 at $\mathscr{O}_{U,y}$ is non-zero. So $\sigma_1 = \widetilde{\sigma}_1$ since $\mathscr{O}_{U,y}$ is an integral domain. This proves (2.6.2). $\square$

Theorem 2.6.2 (Oka's coherence theorem). *For every complex space X, $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module.*

Proof. We prove the coherence of $\mathscr{O}_{\mathbb{C}^m}$ by induction on m. The case $m = 0$ is obvious. Assume that $\mathscr{O}_{\mathbb{C}^m}$ is coherent. Let us prove that $\mathscr{O}_{\mathbb{C}^{m+1}}$ is coherent.

By Lemma 2.6.1, it suffices to show that for each open connected $U \subset \mathbb{C}^{m+1}$ and non-zero $h \in \mathscr{O}(U)$, if we write $Y = \mathrm{Specan}(\mathscr{O}_U / h\mathscr{O}_U)$ then $\mathscr{O}_Y$ is a coherent $\mathscr{O}_Y$-module. Let $\mathscr{K}$ be the kernel of a morphism $\mathscr{O}_Y^N \to \mathscr{O}_Y$. Then we have an exact sequence of $\mathscr{O}_Y$-modules

$$
0 \to \mathscr{K} \to \mathscr{O}_Y^N \to \mathscr{O}_Y.
$$

We need to show that for each $x \in U$, say $x = 0$, after shrinking U to a neighborhood of x, $\mathscr{K}$ is $\mathscr{O}_U$-generated by finitely many elements of $\mathscr{K}(U)$.

The germ of h in $\mathscr{O}_{U,x}$ is non-zero by the Identitätssatz 1.1.3. Thus, by choosing a new set of coordinates $(z, t_1, \dots, t_m)$ of U such that $x = 0$, we may assume that the germ of h at 0, which is an element of $\mathbb{C}\{z, t_1, \dots, t_m\}$, has finite order n in z. (Cf. the proof of Thm. 1.5.5). Thus, by WPT, after shrinking U to a smaller neighborhood of 0 we may assume that $h \in \mathbb{C}\{t_\bullet\}[z]$ is a Weierstrass polynomial of degree=order n in z.

We assume $U = V \times W$ where $V \subset \mathbb{C}$ and $W \subset \mathbb{C}^m$ are neighborhoods of 0. By Rem. 1.5.2, we may assume that $N(h) = \{(z, t_\bullet) \in \mathbb{C} \times W : h(z, t_\bullet) = 0\}$ is like

Fig. 1.5.1: for each $(t_\bullet) \in W$, the polynomial $h(z, t_\bullet)$ of z has n zeros in V counting multiplicities. Thus $N(h) \subset U$. Therefore

$$\mathscr{O}_U / h\mathscr{O}_U = \mathscr{O}_{\mathbb{C}\times W} / h\mathscr{O}_{\mathbb{C}\times W}.$$

So the projection of $\pi : Y \to W$ (inherited from $\mathbb{C} \times W \to W$) is a Weierstrass map. By the Fundamental Thm. 2.5.4 of Weierstrass maps, $\pi_*\mathscr{O}_Y$ and hence $\pi_*(\mathscr{O}_Y^N) = (\pi_*\mathscr{O}_Y)^N$ are $\mathscr{O}_W$-free. So they are $\mathscr{O}_W$-coherent *by our assumption that $\mathscr{O}_{\mathbb{C}^m}$ is coherent*. Therefore $\pi_*\mathscr{K}$ is $\mathscr{O}_W$-coherent by Cor. 2.1.5 and the exactness of

$$0 \to \pi_*\mathscr{K} \to \pi_*\mathscr{O}_Y^N \to \pi_*\mathscr{O}_Y.$$

So $\mathscr{K}$ is $\mathscr{O}_Y$-finite-type by the following lemma. □

Lemma 2.6.3. *Let $\pi : X \to S$ be a finite morphism of $\mathbb{C}$-ringed spaces, and let $\mathscr{E}$ be an $\mathscr{O}_X$-module. If $\pi_*\mathscr{E}$ is $\mathscr{O}_S$-finite-type, then $\mathscr{E}$ is $\mathscr{O}_X$-finite-type.*

Proof. Choose any $t \in S$. By shrinking S to a neighborhood of t (and shrinking X to $\pi^{-1}(S)$), we can find $\sigma_1, \dots, \sigma_k \in \mathscr{E}(X) = (\pi_*\mathscr{E})(S)$ which $\mathscr{O}_S$-generate $\pi_*\mathscr{E}$. For each $x \in X$, by Prop. 2.4.5, $\mathscr{E}_x$ is a direct summand of the $\mathscr{O}_{S,\pi(x)}$-module $(\pi_*\mathscr{E})_{\pi(x)}$. So $\mathscr{E}_x$ is $\mathscr{O}_{S,\pi(x)}$-generated (and hence $\mathscr{O}_{X,x}$-generated) by $\sigma_1, \dots, \sigma_k$. This proves that $\mathscr{E}$ is $\mathscr{O}_X$-generated by $\sigma_1, \dots, \sigma_k$. □

Corollary 2.6.4. *Let X be a complex space. An ideal of $\mathscr{O}_X$ is finite-type if and only if it is coherent.*

2.7 Finite mapping theorem

The following two theorems are the main results of this section.

Theorem 2.7.1 (Finite mapping theorem). *Let $\pi : X \to S$ be a finite holomorphic map of complex spaces, and let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the following are equivalent.*

(1) $\mathscr{E}$ is $\mathscr{O}_X$-coherent.

(2) $\pi_*\mathscr{E}$ is $\mathscr{O}_S$-coherent.

Theorem 2.7.2. *Let $\pi : X \to S$ be a holomorphic map of complex spaces. Let $t \in S$, and assume that $x \in \pi^{-1}(t)$ is an isolated point of $\pi^{-1}(t)$. Then there are neighborhoods $U \subset X$ of x and $W \subset S$ of $\pi(U)$ such that π restricts to a finite holomorphic map $\pi : U \to W$.*

Remark 2.7.3. It follows immediately from Thm. 2.7.2 that if $\pi : X \to S$ is holomorphic and if $t \in S$ is such that $\pi^{-1}(t)$ is a finite set, then there are neighborhoods $U \subset X$ of $\pi^{-1}(t)$ and $W \subset S$ of $\pi(U)$ such that the restriction $\pi : U \to W$ is finite.

Corollary 2.7.4. *Let X be a complex space which, as a set, is $\{x\}$. Then an $\mathscr{O}_X$-module $\mathscr{E}$ is $\mathscr{O}_X$-coherent if and only if $\mathscr{E}$ (or more precisely $\mathscr{E}(\{x\})$) is a finite-dimensional vector space.*

Proof. Let $\pi : X \to \{0\}$ be the obvious map where $\{0\}$ is the reduced single point. (Note that x is not assumed to be reduced.) Then π is clearly finite. That $\mathscr{E}$ is finite-dimensional is equivalent to that $\pi_*\mathscr{E}$ is $\mathscr{O}_{\{0\}}$-coherent. Thus the proof is finished by Thm. 2.7.1. □

2.7.1 Proof of the main results

We begin with the following preliminary lemma.

Lemma 2.7.5. *Given a finite holomorphic $\pi : X \to S$, if $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent, then for each coherent $\mathscr{O}_X$-module $\mathscr{E}$, $\pi_*\mathscr{E}$ is $\mathscr{O}_S$-coherent.*

Proof. Choose any $t \in S$. By Lemma 2.4.8, we can shrink S to a neighborhood of t and shrink X to $\pi^{-1}(S)$ so that $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{O}_X^m \to \mathscr{O}_X^n)$ for a morphism $\mathscr{O}_X^m \to \mathscr{O}_X^n$. Thus, by the (right) exactness of π_* (Cor. 2.4.6), $\pi_*\mathscr{E} \simeq \mathrm{Coker}(\pi_*\mathscr{O}_X^m \to \pi_*\mathscr{O}_X^n)$, which is coherent since $\pi_*\mathscr{O}_X$ is coherent. □

The crucial part of the proof is the following lemma.

Lemma 2.7.6. *Choose open subsets $R \subset \mathbb{C}^k$ and $S \subset \mathbb{C}^m$. Let $X = \mathrm{Specan}(\mathscr{O}_{R\times S}/\mathcal{I})$ where $\mathcal{I}$ is a coherent ideal of $\mathscr{O}_{R\times S}$. Let $\pi : X \to S$ be the holomorphic map restricted from the projection $R \times S \to S$. Let $t \in S$ and assume that $x \in \pi^{-1}(t)$ is an isolated point of $\pi^{-1}(t)$. Then there are neighborhoods $U \subset R$ of x and $W \subset S$ of $\pi(U)$ such that the restriction $\pi : (U \times W) \cap X \to W$ is finite, and that $\pi_*\mathscr{O}_{(U\times W)\cap X}$ is $\mathscr{O}_W$-coherent.*

We assume $x = 0_R$ and $t = 0_S$ for simplicity, and prove the lemma by induction on k.

Proof for the case $k = 1$. Shrink R to a neighborhood of 0_R such that $\pi^{-1}(0_S) = (R\times 0_S)\cap N(\mathcal{I})$ is $\{0\}$. So we may shrink R further so that we can find $f \in \mathcal{I}(R\times S)$ such that $(R \times 0_S) \cap N(f) = \{0\}$. So f, as an element of $\mathbb{C}\{z, t_1, \dots, t_m\}$, has finite order in z. So we may shrink R, S further and replace f by a Weierstrass polynomial of z, which we still denote by f.

Let $\mathcal{J} = f\mathscr{O}_{R\times S}$ and $Y = \mathrm{Specan}(\mathscr{O}_{R\times S}/\mathcal{J})$. Let $\widetilde{\pi} : Y \to S$ be the restriction of $R\times S \to S$ to Y. As in the proof of Oka's coherence theorem, we may shrink R and S so that Fig. 1.5.1 holds, and hence that $\widetilde{\pi}$ is a Weierstrass map. So $\pi = \widetilde{\pi} \circ \iota_{X,Y}$ is finite since both $\widetilde{\pi}$ and the inclusion map $\iota = \iota_{X,Y}$ are finite.

By the Fundamental Thm. 2.5.4 of Weierstrass maps (and Oka's coherence theorem), $\widetilde{\pi}_*\mathscr{O}_Y$ is $\mathscr{O}_S$-coherent. So by Lemma 2.7.5, $\widetilde{\pi}_*$ sends coherent $\mathscr{O}_Y$-modules to coherent $\mathscr{O}_S$-modules. But $\iota_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent by Extension principle 2.1.12. So $\pi_*\mathscr{O}_X = \widetilde{\pi}_*\iota_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent. □

Proof that case $k \Rightarrow$ case $k+1$. Assume that case k is true. Now assume R is an open subset of $\mathbb{C}^{k+1}$. By shrinking R to a neighborhood of 0_R we assume $R = U \times V$ where $U \subset \mathbb{C}$ and $V \subset \mathbb{C}^k$ are open subsets containing $0_{\mathbb{C}}$ and $0_{\mathbb{C}^k}$ respectively, and that $\pi^{-1}(0_S) = (U \times V \times 0_S) \cap N(\mathscr{I})$ equals $\{0\}$.

Let $\alpha : X \to V \times S$ be the restriction of the projection $U \times V \times S \to V \times S$. Then $\alpha^{-1}(0_{V\times S}) = (U \times 0_V \times 0_S) \times N(\mathscr{I})$ is $\{0\}$. So by the case $k = 1$, we may shrink U, V, S to smaller neighborhoods of $0_U, 0_V, 0_S$ respectively so that α is finite and $\alpha_*\mathscr{O}_X$ is $\mathscr{O}_{V\times S}$-coherent. By Def. 2.3.8, we can define the image space $\alpha(X)$ whose underlying topological space is $\mathrm{Im}(\alpha)$, and by Prop. 2.3.12, α factors as the composition of a holomorphic $\widetilde{\alpha} : X \to \alpha(X)$ and the inclusion $\alpha(X) \hookrightarrow V \times S$. We thus obtain a commutative diagram

$$\begin{tikzcd}
 & V \times S \arrow[dr, "\mathrm{pr}_S"] & \\
X = \mathrm{Specan}(\mathscr{O}_{U\times V\times S}/\mathscr{I}) \arrow[ur, "\alpha"] \arrow[dr, "\widetilde{\alpha}"'] & & S \\
 & \alpha(X) \arrow[uu, hook, "\iota"] \arrow[ur, "\widetilde{\pi}"'] &
\end{tikzcd}$$

where $\widetilde{\pi}$ is the restriction of pr_S to $\alpha(X)$. We have $\pi = \mathrm{pr}_S \circ \alpha = \widetilde{\pi} \circ \widetilde{\alpha}$.

Clearly $\widetilde{\pi}^{-1}(0_S) = \{0_{V\times S}\}$. Thus, by our assumption on case k, we may shrink V, S so that $\widetilde{\pi}$ is finite and (by Lemma 2.7.5) $\widetilde{\pi}_*$ sends coherent $\mathscr{O}_{\alpha(X)}$-modules to coherent $\mathscr{O}_S$-modules. Note that we still have that α is finite and $\iota_*\widetilde{\alpha}_*\mathscr{O}_X = \alpha_*\mathscr{O}_X$ is $\mathscr{O}_{V\times S}$-coherent after shrinking V, S (but not shrinking U). So $\widetilde{\alpha}$ is finite, and by Extension principle 2.1.12, $\widetilde{\alpha}_*\mathscr{O}_X$ is $\mathscr{O}_{\alpha(X)}$-coherent. So $\pi = \widetilde{\pi} \circ \widetilde{\alpha}$ is finite, and $\pi_*\mathscr{O}_X = \widetilde{\pi}_*\widetilde{\alpha}_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent. We are done with the proof of Lemma 2.7.6. □

We are now ready to prove Thm. 2.7.2 and more:

Lemma 2.7.7. *Thm. 2.7.2 is true. Moreover, in Thm. 2.7.2, U and W can be chosen so that (besides that π is finite) $\pi_*\mathscr{O}_U$ is also $\mathscr{O}_W$-coherent.*

Proof. It suffices to assume that X is a model space, say a closed subspace of an open $R \subset \mathbb{C}^k$. We first assume S is an open subset of $\mathbb{C}^m$. Define $\varphi : X \to R \times S$ so that the following triangular diagram commutes

$$\begin{tikzcd}
 & X \times S \arrow[dr, "\iota_{X,R}\times\mathbf{1}"] & & \\
X \arrow[ur, "\mathbf{1}\vee\pi"] \arrow[rr, "\varphi"'] & & R \times S \arrow[r, "\mathrm{pr}_S"'] & S
\end{tikzcd}$$

By Prop. 1.13.6 and Prop. 1.12.5, $\mathbf{1} \vee \pi$ and $\iota_{X,R} \vee \mathbf{1}$ are closed embeddings. So their composition φ is a closed embedding (Cor. 1.7.6). By Prop. 1.11.6,

$$\mathrm{pr}_S \circ \varphi = \mathrm{pr}_S \circ (\iota \times \mathbf{1}) \circ (\mathbf{1} \vee \pi) = \mathrm{pr}_S \circ (\iota \vee \pi) = \pi.$$

Thus, by identifying X with $\varphi(X)$, the assumptions of Lemma 2.7.6 are satisfied. The conclusions of Lemma 2.7.6 prove what we want to prove.

In the general case, we may shrink S (and shrink X accordingly) so that S is a closed subspace of an open $\Omega\subset\mathbb{C}^m$. Let $\iota:S\to\Omega$ be the inclusion. Then by shrinking X and Ω (and S accordingly) to neighborhoods of any given points, $\iota\circ\pi:X\to\Omega$ is finite and $\iota_*\pi_*\mathscr{O}_X$ is $\mathscr{O}_\Omega$-coherent. Clearly π is finite, and by Extension principle 2.1.12, $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent. $\square$

Proof of Thm. 2.7.1, (1)⇒(2). Let us prove that $\pi_*\mathscr{O}_X$ is coherent. Choose any $t\in S$. By Lemma 2.7.7, for each $x\in\pi^{-1}(t)$ we can choose neighborhoods $U_x\ni x$ and $W_x\supset\pi(U_x)$ such that $\pi_*\mathscr{O}_{U_x}$ is $\mathscr{O}_{W_x}$-coherent, and that $U_x\cap U_{x'}=\varnothing$ if $x\neq x'$. So for each open $W\subset\bigcap_{x\in\pi^{-1}(t)}W_x$, we have that $\pi_*\mathscr{O}_{U_x\cap\pi^{-1}(W)}$ is $\mathscr{O}_W$-coherent. Therefore, if we set $U=\bigcup_{x\in\pi^{-1}(t)}U_x$, then

$$\pi_*\mathscr{O}_{U\cap\pi^{-1}(W)}\simeq\bigoplus_{x\in\pi^{-1}(t)}\pi_*\mathscr{O}_{U_x\cap\pi^{-1}(W)}$$

is $\mathscr{O}_W$-coherent.

Since $\pi:X\to S$ is finite, by Prop. 2.4.1, there is a neighborhood $W\ni t$ inside $\bigcap_{x\in\pi^{-1}(t)}W_x$ such that $\pi^{-1}(W)=U\cap\pi^{-1}(W)$. So $\pi_*\mathscr{O}_{\pi^{-1}(W)}=(\pi_*\mathscr{O}_X)|_W$ is $\mathscr{O}_W$-coherent. $\square$

The proof of (2)⇒(1) is similar to that of Oka's coherence Thm. 2.6.2:

Proof of Thm. 2.7.1, (2)⇒(1). Assume that $\pi_*\mathscr{E}$ is coherent. Then $\mathscr{E}$ is $\mathscr{O}_X$-finite-type by Lemma 2.6.3. Let us show that the sheaves of relations of $\mathscr{E}$ are finite-type. By Prop. 2.4.1 or Rem. 2.4.4, we have a neighborhood W of t such that

$$\pi^{-1}(W)=\coprod_{x\in\pi^{-1}(t)}U_x$$

where each U_x is a small enough neighborhood of y. Shrink Y to W and X to $\pi^{-1}(W)$. So we have an equivalence of $\mathscr{O}_W$-modules

$$\pi_*\mathscr{E}\simeq\bigoplus_{x\in\pi^{-1}(t)}\pi_*(\mathscr{E}|_{U_x}).$$

Suppose $\alpha:\mathscr{O}_{U_x}^N\to\mathscr{E}_{U_x}$ is a morphism of $\mathscr{O}_{U_x}$-modules. Let $\mathscr{K}=\mathrm{Ker}(\alpha)$ so that we have an exact

$$0\to\mathscr{K}\to\mathscr{O}_{U_x}^N\to\mathscr{E}_{U_x}.$$

We regard $\mathscr{K},\mathscr{O}_{U_x},\mathscr{E}_{U_x}$ as $\mathscr{O}_X$-modules by identifying them with their direct images under $U_x\hookrightarrow X$. Clearly $\mathscr{O}_{U_x}$ is $\mathscr{O}_X$-coherent. So $\pi_*\mathscr{O}_{U_x}$ is $\mathscr{O}_S$-coherent. Also

$\pi_*\mathscr{E}_{U_x}$ is $\mathscr{O}_S$-coherent since it is a direct summand of the coherent sheaf $\pi_*\mathscr{E}$ (cf. Cor. 2.1.4). Thus, the exact sequence of $\mathscr{O}_S$-modules

$$0 \to \pi_*\mathscr{K} \to \pi_*\mathscr{O}_{U_x}^N \to \pi_*\mathscr{E}_{U_x}$$

together with Cor. 2.1.5 show that $\pi_*\mathscr{K}$ is $\mathscr{O}_S$-coherent. Therefore, by Lemma 2.6.3, $\mathscr{K}$ is $\mathscr{O}_X$-finite-type. □

We are done with the proofs of Thm. 2.7.1 and 2.7.2. In the following, we give some applications.

2.7.2 Applications

Corollary 2.7.8. *Let $\varphi : X \to Y$ be a holomorphic map of complex spaces. Then the following are equivalent.*

(1) φ is a closed embedding.

(2) φ is an immersion of complex spaces, and it is a closed and injective map of topological spaces.

Proof. (1)⇒(2) is obvious. Assume (2). Then as φ is finite, $\varphi_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent. By (2.3.6), the coherent ideal

$$\mathscr{I} = \mathscr{A}nn_{\mathscr{O}_Y}(\varphi_*\mathscr{O}_X)$$

satisfies the assumptions in Prop. 1.7.3. Thus (1) follows from Prop. 1.7.3. □

Rem. 1.13.8 tells us that any holomorphic map factors as the composition of a closed embedding and the projection of a direct product. When the holomorphic map is finite, such decomposition might not be useful because, although closed embeddings are finite, projections are usually not. The following proposition gives a refinement of this decomposition. It says that any finite holomorphic map locally factors as the composition of a closed embedding and a Weierstrass map. This result will be used e.g. in the proof of Base change Thm. 2.8.2.

Proposition 2.7.9. *Let $\pi : X \to S$ be a finite holomorphic map of complex spaces. Then each $t \in S$ is contained in a neighborhood $W \subset S$ such that the restriction $\pi : \pi^{-1}(W) \to W$ is equivalent to the restriction of a Weierstrass map. More precisely, there exist a Weierstrass map $\kappa : Y \to W$ and a closed embedding $\varphi : \pi^{-1}(W) \to Y$ such that the following diagram commutes.*

$$\begin{tikzcd}
\pi^{-1}(W) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\kappa"] \\
& W &
\end{tikzcd} \tag{2.7.1}$$

Proof-Step 1. By Finite mapping theorem, $\pi_*\mathscr{O}_X$ is coherent. So we may shrink S to a neighborhood of t and shrink X accordingly (i.e. replace X by the new $\pi^{-1}(S)$) so that $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-generated by $f_1,\dots,f_k\in\mathscr{O}(X)$. Consider $F=(f_1,\dots,f_k)$ as a holomorphic map $F:X\to\mathbb{C}^k$ (Thm. 1.4.1). Then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "F\vee\pi"] \arrow[dr, "\pi"'] & & \mathbb{C}^k\times S \arrow[dl, "\mathrm{pr}_S"] \\
& S &
\end{tikzcd} \tag{2.7.2}$$

We want to show that $F\vee\pi$ is a closed embedding.

Since π is closed, one checks easily using (2.7.2) that $F\vee\pi$ is closed. To show that $F\vee\pi$ is injective, it suffices to show that F is injective when restricted to each fiber $\pi^{-1}(\tau)$ (where $\tau\in S$). By Prop. 2.4.5, we have

$$(\pi_*\mathscr{O}_X)_\tau\simeq\bigoplus_{x\in\pi^{-1}(\tau)}\mathscr{O}_{X,x} \tag{2.7.3}$$

which is $\mathscr{O}_{S,s}$-generated by $f_1,\dots,f_k$. If $x,x'\in\pi^{-1}(\tau)$ and $x\neq x'$, then an $\mathscr{O}_{S,\tau}$-linear combination of $f_1,\dots,f_k$ is 1 in $\mathscr{O}_{X,x}$ and 0 in $\mathscr{O}_{X,x'}$. So a $\mathbb{C}$-linear combination of $f_1,\dots,f_k$ takes value 1 at x and 0 at x'. So $F(x)\neq F(x')$. To show that $F\vee\pi$ is an immersion, note that by (2.7.3), the $\mathbb{C}$-algebra morphism

$$F^\#:\mathscr{O}_{\mathbb{C}^k,F(x)}\to\mathscr{O}_{X,x}$$

sends $z_1,\dots,z_k$ to (the germs at x of) $f_1,\dots,f_k$ respectively. So the morphism

$$(F\vee\pi)^\#:\mathscr{O}_{\mathbb{C}^k\times S,x\times\tau}=\mathscr{O}_{\mathbb{C}^k,x}\widehat{\otimes}\mathscr{O}_{S,\tau}\longrightarrow\mathscr{O}_{X,x}$$

sends $z_i\otimes h$ (where $h\in\mathscr{O}_{S,\tau}$) to $h\cdot f_i$. Thus, this morphism is surjective since $\mathscr{O}_{X,x}$ is $\mathscr{O}_{S,\tau}$-generated by $f_1,\dots,f_k$. So $F\vee\pi$ is an immersion. By Cor. 2.7.8, $F\vee\pi$ is a closed embedding. $\square$

Proof-Step 2. Since $(\pi_*\mathscr{O}_X)_t$ is a finitely generated module of the Noetherian ring $\mathscr{O}_{S,t}$, for each i, the $\mathscr{O}_{S,t}$-submodule of $(\pi_*\mathscr{O}_X)_t$ generated by all non-negative powers of f_i is finitely generated. So f_i is **integral over** $\mathscr{O}_{S,t}$. Namely, we may find $n_i\in\mathbb{Z}_+$ such that

$$a_{i,0}+a_{i,1}f_i+\cdots+a_{i,n_i-1}f_i^{n_i-1}+f_i^{n_i}=0 \tag{2.7.4}$$

where each $a_{i,j}\in\mathscr{O}_{S,t}$.

Shrink S to a neighborhood of t (and shrink X to $\pi^{-1}(S)$) so that all $a_{i,j}$ are elements of $\mathscr{O}(S)$, and that (2.7.4) holds at the level of $\mathscr{O}(X)$. Then

$$p_i(z_i)=a_{i_0}+a_{i,1}z_i+\cdots+a_{i,n_i-1}z_i^{n_i-1}+z_i^{n_i}$$

is a monic polynomial of z_i, viewed as in $\mathscr{O}(\mathbb{C}^k \times S)$. Note that $F \vee \pi$ is still a closed embedding. We let $\mathcal{I}$ be the ideal of $\mathscr{O}_{\mathbb{C}^k \times S}$ generated by $p_1, \dots, p_k$, and let $Y = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k \times S}/\mathcal{I})$. Then $\mathrm{pr}_S : \mathbb{C}^k \times S \to S$ restricts to a Weierstrass map $\kappa : Y \to S$. By Thm. 1.4.8, $F \vee \pi : X \to \mathbb{C}^k \times S$ restricts to $\varphi : X \to Y$, which is clearly a closed embedding. And we clearly have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\kappa"] \\ & S & \end{tikzcd}$$

This finishes the proof. □

2.8 Base change theorem for finite holomorphic maps

In algebraic geometry, if X, Y, S are affine schemes, then $\mathscr{O}(X \times_S Y) \simeq \mathscr{O}(X) \otimes_{\mathscr{O}(S)} \mathscr{O}(Y)$. In complex analytic geometry, fiber products are in general related to completed tensor products. But in the case that one holomorphic map is finite, the usual (algebraic) tensor products are sufficient. The goal of this section is to explore the relationship between $X \times_S Y$ and tensor products in the analytic setting and at the level of stalks. This goal will be achieved in Cor. 2.8.4 which is crucial to the future proof that "flatness of holomorphic maps is preserved by base change". We shall prove Cor. 2.8.4 as a consequence of the Base change theorem of finite holomorphic maps.

2.8.1 The setting

Consider a Cartesian square of holomorphic maps of complex spaces.

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"] \\ S & Y \arrow[l, "\psi"] \end{tikzcd} \tag{2.8.1}$$

Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then we have an $\mathscr{O}_Y$-module morphism

$$\Psi : \psi^* \pi_* \mathscr{E} \longrightarrow \mathrm{pr}_{Y,*} \mathrm{pr}_X^* \mathscr{E}, \tag{2.8.2}$$

namely, a morphism

$$\Psi : (\pi_* \mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_Y \longrightarrow \mathrm{pr}_{Y,*}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_{X \times_S Y}) \tag{2.8.3}$$

such that for each open $V \subset Y$ and each open $W \subset S$ containing $\psi(V)$, Ψ sends

$$\sigma \otimes g \qquad \in \mathscr{E}(\pi^{-1}(W)) \otimes_{\mathscr{O}_S(W)} \mathscr{O}_Y(V) \tag{2.8.4}$$

to

$$\sigma \otimes \mathrm{pr}_Y^\# g \qquad \in \mathscr{E}(\pi^{-1}(W)) \otimes_{\mathscr{O}_X(\pi^{-1}(W))} \mathscr{O}_{X\times_S Y}(\mathrm{pr}_Y^{-1}(V)). \tag{2.8.5}$$

(Note that $\mathrm{pr}_X(\mathrm{pr}_Y^{-1}(V)) \subset \pi^{-1}(W)$.) It is easy to see that Ψ is natural. We call Ψ the **base change morphism**.

Remark 2.8.1. The stalk map of Ψ at each $y \in Y$ is the $\mathscr{O}_{Y,y}$-module morphism determined by

$$\begin{aligned} \Psi : (\pi_*\mathscr{E})_{\psi(y)} \otimes_{\mathscr{O}_{S,\psi(y)}} \mathscr{O}_{Y,y} &\longrightarrow \mathrm{pr}_{Y,*}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_{X\times_S Y})_y \\ \sigma \otimes 1 \quad &\mapsto \quad \sigma \otimes 1 \end{aligned} \tag{2.8.6}$$

2.8.2 Base change theorem

The following theorem is the main result of this section. Note that in the Cartesian square (2.8.1), if π is finite then pr_Y is finite (Prop. 2.4.11).

Theorem 2.8.2 (Base change theorem). *In the setting of Subsec. 2.8.1, assume that $\pi : X \to S$ is finite and $\mathscr{E}$ is a coherent $\mathscr{O}_X$-module. Then the base change morphism Ψ (cf. (2.8.3)) is an isomorphism of $\mathscr{O}_Y$-modules.*

Note that this theorem is local by nature. Namely, in the proof we may shrink S to a neighborhood of any given point, and replace X by $\pi^{-1}(S)$ and Y by $\psi^{-1}(S)$.

In the special case that $\mathscr{E} = \mathscr{O}_X$, we have:

Corollary 2.8.3. *Let (2.8.1) be a Cartesian square of holomorphic maps of complex spaces. Assume that $\pi : X \to S$ is finite. Then we have an $\mathscr{O}_Y$-module isomorphism*

$$\Psi : (\pi_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y \xrightarrow{\simeq} \mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y} \tag{2.8.7}$$

whose stalk map at each $y \in Y$ is an $\mathscr{O}_{Y,y}$-module morphism determined by

$$\begin{aligned} \Psi : (\pi_*\mathscr{O}_X)_{\psi(y)} \otimes_{\mathscr{O}_{S,\psi(y)}} \mathscr{O}_{Y,y} &\longrightarrow \mathrm{pr}_{Y,*}(\mathscr{O}_{X\times_S Y})_y \\ f \otimes 1 \quad &\mapsto \quad \mathrm{pr}_X^\# f \end{aligned} \tag{2.8.8}$$

Clearly (2.8.7) is also an isomorphism of $\mathscr{O}_S$-algebras.

Corollary 2.8.4. *Let (2.8.1) be a Cartesian square, and assume that $\pi : X \to S$ is finite. Then for each $x \in X$ and $y \in Y$ such that $\pi(x)$ equals $t = \psi(y)$, there is an isomorphism of $\mathscr{O}_{S,t}$-algebras*

$$\begin{aligned} \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,t}} \mathscr{O}_{Y,y} &\xrightarrow{\simeq} \mathscr{O}_{X\times_S Y, x\times y} \\ f \otimes g \quad &\mapsto \quad \mathrm{pr}_X^\# f \cdot \mathrm{pr}_Y^\# g \end{aligned} \tag{2.8.9}$$

First Proof. By Thm. 2.7.2, we may shrink X and S to neighborhoods of x and t respectively, and shrink Y to $\psi^{-1}(S)$, so that $\pi^{-1}(t) = \{x\}$ (as sets) and π is still finite. Then in view of Prop. 2.4.5, we see that (2.8.8) becomes exactly (2.8.9). □

Second Proof. By Prop. 2.4.5, for each y and $t = \psi(y)$, (2.8.8) is precisely the direct sum of (2.8.9) over all $x \in \pi^{-1}(t) = \mathrm{pr}_Y^{-1}(y)$. □

The second proof shows that Cor. 2.8.3 and Cor. 2.8.4 are indeed equivalent.

2.8.3 Proof of Base change Thm. 2.8.2

Lemma 2.8.5. *Assume that Thm. 2.8.2 holds when $\mathscr{E} = \mathscr{O}_X$. Then Thm. 2.8.2 holds for any coherent $\mathscr{O}_X$-module $\mathscr{E}$.*

Proof. If Thm. 2.8.2 holds when $\mathscr{E} = \mathscr{O}_X$, then it holds when $\mathscr{E}$ is $\mathscr{O}_X$-free. Now in the general case, by Lemma 2.4.8 we can assume that S is so small that there is an exact sequence of $\mathscr{O}_X$-modules

$$\mathscr{F} \to \mathscr{G} \to \mathscr{E} \to 0$$

where $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-free. By the right exactness of ψ^* and π_* (Cor. 2.4.6), we have an exact sequence

$$\psi^*\pi_*\mathscr{F} \to \psi^*\pi_*\mathscr{G} \to \psi^*\pi_*\mathscr{E} \to 0.$$

Since the base change map Ψ is natural, we have a commutative diagram

$$\begin{tikzcd} \psi^*\pi_*\mathscr{F} \arrow[r] \arrow[d, "\Psi"', "\simeq"] & \psi^*\pi_*\mathscr{G} \arrow[r] \arrow[d, "\Psi"', "\simeq"] & \psi^*\pi_*\mathscr{E} \arrow[r] \arrow[d, "\Psi"'] & 0 \\ \mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{F} \arrow[r] & \mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{G} \arrow[r] & \mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{E} \arrow[r] & 0 \end{tikzcd}$$

where the first two Ψ are isomorphisms by assumption. So the third Ψ is an isomorphism by Five Lemma. □

Lemma 2.8.6. *Cor. 2.8.3 holds if $\pi : X \to S$ is a Weierstrass map.*

Proof. By Prop. 2.5.3, we may assume that $\mathrm{pr}_Y : X \times_S Y \to Y$ is a Weierstrass map. More precisely, we may assume that (2.8.1) factors as

$$\begin{tikzcd} X & X \times_S Y \arrow[l] \\ \mathbb{C}^k \times S \arrow[u, hookleftarrow] & \mathbb{C}^k \times Y \arrow[l] \arrow[u, hookleftarrow] \\ S & Y \arrow[l] \end{tikzcd}$$

$$\begin{array}{ccc} X & \longleftarrow & X \times_S Y \\ \downarrow & & \downarrow \\ \mathbb{C}^k \times S & \longleftarrow & \mathbb{C}^k \times Y \\ \downarrow & & \downarrow \\ S & \longleftarrow & Y \end{array}$$

where the two small squares are Cartesian. By the Fundamental Thm. 2.5.4 of Weierstrass maps, $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-freely generated by (2.5.3), and so $(\pi_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y$ is $\mathscr{O}_Y$-freely generated by (2.5.3) $\otimes 1$. Also, $\mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y}$ is $\mathscr{O}_Y$-freely generated by (2.5.3). Using e.g. (2.8.8) one checks that Ψ sends the given free generators of $(\pi_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y$ bijectively to those of $\mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y}$. So Ψ must be an isomorphism. □

***Proof of Thm.* 2.8.2.** By Lemma 2.8.5, it suffices to prove Cor. 2.8.3. By Prop. 2.7.9, we may assume S is so small that $\pi : X \to S$ factors as $X \hookrightarrow Z \xrightarrow{\widetilde{\pi}} S$ where X is a closed subspace of Z and $\widetilde{\pi}$ is equivalent to a Weierstrass map. Thus, (2.8.1) factors as the combination of two Cartesian squares

$$\begin{array}{ccc} X & \longleftarrow & X \times_S Y \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota\times\mathbf{1}} \\ Z & \longleftarrow & Z \times_S Y \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\widetilde{\mathrm{pr}}_Y} \\ S & \xleftarrow{\ \psi\ } & Y \end{array} \tag{2.8.10}$$

where $\mathrm{pr}_Y : X \times_S Y \to Y$ is $\widetilde{\mathrm{pr}}_Y \circ (\iota \times \mathbf{1})$.

We have proved that Cor. 2.8.3 holds (and hence Thm. 2.8.2 holds, cf. Lemma 2.8.5) for the lower Cartesian square. Apply Thm. 2.8.2 to the lower square and the coherent $\mathscr{O}_Z$-module $\iota_*\mathscr{O}_X$: The domain of the isomorphism Ψ is

$$(\widetilde{\pi}_*\iota_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y = \pi_*\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_Y$$

and the codomain is

$$\widetilde{\mathrm{pr}}_{Y,*}(\iota_*\mathscr{O}_X \otimes_{\mathscr{O}_Z} \mathscr{O}_{Z\times_S Y}) \simeq \widetilde{\mathrm{pr}}_{Y,*}\big((\iota\times\mathbf{1})_*\mathscr{O}_{X\times_S Y}\big) = \mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y}.$$

By checking stalkwisely with the help of (2.8.6) and (2.8.8) (and possibly Prop. 2.4.5), one sees that this morphism (i.e. the base change map for the lower square of (2.8.10) and the $\mathscr{O}_Z$-module $\iota_*\mathscr{O}_X$) agrees with the morphism Ψ in Cor. 2.8.3. So the latter must be an isomorphism. □

2.9 Analytic spectra Specan

We fix a complex space S.

2.9.1 Main results

Definition 2.9.1. A **morphism** from a finite holomorphic map $\pi : X \to S$ to a finite holomorphic $\kappa : Y \to S$ is a holomorphic map $\varphi : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\kappa"] \\ & S & \end{tikzcd} \tag{2.9.1}$$

The set of morphisms is denoted by $\mathrm{Mor}_S(X, Y)$. This defines the **category of finite holomorphic maps to** S.

Definition 2.9.2. An $\mathscr{O}_S$**-algebra** is an S-sheaf of $\mathbb{C}$-algebras $\mathscr{A}$ together with a morphism of sheaves of $\mathbb{C}$-algebras $\mathscr{O}_S \to \mathscr{A}$. Since $\mathscr{A}$ is an $\mathscr{A}$-module, it becomes an $\mathscr{O}_S$-module. We say that $\mathscr{A}$ is a **coherent** $\mathscr{O}_S$**-algebra** if it is an $\mathscr{O}_S$-algebra which is coherent as an $\mathscr{O}_S$-module.

A **morphism** of $\mathscr{O}_S$-algebras from $\mathscr{B}$ to $\mathscr{A}$ is by definition a morphism $\Phi : \mathscr{B} \to \mathscr{A}$ of sheaves of $\mathbb{C}$-algebras such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{A} & & \mathscr{B} \arrow[ll, "\Phi"'] \\ & \mathscr{O}_S \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{2.9.2}$$

The commutativity of (2.9.2) is equivalent to saying that the morphism of sheaves of $\mathbb{C}$-algebras Φ is also a morphism of $\mathscr{O}_S$-modules. The set of morphisms is denoted by $\mathrm{Mor}_{\mathscr{O}_S}(\mathscr{B}, \mathscr{A})$. This defines the **category of coherent** $\mathscr{O}_S$**-algebras**. $\square$

We have avoided using the symbol $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{B}, \mathscr{A})$, which is the set of $\mathscr{O}_S$-module morphisms but not $\mathscr{O}_S$-algebra morphisms.

Theorem 2.9.3. *The contravariant functor $\mathfrak{F}$ from the category of finite holomorphic maps to S to the category of coherent $\mathscr{O}_S$-algebras is an antiequivalence of categories. The functor $\mathfrak{F}$ sends each finite holomorphic map $\pi : X \to S$ to the coherent $\mathscr{O}_S$-algebra $\pi_*\mathscr{O}_X$. At the level of morphisms the functor is*

$$\mathfrak{F} : \mathrm{Mor}_S(X, Y) \to \mathrm{Mor}_{\mathscr{O}_S}(\kappa_*\mathscr{O}_Y, \pi_*\mathscr{O}_X), \qquad \varphi \mapsto \varphi^{\#}. \tag{2.9.3}$$

Thus, for each coherent $\mathscr{O}_S$-algebra $\mathscr{A}$ there is, *up to isomorphisms*, a unique finite holomorphic map $\pi : X \to S$ such that $\pi_*\mathscr{O}_X = \mathscr{A}$. We write this map as $\mathrm{Specan}(\mathscr{A}) \to S$ and call this map (or simply call the complex space $\mathrm{Specan}(\mathscr{A})$) the **analytic spectrum** of $\mathscr{A}$.

Note that when $\mathscr{A} = \mathscr{O}_S/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathscr{O}_S$, as before, $\mathrm{Specan}(\mathscr{A})$ denotes the unique analytic spectrum as a closed subspace of S. For a general $\mathscr{A}$, $\mathrm{Specan}(\mathscr{A})$ is not unique.

Corollary 2.9.4. *Let $\psi : Z \to S$ be a holomorphic map of complex spaces. Then*

$$\mathrm{Specan}(\mathscr{A} \otimes_{\mathscr{O}_S} \mathscr{O}_Z) \simeq \mathrm{Specan}(\mathscr{A}) \times_S Z$$

Proof. This is just a rephrasing of Cor. 2.8.3. □

2.9.2 Proof of Thm. 2.9.3

***Proof that* (2.9.3) *is injective*.** Let $\varphi, \psi \in \mathrm{Mor}_S(X, Y)$ such that $\psi^*, \varphi^* : \kappa_*\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ are equal. By Prop. 2.4.5, for each $t \in S$, $\varphi^\# : (\kappa_*\mathscr{O}_Y)_t \to (\pi_*\mathscr{O}_X)_t$ is an $\mathscr{O}_{S,t}$-module morphism of the form

$$\varphi^\# : \bigoplus_{y \in \kappa^{-1}(t)} \mathscr{O}_{Y,y} \to \bigoplus_{x \in \pi^{-1}(t)} \mathscr{O}_{X,x}$$

whose restriction to $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is non-zero iff $y = \varphi(x)$. A similar description holds for $\psi^\#$. It follows that φ and ψ must be equal, first of all as maps of sets, and then clearly as holomorphic maps. □

***Proof that* (2.9.3) *is surjective*.** Choose any $\Phi \in \mathrm{Mor}_{\mathscr{O}_S}(\kappa_*\mathscr{O}_Y, \pi_*\mathscr{O}_X)$. It suffices to show that Φ is locally realized by φ_W, i.e., that each $t \in S$ is contained in a neighborhood $W \subset S$ such that, after shrinking S to W, X to $\pi^{-1}(X)$, and Y to $\kappa^{-1}(X)$, Φ equals $\varphi_W^\#$. Then by the injectivity of (2.9.3), φ_W and $\varphi_{W'}$ agree on $W \cap W'$. So these φ_W can be glued together to realize Φ globally.

To find φ locally, we first assume that κ is a Weierstrass map, which factors as $\kappa : Y \hookrightarrow \mathbb{C}^k \times S \xrightarrow{\mathrm{pr}_S} S$. Consider $z_1, \dots, z_k$ as elements of $\mathscr{O}(\mathbb{C}^k \times S)$ and also of $\mathscr{O}(Y) = (\kappa_*\mathscr{O}_Y)(S)$ by restriction. Let $f_i = \Phi(z_i)$, which is an element of $(\pi_*\mathscr{O}_X)(S) = \mathscr{O}(X)$. Regard $F = (f_1, \dots, f_k)$ as a holomorphic map $X \to \mathbb{C}^k$ (Thm. 1.4.1). Then by Thm. 1.4.8, the holomorphic map $F \vee \mathrm{pr}_S : X \to \mathbb{C}^k \times S$ restricts to a holomorphic $\varphi : X \to Y$. (This is similar to the Proof-Step 2 of Prop. 2.7.9. Note that one needs the commutativity of (2.9.2) to check condition (b) of Thm. 1.4.8!) Then (2.9.1) commutes because $\kappa \circ \varphi = \mathrm{pr}_S \circ (F \vee \pi) = \pi$. Both $\varphi^\#$ and Φ send each $z_i \in (\kappa_*\mathscr{O}_Y)(S)$ to f_i. So $\varphi^\# = \Phi$ because the powers of $z_1, \dots, z_k$ generate the $\mathscr{O}_S$-module $\kappa_*\mathscr{O}_Y$ by Thm. 2.5.4.

Now, in the general case, by Prop. 2.7.9 we may assume S is small enough such that κ factors as

$$\kappa : Y \hookrightarrow Z \xrightarrow{\varpi} S$$

where $\varpi : Z \to S$ is a isomorphic to a Weierstrass map and $Y = \mathrm{Specan}(\mathscr{O}_Z/\mathscr{J})$ is a closed subspace of Z. We have a sequence of morphisms of $\mathscr{O}_S$-algebras

$$\pi_*\mathscr{O}_X \xleftarrow{\Phi} \kappa_*\mathscr{O}_Y \xleftarrow{\iota^\#} \varpi_*\mathscr{O}_Z.$$

By the previous paragraph, there is $\psi \in \mathrm{Mor}_S(X,Z)$ such that $\psi^\# : \varpi_*\mathscr{O}_Z \to \pi_*\mathscr{O}_X$ equals $\Phi \circ \iota^\#$ and hence vanishes on $\varpi_*\mathscr{J}$. Thus, by Prop. 2.4.5, for each $x \in X$, $\psi^\# : \mathscr{O}_{Z,\psi(x)} \to \mathscr{O}_{X,x}$ vanishes on $\mathscr{J}_{\psi(x)}$. So Thm. 1.4.8 tells us that ψ restricts to a holomorphic $\varphi : X \to Y$. Namely $\psi = \iota \circ \varphi$. Clearly $\varphi \in \mathrm{Mor}_S(X,Y)$.

We have $\varphi^\# \circ \iota^\# = \psi^\# = \Phi \circ \iota^\#$. Thus, to show that $\varphi^\# = \Phi$, it suffices to show that $\iota^\# : \varpi_*\mathscr{O}_Z \to \kappa_*\mathscr{O}_Y$ is surjective. This is clear from Prop. 2.4.5 and the fact that Y is a closed subspace of Z. □

The above two proofs together show that $\mathfrak{F}$ is fully faithful.

Proof that $\mathfrak{F}$ is essentially surjective. Given any coherent $\mathscr{O}_S$-algebra $\mathscr{A}$, our goal is to find a finite holomorphic map $\pi : X \to S$ (for some complex space X) such that $\pi_*\mathscr{O}_X$ is equivalent to $\mathscr{A}$ as $\mathscr{O}_S$-algebras.

We first show that the construction of π is local by nature. Suppose that we have an open cover $(S_i)_{i\in I}$ of S such that for each i we have a finite holomorphic $\pi_i : X_i \to S_i$ such that there is an isomorphism of $\mathscr{O}_{S_i}$-algebras

$$\Phi_i : \pi_{i,*}\mathscr{O}_{X_i} \xrightarrow{\simeq} \mathscr{A}|_{S_i}.$$

Write $S_{ij} = S_i \cap S_j$, $X^i_{ij} = \pi_i^{-1}(S_{ij})$, and let $\pi^i_{ij} : X^i_{ij} \to S_{ij}$ be the restriction of π_i. Then by the full-faithfulness of $\mathfrak{F}$, there is a unique isomorphism $\gamma_{j,i} \in \mathrm{Mor}_{S_{ij}}(X^i_{ij}, X^j_{ij})$ such that $\gamma^\#_{j,i} : \pi^j_{ij,*}\mathscr{O}_{X^j_{ij}} \to \pi^i_{ij,*}\mathscr{O}_{X^i_{ij}}$ equals $\Phi_i^{-1}|_{S_{ij}} \circ \Phi_j|_{S_{ij}}$. One checks easily that these $\gamma_{j,i}$ satisfy the cocycle condition so that they can be used as the gluing maps to glue all π_i together and form a desired $\pi : X \to S$.

Let us construct π locally. Choose $t \in S$. Using the methods in the proof of Prop. 2.7.9, one shows that if S is sufficiently small then there exist a Weierstrass map $\kappa : Y \to S$ and $\Phi : \mathrm{Mor}_{\mathscr{O}_S}(\kappa_*\mathscr{O}_Y, \mathscr{A})$ which is surjective as an $\mathscr{O}_S$-module morphism. $\mathcal{T} = \mathrm{Ker}(\Phi)$ is an ideal of $\kappa_*\mathscr{O}_Y$, i.e., an $\mathscr{O}_S$-submodule of $\kappa_*\mathscr{O}_Y$ whose stalks at each $\tau \in S$ is invariant under $(\kappa_*\mathscr{O}_Y)_\tau$. So $\mathcal{T}_\tau = \mathcal{T}_\tau \cdot (\kappa_*\mathscr{O}_Y)_\tau$. Thus, by Prop. 2.4.5, we have an $(\kappa_*\mathscr{O}_Y)_\tau$-module isomorphism

$$\kappa_*(\mathcal{T}\mathscr{O}_Y)_\tau \simeq \bigoplus_{y\in\kappa^{-1}(\tau)} (\mathcal{T}\mathscr{O}_Y)_y = \bigoplus_{y\in\kappa^{-1}(\tau)} \mathcal{T}_\tau\mathscr{O}_{Y,y} \simeq \mathcal{T}_\tau \cdot (\kappa_*\mathscr{O}_Y)_\tau = \mathcal{T}_\tau$$

such that each $\sigma \in \mathcal{T}_\tau$ corresponds to $\sigma \cdot 1$ on the LHS.

$\mathcal{T}\mathscr{O}_Y$ is a finite-type ideal of $\mathscr{O}_Y$ since $\mathcal{T}$ is $\mathscr{O}_S$-coherent by Cor. 2.1.5. Define $X = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{T}\mathscr{O}_Y)$, and let $\pi : X \to S$ be the restriction of κ. This gives the desired finite holomorphic map since, by the exactness of κ_*, we have an $\kappa_*\mathscr{O}_Y$-module isomorphism

$$\pi_*\mathscr{O}_X = \kappa_*(\mathscr{O}_Y/\mathcal{T}\mathscr{O}_Y) \simeq \kappa_*\mathscr{O}_Y/\kappa_*(\mathcal{T}\mathscr{O}_Y) \simeq \kappa_*\mathscr{O}_Y/\mathcal{T} \simeq \mathscr{A}.$$

(These isomorphisms are explicit at the level of stalks.) $\square$

2.10 Nullstellensatz

In this section, we give another application of Finite mapping Thm. 2.7.1 and Thm. 2.7.2: we prove the complex analytic version of Hilbert Nullstellensatz, called Rückert Nullstellensatz in [GR-b] and [GPR]. Nullstellensatz will be used in an essential way to prove that every complex space X has an associated reduced complex space X_{red}, and that if X is reduced at x then X is reduced near x.

2.10.1 Equivalent forms of Nullstellensatz

Theorem 2.10.1 (Nullstellensatz). *Let X be a complex space. If $f \in \mathscr{O}(X)$ satisfies that $f(x) = 0$ for all $x \in X$, then the germ of f at each $x \in X$ is a nilpotent element of $\mathscr{O}_{X,x}$.*

The converse is clearly true: If f is nilpotent at $\mathscr{O}_{X,x}$ for each x, then f a zero continuous function.

Recall that if I is an ideal of a commutative ring A, then its **radical** $\sqrt{I}$ is

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n \in \mathbb{Z}_+\}.$$

Similarly:

Definition 2.10.2. If X is a $\mathbb{C}$-ringed space and $\mathcal{I}$ is an ideal of $\mathscr{O}_X$, then the **radical of $\mathcal{I}$** is the ideal $\sqrt{\mathcal{I}}$ of $\mathscr{O}_X$ defined by

$$\sqrt{\mathcal{I}}(U) = \{f \in \mathscr{O}(U) : f \in \sqrt{\mathcal{I}_x} \text{ for all } x \in U\}.$$

So $\mathcal{I}$ is determined by $(\sqrt{\mathcal{I}})_x = \sqrt{\mathcal{I}_x}$ for all $x \in X$.

Then there is an equivalent way of stating Nullstellensatz:

Theorem 2.10.3 (Nullstellensatz). *Let X be a complex space. Then the kernel of the reduction map* $\mathrm{red}: \mathscr{O}_X \to \mathscr{C}_X$ *(where $\mathscr{C}_X$ is the sheaf of germs of continuous functions) equals $\sqrt{0_X}$, the radical of the zero ideal of $\mathscr{O}_X$.*

We call $\sqrt{0_X}$ the **nilradical** of $\mathscr{O}_X$ (or of X).

Remark 2.10.4. There are some other equivalent statements of **Nullstellensatz**:

1. Let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$. Then $f \in \mathscr{O}(X)$ vanishes on the subset $N(\mathcal{I})$ if and only if $f \in \sqrt{\mathcal{I}}$.

2. Let $\mathscr{O}_{X,x}$ be an analytic local $\mathbb{C}$-algebra, and let I be an ideal. Then $f \in \mathscr{O}_{X,x}$ is an nilpotent element of I if and only if f vanishes on the $\mathrm{Specan}(\mathscr{O}_{X,x}/I)$, the **germ of complex subspace** of X defined by I.

3. If $\mathscr{E}$ is a coherent sheaf on a complex space X. Then $f \in \mathscr{O}(X)$ vanishes on the subset $\mathrm{Supp}(\mathscr{E})$ if and only if for each $x \in X$ there is $n \in \mathbb{Z}_+$ such that $f^n\mathscr{E}_x = 0$.

Proof. 1⇔Thm. 2.10.1: Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then $f \in \mathscr{O}_{X,x}$ belongs to $\sqrt{\mathcal{I}_x}$ iff the residue class of f in $\mathscr{O}_{Y,x} = \mathscr{O}_{X,x}/\mathcal{I}_x$ is nilpotent.

1⇔2: Obvious. 3⇒ 1: Take $\mathscr{E} = \mathscr{O}_X/\mathcal{I}$. 1⇒3: Take $\mathcal{I} = \mathscr{A}nn_{\mathscr{O}_X}(\mathscr{E})$. □

2.10.2 Proof of Nullstellensatz

We start by proving a special case.

Lemma 2.10.5. *Let X be a neighborhood of $0 \in \mathbb{C}^{m+1}$ where $m \in \mathbb{N}$. Let $(z, w, t_2, \dots, t_m)$ be the standard coordinates of $\mathbb{C}^{m+1}$. Let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$ such that*

$$N(\mathcal{I}) \subset \{(z, w, t_\bullet) \in X : z = 0\}.$$

Then (the germ at 0 of) z is an element of $\sqrt{\mathcal{I}_0}$.

Proof. We prove by induction on $m \in \mathbb{N}$. The base case $m = 0$ is elementary and is hence omitted. Assume the lemma holds for $m - 1$ where $m \geqslant 1$. Let us prove it for m. Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$.

We first assume that $\mathcal{I}_0$ contains

$$h(z, w, t_\bullet) = \sum_{n=0}^{\infty} a_n(w, t_\bullet)z^n \tag{2.10.1}$$

where $a_0 \neq 0$. Then as in the proof of Thm. 1.5.5, we may choose a new set of coordinates $(w, t_\bullet)$ for $\mathbb{C}^m$ such that $a_0(w, t_\bullet) = h(0, w, t_\bullet)$ has finite order in w, i.e.

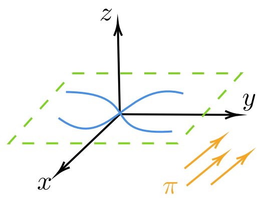

Figure 2.10.1

$a(w,0)$ is non-zero. So $0_{\mathbb{C}^{m+1}}$ is an isolated point of the fiber at $0_{\mathbb{C}^m}$ of the holomorphic map $\pi : Y \to \mathbb{C}^m$ defined by the restriction of $\mathbb{C}^{m+1} \to \mathbb{C}^m$, $(z,w,t_\bullet) \mapsto (z,t_\bullet)$. We shrink X to a neighborhood of 0 so that $0_{\mathbb{C}^{m+1}}$ is the only point of that fiber, and that (by Thm. 2.7.2) $\pi : Y \to V$ is finite where V is a neighborhood of $0 \in \mathbb{C}^m$. See Fig. 2.10.1.

By Finite mapping Thm. 2.7.1, $\pi_*\mathscr{O}_Y$ is a coherent $\mathscr{O}_V$-module. By assumption, the Nullstellensatz holds for any coherent ideal $\mathscr{J}$ of $\mathscr{O}_V$ such that

$$N(\mathscr{J}) \subset \{(z,t_\bullet) \in V : z = 0\}.$$

Choose $\mathscr{J} = \mathscr{A}nn_{\mathscr{O}_V}(\pi_*\mathscr{O}_Y)$. Then the assumption tells us that there is $n \in \mathbb{Z}_+$ such that $z^n \in \mathscr{O}_{\mathbb{C}^m,0}$ kills the stalk $(\pi_*\mathscr{O}_Y)_0 \simeq \mathscr{O}_{Y,0}$ (Prop. 2.4.5). So $\pi^\# z^n$ (or simply z^n as an element of $\mathscr{O}(\mathbb{C}^{m+1})$) kills $\mathscr{O}_{Y,0} = \mathscr{O}_{\mathbb{C}^{m+1},0}/\mathcal{I}_0$. Therefore $z^n \in \mathcal{I}_0$.

Now, in the general case, note that it suffices to prove that z is nilpotent in $z^{-k}\mathcal{I}_0 = \{f \in \mathscr{O}_{\mathbb{C}^{m+1},0} : z^k f \in \mathcal{I}_0\}$ for some $k \in \mathbb{N}$. This statement is true if we can find k and $h \in z^{-k}\mathcal{I}_0$ whose series expansion as in (2.10.1) has non-zero constant term. This follows by choosing a non-zero $g \in \mathcal{I}_0$, letting k be the smallest power of z such that the series expansion of g in z has non-zero coefficient before z^k, and setting $h = z^{-k}g$. $\square$

Proof of Nullstellensatz. Let X be a complex space, and assume that $f \in \mathscr{O}(X)$ vanishes at every $x \in X$. We now fix $x \in X$ and show that f is nilpotent in $\mathscr{O}_{X,x}$. Consider the graph $\mathfrak{G}(f)$ of f, namely the image of the closed embedding $f \vee \mathbf{1} : X \to \mathbb{C} \times X$ (cf. Prop. 1.13.6). Assume X is a small enough neighborhood of x so that X is a closed subspace of an open $U \subset \mathbb{C}^m$ and $x = 0_{\mathbb{C}^m}$. Then $\mathfrak{G}(f)$ is a closed subspace of $\mathbb{C} \times U$.

As a set, $\mathfrak{G}(f)$ is contained in $0 \times U$. Let $z \in \mathscr{O}(\mathbb{C})$ be the standard coordinate of $\mathbb{C}$. Then by Lemma 2.10.5, $z \otimes 1 \in \mathscr{O}_{\mathbb{C}\times U,0\times 0}$ is nilpotent in $\mathscr{O}_{\mathfrak{G}(f),0\times 0}$. But the restriction $f \vee \mathbf{1} : X \to \mathfrak{G}(f)$ is a biholomorphism, and it pulls $z \otimes 1 = \mathrm{pr}_{\mathbb{C}}^\# z$ (where $\mathrm{pr}_{\mathbb{C}} : \mathbb{C} \times U \to \mathbb{C}$ is the projection) back to $z \circ \mathrm{pr}_{\mathbb{C}} \circ (f \vee \mathbf{1}) = z \circ f = f$. So f is nilpotent in $\mathscr{O}_{X,0}$. $\square$

2.A Kernels and cokernels in categories of modules

In a category $\mathcal{C}$, if $\varphi, \psi : X \to Y$ are two morphisms of objects, then the equalizers of double arrow $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$ can be defined in the same way as in Def. 1.8.1. Likewise, a **coequalizer** of this double arrow is an object $C \in \mathcal{C}$ and a morphism $\pi : Y \to C$ such that $\pi \circ \varphi = \pi \circ \psi$, and that for every object T and morphism $\nu : Y \to T$ satisfying $\nu \circ \varphi = \nu \circ \psi$ there is a unique morphism $\widetilde{\nu} : C \to T$ such that $\nu = \widetilde{\nu} \circ \pi$.

$$\begin{tikzcd} X \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & Y \arrow[r, "\pi"] \arrow[dr, "\nu"'] & C \arrow[d, dashed, "\widetilde{\nu}"] \\ & & T \end{tikzcd} \tag{2.A.1}$$

Thus, if a functor (resp. contravariant functor) $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ is an equivalence (resp. antiequivalence) of categories (cf. Thm. 1.6.2 and 2.2.2), then $\mathfrak{F}$ sends the (co)equalizer of a double arrow $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$ (on the $\mathcal{C}$ side) to a (co)equalizer of $\mathfrak{F}(X) \underset{\mathfrak{F}(\psi)}{\overset{\mathfrak{F}(\varphi)}{\rightrightarrows}} \mathfrak{F}(Y)$ (resp. sends equalizers to coequalizers of $\mathfrak{F}(Y) \underset{\mathfrak{F}(\psi)}{\overset{\mathfrak{F}(\varphi)}{\rightrightarrows}} \mathfrak{F}(X)$ and coequalizers to equalizers).

The category of modules of a commutative rings and the one of (coherent) $\mathscr{O}_X$-modules (where X is a $\mathbb{C}$-ringed space) are both additive categories, which means roughly that one can take direct sums, that the morphism spaces are abelian groups, and that there is a zero object. A functor between additive functions, called an **additive functor**, is assumed to preserve the abelian group structures of the morphism spaces.

Moreover, the above categories are **abelian categories**. This means that the kernel of a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ (which is an object together with the "inclusion" morphism) is equivalent to the equalizer of $\mathcal{M} \underset{0}{\overset{\varphi}{\rightrightarrows}} \mathcal{N}$ and that the cokernel is equivalent to the coequalizer of this double arrow. From this, it is clear that if a functor (resp. contravariant functor) $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ is an equivalence (resp. antiequivalence) of abelian categories, then *$\mathfrak{F}$ must be an exact functor*, because $\mathfrak{F}$ commutes with kernels and cokernels (resp. sends kernels to cokernels and cokernels to kernels).

We refer the readers to [Vak17, Chapter 1] for a more detailed introduction to abelian categories.

Chapter 3

Dimensions and local geometry of complex spaces

3.1 Prime decomposition

We fix a commutative ring $\mathcal{A}$. Recall that $\mathcal{A}$ is called **reduced** if $\mathcal{A}$ has no non-zero nilpotent elements. This is equivalent to saying that $\{0\} = \sqrt{\{0\}}$. If I is an ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is reduced iff $\sqrt{I} = I$.

Remark 3.1.1. Recall the general fact that for any ideals $I_1, \ldots, I_k$ of $\mathcal{A}$ we have

$$\sqrt{I_1 \cdots I_k} = \sqrt{I_1 \cap \cdots \cap I_k} = \sqrt{I_1} \cap \cdots \cap \sqrt{I_k}. \tag{3.1.1}$$

In view of Nullstellensatz, the first equality says that "the zero sets defined by $I_1 \cdots I_k$ and defined by $I_1 \cap \cdots \cap I_k$ are equal" (namely, they are equal to the union of the zero sets of $I_1, \ldots, I_k$). The second equality implies that if $I_i = \sqrt{I_i}$ for each i, then $I_1 \cap \cdots \cap I_k$ is its own radical.

Proof. The two equalities in (3.1.1) are clearly $\subset$. If $f \in \cap_i \sqrt{I_i}$ then $f^{n_i} \in I_i$ for some $n_i \in \mathbb{Z}_+$. Then $f^{n_1+\cdots+n_k} \in I_1 \cdots I_k$, and hence $f \in \sqrt{I_1 \cdots I_k}$. This proves (3.1.1). $\square$

Proposition 3.1.2. *Let $\mathfrak{p} \subsetneq \mathcal{A}$ be an ideal.*[1] *Then the following are equivalent.*

(a) $\mathfrak{p}$ is a prime ideal. Equivalently, $\mathcal{A}/\mathfrak{p}$ is an integral domain.

(b) $\mathfrak{p} = \sqrt{\mathfrak{p}}$. Moreover, if $\mathfrak{p} = I_1 \cap I_2$ where I_1, I_2 are ideals of $\mathcal{A}$, then $I_1 = \mathfrak{p}$ or $I_2 = \mathfrak{p}$.

(c) $\mathfrak{p} = \sqrt{\mathfrak{p}}$. Moreover, if $\mathfrak{p} = I_1 \cap I_2$ where I_1, I_2 are ideals of $\mathcal{A}$ satisfying $I_1 = \sqrt{I_1}$ and $I_2 = \sqrt{I_2}$, then $I_1 = \mathfrak{p}$ or $I_2 = \mathfrak{p}$.

[1] Recall that a prime ideal $\mathfrak{p}$ is required not to be the whole ring $\mathcal{A}$.

We leave it to the readers to figure out the geometric meaning of this lemma (in the case that $\mathcal{A}$ is an analytic $\mathbb{C}$-algebra).

Proof. By replacing $\mathcal{A}$ by $\mathcal{A}/\mathfrak{p}$, we may assume $\mathfrak{p} = \{0\}$. Clearly (b)$\Rightarrow$(c).

(a)$\Rightarrow$(b): Assume $\{0\}$ is prime. Then clearly $\{0\} = \sqrt{\{0\}}$. Suppose $\{0\} = I_1 \cap I_2$ and $I_1, I_2 \neq \{0\}$. Then we may choose non-zero $f_i \in I_i$. And $f_1 f_2 \in I_1 \cdot I_2 \subset I_1 \cap I_2 = \{0\}$. So $f_1 f_2 = 0$, contradicting that $\{0\}$ is prime. So (b) follows.

(c)$\Rightarrow$(a). Assume (c). Suppose that there are non-zero $f, g \in \mathcal{A}$ such that $fg \in \{0\}$, i.e. $fg = 0$. Then as $\mathcal{A}$ is reduced, $\{0\} = \sqrt{\{0\}} = \sqrt{f\mathcal{A} \cdot g\mathcal{A}} = \sqrt{f\mathcal{A}} \cap \sqrt{g\mathcal{A}}$. This contradicts (c). $\square$

Theorem 3.1.3. *If $\mathcal{A}$ is Noetherian and reduced, then there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ of $\mathcal{A}$ such that*

$$\{0\} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N \tag{3.1.2}$$

and that for each $1 \leqslant i \leqslant N$,

$$\{0\} \neq \bigcap_{j \neq i} \mathfrak{p}_j. \tag{3.1.3}$$

*Moreover the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ satisfying (3.1.2) and (3.1.3) are unique. We call this unique decomposition the **prime decomposition of** $\{0\} \subset \mathcal{A}$.*

The geometric meaning of (3.1.2) is that an element $f \in \mathcal{A}$ is zero iff f restricts to zero on $\mathcal{A}/\mathfrak{p}_i$ (i.e. "f vanishes on the zero set $N(\mathfrak{p}_i)$") for all i.

Note that if $\mathfrak{a} = \sqrt{\mathfrak{a}}$ is an ideal of a Noetherian ring $\mathcal{A}$, then Thm. 3.1.3 applied to $\mathcal{A}/\mathfrak{a}$ says that there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ of $\mathcal{A}$ such that

$$\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N \tag{3.1.4a}$$

$$\mathfrak{a} \neq \bigcap_{j \neq i} \mathfrak{p}_j \qquad \forall 1 \leqslant i \leqslant N \tag{3.1.4b}$$

called the **prime decomposition of** $\mathfrak{a}$.

Proof of the existence. We first note that if we can find prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ satisfying (3.1.2), then by discarding some members of these ideals so that the intersection of the remaining ones is still $\{0\}$ until we cannot do this anymore, (3.1.3) is automatically satisfied. So we only need to find prime ideals satisfying (3.1.2).

Let $\mathfrak{A}$ be the set of all ideals $\mathfrak{a}$ not equal to $\mathcal{A}$ such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$ and that $\mathfrak{a} \subset \mathcal{A}$ has no prime decomposition (equivalently, $\mathfrak{a}$ is not a finite intersection of prime ideals). Note that if $\mathfrak{a} \in \mathfrak{A}$, then $\mathfrak{a} = \sqrt{\mathfrak{a}}$ and $\mathfrak{a}$ is not prime. So by Prop. 3.1.2, $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ where the ideals $\mathfrak{b}, \mathfrak{c}$ are not $\mathfrak{a}$ and are the radicals of themselves. One of $\mathfrak{b}, \mathfrak{c}$ is not a finite intersection of prime ideals, otherwise $\mathfrak{a}$ is a finite intersection of prime ideals. So one of $\mathfrak{b}, \mathfrak{c}$ is in $\mathfrak{A}$.

The above argument shows that if $\mathfrak{a}_1 = \{0\}$ belongs to $\mathfrak{A}$, then we can construct a strictly increasing infinite chain of elements of $\mathfrak{A}$: $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \cdots$, contradicting that $\mathcal{A}$ is Noetherian. So $\{0\} \notin \mathfrak{A}$. $\square$

Remark 3.1.4. In Thm. 3.1.3, (3.1.2) and (3.1.3) imply that

$$\bigcap_{j\neq i}\mathfrak{p}_j \Big\backslash \mathfrak{p}_i \neq \varnothing.$$

This means that we can find $f \in \mathcal{A}$ which is non-zero when restricted to $\mathcal{A}/\mathfrak{p}_i$ (i.e. "non-zero on $N(\mathfrak{p}_i)$") and zero in the other $\mathcal{A}/\mathfrak{p}_j$. Thus, by taking linear combinations, we see that there always exists $f \in \mathcal{A}$ which is non-zero precisely when restricted to the given ones of $\mathcal{A}/\mathfrak{p}_1, \ldots, \mathcal{A}/\mathfrak{p}_N$.

We remark that when $\mathcal{A}$ is not necessarily reduced, there is a generalization called **primary decomposition**, cf. [AM]. We will not use this notion in out notes.

To prove the uniqueness part of Thm. 3.1.3 we first need:

Lemma 3.1.5. *In Thm. 3.1.3, for each $f \in \mathcal{A}$, the annihilator $\mathrm{Ann}_{\mathcal{A}}(f)$ equals*

$$\mathrm{Ann}_{\mathcal{A}}(f) = \bigcap_{\substack{1\leqslant i\leqslant N\\ f\notin\mathfrak{p}_i}} \mathfrak{p}_i \tag{3.1.5}$$

Recall that $\mathrm{Ann}_{\mathcal{A}}(f) = \mathrm{Ann}_{\mathcal{A}}(f\mathcal{A})$ is the ideal of all $a \in \mathcal{A}$ satisfying $af = 0$ (Def. 2.3.1). Then (3.1.5) says that $af = 0$ iff a "vanishes on all $N(\mathfrak{p}_i)$ where f is non-zero on $N(\mathfrak{p}_i)$".

Proof. Suppose $a \in \mathcal{A}$ and $af = 0$. Then af restricts to 0 on the integral domain $\mathcal{A}/\mathfrak{p}_i$. If $f \notin \mathfrak{p}_i$ then f is nonzero in $\mathcal{A}/\mathfrak{p}_i$. So a is 0 in $\mathcal{A}/\mathfrak{p}_i$. Hence $a \in \mathfrak{p}_i$. Conversely, if $a \in \mathfrak{p}_i$ for all i such that $f \notin \mathfrak{p}_i$, then af belongs to $\mathfrak{p}_i$ for all $1 \leqslant i \leqslant N$. So $af \in \cap_i \mathfrak{p}_i = \{0\}$. $\square$

Note that f is a non zero-divisor iff $\mathrm{Ann}_{\mathcal{A}}(f) = \{0\}$. Therefore:

Corollary 3.1.6. *In Thm. 3.1.3, $f \in \mathcal{A}$ is a non zero-divisor if and only if $f \notin \mathfrak{p}_i$ for all $1 \leqslant i \leqslant N$.*

Now the uniqueness of prime decomposition follows immediately from the following fact:

Proposition 3.1.7. *In Thm. 3.1.3, $\mathfrak{p}_1, \ldots, \mathfrak{p}_N$ are precisely the **associated primes** of $\mathcal{A}$, i.e. prime ideals of the form $\mathrm{Ann}_{\mathcal{A}}(f)$ for some $f \in \mathcal{A}$.*

Proof. We first note that an intersection of more than one members of $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ is not prime. This together with Lemma 3.1.5 would imply that $\mathrm{Ann}_{\mathcal{A}}(f)$ is prime only if $\mathrm{Ann}_{\mathcal{A}}(f) = \mathfrak{p}_i$ for some i, and hence that the associated primes are among $\mathfrak{p}_1, \dots, \mathfrak{p}_N$. To prove the claim, consider for instance $\mathfrak{p} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$ where $k > 1$. Suppose $\mathfrak{p}$ is prime. Then by Prop. 3.1.2, either $\mathfrak{p}_1$ or $\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$ equals $\mathfrak{p}$, contradicting (3.1.3). So $\mathfrak{p}$ cannot be prime.

For each i, by Rem. 3.1.4 we can choose $f \in \mathcal{A}$ non-zero on $\mathcal{A}/\mathfrak{p}_i$ but zero on $\mathcal{A}/\mathfrak{p}_j$ whenever $j \neq i$. Then $\mathfrak{p}_i = \mathrm{Ann}_{\mathcal{A}}(f)$ by Lemma 3.1.5, which shows that $\mathfrak{p}_i$ must be an associated prime. $\square$

3.2 Reduction $\mathrm{red}(X)$ and coherence of $\sqrt{\mathcal{I}}$

In this section we study the reduction of complex spaces. The main results Thm. 3.2.1 and equivalently Thm. 3.2.2 are originally due to Oka and H. Cartan. Some key ingredients of the proof are prime decomposition, Nullstellensatz, and the ranks of Jacobian matrices (which are a guise for embedding dimensions to be studied later). Our approach follows [GPR].

3.2.1 Main results and consequences

Theorem 3.2.1. *Let X be a complex space reduced at a point x. There X is reduced on a neighborhood U of x.*

This theorem is equivalent to:

Theorem 3.2.2. *Let X be a complex space. Then for each coherent ideal $\mathcal{I}$ of $\mathscr{O}_X$, its radical $\sqrt{\mathcal{I}}$ is coherent.*

Remark 3.2.3. Note that Thm. 3.2.2 is equivalent to the seemingly special case that for each complex space X, $\sqrt{0_X}$ is coherent. Indeed, if this special case is true, let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then $\sqrt{0_Y}$ is (or more precisely, $\iota_{Y,X,*}\sqrt{0_Y}$ is)

$$\sqrt{0_Y} = \sqrt{\mathcal{I}/\mathcal{I}} = \sqrt{\mathcal{I}}/\mathcal{I}. \tag{3.2.1}$$

So $\sqrt{\mathcal{I}}/\mathcal{I}$ is coherent, and hence $\sqrt{\mathcal{I}}$ is coherent. Therefore Thm. 3.2.2 holds.

Proof that Thm. 3.2.1 and 3.2.2 are equivalent. Assume Thm. 3.2.2. Assume X is reduced at x. Then $\sqrt{\mathcal{I}}$ is coherent and its stalk at x is 0. So its stalk is zero on a neighborhood U of x. Then X is reduced everywhere on U.

Assume Thm. 3.2.1. Choose any complex space X and coherent ideal $\mathcal{I}$. Choose $x \in X$. Since $\mathscr{O}_{X,x}$ is Noetherian, $\sqrt{\mathcal{I}_x}$ is generated by finitely many elements $f_1, f_2, \dots$. By shrinking X to a neighborhood of x, we assume $f_1, f_2, \dots \in$

$\sqrt{\mathcal{I}}(X)$. Let $\mathcal{J}$ be the ideal generated by $f_1, f_2, \dots$. Then $\mathcal{J} \subset \sqrt{\mathcal{I}}$ and $\mathcal{J}_x = \sqrt{\mathcal{I}}_x$. This implies that $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{J})$ is reduced at x (since $\sqrt{0_{Y,x}} = \sqrt{\mathcal{J}_x}/\mathcal{J}_x$).

$\mathcal{J}_x = \sqrt{\mathcal{I}}_x$ also implies $\mathcal{I}_x \subset \mathcal{J}_x$. Therefore, since $\mathcal{I}$ is coherent, by Rem. 1.2.16 we may shrink X so that $\mathcal{I} \subset \mathcal{J}$. We conclude that

$$\mathcal{I} \subset \mathcal{J} \subset \sqrt{\mathcal{I}} \subset \sqrt{\mathcal{J}}.$$

By Thm. 3.2.1, we may shrink X so that Y is reduced everywhere on X. This means $\mathcal{J} = \sqrt{\mathcal{J}}$, which proves that $\sqrt{\mathcal{I}}$ equals $\mathcal{J}$ and is therefore coherent. □

Corollary 3.2.4. *Let X be a complex space. Then for each analytic subset A of X, the **ideal associated to** A defined by*

$$\mathscr{I}_A(U) = \{f \in \mathscr{O}_X(U) : f(x) = 0 \quad \forall x \in A \cap U\} \tag{3.2.2}$$

(for all open $U \subset X$) is coherent.

Proof. If $A = N(\mathcal{I})$ for some coherent ideal $\mathcal{I}$ then

$$\mathscr{I}_A = \sqrt{\mathcal{I}}. \tag{3.2.3}$$

□

Remark 3.2.5. Let X be a reduced complex space. By Nullstellensatz, we have a bijection

$$\begin{gathered}\{\text{Analytic subsets of } X\} \xleftrightarrow{\simeq} \{\text{Coherent ideals } \mathcal{I} \subset \mathscr{O}_X \text{ satisfying } \mathcal{I} = \sqrt{\mathcal{I}}\} \\ A \mapsto \mathscr{I}_A \qquad N(\mathcal{I}) \leftarrow \mathcal{I}\end{gathered} \tag{3.2.4}$$

If A, B are analytic subsets of X then clearly

$$A \subset B \quad \Longleftrightarrow \quad \mathscr{I}_A \supset \mathscr{I}_B$$

$A \cap B$ and $A \cup B$ are both analytic subsets of X, and we indeed have

$$\mathscr{I}_{A\cap B} = \sqrt{\mathscr{I}_A + \mathscr{I}_B} \qquad \mathscr{I}_{A\cup B} = \mathscr{I}_A \cap \mathscr{I}_B = \sqrt{\mathscr{I}_A \cdot \mathscr{I}_B} \tag{3.2.5}$$

Proof. It is clear that the coherent ideals (cf. Cor. 2.1.6 for the coherence) $\mathscr{I}_A + \mathscr{I}_B$ has zero set $A \cap B$ and $\mathscr{I}_A \cdot \mathscr{I}_B$ has zero set $A \cup B$. And $\sqrt{\mathscr{I}_A \cdot \mathscr{I}_B} = \mathscr{I}_A \cap \mathscr{I}_B$ by Rem. 3.1.1. □

Remark 3.2.6. We often identify an analytic subset A with the corresponding reduced complex subspace $\mathrm{Specan}(\mathscr{O}_X/\mathscr{I}_A)$. In that case "analytic subsets" and "reduced complex subspaces" are synonymous. But there is one exception. The intersection of analytic subsets $A \cap B$ is usually not the intersection of two (reduced) complex spaces (as defined in Exp. 1.12.4): In the former case $A \cap B$ is determined by the ideal $\mathscr{I}_{A\cap B} = \sqrt{\mathscr{I}_A + \mathscr{I}_B}$ and the latter case $\mathscr{I}_A + \mathscr{I}_B$. *So we will make distinctions between analytic subsets and reduced complex subspaces when taking intersections.*

There is no such a problem when taking unions: We haven't defined unions for closed complex subspaces, since both $\mathcal{I}_1 \cap \mathcal{I}_2$ and $\mathcal{I}_1 \cdot \mathcal{I}_2$ are reasonable ideals for defining the union. Certainly, for analytic subspaces, $\mathscr{I}_{A\cap B}$ is the correct ideal defining the union. □

Corollary 3.2.7. *Let X be a complex space. Then the set of all non-reduced points of X is an analytic subset of X.*

Proof. $x \in X$ is not reduced iff $x \in \mathrm{Supp}(\sqrt{0_X})$. □

Corollary 3.2.8. *Let A be a subset of a complex space X. Then the following are equivalent:*

(1) A is an analytic subset of X. (Recall this means that $A = N(\mathcal{I})$ for a coherent ideal $\mathcal{I} \subset \mathscr{O}_X$.)

(2) Each $x \in X$ is contained in a neighborhood U such that $A \cap U$ is analytic in U.

Therefore A is analytic iff each $x \in X$ is contained in a neighborhood U such that $A \cap U$ is the zero set of finitely many elements of $\mathscr{O}(U)$.

Proof. Clearly (1)⇒(2). Assume (2). Let $\mathscr{I}_A$ be defined by (3.2.2). For each $x \in X$ there is a neighborhood U of x such that $A \cap U$ is analytic, i.e. $A \cap U = N(\mathcal{I}_U)$ for a coherent ideal $\mathcal{I}_U$ of $\mathscr{O}_U$. Then $\mathscr{I}_A|_U$ equals $\mathscr{I}_{A\cap U} = \sqrt{\mathcal{I}_U}$ which is coherent. Therefore $\mathscr{I}_A$ is coherent. We have $N(\mathscr{I}_A) = A$ since $N(\mathscr{I}_A) \cap U = N(\mathscr{I}_{A\cap U}) = A \cap U$. So A is analytic. □

Definition 3.2.9. Let X be a complex space. Then the reduced space

$$\mathrm{red}(X) = \mathrm{Specan}(\mathscr{O}_X/\sqrt{0_X})$$

is called the **reduction** of X.

3.2.2 Proof of Thm. 3.2.1

Definition 3.2.10. We say that a complex space X is **irreducible at** x if $\mathscr{O}_{X,x}$ is an integral domain. (Note that if X is irreducible at x then X is reduced at x.) We say that X is **locally irreducible** if X is irreducible at every point of X. If X is not irreducible at x, we say that X is **reducible at** x. (Note that "reducible"≠"reduced"!)

Lemma 3.2.11. *Suppose that Thm. 3.2.1 holds whenever X is irreducible at x. Then Thm. 3.2.1 holds in general.*

Proof. Assume $\mathscr{O}_{X,x}$ is reduced. Apply prime decomposition (Thm. 3.1.3) to $A = \mathscr{O}_{X,x}$ to get $\{0\} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$. By shrinking X to a neighborhood of x we assume each $\mathfrak{p}_i$ is the stalk $\mathscr{P}_{i,x}$ of a coherent ideal $\mathscr{P}_i$ of $\mathscr{O}_X$. Let $Y_i = \mathrm{Specan}(\mathscr{O}_X/\mathscr{P}_i)$. Then Y_i is irreducible at x. Since $\bigcap_{i=1}^N \mathscr{P}_i$ is $\mathscr{O}_X$-coherent (Cor. 2.1.6), we may shrink X so that $\bigcap_i \mathscr{P}_{i,y} = \{0\}$ for all $y \in X$.

By assumption, we can shrink X further so that each Y_i is reduced everywhere. This means that for each $y \in X$ we have $\mathscr{P}_{i,y} = \sqrt{\mathscr{P}_{i,y}}$. Therefore by Rem. 3.1.1, the zero ideal of $\mathscr{O}_{X,y}$ is its own radical. So $\mathscr{O}_{X,y}$ is reduced. □

Lemma 3.2.12. *Let X be a model space irreducible at $0 \in X$. Then after shrinking X to a neighborhood of 0, there exists $\Delta \in \mathscr{O}(X)$ whose germ at 0 is non-zero such that X is smooth outside $N(\Delta)$.*

Proof of Thm. 3.2.1. By Lemma 3.2.11, it suffices to assume that X is a complex model space irreducible (and hence reduced) at 0. Assume that the statement in Lemma 3.2.12 holds. Since Δ is non-zero in the integral domain $\mathscr{O}_{X,0}$, Δ is a non zero-divisor of $\mathscr{O}_{X,0}$. Therefore, by Prop. 2.3.13, we may shrink X to a neighborhood of 0 so that Δ is a non zero-divisor of $\mathscr{O}_{X,x}$ for all $x \in X$.

Choose any open $V \subset X$ and $f \in \sqrt{0_X}(V)$. Since $X \backslash N(\Delta)$ is a complex manifold, $\sqrt{0_{X\backslash N(\Delta)}} = 0$. So the support of f, or more precisely $\mathrm{Supp}(f\mathscr{O}_V)$, is inside $N(\Delta)$. So Δ vanishes on $\mathrm{Supp}(f\mathscr{O}_V)$. Therefore, by Nullstellensatz (Rem. 2.10.4-3), for each $x \in V$ there is $n \in \mathbb{N}$ such that $f\Delta^n = 0$ in $\mathscr{O}_{X,x}$. This proves $f = 0$ in $\mathscr{O}_{X,x}$ because Δ is a non zero-divisor. Therefore $\sqrt{0_X} = 0$. □

The proof of Lemma 3.2.12 is postponed to Sec. 3.5. We first give a preliminary lemma which will aid in the proof of Lemma 3.2.12.

Lemma 3.2.13. *Let $(w_1, \dots, w_m, z_1, \dots, z_n)$ be the standard coordinates of $\mathbb{C}^{m+n}$. Let I be an ideal of $\mathscr{A} = \mathscr{O}_{\mathbb{C}^{m+n},0}$ such that $I \neq \mathscr{A}$. Then the following are equivalent.*

(1) $I \subset w_1\mathscr{A} + \cdots + w_m\mathscr{A}$.

(2) $\partial_{z_j} I \subset I$ for all $1 \leqslant j \leqslant n$.

Proof. By taking power series expansions one sees immediately (1)⇒(2). Now let us assume (2) and prove (1). Note that $I \neq \mathscr{A}$ means that all elements of I vanish at 0. Now (2) implies that all higher partial derivatives over $z_1, \dots, z_n$ of $f \in I$ are in I, and hence vanish at 0. Therefore the restriction of f to $0_{\mathbb{C}^m} \times \mathbb{C}^n$ must be constantly zero, since its power series expansion at 0 is zero. But the ideal of elements of $\mathscr{A}$ vanishing on $0 \times \mathbb{C}^n$ is precisely $w_1\mathscr{A} + \cdots + w_m\mathscr{A}$. This proves (1). □

3.3 Local decomposition of reduced complex spaces

3.3.1 Germs of analytic subsets and ideals

Fix a complex space X. Suppose that X is reduced and $x \in X$. Then similar to Rem. 3.2.5, we have a bijection $A \mapsto I_A, N(I) \leftarrow I$:

(1) Germs of analytic subsets of X at x.

(2) Ideals $I \subset \mathscr{O}_{X,x}$ satisfying $I = \sqrt{I}$.

Indeed, (1) are precisely the germs of closed reduced complex subspaces of X passing through x, and (2) are precisely the germs of coherent ideals $\mathcal{I} \subset \mathscr{O}_X$ at x satisfying $\mathcal{I} = \sqrt{\mathcal{I}}$ (cf. Thm. 2.2.2).

Remark 3.3.1. To be more explicit, if a germ A in (1) is represented by an analytic subset A closed in a neighborhood U of x, then the stalk at x of $\mathscr{I}_A = \{f \in \mathscr{O}_U : f(y) = 0, \forall y \in A\}$ gives the corresponding ideal I_A in (2). Conversely, given an ideal I in (2) which is finitely generated because $\mathscr{O}_{X,x}$ is Noetherian, let $f_1, \dots, f_k \in I$ generate I, and choose a neighborhood $U \subset X$ of x such that $f_1, \dots, f_k \in \mathscr{O}_X(U)$. Then the germ at x of $N(f_1\mathscr{O}_U + \cdots + f_k\mathscr{O}_U)$ gives the germ $N(I)$ in (1).

Remark 3.3.2. We list some easy but useful facts about this correspondence. Let (X, x) be a germ of reduced complex space.

- $I_{A\cup B} = I_A \cap I_B = \sqrt{I_A \cdot I_B}$.
- By Prop. 3.1.2-(c), $\mathscr{O}_{X,x}$ is an integral domain if and only if (X, x) is an irreducible germ, namely if $(X, x) = (A, x) \cup (B, x)$ where $(A, x), (B, x)$ are germs of analytic subsets then $(A, x) = (X, x)$ or $(B, x) = (X, x)$.
  - More precisely, $\mathscr{O}_{X,x}$ is an integral domain iff for every neighborhood U of x written as $U = A \cup B$ where A, B are analytic subsets of U, one of A and B contains a neighborhood of $x \in X$.

3.3.2 Local decomposition

Theorem 3.3.3. *Let X be a reduced complex space and $x \in X$. Then after shrinking X to a neighborhood of x, we have*

$$X = X_1 \cup \cdots \cup X_N \tag{3.3.1}$$

where each X_i is an analytic subset of X which is irreducible at x, and for each $1 \leqslant i \leqslant N$,

$$\bigcup_{j\neq i} X_j \quad \text{contains no neighborhoods of } x \in X. \tag{3.3.2}$$

Such decomposition of X is unique up to shrinking X to smaller neighborhoods of x. We call it the ***local decomposition of X at*** *x. Moreover, we have*

$$\{0\} = \mathscr{I}_{X_1,x} \cap \cdots \cap \mathscr{I}_{X_N,x} \tag{3.3.3}$$

which gives the prime decomposition of $\{0\} \subset \mathscr{O}_{X,x}$.

Note that (assuming (3.3.1) then) (3.3.2) is equivalent to saying that

$$X \Big\backslash \bigcup_{j \neq i} X_j = X_i \Big\backslash \bigcup_{j \neq i} X_j \quad \text{intersects every neighborhood of } x \in X. \tag{3.3.4}$$

Proof. Uniqueness: Every local decomposition (3.3.1) clearly gives a prime decomposition (3.3.3), where the condition $\bigcap_{j \neq i} \mathscr{I}_{X_j,x} \neq 0$ corresponds precisely to (3.3.2). The uniqueness of prime decomposition implies the uniqueness of local decomposition.

Existence: Let $\{0\} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$ be the prime decomposition of $\{0\} \subset \mathscr{O}_{X,x}$. By shrinking X, for each i we may find a coherent ideal $\mathscr{P}_i$ whose stalk at x is $\mathfrak{p}_i$. Since $\mathscr{P}_1 \cap \cdots \cap \mathscr{P}_N$ is coherent (Cor. 2.1.6), we can shrink X further so that $\mathscr{P}_1 \cap \cdots \cap \mathscr{P}_N = 0_X$. So by Rem. 3.1.1,

$$X = N(0_X) = N(\mathscr{P}_1 \cap \cdots \cap \mathscr{P}_N) = N(\mathscr{P}_1 \cdots \mathscr{P}_N) = X_1 \cup \cdots \cup X_N.$$

This gives a local decomposition. □

Property (3.3.2) can be upgraded to the following form:

Theorem 3.3.4. *Let $X = X_1 \cup \cdots \cup X_N$ be a local decomposition of a reduced complex space X at x. Then after shrinking X to a neighborhood of x, for each $i \neq j$,*

$$X_i \cap X_j \ \text{ is nowhere dense in } X_i \tag{3.3.5}$$

In that case, X is reducible at each point of $X_i \cap X_j$ where $i \neq j$.

Note that (3.3.5) implies, for instance, that if $1 \leqslant k < N$ then $(X_1 \cup \cdots \cup X_k) \cap (X_{k+1} \cup \cdots \cup X_N)$ is nowhere dense in every X_i. Hence it is nowhere dense in any union of subclass of $X_1, \ldots, X_N$.

We will prove Thm. 3.3.4 in Sec. 3.4. Note that (cf. Rem. 3.2.6) here $X_i \cap X_j$ means set-theoretic intersection (i.e. intersection of analytic subsets), not intersection of complex spaces. But this is not really a big issue here; we are just reminding the readers of the conventions we set before.

It is easy to see that if $X_1, \ldots, X_N$ are irreducible at x and if (3.3.5) is satisfied for all $i \neq j$, then (3.3.2) is satisfied, and hence $X = X_1 \cup \cdots \cup X_N$ is the unique local decomposition of X at x. This observation can be generalized:

Proposition 3.3.5. *Let $X = X_1 \cup \cdots \cup X_N$ be a decomposition of reduced complex space X into analytic subsets. Choose $x \in X_1 \cap \cdots \cap X_N$. Assume X is small enough such that for each $1 \leqslant i \leqslant N$, X_i has a local decomposition*

$$X_i = X_{i,1} \cup X_{i,2} \cup \cdots$$

at x. Assume that (3.3.5) holds for all $1 \leqslant i \neq j \leqslant N$. Then

$$X = \bigcup_{i,k} X_{i,k}$$

is the local decomposition of X at x.

Proof. It suffices to show that, after shrinking X to a neighborhood of x, $X_{i,k} \cap X_{j,l}$ is nowhere dense in $X_{i,k}$ if $(i,k) \neq (j,l)$. By Thm. 3.3.4, we may shrink X so that this is true whenever $i = j$. So let us assume $i \neq j$. Suppose that $X_{i,k} \cap X_{j,l}$ contains a non-empty open subset $U \subset X_{i,k}$. Since $X_{i,k} \cap (\bigcup_{k' \neq k} X_{i,k'})$ is nowhere dense in $X_{i,k}$, it does not contain U. Thus U intersects $X_{i,k} \backslash \bigcup_{k' \neq k} X_{i,k'}$, where the latter equals $X_i \backslash \bigcup_{k' \neq k} X_{i,k'}$ and is hence an open subset of X_i. This shows that $X_{i,k} \cap X_{j,l}$ contains an open subset of X_i, contradicting the fact that $X_i \cap X_j$ is nowhere dense in X_i. □

3.4 Non zero-divisors and nowhere dense analytic subsets

As an application of local decomposition, we give a useful method for showing that an analytic subset is nowhere dense:

Proposition 3.4.1. *Let X be a reduced complex space and $x \in X$. Choose $f \in \mathscr{O}(X)$. Then the following are equivalent.*

(1) f is a non zero-divisor of $\mathscr{O}_{X,x}$.

(2) There is a neighborhood $U \subset X$ of x such that $N(f) \cap U$ is nowhere dense in U.

Proof. Assume (1) is true. Then by Prop. 2.3.13, after shrinking X to a neighborhood of x, f is a non-zero divisor of $\mathscr{O}_{X,p}$ for all $p \in X$. If $N(f)$ contains an open subset V of X, then f takes value zero everywhere on V. So $f|_V = 0$ because X is reduced, contradicting the fact that f is a non zero-divisor of $\mathscr{O}_{V,p}$ when $p \in V$. So (2) must be true.

Assume that (1) is not true. By shrinking X, we may find a local decomposition $X = X_1 \cup \cdots \cup X_N$ at x. By Cor. 3.1.6, the germ of f at x belongs to $\mathscr{I}_{X_i,x}$ for some i. Shrink X so that $f \in \mathscr{I}_{X_i}(X)$. Then f vanishes on X_i. Thus, by (3.3.4), $N(f) \supset X_i$ contains the non-empty open subset $X \backslash \bigcup_{j \neq i} X_j$ of X. So (2) is not true. □

In the case that X is irreducible at x, Condition-(2) of Prop. 3.4.1 is equivalent to the seemingly weaker statement that $N(f)$ contains no neighborhoods of $x \in X$:

Corollary 3.4.2. *Let X be a complex space irreducible at x. If A is an analytic subset of X containing no neighborhoods of $x \in X$, then there is a neighborhood U of $x \in X$ such that $A \cap U$ is nowhere dense in U.*

Proof. Since A contains no neighborhoods of $x \in X$, the germs (A, x) and (X, x) are not equal. So $\mathscr{I}_{A,x}$ is not equal to $\mathscr{I}_{X,x} = 0$. Thus, after shrinking X to a neighborhood of x, we may find $f \in \mathscr{I}_A(X)$ non-zero in the integral domain $\mathscr{O}_{X,x}$. By Prop. 3.4.1, we may shrink X further so that $N(f)$ is nowhere dense in X. So $A \subset N(f)$ is nowhere dense. □

Remark 3.4.3. Prop. 3.4.1 can be used in the following way.

- Suppose A is an analytic subset of a reduced space X. To show that A is nowhere dense, it suffices to prove that for each $x \in A$ there is a non zero-divisor $f \in \mathscr{O}_{X,x}$ vanishing on $A \cap U$ for a neighborhood U of x. Then after shrinking U, $N(f) \cap U$ is nowhere dense. So its subset $A \cap U$ is nowhere dense.

Actually, if A is expected to be nowhere dense, then one must be able to find such f due to the following generalization of Prop. 3.4.1:

Proposition 3.4.4. *Let X be a reduced complex space and $\mathcal{I}$ a coherent ideal of $\mathscr{O}_X$. Let $A = N(\mathcal{I})$. The following are equivalent.*

(1) A is nowhere dense in X.

(2) For each $x \in X$, $\mathcal{I}_x$ contains a non zero-divisor of $\mathscr{O}_{X,x}$.

Another description of nowhere dense analytic subsets is given by Ritt's lemma 3.10.6.

Proof. (2)⇒(1) is already explained in Rem. 3.4.3. Let us prove (1)⇒(2).

Assume that A is nowhere dense. By shrinking X to a neighborhood of x we may find a local decomposition $X = X_1 \cup \cdots \cup X_N$ at x. For each i, we have $(X_i, x) \not\subset (A, x)$, namely, we cannot find any neighborhood $U \subset X$ of x such that $X_i \cap U \subset A \cap U$: Otherwise, by (3.3.4), X_i contains an open subset (namely $X_i \backslash \bigcap_{j \neq i} X_j$) which intersects U, contradicting the fact that A is nowhere dense.

Therefore, we have $\mathscr{I}_{A,x} \not\subset \mathscr{I}_{X_i,x}$ for all i. Since $\sqrt{\mathcal{I}_x} = \mathscr{I}_{A,x}$ and $\mathscr{I}_{X_i,x}$ is its own radical, we have $\mathcal{I}_x \not\subset \mathscr{I}_{X_i,x}$. The existence of a non zero-divisor follows from the next lemma. □

Lemma 3.4.5. *Let $X = X^1 \cup \cdots \cup X^N$ be a decomposition of reduced complex space X into analytic subsets. Let $x \in X$, and assume that each X^j has a local decomposition at x:*

$$X^j = X^j_1 \cup X^j_2 \cup \cdots$$

Suppose that we have a linear subspace $\mathscr{W} \subset \mathscr{O}_{X,x}$ such that

$$\mathscr{W} \not\subset \mathscr{I}_{X^j_i,x} \qquad (\forall i,j)$$

Then there is an element of $\mathscr{W}$ which is a non zero-divisor of $\mathscr{O}_{X^1,x},\dots,\mathscr{O}_{X^N,x}$.

Proof. Since each $\mathscr{W} \cap \mathscr{I}_{X^j_i,x}$ is not the full space $\mathscr{W}$, the *finite* union $\bigcup_{i,j}(\mathscr{W} \cap \mathscr{I}_{X^j_i,x}) = \mathscr{W} \cap \big(\bigcup_{i,j}\mathscr{I}_{X^j_i,x}\big)$ is not $\mathscr{W}$. So there is an element $f \in \mathscr{W}$ which is not in $\bigcup_{i,j}\mathscr{I}_{X^j_i,x}$. By Cor. 3.1.6, f is a non zero-divisor of each $\mathscr{O}_{X^j,x}$. □

Note that in the above proof we have used the fact that $\mathbb{C}$ is an infinite field. Over a finite field, a finite union of proper linear subspaces might be the full linear space.

We are now ready to prove Thm. 3.3.4.

Proof of Thm. 3.3.4. We set $A = X_i$, $B = X_j$ for simplicity. In view of Prop. 3.4.4, proving (3.3.5) means proving the following claim: After shrinking X to a neighborhood of x, for each $y \in A \cap B$, $\mathscr{I}_{A\cap B,y}$ contains a non zero-divisor of $\mathscr{O}_{A,y}$.

Note that $\mathscr{I}_{A\cap B} \supset \mathscr{I}_A + \mathscr{I}_B$. Since $\{0\} = \mathscr{I}_{X_1,x} \cap \cdots \cap \mathscr{I}_{X_N,x}$ is a prime decomposition, we have $\mathscr{I}_{B,x} \not\subset \mathscr{I}_{A,x}$. Therefore $(\mathscr{I}_{A,x} + \mathscr{I}_{B,x})\backslash\mathscr{I}_{A,x}$ is non-empty. Choose any element f of this set. Then since $\mathscr{I}_{A,x}$ is prime, f is a non zero-divisor of $\mathscr{O}_{A,x}$. By shrinking X to a neighborhood of x, we have that $f \in (\mathscr{I}_A + \mathscr{I}_B)(U)$ and that (by Prop. 2.3.13) f is a non zero-divisor of $\mathscr{O}_{A,y}$ for all $y \in X$. This proves the claim.

Now assume that (3.3.5) holds for all $i \neq j$. Let us prove the last sentence of Thm. 3.3.4. Let $X_2' = X_2 \cup X_3 \cup \cdots \cup X_N$. Then $X_1 \cap X_2'$ is nowhere dense in X_1 and in X_2'. Therefore we have decomposition $X = X_1 \cup X_2'$, and for each $y \in X_1 \cap X_2 \subset X_1 \cap X_2'$, X_1 and X_2' contain no neighborhoods of y in X. So by Rem. 3.3.2, the germ (X,y) is not reducible when $y \in X_1 \cap X_2$, and similarly when $y \in X_i \cap X_j$ for all $i \neq j$. □

3.5 Ranks of Jacobian matrices and singular loci

The goal of this section is to prove Lemma 3.2.12, a crucial ingredient in the proof that any complex space reduced at a point is reduced near that point (Thm. 3.2.1). Indeed, even if we assume that a complex space is reduced everywhere, this lemma still tells us something interesting: it says that if X is irreducible at 0

then, after shrinking X to a neighborhood of 0, X is smooth outside a nowhere dense analytic subset (due to Prop. 3.4.1).

The proof of Lemma 3.2.12 relies on Jacobian matrices, which are very useful for determining the singular locus of a complex space.

Definition 3.5.1. If X is a complex space, we define the **singular locus** of X to be the closed (cf. Cor. 1.6.5) subset

$$\mathrm{Sg}(X) = \{x \in X : X \text{ is not smooth at } x\}.$$

3.5.1 Jacobian matrices

Assume $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is a closed subspace of an open $U \subset \mathbb{C}^m$, where $\mathcal{I}$ is generated by $f^1, \dots, f^n \in \mathscr{O}(U)$. Let $(z_1, \dots, z_m)$ be the standard coordinates of $\mathbb{C}^m$, and consider the Jacobian matrix function

$$\partial_{z_\bullet}(f^\bullet) = \Big(\partial_{z_i} f^j\Big)_{1\leqslant i\leqslant m}^{1\leqslant j\leqslant n}$$

which is an $m \times n$ matrix valued function on U whose $i \times j$ entry is $\partial_{z_i} f^j$.

For each $k \in \mathbb{N}$, let

$$Z_k = \{x \in U : \mathrm{rank}\, \partial_{z_\bullet}(f^\bullet)(x) \leqslant k\}. \tag{3.5.1}$$

Then clearly

$$Z_0 \subset Z_1 \subset \cdots \subset Z_{m-1} \subset Z_m = Z_{m+1} = Z_{m+2} = \cdots = U. \tag{3.5.2}$$

Each Z_k is an analytic subset of U, because

$$Z_k = \bigcap_{\substack{1\leqslant i_1<\cdots<i_{k+1}\leqslant m\\ 1\leqslant j_1<\cdots<j_{k+1}\leqslant n}} N\left(\det\ \partial_{z_\bullet}(f^\bullet)\Big|_{i=i_1,\dots,i_{k+1}}^{j=j_1,\dots,j_{k+1}}\right) \tag{3.5.3}$$

3.5.2 Proof of Lemma 3.2.12

***Proof-Step 1**.* Assume the setting of Subsec. 3.5.1, and assume $0 \in X$. In this first step, we construct Δ. Fix $r \in \mathbb{N}$ to be

$$r = \text{“the smallest number such that } (Z_r \cap X, 0) = (X, 0)\text{”}$$

where $(Z_r \cap X, 0), (X, 0)$ are germs of sets at 0. Namely, r is the smallest number such that $Z_r \cap X$ contains a neighborhood of $0 \in X$. Thus, we may shrink[2] U so that

$$X \subset Z_r$$

[2]This is the only place we shrink U in Step 1 and 2 of the proof.

at the level of sets. More precisely, $N(\mathcal{I}) \subset Z_r$.

Since $Z_{r-1} \cap X$ containes no neighborhoods of $0 \in X$, by (3.5.3) we can choose an $r \times r$-submatrix, say the first r rows and the first r columns:

$$\partial_{z_\bullet}(f^\bullet)\Big|_{\leqslant r}^{\leqslant r} = \left(\partial_{z_i} f^j\right)_{1\leqslant i\leqslant r}^{1\leqslant j\leqslant r}$$

such that the zero set of its determinant

$$\varDelta = \det\ \partial_{z_\bullet}(f^\bullet)\Big|_{\leqslant r}^{\leqslant r} \qquad \in \mathscr{O}(U)$$

intersected with X contains no neighborhoods of $0 \in X$. (Note that $Z_{r-1} \subset N(\varDelta)$.) This implies that $\varDelta$ is non-zero in $\mathscr{O}_{X,0}$. Our goal is to show that $X\backslash N(\varDelta)$ is smooth. $\square$

***Proof-Step* 2.** Set

$$w_1 = f^1, \dots, w_r = f^r, \qquad w_{r+1} = z_{r+1}, \dots, w_m = z_m.$$

Then by inverse function theorem, each point $x \in U\backslash N(\varDelta)$ has a neighborhood on which $w_1, \dots, w_m$ are a set of coordinates. Recall that $\mathcal{I}_x$ is generated by $w_1, \dots, w_r$ and $f^{r+1}, \dots, f^n$. If we can show for each $x \in X\backslash N(\varDelta)$ that $\mathcal{I}_x$ is generated by $w_1, \dots, w_r$, then X is smooth at x, since X is near x the $(m-r)$-dimensional submanifold defined by $w_1 = \cdots = w_r = 0$. Thus $\mathrm{Sg}(X) \subset N(\varDelta)$.

- Claim: After possibly shrinking X to a neighborhood of 0, for each $x \in X\backslash N(\varDelta)$ we have

$$\partial_{w_i} f^j \in \mathcal{I}_x \qquad (\forall i, j > r)$$

If this is proved, then for each $i > r$, $\partial_{w_i} f^j$ belongs to $\mathcal{I}_x$ for all j since it is zero when $j \leqslant r$. Then $\partial_{w_i}\mathcal{I}_x \subset \mathcal{I}_x$. Thus by Lemma 3.2.13, $\mathcal{I}_x$ is generated by $w_1, \dots, w_r$, finishing the proof. (We warn the reader that ∂_{w_i} is not equal to ∂_{z_i} even if $i > r$, and is not defined on $N(\varDelta)$.)

Let us take a closer look at the relationship between the Jacobians of $(f^\bullet)$ over $z_\bullet$ and over $w_\bullet$. On $U\backslash N(\varDelta)$ we have

$$\partial_{z_\bullet}(f^\bullet) = \underbrace{\left[\begin{array}{c|c} \partial_{z_\bullet}(f^\bullet)\Big|_{\leqslant r}^{\leqslant r} & 0 \\ \hline * & I_{(m-r)\times(m-r)} \end{array}\right]}_{\partial_{z_\bullet}(w_\bullet)} \cdot \partial_{w_\bullet}(f^\bullet) \tag{3.5.4}$$

and also

$$\partial_{w_\bullet}(f^\bullet) = \left[\begin{array}{c|c} I_{r\times r} & \clubsuit \\ \hline 0 & \partial_{w_\bullet}(f^\bullet)\Big|_{>r}^{>r} \end{array}\right] \tag{3.5.5}$$

where $* \in \mathscr{O}(U)$ and $\clubsuit \in \mathscr{O}(U\backslash N(\varDelta))$. From these two relations we observe:

Ob 1. $\partial_{z_\bullet}(f^\bullet)\big|_{\leqslant r}^{\leqslant r}\cdot\clubsuit$ equals the upper right block of $\partial_{z_\bullet}(f^\bullet)$ which is holomorphic on U. So by Cramer's rule, $\varDelta\cdot\clubsuit$ can be extended to an element of $\mathscr{O}(U)$. So the same can be said about $\varDelta\cdot\partial_{w_\bullet}(f^\bullet)\big|_{>r}^{>r}$. (Look at the lower right block of $\partial_{z_\bullet}(f^\bullet)$.) We conclude

$$\partial_{w_i}f^j=h_i^j/\varDelta\qquad\text{for some } h_i^j\in\mathscr{O}(U)\qquad(\forall i,j>r)$$

Ob 2. At each $x\in X\backslash N(\varDelta)\subset Z_r\backslash Z_{r-1}$, the rank of $\partial_{w_\bullet}(f^\bullet)$ equals that of $\partial_{z_\bullet}(f^\bullet)$, which is r. Therefore, by (3.5.5), for all $i,j>r$, $\partial_{w_i}f^j$ vanishes on $X\backslash N(\varDelta)$, and hence h_i^j vanishes on $X\backslash N(\varDelta)$.

$\square$

Observation 2 shows that if we already know that X is reduced, then every holomorphic function vanishing on $X\backslash N(\varDelta)$, in particular $\partial_{w_i}f^j$ where $i,j>r$, must be an element of $\mathcal{I}(X\backslash N(\varDelta))$. Then the Claim in Step 2 follows and hence $\mathrm{Sg}(X)\subset N(\varDelta)$. But since we cannot assume what we want to prove, we need a little more effort to prove the Claim.

In Step 1 and 2, we have not used the fact that X is irreducible at x. This condition enters Step 3 of the proof. Indeed, we only need the weaker condition that X is reduced at x.

Proof-Step 3. Assume that $\mathscr{O}_{X,0}$ is an integral domain, and hence reduced. For each $i,j>r$, the two observations in Step 2 show that the holomorphic function $\varDelta\cdot h_i^j$ on U takes value zero at every point of X. So its germ at 0 is a nilpotent element of $\mathscr{O}_{X,0}$ by Nullstellensatz, and hence is zero. We can thus shrink U to a neighborhood of 0 so that $\varDelta\cdot h_i^j$ is zero in $\mathscr{O}_X(X)$ for all $i,j>r$. If $x\in X\backslash N(\varDelta)$, then $\varDelta(x)\neq 0$ and hence $\varDelta$ is invertible in $\mathscr{O}_{X,x}$. Therefore in $\mathscr{O}_{X,x}$ we have $h_i^j=0$ and hence $\partial_{w_i}f^j=0$ if $i,j>r$. This proves the claim in Step 2 that $\partial_{w_i}f^j$ is in $\mathcal{I}_x$. $\square$

We are done with the proof of Lemma. 3.2.12.

3.5.3 Additional comments

Assume the setting of Subsec. 3.5.1, and assume moreover that X is reduced. Assume U is small enough so that $X\subset Z_r$. Then Proof-Step 1&2 show that $\mathrm{Sg}(X)\subset X\cap N(\varDelta)$ (see the comments before Step 3), and that $X\backslash N(\varDelta)$ is an $m-r$ dimensional complex manifold. Note that in the proof we take $\varDelta$ to be the determinant of one $r\times r$ submatrix of $\partial_{z_\bullet}f^\bullet$, and we may well take other submatrices. By (3.5.3), Z_{r-1} is the intersection of $N(\varDelta)$ where $\varDelta$ runs through the determinants of all $k\times k$ submatrices of $\partial_{z_\bullet}f^\bullet$. Therefore $\mathrm{Sg}(X)\subset X\cap Z_{r-1}$.

It is natural to ask if we have $\mathrm{Sg}(X) = X \cap Z_{r-1}$. In Sec. 3.6, we will prove Lemma 3.5.2 saying that this is indeed true if $X \cap Z_{r-1}$ is nowhere dense in X. Note that if X is irreducible at 0, then Δ is non-zero in $\mathscr{O}_{X,0}$ and hence is a non zero-divisor. Thus, by Prop. 3.4.1, we can shrink X to a neighborhood of 0 so that $X \cap N(\Delta)$ and hence $X \cap Z_{r-1}$ are nowhere dense in X.

Lemma 3.5.2. *Assume the setting of Subsec. 3.5.1.*

(1) Assume that X is reduced, that $X \subset Z_r$, and that $X \cap Z_{r-1}$ is nowhere dense in X. Then

$$\mathrm{Sg}(X) = X \cap Z_{r-1}$$

and $X \backslash Z_{r-1}$ is an $(m-r)$-dimensional complex manifold.

(2) If the X in Subsec. 3.5.1 is irreducible at $0 \in X$, then we can shrink U to a neighborhood of $0 \in U$ (and replace X by $X \cap U$) so that the assumptions in (1) are satisfied for some $r \in \mathbb{N}$.

The only thing in Lemma 3.5.2 unproved so far is $\mathrm{Sg}(X) \supset X \cap Z_{r-1}$.

3.6 Embedding dimensions and singular loci

The rank of $\partial_{z_\bullet} f^\bullet$ in Subsec. 3.5.1 depends on how X is embedded into an open subset of a number space. Using Jacobi criterion, we can relate this rank to intrinsic numbers of X call embedding dimensions.

3.6.1 Embedding dimensions

Definition 3.6.1. Let X be a complex space and $x \in X$. The **embedding dimension** of X at x, denoted by $\mathrm{emb}_x X$ or $\mathrm{emb}\mathscr{O}_{X,x}$, is the smallest n such that a neighborhood U of x can be closely embedded to an open subset of $\mathbb{C}^n$.

Equivalently (Prop. 1.7.2), $\mathrm{emb}_x X$ is the smallest n such that there is a neighborhood U of x and a holomorphic $f : U \to \mathbb{C}^n$ which is an immersion at x. □

Proposition 3.6.2. *For each complex space X and $x \in X$,*

$$\mathrm{emb}_x X = \mathrm{emb}\mathscr{O}_{X,x} = \dim_{\mathbb{C}} \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2. \tag{3.6.1}$$

Proof. If $\varphi : X \to \mathbb{C}^n$ is an immersion at x, then by Thm. 1.7.8, $n \geqslant \dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. We can choose n to be $\dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ by shrinking X to a neighborhood of x, and choosing $f_1, \dots, f_n \in \mathscr{O}(X)$ forming a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. Then $\varphi = (f_1, \dots, f_n)$ is an immersion at x due to Thm. 1.7.8. □

As an immediate consequence of Prop. 3.6.2, $\mathbb{C}^n$ has embedding dimension n everywhere. Thus, for complex manifolds, embedding dimensions agree with the usual dimensions.

Proposition 3.6.3. *Let Z be a complex space and $\mathcal{I}$ a coherent ideal of $\mathscr{O}_Z$. Let $X = \mathrm{Specan}(\mathscr{O}_Z/\mathcal{I})$ and $x \in X$, and define the quotien map $d_x : \mathfrak{m}_{Z,x} \to \mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2$ (the differential map of Z at x). Then*

$$\mathrm{emb}_x X + \dim_{\mathbb{C}} d_x(\mathcal{I}_x) = \mathrm{emb}_x Z. \tag{3.6.2}$$

Proof. We have an exact sequence

$$0 \to \frac{\mathfrak{m}_{Z,x}^2 + \mathcal{I}_x}{\mathfrak{m}_{Z,x}^2} \to \frac{\mathfrak{m}_{Z,x}}{\mathfrak{m}_{Z,x}^2} \to \frac{\mathfrak{m}_{Z,x}}{\mathfrak{m}_{Z,x}^2 + \mathcal{I}_x} \to 0$$

where $\frac{\mathfrak{m}_{Z,x}}{\mathfrak{m}_{Z,x}^2+\mathcal{I}_x} = \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ since $\mathfrak{m}_{X,x} = \mathfrak{m}_{Z,x}/\mathcal{I}_x$. Thus (3.6.2) follows. □

Corollary 3.6.4 (Jacobi criterion). *Let U be an open subset of $\mathbb{C}^m$, let $\mathcal{I}$ be the ideal of $\mathscr{O}_U$ generated by $f^1, \dots, f^n \in \mathscr{O}(U)$, and let $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$. Then for each $x \in X$,*

$$\mathrm{emb}_x X + \mathrm{rank}_x\big(\partial_{z_\bullet} f^\bullet\big) = m. \tag{3.6.3}$$

Proof. One checks easily that $\mathrm{rank}_x\big(\partial_{z_\bullet} f^\bullet\big)$ equals $\dim_{\mathbb{C}} d_x(\mathcal{I}_x)$. □

As an easy application of Jacobi criterion, we prove:

Proposition 3.6.5. *Let X, Y be complex spaces and $x \in X, y \in Y$. Then*

$$\mathrm{emb}_{x\times y} X \times Y = \mathrm{emb}_x X + \mathrm{emb}_y Y. \tag{3.6.4}$$

Proof. Let $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ be open subsets such that $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ and $Y = \mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$, where $\mathcal{I}$ is an ideal of $\mathscr{O}_U$ generated by finitely many $f^1, f^2, \dots \mathscr{O}(U)$, and $\mathcal{J}$ is an ideal of $\mathscr{O}_V$ generated by finitely many $g^1, g^2, \dots \in \mathscr{O}(V)$. Let $z_\bullet$ be the set of coordinates of $\mathbb{C}^m$ and $w_\bullet$ the set of coordinates of $\mathbb{C}^n$. Then by Rem. 1.12.6, $X \times Y$ is the closed subspace of $U \times V$ defined by the ideal of $\mathscr{O}_{U\times V}$ generated by $f^1(z_\bullet), f^2(z_\bullet), \dots$ and $g^1(w_\bullet), g^2(w_\bullet), \dots$. By Jacobi criterion,

$$\begin{aligned}
\mathrm{emb}_{x\times y} X \times Y &= m + n - \mathrm{rank}_{x\times y}\, \partial_{(z_\bullet, w_\bullet)}\big(f^\bullet(z_\bullet), g^\bullet(w_\bullet)\big)\\
&= m + n - \mathrm{rank}_x\, \partial_{z_\bullet} f^\bullet(z_\bullet) - \mathrm{rank}_y\, \partial_{w_\bullet} g^\bullet(w_\bullet) = \mathrm{emb}_x X + \mathrm{emb}_y Y.
\end{aligned}$$

□

3.6.2 Analysis of singular loci

Proof of Lemma 3.5.2. Under the assumptions of (1), we need to show that each $x \in X \cap Z_{r-1}$ is a singular point. If x is smooth, we can find a neighborhood $W \subset X$ of x which is a complex manifold. In particular, the embedding dimensions of W must be constant on W. Thus, by Jacobi criterion, the ranks of $\partial_{z_\bullet} f^\bullet$ are constant on W.

Notice the assumptions in (1) that $X \cap Z_{r-1}$ is nowhere dense in X. So $W \not\subset X \cap Z_{r-1}$. From the definition of $Z_\bullet$, we know that the ranks of $\partial_{z_\bullet} f^\bullet$ on Z_{r-1} (and in particular at $x \in W$) are $\leqslant r-1$, and that the rank on the non-empty set $W \backslash Z_{r-1}$ is r (since $X \subset Z_r$). This is impossible. So x is singular. □

Lemma 3.5.2 shows that if X is irreducible at 0, then the singular locus of a neighborhood of $0 \in X$ is a nowhere dense analytic subset of that neighborhood. This property can be generalized.

Proposition 3.6.6. *Let X be a complex space reduced at x. Then after shrinking X to a neighborhood of x, there is a local decomposition $X = X_1 \cup \cdots \cup X_N$ at x such that*

$$\mathrm{Sg}(X) = \Big(\bigcup_{i \neq j} X_i \cap X_j\Big) \cup \Big(\bigcup_i \mathrm{Sg}(X_i)\Big). \tag{3.6.5}$$

In particular, after shrinking X further, $\mathrm{Sg}(X)$ is a nowhere dense analytic subset of X.

Note that each $\mathrm{Sg}(X_i)$ can be described by Lemma 3.5.2. We thus have an explicit local description of singular loci of reduced complex spaces.

Proof. Clearly we have $\subset$ in (3.6.5). To show $\supset$ we only need to show that $\mathrm{Sg}(X) \supset X_i \cap X_j$ if $i \neq j$ and after shrinking X. This is due to Thm. 3.3.4, since a reducible point must be singular. This proves (3.6.5). Thm. 3.3.4 says that $X_i \cap X_j$ is nowhere dense in X. By Lemma 3.5.2, $\mathrm{Sg}(X_i)$ is nowhere dense in X_i (and hence in X) after shrinking X. So $\mathrm{Sg}(X)$ is nowhere dense. □

Theorem 3.6.7. *Let X be a complex space. Then $\mathrm{Sg}(X)$ is an analytic subset of X. If X is reduced, then $\mathrm{Sg}(X)$ is nowhere dense in X.*

Proof. Prop. 3.6.6 shows that if X is reduced, then each $x \in X$ is contained in a neighborhood $U_x \subset X$ such that $\mathrm{Sg}(X) \cap U_x$ is analytic and nowhere sense in U_x. Therefore by Cor. 3.2.8, $\mathrm{Sg}(X)$ is analytic and nowhere dense in X. In the general case, $X = X' \cup (X \backslash X')$ where X' is the set of non-reduced points of X, which is an analytic subset by Cor. 3.2.7. Clearly

$$\mathrm{Sg}(X) = X' \cup \mathrm{Sg}(X \backslash X'). \tag{3.6.6}$$

So $\mathrm{Sg}(X)$ must be analytic. □

3.7 Products of reduced spaces are reduced

In this section, we give our first application of Thm. 3.6.7: We study the reducedness of complex spaces with the help of their singular loci.

Proposition 3.7.1. *Let X be a complex space and $x \in X$. Let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$ such that $N(\mathcal{I}) = \mathrm{Sg}(X)$. (For instance, $\mathcal{I} = \mathscr{I}_{\mathrm{Sg}(X)}$.) Then the following are equivalent.*

(1) X is reduced at x.

(2) $\mathcal{I}_x$ contains a non zero-divisor of $\mathscr{O}_{X,x}$.

Proof. Assume (1). By Thm. 3.2.1, we may shrink X to a neighborhood of x so that X is reduced. Then by Thm. 3.6.7, $N(\mathcal{I})$ is nowhere dense in X. Thus (2) follows from Prop. 3.4.4.

Assume (2). Shrink X so that there is $f \in \mathcal{I}(X)$ which is a non zero-divisor of $\mathscr{O}_{X,x}$. To prove (1), we need to show that every $g \in \sqrt{0_{X,x}}$ is zero. Shrink X further so that $g \in \mathscr{O}(X)$ and g^n is zero in $\mathscr{O}(X)$ for some $n \in \mathbb{Z}_+$. Since $X \backslash N(X)$ is smooth, $g|_{X\backslash N(X)} = 0$. So $\mathrm{Supp}(g) = \mathrm{Supp}(g\mathscr{O}_X)$ is inside $N(\mathcal{I})$. Since f vanishes on $N(\mathcal{I})$, by Nullstellensatz (Rem. 2.10.4-3), there is $k \in \mathbb{Z}_+$ such that in $\mathscr{O}_{X,x}$ we have $f^k g = 0$, and hence $g = 0$ because f is a non zero-divisor. □

Note that the proof of (2)⇒(1) is similar to that of Thm. 3.2.1. (See the proof above Lemma 3.2.13.)

We shall prove that the direct product of two reduced complex spaces is reduced. To prove this fact, we first need a result on completed tensor product of non zero-divisors.

Proposition 3.7.2. *Let X, Y be complex spaces and $x \in X, y \in Y$. Let $f \in \mathscr{O}_{X,x}$ be a non zero-divisor of $\mathscr{O}_{X,x}$ and $g \in \mathscr{O}_{X,x}$ be a non zero-divisor of $\mathscr{O}_{Y,y}$. Then $f \otimes g$ is a non zero-divisor of $\mathscr{O}_{X,x} \widehat{\otimes} \mathscr{O}_{Y,y} = \mathscr{O}_{X\times Y, x\times y}$.*

Recall the meaning of $f \otimes g$ in (1.13.1). Since $f \otimes g = (f \otimes 1)(1 \otimes g)$ and the product of two non zero-divisors is a non zero-divisor, it suffices to prove that $f \otimes 1$ is a non zero-divisor.

A different proof of this proposition is given in Sec. 5.5, after Cor. 5.5.8.

Proof-Step 1. We prove Prop. 3.7.2 under the assumption that y is the only point of Y. Then the obvious projection $Y \to \{0\}$, where $\{0\}$ is the *reduced* single point, is finite. Therefore, by Cor. 2.8.4, we have a canonical equivalence

$$\mathscr{O}_{X\times Y, x\times y} \simeq \mathscr{O}_{X,x} \otimes_{\mathbb{C}} \mathscr{O}_{Y,y}.$$

Note that by Thm. 2.7.1, $\mathscr{O}_{Y,y}$ is a finite-dimensional vector space. Then one checks easily that $f \otimes 1$ is a non zero-divisor: choose any element of $\mathscr{O}_{X,x} \otimes_{\mathbb{C}} \mathscr{O}_{Y,y}$ and write it as a finite sum $h = \sum_i h_i \otimes e_i$ where $\{e_i\}$ is a basis of $\mathscr{O}_{Y,y}$. If $(f \otimes 1)h = \sum_i fh_i \otimes e_i$ is zero, then each $fh_i = 0$, and hence $h_i = 0$. □

Proof-Step 2. We now prove the general case. Choose any $h \in \mathscr{O}_{X\times Y,x\times y}$ such that $(f \otimes 1)h = 0$. We shall prove that $h \in \mathfrak{m}_{Y,y}^k \cdot \mathscr{O}_{X\times Y,x\times y}$ for all $k \in \mathbb{N}$. Then since $\mathfrak{m}_{Y,y}^k \cdot \mathscr{O}_{X\times Y,x\times y} \subset \mathfrak{m}_{X\times Y,x\times y}^k$, we have $h = 0$ by Krull's intersection Thm. 1.4.4.

Let $\mathcal{J}$ be $\mathscr{I}_{\{y\}}$, the ideal sheaf of all sections of $\mathscr{O}_Y$ vanishing at y. Then $\mathcal{J}_y = \mathfrak{m}_{Y,y}$. Thus, what we need to prove is that h is zero in $\mathscr{O}_{X\times Y,x\times y}/\mathcal{J}_y^k\mathscr{O}_{X\times Y,x\times y}$ for all k. Let $Y^k = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J}^k)$ whose underlying topological space is $\{y\}$ but might be non-reduced. Let $\mathrm{pr}_Y : X \times Y \to Y$ be the projection. Then by Prop. 1.12.1 and 1.12.5, $\mathscr{O}_{X\times Y,x\times y}/\mathcal{J}_y^k\mathscr{O}_{X\times Y,x\times y}$ is the stalk at $x \times y$ of

$$\mathscr{O}_{X\times Y}/\mathcal{J}^k\mathscr{O}_{X\times Y} = \mathscr{O}_{\mathrm{pr}_Y^{-1}(Y^k)} = \mathscr{O}_{X\times Y^k}.$$

Note that by Prop. 1.12.5, the inclusion $\iota_{X\times Y^k,X\times Y}$ equals $\mathbf{1}_X \times \iota_{Y^k,Y}$. Thus, the residue class of $f \otimes 1_{\mathscr{O}_{Y,y}} = \mathrm{pr}_{X\times Y,X}^* f$ in $\mathscr{O}_{X\times Y^k,x\times y}$ is

$$(\mathbf{1}_X \times \iota_{Y^k,Y})^*\mathrm{pr}_{X\times Y,X}^* f = \mathrm{pr}_{X\times Y^k,X}^* f = f \otimes 1_{\mathscr{O}_{Y^k,y}}$$

which, by Step 1, is a non zero-divisor of $\mathscr{O}_{X\times Y^k,x\times y}$. So h is 0 in $\mathscr{O}_{X\times Y^k,x\times y}$. This finishes the proof. □

Theorem 3.7.3. *Let X, Y be reduced complex spaces. Then the direct product $X \times Y$ is reduced.*

Proof. Since reducedness is a local property, we may assume that X and Y are small enough. Choose any $x \in X$ and $y \in Y$. Since X, Y are reduced, by Prop. 3.7.1, we may shrink X, Y to neighborhoods of x, y respectively so that we can find $f \in \mathscr{I}_{\mathrm{Sg}(X)}(X)$ which is a non zero-divisor of $\mathscr{O}_{X,x}$, and find $g \in \mathscr{I}_{\mathrm{Sg}(Y)}(Y)$ which is a non zero-divisor of $\mathscr{O}_{Y,y}$. Since f takes value zero on $\mathrm{Sg}(X)$, $f \otimes 1$ takes value zero on $\mathrm{Sg}(X) \times Y$, and similarly $1 \otimes g$ takes value zero on $X \times \mathrm{Sg}(Y)$. Thus $f \otimes g = (f \otimes 1)(1 \otimes g)$ vanishes on

$$(\mathrm{Sg}(X) \times Y) \cup (X \times \mathrm{Sg}(Y)) \supset \mathrm{Sg}(X \times Y). \tag{3.7.1}$$

The above $\supset$ is due to the fact that the product of smooth spaces is smooth, according to Exp. 1.13.3.

Now we have $f \otimes g \in \mathscr{I}_{\mathrm{Sg}(X\times Y)}(X \times Y)$. By Prop. 3.7.2, $f \otimes g$ is a non zero-divisor of $\mathscr{O}_{X\times Y,x\times y}$. So by 3.7.1, $X \times Y$ is reduced at $x \times y$. □

We remark that the "$\supset$" in (3.7.1) is actually "$=$". See Cor. 3.10.10.

3.8 Non locally-free loci of coherent sheaves

In this section, we use (co)rank functions to study the non locally-free loci of coherent sheaves.

Definition 3.8.1. Let X be a complex space and $\mathscr{E}$ an $\mathscr{O}_X$-module. We say that $\mathscr{E}$ is **locally free at** x if there is a neighborhood $U \subset X$ of x such that $\mathscr{E}|_U$ is $\mathscr{O}_U$-free. (Recall our convention that free sheaves are assumed to have finite ranks). When $\mathscr{E}$ is $\mathscr{O}_X$-coherent, then this is equivalent to saying that $\mathscr{E}_x$ is a free $\mathscr{O}_{X,x}$-module (Thm. 2.2.2).

The (clearly closed) subset of all $x \in X$ at which $\mathscr{E}$ is not locally free is called the **non locally-free locus** of $\mathscr{E}$. □

Lemma 3.8.2. *Let $\mathcal{A}$ be a commutative Noetherian local ring and $\mathcal{M}$ an $\mathcal{A}$-module together with a surjective morphism of $\mathcal{A}$-modules $\varphi : \mathcal{A}^n \to \mathcal{M}$. Then $\mathcal{M}$ is $\mathcal{A}$-free if and only if the morphism*

$$\varphi_* : \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{A}^n) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}), \qquad \alpha \mapsto \varphi \circ \alpha$$

is surjective.

Proof. If $\mathcal{M}$ is free then φ_* is surjective because $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, -)$ is right exact. Conversely, if φ_* is surjective, then the fact that $\mathbf{1}_{\mathcal{M}}$ is in the image of φ_* means that there is a lift $\psi : \mathcal{M} \to \mathcal{A}^n$ such that $\varphi \circ \psi = \mathbf{1}_{\mathcal{M}}$. This proves that $\mathcal{M}$ is a direct summand of $\mathcal{A}^n$. Therefore $\mathcal{M}$ is a projective $\mathcal{A}$-module by Prop. 5.3.7, and hence is free of finite rank by Thm. 5.5.11. □

Theorem 3.8.3. *Let X be a complex space and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Then*

$$E = \{x \in X : \mathscr{E} \text{ is not locally free at } x\}$$

is an analytic subset of X. If X is reduced, then E is nowhere dense.

Proof-Step 1. Let us prove that E is analytic. By Cor. 3.2.8, it suffices to show that each $x \in X$ is contained in a neighborhood U such that $E \cap U$ is analytic in U. So let us assume X is so small that there is a surjective $\mathscr{O}_X$-module morphism $\varphi : \mathscr{O}_X^n \to \mathscr{E}$. This yields a morphism of coherent modules (cf. Cor. 2.2.5)

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X^n) \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}).$$

The support of the cokernel of this morphism is, by Lemma 3.8.2, the set of all x such that $\mathscr{E}_x$ is not $\mathscr{O}_{X,x}$-free, namely E. So E is analytic since it is (as a set) the support of a coherent sheaf. □

Proof-Step 2. Assume that X is reduced. We need to show that E contains no nonempty open subsets of X. By shrinking X, it suffices to prove that $E \neq X$. So let us assume $E = X$ and find a contradiction.

Now our assumption is that $\mathscr{E}$ is nowhere locally free on X. By shrinking X, we assume that

$$\mathscr{E} \simeq \mathrm{Coker}(\varphi : \mathscr{O}_X^m \to \mathscr{O}_X^n).$$

Let $\xi_1 = \varphi(1,0,\dots,0),\dots,\xi_m = \varphi(0,0,\dots,1)$, which are elements of $\mathscr{O}(X)^n$. Then $F = (\xi_1,\dots,\xi_m)$ can be viewed as an element of $\mathscr{O}(X)^{n\times m}$, i.e. an $n\times m$ matrix-valued holomorphic function on X. And for each $x\in X$, setting $\mathbb{C}_x = \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}\mathscr{O}_{X,x}$, we have

$$\begin{aligned} n - \mathrm{rank} F(x) &= \dim \mathrm{Coker}(\varphi(x): \mathbb{C}_x^m \to \mathbb{C}_x^n) \\ &= \dim \mathrm{Coker}(\varphi\otimes 1 : \mathscr{O}_X^m \otimes_{\mathbb{C}} \mathbb{C}_x \to \mathscr{O}_X^n \otimes_{\mathbb{C}} \mathbb{C}_x) \\ &= \dim \mathrm{Coker}(\varphi : \mathscr{O}_X^m \to \mathscr{O}_X^n)\otimes_{\mathbb{C}} \mathbb{C}_x = \dim(\mathscr{E}|x). \end{aligned}$$

As in Subsec. 3.5.1, for each $k\in\mathbb{N}$, the set

$$\Gamma_k = \{x\in X : \mathrm{rank} F(x) \leqslant k\}$$

is an analytic subset of X. We let r be the smallest number such that Γ_r contains a nonempty open subset of X. Then $\Gamma_r\backslash\Gamma_{r-1}$ also contains a non-empty open subset $U\subset X$. By restricting X to U, we assume that $X=\Gamma_r$. So the dimensions of the fibers $\dim(\mathscr{E}|x)$ are constant on X. Therefore, since X is reduced, Prop. 1.3.14 implies that $\mathscr{E}$ is locally free on X. Impossible. □

Exercise 3.8.4. Let X be a reduced complex space irreducible at $x\in X$. Show that after shrinking X to a neighborhood of x, there is $r\in\mathbb{N}$ such that $X=\Gamma_r$ and that Γ_{r-1} is nowhere dense in X. Show that if $X=\Gamma_r$ and if Γ_{r-1} is nowhere dense then Γ_{r-1} is the non locally-free locus of $\mathscr{E}$.

3.9 Dimensions

Definition 3.9.1. Let X be a complex space and $x\in X$. The **(Chevalley) dimension of $\mathscr{O}_{X,x}$**, also called the **dimension of X at x** and denoted by $\dim\mathscr{O}_{X,x}$ or equivalently $\dim_x X$, is the smallest $n\in\mathbb{N}$ such that:

- There exists a neighborhood U of x and $f_1,\dots,f_n\in\mathscr{O}(U)$ such that x is an isolated point of $N(f_1,\dots,f_n)$.

It is clear that $\dim_x X = 0$ iff x is an isolated point of X. We set $\dim_x\varnothing = -\infty$.

The **global dimension** is defined to be

$$\dim X = \sup_{x\in X}\dim_x X.$$

We say that X is **(resp. locally) pure dimensional** if $x\in X\mapsto \dim_x X$ is a (resp. locally) constant function. We say that X has **pure dimension n at x** if X has dimension n at every point of a neighborhood of x. □

3.9.1 Basic facts about dimensions

Proposition 3.9.2. *Let X be a complex space and $x \in X$. Then*

$$\dim_x X = \dim_x \mathrm{red}(X).$$

Equivalently, for $\mathscr{A} = \mathscr{O}_{X,x}$ we have

$$\dim \mathscr{A} = \dim \mathscr{A}/\sqrt{0_{\mathscr{A}}}.$$

Proof. If X is small enough such that $f_1, \dots, f_n \in \mathscr{O}(X)$ makes x an isolated point of $N(f_\bullet)$, then their restrictions to $\mathrm{red}(X)$ (i.e. their residue classes in $\mathrm{red}(X)$) also make x an isolated point of the zero set.

Conversely, if $f_1, \dots, f_n \in \mathscr{O}(\mathrm{red}(X))$ makes x an isolated point of $N(f_\bullet)$, then after shrinking X to a neighborhood of x, we can assume $f_1, \dots f_n$ are the restrictions of elements of $\mathscr{O}(X)$, whose zero set also has x as an isolated point. □

Proposition 3.9.3. *We have $\dim_x X \leqslant n$ if and only if there exist a neighborhood $U_x \subset X$ of x, an open subset $V \subset \mathbb{C}^n$, and a finite holomorphic map $F : U_x \to V$.*

Proof. The "if" part is clear. The "only if" part follows by choosing $F = (f_1, \dots, f_n)$ (where $f_\bullet$ are in Def. 3.9.1) and applying Thm. 2.7.2 to deduce the finiteness. □

Corollary 3.9.4. *For each complex space X, the dimension function*

$$X \to \mathbb{N} \qquad x \mapsto \dim_x X$$

is upper semicontinuous.

Proof. Fix any $p \in X$ and let $n = \dim_p X$. Then by Prop. 3.9.3, we may shrink X to a neighborhood of p and find an open subset $V \subset \mathbb{C}^n$ such that there is a finite map $\varphi : X \to \mathbb{C}^n$. Then clearly $\dim_x X \leqslant n$ for each $x \in X$. □

Corollary 3.9.5. *Let $\varphi : X \to Y$ be a finite holomorphic map. Then for each $x \in X$,*

$$\dim_x X \leqslant \dim_{\varphi(x)} Y. \tag{3.9.1}$$

Proof. Let $y = \varphi(x)$ and $n = \dim_y Y$. By Prop. 3.9.3, after shrinking Y and replacing X by $\varphi^{-1}(Y)$, we have a finite holomorphic map $\pi : Y \to V$ where V is an open subset of $\mathbb{C}^n$. Since $F \circ \varphi : X \to V$ is finite, by Prop. 3.9.3 we conclude $\dim_x X \leqslant n$. □

Proposition 3.9.6. *Let X be a complex space and $x \in X$. The following are equivalent.*

(1) $\dim_x X = 0$*, namely, x is an isolated point of X.*

(2) $\mathscr{O}_{X,x}$ *is a finite-dimensional vector space.*

(3) $\mathscr{O}_{X,x}$ *is an Artinian ring.*

(4) *There exists* $k \in \mathbb{Z}_+$ *such that* $\mathfrak{m}_{X,x}^k = 0$*.*

Proof. (1)⇒(2): By shrinking X, we assume x is the only point of X. Let $\{0\}$ be the reduced single point (whose structure sheaf is $\mathbb{C}$). Then the obvious holomorphic map $X \to \{0\}$ is finite. Therefore, by Thm. 2.7.1, $\mathscr{O}_{X,x}$ is $\mathbb{C}$-coherent, i.e., $\mathbb{C}$-finite-dimensional.

(2)⇒(3): Obvious.

(3)⇒(4): The decreasing chain $\mathfrak{m}_{X,x} \supset \mathfrak{m}_{X,x}^2 \supset \mathfrak{m}_{X,x}^3 \supset \cdots$ must be stationary as $\mathscr{O}_{X,x}$ is Artinian. So there is $k \in \mathbb{Z}_+$ such that $\mathfrak{m}_{X,x}^k = \mathfrak{m}_{X,x}^{k+1}$. So $\mathfrak{m}_{X,x}^k = 0$ by Nakayama's lemma 1.2.15.

(4)⇒(1): $\mathfrak{m}_{X,x}$ is the stalk at x of $\mathscr{I}_{\{x\}} = \{f \in \mathscr{O}_X : f(x) = 0\}$. Suppose $\mathfrak{m}_{X,x}^k = 0$. Then after shrinking X to a neighborhood of x, we have $\mathscr{I}_{\{x\}}^k = 0$. So $\{x\} = N(\mathscr{I}_{\{x\}}) = N(\mathscr{I}_{\{x\}}^k) = N(0_X) = X$. □

3.10 Active lemma for dimensions

Let X be a complex space.

3.10.1 Active lemma

Definition 3.10.1. An element $f \in \mathscr{O}(X)$ is called **active at** x or **active in** $\mathscr{O}_{X,x}$ if f (or more precisely $\mathrm{red}(f)$) is a non zero-divisor of $\mathscr{O}_{\mathrm{red}(X),x} = \mathscr{O}_{X,x}/\sqrt{0_{X,x}}$.

Non zero-divisors are always active, but the converse is not true.

Proposition 3.10.2. *If* $f \in \mathscr{O}(X)$ *is a non zero-divisor of* $\mathscr{O}_{X,x}$*, then* f *is active in* $\mathscr{O}_{X,x}$*.*

Proof. Let $\mathscr{A} = \mathscr{O}_{X,x}$. Suppose that f is not active at x, i.e. f is a zero-divisor of $\mathscr{A}/\sqrt{0}$. Then $fg \in \sqrt{0}$ for some $g \in \mathscr{A}$ and $g \notin \sqrt{0}$. So for some $n \in \mathbb{Z}_+$ we have $f^n g^n = 0$ in $\mathscr{A}$. Notice that $g^n \neq 0$. So we can find $k \in \mathbb{N}$ such that in $\mathscr{A}$ we have $f^k g^n \neq 0$ and $f \cdot f^k g^n = 0$. Therefore f is a zero-divisor of $\mathscr{A}$. □

Theorem 3.10.3 (Active lemma). *Let* $f \in \mathscr{O}(X)$ *and* $x \in N(f)$*. If* f *is a non zero-divisor of* $\mathscr{O}_{X,x}$*, then*

$$\dim_x N(f) = \dim_x X - 1. \tag{3.10.1}$$

Thus, by Prop. 3.9.2, if f *is active at* x *then* (3.10.1) *also holds.*

One may compare Active lemma with Prop. 3.6.3.

Proof. Let $m = \dim_x N(f)$ and $n = \dim_x X$. Then, after shrinking X to a neighborhood of x, there are $g_1, \ldots, g_m \in \mathscr{O}(X)$ such that $N(f) \cap N(g_1, \ldots, g_m) = \{x\}$. Thus $n \leqslant m + 1$.

Let us prove $m \leqslant n - 1$. Let $A = N(f)$. By Prop. 3.9.3, we may shrink X and find a finite holomorphic map $\varphi : X \to V$ sending x to 0, where $V \subset \mathbb{C}^n$ is open. By Exe. 2.3.11, $\varphi(A)$ is an analytic subset of V. So $m \leqslant \dim_0 \varphi(A)$ by Cor. 3.9.5. Therefore, it suffices to prove $\dim_0 \varphi(A) \leqslant n - 1$.

By Thm. 2.7.1, $\varphi_* \mathscr{O}_X$ is $\mathscr{O}_V$-coherent. Therefore $\mathscr{O}_{X,x}$ is a finitely-generated $\mathscr{O}_{V,0}$-module since it is a direct summand of $(\varphi_* \mathscr{O}_X)_x$ (cf. Prop. 2.4.5). Thus, as $\mathscr{O}_{X,x}$ is Noetherian, the germ of f in $\mathscr{O}_{X,x}$ is integral over $\mathscr{O}_{V,0}$ (see the argument for (2.7.4)), i.e.

$$f^N + a_{N-1} f^{N-1} + \cdots + a_k f^k = 0$$

for some $a_k, \ldots, a_{N-1} \in \mathscr{O}_{V,0}$ where a_k is non-zero in $\mathscr{O}_{V,0}$. Since f is a non zero-divisor of $\mathscr{O}_{X,x}$, we conclude that a_k (or more precisely $\varphi^{\#} a_k$) equals

$$-f^{N-k} - a_{N-1} f^{N-k-1} - \cdots - a_{k+1} f$$

in $\mathscr{O}_{X,x}$. Therefore, after shrinking V to a neighborhood of 0 and replacing X by $\varphi^{-1}(V)$, we have $a_k \in \mathscr{O}(V)$ and that $\varphi^{\#} a_k$ takes value zero on $N(f)$. Therefore $\varphi(A) \subset N(a_k)$. So it suffices to prove $\dim_0 N(a_k) \leqslant n - 1$.

Since a_k is non-zero in $\mathscr{O}_{V,0}$, as in the proof of Thm. 1.5.5 we may choose a new set of coordinates $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ such that a_k has finite order in z_1. So $0_{\mathbb{C}^n}$ is an isolated point of the fiber $\pi^{-1}(0_{\mathbb{C}^{n-1}})$, where $\pi : N(a_k) \to \mathbb{C}^{n-1}$ is the restriction of $\mathrm{pr}_{\mathbb{C}^{n-1}} : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$. This proves $\dim_0 N(a_k) \leqslant n - 1$. $\square$

Remark 3.10.4. Assume $\dim_x X > 0$. By taking local decomposition of $\mathrm{red}(X)$ at x and using Cor. 3.1.6 or Lemma 3.4.5, we can find $f \in \mathfrak{m}_{X,x}$ which is an active germ of X at x. By Active lemma, we can repeat this procedure to obtain $f_1, \ldots, f_n \in \mathfrak{m}_{X,x}$ such that, after shrinking X to a neighborhood of x, each f_i is in $\mathscr{O}(X)$ and is an active germ of $N(f_1, \ldots, f_{i-1})$ at x. And x is an isolated point of $N(f_1, \ldots, f_n)$.

Contrary to active elements, if $\mathscr{O}_{X,x}$ is not reduced then $\mathfrak{m}_{X,x}$ might not contain a non zero-divisor of $\mathscr{O}_{X,x}$. Thus, we may not be able to find $f_1, \ldots, f_n \in \mathfrak{m}_{X,x}$ such that each f_i is a non zero-divisor of $\mathscr{O}_{X,x}/(f_1 \mathscr{O}_{X,x} + \cdots + f_{i-1} \mathscr{O}_{X,x})$. In the case that we can, we will call $\mathscr{O}_{X,x}$ a **Cohen-Macaulay ring**.

3.10.2 Consequences of Active lemma

Corollary 3.10.5. *If $x \in \mathbb{C}^n$ then*

$$\dim_x \mathbb{C}^n = n.$$

Proof. This is clear when $n = 0$. That $\dim_x \mathbb{C}^n = n$ implies $\dim_x \mathbb{C}^{n+1} = n+1$ follows from Active lemma. $\square$

Corollary 3.10.6 (Ritt's lemma). *Let A be an analytic subset of a complex space X. The following are equivalent.*

(1) A is nowhere dense in X.

(2) $\dim_x A < \dim_x X$ for all $x \in X$.

Proof. By Prop. 3.9.2, it suffices to assume that X is reduced. Clearly (2)⇒(1). Assume (1). Then by Prop. 3.4.4, for each x there is a non zero-divisor $f \in \mathscr{O}_{X,x}$ vanishing on the germ (A, x). Therefore $\dim_x A \leqslant \dim_x N(f) = \dim_x X - 1$ by Active lemma. This proves (2). $\square$

Proposition 3.10.7. *Let $X = A_1 \cup \cdots \cup A_N$ be a union of analytic subsets. Then*

$$\dim_x X = \sup_{1 \leqslant i \leqslant N} \dim_x A_i \tag{3.10.2}$$

Proof. By Prop. 3.9.2, we may assume that X is reduced. "$\geqslant$" clearly holds. We prove "$\leqslant$" by induction on $m = \sup_i \dim_x A_i$. We may assume that x is in each A_i and hence $\dim_x A_i \geqslant 0$.

The base case $m = 0$ is obvious. Assume that (3.10.2) holds for any decomposition such that $\sup_i \dim_x A_i = m - 1$. Now assume $\sup_i \dim_x A_i = m > 0$. We may shrink X to a neighborhood of x and discard those i satisfying $\dim_x A_i = 0$. Thus, we may assume $\dim_x A_i > 0$ for all i. Shrink X further so that each A_i has a local decomposition $A_i = B_{i,1} \cup B_{i,2} \cup \cdots$ at x. Then for each i, j, clearly x is not an isolated point of $B_{i,j}$. This implies $\mathfrak{m}_{X,x} \not\subset \mathscr{I}_{B_{i,j},x}$. Therefore, by Lemma 3.4.5, we can find $f \in \mathfrak{m}_{X,x}$ which is a non zero-divisor of $\mathscr{O}_{X,x}$ and of every $\mathscr{O}_{A_i,x}$. Thus, by Active lemma, $\dim_x N(f) = \dim_x X - 1$ and $\sup_i \dim_x N(f) \cap A_i = m - 1$. These two numbers are equal by assumption on case $m - 1$. So $\dim_x X = m$. $\square$

Proposition 3.10.8. *Let X, Y be complex spaces and $x \in X, y \in Y$. Then*

$$\dim_{x \times y} X \times Y = \dim_x X + \dim_y Y. \tag{3.10.3}$$

Proof. We prove this by induction on $m = \dim_x X$. The case $\dim_x X = 0$ is obvious. Suppose (3.10.3) holds whenever $\dim_x X = m - 1$. In the case that $\dim_x X = m$, choose $f \in \mathfrak{m}_{X,x}$ active in $\mathscr{O}_{X,x}$ (Rem. 3.10.4). Then by Prop. 3.7.2, $f \otimes 1 \in \mathfrak{m}_{X \times Y, x \times y}$ is active in $\mathscr{O}_{X \times Y, x \times y}$. By shrinking X to a neighborhood of x we may assume $f \in \mathscr{O}(X)$. Therefore, by Active lemma, $\dim_{x \times y} N(f) \times Y = \dim_{x \times y} N(f \otimes 1) = \dim_{x \times y}(X \times Y) - 1$, and $\dim_x N(f) = \dim X - 1$. By assumption, $\dim_{x \times y} N(f) \times Y = \dim_x N(f) + \dim_y Y$. This proves (3.10.3). $\square$

Note that in the above proof, one can also use Prop. 3.4.1 to show that $f \otimes 1$ is active if f is so.

3.10.3 Comparing different versions of dimensions

We first compare (Chevalley) dimensions and embedding dimensions.

Proposition 3.10.9. *Let X be a complex space and $x \in X$. Then $\dim_x X \leqslant \mathrm{emb}_x X$. Moreover, X is smooth at x if and only if $\dim_x X = \mathrm{emb}_x X$.*

Proof. Clearly $\dim_x X \leqslant \mathrm{emb}_x X$ in general (Recall Def. 3.6.1) and $\dim_x X = \mathrm{emb}_x X$ if X is smooth at x. We now assume $n := \mathrm{emb}_x X$ equals $\dim_x X$ and prove that X is smooth at x.

By Def. 3.6.1, after shrinking X to a neighborhood of x, we may view X as a closed subspace of an open subset V of $\mathbb{C}^n$. Write $X = \mathrm{Specan}(\mathscr{O}_V/\mathcal{I})$. We claim that $\mathcal{I}_x = 0$. Then we can choose a neighborhood $W \subset V$ of x such that $\mathcal{I}|_W = 0$ (Rem. 1.2.16), and clearly the complex subspace $X \cap W$ of X is smooth. Hence X is smooth at x.

Suppose $\mathcal{I}_x \neq 0$. Then $\mathcal{I}_x$ contains a nonzero element f, which is a non zero-divisor of the integral domain $\mathscr{O}_{\mathbb{C}^n,0}$. Since f vanishes on the germ $(X, 0)$, by Active lemma we have $\dim_x X \leqslant n - 1$, which is impossible. □

Corollary 3.10.10. *Let X, Y be complex spaces. Then*

$$\mathrm{Sg}(X \times Y) = (\mathrm{Sg}(X) \times Y) \cup (X \times \mathrm{Sg}(Y)). \tag{3.10.4}$$

Proof. We need to prove that for every $x \in X$ and $y \in Y$, $X \times Y$ is smooth at $x \times y$ iff X is smooth at x and Y is smooth at y. This is immediate from Prop. 3.6.5, 3.10.8, and 3.10.9. □

In algebraic geometry, the dimension of a commutative ring usually means Krull dimension. Fortunately, it agrees with Chevalley dimension when the ring is an analytic local $\mathbb{C}$-algebra.

Definition 3.10.11. Let $\mathcal{A}$ be a commutative ring. The **Krull dimension** of $\mathcal{A}$ is the largest $n \in \mathbb{N}$ such that there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals of $\mathcal{A}$. If such n can be arbitrarily large, we set the Krull dimension to be $+\infty$.

Lemma 3.10.12. *Let $\mathfrak{p}' \subsetneq \mathfrak{p}$ be prime ideals of $\mathscr{O}_{X,x}$. Then*

$$\dim \mathscr{O}_{X,x}/\mathfrak{p} < \dim \mathscr{O}_{X,x}/\mathfrak{p}'. \tag{3.10.5}$$

Recall that a prime ideal of $\mathscr{O}_{X,x}$ is not equal to $\mathscr{O}_{X,x}$, and hence is contained in $\mathfrak{m}_{X,x}$.

Proof. By replacing $\mathscr{O}_{X,x}$ by $\mathscr{O}_{X,x}/\mathfrak{p}'$ (and replacing X by a closed subspace of a neighborhood of x), it suffices to assume that $\mathfrak{p}' = 0$ and that $\mathscr{A} = \mathscr{O}_{X,x}$ is an integral domain. Then $0 \subsetneq \mathfrak{p} \subsetneq \mathscr{A}$. Choose a non-zero $f \in \mathfrak{p}$. Then f is a non zero-divisor of $\mathscr{A}$. Thus by Active lemma, $\dim \mathscr{A}/\mathfrak{p} \leqslant \dim \mathscr{A}/f\mathscr{A} = \dim \mathscr{A} - 1$. This proves (3.10.5). □

Proposition 3.10.13. $\dim_x X$ *equals the Krull dimension of* $\mathscr{O}_{X,x}$.

Proof. Lemma 3.10.12 shows that $n = \dim_x X$ is no less than the Krull dimension of $\mathscr{O}_{X,x}$. To prove the equality, we need to show the existence of a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$. We prove this by induction on n. The case $n = 0$ is obvious. Assume this is true whenever $\dim_x X = n-1$. Now assume $\dim_x X = n$. By Rem. 3.10.4, we can find an active germ $f \in \mathfrak{m}_{X,x}$ at x. Then $\dim_x N(f) = n-1$ by Active lemma. By Prop. 3.10.7, in the prime decomposition of $f\mathscr{O}_{X,x} \subset \mathscr{O}_{X,x}$ there is a prime component $\mathfrak{p}_0$ such that $\dim \mathscr{O}_{X,x}/\mathfrak{p}_0 = n-1$. By assumption, we have a strictly increasing chain of n prime ideals of $\mathscr{O}_{X,x}/\mathfrak{p}_0$. These are prime ideals of $\mathscr{O}_{X,x}$ containing $\mathfrak{p}_0$. In this way, we get a strictly increasing chain of $n+1$ prime ideals of $\mathscr{O}_{X,x}$. $\square$

3.11 Noether property for coherent sheaves

Let X be a complex space. In this section, we use dimension theory and Active lemma to prove the Noether property for coherent sheaves of X. This result will be used in the proof of Grauert comparison theorem. (See the proof of Lemma 6.5.5.)

Definition 3.11.1. Let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module. An **ascending chain of coherent $\mathscr{O}_X$-submodules** of $\mathscr{E}$ is a collection $(\mathscr{E}_i)_{i\in\mathfrak{I}}$ where $\mathfrak{I}$ is a directed set, and $\mathscr{E}_i \subset \mathscr{E}_j$ if $i \leqslant j$. We say that the chain is **stationary at** $x \in X$ if there is a neighborhood $U \subset X$ of x and $i \in \mathfrak{I}$ such that $\mathscr{E}_i|_U = \mathscr{E}_j|_U$ for all $j \geqslant i$. We say that $\mathscr{E}$ satisfies **Noether property at** x every every ascending chain of coherent submodules of $\mathscr{E}$ is stationary at x. We say that $\mathscr{E}$ satisfies **Noether property** if it satisfies Noether property at every $x \in X$.

It is clear that if $\mathscr{E}$ satisfies Noether property, then any ascending chain of coherent submodules of $\mathscr{E}$ is stationary on any precompact open subset of X.

Lemma 3.11.2. *Let* $0 \to \mathscr{E} \to \mathscr{F} \xrightarrow{\psi} \mathscr{G} \to 0$ *be an exact sequence of* $\mathscr{O}_X$*-modules. If* $\mathscr{F}$ *satisfies Noether property at* x *then so does* $\mathscr{G}$*. If* $\mathscr{E}$ *and* $\mathscr{G}$ *satisfy Noether property at* x *then so does* $\mathscr{F}$*.*

Proof. Assume that $\mathscr{F}$ satisfies Noether property at x. Let $(\mathscr{G}_i)_{i\in\mathfrak{I}}$ be an ascending chain of coherent submodules of $\mathscr{G}$. By Cor. 2.1.8, $\psi^{-1}(\mathscr{G}_i)$ is $\mathscr{O}_X$-coherent. So we have an ascending chain $(\mathscr{G}_i)_{i\in\mathfrak{I}}$ which is stationary at x. Then $(\mathscr{G}_i)_{i\in\mathfrak{I}} = (\psi(\psi^{-1}(\mathscr{G}_i)))_{i\in\mathfrak{I}}$ is stationary at x.

Now assume that $\mathscr{E}$ and $\mathscr{G}$ satisfy Noether property at x. Let $(\mathscr{F}_i)_{i\in\mathfrak{I}}$ be an ascending chain of coherent submodules of $\mathscr{F}$. We regard $\mathscr{E}$ as a submodule of $\mathscr{F}$. Then $(\psi(\mathscr{F}_i))_{i\in\mathfrak{I}}$ and $(\mathscr{E} \cap \mathscr{F}_i)_{i\in\mathfrak{I}}$ are ascending chains of coherent submodules of $\mathscr{G}$ and $\mathscr{E}$ respectively, where the coherence is due to Cor. 2.1.5 and 2.1.6. So they are stationary at x. From this one deduces that $(\mathscr{F}_i)_{i\in\mathfrak{I}}$ is stationary at x. $\square$

Lemma 3.11.3. *Let $\varphi : X \to Y$ be a finite holomorphic map of complex spaces, and let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Let $x \in X$. If $\varphi_*\mathscr{E}$ satisfies Noether property at $y = \varphi(x)$, then $\mathscr{E}$ satisfies Noether property at x.*

Proof. Let $(\mathscr{E}_i)_{i\in\mathfrak{I}}$ be an ascending chain of coherent submodules of $\mathscr{E}$. Then $(\varphi_*\mathscr{E}_i)_{i\in\mathfrak{I}}$ is an ascending chain of coherent (Thm. 2.7.1) submodules of $\varphi_*\mathscr{E}$. By assumption, after shrinking Y to a neighborhood of y and shrinking X to $\varphi^{-1}(Y)$, there is $i \in \mathfrak{I}$ such that for all $j \geqslant i$ we have $\varphi_*\mathscr{E}_i = \varphi_*\mathscr{E}_j$, namely $(\varphi_*\mathscr{E}_i)_{\mathfrak{y}} = (\varphi_*\mathscr{E}_j)_{\mathfrak{y}}$ for all $\mathfrak{y} \in Y$. This means, by Prop. 2.4.5, that $\mathscr{E}_{i,\mathfrak{x}} = \mathscr{E}_{j,\mathfrak{x}}$ for all $\mathfrak{x} \in \varphi^{-1}(\mathfrak{y})$ and $j \geqslant i$. So $\mathscr{E}_i = \mathscr{E}_j$ when $j \geqslant i$. □

Lemma 3.11.4. *Let X be a connected complex manifold, and let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$. If $\mathcal{I} \neq 0_X$, then $\mathcal{I}_x \neq 0$ for every $x \in X$.*

Proof. We know that $\mathrm{Supp}(\mathcal{I})$ is a (closed) analytic subset of X. If we can show that $\mathrm{Supp}(\mathcal{I})$ is open in X, then $\mathrm{Supp}(\mathcal{I}) = X$, which finishes the proof of the lemma.

Choose any $x \in \mathrm{Supp}(\mathcal{I})$. Then $\mathcal{I}_x \neq 0$. There exist a connected neighborhood U of x and $f \in \mathcal{I}(U)$ such that f is non-zero in $\mathscr{O}_{U,x}$. By Identitätssatz 1.1.3, f is non-zero in $\mathscr{O}_{U,p}$ for all $p \in U$, which shows $U \subset \mathrm{Supp}(\mathcal{I})$. □

Theorem 3.11.5. *Let X be a complex space and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Then $\mathscr{E}$ satisfies Noether property.*

Proof. We need to prove that any coherent module $\mathscr{E}$ satisfies Noether property at x. We prove this by induction on $\dim_x X$. Then case $\dim_x X = 0$ is obvious since $\mathscr{E}$ is a finite-dimensional vector space (Cor. 2.7.4). Now assume that coherent sheaves satisfy Noether property at x (for all X and $x \in X$) whenever $\dim_x X \leqslant n-1$ and $n \in \mathbb{Z}_+$. Let us prove that this is true when $\dim_x X \leqslant n$.

By Prop. 3.9.3, after shrinking X to a neighborhood of x, we may find a finite map from X to an open subset of $\mathbb{C}^n$. Therefore, by Lemma 3.11.3, it suffices to assume that X is an open subset of $\mathbb{C}^n$. Let $\mathscr{E}$ be a coherent $\mathscr{O}_{X,x}$-module. After shrinking X further, $\mathscr{E}$ is the cokernel of a morphism of free $\mathscr{O}_X$-modules (Thm. 2.1.9). Thus, by Lemma 3.11.2, it suffices to assume that $\mathscr{E}$ is $\mathscr{O}_X$-free, and hence that $\mathscr{E} = \mathscr{O}_X$.

Let $(\mathcal{I}_i)_{i\in\mathfrak{I}}$ be an ascending chain of ideals of $\mathscr{O}_X$. We need to show that it is stationary at x. It suffices to assume that $\mathcal{I}_k \neq 0_X$ for some k. By Lemma 3.11.4, $\mathcal{I}_{k,x} \neq 0$. Shrink X so that we can find $f \in \mathcal{I}_k(X)$ non-zero in $\mathscr{O}_{X,x}$. By discarding all $i < k$, we assume that $f\mathscr{O}_X \subset \mathcal{I}_i$ for all $i \in \mathfrak{I}$. Let $\mathcal{J}_i = \mathcal{I}_i/f\mathscr{O}_X$, which is an $\mathscr{O}_X$-submodule of $\mathscr{O}_X/f\mathscr{O}_X$. Identify $\mathcal{J}_i$ with its restriction to $Y = \mathrm{Specan}(\mathscr{O}_X/f\mathscr{O}_X)$. Then $(\mathcal{J}_i)_{i\in\mathfrak{I}}$ is an ascending chain of coherent ideals of $\mathscr{O}_Y$. Since $\mathscr{O}_{X,x}$ is an integral domain, we have $\dim \mathscr{O}_{Y,x} = \dim \mathscr{O}_{X,x} - 1 = n-1$ by Active lemma. Thus, by assumption, $(\mathcal{J}_i)_{i\in\mathfrak{I}}$ is stationary at x. Therefore $(\mathcal{I}_i)_{i\in\mathfrak{I}}$ is stationary at x. □

3.12 Openness and dimensions of fibers I

Definition 3.12.1. Let $\varphi : X \to Y$ be a continuous map of topological spaces. We say that φ is **open at** $x \in X$ if for each neighborhood $U \subset X$ of x, $\varphi(U)$ contains a neighborhood of $\varphi(x)$. We say that φ is **open (on X)** if φ is open at every point of X.

It is clear that φ is open at x iff

$$\{U \subset X : U \text{ is a neighborhood of } X \text{ and } \varphi(U) \text{ is open in } Y\}$$

is a base of neighborhoods of x. □

3.12.1 Dimension Formula (3.12.2)

In the following, $\varphi : X \to Y$ always denotes a holomorphic map of complex spaces. For each $y \in Y$, the **fiber** X_y means the inverse image $\varphi^{-1}(y)$ (cf. Prop. 1.12.1), namely

$$X_y = \varphi^{-1}(y) = \mathrm{Specan}(\mathscr{O}_X/\mathscr{I}_{\{y\}}\mathscr{O}_X).$$

Recall that $\mathscr{I}_{\{y\}}$ is the ideal of all $g \in \mathscr{O}(Y)$ vanishing at y, and can also be written as $\mathfrak{m}_{Y,y}$ by abuse of notations.

Proposition 3.12.2. *The following are true for $\varphi : X \to Y$.*

(1) For each $x \in X$,

$$\dim_x X_{\varphi(x)} \geqslant \dim_x X - \dim_{\varphi(x)} Y. \tag{3.12.1}$$

(2) The function $x \in X \mapsto \dim_x X_{\varphi(x)}$ is upper semicontinuous.

Note that part (1) generalizes Cor. 3.9.5, and part (2) generalizes Cor. 3.9.4.

Proof. Fix $x \in X$. Let $m = \dim_x X_{\varphi(x)}$ and $n = \dim_{\varphi(x)} Y$. By the definition of dimensions, we may shrink Y to a neighborhood of $y = \varphi(x)$ and X to a neighborhood of x inside $\varphi^{-1}(Y)$ such that there exist $f_1, \dots, f_m \in \mathscr{O}(X)$ such that x is the only point of $N(f_1, \dots, f_m) \cap X_y$. Consider $(f_\bullet) \in \mathscr{O}(X)^m$ as a holomorphic map $X \to \mathbb{C}^m$, and let $\Psi = (f_\bullet) \vee \varphi : X \to \mathbb{C}^m \times Y$. Then x is the only point of $\Psi^{-1}(\Psi(x)) = \Psi^{-1}(0 \times y)$. Therefore, by Thm. 2.7.2, we may shrink X and Y further so that Ψ is finite. Then by Cor. 3.9.5 and Prop. 3.10.8,

$$\dim_x X \leqslant \dim_{0\times y} \mathbb{C}^m \times Y = m + n.$$

This proves (1) .

Since Ψ is finite, each $p \in X$ is an isolated point of $\Psi^{-1}(\Psi(p))$. Since

$$\Psi^{-1}(\Psi(p)) = N(f_1 - f_1(p), \dots, f_m - f_m(p)) \cap X_{\varphi(p)}$$

we must have $\dim_p X_{\varphi(p)} \leqslant m$. This proves (2). □

Our main goal of this section and the next one is to understand when the following **Dimension Formula** holds:

$$\boxed{\dim_x X_{\varphi(x)} = \dim_x X - \dim_{\varphi(x)} Y} \tag{3.12.2}$$

More precisely, our goal is to understand the following result of Remmert which relates the openness of φ and (3.12.2).

Corollary 3.12.3. *Assume that Y is locally irreducible. Then $\varphi : X \to Y$ is open if and only if Dimension Formula (3.12.2) holds for all $x \in X$.*

Proof. This follows from Thm. 3.13.1 and 3.13.3, together with the fact that every locally irreducible space is locally pure dimensional (Thm. 3.14.9). □

The following proposition is helpful for the proof of Thm. 3.13.1 and 3.13.3.

Proposition 3.12.4. *Assume that X and Y are locally pure dimensional. If Dimension Formula (3.12.2) holds at $x_0 \in X$, then it holds everywhere on a neighborhood of x_0.*

Proof. Since X and Y are locally pure dimensional, we may shrink X to a neighborhood of x_0 so that the RHS of (3.12.2) is constant over all $x \in X$. By Prop. 3.12.2-(2), after further shrinking X, $\dim_x X_{\varphi(x)} \leqslant \dim_{x_0} X_{\varphi(x_0)}$ for all $x \in X$. Assume that (3.12.2) holds at x_0. Then by Prop. 3.12.2-(1), $\dim_x X_{\varphi(x)} \geqslant \dim_{x_0} X_{\varphi(x_0)}$ for all $x \in X$. So "$=$" holds. So $\dim_x X_{\varphi(x)}$ is constant over $x \in X$. Therefore (3.12.2) holds for all $x \in X$. □

3.12.2 Openness and Dimension Formula: the finite case

In this subsection, we study the relation between Dimension Formula and openness when φ is finite.

Lemma 3.12.5. *Assume that $\varphi : X \to Y$ is finite. Let $x \in X$ and $y = \varphi(x)$, and assume that x is the only point of $\varphi^{-1}(y)$. Then the following are equivalent.*

(1) φ is open at x.

(2) The set $\varphi(X)$ contains a neighborhood of y in Y.

Proof. Clearly (1)⇒(2). Assume (2). By Prop. 2.4.1, for each neighborhood $U \subset X$ of x there is a neighborhood $V \subset Y$ of y such that $\varphi^{-1}(V) \subset U$. Then $\varphi(U)$ contains $\varphi(\varphi^{-1}(V)) = V \cap \varphi(X)$, and $V \cap \varphi(X)$ contains a neighborhood of $y \in Y$ by (2). This proves (1). □

Theorem 3.12.6. *Assume that $\varphi : X \to Y$ is finite, and let $x \in X$. Consider the following statements:*

(1) φ is open at x.

(2) Dimension Formula (3.12.2) holds at x, namely

$$\dim_x X = \dim_{\varphi(x)} Y. \tag{3.12.3}$$

Then (1)⇒(2). If Y is irreducible at $\varphi(x)$, then (2)⇒(1).

Proof of (1)⇒(2). Assume for simplicity that X, Y are reduced, and that (by Thm. 2.7.2) x is the only point of $\varphi^{-1}(y)$ where $y = \varphi(x)$. Assume (1). If $\dim_x X = 0$ then y contains a neighborhood of itself. So y is isolated in Y and hence (3.12.3) holds.

Now assume $\dim_x X > 0$. By Prop. 2.4.1, after shrinking Y to a neighborhood of y and shrinking X to $\varphi^{-1}(Y)$, we may assume that X has local decomposition $X = X_1 \cup \cdots \cup X_N$ at x. Then x is not an isolated point of any X_i, otherwise (3.3.4) does not hold. Thus $\dim_x X_i > 0$.

Recall that each $Y_i = \varphi(X_i)$ is an analytic subset (i.e. reduced complex subspace) of Y, cf. Exe. 2.3.11. By Def. 2.3.8 and Exe. 2.3.11,

$$\mathscr{O}_{Y_i,y} = \big(\mathscr{O}_Y / \mathscr{Ann}(\varphi_* \mathscr{O}_{X_i})\big)_y = \mathscr{O}_{Y,y}/\mathrm{Ann}_{\mathscr{O}_{Y,y}}(\mathscr{O}_{X_i,x}),$$

where we have used Prop. 2.4.5 in the last equality. Thus $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X_i,x}$ restricts to an injective map $\varphi^\# : \mathscr{O}_{Y_i,y} \to \mathscr{O}_{X_i,x}$. Therefore $\mathscr{O}_{Y_i,y}$ is an integral domain since $\mathscr{O}_{X_i,x}$ is so.

By Cor. 3.9.5, $\dim_y Y_i > 0$. Therefore $\mathfrak{m}_{Y,y} \not\subset \mathscr{I}_{Y_i,y}$ for each i. By Lemma 3.4.5, there exists $g \in \mathfrak{m}_{Y,y}$ which is a non-zero in $\mathscr{O}_{Y_i,y}$ for all i. Thus $g \circ \varphi$ is non-zero in $\mathscr{O}_{X_i,x}$. Shrink Y and shrink X to $\varphi^{-1}(X)$ so that $g \in \mathscr{O}(Y)$. Then, by Active lemma, $\dim_y N(g) \cap Y_i = \dim Y_i - 1$ and $\dim_y N(g \circ \varphi) \cap X_i = \dim_x X_i - 1$. Thus, by Prop. 3.10.7, $\dim_y N(g) \cap \varphi(X) = \dim_y \varphi(X) - 1$ and

$$\dim_x N(g \circ \varphi) = \dim_x X - 1.$$

By assumption (1), $\varphi(X)$ contains a neighborhood of $y \in Y$. So

$$\dim_y N(g) = \dim_y Y - 1.$$

Clearly the restriction $\varphi : N(g \circ \varphi) \to N(g)$ is finite and open. Therefore, if (2) holds for this restriction then (2) holds for $\varphi : X \to Y$. This proves (2) by induction on $\dim_x X$. □

Proof of (2)⇒(1). Assume that (1) is not true and that Y is irreducible at $y = \varphi(x)$. By Thm. 2.7.2, we may assume that x is the only point of $\varphi^{-1}(y)$. By Lemma 3.12.5, the analytic subset $\varphi(X)$ contains no neighborhoods of $y \in Y$. Equivalently, the germs of sets $(\varphi(X), y)$ and (Y, y) are not equal. So $\mathscr{I}_{\varphi(X),y}$ is not 0 in $\mathscr{O}_{Y,y}$ (cf. Subsec. 3.3.1). After shrinking Y and shrinking X to $\varphi^{-1}(Y)$, we may find

$g \in \mathscr{O}(Y)$ non-zero in $\mathscr{O}_{Y,y}$ and vanishing on $\varphi(X)$. Since Y is irreducible at y, g is a non zero-divisor of $\mathscr{O}_{Y,y}$. So by Cor. 3.9.5 and Active lemma,

$$\dim_x X \leqslant \dim_y \varphi(X) \leqslant \dim_y N(g) = \dim_y Y - 1.$$

This disproves (2). □

Remark 3.12.7. In the proof of (1)⇒(2), we have shown that if $\varphi : X \to Y$ is a finite holomorphic map of complex spaces and if A is an analytic subset of X which is irreducible at $x \in X$, then the analytic subset $\varphi(A)$ of Y is irreducible at $\varphi(x)$. This also follows from the geometric description of the irreducibility in Rem. 3.3.2.

Example 3.12.8. Consider analytic subsets $X = 0 \times 0 \times \mathbb{C}^2$ and $Y = (\mathbb{C}^2 \times 0 \times 0) \cup (0 \times 0 \times \mathbb{C}^2)$ of $\mathbb{C}^4$, viewed as reduced complex spaces. Then Y is pure dimensional but is reducible at 0. Dimension Formula (3.12.2) holds for the inclusion map $\iota_{X,Y}$ at every point of X, but $\iota_{X,Y}$ is not open at 0. Therefore, in Thm. 3.12.6, to deduce (2)⇒(1) one cannot remove the irreducibility condition.

Corollary 3.12.9 (Invariance of dimensions). *Assume that $\varphi : X \to Y$ is finite, and let $y \in Y$. Then*

$$\dim_y \varphi(X) = \sup_{x \in X_y} \dim_x X. \tag{3.12.4}$$

Proof. By Rem. 2.4.4, we may assume Y is small enough such that $\varphi^{-1}(Y)$ is a disjoint union $\coprod_{x \in X_y} U_x$ where each U_x is a neighborhood of $x \in X$, and each restriction $\varphi : U_x \to Y$ is finite. Then $\varphi(X) = \bigcup_{x \in X_y} \varphi(U_x)$, and so by Prop. 3.10.7, we have $\dim_y \varphi(X) = \sup_{x \in X_y} \dim_y \varphi(U_x)$. By Lemma 3.12.5, $\varphi : U_x \to \varphi(U_x)$ is open at x. Thus, by Thm. 3.12.6, $\dim_y \varphi(U_x) = \dim_x U_x = \dim_x X$. □

3.13 Openness and dimensions of fibers II

We fix a holomorphic map of complex spaces $\varphi : X \to Y$.

3.13.1 Openness and Dimension Formula: the general case

The following theorem generalizes the part (2)⇒(1) of Thm. 3.12.6.

Theorem 3.13.1. *Let $x \in X$, and assume that Y is irreducible at $\varphi(x)$. If Dimension Formula (3.12.2) holds at x, then φ is open at x.*

Proof. Let $y = \varphi(x)$ and $\dim_x X_y = m$, and assume that (3.12.2) holds at x. We may shrink X to a neighborhood of x so that there exist $f_1, \dots, f_m \in \mathscr{O}(X)$ such that $N(f_1, \dots, f_m) \cap X_y = \{x\}$. Consider $F = (f_1, \dots, f_m)$ as a holomorphic map

$X \to \mathbb{C}^m$. Then $\Upsilon = \varphi \vee F : X \to Y \times \mathbb{C}^m$ satisfies that x is the only point of $\Upsilon^{-1}(y \times 0)$. Since the projection $\mathrm{pr}_Y : Y \times \mathbb{C}^m \to Y$ is open and $\varphi = \mathrm{pr}_Y \circ \Upsilon$, in order to show that φ is open at x it suffices to show that Υ is open at x.

By Thm. 2.7.2, we may shrink X further so that Υ is finite map from X to a neighborhood of $y \times 0$ in $Y \times \mathbb{C}^m$. By assumption, Dimension formula holds for Υ at x. Thus, Υ is open at x by Thm. 3.12.6 and the fact that $Y \times \mathbb{C}^m$ is irreducible at $y \times 0$ (due to Lemma 3.13.2). $\square$

Lemma 3.13.2. *If Y is irreducible at y, then $Y \times \mathbb{C}^m$ is irreducible at $y \times 0$.*

In fact, the product of any two irreducible points is irreducible. See Cor. 4.11.5.

Proof. By induction, it suffices to assume $m = 1$. Let $\mathscr{A} = \mathscr{O}_{Y,y}$ and $\mathscr{B} = \mathscr{O}_{Y \times \mathbb{C}, y \times 0}$. It suffices to prove that there is a monomorphism of $\mathbb{C}$-algebras $\mathscr{B} \to \mathscr{A}[[z]]$ where $\mathscr{A}[[z]]$ is the algebra of formal power series of z whose coefficients are elements of $\mathscr{A}$. Then since $\mathscr{A}$ is an integral domain, $\mathscr{A}[[z]]$ is clearly also an integral domain, and so is $\mathscr{B}$.

We may write $\mathscr{A} = \mathscr{O}_{\mathbb{C}^n,0}/J$ where J is an ideal of $\mathscr{O}_{\mathbb{C}^n,0}$, and write $y = 0$ (in $\mathbb{C}^n$). Let $(w_1, \dots, w_n, z)$ be the set of coordinates of $\mathbb{C}^{n+1}$. Then we have a $\mathbb{C}$-algebra monomorphism $\Phi : \mathscr{O}_{\mathbb{C}^{n+1},0} \to \mathscr{O}_{\mathbb{C}^n,0}[[z]]$ defined by taking power series expansions with respect to z. More precisely, if $f(w_\bullet, z)$ is in $\mathscr{O}_{\mathbb{C}^{n+1},0}$, then $\Phi(f) = \sum_{k \in \mathbb{N}} a_k(w_\bullet) z^k$ where

$$a_k(w_\bullet) = \mathrm{Res}_{z=0} f(w_\bullet, z) z^{-k-1} dz.$$

This formula shows that if $f(w_\bullet, z)$ belongs to $J\mathscr{O}_{\mathbb{C}^{n+1},0}$ (i.e. f is a (finite) $\mathscr{O}_{\mathbb{C}^{n+1},0}$-linear combination of elements $g(w_\bullet) \in J$) then each coefficient a_k belongs to J. Thus Φ restricts to a morphism

$$\Psi : \mathscr{B} = \mathscr{O}_{\mathbb{C}^{n+1},0}/J\mathscr{O}_{\mathbb{C}^{n+1},0} \to \mathscr{A}[[z]] = (\mathscr{O}_{\mathbb{C}^n,0}/J)[[z]]$$

If Ψ sends the residue class $[f] \in \mathscr{B}$ of $f \in \mathscr{O}_{\mathbb{C}^{n+1},0}$ to the zero element of $\mathscr{A}[[z]]$, then each a_k belongs to J. The power series expansion $f = \sum_k a_k z^k$ shows that $[f]$ belongs to $z^k\mathscr{B} \subset \mathfrak{m}^k_{Y \times \mathbb{C},0}$ for all $k \in \mathbb{N}$. Thus $[f] = 0$ by Krull's intersection Thm. 1.4.4. This proves that Ψ is injective. $\square$

Theorem 3.13.3. *Assume that Y is locally pure dimensional. If $\varphi : X \to Y$ is open, then Dimension Formula* (3.12.2) *holds for every $x \in X$.*

This theorem can not be proved by Thm. 3.12.6. Instead, a prototype of this theorem is Prop. 3.13.5-A which can be proved before we prove Thm. 3.13.3.

Proof. Assume that Y has pure dimension n. Fix $x \in X$ and $y = \varphi(x)$. Then (3.12.2) obviously holds when $n = 0$. Now assume $n > 0$. To prove (3.12.2) by induction on n, it suffices to show that after shrinking Y to a neighborhood of y and X to $\varphi^{-1}(Y)$, there exists $g \in \mathscr{O}(Y)$ with $g(y) = 0$ such that

(a) $N(g)$ has pure dimension $n - 1$.

(b) $\dim_x N(g \circ \varphi) = \dim_x X - 1$.

Then (3.12.2) holds at x for the restriction of φ to $N(g \circ \varphi) \to N(g)$ (since it is clearly open), and hence holds for $\varphi : X \to Y$.

By Rem. 3.10.4, we may shrink Y (and shrink X accordingly) so that there exists $g \in \mathscr{O}(Y)$ with $g(y) = 0$ such that g is active in $\mathscr{O}_{Y,y}$. Then by Active lemma, Dimension Formula (3.12.2) holds for $g : Y \to \mathbb{C}$ at y. Thus, by Prop. 3.12.4, (3.12.2) holds for g at every point of Y. This proves (a). It also proves, together with Thm. 3.13.1, that $g : Y \to \mathbb{C}$ is open. So $g \circ \varphi : X \to \mathbb{C}$ is open. Thus, by Prop. 3.13.5-A to be proved in the next subsection, $g \circ \varphi$ is active in $\mathscr{O}_{X,x}$. This proves (b) by Active lemma. □

Corollary 3.13.4. *Let $\varphi : X \to S$ and $\psi : Y \to S$ be open holomorphic maps of complex spaces. Assume that S is locally pure dimensional. Then for each $x \in X, y \in Y$ such that $s = \varphi(x)$ equals $\psi(y)$,*

$$\dim_{x \times y} X \times_S Y = \dim_x X + \dim_y Y - \dim_s S. \tag{3.13.1}$$

Proof. By Thm. 3.13.3, Dimension Formula (3.12.2) holds at $x \times y$ for both $\varphi \times \psi : X \times Y \to S \times S$ (which, together with Prop. 3.10.8, shows that as an analytic subset of $X \times Y$ the fiber $X_s \times Y_s = (X \times_S Y)_s$ has dimension $\dim_x X + \dim_y Y - 2\dim_s S$ at $x \times y$) and $X \times_S Y \to S$ (which shows that $\dim_{x \times y} X \times_S Y$ equals $\dim_{x \times y}(X \times_S Y)_s + \dim_s S$). □

3.13.2 Openness and active elements

Active lemma tells us that Dimension Formula (3.12.2) holds if $Y = \mathbb{C}$ and $\varphi : X \to \mathbb{C}$ (considered as a holomorphic function) satisfies that $\varphi - \varphi(x)$ is active at x. This suggests that active elements are related to openness. Let us give a result indicating their relationship.

Proposition 3.13.5. *Let $f \in \mathscr{O}(X)$. Consider the following conditions for $x \in X$.*

(1) The holomorphic map $f : X \to \mathbb{C}$ is open at x.

(2) $f - f(x)$ is active in $\mathscr{O}_{X,x}$.

Then the following are true.

$\boxed{A}$ *If (1) holds for all $x \in X$ then (2) holds for all $x \in X$.*

$\boxed{B}$ *If (2) holds for a given $x \in X$ then (1) holds for the same point x.*

Proof of $\boxed{A}$. Assume that f is open on X. Choose any $x \in X$. Let us prove (2) for x. Assume for simplicity that X is reduced and $f(x) = 0$. If f is not active, then after shrinking X to a neighborhood of x, there exists a non-zero $g \in \mathscr{O}(X)$ such that $fg = 0$ in $\mathscr{O}(X)$. Then $U = \{p \in X : g(p) \neq 0\}$ is a non-empty open subset of X, and $f(U) = \{0\}$. This contradicts the fact that f is open. □

Proof of $\boxed{B}$. Assume that (2) is true for a given $x \in X$. Then by Active lemma, (3.12.2) holds for $f : X \to \mathbb{C}$ at x. Thus f is open at x by Thm. 3.13.1. □

The proof of B shows that for any $f \in \mathscr{O}(X)$,

$$\begin{gathered} f - f(x) \text{ is active in } \mathscr{O}_{X,x} \\ \Downarrow \\ \text{Dimension Formula (3.12.2) holds for } f : X \to \mathbb{C} \text{ at } x \\ \Downarrow \\ f : X \to \mathbb{C} \text{ is open at } x \end{gathered} \tag{3.13.2}$$

The following example shows that in Thm. 3.13.3, knowing that φ is open at a point x is not sufficient to imply Dimension Formula (3.12.2) at x. It also shows that in Prop. 3.13.5, knowing that (1) holds at x is not sufficient to imply (2) at x.

Example 3.13.6. Let X be reduced with local decomposition $X = X_1 \cup \cdots \cup X_N$ at x where $N \geqslant 2$. Let $n = \dim_x X > 0$. By Rem. 3.1.4, we may shrink X to a neighborhood of x and find $f \in \mathscr{O}(X)$ which belongs to $\mathscr{I}_{X_2,x}, \dots, \mathscr{I}_{X_N,x}$ but not to $\mathscr{I}_{X_1,x}$. Then $f|_{X_1}$ is active in the integral domain $\mathscr{O}_{X_1,x} = \mathscr{O}_{X,x}/\mathscr{I}_{X_1,x}$, and hence $f(X_1)$ contains a neighborhood of $0 = f(x)$ in $\mathbb{C}$ by Prop. 3.13.5-B. This shows that f is open at x. However, by Cor. 3.1.6, f is not active in $\mathscr{O}_{X,x}$.

Now we assume that $\dim_x X_1 = \dim_x X_2 = n$. By Active lemma, $\dim_x N(f) \cap X_1 = n-1$ and $\dim_x N(f) \cap X_2 = n$. Apply Prop. 3.10.7 to $N(f) = \bigcup_i (N(f) \cap X_i)$. Then we see that $\dim_x N(f) = n$. So Dimension Formula (3.12.2) does not hold for $f : X \to \mathbb{C}$ at x.

Corollary 3.13.7 (Open mapping theorem). *Assume that X is reduced, and choose $f \in \mathscr{O}(X)$ and $x \in X$. If f is not a constant function on any neighborhood of $x \in X$, then $f : X \to \mathbb{C}$ is open at x.*

Note that the condition that f is not constant on neighborhoods of x means precisely that $f - f(x)$ is not zero in $\mathscr{O}_{X,x}$.

Proof. We may assume X is small enough such that there is a local decomposition $X = X_1 \cup \cdots \cup X_N$ at x, corresponding to the prime decomposition $\{0\} = \mathscr{I}_{X_1,x} \cap \cdots \cap \mathscr{I}_{X_N,x}$. Since $f - f(x)$ is not zero in $\mathscr{O}_{X,x}$, it does not belong to $\mathscr{I}_{X_i,x}$ for some i. So $f - f(x)$ is active in $\mathscr{O}_{X_i,x} = \mathscr{O}_{X,x}/\mathscr{I}_{X_i,x}$. Therefore, for each neighborhood U of $x \in X$, Prop. 3.13.5 implies that $f(X_i \cap U)$ contains a neighborhood of $f(x) \in \mathbb{C}$. So $f(U)$ contains a neighborhood of $f(x)$. f is open at x. □

3.14 Openness and torsion sheaves; irreducible and pure dimensional

The goal of this section is to use torsion sheaves to establish a relationship between irreducibility and pure dimensionality.

3.14.1 Coherence of torsion sheaves

Definition 3.14.1. Let $\mathcal{A}$ be a commutative ring and $\mathcal{M}$ an $\mathcal{A}$-module. A **torsion element** of $\mathcal{M}$ is an element $\xi \in \mathcal{M}$ such that $a\xi = 0$ for a non zero-divisor $a \in \mathcal{A}$ of $\mathcal{A}$. The set of torsion elements clearly form an $\mathcal{A}$-submodule, and is denoted by $T_{\mathcal{A}}(\mathcal{M})$ or simply $T(\mathcal{M})$ and called the **torsion module** of $\mathcal{M}$. We say that $\mathcal{M}$ is **torsion free** if $T(\mathcal{M}) = 0$. In general, $\mathcal{M}/T(\mathcal{M})$ is always torsion free.

Definition 3.14.2. Let X be a complex space and $\mathscr{E}$ an $\mathscr{O}_X$-module. The **torsion sheaf** of $\mathscr{E}$, denoted by $\mathscr{T}_{\mathscr{O}_X}(\mathscr{E})$ or simply $\mathscr{T}(\mathscr{E})$, is the sheaf associating to each open $U \subset X$:

$$\mathscr{T}_{\mathscr{O}_X}(\mathscr{E})(U) = \{s \in \mathscr{E}(U) : \text{the stalk } s_x \in T_{\mathscr{O}_{X,x}}(\mathscr{E}_x), \forall x \in U\}$$

We have a canonical equivalence

$$\mathscr{T}_{\mathscr{O}_X}(\mathscr{E})_x \simeq T_{\mathscr{O}_{X,x}}(\mathscr{E}_x). \tag{3.14.1}$$

Note that to show (3.14.1) one needs the fact that any $s \in \mathscr{T}(\mathscr{E})$ torsion in $\mathscr{E}_x$ is torsion in $\mathscr{E}_p$ for all p in a neighborhood of x. This follows from Prop. 2.3.13.

There is a geometric description of torsion elements:

Proposition 3.14.3. *Let X be a reduced complex space, $\mathscr{E}$ a coherent $\mathscr{O}_X$-module, and $s \in \mathscr{E}(X)$. Then s belongs to $\mathscr{T}(\mathscr{E})(X)$ if and only if* $\mathrm{Supp}(s) := \mathrm{Supp}(s\mathscr{O}_X)$ *is nowhere dense in X.*

Applying this proposition to sufficiently small neighborhoods of x, we see that the stalk s_x belongs to $T(\mathscr{E}_x)$ iff $\mathrm{Supp}(s) \cap U$ is nowhere dense in U for a neighborhood U of x. When X is irreducible at x, then by Cor. 3.4.2, $s_x \in T(\mathscr{E}_x)$ iff $\mathrm{Supp}(s)$ contains no neighborhoods of $x \in X$.

Proof. Assume that $s \in \mathscr{T}(\mathscr{E})(X)$. Then each $x \in X$ is contained in a neighborhood U such that there is $f \in \mathscr{O}(U)$ such that $fs = 0$ and that f is a non zero-divisor of $\mathscr{O}_{X,x}$. Then $\mathrm{Supp}(s) \cap U \subset N(f)$, and by Prop. 3.4.1, $N(f)$ is nowhere dense after shrinking U to a smaller neighborhood of x. This proves that $\mathrm{Supp}(s)$ is nowhere dense.

Conversely, suppose that $\mathrm{Supp}(s\mathscr{O}_X)$ is nowhere dense. Recall that $\mathrm{Supp}(s)$ is the zero set of $\mathscr{A}nn(s\mathscr{O}_X)$. Thus, by Prop. 3.4.4, there is a non zero-divisor

$f \in \mathscr{O}_{X,x}$ such that $f \in \mathscr{Ann}(s\mathscr{O}_X)_x = \mathrm{Ann}(s\mathscr{O}_{X,x})$. So $fs = 0$. Therefore s belongs to $T(\mathscr{E}_x)$ for every x. $\square$

In the following discussion of torsion sheaves, we are mainly interested in integral domains and locally irreducible spaces.

Proposition 3.14.4. *Let $\mathcal{M}$ be a finitely generated module of an integral domain $\mathcal{A}$. Then $T_{\mathcal{A}}(\mathcal{M})$ is the kernel of the canonical morphism $\mathcal{M} \to \mathcal{M}^{\vee\vee}$, where $\mathcal{M}^{\vee\vee}$ is the double dual of $\mathcal{M}$.*

Proof. Choose any $\xi \in \mathcal{M}$. Then ξ belongs to the kernel of $\Phi : \mathcal{M} \to \mathcal{M}^{\vee\vee}$ iff $\psi(\xi) = 0$ for all $\psi \in \mathcal{M}^\vee = \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$. If $\xi \in T(\mathcal{M})$ then $a\xi = 0$ for a nonzero $a \in \mathcal{A}$. So $a\psi(\xi) = \psi(a\xi) = 0$, and hence $\psi(\xi) = 0$ because a is a non zero-divisor.

Conversely, choose any $\xi \in \mathcal{M} \backslash T(\mathcal{M})$. We need to show that there exists $\psi \in \mathcal{M}^\vee$ such that $\psi(\xi) \neq 0$. Let $(\mathcal{A}^\times)^{-1}\mathcal{M}$ be the **localization** of $\mathcal{M}$ by $\mathcal{A}^\times = \{a \in \mathcal{A} : a \neq 0\}$, which is a vector space over the factional field $\mathcal{Q} = (\mathcal{A}^\times)^{-1}\mathcal{A}$. (So elements of $(\mathcal{A}^\times)^{-1}\mathcal{M}$ are of the form $\xi/a, \eta/b, \dots$ where $\xi, \eta \in \mathcal{M}$ and $a, b \in \mathcal{A}^\times$. $\xi/a = \eta/b$ iff $ca\xi = cb\eta$ for some $c \in \mathcal{A}^\times$. See [AM, Chapter 3] for details). So $\xi/1$ is not zero in $(\mathcal{A}^\times)^{-1}\mathcal{M}$. We can thus choose a $\mathcal{Q}$-linear functional $\lambda : (\mathcal{A}^\times)^{-1}\mathcal{M} \to \mathcal{Q}$ such that $\lambda(\xi/1) \neq 0$.

Since $\mathcal{M}$ is $\mathcal{A}$-generated by finitely many elements $\eta_1, \eta_2, \dots$, we may find $a \neq 0$ in $\mathcal{A}$ such that $a\lambda(\eta_i) \in \mathcal{A}$ for each i. Then

$$\psi : \mathcal{M} \to (\mathcal{A}^\times)^{-1}\mathcal{M} \xrightarrow{a\lambda} \mathcal{A}$$

is an $\mathcal{A}$-module morphism non-zero at ξ. $\square$

From this proposition it follows immediately that:

Corollary 3.14.5. *Let X be a locally irreducible complex space and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Then $\mathscr{T}_{\mathscr{O}_X}(\mathscr{E})$ is the kernel of the canonical morphism $\mathscr{E} \to \mathscr{E}^{\vee\vee}$. Consequently, $\mathscr{T}_{\mathscr{O}_X}(\mathscr{E})$ is $\mathscr{O}_X$-coherent.*

Remark 3.14.6. In Cor. 3.14.5, note that the support of $\mathscr{T}(\mathscr{E})$ (as a set) is an analytic subset of X (Rem. 2.3.5). It is clearly inside the non locally-free locus of $\mathscr{E}$. Therefore, by Thm. 3.8.3, the support of $\mathscr{T}(\mathscr{E})$ is nowhere dense in X.

3.14.2 Openness and torsion sheaves

Proposition 3.14.7. *Let $\varphi : X \to Y$ be a finite holomorphic map of complex spaces. Let $y \in Y$. Consider the following statements:*

(1) $(\varphi_*\mathscr{O}_X)_y$ is a torsion free $\mathscr{O}_{Y,y}$-module.

(2) φ is open at every $x \in X_y = \varphi^{-1}(y)$.

Then the following are true.

$\boxed{A}$ *If Y is irreducible at $y \in Y$ then (1)⇒(2).*

$\boxed{B}$ *If X is irreducible at every $x \in X_y$ then (2)⇒(1).*

Recall that $\varphi_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent by Thm. 2.7.1. Also, note that $(\varphi_*\mathscr{O}_X)_y = \bigoplus_{x\in X_y}\mathscr{O}_{X,x}$ by Prop. 2.4.5. Thus $(\varphi_*\mathscr{O}_X)_y$ is $\mathscr{O}_{Y,y}$-torsion-free iff $\mathscr{O}_{X,x}$ is $\mathscr{O}_{Y,y}$-torsion-free for every $x \in X_y$.

Proof of $\boxed{A}$. Assume that Y is irreducible at $y \in Y$ and (2) is not true. Then φ is not open at some $x \in X$. By Thm. 2.7.2 there is a neighborhood U of x and V of $\varphi(U)$ so that φ restricts to a finite holomorphic map $\varphi : U \to W$ such that x is the only point of $U_y = U \cap \varphi^{-1}(y)$. By Lemma 3.12.5, the germ of analytic subset $(\varphi(U), y)$ does not equal (W, y). Thus, after shrinking U and W to smaller neighborhoods of x and y respectively, we can find $g \in \mathscr{O}(W)$ non-zero in the integral domain $\mathscr{O}_{Y,y}$ such that g vanishes on $\varphi(U)$. So $g \circ \varphi$ takes value zero on U. Thus, by Nullstellensatz, g^n annihilates $1 \in \mathscr{O}_{X,x}$ for some $n \in \mathbb{Z}_+$. Thus $\mathscr{O}_{X,x}$ is not $\mathscr{O}_{Y,x}$-torsion-free. Therefore (1) is not true. □

Proof of $\boxed{B}$. Assume that X is irreducible at every $x \in X_y$ and (2) is true. If $\mathscr{O}_{X,x}$ is not $\mathscr{O}_{X,x}$-torsion free for some $x \in X_y$, then we can find a non zero-divisor $g \in \mathscr{O}_{Y,y}$ and nonzero $f \in \mathscr{O}_{X,x}$ such that $fg = 0$. More precisely: $(g \circ \varphi) \cdot f = 0$. Since $\mathscr{O}_{X,x}$ is an integral domain, $g \circ \varphi$ is zero in $\mathscr{O}_{X,x}$. Shrink Y to a neighborhood of y so that $g \in \mathscr{O}(Y)$, and shrink X to $\varphi^{-1}(Y)$. Then we can find a neighborhood U of x such that $g \circ \varphi$ is zero in $\mathscr{O}(U)$. Therefore g takes zero value on $\varphi(U)$. Since φ is open at x, $\varphi(U)$ contains a neighborhood of y. Therefore, by Nullstellensatz, g is nilpotent in $\mathscr{O}_{Y,y}$, contrary to the fact that g is a non zero-divisor. So (1) must be true. □

Example 3.14.8. Every Weierstrass map $\pi : X \to S$ is open. One can check that this follows from Rem. 1.5.2. But it also follows from Prop. 3.14.7, as explained below.

Proof. By Thm. 2.5.4, $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-free. So by Prop. 3.14.7, π is open if S is smooth. In the general case, we may assume that S is small enough such that it is a closed subspace of an open subset Ω of $\mathbb{C}^n$. Then by Prop. 2.5.3, we have a Cartesian square

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\pi"'] & Y \arrow[d, "\varpi"] \\ S \arrow[r, hook] & \Omega \end{tikzcd} \tag{3.14.2}$$

(so $\varpi^{-1}(S) = X$, cf. Prop. 1.12.1) where ϖ is a Weierstrass map. So ϖ is open. Choose any open $U \subset X$. Then $U = X \cap V$ for an open $V \subset Y$. $\varpi(V)$ is open in Ω. So $\varpi(V) \cap S = \varpi(V \cap \varpi^{-1}(S)) = \varpi(U) = \pi(U)$ is open in S. This proves that π is open. □

3.14.3 Irreducible and pure dimensional

Theorem 3.14.9. *Let X be a complex space irreducible at x. Then X is pure dimensional at x.*

Recall Def. 3.9.1 for the definition of pure dimensionality at a point.

Proof. By Thm. 3.2.1, we may shrink X so that X is reduced. Let $n = \dim_x X$. Then after shrinking X further, we may find a finite holomorphic map $\varphi : X \to V$ such that V is open in $\mathbb{C}^n$, $\varphi(x) = 0$, and x is the only point of $\varphi^{-1}(0)$ (due to Thm. 2.7.2). By Thm. 3.12.6, φ is open at x. So by Prop. 3.14.7, $(\varphi_*\mathscr{O}_X)_0$ is $\mathscr{O}_{V,0}$-torsion-free. By Cor. 3.14.5, $\mathscr{T}(\varphi_*\mathscr{O}_X)$ is $\mathscr{O}_V$-coherent. Thus, after shrinking V to a neighborhood of 0 and replacing X by $\varphi^{-1}(V)$, $\varphi_*\mathscr{O}_X$ is $\mathscr{O}_V$-torsion-free. By Prop. 3.14.7, φ is open at every point of X. Therefore, by Thm. 3.12.6, X has pure dimension n. □

Corollary 3.14.10. *Let X be a complex space and $x \in X$. Let $n \in \mathbb{N}$. Then the following are equivalent.*

(1) In the prime decomposition $\sqrt{0_{X,x}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$ of $\sqrt{0_{X,x}} \subset \mathscr{O}_{X,x}$ we have

$$\dim \mathscr{O}_{X,x}/\mathfrak{p}_i = n$$

for all $1 \leqslant i \leqslant N$.

(2) X has pure dimension n at x.

Proof. This is clear from Thm. 3.14.9, Prop. 3.10.7, and (3.3.4) (for the direction (2)⇒(1)). □

Example 3.14.11. Let $\varphi : X \to Y$ be a finite holomorphic map of complex spaces. Assume that Y is locally irreducible. Let $x \in X$ and $y = \varphi(x)$. If X is pure dimensional at x and φ is open at x, then φ is open on a neighborhood of X.

Proof. By Thm. 3.14.9, Y is locally pure dimensional. By Thm. 2.7.2, we may shrink X and Y to neighborhoods of x and y respectively such that X has pure dimension m and Y has pure dimension n, and that φ is still finite. By Thm. 3.12.6, Dimension Formula (3.12.2) holds for φ at x. Thus $m = n$. By Prop. 3.12.4, we may shrink X and Y further so that φ is finite and (3.12.2) holds at every point of X. So φ is open by Thm. 3.12.6. □

As an application of Thm. 3.14.9, we use global dimensions of complex manifolds (i.e. the largest dimensions of connected components, recall Def. 3.9.1) to describe dimensions of points of complex spaces:

Proposition 3.14.12. *Let X be a reduced complex space and $x \in X$. Then*

$$\dim_x X = \inf \left\{ \dim U \backslash \mathrm{Sg}(X) : U \text{ is a neighborhood of } x \right\}$$

Proof. We have $\geqslant$ by Cor. 3.9.4. To show $\leqslant$, we need to show that there exist smooth points arbitrarily close to x whose dimension is $n = \dim_x X$. We first shrink X to a neighborhood of x so that X has local decomposition $X = X_1 \cup \cdots \cup X_N$ at x. By Prop. 3.10.7, we have $\dim_x X_i = n$ for some i. By Thm. 3.14.9, we can shrink X further so that X_i has pure dimension n. By (3.3.4), for each neighborhood U of $x \subset X$, $V = U \cap (X_i \backslash \bigcup_{j \neq i} X_j)$ is non-empty. Since $\mathrm{Sg}(V)$ is nowhere dense in V, $V \backslash \mathrm{Sg}(V)$ is a non-empty open complex submanifold of $U \backslash \mathrm{Sg}(X)$ with pure dimension n. This finishes the proof. $\square$

Chapter 4

Normalization and global irreducibility

4.1 Sheaves of meromorphic functions $\mathscr{M}_X$

We fix a *reduced* complex space X. So non zero-divisors and active elements are synonymous.

Definition 4.1.1. The **sheaf of (germs of) densely defined holomorphic functions** of X is the sheaf $\mathfrak{W}_X$ associated to presheaf $\mathfrak{W}_X^{\mathrm{pre}}$ such that for each open $U \subset X$,

$$\mathfrak{W}_X^{\mathrm{pre}}(U) = \varinjlim_{\substack{\text{nowhere dense} \\ \text{analytic subsets } A \subset U}} \mathscr{O}(U \backslash A)$$

where the direct limit is defined by the obviously injective inclusion maps $\mathscr{O}(U\backslash A) \to \mathscr{O}(U\backslash B)$ if $A, B \subset U$ are nowhere dense analytic subsets and $A \subset B$.

$\mathfrak{W}_X$ clearly contains $\mathscr{O}_X$ and, more generally, contains $\mathscr{O}_{X\backslash A}$ as subsheaves where A is any nowhere dense analytic subset of X. Moreover, we have an obvious identification

$$\mathfrak{W}_X(X) = \mathfrak{W}_X(X\backslash A) \tag{4.1.1}$$

□

Remark 4.1.2. $\mathfrak{W}_X$ is a torsion-free $\mathscr{O}_X$-module.

Proof. Choose $x \in X$, $f \in \mathfrak{W}_{X,x}$, and a non zero-divisor $v \in \mathscr{O}_{X,x}$ such that $vf = 0$. By shrinking X to a neighborhood of x, we may assume that $v \in \mathscr{O}(X)$, that $f \in \mathscr{O}(X\backslash A)$ where $A \subset X$ is a nowhere dense analytic subset, and that (by Prop. 3.4.1) $N(v)$ is nowhere dense in X. So f must be zero outside $A \cup N(v)$. Thus f is zero in $\mathscr{O}(X\backslash A)$. □

4.1.1 The sheaf of meromorphic functions $\mathscr{M}_X$

Definition 4.1.3. The **sheaf of (germs of) meromorphic functions on** X is the subsheaf $\mathscr{M}_X$ of $\mathfrak{W}_X$ defined by

$$\begin{aligned}\mathscr{M}_X(U) = \{f \in \mathfrak{W}_X(U) : \forall x \in U \text{ there is an active } v \in \mathscr{O}_{X,x}\\ \text{such that } vf_x \in \mathscr{O}_{X,x}\}\end{aligned}$$

where $U \subset X$ is open and f_x denotes the stalk of f at x.

If $\mathcal{A}$ a commutative ring, we let

$$\mathrm{Nzd}(\mathcal{A}) = \{\text{Non zero-divisors of } \mathcal{A}\}. \tag{4.1.2}$$

Recall that if $\mathcal{M}$ is an $\mathcal{A}$-module, then the **localization** of $\mathcal{M}$ by $\mathrm{Nzd}(\mathcal{A})$, which is denoted by $\mathrm{Nzd}(\mathcal{A})^{-1}\mathcal{M}$, is the set of elements of the form s/u where $s \in \mathcal{M}$ and $u \in \mathrm{Nzd}(\mathcal{A})$, and $s/u = s'/u'$ iff $u's - us'$ is annihilated by an element of $\mathrm{Nzd}(\mathcal{A})$. In the case that $\mathcal{M}$ is torsion free (e.g. $\mathcal{A} = \mathcal{M} = \mathscr{O}_{X,x}$), $s/u = s'/u'$ iff $u's = us'$.

Remark 4.1.4. Note that for any active $v \in \mathscr{O}_{X,x}$ one can find a neighborhood $V \subset X$ of x so that $v \in \mathscr{O}(V)$ and $N(v)$ is nowhere dense in V (Prop. 3.4.1). From this, it is clear that each f/v where $v \in \mathrm{Nzd}(\mathscr{O}_{X,x})$ and $f \in \mathscr{O}_{X,x}$ can be extended to an element of $\mathscr{M}_X(U)$. Therefore, we have a canonical equivalence

$$\mathscr{M}_{X,x} \simeq \mathrm{Nzd}(\mathscr{O}_{X,x})^{-1}\mathscr{O}_{X,x} \tag{4.1.3}$$

In particular, if X is irreducible at x, then $\mathscr{M}_{X,x}$ is the field of fractions of $\mathscr{O}_{X,x}$.

Proposition 4.1.5. *Every finite-type $\mathscr{O}_X$-submodule of $\mathscr{M}_X$ is $\mathscr{O}_X$-coherent.*

Proof. Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-submodule of $\mathscr{M}_X$. It suffices to show that the sheaves of relations of $\mathscr{E}$ are finite-type. Choose any open $U \subset X$ and $s_1, \dots, s_n \in \mathscr{E}(U)$. Let us show that for each $x \in U$, after shrinking U to a smaller neighborhood of x, $\mathscr{R}el(s_1, \dots, s_n)$ is $\mathscr{O}_U$-coherent. It is clear that we can shrink U and find $v \in \mathscr{O}(U)$ which is a non zero-divisor of $\mathscr{O}_{X,x}$ such that $vs_1, \dots, vs_n \in \mathscr{O}(U)$. By Prop. 3.4.1, we may shrink U further so that $N(v)$ is nowhere dense in U. Then it is clear that $\mathscr{R}el(s_1, \dots, s_n)$ equals $\mathscr{R}el(vs_1, \dots, vs_n)$, which is locally finitely generated because $\mathscr{O}_X$ is coherent. $\square$

Example 4.1.6. Let X be a reduced complex space with local decomposition $X = X_1 \cup \cdots \cup X_N$ at x such that Thm. 3.3.4 holds. Then we can define the **characteristic function**

$$\begin{gathered}\chi_{X_k} \in \mathscr{O}\Big(X \Big\backslash \bigcup_{1 \leqslant i < j \leqslant N} X_i \cap X_j\Big) \qquad \subset \mathfrak{W}_X(X)\\ \chi_{X_k}(p) = \begin{cases} 1 & \text{if } p \in X_k \backslash \bigcup_{1 \leqslant i \leqslant N} X_i \\ 0 & \text{otherwise}\end{cases}\end{gathered} \tag{4.1.4}$$

Then the stalk of χ_{X_k} at x belongs to $\mathscr{M}_{X,x}$.

Proof. When $N = 1$, $\chi_{X_k} = 1$ in $\mathscr{O}_X(X)$. So let us assume $N > 1$. We assume $k = 1$ for simplicity. Apply Rem. 3.1.4 to the prime decomposition $0 = \mathscr{I}_{X_1,x} \cap \cdots \cap \mathscr{I}_{X_N,x}$ of $0 \subset \mathscr{O}_{X,x}$. Then we can find

$$f \in \bigcap_{i>1} \mathscr{I}_{X_i} \Big\backslash \mathscr{I}_{X_1,x} \qquad g \in \mathscr{I}_{X_1,x} \Big\backslash \bigcup_{i>1} \mathscr{I}_{X_i,x}$$

Then $f + g \notin \mathscr{I}_{X_j}$ for all $1 \leqslant j \leqslant N$. Therefore $f + g \in \mathrm{Nzd}(\mathscr{O}_{X,x})$ by Cor. 3.1.6. We can thus shrink X to a neighborhood of x to get $f, g \in \mathscr{O}(X)$ satisfying that f vanishes on $X_2 \cup \cdots \cup X_N$ and that g vanishes on X_1. Then $(f + g)\chi_{X_1}$ and f are equal on $X \Big\backslash \bigcup_{1 \leqslant i < j \leqslant N} X_i \cap X_j$, and hence are equal as elements of $\mathfrak{W}_X(X)$. This proves $\chi_{X_k,x} \in \mathscr{M}_{X,x}$. $\square$

4.1.2 Polar sets $P(f)$

According to the definition of $\mathfrak{W}_X$, given an element $f \in \mathfrak{W}_X(X)$, one may not be able to find a nowhere dense analytic subset $A \subset X$ such that $f \in \mathscr{O}(X \backslash A)$. Namely, f is not necessarily in $\mathfrak{W}_X^{\mathrm{pre}}(X)$. But we can find such A whenever $f \in \mathscr{M}_X(X)$. Equivalently, we have $\mathscr{M}_X(X) \subset \mathfrak{W}_X^{\mathrm{pre}}(X)$.

Definition 4.1.7. Let $f \in \mathfrak{W}_X(X)$. The **polar set** of f is defined to be

$$P(f) = \{x \in X : f_x \notin \mathscr{O}_{X,x}\} \tag{4.1.5}$$

which is clearly a closed subset of X.

We shall show that if $f \in \mathscr{M}_X(X)$ then $P(f)$ is a nowhere dense analytic subset of X. Therefore $f \in \mathscr{O}(X \backslash P(f))$. We first discuss some general facts about coherent sheaves.

Remark 4.1.8. In the general case that X is not necessarily reduced, if $\mathscr{E}, \mathscr{F}$ are coherent $\mathscr{O}_X$-submodules of a coherent $\mathscr{O}_X$-module $\mathscr{G}$, then as sets,

$$\mathrm{Supp}(\mathscr{E}/\mathscr{E} \cap \mathscr{F}) = \{x \in X : \mathscr{E}_x \not\subset \mathscr{F}_x\}, \tag{4.1.6}$$

which are analytic subsets of X because $\mathscr{E}/\mathscr{E} \cap \mathscr{F}$ is coherent due to Cor. 2.1.6. With the help of this observation, we now show:

Proposition 4.1.9. *Let $f \in \mathscr{M}_X(X)$. Then $P(f)$ is a nowhere dense analytic subset of X.*

Proof. By (4.1.6), we have (at the level of sets) that

$$P(f) = \mathrm{Supp}(\mathscr{O}_X f / \mathscr{O}_X f \cap \mathscr{O}_X) \tag{4.1.7}$$

which is analytic because, by Prop. 4.1.5, $f\mathscr{O}_X$ and $f\mathscr{O}_X + \mathscr{O}_X$ are coherent $\mathscr{O}_X$-submodules of $\mathscr{M}_X$. Each $x \in X$ is contained in a neighborhood $U \subset X$ such that one can find $v \in \mathscr{O}(X)$ satisfying that $vf \in \mathscr{O}(U)$ and that v is a non zero-divisor of $\mathscr{O}_{X,x}$. By Prop. 3.4.1, we may shrink U to a smaller neighborhood of x so that $N(v)$ is nowhere dense in U. Since the stalks of v are invertible outside $N(v)$, we have $U \cap P(f) \subset U \cap N(v)$, which shows that $U \cap P(f)$ is nowhere dense in U. Therefore $P(v)$ is nowhere dense in X. □

4.2 Sheaves of weakly holomorphic functions $\widehat{\mathscr{O}}_X$

We fix a reduced complex space X.

Definition 4.2.1. We say that $f \in \mathfrak{W}_X(X)$ is **locally bounded at** x if there is a neighborhood $U \subset X$ of x and a nowhere dense analytic subset $A \subset U$ such that $f|_{U\backslash A} \in \mathscr{O}(U\backslash A)$, and that

$$\sup_{p\in U\backslash A} |f(p)| < +\infty.$$

We say that $f \in \mathfrak{W}_X(X)$ is a **weakly holomorphic function** if f is locally bounded at every point of X. The $\mathscr{O}_X$-module $\widehat{\mathscr{O}}_X$ defined by

$$\widehat{\mathscr{O}}_X(U) = \{f \in \mathfrak{W}_X(U) : f \text{ is locally bounded at every } x \in U\}$$

(for any open $U \subset X$) is called the **sheaf of (germs of) weakly holomorphic functions**.

Let us consider the question of whether a holomorphic map of reduced complex spaces induces a morphism of $\mathfrak{W}$-sheaves or $\mathscr{M}$-sheaves or $\widehat{\mathscr{O}}$-sheaves.

Proposition 4.2.2. *Let $\varphi : X \to Y$ be a holomorphic map of reduced complex spaces. Assume that for any open subset $V \subset Y$ and nowhere dense analytic subset $B \subset V$, the analytic subset $A = \varphi^{-1}(B)$ is nowhere dense in $U = \varphi^{-1}(V)$. Then the map $\varphi^\# : \mathscr{O}_Y(V\backslash B) \to \mathscr{O}_X(U\backslash A)$ induces a morphism of $\mathscr{O}_Y$-algebras*

$$\varphi^\# : \mathfrak{W}_Y \to \varphi_*\mathfrak{W}_X \tag{4.2.1a}$$

which restricts to morphisms of $\mathscr{O}_Y$-algebras

$$\varphi^\# : \mathscr{M}_Y \to \varphi_*\mathscr{M}_X \tag{4.2.1b}$$

$$\varphi^\# : \widehat{\mathscr{O}}_Y \to \varphi_*\widehat{\mathscr{O}}_X \tag{4.2.1c}$$

Proof. $\varphi^\# : \mathscr{O}_Y(V\backslash B) \to \mathscr{O}_X(U\backslash A)$, when passing to the direct limit over all nowhere dense analytic $B \subset V$, gives $\mathfrak{W}_Y^{\mathrm{pre}}(V) \to \mathfrak{W}_X^{\mathrm{pre}}(U)$, hence $\mathfrak{W}_Y^{\mathrm{pre}}(V) \to \mathfrak{W}_X(U) = \varphi_*\mathfrak{W}_X(V)$, hence $\mathfrak{W}_Y^{\mathrm{pre}} \to \varphi_*\mathfrak{W}_X$, and hence (4.2.1a). It clearly restricts to (4.2.1c).

If $g \in \mathscr{M}_Y(V)$, then by shrinking V to a neighborhood of $y = \varphi(x)$ for any $x \in U$, we can find $v \in \mathscr{O}_Y(V)$ such that $N(v)$ is nowhere dense in V, and that $vf \in \mathscr{O}_Y(V)$. Then $\varphi^\#(v)\varphi^\#(f) \in \mathscr{O}_X(U)$, and $N(\varphi^\#(v)) = \varphi^{-1}(N(v))$ is nowhere dense in $U = \varphi^{-1}(V)$. By Prop. 3.4.1, the stalk $\varphi^\#(v)_p$ at every $p \in U$ is a non zero-divisor of $\mathscr{O}_{X,p}$. Therefore $\varphi^\#(f) \in \mathscr{M}_X(U)$. This gives (4.2.1b). $\square$

Exercise 4.2.3. Under the assumption of Prop. 4.2.2, suppose moreover that φ is surjective. Show that $\varphi^\# : \mathfrak{W}_Y \to \varphi_*\mathfrak{W}_X$ is injective.

Theorem 4.2.4. *Assume $X = X_1 \cup \cdots \cup X_N$ where $X_1, \dots, X_N$ are analytic subsets of X, and assume for each $1 \leqslant i \neq j \leqslant N$ that $X_i \cap X_j$ is nowhere dense in X_i. Then the closed embedding $\iota_i : X_i \hookrightarrow X$ satisfies the assumption in Prop. 4.2.2. Moreover, we have isomorphisms of $\mathscr{O}_X$-algebras*

$$\bigoplus_i \iota_i^\# : \quad \mathfrak{W}_X \xrightarrow{\simeq} \bigoplus_{1\leqslant i\leqslant N} \mathfrak{W}_{X_i} \tag{4.2.2a}$$

$$\bigoplus_i \iota_i^\# : \quad \mathscr{M}_X \xrightarrow{\simeq} \bigoplus_{1\leqslant i\leqslant N} \mathscr{M}_{X_i} \tag{4.2.2b}$$

$$\bigoplus_i \iota_i^\# : \quad \widehat{\mathscr{O}}_X \xrightarrow{\simeq} \bigoplus_{1\leqslant i\leqslant N} \widehat{\mathscr{O}}_{X_i} \tag{4.2.2c}$$

Proof-Step 1. Let us show that ι_i satisfies the assumption in Prop. 4.2.2. Let $U \subset X$ be open and A be a nowhere dense analytic subset of U. Then we need to show that $A \cap X_i$ is nowhere dense in $U \cap X_i$. Set $Y = U$ and $Y_i = U \cap X_i$, which is an analytic subset of Y. Then A is a nowhere dense analytic subset of Y, and we need to show that $A \cap Y_i$ is nowhere dense in Y_i. We assume for simplicity that $i = 1$.

Consider the open subset $Y_1^\circ = Y_1\backslash(Y_2 \cup \cdots \cup Y_N) = Y\backslash(Y_2 \cup \cdots \cup Y_N)$ of Y. Then $A \cap Y_1^\circ$ contains no open subsets of Y_1°. If Ω is an open subset of Y_1 contained inside $A \cap Y_1$, then $\Omega \cap Y_1^\circ$ is an open subset of Y_1° contained inside $A \cap Y_1^\circ$, which is empty. Thus $\Omega \subset Y_1'$ where $Y_1' = Y_1\backslash Y_1^\circ$. But $Y_1' = \bigcup_{j>1} Y_1 \cap Y_j$ is nowhere dense in Y_1 since $\bigcup_{j>1} X_1 \cap X_j$ is nowhere dense in X_1. So Ω is empty. Thus $A \cap Y_1$ is nowhere dense in Y_1. $\square$

Proof-Step 2. That (4.2.2a) and (4.2.2c) are isomorphisms is not hard to check and is left to the readers. Since (4.2.2b) is the restriction of (4.2.2a), (4.2.2b) is injective. Let us show that the stalk map of (4.2.2b) at any $x \in X$ is surjective. By discarding those X_i not containing x, we assume $x \in \bigcap_{i=1}^N X_i$. Also, if (4.2.2b) is surjective (and hence isomorphic) in the special case that each X_i is irreducible at x, then it is isomorphic in the general case due to Prop. 3.3.5 and the special case. So we may well assume that each X_i is irreducible at x.

It suffices to show that each $\mathscr{M}_{X_i,x}$ (which is inside $\mathfrak{W}_{X_i,x} \subset \bigoplus_j \mathfrak{W}_{X_j,x} \simeq \mathfrak{W}_{X,x}$) belongs to $\mathscr{M}_{X,x}$. Set $i = 1$ for simplicity. Then we need to show that the zero-extension of each $f_1 \in \mathscr{M}_{X_1,x}$ from the germ (X_1, x) to (X, x), still denoted by f_1 but now belonging to $\mathfrak{W}_{X,x}$, is inside $\mathscr{M}_{X,x}$.

Choose $v_1 \in \mathrm{Nzd}(\mathscr{O}_{X_1,x})$ such that $g_1 := v_1 f_1 \in \mathscr{O}_{X_1,x}$. Since $\mathscr{O}_{X_1,x} = \mathscr{O}_{X,x}/\mathscr{I}_{X_1,x}$, we can lift (i.e. extend) v_1 and g_1 to elements

$$\widetilde{v}_1, \widetilde{g}_1 \in \mathscr{O}_{X,x}.$$

We add $\widetilde{\ }$ since $\widetilde{v}_1$ and $\widetilde{g}_1$ are not necessarily the *zero*-extensions of v_1 and g_1. By contrast, $\widetilde{v}_1 f_1$ (as an element of $\mathfrak{W}_{X,x}$) is the zero-extension of g_1.

We write the characteristic function χ_{X_i} (cf. Exp. 4.1.6) as χ_i, which is in $\mathscr{M}_{X,x}$. Then $\chi_1 \widetilde{v}_1 f_1 = \chi_1 \widetilde{g}_1$ in $\mathfrak{W}_{X,x}$. Also, it is clear that $\chi_j f_1 = 0$ in $\mathfrak{W}_{X,x}$ if $j > 1$. Let

$$u = \chi_1 \widetilde{v}_1 + \chi_2 + \cdots + \chi_N \qquad \in \mathscr{M}_{X,x}$$

Then $u f_1 = \chi_1 \widetilde{g}_1$ holds in $\mathfrak{W}_{X,x}$. Since $\widetilde{g}_1 \in \mathscr{O}_{X,x}$ and $\chi_1 \in \mathscr{M}_{X,x}$, we conclude $u f_1 \in \mathscr{M}_{X,x}$.

To show that $f_1 \in \mathscr{M}_{X,x}$, it now remains to show that u is a unit of $\mathscr{M}_{X,x}$, namely, u is the quotient of two elements of $\mathrm{Nzd}(\mathscr{O}_{X,x})$. Shrink X to a neighborhood of x so that $\widetilde{v}_1 \in \mathscr{O}_X(X)$ and $\chi_1, \dots, \chi_N \in \mathscr{M}_X(X)$. Then $u \in \mathscr{M}_X(X)$. u equals $v_1 \in \mathscr{O}_{X_1}(X_1)$ on $X_1 \backslash X_1'$ where $X_1' = X_2 \cup \cdots \cup X_N$, and equals 1 on $X_1' \backslash X_1$. Since v_1 is a non zero-divisor of $\mathscr{O}_{X_1,x}$, by Prop. 3.4.1, we can shrink X further so that $N(v_1)$ is nowhere dense in X_1. Choose $w \in \mathrm{Nzd}(\mathscr{O}_{X,x})$ such that $wu \in \mathscr{O}_{X,x}$, and shrink X so that $w \in \mathscr{O}(X)$ and that $N(w)$ is nowhere dense in X (again by Prop. 3.4.1). Then

$$N(wu) \subset (N(v_1) \cap (X_1 \backslash X_1')) \cup (X_1 \cap X_1') \cup N(w)$$

is nowhere dense in X, which implies that wu is a non zero-divisor of $\mathscr{O}_{X,x}$. □

A main goal of this chapter is to show *that $\widehat{\mathscr{O}}_X$ is a coherent $\mathscr{O}_X$-module, that $\widehat{\mathscr{O}}_X \subset \mathscr{M}_X$, and that $\widehat{\mathscr{O}}_{X,x}$ is the integral closure of $\mathscr{O}_{X,x}$ in $\mathfrak{W}_{X,x}$ (and hence in $\mathscr{M}_{X,x}$).* We first recall some facts about integral elements.

Recall that if $\mathcal{A}$ is a commutative ring and $\mathcal{B}$ is a commutative $\mathcal{A}$-ring, i.e. a commutative ring with a homomorphism $\mathcal{A} \to \mathcal{B}$, an element $x \in \mathcal{B}$ is called **integral over** $\mathcal{A}$ if

$$x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{4.2.3}$$

for some $n \in \mathbb{Z}_+$ and $a_0, \dots, a_{n-1} \in \mathcal{A}$. We collect some facts about integral elements.

Proposition 4.2.5. *Assume that $\mathcal{A}$ is Noetherian.*

1. *$x \in \mathcal{B}$ is integral over $\mathcal{A}$ if and only if x is contained in an $\mathcal{A}$-subring $\mathcal{C} \subset \mathcal{B}$ which is a finitely-generated $\mathcal{A}$-module.*

2. *Let $\mathfrak{A} \subset \mathcal{B}$ be an $\mathcal{A}$-subring of $\mathcal{B}$. Assume that $\mathfrak{A}$ is a finitely-generated $\mathcal{A}$-module. Then an element $x \in \mathcal{B}$ is integral over $\mathcal{A}$ if and only if x is integral over $\mathfrak{A}$.*

3. *If $x_1, \ldots, x_n \in \mathcal{B}$ are integral over $\mathcal{A}$, then $\mathcal{A}[x_1, \ldots, x_n]$ (the $\mathcal{A}$-subring of $\mathcal{B}$ generated by $x_1, \ldots, x_n$) is a finitely generated $\mathcal{A}$-module.*

Note that an $\mathcal{A}$-subring of $\mathcal{B}$ is a subset of $\mathcal{B}$ closed under multiplications and $\mathcal{A}$-linear combinations.

Proof. 1. Let $\mathcal{X}$ be the $\mathcal{A}$-subring generated by x, namely $\mathcal{X} = \mathcal{A}[x]$. Then x being integral means precisely that $\mathcal{X}$ is a finitely-generated $\mathcal{A}$-module. Then part 1 is obvious, because $\mathcal{A}$ is Noetherian.

2. The "only if" part is obvious. Suppose that x is integral over $\mathfrak{A}$. Then $\mathfrak{A}[x]$ is a finitely-generated $\mathfrak{A}$-module. Since $\mathfrak{A}$ is $\mathcal{A}$-finitely generated, $\mathfrak{A}[x]$ is clearly $\mathcal{A}$-finitely generated. Thus x is integral over $\mathcal{A}$ due to part 1.

3. Induction on n. The case $n = 1$ is clear. Assume case $n - 1$ is true. Let $x_1, \ldots, x_n \in \mathcal{B}$ be integral over $\mathcal{A}$. Then $\mathcal{X} = \mathcal{A}[x_1, \ldots, x_{n-1}]$ is $\mathcal{A}$-finitely generated, and (since x_n is clearly integral over $\mathcal{X}$) $\mathcal{A}[x_1, \ldots, x_n] = \mathcal{X}[x_n]$ is $\mathcal{X}$-finitely generated. Therefore $\mathcal{X}[x_n]$ is $\mathcal{A}$-finitely-generated. $\square$

Definition 4.2.6. Assume that $\mathcal{A}$ is Noetherian. The set of all elements of $\mathcal{B}$ which are integral over $\mathcal{A}$ is called the **integral closure** of $\mathcal{A}$ in $\mathcal{B}$, which is an $\mathcal{A}$-subring of $\mathcal{B}$ by Prop. 4.2.5. If $\mathcal{A}$ is the integral closure of $\mathcal{A}$, we say $\mathcal{A}$ is **integrally closed** in $\mathcal{B}$. If $\mathcal{A}$ is integrally closed in $\mathrm{Nzd}(\mathcal{A})^{-1}\mathcal{A}$, we say that $\mathcal{A}$ is a **normal ring**.

Remark 4.2.7. Assume $\mathcal{A}$ is Noetherian, and let $\widehat{\mathcal{A}}$ be the integral closure of $\mathcal{A}$ in $\mathcal{B}$. Then $\widehat{\mathcal{A}}$ is integrally closed in $\mathcal{B}$.

Proof. Let $x \in \mathcal{B}$ be integral over $\widehat{\mathcal{A}}$. Then we can find $n \in \mathbb{N}$ and $c_0, \ldots, c_{n-1} \in \widehat{\mathcal{A}}$ such that $x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$. Let $\mathcal{C} = \mathcal{A}[c_0, c_1, \ldots, c_{n-1}]$, which is a finitely-generated $\mathcal{A}$-module by Prop. 4.2.5. Clearly $\mathcal{C}[x]$ is $\mathcal{C}$-finitely-generated, and hence $\mathcal{A}$-finitely-generated. So x is integral over $\mathcal{A}$. $\square$

Example 4.2.8. Let $\mathcal{A}$ be a Noetherian integral domain with field of fractions $\mathbb{K} = \mathrm{Nzd}(\mathcal{A})^{-1}\mathcal{A}$, and let $\widehat{\mathcal{A}}$ be the integral closure of $\mathcal{A}$ in $\mathbb{K}$. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \in \mathcal{A}[z]$. Suppose that $p(z) = p_1(z) \cdots p_N(z)$ where each $p_i(z) \in \mathbb{K}[z]$ is monic. (For instance, since $\mathbb{K}[z]$ is a UFD (unique factorization domain), we may take the irreducible decomposition of $p(z)$ in $\mathbb{K}[z]$.) Then $p_i(z) \in \widehat{\mathcal{A}}[z]$. In particular, if $\mathcal{A}$ is normal, then $p_i(z) \in \mathcal{A}[z]$.

Proof. Let $\overline{\mathbb{K}}$ be a field extension of $\mathbb{K}$ in which $p(z)$ splits as $p(z) = (z - b_1) \cdots (z - b_n)$ where each $b_i \in \overline{\mathbb{K}}$. Since $p(b_i) = 0$, each b_i is integral over $\mathcal{A}$. The coefficients of $p_i(z)$ are contained in the $\mathcal{A}$-subring of $\overline{\mathbb{K}}$ generated by $b_1, \ldots, b_n$, and hence are integral over $\mathcal{A}$ due to Prop. 4.2.5. So these coefficients are in $\widehat{\mathcal{A}}$. □

From this example, we see immediately that

Corollary 4.2.9. *Assume that $\mathcal{A}$ is a Noetherian and normal integral domain, and let $\mathbb{K}$ be its field of fractions. Then a monic polynomial $p(z) \in \mathcal{A}[z]$ is irreducible in $\mathcal{A}[z]$ if and only if it is irreducible in $\mathbb{K}[z]$.*

We have promised to prove that $\widehat{\mathscr{O}}_{X,x}$ is the integral closure of $\mathscr{O}_{X,x}$ in $\mathfrak{W}_{X,x}$. Now we prove a half of this result.

Lemma 4.2.10. *Let $x \in X$. Then the integral closure of $\mathscr{O}_{X,x}$ in $\mathfrak{W}_{X,x}$ is contained inside $\widehat{\mathscr{O}}_{X,x}$.*

Proof. Let $f \in \mathfrak{W}_{X,x}$ be integral over $\mathscr{O}_{X,x}$. Then by shrinking X to a neighborhood of x, we may find a nowhere dense analytic $A \subset X$ and $a_0, a_1, \ldots, a_{n-1} \in \mathscr{O}(X)$ such that $f \in \mathscr{O}(X \backslash A)$ and that on $X \backslash A$ we have

$$f^n = a_0 + a_1 f + \cdots + a_{n-1} f^{n-1}.$$

By further shrinking X, we find $M > 0$ such that for all $0 \leqslant i \leqslant n-1$ we have $\sup_{p \in X} a_i(p) \leqslant M$. Therefore, if $p \in X \backslash A$ is such that $|f(p)| = R > 1$, then

$$R^n \leqslant M(1 + R + \cdots + R^{n-1}) \leqslant nMR^{n-1},$$

and hence $R \leqslant nM$. This shows $|f(p)| \leqslant \max\{1, nM\}$ for all $p \in X \backslash A$, and hence $f \in \widehat{\mathscr{O}}_X(X)$. □

4.3 Riemann extension theorems; $\mathscr{O}_{\mathbb{C}^n,0}$ is normal

Let X be a reduced complex space. Recall that the singular locus $\mathrm{Sg}(X)$ is a nowhere dense analytic subset of X by Thm. 3.6.7.

Theorem 4.3.1 (First Riemann extension theorem). *If X is smooth, then*

$$\widehat{\mathscr{O}}_X = \mathscr{O}_X$$

It follows that for a general reduced complex space, we have for every open $U \subset X$ that $\widehat{\mathscr{O}}_X(U) \subset \mathscr{O}_X(U \backslash \mathrm{Sg}(X))$. And hence

$$\widehat{\mathscr{O}}_X \subset \mathscr{O}_{X \backslash \mathrm{Sg}(X)} \tag{4.3.1}$$

Proof. We need to prove that for any (small enough) pure n-dimensional complex manifold X and any nowhere dense analytic subset A, if $f \in \mathscr{O}(X\backslash A)$ is locally bounded at every point of A, then f can be extended (necessarily uniquely) to an element of $\mathscr{O}(X)$. We prove this by induction on $\dim A$. The case $\dim A = -\infty$ (i.e. $A = \varnothing$) is obvious. Assume the case $\dim A \leqslant m - 1$ is true. Consider the case $\dim A = m$. Note that $m < n$ by Ritt's lemma 3.10.6. It suffices to prove that any locally bounded $f \in \mathscr{O}(X\backslash A)$ can be extended to an element of $\mathscr{O}(X\backslash \mathrm{Sg}(A))$. Then since $\dim \mathrm{Sg}(A) \leqslant m-1$ (due to Thm. 3.6.7 and Ritt's lemma 3.10.6), we can apply the assumption on case $\leqslant m - 1$ to conclude $f \in \mathscr{O}(X)$.

Thus, by replacing X by $X\backslash \mathrm{Sg}(A)$, it suffices to assume that A is an m-dimensional smooth complex subspace of X. Since what we want to prove is local by nature, in view of Rem. 1.7.9, we may choose any $x \in X$ and shrink X to a neighborhood of x so that X is an open subset of $\mathbb{C}^n$ with coordinates $(z_\bullet, \zeta) = (z_1, \dots, z_{n-1}, \zeta)$, that $x = 0$, and that $A = \{(z_\bullet, \zeta) \in X : z_{m+1} = \cdots = z_{n-1} = \zeta = 0\}$. We assume moreover that X is of the form $U \times \mathbb{D}_{2r}$ where U is a neighborhood of $0 \in \mathbb{C}^{n-1}$ and $\mathbb{D}_{2r} = \{\zeta \in \mathbb{C} : |\zeta| < 2r\}$.

Define

$$\widetilde{f}(z_\bullet, \zeta) = \oint\limits_{|w|=r} \frac{f(z_\bullet, w)}{w - \zeta} \frac{dw}{2\mathrm{i}\pi}$$

which is a holomorphic function on $U \times \mathbb{D}_r$. The proof is finished if we can show that $\widetilde{f}$ equals f on $U \times \mathbb{D}_r^\times$. Namely, we shall show that for each fixed $z_\bullet \in U$, $\widetilde{g}(\zeta) = \widetilde{f}(z_\bullet, \zeta)$ and $g(\zeta) = f(z_\bullet, \zeta)$ are the same holomorphic function on $\mathbb{D}_r^\times$. This is clear, because g is locally bounded at $0 \in \mathbb{C}$, and is hence a holomorphic function on $\mathbb{D}_r$. So $\widetilde{g} = g$ by Cauchy's integral formula. □

Corollary 4.3.2. *$\mathscr{O}_{\mathbb{C}^n,0}$ is a normal ring.*

Proof. By Lemma 4.2.10, the integral closure of $\mathscr{O}_{\mathbb{C}^n,0}$ in $\mathrm{Nzd}(\mathscr{O}_{\mathbb{C}^n,0})^{-1}\mathscr{O}_{\mathbb{C}^n,0}$ is between $\mathscr{O}_{\mathbb{C}^n,0}$ and $\widehat{\mathscr{O}}_{\mathbb{C}^n,0}$, which are equal by Thm. 4.3.1. □

Definition 4.3.3. A closed subset $A \subset X$ is called **thin** if each $x \in X$ (equivalently, each $x \in A$) has a neighborhood U_x such that $A \cap U_x$ is contained in a nowhere dense analytic subset $\widetilde{A}_x$ of U, whose dimension at x is necessarily less than that of U_x by Ritt's lemma 3.10.6. We say that A is **thin of order** k if for each x we can find $\widetilde{A}_x$ such that $\dim_x U_x - \dim_x \widetilde{A}_x \geqslant k$.

Corollary 4.3.4. *Assume that X is smooth and A is a thin subset of X. Then X is connected if and only if $X\backslash A$ is connected.*

Proof. If X is not connected, then X is a disjoint union of non-empty open subsets $X = U \cup V$. Then $X\backslash A$ is the disjoint union of $U\backslash A$ and $V\backslash A$ which are non-empty because A is nowhere dense in U and in V. So $X\backslash A$ is disconnected.

Conversely, assume that $X\backslash A$ is disconnected, and write it as a disjoint union of non-empty open subsets $X\backslash A = U \cup V$. Define $f \in \mathscr{O}(U \cup V)$ to be constantly 1 on U and 0 on V. Then $f \in \widehat{\mathscr{O}}_X(X)$, and hence $f \in \mathscr{O}_X(X)$ by Thm. 4.3.1. Namely, f can be extended to a holomorphic function on X. Since A is nowhere dense in X, $X\backslash A$ is dense in X. So the range of the continuous function $f : X \to \mathbb{C}$ is $\{0, 1\}$. Therefore X is not connected, otherwise the intermediate value theorem is violated. □

We give an interesting application of Cor. 4.3.2.

Theorem 4.3.5. *$\mathscr{O}_{\mathbb{C}^n,0}$ is a UFD.*

Proof. To prove that $\mathscr{O}_{\mathbb{C}^n,0}$ is a UFD, we need to show that each non-zero $f \in \mathscr{O}_{\mathbb{C}^n,0}$ factors as the product of a unit and some prime elements of $\mathscr{O}_{\mathbb{C}^n,0}$. This is clearly true when $n = 0$. So let us assume $n > 0$.

Since $f \neq 0$, we may change the coordinate of $\mathbb{C}^n$ to a new one $(w_\bullet, z) = (w_1, \dots, w_{n-1}, z)$ such that f has finite order in z. (Cf. the proof of Thm. 1.5.5). Thus, by WPT, we may write $f = uq$ where $u \in \mathscr{O}_{\mathbb{C}^{n-1},0}[z]$ is a unit and q is a Weierstrass polynomial. In particular, q is monic. So, as $\mathscr{M}_{\mathbb{C}^{n-1},0}[z]$ is a UFD, we have $q(z) = p_1(z) \cdots p_N(z)$ where each $p_i(z) \in \mathscr{M}_{\mathbb{C}^{n-1},0}[z]$ is monic and irreducible. By Cor. 4.3.2, the Noetherian integral domain $\mathscr{O}_{\mathbb{C}^{n-1},0}$ is normal. So by Rem. 4.2.8, each $p_i(z)$, which is irreducible in $\mathscr{M}_{\mathbb{C}^{n-1},0}[z]$, is a monic polynomial in $\mathscr{O}_{\mathbb{C}^{n-1},0}[z]$. It remains to prove that each $p_i(z)$ is a prime element of $\mathscr{O}_{\mathbb{C}^n,0}$. This follows from Lemma. 4.3.6. □

Lemma 4.3.6. *Let X be a complex space irreducible at $x \in X$, let $\mathbb{K}$ be the field of fractions of $\mathscr{A} = \mathscr{O}_{X,x}$, and let $p(z)$ be a monic polynomial in $\mathscr{A}[z]$ which is irreducible in $\mathbb{K}[z]$. Then $p(z)$ is a prime element of $\mathscr{B} = \mathscr{O}_{X\times\mathbb{C},x\times 0}$.*

Proof. Since $p(z)$ is monic, it has finite order k in z. We need to prove that if $a, b \in \mathscr{B}$ and $p(z)$ divides $a(z)b(z)$ in $\mathscr{B}$, then p divides one of a, b in $\mathscr{B}$. By WDT, $\mathscr{B}/p(z)\mathscr{B}$ is $\mathscr{A}$-generated by $1, z, \dots, z^{k-1}$. Thus, it suffices to assume that $a(z), b(z)$ are polynomials in $\mathscr{A}[z]$ of degrees $< k$.

We claim that $p(z)$ divides ab in $\mathscr{A}[z]$. Then it follows that $p(z)$ divides one of a, b in $\mathbb{K}[z]$ because, in the UFD $\mathbb{K}[z]$, $p(z)$ is irreducible and hence prime. Let's say $p(z)$ divides $a(z)$ in $\mathbb{K}[z]$. Since the degree of $p(z)$ is larger than that of $a(z)$, $a(z)$ must be zero. Then clearly $p(z)$ divides $a(z)$ in $\mathscr{A}[z]$, which finishes the proof.

By Euclidean division (which is available because $p(z)$ is monic), $ab = gp + r$ where $g(z), r(z) \in \mathscr{A}[z]$ and $r(z)$ has degree $< k$. This gives the unique Weierstrass division of ab by p (cf. Thm. 1.5.3). Since p divides ab in $\mathscr{B}$, we have $ab = hp$ for some $h \in \mathscr{B}$, which also gives the Weierstrass division. So $h = g \in \mathscr{A}[z]$. This proves the claim. □

Theorem 4.3.7 (Second Riemann extension theorem). *If X is smooth and A is a thin subset of X of order 2, then*

$$\mathscr{O}_{X\backslash A} = \mathscr{O}_X$$

Proof. We shall prove that $\mathscr{O}(U\backslash A) = \mathscr{O}(U)$ for any sufficiently small neighborhood U of any $x \in X$. In that case, $A \cap U$ is contained in a thin (i.e. nowhere dense) analytic subset of U, and we may well assume that $A \cap U$ is that analytic subset. Thus, by shrinking X to a neighborhood of x and extending A, we assume A is a thin analytic subset of X.

As in the proof of Thm. 4.3.1, by an inductive argument, it suffices to assume that X is an open subset of $\mathbb{C}^n$ with coordinates $(z_\bullet, \zeta_1, \zeta_2) = (z_1, \dots, z_{n-2}, \zeta_1, \zeta_2)$, and that $A = \{(z_\bullet, \zeta_1, \zeta_2) \in X : z_{m+1} = \cdots = z_{n-2} = \zeta_1 = \zeta_2 = 0\}$. Note that $m \leqslant n-2$. We assume moreover that X is of the form $U \times \mathbb{D}_{2r} \times \mathbb{D}_{2r}$ where U is a neighborhood of $0 \in \mathbb{C}^{n-2}$ and $\mathbb{D}_{2r} = \{\zeta \in \mathbb{C} : |\zeta| < 2r\}$. To show that $f \in \mathscr{O}(X\backslash A)$ is actually in $\mathscr{O}(X)$, by Thm. 4.3.1, it suffices to show that f is locally bounded at any point of A.

For each $z_\bullet$ in a precompact subset $V \subset U$ and $\zeta_1 \in \mathbb{D}_r^\times = \mathbb{D}_r\backslash\{0\}$, applying the maximal principle to the holomorphic function $f(z_\bullet, \zeta_1, \zeta_2)$ of ζ_2 (defined on $\mathbb{D}_{2r}$ since $\zeta_1 \neq 0$), we have for all $|\zeta_2| < r$ that

$$|f(z_\bullet, \zeta_1, \zeta_2)| \leqslant \sup_{|w_2|=r} |f(z_\bullet, \zeta_1, w_2)| \leqslant M$$

where

$$M = \sup_{\gamma_\bullet \in V, |w_1| \leqslant r, |w_2|=r} |f(\gamma_\bullet, w_1, w_2)| < +\infty.$$

□

The study of $\widehat{\mathscr{O}}_X$ for singular (reduced) complex spaces is more difficult and relies on the notion of branched coverings.

4.4 Resultants and discriminants

Let $\mathcal{A}$ be a commutative ring. In this section, we collect some facts about polynomials that will be helpful for the subsequent study of branched coverings.

Definition 4.4.1. Let $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ and $g(z) = b_0 + b_1 z + \cdots + b_n z^n$ be polynomials in $\mathcal{A}[z]$ of degree m, n respectively. Then the **resultant** $\mathrm{res}(f, g)$ of

f and g is the determinant of the $(m+n) \times (m+n)$ matrix

$$\left[\begin{array}{ccccccccc} a_0 & a_1 & \cdots & \cdots & a_m & & & & \\ & a_0 & a_1 & \cdots & \cdots & a_m & & & \\ & & & \cdots & \cdots & & & & \\ & & & a_0 & a_1 & \cdots & \cdots & a_m & \\ \hline b_0 & b_1 & \cdots & \cdots & b_n & & & & \\ & b_0 & b_1 & \cdots & \cdots & b_n & & & \\ & & & \cdots & \cdots & & & & \\ & & & b_0 & b_1 & \cdots & \cdots & b_n & \end{array}\right] \tag{4.4.1}$$

where the first block has n rows and the second one has m rows. Let $f'(z)$ be the derivative of $f(z)$. Then

$$D(f) = \mathrm{res}(f, f')$$

is called the **discriminant** of f.[1]

Now we assume $\mathcal{A} = \mathbb{K}$ is a field.

Proposition 4.4.2. *Let $\overline{\mathbb{K}}$ be any field extension of $\mathbb{K}$ in which $f(z)$ and $g(z)$ split. Then the following are equivalent.*

(a) $\mathrm{res}(f, g) \neq 0$.

(b) f and g have no common zeros in $\overline{\mathbb{K}}$.

(c) 1 is a gcd (greatest common divisor) of f, g in $\mathbb{K}[z]$.

Proof. Recall that a gcd of f, g in $\mathcal{A} = \mathbb{K}[z]$ is equivalently an element in $f\mathcal{A} + g\mathcal{A}$ dividing f and g in $\mathcal{A}$, which is therefore also a gcd of f, g in $\overline{\mathbb{K}}[z]$. So (b)$\Leftrightarrow$(c). In the following, we prove (a)$\Leftrightarrow$(b), and it suffices to assume that f, g split in $\mathbb{K}$.

Clearly, f, g have common zeros in $\mathbb{K}$ iff there exist $u(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$ and $v(z) = d_0 + d_1 z + \cdots + d_{m-1} z^{m-1}$ in $\mathbb{K}[z]$ such that $uf = -vg$. This is equivalent to that $\det$ (4.4.1) $= 0$, because $uf + vg = 0$ iff

$$(c_0, c_1, \ldots, c_{n-1}, d_0, d_1, \ldots, d_{m-1}) \cdot (4.4.1) = 0.$$

$\square$

Corollary 4.4.3. *Let $\overline{\mathbb{K}}$ be a field extension of $\mathbb{K}$ in which $f \neq 0$ splits. Then $D(f) \neq 0$ if and only if each zero of f in $\overline{\mathbb{K}}$ has multiplicity 1.*

[1]Our definition of $D(f)$ is a non-zero constant times the usual definition of $D(f)$. Such difference is unimportant for the purpose of our notes.

Proof. This follows from Prop. 4.4.2, because each zero of f in $\overline{\mathbb{K}}$ has multiplicity 1 iff 1 is a gcd of f and f' in $\overline{\mathbb{K}}[z]$. □

Definition 4.4.4. If $f(z) \in \mathbb{K}[z]$ is monic, we define its **reduction** $\mathrm{red}(f) \in \mathbb{K}[z]$ as follows. Since $\mathbb{K}[z]$ is a UFD, we can write $f(z) = p_1(z)^{n_1} \cdots p_N(z)^{n_N}$ in a unique way where $n_1, \dots, n_N \in \mathbb{Z}_+$, $p_1, \dots, p_N \in \mathbb{K}[z]$ are monic and irreducible, and $p_i \neq p_j$ if $i \neq j$. We set

$$\mathrm{red}(p)(z) = p_1(z) \cdots p_N(z) \tag{4.4.2}$$

Remark 4.4.5. The discriminant $D(\mathrm{red}(p))$ is a non-zero element of $\mathbb{K}$. Equivalently (by Cor. 4.4.3), the multiplicity of any zero of p in $\overline{\mathbb{K}}$ is 1.

Proof. By Prop. 4.4.2. it suffices to show that 1 is a gcd of $\mathrm{red}(p)$ and $\mathrm{red}(p)'$ in $\mathbb{K}[z]$. If not, then a gcd must be divided by p_i for some i, say divided by p_1. So p_1 divides $\mathrm{red}(p)' = p_1' p_2 \cdots p_N + p_1 p_2' \cdots p_N + \cdots + p_1 p_2 \cdots p_N'$, and hence divides $p_1' p_2 \cdots p_N$. Since all p_i are irreducible and $p_1 \neq p_i$ if $i > 1$, we must have that p_1 divides p_1' (in $\mathbb{K}[z]$), which is impossible because the degree of p_1' is less than that of p_1. □

Remark 4.4.6. Clearly $\mathrm{red}(p)$ and p have the same zero sets in any field extension $\overline{\mathbb{K}}$ in which p splits. Thus, if $p(z) = (z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$ in $\overline{\mathbb{K}}[z]$, then by Rem. 4.4.5,

$$\mathrm{red}(p)(z) = (z - z_1) \cdots (z - z_k).$$

In particular, the expression of $\mathrm{red}(p)$ is unchanged if we replace $\mathbb{K}$ by any field extension of $\mathbb{K}$.

4.5 Branched coverings

In this section, unless otherwise stated, all complex spaces are reduced.

Definition 4.5.1. A holomorphic map of complex spaces $\varphi : X \to Y$ is called a **local biholomorphism at** $x \in X$ if there is a neighborhood U of x such that $V = \varphi(U)$ is open in Y and that the restriction $\varphi : U \to V$ is a biholomorphism; equivalently (cf. Cor. 1.6.3), $\varphi^\# : \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$ is an isomorphism of local $\mathbb{C}$-algebras. We say that φ is a **local biholomorphism** if it is so at every point of X.

Definition 4.5.2. A finite surjective holomorphic map $\pi : X \to S$ is called a **branched covering** (of S) if there is a thin subset $\Delta \subset S$ such that $\pi^{-1}(\Delta)$ is thin in X, and that the restriction $\pi : X \backslash \pi^{-1}(\Delta) \to S \backslash \Delta$ is a local biholomorphism. We say that Δ is the **branch locus** of π. Then if $V \subset Y$ is open, $\pi : \pi^{-1}(V) \to V$ is clearly a branched covering with branch locus $V \cap \Delta$.

If $\Delta = \emptyset$, we say that π is an **unbrached covering**. □

Remark 4.5.3. The restriction $\pi : X\backslash\pi^{-1}(\Delta) \to S\backslash\Delta$ is clearly an unbranched covering. Using Prop. 2.4.1, it is easy to see that each $y \in S\backslash\Delta$ is contained in a neighborhood V such that $\pi^{-1}(V)$ is disjoint union of open subsets $U_1 \cup \cdots \cup U_N$, where each $\pi : U_i \to V$ is a biholomorphism. In particular, this map is a covering map of topological spaces.

Remark 4.5.4. It is clear that a branched covering $\pi : X \to Y$ satisfies the assumption in Prop. 4.2.2. In particular, the inverse image under π of a thin subset of Y is a thin subset of X. This means any thin subset of Y containing a given branch locus Δ can also be a branch locus. Also, it is easy to see that π sends thin subsets of X to thin subsets of Y.

Remark 4.5.5. Let $\pi : X \to S$ be a finite holomorphic map. Let Ω be the set of all $t \in S$ such that $\pi : \pi^{-1}(V) \to V$ is a local biholomorphism (equivalently, an unbranched covering) for some neighborhood $V \subset S$ of t. It is clear that π is a branched covering if and only if the (obviously) closed subset $S\backslash\Omega$ is thin in S, and $\pi^{-1}(S\backslash\Omega)$ is thin in X. In that case, $S\backslash\Omega$ is the smallest branch locus.

Remark 4.5.6. By Rem. 4.5.5, it is clear that the property of being a branched covering is local with respect to the base space: If S is covered by some open subsets $(V_\alpha)_{\alpha\in\mathfrak{A}}$ such that the restriction $\pi : \pi^{-1}(V_\alpha) \to V_\alpha$ is a branched covering for every α, then $\pi : X \to S$ is a branched covering.

When constructing branched coverings, once one has found a thin $\Delta \subset S$ and know that π is unbranched outside Δ, one can use the following criterion to show that π is surjective and that $\pi^{-1}(\Delta)$ is thin:

Lemma 4.5.7. *Let $\pi : X \to S$ be a finite holomorphic map of reduced complex spaces. If $\pi^\# : \mathscr{O}_{S,t} \to (\pi_*\mathscr{O}_X)_t$ is injective for each $t \in S$, then π is surjective. If $(\pi_*\mathscr{O}_X)_t$ is $\mathscr{O}_{S,t}$-torsion-free for each $t \in S$, then for any thin subset $\Delta \subset S$, $\pi^{-1}(\Delta)$ is thin in X.*

Proof. The surjectivity of π is due to Rem. 2.3.10. If $\pi^{-1}(\Delta)$ is not thin, then there is $x \in X$ such that $\pi^{-1}(\Delta)$ contains a neighborhood U of x. Let $t = \pi(x)$. Since Δ is thin, by Prop. 3.4.1 we can find $g \in \mathrm{Nzd}(\mathscr{O}_{S,t})$ vanishing on the germ (Δ, t). So $\pi^\# g$ vanishes in $\mathscr{O}_{X,x}$. By Prop. 2.4.5, $(\pi_*\mathscr{O}_X)_t = \bigoplus_{y\in\pi^{-1}(t)} \mathscr{O}_{X,y}$. Define $f \in (\pi_*\mathscr{O}_X)_t$ to be 1 in $\mathscr{O}_{X,x}$ and 0 in $\mathscr{O}_{X,y}$ whenever $y \in \pi^{-1}(t)\backslash\{x\}$. Then $gf = 0$. So $(\pi_*\mathscr{O}_X)_t$ is not $\mathscr{O}_{S,t}$-torsion-free. $\square$

4.5.1 Main results

The goal of this section is to prove:

Theorem 4.5.8. *Let X, S be pure n-dimensional reduced complex spaces, and let $\pi : X \to S$ be a finite holomorphic map which is surjective. Then π is a branched covering.*

Corollary 4.5.9. *Let $\pi : X \to S$ be a finite holomorphic map which is surjective and open. Assume that S is locally irreducible. Then π is a branched covering.*

Proof. By Thm. 3.14.9, we may shrink S and assume that S has pure dimension n. Then since π is open, finite, and surjective, we see that X has dimension n everywhere due to Thm. 3.12.6. $\square$

A converse of Thm. 4.5.8 is easy to prove:

Proposition 4.5.10. *Let $\pi : X \to S$ be a branched covering, and assume that S has pure dimension n. Then X also has pure dimension n.*

Proof. Let $\Delta \subset S$ be a branch locus. Then X clearly has pure dimension n outside the thin subset $\pi^{-1}(\Delta)$. If $x \in \pi^{-1}(\Delta)$, then $\dim_x X \leqslant n$ by Prop. 3.9.5, and $\dim_x X \geqslant n$ by the upper-semicontinuity of dimensions (Cor. 3.9.4). $\square$

4.5.2 Weierstrass branched coverings

Assume the setting of Def. 2.5.1. So $\pi : X \to S$ is a Weierstrass map defined by polynomials $p_1(z_1), \ldots, p_k(z_k)$. We assume that S is reduced. We do not assume that X is reduced. Then we have discriminants

$$D(p_i) \in \mathscr{O}(S).$$

We set

$$\Delta = \bigcup_{i=1}^{k} N(D(p_i)) = N\big(D(p_1)\cdots D(p_k)\big)$$

Lemma 4.5.11. *The restriction $\pi : X \backslash \pi^{-1}(\Delta) \to S\backslash\Delta$ is a local biholomorphism. In particular, $X\backslash\pi^{-1}(\Delta)$ is reduced.*

Proof. Let $x \in X$ such that $t = \pi(x)$ is not in Δ. Then for each i, $D(p_i(t)) = D(p_i)(t) \neq 0$. By Prop. 4.4.3, (for the fixed t) each zero of $p_i(t,z)$ has multiplicity 1. Thus, if we write $x = (t, \zeta_1, \ldots, \zeta_k)$, then ζ_i is a zero of $p_i(t,z)$ with multiplicity 1. Assume for simplicity that $\zeta_1 = \cdots = \zeta_k = 0$. Then by WPT, in $\mathscr{O}_{S\times\mathbb{C}^k,x}$, p_i is a unit times $q_i = z_i - b_i$ where $b_i \in \mathfrak{m}_{S,t}$. Therefore

$$\mathscr{O}_{X,x} = \mathscr{O}_{S\times\mathbb{C}^k,x}\Big/\sum_{i=1}^{k} q_i\mathscr{O}_{S\times\mathbb{C}^k,x}$$

which, by Thm. 2.5.4, is a free $\mathscr{O}_{S,t}$-module generated by 1. Therefore $\pi^\# : \mathscr{O}_{S,t} \to \mathscr{O}_{X,x}$ is an isomorphism of local $\mathbb{C}$-algebras. So π is a local biholomorphism at x. $\square$

Proposition 4.5.12. *Assume that Δ is nowhere dense in S (equivalently, that each $N(D(p_i))$ is nowhere dense in S). Then X is reduced, and the Weierstrass map $\pi : X \to S$ is a branched covering with branch locus Δ.*

The branched covering π in Prop. 4.5.12 is called a **Weierstrass (branched) covering**.

Proof. By Lemma 4.5.11, X is reduced at $x \in X$ if $\pi(x) \neq \Delta$. Now assume $\pi(x) = \Delta$. To show that X is reduced at x, by Prop. 3.7.1, it suffices to show that $\mathscr{I}_{\mathrm{Sg}(X),x}$ contains a non zero-divisor of $\mathscr{O}_{X,x}$.

By Lemma 4.5.11, we have $\mathrm{Sg}(X) \subset \pi^{-1}(B)$ where

$$B = \mathrm{Sg}(Y) \cup \Delta.$$

Since Y is reduced, by Thm. 3.6.7, $\mathrm{Sg}(Y)$ is nowhere dense in B. Since Δ is nowhere dense by assumption, B is also nowhere dense in Y. Thus, by Prop. 3.4.4, we can find $g \in \mathscr{I}_{B,y}$ which is a non zero-divisor of $\mathscr{O}_{Y,y}$. By Thm. 2.5.4, $\mathscr{O}_{X,x}$ is a free $\mathscr{O}_{S,\pi(x)}$-module. (We only need the torsion freeness.) Therefore $\pi^{\#}g$ is a non zero-divisor of $\mathscr{O}_{X,x}$. This proves that $\mathscr{O}_{X,x}$ is reduced, because $\pi^{\#}g \in \mathscr{I}_{\pi^{-1}(B),x} \subset \mathscr{I}_{\mathrm{Sg}(X),x}$.

Since any Weierstrass map is open (cf. Exp. 3.14.8), and since Δ is nowhere dense in Y, $\pi^{-1}(\Delta)$ is nowhere dense in X. So π is a branched covering by Lemma 4.5.11. $\square$

Theorem 4.5.13. *Let $t \in S$, and assume that $\mathscr{O}_{S,t}$ is a normal integral domain (e.g. S is smooth, cf. Cor. 4.3.2). Then we can shrink S to a neighborhood of $t \in S$ and replace X by $\pi^{-1}(S)$ so that the restriction $\pi : \mathrm{red}(X) \to S$ is a Weierstrass covering.*

Proof. We may assume that $p_1, \dots, p_k$ are monic. Let $\mathbb{K} = \mathscr{M}_{S,t}$ be the field of fractions of $\mathscr{O}_{S,t}$, and view each $p_i(z_i) \in \mathscr{O}_{S,t}[z_i]$ as a polynomial in $\mathbb{K}[z_i]$. As in Def. 4.4.4, we have irreducible decomposition

$$p_i = p_{i,1}^{\bullet} p_{i,2}^{\bullet} \cdots \tag{4.5.1}$$

in $\mathbb{K}[z_i]$ where $\bullet$ denote elements of $\mathbb{Z}_+$, each $p_{i,*} \in \mathbb{K}[z_i]$ is monic and irreducible, and $p_{i,j} \neq p_{i,l}$ if $j \neq l$. Then $q_i = \mathrm{red}(p_i)$ equals

$$q_i = p_{i,1} p_{i,2} \cdots \tag{4.5.2}$$

Since $\mathscr{O}_{S,t}$ is normal, by Exp. 4.2.8, we have $p_{i,*} \in \mathscr{O}_{S,t}[z_i]$.

Shrink S to a neighborhood of t (and shrink X accordingly to $\pi^{-1}(S)$) so that $p_{i,*} \in \mathscr{O}(S)[z_i]$ for all i, and that (4.5.1) and (4.5.2) hold in $\mathscr{O}_S[z_i]$. Then from these two formulas, it is clear that $N(p_i) = N(q_i)$. Thus, $\mathrm{red}(X)$ (as an analytic subset of X) equals $N(q_1, \dots, q_k)$. Let $Y = \mathrm{Specan}(\mathscr{O}_{S \times \mathbb{C}^k} / \sum_i q_i \mathscr{O}_{S \times \mathbb{C}^k})$. Then we have a Weierstrass map $\pi : Y \to S$ such that the underlying set of Y equals that of X.

We now show that, after shrinking S further, Y is reduced and $\pi : Y \to S$ is a branched covering. This will imply that $\pi : Y \to S$ equals $\pi : \mathrm{red}(X) \to S$, finishing the proof. Indeed, by Rem. 4.4.5, the discriminant $D(q_i)$ (which is an element of $\mathscr{O}_{S,t} \subset \mathbb{K}$ since the coefficients of q_i are in $\mathscr{O}_{S,t}$) is non-zero. So $D(q_1)\cdots D(q_k)$ is non-zero in the integral domain $\mathscr{O}_{S,t}$, and hence is a non zero-divisor. So by Prop. 3.4.1, we may shrink S further so that $\Delta = N(D(q_1)\cdots D(q_k))$ is nowhere dense in S. This proves the claim with the help of Prop. 4.5.12. □

From the above proof, we conclude a useful criterion on reducedness:

Corollary 4.5.14. *Let $t \in S$, and assume that $\mathscr{O}_{S,t}$ is a normal integral domain. For each $1 \leqslant i \leqslant k$, assume that $p_i(z_i) \in \mathscr{O}_{S,t}[z_i]$ is monic, and view it as a polynomial in $\mathscr{M}_{S,t}[z_i]$ so that we can define its reduction $\mathrm{red}(p_i) \in \mathscr{M}_{S,t}[z_i]$. Then $\mathrm{red}(p_i) \in \mathscr{O}_{S,t}[z_i]$, and for each $x \in \mathrm{pr}_S^{-1}(t)$ (where $\mathrm{pr}_S : S \times \mathbb{C}^k \to S$ is the projection), the ring*

$$\mathscr{O}_{S\times\mathbb{C}^k,x}\Big/\sum_{i=1}^k \mathrm{red}(p_i)\cdot\mathscr{O}_{S\times\mathbb{C}^k,x}$$

is reduced.

4.5.3 Proof of Thm. 4.5.8

Lemma 4.5.15. *Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be surjective finite holomorphic maps of reduced complex spaces. Assume that ψ and $\psi\circ\varphi : X \to Z$ are branched coverings. Then φ is a branched covering.*

Proof. The branch loci of ψ and $\psi\circ\varphi$ are both thin subsets of Z. So their union Δ is also thin in Z. By Rem. 4.5.4, we can enlarge a branch locus to any larger thin subset. So we may assume that ψ and $\psi\circ\varphi$ have common branch locus Δ. Clearly $\varphi : X\backslash\varphi^{-1}\psi^{-1}(\Delta) \to Y\backslash\psi^{-1}(\Delta)$ is a local biholomorphism. So φ is a branched covering with branch locus $\psi^{-1}(\Delta)$. □

Proof of Thm. 4.5.8. In view of Rem. 4.5.6, it suffices to choose any $t \in S$ and show that we can shrink S to a neighborhood of t and shrink X to $\pi^{-1}(S)$ so that π is a branched covering.

We first consider the case when S is smooth. By Prop. 2.7.9, we may shrink S and X so that there is a Weierstrass map $\psi : Y \to S$ and a closed embedding $\alpha : X \to Y$ such that $\pi = \psi\circ\alpha$. By Thm. 4.5.13, we may shrink S, and shrink X to $\pi^{-1}(S)$ and Y to $\psi^{-1}(S)$, so that $\psi : \mathrm{red}(Y) \to S$ is a Weierstrass covering. Thus, as $\alpha(X)$ (which is reduced, cf. Exe. 2.3.11) is a closed subspace of $\mathrm{red}(Y)$, we may replace Y by $\mathrm{red}(Y)$ so that (Y is reduced and) $\psi : Y \to S$ is a Weierstrass covering. Let Δ be a branch locus.

By Thm. 3.12.6, π is open. Therefore $\pi^{-1}(\Delta)$ is a thin subset of X. To prove that π is a branched covering with branch locus Δ, it suffices to show that π is a biholomorphism at every $x \in X \backslash \pi^{-1}(\Delta)$. Let $y = \alpha(x)$ and $s = \pi(x)$. Using the fact that ψ is a biholomorphism from a neighborhood of $y \in Y$ to a neighborhood of $s \in S$ and the fact that π is open at x, it is easy to see that the closed embedding α is open at x. Thus, the reduced subspace $\alpha(X)$ of Y contains a neighborhood of y. So the germs of analytic sets $(\alpha(X), y)$ and (Y, y) are equal. Namely, the inclusion of reduced complex spaces $\iota : \alpha(X) \to Y$ is a local biholomorphism at y. So α is a local biholomorphism at x. This finishes the proof of the smooth case.

Now we consider the general case. Since S has pure dimension n, by Prop. 3.9.3, we can shrink S and X so that there is a finite holomorphic map $\varpi : S \to W$ where W is an open subset of $\mathbb{C}^n$. By the smooth case, both ϖ and $\varpi \circ \pi$ are branched coverings. Thus, by Lemma 4.5.15, π is a branched covering. □

4.6 $\widehat{\mathscr{O}}_{X,x}$ is the integral closure of $\mathscr{O}_{X,x}$ in $\mathscr{M}_{X,x}$

Let X be a reduced complex space. The main result of this section (Cor. 4.6.8) is indicated in the title. This is an immediate consequence of Thm. 4.6.7 which says that the two equivalent conditions in Lemma 4.6.1 always hold.

Lemma 4.6.1. *Let $x \in X$. Then the following are equivalent.*

*(1) There exists $\delta \in \mathrm{Nzd}(\mathscr{O}_{X,x})$ satisfying that $\delta \cdot \widehat{\mathscr{O}}_{X,x} \subset \mathscr{O}_{X,x}$. We call δ a **universal denominator** of $\widehat{\mathscr{O}}_{X,x}$.*

(2) $\widehat{\mathscr{O}}_{X,x}$ is a finitely-generated $\mathscr{O}_{X,x}$-submodule of $\mathscr{M}_{X,x}$.

Proof. Assume (1). Then clearly $\widehat{\mathscr{O}}_{X,x} \subset \mathscr{M}_{X,x}$. The $\mathscr{O}_{X,x}$-module morphism $\mathscr{M}_{X,x} \xrightarrow{\times\delta} \mathscr{M}_{X,x}$ is injective, and it sends $\widehat{\mathscr{O}}_{X,x}$ to $\delta\widehat{\mathscr{O}}_{X,x}$ which is an ideal of $\mathscr{O}_{X,x}$ and hence $\mathscr{O}_{X,x}$-finitely-generated because $\mathscr{O}_{X,x}$ is Noetherian. Therefore $\widehat{\mathscr{O}}_{X,x}$ is $\mathscr{O}_{X,x}$-finitely-generated. (2) is true.

Assume (2). Then $\widehat{\mathscr{O}}_{X,x}$ is $\mathscr{O}_{X,x}$-generated by $f_1, \dots, f_n \in \widehat{\mathscr{O}}_{X,x}$. Since each $f_i\mathscr{O}_{X,x}$ belongs to $\mathscr{M}_{X,x}$, there is $\delta_i \in \mathrm{Nzd}(\mathscr{O}_{X,x})$ such that $\delta_i f_i \in \mathscr{O}_{X,x}$. Then $\delta = \delta_1 \cdots \delta_n$ is a universal denominator. □

4.6.1 Primitive elements

Definition 4.6.2. A branched covering $\pi : X \to S$ is called a b**-sheeted (branched) covering** (where $b \in \mathbb{Z}_+$) if for each $t \in S\backslash\Delta$, $\pi^{-1}(t)$ has b distinct elements. Note that the function

$$t \in S\backslash\Delta \mapsto |\pi^{-1}(t)|$$

is clearly locally constant. Therefore, if a branched covering $\pi : X \to S$ satisfies that $S \backslash \Delta$ is connected, then π is b-sheeted for some b.

In the following part of this section, we assume that $\pi : X \to S$ is a branched covering and S is a connected complex manifold. Then by Cor. 4.3.4, $S \backslash \Delta$ is connected. So π is b-sheeted for some b.

Choose $e \in \mathscr{O}(X)$. We can define $\gamma_e(z) \in \mathscr{O}(S \backslash \Delta)[z]$ such that for each $t \in S \backslash \Delta$,

$$\gamma_e(t, z) = \prod_{x \in \pi^{-1}(t)} (z - e(x)). \tag{4.6.1}$$

Clearly $\gamma_e(z)$ is a monic polynomial with degree b.

Lemma 4.6.3. *$\gamma_e(z)$ is an element of $\mathscr{O}(S)[z]$.*

Proof. e is (uniformly) bounded on any compact subset of X. Since π is finite and hence proper (Prop. 2.4.10), the coefficients of $\gamma_e(z)$ are bounded on $V \backslash \Delta$ for each precompact open subset $V \subset S$. So these coefficients belong to $\widehat{\mathscr{O}}_S(S)$, and hence belong to $\mathscr{O}_S(S)$ by Riemann extension Thm. 4.3.1. □

Definition 4.6.4. We say that e is a **primitive element** of $\mathscr{O}(X)$ over $\mathscr{O}(S)$ if the discriminant $D(\gamma_e(z)) \in \mathscr{O}(S)$ is non-zero at some $t \in S$. In that case, by Identitätssatz 1.1.3, the zero set of $D(\gamma_e(z))$ is nowhere dense in S. By Cor. 4.4.3, e is primitive if and only if there is some $t \in S \backslash \Delta$ such that the restriction $e : \pi^{-1}(t) \to \mathbb{C}$ is injective.

Lemma 4.6.5. *If X is biholomorphic to a closed analytic subset of an open subset U of $\mathbb{C}^N$, then there exists a primitive element $e \in \mathscr{O}(X)$ over $\mathscr{O}(S)$.*

Proof. Let X be a closed analytic subset of $U \subset \mathbb{C}^N$. Let $(z_1, \dots, z_N)$ be the standard coordinates of $\mathbb{C}^N$. Choose any $t \in S \backslash \Delta$. Then one can easily find $a_1, \dots, a_N \in \mathbb{C}$ such that $e = a_1 z_1 + \cdots + a_N z_N$ is injective on $\pi^{-1}(t)$. The restriction of e to X is a primitive element. □

4.6.2 Main results

Theorem 4.6.6. *Let $X \to S$ be a branched covering where X is reduced and S is a connected complex manifold. Assume that there is a primitive element $e \in \mathscr{O}(X)$ over $\mathscr{O}(S)$. Then $\mathscr{O}(X)$ contains an element whose stalk at each $x \in X$ is a universal denominator of $\widehat{\mathscr{O}}_{X,x}$.*

Proof. Let π be b-sheeted, and let $\Delta \subset S$ be a branch locus. For each $t \in S \backslash \Delta$, write $\pi^{-1}(t) = \{x_1, \dots, x_b\}$, and let

$$M(t) = \det \begin{bmatrix} 1 & e(x_1) & e(x_1)^2 & \cdots & e(x_1)^{b-1} \\ 1 & e(x_2) & e(x_2)^2 & \cdots & e(x_2)^{b-1} \\ & & \vdots & & \\ 1 & e(x_b) & e(x_b)^2 & \cdots & e(x_b)^{b-1} \end{bmatrix} \tag{4.6.2}$$

be the Vandermonde determinant. Then $M(t)^2$ is independent of the order $x_1, \ldots, x_b$ of elements of $\pi^{-1}(t)$. Therefore, by varying $t \in S\backslash\Delta$, we get $\delta \in \mathscr{O}(S\backslash\Delta)$ such that

$$\delta(t) = M(t)^2$$

for all $t \in S\backslash\Delta$. Since e is continuous on X and hence bounded on compact subsets of X, and since π is proper (Prop. 2.4.10), δ must belong to $\widehat{\mathscr{O}}_S(S)$. Thus, by Riemann extension Thm. 4.3.1, $\delta \in \mathscr{O}(S)$. Clearly $N(\delta)$ is contained in Δ (because e is primitive), so $N(\pi^\#\delta)$ is contained in the thin subset $\pi^{-1}(\Delta)$ of X. Thus, for each $x \in X$, $(\pi^\#\delta)_x \in \mathrm{Nzd}(\mathscr{O}_{X,x})$ by Prop. 3.4.1.

Let us show that $\pi^\#\delta \in \mathscr{O}(X)$ is a universal denominator of $\widehat{\mathscr{O}}_{X,x}$. Choose any $f \in \widehat{\mathscr{O}}_{X,x}$. By Prop. 2.4.1, we may shrink S to a neighborhood of $\pi(x)$ and shrink X to $\pi^{-1}(S)$ so that $f \in \mathscr{O}(X\backslash A)$ for some thin subset $A \subset X$, and that f is bounded on $X\backslash A$. Note that $\pi(A)$ is thin in S (Rem. 4.5.4). For each $1 \leqslant j \leqslant b$ and $t \in S\backslash(\Delta \cup \pi(A))$, let $K_j(t)$ be the determinant of the matrix defined by replacing the j-th column of the Vandermonde matrix in (4.6.2) with $(f(x_1), \ldots, f(x_b))^{\mathrm{t}}$. Then

$$\omega_j(t) = M(t)K_j(t)$$

is independent of the order $x_1, \ldots, x_b$. Thus, by varying t, ω_j becomes a (clearly bounded) holomorphic function on $S\backslash(\Delta \cup \pi(A))$. Thus, we have $\omega_j \in \mathscr{O}(S)$, again by Riemann extension Thm. 4.3.1. By Cramer's rule, for each $x_i \in \pi^{-1}(t)$,

$$\delta(t) \cdot f(x_i) = \sum_{j=1}^{b} \omega_j(t) \cdot e(x_i)^{j-1}$$

Therefore, the following relation holds in $\mathscr{O}(X\backslash(\pi^{-1}(\Delta) \cup A))$

$$\pi^\#\delta \cdot f = \sum_{j=1}^{b} \pi^\#\omega_j \cdot e^{j-1} \tag{4.6.3}$$

where the RHS is an element of $\mathscr{O}(X)$. $\square$

Theorem 4.6.7. *Let X be a reduced complex space and $x \in X$. Then $\widehat{\mathscr{O}}_{X,x}$ is a finitely-generated $\mathscr{O}_{X,x}$-submodule of $\mathscr{M}_{X,x}$.*

Proof. By Thm. 4.2.4 and local decomposition (Thm. 3.3.4), it suffices to assume that X is irreducible at x. Thus, we can shrink X to a neighborhood of x so that (by Thm. 3.14.9) X has pure dimension n and that (by Prop. 3.9.3) there is a finite map $\pi : X \to S$ where S is a connected open subset of $\mathbb{C}^n$. By Thm. 3.12.6, π is an open map. Thus, we may replace S by $\pi(X)$ so that π is surjective (and clearly still finite). By Thm. 4.5.8, π is a branched covering. By Prop. 2.4.1, we may shrink S further and shrink X to $\pi^{-1}(S)$ so that X is biholomorphic to a model space. Therefore, by Lemma 4.6.5, there is a primitive element $e \in \mathscr{O}(X)$ over $\mathscr{O}(S)$. So by Thm. 4.6.6, there is a universal denominator of $\mathscr{O}_{X,x}$. This proves the theorem with the help of Lemma 4.6.1. $\square$

Corollary 4.6.8. *For each reduced complex space X and each $x \in X$, $\widehat{\mathscr{O}}_{X,x}$ is the integral closure of $\mathscr{O}_{X,x}$ in $\mathscr{M}_{X,x}$.*

Proof. Thm. 4.6.7 shows that $\widehat{\mathscr{O}}_{X,x}$ is included in the integral closure of $\mathscr{O}_{X,x}$. That it contains the integral closure is already shown in Lemma 4.2.10. $\square$

The proof of Thm. 4.6.6 implies the following generalization of Second Riemann extension Thm. 4.3.7. It will be used in Prop. 4.9.2 to obtain global decomposition of reduced complex spaces.

Theorem 4.6.9. *Let X be a reduced locally pure-dimensional complex space and let A be a thin subset of X of order 2. Then $\mathscr{O}_{X\backslash A} \subset \widehat{\mathscr{O}}_X$.*

Proof. We may shrink X so that it has pure dimension n. Let us check that $\mathscr{O}_{X\backslash A,x} \subset \widehat{\mathscr{O}}_{X,x}$ for each $x \in X$. As in the proof of Thm. 4.6.7, we may shrink X to a neighborhood of x to get a b-sheeted branched covering $\pi : X \to S$ where S is a connected open subset of $\mathbb{C}^n$ and there is a primitive $e \in \mathscr{O}(X)$. As in the proof of Thm. 4.6.6, we have $\delta \in \mathscr{O}(S)$. Let us show that $\pi^\#\delta \cdot \mathscr{O}_{X\backslash A,x} \subset \mathscr{O}_{X,x}$. Then the argument in Lemma 4.6.1 shows that $\mathscr{O}_{X\backslash A,x}$ belongs to the integral closure of $\mathscr{O}_{X,x}$ in $\mathscr{M}_{X,x}$, namely $\mathscr{O}_{X\backslash A,x} \subset \widehat{\mathscr{O}}_{X,x}$.

If $f \in \mathscr{O}_{X\backslash A,x}$, we may shrink X and S so that $f \in \mathscr{O}(X\backslash A)$. Then $\omega_j \in \mathscr{O}(S\backslash(\Delta \cup \pi(A)))$ is locally bounded at each point of $\Delta\backslash\pi(A)$, because f is continuous at each point of $\pi^{-1}(\Delta)\backslash A$. Therefore, ω_j is holomorphic on $S\backslash\pi(A)$ by First Riemann extension Thm. 4.3.1. Since A is thin in X of order 2, by Cor. 3.12.9, $\pi(A)$ is thin in S of order 2. Therefore $\omega_j \in \mathscr{O}(S)$ by Second Riemann extension Thm. 4.3.7. Thus $\pi^\#\delta \cdot f$ belongs to $\mathscr{O}_{X,x}$ by (4.6.3). $\square$

4.7 Coherence of $\widehat{\mathscr{O}}_X$; the normalization $\widehat{X}$

Let X be a reduced complex space. We say that $x \in X$ is **normal** if $\mathscr{O}_{X,x}$ is normal, i.e. $\mathscr{O}_{X,x} = \widehat{\mathscr{O}}_{X,x}$ (cf. Cor. 4.6.8). We say that X is a **normal (reduced) complex space** if every point of X is normal.

The first goal of this section is to prove:

Theorem 4.7.1. *$\widehat{\mathscr{O}}_X$ is a coherent $\mathscr{O}_X$-module.*

Corollary 4.7.2. *The set of non-normal points of X is a nowhere dense analytic subset of X.*

Proof. The non-normal locus of X is the support of the coherent sheaf $\widehat{\mathscr{O}}_X/\mathscr{O}_X$. $\square$

The construction of the normalization $\widehat{X}$ (defined by $\mathrm{Specan}\widehat{\mathscr{O}}_X$) will be an immediate consequence of (the proof of) Thm. 4.7.1.

4.7.1 Non-normal loci

It turns out that in order to prove Thm. 4.7.1 we need to first prove Cor. 4.7.2. In fact, we only need the fact that the set of normal points are open. Cor. 4.7.2 follows easily from the following criterion.

Theorem 4.7.3. *The set of non-normal points of X is equal to the support of*

$$\frac{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{I}_{\mathrm{Sg}(X)})}{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{I}_{\mathrm{Sg}(X)}) \cap \mathscr{O}_X}$$

The following proof actually shows that the theorem still holds if we replace $\mathscr{I}_{\mathrm{Sg}(X)}$ by $\mathscr{I}_A$ where A is an arbitrary nowhere dense analytic subset of X containing $\mathrm{Sg}(X)$.

Proof. Choose any $x \in X$. We need to show that

$$\widehat{\mathscr{O}}_{X,x} = \mathscr{O}_{X,x} \qquad \Longleftrightarrow \qquad \mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{I}_{\mathrm{Sg}(X),x}) \subset \mathscr{O}_{X,x} \tag{4.7.1}$$

Part 1. Assume $\widehat{\mathscr{O}}_{X,x} \neq \mathscr{O}_{X,x}$. Choose $f \in \widehat{\mathscr{O}}_{X,x}$ not in $\mathscr{O}_{X,x}$. Shrink X to a neighborhood of x so that $f \in \widehat{\mathscr{O}}_X(X) \subset \mathscr{M}_X(X)$, and that $\mathscr{I}_{\mathrm{Sg}(X)}$ is generated by finitely many sections $g_1, g_2, \dots \in \mathscr{I}_{\mathrm{Sg}(X)}(X)$. The polar set $P(f)$ is contained in $\mathrm{Sg}(X)$ due to First Riemann extension Thm. 4.3.1. By Prop. 4.1.5, $\mathscr{O}_X f$ is a coherent $\mathscr{O}_X$-submodule of $\mathscr{M}_X$. So $\mathscr{O}_X f/(\mathscr{O}_X f \cap \mathscr{O}_X)$ is coherent, and its support is $P(f)$ (cf. (4.1.7)) on which the functions $g_1, g_2, \dots$ vanish.

By applying Nullstellensatz (Rem. 2.10.4-3) to $\mathscr{O}_X f/(\mathscr{O}_X f \cap \mathscr{O}_X)$, we see that there is $n \in \mathbb{Z}_+$ such that $(g_i^n f)_x \in \mathscr{O}_{X,x}$ for all i, thus $f\mathscr{I}^n_{\mathrm{Sg}(X),x} \subset \mathscr{O}_{X,x}$. Since $f \notin \mathscr{O}_{X,x}$, we can find the smallest $n \in \mathbb{Z}_+$ such that $f\mathscr{I}^n_{\mathrm{Sg}(X),x} \subset \mathscr{O}_{X,x}$. Choose any $\widetilde{f} \in f\mathscr{I}^{n-1}_{\mathrm{Sg}(X),x}$ not in $\mathscr{O}_{X,x}$. (But note that $\widetilde{f}$ belongs to $\widehat{\mathscr{O}}_{X,x}$ since f does.) Then $\widetilde{f}\mathscr{I}_{\mathrm{Sg}(X),x} \subset \mathscr{O}_{X,x}$. We claim that $\widetilde{f}\mathscr{I}_{\mathrm{Sg}(X),x} \subset \mathscr{I}_{\mathrm{Sg}(X),x}$. Then the multiplication of $\widetilde{f}$ gives an element of $\mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{I}_{\mathrm{Sg}(X),x})$ not inside $\mathscr{O}_{X,x}$, which disproves the RHS of (4.7.1).

Shrink X further so that $\widetilde{f} \in \widehat{\mathscr{O}}_X(X)$ and that $\widetilde{f}g_i \in \mathscr{O}(X)$ for each i. By Thm. 4.3.1 again, $\widetilde{f}$ belongs to $\mathscr{O}(S\backslash\mathrm{Sg}(X))$ and is locally bounded on X. Therefore, each $\widetilde{f}g_i$ vanishes on $\mathrm{Sg}(X)$, and hence must belong to $\mathscr{I}_{\mathrm{Sg}(X)}(X)$. This proves the claim.

Part 2. Assume $\widehat{\mathscr{O}}_{X,x} = \mathscr{O}_{X,x}$. Choose any $\mathscr{O}_{X,x}$-module endomorphism α of $\mathscr{I}_{\mathrm{Sg}(X),x}$. Then α is the multiplication by f for some $f \in \mathscr{M}_{X,x}$. Indeed, since $\mathrm{Sg}(X)$ is thin in X (Thm. 3.6.7), by Prop. 3.4.4 we can find $g \in \mathscr{I}_{\mathrm{Sg}(X),x}$ which is a non zero-divisor of $\mathscr{O}_{X,x}$ and hence of $\mathscr{M}_{X,x}$. Set $f = \frac{\alpha(g)}{g}$. Then for each $h \in \mathscr{I}_{\mathrm{Sg}(X),x}$, since α is a homomorphism, we have

$$fh = \frac{\alpha(g)h}{g} = \frac{\alpha(hg)}{g} = \frac{\alpha(h)g}{g} = \alpha(h)$$

which shows that α is the multiplication of f on $\mathscr{M}_{X,x}$.

Since $\mathscr{O}_{X,x}$ is Noetherian and $\mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{I}_{\mathrm{Sg}(X),x})$ is a finitely-generated $\mathscr{O}_{X,x}$-module, the $\mathscr{O}_{X,x}$-submodule generated by $1, \alpha, \alpha^2, \dots$ is finitely-generated. Therefore α is integral over $\mathscr{O}_{X,x}$. Thus, a monic $\mathscr{O}_{X,x}$-polynomial of f multiplied by the non zero-divisor g is zero, which implies that f is integral over $\mathscr{O}_{X,x}$. Thus $f \in \widehat{\mathscr{O}}_{X,x} = \mathscr{O}_{X,x}$. Therefore α is the multiplication of an element of $\mathscr{O}_{X,x}$. This proves the RHS of (4.7.1). □

4.7.2 Proof of Thm. 4.7.1

Proposition 4.7.4. *Let $\pi : X \to S$ be a 1-sheeted branched covering of reduced complex space. Then we have an isomorphism of $\mathscr{O}_S$-algebras (cf. Prop. 4.2.2)*

$$\pi^\# : \widehat{\mathscr{O}}_S \xrightarrow{\simeq} \pi_* \widehat{\mathscr{O}}_X$$

Proof. Outside the branch locus Δ, $\pi : X \backslash \pi^{-1}(\Delta) \to S \backslash \Delta$ is a 1-sheeted unbranched covering, i.e. a biholomorohism. For each $t \in S$, an element $g \in \widehat{\mathscr{O}}_{S,t}$ is of the form $g \in \mathscr{O}_S(V \backslash B)$ where V is a neighborhood of $t \in S$, B is a thin analytic subset of V, and g is bounded. Since g is determined by its values outside the nowhere dense subset $(V \cap \Delta) \cup B$ of V, $\pi^\# g$ is non-zero if g is non-zero. So $\pi^\#$ is injective at t.

If $f \in (\pi_* \widehat{\mathscr{O}}_X)_t$, choose a neighborhood V of t such that $f \in \widehat{\mathscr{O}}_X(\pi^{-1}(V))$. By Prop. 2.4.1, we may shrink V and find a thin analytic subset $A \subset \pi^{-1}(V)$ such that $f \in \mathscr{O}_X(\pi^{-1}(V) \backslash A)$ and f is bounded. So f restricts to a holomorphic map on $\pi^{-1}(V) \backslash (A \cup \pi^{-1}(\Delta))$ which is sent biholomorphically by π to $V \backslash (\pi(A) \cup \Delta)$. Define $g \in \mathscr{O}_S(V \backslash (\pi(A) \cup \Delta))$ to be $f \circ \pi^{-1}$, which is bounded and hence belongs to $\widehat{\mathscr{O}}_S(V)$. Then $\pi^\# g = f$. This shows that $\pi^\#$ is surjective at t. □

We are now ready to give the

***Proof of Thm. 4.7.1**.* Choose any $x \in X$. We show that $\widehat{\mathscr{O}}_X$ is coherent up to shrinking X to a neighborhood of x. By Thm. 4.6.7, $\widehat{\mathscr{O}}_{X,x}$ is $\mathscr{O}_{X,x}$-generated by finitely many elements $f_1, \dots, f_n \in \widehat{\mathscr{O}}_{X,x}$. Since each f_i is integral over $\mathscr{O}_{X,x}$, we can find a monic $\mathscr{O}_{X,x}$-polynomial P_i such that $P_i(f_i) = 0$. Shrink X so that each f_i belongs to $\widehat{\mathscr{O}}_X(X)$, that the coefficients of each P_i belong to $\mathscr{O}(X)$, and that $P_i(f_i) = 0$ holds in $\widehat{\mathscr{O}}_X(X)$.

Let $\mathscr{A}$ be the $\mathscr{O}_X$-subalgebra of $\widehat{\mathscr{O}}_X$ generated by $f_1, \dots, f_n$, namely, it is the unique subsheaf of $\widehat{\mathscr{O}}_X$ whose stalk at each $p \in X$ is the $\mathscr{O}_{X,p}$-subalgebra generated by the stalks of $f_1, \dots, f_n$ at p. Then $P_i(f_i) = 0$ implies that $\mathscr{A}$ is a finite-type $\mathscr{O}_X$-module, and hence coherent by Prop. 4.1.5. Thus, by Thm. 2.9.3, we can define a finite holomorphic map $\psi : Y = \mathrm{Specan}(\mathscr{A}) \to X$ such that the equivalence of $\mathscr{O}_X$-algebras $\psi_*\mathscr{O}_Y \simeq \mathscr{A}$ holds. Clearly each stalk of $\mathscr{A}$ has no non-zero nilpotent elements. So each stalk $\mathscr{O}_{Y,q}$ ($q \in Y$), which is a direct summand of $(\psi_*\mathscr{O}_Y)_{\psi(q)}$ (Prop. 2.4.5), is reduced. Therefore Y is reduced.

Since $\widehat{\mathscr{O}}_X$ equals $\mathscr{O}_X$ outside the thin analytic subset $\Delta = \mathrm{Sg}(X)$ by First Riemann extension Thm. 4.3.1, $\mathscr{A}$ equals $\mathscr{O}_X$ outside Δ. Therefore $Y\backslash\Delta = X\backslash\Delta$. Thus, by Lemma 4.5.7, ψ is a 1-sheeted branched covering. By Prop. 4.7.4, we obtain an isomorphism of $\mathscr{O}_X$-algebras $\widehat{\mathscr{O}}_X \simeq \psi_*\widehat{\mathscr{O}}_Y$.

We know that $(\psi_*\mathscr{O}_Y)_x = \mathscr{A}_x$ is the integral closure $\widehat{\mathscr{O}}_{X,x}$ of $\mathscr{O}_{X,x}$ in $\mathscr{M}_{X,x}$. So by Rem. 4.2.7, $\mathscr{A}_x$ is the integral closure of itself in $\mathscr{M}_{X,x}$. So $\mathscr{A}_x = \bigoplus_{y\in\psi^{-1}(x)} \mathscr{O}_{Y,y}$ is a normal ring. (Note that the elements of $\mathrm{Nzd}(\mathscr{A}_x)^{-1}\mathscr{A}_x$ belong to $\mathscr{M}_{X,x}$.) Therefore $\mathscr{O}_{Y,y}$ is normal for each $y \in \psi^{-1}(x)$. By Cor. 4.7.2 (implied by Thm. 4.7.3), each $y \in \psi^{-1}(x)$ is contained in a normal open subset of Y. Therefore, by Prop. 2.4.1, we may shrink X to a neighborhood of x and shrink Y to $\psi^{-1}(X)$ so that Y is normal. Therefore $\widehat{\mathscr{O}}_X \simeq \psi_*\mathscr{O}_Y$, and $\psi_*\mathscr{O}_Y$ is $\mathscr{O}_X$-coherent by Finite mapping Thm. 2.7.1. □

The normalization $\widehat{X}$

Using the coherence of $\widehat{\mathscr{O}}_X$, we immediately obtain

Theorem 4.7.5. *For any reduced complex space X, there is, up to isomorphisms (in the sense of Def. 2.9.1), a unique 1-sheeted branched covering $\nu : \widehat{X} \to X$ such that $\widehat{X}$ is normal. This covering (or simply the complex space $\widehat{X}$) is called the* ***normalization*** *of X. $\mathrm{Specan}\widehat{\mathscr{O}}_X \to \mathscr{O}_X$ is a normalization.*

Proof. By Prop. 4.7.4, we have an isomorphism of $\mathscr{O}_X$-algebras $\nu_*\mathscr{O}_{\widehat{X}} \simeq \widehat{\mathscr{O}}_X$. So the equivalence class of the $\mathscr{O}_X$-algebra $\nu_*\mathscr{O}_{\widehat{X}}$ is unique. Therefore the normalization is unique due to Thm. 2.9.3.

Let $\widehat{X}$ be $\mathrm{Specan}\widehat{\mathscr{O}}_X$. Then the proof of Thm. 4.7.1 shows that $\widehat{X}$ is reduced and normal, and $\nu : \widehat{X} \to X$ is a 1-sheeting covering with branch locus $\mathrm{Sg}(X)$. □

Remark 4.7.6. Suppose that we have decomposition $X = X_1 \cup \cdots \cup X_N$ of X into analytic subsets such that $X_i \cap X_j$ is nowhere dense in X_i for all $i \neq j$. Then by Thm. 4.2.4, or more precisely by (4.2.2c), $\widehat{X}$ is a disjoint union of open subsets

$$\widehat{X} = \coprod_{i=1}^{N} \widehat{X}_i$$

where each $\widehat{X}_i$ is the normalization of X_i.

4.8 Basic properties of normal complex spaces

Let X be a reduced complex space.

Proposition 4.8.1. *Assume that X is normal. Then X is locally irreducible. In particular, X is locally pure dimensional (by Thm. 3.14.9).*

Proof. Suppose that X is not irreducible at x. Shrink X so that we have local decomposition $X = X_1 \cup \cdots \cup X_N$ at x (where $N \geqslant 2$) and Thm. 3.3.4 holds for all $i \neq j$. The characteristic function χ_{X_i} (cf. Exp. 4.1.6) clearly belongs to $\widehat{\mathscr{O}}_X(X)$. But it cannot be extended to a continuous function on X, otherwise its value at x would be both 1 and 0. So it is not in $\mathscr{O}(X)$. This contradicts the normality $\widehat{\mathscr{O}}_X = \mathscr{O}_X$. □

Theorem 4.8.2. *Assume that X is normal. Then $\mathrm{Sg}(X)$ is thin of order 2.*

Consequently, a reduced complex curve (i.e. reduced 1-dimensional complex space) is smooth iff it is normal.

Proof. Let $Y = \mathrm{Sg}(X)$ and fix $x \in X$. By Prop. 4.8.1, after shrunk to a neighborhood of x, X has pure dimension n. Since Y is nowhere dense, Y has dimension $\leqslant n-1$ at x by Ritt's lemma 3.10.6. Let us assume that $\dim_x Y = n-1$ and find a contradiction.

By Prop. 3.14.12, Y has a smooth point with dimension $n-1$. For the purpose of finding a contradiction, we may assume x is that point. Namely, we assume Y is smooth at x and $\dim_x Y = n-1$.

Step 1. Let $\{f_1, \dots, f_N\}$ be a set of non-zero generators of the ideal $\mathscr{I}_{Y,x}$. Note that they are non zero-divisors of $\mathscr{O}_{X,x}$ because X is irreducible everywhere. We claim that for each i, the germ (Y, x) is a component in the local irreducible decomposition $(N(f_i), x) = \bigcup_k (Z_k, x)$. Indeed, since (Y, x) is irreducible (because $\mathscr{O}_{Y,x} \simeq \mathscr{O}_{\mathbb{C}^n,0}$) and has decomposition $(Y, x) = \bigcup_k (Y \cap Z_k, x)$, by Rem. 3.3.2 we have $(Y, x) = (Y \cap Z_k, x)$ for some k, and hence $(Y, x) \subset (Z_k, x)$. If $(Y, x) \neq (Z_k, x)$

then there is a element of $\mathscr{I}_{Y,x} \backslash \mathscr{I}_{Z_k,x}$. This element, when restricted to (Z_k, x), is non-zero and hence a non zero-divisor of the integral domain $\mathscr{O}_{Z_k,x}$. Thus, by Active lemma,

$$\dim_x Y \leqslant \dim_x Z_k - 1 \leqslant \dim_x N(f_i) - 1 = \dim_x X - 2 = n - 2$$

which is impossible. So $(Y, x) = (Z_k, x)$.

After shrinking X to a neighborhood of x, we have that $f_i \in \mathscr{I}_Y(X)$, that (by Prop. 2.3.13 and Rem. 1.2.16) $f_{1,p}, \dots, f_{N,p}$ are non zero-divisors generating $\mathscr{O}_{X,p}$ for each $p \in X$, and that (by Thm. 3.3.4) all the germs not equal to (Y, x) in the irreducible decomposition of $(N(f_i), x)$ are analytic subsets of X whose intersections with Y are nowhere dense in Y. Thus, we may pick a point $p \in Y$ close to x such that $(N(f_i), p) = (Y, p)$.

Step 2. We assume that for some $L \leqslant N$, the germs of $f_1, \dots, f_L$ form a minimal set of generators of $\mathscr{I}_{Y,p}$. We claim that $L = 1$. If this can be proved, then by Prop. 3.6.3, we have $\mathrm{emb}_p Y \geqslant \mathrm{emb}_p X - 1$ since, in general, $\dim_{\mathbb{C}} d_p(\mathcal{I}_p)$ is bounded by the number of generators of an ideal $\mathcal{I}_p$ of $\mathscr{O}_{X,p}$. Recall that in general the embedding dimensions are no less than the dimensions. Since Y is smooth at p and X is singular at p (because $p \in Y = \mathrm{Sg}(X)$), by Prop. 3.10.9 we have $\mathrm{emb}_p Y = \dim_p Y = n - 1$ and $\mathrm{emb}_p X > \dim_p X = n$. This gives a contradiction, and hence finishes the proof.

Step 3. Let us prove $L = 1$. Assume $L \geqslant 2$. Then by the assumption of minimality on $\{f_1, \dots, f_L\}$, the stalks of f_1/f_2 and f_2/f_1 in $\mathscr{M}_{X,p}$ are not inside $\mathscr{O}_{X,p}$. We assume that there is a sequence $(p_n)_{n \in \mathbb{Z}_+}$ in $X \backslash Y$ converging to p such that

$$\sup_n |f_1(p_n)/f_2(p_n)| < +\infty, \tag{4.8.1}$$

otherwise there must exist such a sequence for f_2/f_1.

Clearly, there exists $k \in \mathbb{Z}_+$ such that $(f_{1,p}/f_{2,p}) \cdot \mathscr{I}_{Y,p}^k \subset \mathscr{O}_{X,p}$. We let k be the smallest such number, and find $g = f_{i_1,p} \cdots f_{i_{k-1},p}$ (where $1 \leqslant i_1, \dots, i_{k-1} \leqslant L$) such that $h = (f_{1,p}/f_{2,p}) \cdot g \notin \mathscr{O}_{X,p}$. For each f_i, by (4.8.1), $hf_i \in \mathscr{O}_{X,p}$ vanishes on p. But $N(hf_i) \subset (Y, p)$ because $(N(f_j), p) = (Y, p)$ for each j. Thus, $N(hf_i)$ is a germ of analytic subset passing through p. Since (by Active lemma) both $(N(hf_i), p)$ and (Y, p) have dimension $n - 1$, these two germs must be equal. (Otherwise there is a non-zero element of the integral domain $\mathscr{O}_{Y,p}$ vanishing on $(N(hf_i), p)$, which contradicts Active lemma.) We conclude that $hf_{i,p} \in \mathscr{I}_{Y,p}$ for all i. Therefore $h\mathscr{I}_{Y,p} \subset \mathscr{I}_{Y,p}$. As in the proof of Thm. 4.7.3, we have $h \in \widehat{\mathscr{O}}_{X,p} = \mathscr{O}_{X,p}$, which is impossible. □

Theorem 4.8.3. *Let $\varphi : X \to Y$ be a holomorphic map of reduced complex spaces. Assume that Y is normal and φ is a homeomorphism. Then φ is a biholomorphism.*

For example, if φ is a holomorphic map of complex manifolds which is a homeomorphism, then φ is a biholomorphism.

Proof. Since Y is locally irreducible, by Cor. 4.5.9, φ is a 1-sheeted branched covering with branch locus $\Delta \subset Y$. Let $\psi : Y \to X$ be the inverse of φ. Then ψ restricts to a holomorphic map $\psi : Y\backslash\Delta \to X\backslash\psi(\Delta)$.

Choose any $y \in \Delta$ and let $x = \psi(y)$. Let us show that ψ is holomorphic on a neighborhood of y. By shrinking X to a neighborhood of x replacing Y by $\varphi(X)$, we assume that X is a closed subspace of an open ball U in $\mathbb{C}^n$. Let $\iota : X \to \mathbb{C}^n$ be the inclusion map. Then $\iota \circ \psi$ can be viewed as a continuous map $Y \to \mathbb{C}^n$ which satisfies $\iota \circ \psi(Y) \subset X$ and is holomorphic outside Δ. By First Riemann extension Thm. 4.3.1, $\iota \circ \varphi|_{Y\backslash\Delta} : Y\backslash\Delta \to \mathbb{C}^n$ can be extended to a holomorphic function $Y \to \mathbb{C}^n$, which must equal $\iota \circ \varphi$ as continuous maps. Thus $\iota \circ \psi : Y \to \mathbb{C}^n$ is holomorphic and satisfies $\iota \circ \psi(Y) \subset X$. Thus, by the reducedness of Y and by Thm. 1.4.8, $\iota \circ \psi$ restricts to a holomorphic map $\widetilde{\psi} : Y \to X$, which clearly equals ψ as set maps. Therefore $\varphi \circ \widetilde{\psi} = \mathbf{1}_Y$ and $\widetilde{\psi} \circ \varphi = \mathbf{1}_X$ as set maps, and hence as holomorphic maps because X and Y are reduced. $\square$

4.9 Global decomposition of reduced complex spaces

Let X be a reduced complex space.

4.9.1 Global decomposition: the normal case

Proposition 4.9.1. *Let X be normal, and let T be a thin subset of X. Then the following are equivalent.*

(1) X is connected.

(2) $X\backslash T$ is connected.

*If X satisfies these conditions, we say that X is **irreducible**.*

Note that in the special case that $T = \mathrm{Sg}(X)$, we have that X is connected iff the complex manifold $X\backslash\mathrm{Sg}(X)$ is so.

Proof. If X is a disjoint union of two non-empty open subsets, the same is true for $X\backslash T$ because T is nowhere dense in X. This shows (2)⇒(1). On the other hand, if $X\backslash T$ is a disjoint union of non-empty open subset $U \sqcup V$, define $f : X\backslash T \to \mathbb{C}$ to be 0 on U and 1 on V. Since X is normal and f is locally bounded, f can be extended to a holomorphic function on X because $\widehat{\mathscr{O}}_X = \mathscr{O}_X$. But the range of this function must be $\{0, 1\}$. So X is not connected. This proves (1)⇒(2). $\square$

Proposition 4.9.2. *Let X be normal. Then X is locally connected. Equivalently, X is a disjoint union of open connected subspaces (which are clearly normal, and hence irreducible)*

$$X = \coprod_{\alpha \in \mathfrak{A}} X_\alpha. \tag{4.9.1}$$

If T is a thin subset of X, then

$$X \backslash T = \coprod_{\alpha \in \mathfrak{A}} X_\alpha \backslash T \tag{4.9.2}$$

is the decomposition of $X\backslash T$ into connected components. Each X_α is the closure of $X_\alpha \backslash T$ in X.

We call (4.9.1) the **global decomposition** of the normal complex space X. It follows that (4.9.2) is the global decomposition of $X\backslash T$.

Proof. That X is locally connected is equivalent to the existence of decomposition into connected components (4.9.1) is a basic fact in point-set topology. Once we have (4.9.1), then we clearly have (4.9.2) where each $X_\alpha\backslash T$ is connected by Prop. 4.9.1. Since $T \cap X_\alpha$ is nowhere dense in X_α, X_α is the closure of $X_\alpha\backslash T$.

To prove that X is locally connected, we choose any $x \in X$ and shrink X to a neighborhood of x so that X is a model space. In particular, the complex manifold $X\backslash \mathrm{Sg}(X)$ is second countable, and hence has countably many irreducible components

$$X\backslash \mathrm{Sg}(X) = \coprod_{n \in \mathbb{Z}_+} \Omega_n.$$

Define $f \in \mathscr{O}(X\backslash \mathrm{Sg}(X))$ to be constantly n on Ω_n. Since X is normal, $\mathrm{Sg}(X)$ is thin of order 2 by Thm. 4.8.2. Therefore, by Thm. 4.6.9, $f \in \mathscr{O}(X)$. By continuity, f has range $\mathbb{Z}_+$.

Let $\Omega_{>1} = \bigcup_{n>1} \Omega_n$. Then $X\backslash \mathrm{Sg}(X) = \Omega_1 \cup \Omega_{>1}$. Hence

$$X = (X\backslash \mathrm{Sg}(X))^{\mathrm{cl}} = \Omega_1^{\mathrm{cl}} \cup \Omega_{>1}^{\mathrm{cl}}.$$

Since $\Omega_1^{\mathrm{cl}} \subset A := N(f-1)$ and $\Omega_{>1}^{\mathrm{cl}} \subset B := \{x \in X : f(x) \neq 1\}$, and since $X = A \sqcup B$, we must have $\Omega_1^{\mathrm{cl}} = A$ and $\Omega_{>1}^{\mathrm{cl}} = B$. This proves that $N(f-1) = \Omega_1^{\mathrm{cl}}$ and is open in X. The same argument shows that for each n, $\Omega_n^{\mathrm{cl}} = N(f-n)$ and is open in X. Note that Ω_n^{cl} is connected because Ω_n is so. We thus have

$$X = \coprod_{n \in \mathbb{Z}_+} N(f-n) = \coprod_{n \in \mathbb{Z}_+} \Omega_n^{\mathrm{cl}}$$

where each Ω_n^{cl} is a connected open subset of X. This proves the existence of (4.9.1). $\square$

4.9.2 Global decomposition: the general case

Proposition 4.9.3. *Let T be a thin subset of X containing $\mathrm{Sg}(X)$. Let $\nu : \widehat{X} \to X$ be the normalization of X. Then the following are equivalent.*

(1) $\widehat{X}$ is irreducible.

(2) The complex manifold $X \backslash T$ is connected.

If X satisfies these conditions, we say that X is ***irreducible****.*

Again, if we set $T = \mathrm{Sg}(X)$, we see that X is irreducible iff $X \backslash \mathrm{Sg}(X)$ is connected.

Proof. Since smooth points of X are normal, ν restricts to a biholomorphism $\nu : \widehat{X} \backslash \nu^{-1}(T) \to X \backslash T$. Since $\nu^{-1}(T)$ is thin in $\widehat{X}$, by Prop. 4.9.1, $\widehat{X} \backslash \nu^{-1}(T)$ is connected iff $\widehat{X}$ is irreducible. $\square$

Remark 4.9.4. If X is irreducible then $\widehat{X}$, which is locally pure dimensional by Prop. 4.8.1, must be pure n-dimensional for some n. Therefore X is also pure n-dimensional due to Cor. 3.12.9.

Proposition 4.9.5. *Let $\nu : \widehat{X} \to X$ be the normalization of X. Let*

$$\widehat{X} = \coprod_{\alpha \in \mathfrak{A}} \widehat{X}_\alpha$$

be the global decomposition of $\widehat{X}$. Let $X_\alpha = \nu(\widehat{X}_\alpha)$. Assume that T is a thin subset of X containing $\mathrm{Sg}(X)$. The following are true.

1. *The restriction $\nu_\alpha : \widehat{X}_\alpha \to X_\alpha$ of ν is the normalization of X_α. $T \cap X_\alpha$ is a branch locus of the 1-sheeted branched covering ν_α.*

2. *Each X_α is the closure of the complex manifold $X_\alpha \backslash T$ in X, and*

$$X \backslash T = \coprod_{\alpha \in \mathfrak{A}} X_\alpha \backslash T$$

is the (disjoint) decomposition of the complex manifold $X \backslash T$ into connected components.

We clearly have $X = \bigcup_{\alpha \in \mathfrak{A}} X_\alpha$. This is called the ***global decomposition*** *of X.*

Note that each X_α is an analytic subset of X (cf. Exe. 2.3.11).

To summarize, Prop. 4.9.5 says that the global decomposition of X can be obtained in two equivalent ways: either by taking the image of the connected components of $\widehat{X}$ under the normalization map ν, or by taking the closures of the connected components of $X \backslash T$.

Proof. Since smooth points are normal, the restriction

$$\nu : \widehat{X}\backslash\nu^{-1}(T) \xrightarrow{\simeq} X\backslash T$$

is a biholomorphism, which further restricts to a biholomorphism

$$\nu : \widehat{X}_\alpha\backslash\nu^{-1}(T) \xrightarrow{\simeq} \nu(\widehat{X}_\alpha\backslash\nu^{-1}(T)) = X_\alpha\backslash T.$$

Since $\widehat{X}_\alpha$ is an open and closed connected component of $\widehat{X}$, $\nu^{-1}(T) \cap \widehat{X}_\alpha$ is nowhere dense in $\widehat{X}_\alpha$. So $\widehat{X}_\alpha\backslash\nu^{-1}(T)$ is dense in $\widehat{X}_\alpha$. Therefore X_α must be the closure of $X_\alpha\backslash T$ in X. So $T \cap X_\alpha$ is a thin subset of X_α. Thus $\nu_\alpha : \widehat{X}_\alpha \to X_\alpha$ is a 1-sheeting covering with branch locus $T \cap X_\alpha$. Since $\widehat{X}_\alpha$ is normal, ν_α is the normalization of X_α.

By Prop. 4.9.2, we have global decomposition

$$\widehat{X}\backslash\nu^{-1}(T) = \coprod_{\alpha\in\mathfrak{A}} \widehat{X}_\alpha\backslash\nu^{-1}(T).$$

ν sends this decomposition biholomorphically to $X\backslash T = \coprod_{\alpha\in\mathfrak{A}} X_\alpha\backslash T$. □

4.10 Basic properties of irreducible complex spaces

Let X be a reduced complex space. In this section, we collect some useful facts about irreducible complex spaces.

Proposition 4.10.1. *If X is irreducible, then X is pure dimensional.*

Proof. By definition of irreducible complex spaces, the normalization $\widehat{X}$ is connected. Thus, by Prop. 4.8.1, $\widehat{X}$ has pure dimension n. Therefore X has pure dimension n by Cor. 3.12.9. □

Proposition 4.10.2. *Assume that X is irreducible. Then any analytic subset A of X is either nowhere dense in X or $A = X$.*

Proof. First assume that X is a connected complex manifold. Then by Lemma 3.11.4, either $\mathscr{I}_A = 0_X$ (namely, $A = X$) or $\mathscr{I}_{A,x} \neq 0$ for each $x \in X$. In the latter case, A is clearly nowhere dense in X.

Now assume that X is irreducible but not necessarily smooth. Since $X\backslash\mathrm{Sg}(X)$ is connected, the smooth case implies that either $A \supset X\backslash\mathrm{Sg}(X)$ or $A\backslash\mathrm{Sg}(X)$ is nowhere dense in $X\backslash\mathrm{Sg}(X)$. In the former case, clearly $A = X$. In the latter case, one checks easily that A is nowhere dense in X. □

Corollary 4.10.3. *Assume that X is irreducible, and let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$. If $\mathcal{I} \neq 0_X$, then $\mathcal{I}_x$ contains a non zero-divisor of $\mathscr{O}_{X,x}$ for every $x \in X$.*

In the special case that $\mathcal{I} = f\mathscr{O}_X$ where $f \in \mathscr{O}(X)$, this corollary implies that if f is not constantly zero on X then f is not zero when restricted to any non-empty open subset of X. Thus, this corollary generalizes Lemmas 1.1.3 and 3.11.4.

Proof. Let $A = N(\mathcal{I})$. If $\mathcal{I} \neq 0_X$, then $A \neq X$. So by Prop. 4.10.2, A is nowhere dense in X. So by Prop. 3.4.4, $\mathcal{I}_x \cap \mathrm{Nzd}(\mathscr{O}_{X,x}) \neq \emptyset$ for each $x \in X$. $\square$

Theorem 4.10.4 (Open mapping theorem). *Assume that X is irreducible and $f \in \mathscr{O}(X)$ is not a constant function. Then the map $f : X \to \mathbb{C}$ is open. In particular, the absolute value function $|f| : X \to [0, +\infty)$ does not achieve maximum when restricted to any open subset of X.*

Proof. Since f is not constant, by Cor. 4.10.3, for each $x \in X$, f_x is not constant in $\mathscr{O}_{X,x}$. Thus f is open at x by Cor. 3.13.7. $\square$

Proposition 4.10.5. *The following are equivalent.*

(1) X is irreducible.

(2) Whenever we have $X = A \cup B$ where A and B are analytic subsets of X, then $X = A$ or $X = B$.

Proof. Assume that X is irreducible. If $X = A \cup B$ where A and B are analytic, then one of A and B must equal X, otherwise both A and B are nowhere dense in X due to Prop. 4.10.2, which is impossible. This proves (1)$\Rightarrow$(2).

Assume that X is not irreducible. Let $\nu : \widehat{X} \to X$ be the normalization. Then $\widehat{X}$ is not connected. We have $\widehat{X} = \widehat{A} \sqcup \widehat{B}$ where $\widehat{A}, \widehat{B}$ are non-empty open and closed subsets of X. $A = \nu(\widehat{A})$ is the closure in X of $A_0 = \nu(\widehat{A} \backslash \nu^{-1}(\mathrm{Sg}(X)))$, and similarly $B = \nu(\widehat{B})$ is the closure of $B_0 = \nu(\widehat{B} \backslash \nu^{-1}(\mathrm{Sg}(X)))$. Since $\nu : \widehat{X} \backslash \nu^{-1}(\mathrm{Sg}(X)) \to X \backslash \mathrm{Sg}(X)$ is a biholomorphism, A_0 and B_0 are disjoint non-empty open subsets of $X \backslash \mathrm{Sg}(X)$. So $A \neq X$ and $B \neq X$. $\square$

Corollary 4.10.6. *Let $\varphi : X \to Y$ be a surjective holomorphic map of reduced complex spaces. If X is irreducible, then Y is also irreducible.*

Proof. Immediate from Cor. 4.10.5. $\square$

Proposition 4.10.7. *Let X and Y be reduced complex spaces. Then $X \times Y$ is irreducible if and only if both X and Y are irreducible.*

Proof. Applying Cor. 4.10.6 to the projections of $X \times Y$ to X and Y shows that if $X \times Y$ is irreducible then X and Y are irreducible. Conversely, assume that X and Y are both irreducible. Then $X \backslash \mathrm{Sg}(X)$ and $Y \backslash \mathrm{Sg}(Y)$ are connected. So $(X \backslash \mathrm{Sg}(X)) \times (Y \backslash \mathrm{Sg}(Y))$ is connected. But this complex manifold is $(X \times Y) \backslash \mathrm{Sg}(X \times Y)$ by Cor. 3.10.10. Therefore $X \times Y$ is irreducible. $\square$

4.11 Normalization and local irreducibility

Let X be a reduced complex space, and let $\nu : \widehat{X} \to X$ be the normalization of X. In this section, we use normalization and global irreducibility to study (local) irreducible points of X.

Proposition 4.11.1. *For each $x \in X$, the number of points in $\nu^{-1}(x)$ is equal to the number of irreducible components in the local decomposition of X at x.*

Proof. It suffices to prove the case that X is irreducible at x, since the general case will follow immediately from Rem. 4.7.6. So let us assume that x is an irreducible point. Suppose that $\nu^{-1}(x)$ contains two distinct points y_1, y_2. Since $(\nu_*\mathscr{O}_{\widehat{X}})_x = \widehat{\mathscr{O}}_{X,x}$ is $\mathscr{O}_{X,x}$-torsion free, by Prop. 3.14.7, ν is open at y_1 and y_2. Choose neighborhoods V_1 of y_1 and V_2 and y_2 such that $V_1 \cap V_2 = \emptyset$. Then $\nu(V_1) \cap \nu(V_2)$ contains a neighborhood U of x. Therefore, for each $x' \in U$, the fiber $\nu^{-1}(x')$ contains at least two different points. This contradicts the fact that ν is a 1-sheeting branched covering. □

Corollary 4.11.2. *$x \in X$ is an irreducible point of X if and only if $\nu^{-1}(x)$ has only one point. When this is true, ν is open at the only point of $\nu^{-1}(x)$.*

Proof. This is immediate from Prop. 4.11.1 and its proof. □

Corollary 4.11.3. *X is locally irreducible if and only if $\nu : \widehat{X} \to X$ is a homeomorphism.*

Proof. If X is locally irreducible, then by Prop. 4.11.2, ν is an open map. Also, by Prop. 4.11.2, ν is bijective. Therefore ν is a homeomorphism.

Conversely, if ν is homeomorphism, then ν is bijective, and Prop. 4.11.2 implies immediately that each point of X is irreducible. □

Theorem 4.11.4. *Let $x \in X$. Then the following are equivalent.*

(1) X is irreducible at x.

(2) Each neighborhood of x in X contains a smaller irreducible neighborhood of x.

Proof. Assume that x is an irreducible point. For each neighborhood $U \subset X$ of x, the normalization $\nu : \widehat{X} \to X$ restricts to $\nu : \widehat{U} \to U$. By Cor. 4.11.2, $\nu^{-1}(x)$ has only one point $\widehat{x}$. Then by global decomposition Prop. 4.9.2, $\widehat{U}$ is a disjoint union of two closed and open subsets $\widehat{U} = W_1 \cup W_2$ where W_1 is the connected component of $\widehat{U}$ containing $\widehat{x}$, and W_2 is the union of the other connected components. Since W_2 is closed in $\widehat{U}$, and since $\nu : \widehat{U} \to U$ is closed (since any finite map is closed), $\nu(W_2)$ is a closed subset of U disjoint from x. Then $V = U\backslash\nu(W_2)$ is a neighborhood of x contained in U. Clearly ν restricts to a biholomorphism $\nu : W_1\backslash\nu^{-1}(\mathrm{Sg}(X)) \to V\backslash\mathrm{Sg}(X)$. Therefore $V\backslash\mathrm{Sg}(X)$ is connected, and hence V is irreducible.

Assume that x is not irreducible. Then we can shrink X to a neighborhood of x such that X has local decomposition $X = X_1 \cup \cdots \cup X_N$ at x (where $N \geqslant 2$) such that Thm. 3.3.4 holds. Let $A = \mathrm{Sg}(X) \cup \bigcup_{i \neq j}(X_i \cap X_j)$, which is nowhere dense in X. Then for each neighborhood $U \subset X$ of x, $U \cap A$ is thin in U, and we have disjoint union $U\backslash A = \coprod_{i=1}^{N}(U\backslash A) \cap X_i$ where each $(U\backslash A) \cap X_i = (U\backslash A)\backslash \bigcup_{j \neq i} X_j$ is a non-empty open subset of $U\backslash A$. So $U\backslash A$ is not connected, and hence U is not irreducible. $\square$

Corollary 4.11.5. *Let X and Y be reduced complex spaces. Let $x \in X$ and $y \in Y$. Then x and y are irreducible points of X and Y respectively if and only if $x \times y$ is an irreducible point of $X \times Y$.*

Proof. By Rem. 3.3.2, any holomorphic map sends irreducible germs of complex spaces to irreducible ones. Therefore if $x \times y$ is irreducible then x and y are irreducible.

Conversely, assume that x and y are irreducible. For each neighborhood of $x \times y$, choose a smaller one of the form $U \times V$ where $U \subset X$ and $V \subset Y$ are neighborhoods of x and y respectively. By Thm. 4.11.4, there are smaller irreducible neighborhoods $U' \ni x$ and $V' \ni y$ respectively. By Prop. 4.10.7, $U' \times V'$ are irreducible neighborhoods of $x \times y$ in $X \times Y$. This proves that $x \times y$ is irreducible, thanks to Thm. 4.11.4. $\square$

Chapter 5

Flatness

5.1 δ-functors

Let $\mathfrak{A}$ be an abelian category, for instance, the category of (finitely-generated) modules of a commutative ring, the category of (coherent) $\mathscr{O}_X$-modules where X is a complex space, or more generally the category of X-sheaves of abelian groups.

In this section, we describe how the homology and cohomology in complex geometry should look like. Roughly speaking, in a cohomology theory, one should be able to get long exact sequences from short ones. For instance, given a short exact sequence of $\mathscr{O}_X$-modules, one can get long exact sequences of vector spaces being the cohomology groups of $\mathscr{O}_X$-modules. Moreover, the process of taking long exact sequences should be compatible with the morphisms of short exact sequences, i.e. a commutative diagram (5.1.2) where the top and the bottom sequences are exact. δ-functors are a precise way to describe such cohomology.

Another question is whether or in which sense the cohomology theories are unique. (Those unique cohomologies are called universal δ-functors.) It turns out that the cohomologies are determined by their degree-zero parts if the objects in the categories (e.g. the $\mathscr{O}_X$-modules) can always be embedded into an acyclic object, i.e., an object whose positive-degree cohomology groups vanish. This is Thm. 5.1.6, the main result of this section.

Definition 5.1.1. A **(cohomological covariant) δ-functor** $(H^\bullet, \delta^\bullet)$ from an abelian category $\mathfrak{A}$ to another one $\mathfrak{B}$ is a collection of additive functors $H^n : \mathfrak{A} \to \mathfrak{B}$ ($n \in \mathbb{N}$) together with $\delta^n : H^n(\mathcal{G}) \to H^{n+1}(\mathcal{E})$ for each short exact sequence in $\mathfrak{A}$

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \tag{5.1.1}$$

such that the following conditions hold.

(1) Each exact sequence of $\mathfrak{A}$-objects (5.1.1) gives a long exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{E}) \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \\ \xrightarrow{\delta^0} H^1(\mathcal{E}) \to H^1(\mathcal{F}) \to H^1(\mathcal{G}) \xrightarrow{\delta^1} H^2(\mathcal{E}) \to \cdots \end{aligned}$$

In particular, the functor $H^0 : \mathfrak{A} \to \mathfrak{B}$ is left exact.

(2) For each morphism of short exact sequences in $\mathfrak{A}$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{G}' \arrow[r] & 0 \end{tikzcd} \tag{5.1.2}$$

and each $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd} H^n(\mathcal{G}) \arrow[r, "\delta^n"] \arrow[d] & H^{n+1}(\mathcal{E}) \arrow[d] \\ H^n(\mathcal{G}') \arrow[r, "\delta^n"] & H^{n+1}(\mathcal{E}') \end{tikzcd}$$

We abbreviate δ^n to δ when no confusion arises.

Definition 5.1.2. Modify the statements in Def. 5.1.1 as follows. For each short exact sequence (5.1.1) and each $n \in \mathbb{N}$. We have $\delta^n : H^n(\mathcal{E}) \to H^{n+1}(\mathcal{G})$ such that the following hold, we say $(H^\bullet, \delta^\bullet)$ is a **(cohomological) contravariant δ-functor,** if

(1) Each exact sequence of $\mathfrak{A}$-objects (5.1.1) gives a long exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{G}) \to H^0(\mathcal{F}) \to H^0(\mathcal{E}) \\ \xrightarrow{\delta^0} H^1(\mathcal{G}) \to H^1(\mathcal{F}) \to H^1(\mathcal{E}) \xrightarrow{\delta^1} H^2(\mathcal{G}) \to \cdots \end{aligned}$$

In particular, the contravariant functor $H^0 : \mathfrak{A} \to \mathfrak{B}$ is left exact.

(2) For each morphism of short exact sequences (5.1.2) and each $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd} H^n(\mathcal{E}) \arrow[r, "\delta^n"] \arrow[d] & H^{n+1}(\mathcal{G}) \arrow[d] \\ H^n(\mathcal{E}') \arrow[r, "\delta^n"] & H^{n+1}(\mathcal{G}') \end{tikzcd}$$

Definition 5.1.3. Modify the statements in Def. 5.1.1 as follows. For each short exact sequence (5.1.1) and each $n \in \mathbb{N}$, we have $\delta_n : H_{n+1}(\mathcal{G}) \to H_n(\mathcal{E})$ such that the following hold. We say $(H_\bullet, \delta_\bullet)$ is a **homological (covariant) δ-functor**, if

(1) Each exact sequence of $\mathfrak{A}$-objects (5.1.1) gives a long exact sequence

$$\begin{aligned} &\cdots \to H_2(\mathcal{G}) \xrightarrow{\delta_1} H_1(\mathcal{E}) \to H_1(\mathcal{F}) \to H_1(\mathcal{G}) \\ &\xrightarrow{\delta_0} H_0(\mathcal{E}) \to H_0(\mathcal{F}) \to H_0(\mathcal{G}) \to 0 \end{aligned}$$

In particular, the functor $H_0 : \mathfrak{A} \to \mathfrak{B}$ is right exact.

(2) For each morphism of short exact sequences (5.1.2) and each $n \in \mathbb{N}$, the following diagram commutes

$$\begin{tikzcd} H_{n+1}(\mathcal{G}) \arrow[r, "\delta_n"] \arrow[d] & H_n(\mathcal{E}) \arrow[d] \\ H_{n+1}(\mathcal{G}') \arrow[r, "\delta_n"] & H_n(\mathcal{E}') \end{tikzcd}$$

Definition 5.1.4. A **morphism of δ-functors** $\Phi^\bullet : (H^\bullet, \delta^\bullet) \to (\check{H}^\bullet, \delta^\bullet)$ associates to each $n \in \mathbb{N}$ and $\mathcal{E} \in \mathfrak{A}$ a morphism of $\mathfrak{B}$-objects $\Phi = \Phi^n : H^n(\mathcal{E}) \to \check{H}^n(\mathcal{E})$ such that:

(1) For each $n \in \mathbb{N}$, Φ^n is natural. Namely, for each morphism $\mathcal{E} \to \mathcal{F}$ of $\mathfrak{A}$-objects, the diagram commutes

$$\begin{tikzcd} H^n(\mathcal{E}) \arrow[r] \arrow[d, "\Phi^n"'] & H^n(\mathcal{F}) \arrow[d, "\Phi^n"] \\ \check{H}^n(\mathcal{E}) \arrow[r] & \check{H}^n(\mathcal{F}) \end{tikzcd}$$

(2) Φ commutes with δ. More precisely, for each $n \in \mathbb{N}$ and each exact sequence (5.1.1) in $\mathfrak{A}$, the diagram commutes

$$\begin{tikzcd} H^n(\mathcal{G}) \arrow[r, "\delta^n"] \arrow[d, "\Phi^n"'] & H^{n+1}(\mathcal{E}) \arrow[d, "\Phi^n"] \\ \check{H}^n(\mathcal{G}) \arrow[r, "\delta^n"] & \check{H}^{n+1}(\mathcal{E}) \end{tikzcd}$$

We leave it to the readers to define morphisms of cohomological contravariant and homological covariant δ-functors.

Hohomology and cohohomology in complex geometry are characterized uniquely by the following property.

Definition 5.1.5. A δ-functor $(H^\bullet,\delta^\bullet)$ from $\mathfrak{A}$ to $\mathfrak{B}$ is called **universal** if for any other δ-functor $(\breve{H}^\bullet,\delta^\bullet)$, any natural morphism of functors $\Phi^0 : H^0 \to \breve{H}^0$ can be extended uniquely to a morphism of δ-functors $\Phi^\bullet : (H^\bullet,\delta^\bullet) \to (\breve{H}^\bullet,\delta^\bullet)$. Universal cohomological contravariant functors are defined in a similar way.

A homological covariant δ-functor $(H_\bullet,\delta_\bullet)$ from $\mathfrak{A}$ to $\mathfrak{B}$ is called **universal** if for any other homological covariant δ-functor $(\breve{H}_\bullet,\delta_\bullet)$, any natural morphism of functors $\Phi_0 : \breve{H}_0 \to H_0$ can be extended uniquely to a morphism of homological δ-functors $\Phi_\bullet : (\breve{H}_\bullet,\delta_\bullet) \to (H_\bullet,\delta_\bullet)$. □

It is clear that any two universal (co)homological covariant/contravariant δ-functors with the same degree-zero part H^0 resp. H_0 are isomorphic.

Theorem 5.1.6. *Suppose that $(H^\bullet,\delta^\bullet)$ is a cohomological covariant δ-functor from $\mathfrak{A}$ to $\mathfrak{B}$, and each $\mathcal{E}\in\mathfrak{A}$ has a monomorphism $\mathcal{E}\hookrightarrow\mathcal{E}^0$ such that $H^{>0}(\mathcal{E}^0)=0$. Then $(H^\bullet,\delta^\bullet)$ is a universal δ-functor.*

The same statement holds for cohomological contravariant and homological covariant δ-functors, except that one assumes instead that each $\mathcal{E}\in\mathfrak{A}$ has an epimorphism $\mathcal{E}_0 \twoheadrightarrow \mathcal{E}$ such that $H^{>0}(\mathcal{E}_0)=0$ resp. $H_{>0}(\mathcal{E}_0)=0$.

Thus, the uniqueness of (co)hohomology in complex geometry is addressed. We will discuss the existence problem in the next section.

Proof. We prove the theorem only for cohomological covariant δ-functors, since the other cases can be treated in a similar way. Choose a δ-functor $(\breve{H}^\bullet,\delta^\bullet)$. We construct Φ^n and verifies the desired properties by induction on n. The case $n=0$ is obvious. Assume the unique natural morphisms $\Phi^0,\dots,\Phi^n$ intertwinined by $\delta^0,\dots,\delta^{n-1}$ are constructed. Let us construct a unique natural Φ^{n+1} such that δ^n intertwines Φ^n and Φ^{n+1}.

Step 1. For each $\mathcal{E}$, find a monomorphism $\mathcal{E}\hookrightarrow\mathcal{E}^0$ such that $H^{>0}(\mathcal{E}^0)=0$. Then we have an exact sequence $0\to\mathcal{E}\to\mathcal{E}^0\to\mathcal{E}^0/\mathcal{E}\to 0$. Since $H^{n+1}(\mathcal{E}^0)=0$, by Rem. 1.2.9, there is a unique morphism $\Phi^{n+1} : H^{n+1}(\mathcal{E}) \dashrightarrow \breve{H}^{n+1}(\mathcal{E})$ which yields a morphism of long exact sequences

$$\begin{tikzcd}
H^n(\mathcal{E}^0) \arrow[r] \arrow[d, "\Phi^n"'] & H^n(\mathcal{E}^0/\mathcal{E}) \arrow[r, "\delta"] \arrow[d, "\Phi^n"'] & H^{n+1}(\mathcal{E}) \arrow[r] \arrow[d, dashed, "\Phi^{n+1}"'] & 0 \\
\breve{H}^n(\mathcal{E}^0) \arrow[r] & \breve{H}^n(\mathcal{E}^0/\mathcal{E}) \arrow[r, "\delta"] & \breve{H}^{n+1}(\mathcal{E}) &
\end{tikzcd} \tag{5.1.3}$$

Step 2. Choose any $\mathcal{E}, \mathcal{F} \in \mathfrak{A}$. Suppose that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] \arrow[d] & \mathcal{E}^0 \arrow[d] \\
\mathcal{F} \arrow[r, hook] & \mathcal{F}^0
\end{tikzcd} \tag{5.1.4}$$

where the horizontal arrows are monomorphisms and $H^{>0}(\mathcal{E}^0) = H^{>0}(\mathcal{F}^0) = 0$. By Rem. 1.2.9 again, this diagram can be extended uniquely to a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r, dash] \arrow[d] & \mathcal{E}^0 \arrow[r] \arrow[d] & \mathcal{E}^0/\mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r, hook] & \mathcal{F}^0 \arrow[r] & \mathcal{F}^0/\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

This gives rise to a diagram where all the vertical arrows are Φ:

$$\begin{tikzcd}
& H^n(\mathcal{E}^0) \arrow[rr] \arrow[dl] \arrow[dd] & & H^n(\mathcal{E}^0/\mathcal{E}) \arrow[rr] \arrow[dl] \arrow[dd] & & H^{n+1}(\mathcal{E}) \arrow[dl] \arrow[dd] \\
H^n(\mathcal{F}^0) \arrow[rr] \arrow[dd] & & H^n(\mathcal{F}^0/\mathcal{F}) \arrow[rr] \arrow[dd] & & H^{n+1}(\mathcal{F}) \arrow[dd] & \\
& \check{H}^n(\mathcal{E}^0) \arrow[rr] \arrow[dl] & & \check{H}^n(\mathcal{E}^0/\mathcal{E}) \arrow[rr] \arrow[dl] & & \check{H}^{n+1}(\mathcal{E}) \arrow[dl] \\
\check{H}^n(\mathcal{F}^0) \arrow[rr] & & \check{H}^n(\mathcal{F}^0/\mathcal{F}) \arrow[rr] & & \check{H}^{n+1}(\mathcal{F}) &
\end{tikzcd} \tag{5.1.5}$$

By the assumption on case n, the middle vertical cell commutes. The right front and the right back rectangles commute due to the construction of Φ^{n+1} in Step 1. The right top and the right bottom horizontal cells commute by the definition of δ-functors. Since $H^{n+1}(\mathcal{E}^0) = 0$, the morphism $H^n(\mathcal{E}^0/\mathcal{E}) \to H^{n+1}(\mathcal{E})$ on the top is surjective. Therefore the rightmost vertical (green) parallelogram commutes. To summarize, we have a commutative diagram

$$\begin{tikzcd}
H^{n+1}(\mathcal{F}) \arrow[d, "\Phi^{n+1}"'] & H^{n+1}(\mathcal{E}) \arrow[l] \arrow[d, "\Phi^{n+1}"] \\
\check{H}^{n+1}(\mathcal{F}) & \check{H}^{n+1}(\mathcal{E}) \arrow[l]
\end{tikzcd} \tag{5.1.6}$$

This proves that Φ^{n+1} is natural, once we have shown that Φ^{n+1} is independent of the choice of inclusions $\mathcal{E} \hookrightarrow \mathcal{E}^0$.

To prove that Φ^{n+1} is well-defined, choose monomorphisms $\alpha : \mathcal{E} \hookrightarrow \mathcal{E}^0$ and $\beta : \mathcal{E} \hookrightarrow \mathcal{E}^1$ such that $H^{>0}(\mathcal{E}^0) = H^{>0}(\mathcal{E}^1) = 0$. Then $H^{>0}(\mathcal{E}^0 \oplus \mathcal{E}^1) = 0$. Let (5.1.4) be

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook, "\alpha \vee \beta"] \arrow[d, "="'] & \mathcal{E}^0 \oplus \mathcal{E}^1 \arrow[d] \\
\mathcal{E} \arrow[r, hook, "\alpha"] & \mathcal{E}^0
\end{tikzcd}$$

where the right vertical arrow is the projection onto the first component. Then the commutativity of the corresponding diagram (5.1.6) shows that the $\Phi^{n+1} : H^{n+1}(\mathcal{E}) \to \check{H}^{n+1}(\mathcal{E})$ defined by $\alpha \vee \beta$ agrees with the one defined by α, and hence similarly agrees with the one defined by β.

Step 3. We now check that δ intertwines Φ^n and Φ^{n+1}. Choose any exact sequence (5.1.1). Choose a monomorphism $\mathcal{F} \hookrightarrow \mathcal{E}^0$ where $H^{>0}(\mathcal{E}^0) = 0$, and let its composition with $\mathcal{E} \to \mathcal{F}$ be the monomorphism $\mathcal{E} \hookrightarrow \mathcal{E}^0$ in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, "="'] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r, hook] & \mathcal{E}^0 \arrow[r] & \mathcal{E}^0/\mathcal{E} \arrow[r] & 0
\end{tikzcd}$$

The first cell commutes. Thus there is a morphism $\mathcal{G} \to \mathcal{E}^0/\mathcal{E}$ making the second cell commute. This morphism of exact sequences gives

$$\begin{tikzcd}
 & H^n(\mathcal{F}) \arrow[rr] \arrow[dl] \arrow[dd] & & H^n(\mathcal{G}) \arrow[rr] \arrow[dl] \arrow[dd] & & H^{n+1}(\mathcal{E}) \arrow[dl] \arrow[dd] \\
H^n(\mathcal{E}^0) \arrow[rr] \arrow[dd] & & H^n(\mathcal{E}^0/\mathcal{E}) \arrow[rr] \arrow[dd] & & H^{n+1}(\mathcal{E}) \arrow[dd] & \\
 & \check{H}^n(\mathcal{F}) \arrow[rr] \arrow[dl] & & \check{H}^n(\mathcal{G}) \arrow[rr] \arrow[dl] & & \check{H}^{n+1}(\mathcal{E}) \arrow[dl] \\
\check{H}^n(\mathcal{E}^0) \arrow[rr] & & \check{H}^n(\mathcal{E}^0/\mathcal{E}) \arrow[rr] & & \check{H}^{n+1}(\mathcal{E}) &
\end{tikzcd} \tag{5.1.7}$$

Due to the naturality of Φ^n and Φ^{n+1}, the vertical cells commute. By the definition of δ-functors, the top right and the bottom right horizontal cells commute. By the construction of Φ^n in Step 1, the right front rectangle commutes. Therefore, since $\check{H}^{n+1}(\mathcal{E}) \to \check{H}^{n+1}(\mathcal{E})$ is the identity, the right back (green) rectangle commutes. $\square$

5.2 Derived functors

Let $\mathfrak{A}$ be an abelian category. Recall that an object $\mathcal{Q} \in \mathfrak{A}$ is called **injective** if the contravariant functor $\mathrm{Hom}(-, \mathcal{Q})$ is right exact (and hence exact), namely, for each monomorphism $\mathcal{E} \hookrightarrow \mathcal{F}$ of objects of $\mathfrak{A}$ and for each morphism $\mathcal{E} \to \mathcal{Q}$ there is a morphism $\mathcal{F} \dashrightarrow \mathcal{Q}$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[dl, dashed] \\ & \mathcal{Q} & \end{tikzcd}$$

$\mathcal{Q}$ is called **projective** if the functor $\mathrm{Hom}(\mathcal{Q}, -)$ is right exact (and hence exact), namely, for each epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{G}$ of objects of $\mathfrak{A}$ and for each morphism $\mathcal{Q} \to \mathcal{G}$, there is a morphism $\mathcal{Q} \dashrightarrow \mathcal{F}$ such that the following diagram commutes

$$\begin{tikzcd} & \mathcal{Q} \arrow[d] \arrow[dl, dashed] & \\ \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & 0 \end{tikzcd}$$

Definition 5.2.1. We say that $\mathfrak{A}$ has **enough injectives** if each object $\mathcal{Q} \in \mathfrak{A}$ has a monomorphism into an injective object. We say that $\mathfrak{A}$ has **enough projectives** if each object $\mathcal{Q}$ has an epimorphism from a projective object.

5.2.1 Main result

Choose another abelian category $\mathfrak{B}$.

Theorem 5.2.2. *If $\mathfrak{A}$ has enough injectives, then any left exact covariant functor $T : \mathfrak{A} \to \mathfrak{B}$ can be extended uniquely (up to isomorphism) to a universal covariant δ-functor. If $\mathfrak{A}$ has enough projectives, then any left exact contravariant functor $T : \mathfrak{A} \to \mathfrak{B}$ can be extended uniquely (up to isomorphism) to a universal contravariant δ-functor. In both cases, this functor is denoted by $R^\bullet T$ and called the **right derived functor** of T.*

*If $\mathfrak{A}$ has enough projectives, then any right exact covariant functor $T : \mathfrak{A} \to \mathfrak{B}$ can be extended uniquely (up to isomorphism) to a universal homological δ-functor $L_\bullet T$, called the **left derived functor** of T.*

In the above three cases, $R^{>0}T$ resp. $R^{>0}T$ resp. $L_{>0}T$ vanish on the injectives resp. projectives resp. projectives of $\mathfrak{A}$.

Remark 5.2.3. By saying $R^\bullet T = (R^n T)_{n \in \mathbb{N}}$ resp. $L_\bullet T = (L_n T)_{n \in \mathbb{N}}$ extends T, we mean $R^0 T = T$ resp. $L_0 T = T$.

We need some preparations for the proof of this theorem.

Definition 5.2.4. Let $\mathcal{E} \in \mathfrak{A}$. A **right resolution** $0 \to \mathcal{E} \to \mathcal{E}^\bullet$ of $\mathcal{E}$ is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}^0 \xrightarrow{\varphi^0} \mathcal{E}^1 \xrightarrow{\varphi^1} \mathcal{E}^2 \xrightarrow{\varphi^2} \cdots \tag{5.2.1}$$

If each $\mathcal{E}^n$ (but not necessarily $\mathcal{E}$) is injective, we call $0 \to \mathcal{E} \to \mathcal{E}^\bullet$ an **injective resolution** of $\mathcal{E}$.

Similarly, a **left resolution** $\mathcal{E}_\bullet \to \mathcal{E} \to 0$ of $\mathcal{E}$ is an exact sequence

$$\cdots \xrightarrow{\varphi_3} \mathcal{E}_2 \xrightarrow{\varphi_2} \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{E}_0 \to \mathcal{E} \to 0 \tag{5.2.2}$$

If each $\mathcal{E}_\bullet$ is projective, we call $\mathcal{E}_\bullet \to \mathcal{E} \to 0$ a **projective resolution** of $\mathcal{E}$. □

Remark 5.2.5. If $\mathfrak{A}$ has enough injectives (resp. projectives), then any $\mathcal{E} \in \mathfrak{A}$ has an injective (resp. projective) resolution.

Indeed, suppose that $\mathfrak{A}$ has enough injectives. Then we have an exact sequence $0 \to \mathcal{E} \to \mathcal{E}^0 \to \mathcal{E}^0/\mathcal{E} \to 0$ where $\mathcal{E}^0$ is injective. Embed $\mathcal{E}^0/\mathcal{E}$ into an inject object $\mathcal{E}^1$. This gives an exact sequnce $0 \to \mathcal{E} \to \mathcal{E}^0 \xrightarrow{\varphi^1} \mathcal{E}^1$. Embed $\mathrm{Coker}(\varphi^1)$ into an injective $\mathcal{E}^2$, and repeat this procedure again and again to obtain the injective resolution. □

5.2.2 Motivations

We now explain the ideas of constructing derived functors. Suppose that a left exact functor $T : \mathfrak{A} \to \mathfrak{B}$ can be extended to a δ-functor $R^\bullet T$. Suppose moreover that $\mathcal{E}$ has a right resolution (5.2.1) such that $R^{>0}T$ vanishes on $\mathcal{E}^\bullet$. Then the short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}^0 \xrightarrow{\varphi^0} \mathrm{Ker}(\varphi^1) \to 0$$

produces a long exact sequence, which yields exact sequences

$$0 \to T(\mathcal{E}) \to T(\mathcal{E}^0) \to T(\mathrm{Ker}\varphi^1) \xrightarrow{\delta} R^1T(\mathcal{E}) \to 0 \tag{5.2.3a}$$

$$R^nT(\mathrm{Ker}\varphi^1) \xrightarrow[\simeq]{\delta} R^{n+1}T(\mathcal{E}) \qquad (n \geqslant 1) \tag{5.2.3b}$$

Thus, by (5.2.3a),

$$R^1T(\mathcal{E}) \simeq \frac{T(\mathrm{Ker}\varphi^1)}{\mathrm{Im}\big(T(\mathcal{E}^0) \xrightarrow{T(\varphi^0)} T(\mathrm{Ker}\varphi^1)\big)} \tag{5.2.4}$$

Since T is left exact, the exactness of $0 \to \mathrm{Ker}\varphi^1 \xrightarrow{\iota^1} \mathcal{E}^1 \xrightarrow{\varphi^1} \mathcal{E}^2$ gives an exact sequence

$$0 \to T(\mathrm{Ker}\varphi^1) \xrightarrow{T(\iota^1)} T(\mathcal{E}^1) \xrightarrow{T(\varphi^1)} T(\mathcal{E}^2)$$

Therefore, $T(\iota^1)$ sends $T(\mathrm{Ker}\varphi^1)$ isomorphically to $\mathrm{Ker}\big(T(\mathcal{E}^1) \to T(\mathcal{E}^2)\big)$, and sends the bottom of the RHS of (5.2.4) isomorphically to the image of $T(\iota^1 \circ \varphi^0) : T(\mathcal{E}^0) \to T(\mathcal{E}^1)$. Thus, by (5.2.4) we obtain an isomorphism

$$R^1T(\mathcal{E}) \simeq \frac{\mathrm{Ker}\big(T(\mathcal{E}^1) \to T(\mathcal{E}^2)\big)}{\mathrm{Im}\big(T(\mathcal{E}^0) \to T(\mathcal{E}^1)\big)} \tag{5.2.5}$$

To compute $R^nT(\mathcal{E})$ when $n > 1$, we use (5.2.3b), which says that it is isomorphic to $R^{n-1}T(\mathrm{Ker}\varphi^1)$. Note that $0 \to \mathrm{Ker}\varphi^1 \to \mathcal{E}^1 \to \mathcal{E}^2 \to \cdots$ is a resolution of $\mathrm{Ker}\varphi^1$ where all the terms after $\mathrm{Ker}\varphi^1$ are killed by $R^{>0}T$. Apply (5.2.3b) again and repeat the same procedure, we obtain

$$R^nT(\mathcal{E}) \simeq R^{n-1}T(\mathrm{Ker}\varphi^1) \simeq R^{n-2}T(\mathrm{Ker}\varphi^2) \simeq \cdots \simeq R^1T(\mathrm{Ker}\varphi^{n-1}). \tag{5.2.6}$$

Apply (5.2.5) to the resolution

$$0 \to \mathrm{Ker}\varphi^{n-1} \to \mathcal{E}^{n-1} \to \mathcal{E}^n \to \mathcal{E}^{n+1} \to \cdots$$

of $\mathrm{Ker}\varphi^{n-1}$, we see that (5.2.6) is isomorphic to

$$R^nT(\mathcal{E}) \simeq \frac{\mathrm{Ker}\big(T(\mathcal{E}^n) \to T(\mathcal{E}^{n+1})\big)}{\mathrm{Im}\big(T(\mathcal{E}^{n-1}) \to T(\mathcal{E}^n)\big)} \tag{5.2.7}$$

Since this is also true when $n = 1$ and when $n = 0$ if we set $\mathcal{E}^{<0} = 0$, we conclude $R^\star T(\mathcal{E}) \simeq \mathcal{H}^\star(T(\mathcal{E}^\bullet))$, namely, that $R^\star T(\mathcal{E})$ is isomorphic to the cohomology of the complex $T(\mathcal{E}^\bullet)$. In the proof of Thm. 5.2.2, we will use injective resolutions and $\mathcal{H}^\star(T(\mathcal{E}^\bullet))$ to construct right derived functors.

Exercise 5.2.6. Under the assumptions at the beginning of Subsec. 5.2.2, assume moreover that $\mathcal{F} \in \mathfrak{A}$ has a right resolution $0 \to \mathcal{F} \to \mathcal{F}^\bullet$ such that $R^{>0}T$ vanishes on $\mathcal{F}^\bullet$. Let $\Phi : \mathcal{E} \to \mathcal{F}$ be a morphism which can be extended to a commutative diagram in $\mathfrak{A}$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}^0 & \xrightarrow{d^0} & \mathcal{E}^1 & \xrightarrow{d^1} & \mathcal{E}^2 & \xrightarrow{d^2} & \cdots \\ & & {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi^0}\downarrow & & {\scriptstyle\Phi^1}\downarrow & & {\scriptstyle\Phi^2}\downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}^0 & \xrightarrow{d^0} & \mathcal{F}^1 & \xrightarrow{d^1} & \mathcal{F}^2 & \xrightarrow{d^2} & \cdots \end{array}$$

For each $q \in \mathbb{N}$, show that under the identification $R^qT(\mathcal{E}) \simeq \mathcal{H}^q(T(\mathcal{E}^\bullet))$ and $R^qT(\mathcal{F}) \simeq \mathcal{H}^q(T(\mathcal{F}^\bullet))$, the morphism $R^qT(\Phi) : R^qT(\mathcal{E}) \to R^qT(\mathcal{F})$ is equal to $\mathcal{H}^q(T(\Phi^\bullet))$.

(Hint: Case $q = 0$: obvious. Case $q = 1$: construct a morphism of exact sequences from (5.2.3a) to a similar one about $\mathcal{F}$ and $\mathcal{F}^\bullet$. Case $q > 1$: by induction and a suitable morphism of exact sequences from (5.2.3b).) □

In the case that $T : \mathfrak{A} \to \mathfrak{B}$ is a right exact functor and can be extended to a homological δ-functor $L_\bullet T$, the argument is slightly different: we use the fact that for any morphisms of $\mathfrak{B}$-objects $C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$ where f is an epimorphism and the exactness of the sequence is not assumed, there is an isomorphism

$$f : \frac{\mathrm{Ker}(C_1 \xrightarrow{g\circ f} C_3)}{\mathrm{Ker}(C_1 \overset{f}{\twoheadrightarrow} C_2)} \xrightarrow{\simeq} \mathrm{Ker}(C_2 \xrightarrow{g} C_3) \tag{5.2.8}$$

Suppose that $\mathcal{E}$ has a left resolution (5.2.2) such that $L_{>0}T$ vanishes on $\mathcal{E}_\bullet$. Then we have a short exact sequence

$$0 \to \mathrm{Im}(\varphi_1) \xrightarrow{\iota_1} \mathcal{E}_0 \to \mathcal{E} \to 0$$

whose long exact sequence gives exact sequences

$$0 \to L_1T(\mathcal{E}) \xrightarrow{\delta} T(\mathrm{Im}\varphi_1) \xrightarrow{T(\iota_1)} T(\mathcal{E}_0) \to T(\mathcal{E}) \to 0 \tag{5.2.9}$$

$$L_{n+1}T(\mathcal{E}) \xrightarrow[\simeq]{\delta} L_nT(\mathrm{Im}\varphi_1) \qquad (n \geqslant 1) \tag{5.2.10}$$

Thus $L_1T(\mathcal{E}) \simeq \mathrm{Ker}\big(T(\mathrm{Im}(\varphi_1)) \xrightarrow{T(\iota_1)} T(\mathcal{E}_0)\big)$. Since T is right exact and $\varphi_1 : \mathcal{E}_1 \twoheadrightarrow \mathrm{Im}\varphi_1$ is surjective, the first morphism in the following non-necessarily exact sequence is surjective

$$T(\mathcal{E}_1) \xrightarrow{T(\varphi_1)} T(\mathrm{Im}\varphi_1) \xrightarrow{T(\iota_1)} T(\mathcal{E}_0)$$

Therefore, by (5.2.8),

$$L_1T(\mathcal{E}) \simeq \frac{\mathrm{Ker}(T(\mathcal{E}_1) \to T(\mathcal{E}_0))}{\mathrm{Ker}(T(\mathcal{E}_1) \xrightarrow{T(\varphi_1)} T(\mathrm{Im}\varphi_1))}$$

where the bottom is clearly equal to $\mathrm{Im}(T(\mathcal{E}_2) \to T(\mathcal{E}_1))$ because the right exact functor T preserves the exactness of $\mathcal{E}_2 \to \mathcal{E}_1 \xrightarrow{\varphi_1} \mathrm{Im}\varphi_1 \to 0$. This implies that

$$L_nT(\mathcal{E}) \simeq \frac{\mathrm{Ker}(T(\mathcal{E}_n) \to T(\mathcal{E}_{n-1}))}{\mathrm{Im}(T(\mathcal{E}_{n+1}) \to T(\mathcal{E}_n))} \tag{5.2.11}$$

holds when $n = 1$. It clearly holds when $n = 0$ if we set $\mathcal{E}_{<0} = 0$. Thus, similar to the previous case of left exact functors, we can use (5.2.10) to show that (5.2.11) holds for all $n \in \mathbb{N}$. Thus $L_\star T(\mathcal{E}) \simeq \mathcal{H}_\star(T(\mathcal{E}_\bullet))$, namely, $L_\star T(\mathcal{E})$ is isomorphic to the homology of the complex $T(\mathcal{E}_\bullet)$.

5.2.3 δ-functors for complexes

We recall some basic facts from homological algebra. They can be found in any textbook on algebraic topology (e.g. [Hat, Sec. 2.1]).

There is a canonical δ-functor $(\mathcal{H}^\bullet, \delta^\bullet)$ from the category $\mathrm{Com}(\mathfrak{B})$ of (cochain) complexes of $\mathfrak{B}$ to $\mathfrak{B}$. (Here we assume $\bullet \in \mathbb{Z}$ instead of $\bullet \in \mathbb{N}$.) If $C^\bullet = (C^n \xrightarrow{d^n} C^{n+1})_{n\in\mathbb{Z}}$ is a complex in $\mathfrak{B}$ (in particular $d^{n+1} \circ d^n = 0$ for all n), then $\mathcal{H}^\star(C^\bullet)$ is the cohomology of this complex, namely

$$\mathcal{H}^n(C^\bullet) = \frac{\mathrm{Ker}(C^n \to C^{n+1})}{\mathrm{Im}(C^{n-1} \to C^n)}$$

Given any morphism of complexes $f^\bullet : B^\bullet \to C^\bullet$, namely, whenever we have commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle f^{n-1}}\big\downarrow & & {\scriptstyle f^n}\big\downarrow & & {\scriptstyle f^{n+1}}\big\downarrow & & \\
\cdots & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow & \cdots
\end{array}$$

we have a canonical morphism $\mathcal{H}^n(f^\bullet) : \mathcal{H}^n(B^\bullet) \to \mathcal{H}^n(C^\bullet)$ for each n.

To finish constructing the δ-functor, we note that any short exact sequence of complexes induces naturally a long exact sequence of their (co)homology. More precisely, suppose we have a short exact sequence of complexes in $\mathfrak{B}$: $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$, namely, a commuting diagram of morphisms in $\mathfrak{B}$ where n runs through all integers, see Fig. 5.2.1. Then we have a long exact sequence

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{n+1} & \longrightarrow & B^{n+1} & \longrightarrow & C^{n+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^n & \longrightarrow & B^n & \longrightarrow & C^n & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{n-1} & \longrightarrow & B^{n-1} & \longrightarrow & C^{n-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Figure 5.2.1

$$\cdots \to \mathcal{H}^{n-1}(C^\bullet) \to \mathcal{H}^n(A^\bullet) \to \mathcal{H}^n(B^\bullet) \to \mathcal{H}^n(C^\bullet) \to \mathcal{H}^{n+1}(A^\bullet) \to \cdots$$

The connecting morphisms $\mathcal{H}^n(C^\bullet) \xrightarrow{\delta^n} \mathcal{H}^{n+1}(A^\bullet)$ are defined by "diagram chasing". Moreover, if we have a morphism of short exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{A}^\bullet \arrow[r] & \widetilde{B}^\bullet \arrow[r] & \widetilde{C}^\bullet \arrow[r] & 0
\end{tikzcd}$$

(namely, if we replace $\bullet$ by each n, then this diagram commutes, and the two horizontal sequences are exact), then we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}^{n-1}(C^\bullet) \arrow[r] \arrow[d] & \mathcal{H}^n(A^\bullet) \arrow[r] \arrow[d] & \mathcal{H}^n(B^\bullet) \arrow[r] \arrow[d] & \mathcal{H}^n(C^\bullet) \arrow[r] \arrow[d] & \mathcal{H}^{n+1}(A^\bullet) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{H}^{n-1}(\widetilde{C}^\bullet) \arrow[r] & \mathcal{H}^n(\widetilde{A}^\bullet) \arrow[r] & \mathcal{H}^n(\widetilde{B}^\bullet) \arrow[r] & \mathcal{H}^n(\widetilde{C}^\bullet) \arrow[r] & \mathcal{H}^{n+1}(\widetilde{A}^\bullet) \arrow[r] & \cdots
\end{tikzcd} \tag{5.2.12}$$

So $(\mathcal{H}^\bullet, \delta^\bullet)$ is a δ-functor where $\bullet \in \mathbb{Z}$.

It is important that homotopic maps of complexes give the same map on (co)homology. To be more precise, let $B^\bullet, C^\bullet$ be a complexes of $\mathfrak{B}$, and let $f, g : B^\bullet \to C^\bullet$ be morphisms of complexes. We say that f and g are **homotopic** if there are morphisms $w = w^n : B^n \to C^{n-1}$ for all n such that $f - g = dw + wd$: more precisely, $f - g : B^n \to C^n$ equals $d^{n-1}w^n + w^{n+1}d^n$ where $d^n : B^n \to B^{n+1}$ and $d^{n-1} : C^{n-1} \to C^n$. Then f and g induce the same map $\mathcal{H}^n(f) = \mathcal{H}^n(g) : \mathcal{H}^n(B^\bullet) \to \mathcal{H}^n(C^\bullet)$ for all n.

If there are morphisms of complexes $\varphi : B^\bullet \to C^\bullet$ and $\psi : C^\bullet \to B^\bullet$ such that $\psi \circ \varphi$ is homotopic to $\mathbf{1}_{\mathcal{B}^\bullet}$ and $\varphi \circ \psi$ is homotopic to $\mathbf{1}_{\mathcal{C}^\bullet}$, we say that $B^\bullet$ and $C^\bullet$ are **homotopic**. In that case, we clearly have an isomorphism

$$\mathcal{H}^\star(\varphi) : \mathcal{H}^\star(\mathcal{B}^\bullet) \xrightarrow{\simeq} \mathcal{H}^\star(\mathcal{C}^\bullet)$$

with inverse $\mathcal{H}^\star(\psi)$.

5.2.4 Proof of Thm. 5.2.2

Step 1 of the following proof is especially important: it gives an explicit way of constructing derived functors using resolutions.

***Proof of Thm.* 5.2.2.** We only prove the first case; the other two cases can be treated in a similar way. Also, the uniqueness of derived functors is clear from the definition of universal δ-functors. So it suffices to prove the existence.

Step 1. Assume that $\mathfrak{A}$ has enough injectives and choose a left exact functor $T : \mathfrak{A} \to \mathfrak{B}$. We construct the functor R^nT for each n. For each $\mathcal{E} \in \mathfrak{A}$, we fix an injective resolution $0 \to \mathcal{E} \to \mathcal{E}^\bullet$, and set $\mathcal{E}^{<0} = 0$ so that $(\mathcal{E}^n)_{n\in\mathbb{Z}}$ is a complex in $\mathfrak{A}$. We define

$$R^nT(\mathcal{E}) = \mathcal{H}^n(T(\mathcal{E}^\bullet)) = \frac{\mathrm{Ker}\big(T(\mathcal{E}^n) \to T(\mathcal{E}^{n+1})\big)}{\mathrm{Im}\big(T(\mathcal{E}^{n-1}) \to T(\mathcal{E}^n)\big)} \tag{5.2.13}$$

Choose any $\mathcal{F} \in \mathfrak{A}$ together with an injective resolution $0 \to \mathcal{F} \to \mathcal{F}^\bullet$. If $\varphi : \mathcal{E} \to \mathcal{F}$ is a morphism, we need to construct $R^nT(\varphi) : R^nT(\mathcal{E}) \to R^nT(\mathcal{F})$ for all $n \in \mathbb{N}$. We construct morphisms $\varphi^n : \mathcal{E}^n \to \mathcal{F}^n$ by induction on $n \in \mathbb{N}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{E}^0 \arrow[r, "d^0"] \arrow[d, "\varphi^0"'] & \mathcal{E}^1 \arrow[r, "d^1"] \arrow[d, "\varphi^1"'] & \mathcal{E}^2 \arrow[r, "d^2"] \arrow[d, "\varphi^2"'] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}^0 \arrow[r, "d^0"] & \mathcal{F}^1 \arrow[r, "d^1"] & \mathcal{F}^2 \arrow[r, "d^2"] & \cdots
\end{tikzcd} \tag{5.2.14}$$

The existence of φ^0 follows easily from that $\mathcal{F}^0$ is injective and that $\mathcal{E} \hookrightarrow \mathcal{E}^0$ is a monomorphism. Suppose $\varphi^0, \dots, \varphi^n$ are constructed. Then the commutativity of

$$\begin{tikzcd}
\mathcal{E}^{n-1} \arrow[r] \arrow[d, "\varphi^{n-1}"'] & \mathcal{E}^n \arrow[d, "\varphi^n"'] \\
\mathcal{F}^{n-1} \arrow[r] & \mathcal{F}^n
\end{tikzcd}$$

implies that φ^n descends to a morphism $\mathrm{Coker}(\mathcal{E}^{n-1} \to \mathcal{E}^n) \to \mathrm{Coker}(\mathcal{F}^{n-1} \to \mathcal{F}^n)$. Thus, by the injectivity of $\mathcal{F}^{n+1}$, there is a morphism φ^{n+1} such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Coker}(\mathcal{E}^{n-1} \to \mathcal{E}^n) \arrow[r] \arrow[d, "\varphi^n"'] & \mathcal{E}^{n+1} \arrow[d, "\varphi^{n+1}"'] \\
0 \arrow[r] & \mathrm{Coker}(\mathcal{F}^{n-1} \to \mathcal{F}^n) \arrow[r] & \mathcal{F}^{n+1}
\end{tikzcd}$$

This finishes the construction of φ^n when $n \geqslant 0$.

Let $\varphi^n = 0$ if $n < 0$. Then we have a morphism of $\mathfrak{A}$-complexes $\varphi^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$, and hence a morphism of $\mathfrak{B}$-complexes $\mathcal{H}^\star(T(\varphi^\bullet)) : \mathcal{H}^\star(T(\mathcal{E}^\bullet)) \to \mathcal{H}^\star(T(\mathcal{F}^\bullet))$. The degree n morphism is simply defined to be

$$R^nT(\varphi) = \mathcal{H}^n(T(\varphi^\bullet)) \tag{5.2.15}$$

Step 2. To verify that R^nT is a functor, we still need to show that R^nT preserves the composition of morphisms. This fact is clearly true if we can show that $R^nT(\varphi)$ is independent of the choice of $\varphi^\bullet$. Thus, it suffices to show that if $\phi^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$

also makes (5.2.14) commutes, then $\varphi^\bullet$ and $\phi^\bullet$ are homotopic (Subsec. 5.2.3). Then $T(\varphi^\bullet)$ and $T(\phi^\bullet)$ will be homotopic and hence $\mathcal{H}^n(T(\varphi^\bullet)) = \mathcal{H}^n(T(\phi^\bullet))$.

Recall that we set $\mathcal{E}^{<0} = \mathcal{F}^{<0} = 0$. So certainly we set $w^n : \mathcal{E}^n \to \mathcal{F}^{n-1}$ to be 0 if $n \leqslant 0$. Since $\varphi^0, \phi^0 : \mathcal{E}^0 \to \mathcal{F}^0$ restrict to the same morphism $\varphi : \mathcal{E} \to \mathcal{F}$, $\varphi^0 - \phi^0$ vanishes on $\mathcal{E}$, and hence restricts to a morphism $\mathcal{E}^0/\mathcal{E} \to \mathcal{F}^0$. The injectivity of $\mathcal{F}^0$ implies that there is a morphism $w^1 : \mathcal{E}^1 \to \mathcal{F}^0$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^0/\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}^1 \arrow[dl, "w^1"] \\
& \mathcal{F}^0 &
\end{tikzcd}$$

commutes. Then clearly $\varphi^0 - \phi^0 = d^{-1}w^0 + w^1 d^0$.

To avoid confusions, we write the coboundary maps of complexes as $d^\bullet_{\mathcal{E}} : \mathcal{E}^\bullet \to \mathcal{E}^{\bullet+1}$ and $d^\bullet_{\mathcal{F}} : \mathcal{F}^\bullet \to \mathcal{F}^{\bullet+1}$. Suppose $w^{\leqslant n+1}$ are constructed and $\varphi^n - \phi^n = d^{n-1}_{\mathcal{F}} w^n + w^{n+1} d^n_{\mathcal{E}}$ where $n \in \mathbb{N}$. Let us construct w^{n+2}. We compute that

$$(\varphi^{n+1} - \phi^{n+1})d^n_{\mathcal{E}} = d^n_{\mathcal{F}}(\varphi^n - \phi^n) = d^n_{\mathcal{F}}(d^{n-1}_{\mathcal{F}} w^n + w^{n+1} d^n_{\mathcal{E}}) = (d^n_{\mathcal{F}} w^{n+1}) d^n_{\mathcal{E}}$$

where we have used the commutativity of (5.2.14) and its analog for $\phi^\bullet$ to derive the first equality. This shows that $\varphi^{n+1} - \phi^{n+1} - d^n_{\mathcal{F}} w^{n+1}$ vanishes on $\mathrm{Im} d^n_{\mathcal{E}}$, and hence descends to a morphism $\mathcal{E}^{n+1}/\mathrm{Im}(d^n_{\mathcal{E}}) \to \mathcal{F}^{n+1}$. Thus, by the injectivity of $\mathcal{F}^{n+1}$, there is a morphism $w^{n+2} : \mathcal{E}^{n+2} \to \mathcal{F}^{n+1}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^{n+1}/\mathrm{Im}(d^n_{\mathcal{E}}) \arrow[r] \arrow[d] & \mathcal{E}^{n+2} \arrow[dl, "w^{n+2}"] \\
& \mathcal{F}^{n+1} &
\end{tikzcd}$$

commutes. This finishes the construction of the homotopy map $w^\bullet$.

Consider the special case that $\mathcal{F} = \mathcal{E}$ and $\varphi = \mathbf{1}_{\mathcal{E}}$ in (5.2.14). (Namely, we choose injective resolutions $0 \to \mathcal{E} \to \mathcal{E}^\bullet$ and $0 \to \mathcal{E} \to \mathcal{F}^\bullet$) Then the existence of homotopy maps shows that $\mathcal{E}^\bullet$ is homotopic to $\mathcal{F}^\bullet$, and hence
$mcE^\bullet$ is homotopic to $\mathcal{F}^\bullet$. Therefore, the equivalence class of $R^nT(\mathcal{E})$ is independent of the choice of injective resolutions of $\mathcal{E}$. In particular, if $\mathcal{E}$ is injective, since $0 \to \mathcal{E} \to \mathcal{E} \to 0 \to \cdots$ is an injective resolution of $\mathcal{E}$, we have $R^{>0}T(\mathcal{E}) = 0$.

Step 3. Given a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, if we find morphisms for complexes such that the sequence $0 \to \mathcal{E}^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to 0$ is exact, then we can define the connecting morphism $\delta : R^\star T(\mathcal{G}) \to R^{\star+1}T(\mathcal{E})$ to be the one $\delta : \mathcal{H}^\star(T(\mathcal{G}^\bullet)) \to \mathcal{H}^{\star+1}(T(\mathcal{E}^\bullet))$. To do this, we need to choose a different injective resolution for $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{E}^0 \oplus \mathcal{G}^0 \to \mathcal{E}^1 \oplus \mathcal{G}^1 \to \mathcal{E}^2 \oplus \mathcal{G}^2 \to \cdots$$

We explain the morphism $\mathcal{F} \to \mathcal{E}^0 \oplus \mathcal{G}^0$, since the others are clear. It is the diagonal map of $\mathcal{F} \to \mathcal{E}^0$ and $\mathcal{F} \to \mathcal{G}^0$. The latter is the composition of $\mathcal{F} \to \mathcal{G}$ and the monomorphism $\mathcal{G} \hookrightarrow \mathcal{G}^0$. The first one is one that makes the following diagram commutes, which exists because $\mathcal{E}^0$ is injective:

$$\begin{array}{ccccc} & & \mathcal{E}^0 & & \\ & & \uparrow & \nwarrow & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} \end{array}$$

Set $\widetilde{\mathcal{F}}^n = \mathcal{E}^n \oplus \mathcal{G}^n$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{E}^1 & \longrightarrow & \widetilde{\mathcal{F}}^1 & \longrightarrow & \mathcal{G}^1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{E}^0 & \longrightarrow & \widetilde{\mathcal{F}}^0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where all the rows are exact, the $\mathcal{E}$-column and the $\mathcal{G}$-column are (clearly) exact. Then it is not hard to check that the middle column is exact. (A quick way to see this is to view the above diagram as a short exact sequence of complexes $0 \to \alpha \to \beta \to \gamma \to 0$, which induces a long exact sequence in which the cohomologies of α and γ are zero. Therefore the cohomology of β vanishes, which means precisely that the $\mathcal{F}$-column in the above diagram is exact.)

Thus, we have $\delta : R^\star T(\mathcal{G}) \to R^{\star+1}T(\mathcal{E})$ which clearly satisfies requirement (2) of Def. 5.1.1. Since the functor $\mathcal{H}^\star \circ T$ preserves the composition of morphisms of complexes, we have a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{H}^{n-1}(T(\mathcal{G}^\bullet)) & \longrightarrow & \mathcal{H}^n(T(\mathcal{E}^\bullet)) & \longrightarrow & \mathcal{H}^n(T(\widetilde{\mathcal{F}}^\bullet)) & \longrightarrow & \mathcal{H}^n(T(\mathcal{G}^\bullet)) & \longrightarrow & \mathcal{H}^{n+1}(T(\mathcal{E}^\bullet)) \\ {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{H}^{n-1}(T(\mathcal{G}^\bullet)) & \longrightarrow & \mathcal{H}^n(T(\mathcal{E}^\bullet)) & \longrightarrow & \mathcal{H}^n(T(\mathcal{F}^\bullet)) & \longrightarrow & \mathcal{H}^n(T(\mathcal{G}^\bullet)) & \longrightarrow & \mathcal{H}^{n+1}(T(\mathcal{E}^\bullet)) \end{array}$$

where the morphisms are either induced by the morphisms of complexes through $H^\star \circ T$, or are the previously defined δ. The first line is exact because $\mathcal{H}^\star$ is a δ-functor (Subsec. 5.2.3). Therefore the second line is also exact. Thus $(R^\star T, \delta)$ also satisfies condition (1) of Def. 5.1.1. So it is a δ-functor. We have shown at the end of Step 2 that $R^{>0}T$ vanishes on injective objects. So $(R^\star T, \delta)$ is universal by Thm. 5.1.6. $\square$

The following observation will be used in the proof of Prop. 5.8.3 and Thm. 5.8.7

Remark 5.2.7. Let $\mathcal{A}$ be a ring and $\mathfrak{A}$ the category of $\mathcal{A}$-modules $\mathrm{Mod}(\mathcal{A})$. Suppose that $T : \mathfrak{A} \to \mathfrak{A}$ is a left/right exact covariant/contravariant functor. We say that T **preserves multiplications** if, for all $a \in \mathcal{A}$ and $\mathcal{E} \in \mathfrak{A}$, if we let $\mu_a : \mathcal{E} \xrightarrow{\times a} \mathcal{E}$ denote the multiplication by a (which is clearly a morphism), then $T(\mu_a)$ is the multiplication of a on $T(\mathcal{E})$. For instance, tensor product and Hom preserve multiplications.

We will see that $\mathfrak{A}$ has enough injectives and projectives. In (5.2.14), if we let $\mathcal{F} = \mathcal{E}$, $\mathcal{F}^\bullet = \mathcal{E}^\bullet$, and let φ be μ_a, then one can clearly choose all $\varphi^\bullet$ to be μ_a. It follows from (5.2.15) that if T preserves multiplications, then R^nT resp. L_nT preserves multiplications for all $n \in \mathbb{N}$. □

5.3 Ext and Tor

We fix a commutative ring $\mathcal{A}$ and let $\mathrm{Mod}(\mathcal{A})$ be the category of $\mathcal{A}$-modules. It is clear that any $\mathcal{A}$-object has an epimorphism from a free $\mathcal{A}$-module. Since *free $\mathcal{A}$-modules are clearly projective objects in $\mathrm{Mod}(\mathcal{A})$*, we see that $\mathrm{Mod}(\mathcal{A})$ has enough projectives.

We shall prove that $\mathrm{Mod}(\mathcal{A})$ has enough injectives, and we shall mainly focus on the case that $\mathcal{A}$ is a $\mathbb{C}$-algebra, since this is enough for the purpose of our notes. First we need a lemma.

Lemma 5.3.1. *Assume that $\mathcal{A}$ is a $\mathbb{C}$-algebra. Then for any $\mathcal{A}$-modules $\mathcal{M}, \mathcal{N}$ and any $\mathbb{C}$-vector space $\mathcal{V}$ we have a canonical equivalence of $\mathcal{A}$-modules*

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{V}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathrm{Hom}_{\mathbb{C}}(\mathcal{N}, \mathcal{V})) \tag{5.3.1}$$

where the $\mathcal{A}$-module structure on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{V})$ resp. $\mathrm{Hom}_{\mathbb{C}}(\mathcal{N}, \mathcal{V})$ is defined by that of $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ resp. $\mathcal{N}$. In particular, taking $\mathcal{N} = \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{M}, \mathcal{V}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{V})). \tag{5.3.2}$$

Proof. The RHS of (5.3.1) is equivalently the $\mathcal{A}$-module $\mathcal{W}$ of $\mathbb{C}$-bilinear maps $T : \mathcal{M} \times \mathcal{N} \to \mathcal{V}$ satisfying $\Phi(a\xi, \eta) = \Phi(\xi, a\eta)$ for all $\xi \in \mathcal{M}, \eta \in \mathcal{N}, a \in \mathcal{A}$. The action of $a \in \mathcal{A}$ on Φ is $\Phi(a\cdot, \cdot)$.

Given a $\mathbb{C}$-linear map $S : \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{V}$, one can compose it with the obvious map $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to get T. The correspondence $S \mapsto T$ is clearly injective. To show that it is surjective, note that if the $\mathcal{A}$-module structure on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{W}, \mathcal{V})$ is defined by that of $\mathcal{W}$, then the map

$$\mathcal{M} \times \mathcal{N} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{W}, \mathcal{V})$$

$$(\xi, \eta) \mapsto \big(T \in \mathcal{W} \mapsto T(\xi, \eta)\big)$$

is clearly $\mathcal{A}$-bilinear, and hence gives rise to an $\mathcal{A}$-module morphism

$$\Phi : \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{W}, \mathcal{V}).$$

Each $T \in \mathcal{W}$ gives rise to a canonical linear map $\mathrm{Hom}_{\mathbb{C}}(\mathcal{W}, \mathcal{V}) \to \mathcal{V}$, whose composition with Φ is the desired S. $\square$

Remark 5.3.2. The above lemma can be easily generalized: assume $\mathcal{A}$ is a $\mathcal{B}$-algebra, namely, $\mathcal{A}, \mathcal{B}$ are rings and a ring homomorphism $\mathcal{B} \to \mathcal{A}$ is fixed. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{A}$-modules and $\mathcal{V}$ be $\mathcal{B}$-modules. Then we have a canonical $\mathcal{A}$-module isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{V}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}))$$

Proposition 5.3.3. $\mathrm{Mod}(\mathcal{A})$ *has enough injectives.*

Proof. We prove the proposition only in the special case that $\mathcal{A}$ is an algebra over a field (say $\mathbb{C}$), and refer the readers to [Lang, Sec. XX.4] for the proof in the general case. For each $\mathcal{E} \in \mathcal{A}$,

$$\begin{gathered} \mathcal{E} \to \mathcal{E}^0 = \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{E}) \\ \xi \mapsto \big(a \in \mathcal{A} \mapsto a\xi\big) \end{gathered}$$

is an $\mathcal{A}$-module monomorphism. Since $\mathrm{Hom}_{\mathbb{C}}(-, \mathcal{E})$ is exact on $\mathrm{Mod}(\mathcal{A})$ (and indeed on the category of $\mathbb{C}$-vector spaces), by Lemma 5.3.1, $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{E}^0)$ is exact. Therefore $\mathcal{E}^0$ is injective. $\square$

Recall that for each $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ is a left exact functor, and $\mathcal{E} \otimes_{\mathcal{A}} -$ is a right exact functor.

Definition 5.3.4. For each $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$, we define the functor $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{E}, -)$ ($n \in \mathbb{N}$) to be the right derived functor of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, -)$, and define the functor $\mathrm{Tor}^{\mathcal{A}}_n(\mathcal{E}, -)$ to be the left derived functor of $\mathcal{E} \otimes_{\mathcal{A}} -$. In particular, we have

$$\mathrm{Ext}^0_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \qquad \mathrm{Tor}^{\mathcal{A}}_0(\mathcal{E}, \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$$

Theorem 5.3.5. *Choose any* $\mathcal{F} \in \mathrm{Mod}(\mathcal{A})$. *Then by defining its action on morphisms and defining* δ, $\mathrm{Ext}^\bullet_{\mathcal{A}}(-, \mathcal{F})$ *can be extended to a right derived contravariant functor of* $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{F})$, *and* $\mathrm{Tor}^{\mathcal{A}}_\bullet(-, \mathcal{F})$ *can be extended to a left derived homological (covariant) functor of* $- \otimes_{\mathcal{A}} \mathcal{F}$.

Proof. We prove the theorem for Ext. Tor can be treated in a similar way. Also, we suppress the subscript $\mathcal{A}$ for simplicity.

Fix an injective resolution $0 \to \mathcal{F} \to \mathcal{F}^\bullet$. For each morphism $\varphi : \mathcal{M} \to \mathcal{N}$ in $\mathrm{Mod}(\mathcal{A})$, we have an obvious morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathcal{M},\mathcal{F}^0) & \longrightarrow & \mathrm{Hom}(\mathcal{M},\mathcal{F}^1) & \longrightarrow & \mathrm{Hom}(\mathcal{M},\mathcal{F}^2) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(\mathcal{N},\mathcal{F}^0) & \longrightarrow & \mathrm{Hom}(\mathcal{N},\mathcal{F}^1) & \longrightarrow & \mathrm{Hom}(\mathcal{N},\mathcal{F}^2) & \longrightarrow & \cdots \end{array}$$

By Subsec. 5.2.2 or Step 1 in Subsec 5.2.4, $\mathrm{Ext}^\star(\mathcal{M},\mathcal{F}) = \mathcal{H}^\star(\mathrm{Hom}(\mathcal{M},\mathcal{F}^\bullet))$ and the same relation holds if we replace $\mathcal{M}$ with $\mathcal{N}$. Thus $\mathcal{H}^\star$ acting on the above morphism of complexes defines a morphism $\mathrm{Ext}^n(\mathcal{M},\mathcal{F}) \leftarrow \mathrm{Ext}^n(\mathcal{N},\mathcal{F})$ for all $n \in \mathbb{N}$.

Suppose $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is an exact sequence in $\mathrm{Mod}(\mathcal{A})$. Since each $\mathcal{F}^n$ is injective, we get a short exact sequence of chain complexes

$$0 \leftarrow \mathrm{Hom}(\mathcal{M},\mathcal{F}^\bullet) \leftarrow \mathrm{Hom}(\mathcal{N},\mathcal{F}^\bullet) \leftarrow \mathrm{Hom}(\mathcal{P},\mathcal{F}^\bullet) \leftarrow 0$$

which yields a long exact sequence through $\mathcal{H}^\star$. In this way, we obtain a connecting morphism $\delta : \mathcal{H}^\star(\mathcal{M},\mathcal{F}^\bullet) \to \mathcal{H}^{\star+1}(\mathcal{P},\mathcal{F}^\bullet)$. This makes $\mathrm{Ext}^\bullet_{\mathcal{A}}(-,\mathcal{F})$ a contavariant δ-functor. One checks easily that $\mathcal{H}^{>0}(\mathrm{Hom}(\mathcal{M},\mathcal{F}^\bullet))$ vanishes when $\mathcal{M}$ is free. Since any $\mathcal{A}$-module has an epimorphism from a free module, we conclude from Thm. 5.1.6 that $\mathrm{Ext}^\bullet_{\mathcal{A}}(-,\mathcal{F})$ is universal. $\square$

Thus, the isomorphism class of $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{E},\mathcal{F})$ can be defined either via the right derived functor of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},-)$ or via the the right derived contravariant functor of $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{F})$. The isomorphism class of $\mathrm{Tor}^{\mathcal{A}}_n(\mathcal{E},\mathcal{F})$ can be defined using the left derived functor of either $\mathcal{E} \otimes_{\mathcal{A}} -$ or $- \otimes_{\mathcal{A}} \mathcal{F}$. Thus, as $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}$, we see immediately that:

Corollary 5.3.6. *For each $n \in \mathbb{N}$ and $\mathcal{E},\mathcal{F} \in \mathrm{Mod}(\mathcal{A})$, we have an isomorphism of $\mathcal{A}$-modules*

$$\mathrm{Tor}^{\mathcal{A}}_n(\mathcal{E},\mathcal{F}) \simeq \mathrm{Tor}^{\mathcal{A}}_n(\mathcal{F},\mathcal{E})$$

Using Ext, we give a criterion on projectivity.

Proposition 5.3.7. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$. Then the following are equivalent.*

(1) $\mathcal{E}$ is projective.

(2) $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},-)$ is zero on $\mathrm{Mod}(\mathcal{A})$. (It then follows automatically that $\mathrm{Ext}^{>0}_{\mathcal{A}}(\mathcal{E},-)$ is trivial.)

(3) $\mathcal{E}$ is a direct summand of a free $\mathcal{A}$-module.

Proof. Suppose (1) is true. For each $\mathcal{F} \in \mathrm{Mod}(\mathcal{A})$, $\mathrm{Ext}^{\bullet}_{\mathcal{A}}(-,\mathcal{F})$ is a right derived contravariant functor. So by Thm. 5.2.2, $\mathrm{Ext}^{>0}_{\mathcal{A}}(-,\mathcal{F})$ vanishes on projective objects. This proves (2). Conversely, assume $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},-)$ is zero. Then since each short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathrm{Mod}(\mathcal{A})$ gives a long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{F}') \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{F}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{F}'') \to \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},\mathcal{F}'),$$

$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},-)$ is exact, and hence $\mathcal{E}$ is projective. We have finished proving (1)⇔(2).

(1)⇒(3): Assume (1). Choose an epimorphism $\alpha : \mathcal{E}' \twoheadrightarrow \mathcal{E}$ where $\mathcal{E}'$ is a free $\mathcal{A}$-module. Since $\mathcal{E}$ is projective, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E}') \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E})$ is surjective. Choose $\beta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E}')$ sent to $\mathbf{1}_{\mathcal{E}} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E})$. This means that $\alpha \circ \beta = \mathbf{1}_{\mathcal{E}}$. So α splits. Therefore $\mathcal{E}' \simeq \mathcal{E} \oplus \mathrm{Ker}\alpha$. This proves (3).

(3)⇒(2): Assume that $\mathcal{E}' \simeq \mathcal{E} \oplus \mathcal{E}_0$ where $\mathcal{E}'$ is free. Then for each $\mathcal{F} \in \mathrm{Mod}(\mathcal{A})$,

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},\mathcal{F}) \oplus \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E}_0,\mathcal{F}) \simeq \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E}',\mathcal{F}) = 0.$$

So $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},\mathcal{F}) = 0$. □

Remark 5.3.8. It is not hard to check that taking direct limit is an exact functor from the category of direct systems of $\mathcal{A}$-modules to $\mathrm{Mod}(\mathcal{A})$. Namely, if $(\mathcal{E}_i)_{i\in I}, (\mathcal{F}_i)_{i\in I}, (\mathcal{G}_i)_{i\in I}$ are direct systems in $\mathrm{Mod}(\mathcal{A})$, and if we have an exact sequence of morphisms of direct systems

$$0 \to \mathcal{E}_\bullet \to \mathcal{F}_\bullet \to \mathcal{G}_\bullet \to 0$$

then we have an exact sequence

$$0 \to \varinjlim_{i\in I} \mathcal{E}_i \to \varinjlim_{i\in I} \mathcal{F}_i \to \varinjlim_{i\in I} \mathcal{G}_i \to 0$$

Using this fact, we prove:

Proposition 5.3.9. *Let $(\mathcal{E}_i)_{i\in I}$ be a direct system of $\mathcal{A}$-modules and let $\mathcal{F}$ be an $\mathcal{A}$-module. Then for each $n \in \mathbb{N}$, we have a natural isomorphism*

$$\varinjlim_{i\in I} \mathrm{Tor}^{\mathcal{A}}_n(\mathcal{E}_i,\mathcal{F}) \simeq \mathrm{Tor}^{\mathcal{A}}_n\Big(\varinjlim_{i\in I} \mathcal{E}_i,\mathcal{F}\Big) \tag{5.3.3}$$

Proof. Choose a projective resolution $\mathcal{F}_\bullet \to \mathcal{F} \to 0$. Then we have a chain complex of systems in $\mathcal{A}$

$$\cdots \to \mathcal{E}_\bullet \otimes_{\mathcal{A}} \mathcal{F}_2 \to \mathcal{E}_\bullet \otimes_{\mathcal{A}} \mathcal{F}_1 \to \mathcal{E}_\bullet \otimes_{\mathcal{A}} \mathcal{F}_0 \to 0$$

More precisely, if $i, j \in I$ and $i \leqslant j$, we have a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{E}_i \otimes_{\mathcal{A}} \mathcal{F}_2 & \longrightarrow & \mathcal{E}_i \otimes_{\mathcal{A}} \mathcal{F}_1 & \longrightarrow & \mathcal{E}_i \otimes_{\mathcal{A}} \mathcal{F}_0 & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathcal{E}_j \otimes_{\mathcal{A}} \mathcal{F}_2 & \longrightarrow & \mathcal{E}_j \otimes_{\mathcal{A}} \mathcal{F}_1 & \longrightarrow & \mathcal{E}_j \otimes_{\mathcal{A}} \mathcal{F}_0 & \longrightarrow 0
\end{array}$$

By Rem. 5.3.8, taking direct limit commutes with taking kernels and cokernels. Therefore it commutes with talking homology. This proves

$$\varinjlim_{i\in I} \mathcal{H}_n(\mathcal{E}_i \otimes_{\mathcal{A}} \mathcal{F}_\bullet) \simeq \mathcal{H}_n\Big(\varinjlim_{i\in I}\big(\mathcal{E}_i \otimes_{\mathcal{A}} \mathcal{F}_\bullet\big)\Big) \simeq \mathcal{H}_n\Big(\Big(\varinjlim_{i\in I}\mathcal{E}_i\Big) \otimes_{\mathcal{A}} \mathcal{F}_\bullet\Big)$$

where the second equivalence is due to the fact that direct limit commutes with tensor product (Rem. 1.9.2). This proves (5.3.3). $\square$

Example 5.3.10. Let I, J be ideals of a ring $\mathcal{A}$. Let us compute $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{A}/I, \mathcal{A}/J)$. Tensoring $\mathcal{A}/I$ with the short exact sequence

$$0 \to J \to \mathcal{A} \to \mathcal{A}/J \to 0$$

we get a long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{A}/I, \mathcal{A}/J) \to (\mathcal{A}/I) \otimes_{\mathcal{A}} J \to (\mathcal{A}/I) \otimes_{\mathcal{A}} \mathcal{A} \to (\mathcal{A}/I) \otimes_{\mathcal{A}} (\mathcal{A}/J) \to 0$$

Since tensor products commute with cokernels, we have natural equivalences $(\mathcal{A}/I)\otimes_{\mathcal{A}}\mathcal{A} \simeq \mathcal{A}/I$, $(\mathcal{A}/I)\otimes_{\mathcal{A}} J \simeq J/IJ$, and $(\mathcal{A}/I)\otimes_{\mathcal{A}}(\mathcal{A}/J) \simeq \big(\mathcal{A}/J\big)/\big((I+J)/J\big) \simeq \mathcal{A}/(I+J)$ so that the above long exact sequence is equivalent to

$$0 \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{A}/I, \mathcal{A}/J) \to J/IJ \to \mathcal{A}/I \to \mathcal{A}/(I+J) \to 0$$

Therefore $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{A}/I, \mathcal{A}/J)$ is equivalent to the kernel of $J/IJ \to \mathcal{A}/I$, which is $(I \cap J)/IJ$. We conclude

$$\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{A}/I, \mathcal{A}/J) \simeq \frac{I \cap J}{IJ} \tag{5.3.4}$$

Example 5.3.11. Let $\mathcal{E}$ be an $\mathcal{A}$-module, and let $I \subset \mathcal{A}$ be an ideal. Since $\mathcal{A}$ is $\mathcal{A}$-free and hence projective, $\mathrm{Tor}_1^{\mathcal{A}}(-, \mathcal{A}) = 0$. So we have a long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{A}/I) \to \mathcal{E} \otimes_{\mathcal{A}} I \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A} \tag{5.3.5}$$

which shows that $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{A}/I) = 0$ iff $\mathcal{E} \otimes_{\mathcal{A}} I \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}$ is injective (i.e. the multiplication map $\mathcal{E} \otimes_{\mathcal{A}} I \to \mathcal{E}$ is injective).

5.4 Flatness is preserved by base change

In this section, $\mathcal{A}, \mathcal{B}$ denote commutative rings, and X, Y, S denote complex spaces. Recall that by saying that $\mathcal{A}$ is a $\mathcal{B}$-algebra, we mean that a morphism of rings $\mathcal{B} \to \mathcal{A}$ is fixed so that any $\mathcal{A}$-module is also a $\mathcal{B}$-module. We write $(\mathcal{A}, \mathfrak{m})$ when we mean that $\mathcal{A}$ is a local ring with maximal ideal $\mathfrak{m}$. Recall that by definition, a morphism of local rings $(\mathcal{B}, \mathfrak{n}) \to (\mathcal{A}, \mathfrak{m})$ is a ring homomorphism sending $\mathfrak{n}$ into $\mathfrak{m}$.

Proposition 5.4.1. *Let $\mathcal{E}$ be an $\mathcal{A}$-module. Then the following statements are equivalent.*

(1) The functor $\mathcal{E} \otimes_{\mathcal{A}} -$ is exact on $\mathrm{Mod}(\mathcal{A})$.

(2) $\mathrm{Tor}_n^{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = 0$ for each $n > 0$ and $\mathcal{F} \in \mathrm{Mod}(\mathcal{A})$.

(3) $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = 0$ for each $\mathcal{F} \in \mathrm{Mod}(\mathcal{A})$.

(4) $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{A}/I) = 0$ for each ideal $I \subset \mathcal{A}$.

If one of these statements holds, we say that $\mathcal{E}$ is a ***flat $\mathcal{A}$-module****.*

Proof. (1)⇒(2): Suppose (1) is true. If we let T be the functor $\mathcal{E} \otimes_{\mathcal{A}} -$, then by Thm. 5.1.6, $L_\bullet T$ is the universal δ-functor extending T if we set $L_0 T = T$ and $L_{>0} T = 0$. So $L_\bullet T = \mathrm{Tor}_\bullet^{\mathcal{A}}(\mathcal{E}, -)$. This proves (2).

(2)⇒(3): Obvious.

(3)⇒(1): If (3) is true, then any short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ in $\mathrm{Mod}(\mathcal{A})$ induces a long exact sequence

$$\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{P}) \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{N} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{P} \to 0$$

where the first term is 0. So (1) follows.

(3)⇒(4): Obvious.

(4)⇒(3): Assume (3). Since any $\mathcal{A}$-module is a union (i.e. direct limit) of its finitely-generated $\mathcal{A}$-submodules, by Prop. 5.3.9, it suffices to show that $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{F}) = 0$ whenever $\mathcal{F}$ is finitely generated. We prove this by induction on n, the minimal number of elements generating $\mathcal{F}$. The case $n = 0$ is trivial. Assume case $\leqslant n - 1$ is proved. Let $\mathcal{F}$ be $\mathcal{A}$-generated by n elements, and let $\mathcal{F}'$ be its submodule generated by the first $n - 1$ elements. Denote the last element by x. Then we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{A}/I \to 0$$

where $I = \{a \in \mathcal{A} : ax \in \mathcal{F}'\}$. We obtain an exact sequence

$$\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{F}') \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{E}, \mathcal{A}/I)$$

where the first term vanishes by induction and the third term vanishes by (4). So the middle term vanishes. This proves (3). □

Definition 5.4.2. Let $\varphi : X \to Y$ be a holomorphic map and $\mathscr{E}$ be an $\mathscr{O}_X$-module. If $x \in X$, we say that $\mathscr{E}$ is **flat (over** Y**) at** x or φ**-flat at** x, if $\mathscr{E}_x$ is a flat $\mathscr{O}_{Y,\varphi(x)}$-module. If $\mathscr{E}$ is φ-flat for all $x \in X$, we say that $\mathscr{E}$ is **flat over** Y or that $\mathscr{E}$ is φ**-flat**.

If $\mathscr{O}_X$ is flat over Y at x, we say that φ is **flat at** x. If $\mathscr{O}_X$ is flat over Y, we say that φ is a **flat holomorphic map**. □

Example 5.4.3. If Y is a reduced point, then $\mathscr{O}_Y = \mathbb{C}$. So any $\mathscr{O}_X$-module is clearly flat over Y.

Example 5.4.4. Since free modules are flat, by Thm. 2.5.4, Weierstrass maps are flat.

Example 5.4.5. Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be holomorphic maps of complex spaces. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Let $x \in X$. Suppose that ψ is flat at $\varphi(x)$ and $\mathscr{E}$ is φ-flat at x. The $\mathscr{E}$ is clearly $(\psi \circ \varphi)$-flat at x.

Example 5.4.6. Let $\mathcal{I}$ be an idea of $\mathscr{O}_Y$, and let $X = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{I})$. Suppose $x \in X$ (i.e. $x \in N(\mathcal{I})$) and $\mathcal{I}_x \neq 0$, then $\iota : X \to Y$ is not flat at x.

Indeed, if ι is flat at x, then $\mathscr{O}_{X,x} = \mathscr{O}_{Y,x}/\mathcal{I}_x$ is not $\mathscr{O}_{Y,x}$-flat. By Exp. 5.3.10,

$$0 = \mathrm{Tor}_1^{\mathscr{O}_{Y,x}}(\mathscr{O}_{Y,x}/\mathcal{I}_x, \mathscr{O}_{Y,x}/\mathcal{I}_x) = \mathcal{I}_x/\mathcal{I}_x^2.$$

Therefore $\mathcal{I}_x^2 = \mathcal{I}_x$, and hence $\mathfrak{m}_{X,x}\mathcal{I}_x = \mathcal{I}_x$. So $\mathcal{I}_x = 0$ by Nakayama's lemma 1.2.15. This contradicts the assumption $\mathcal{I}_x \neq 0$. □

The goal of this section is to show that flatness is preserved by base change (Thm. 5.4.9). Its proof relies on the following crucial theorem, which allows us to reduce the study of arbitrary base changes to finite ones.

Theorem 5.4.7. *Let $(\mathcal{B}, \mathfrak{n}) \to (\mathcal{A}, \mathfrak{m})$ be a morphism of Noetherian local rings. Let $\mathcal{E}$ be a finitely-generated $\mathcal{A}$-module. Assume that there exists $k_0 \in \mathbb{N}$ such that $\mathcal{E} \otimes_{\mathcal{B}} (\mathcal{B}/\mathfrak{n}^k)$ is $(\mathcal{B}/\mathfrak{n}^k)$-flat for all $k \geqslant k_0$. Then $\mathcal{E}$ is $\mathcal{B}$-flat.*

Proof. We shall prove $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{B}/J) = 0$ for each ideal $J \subset \mathcal{B}$. Namely (Exp. 5.3.11), we shall prove that $\mathcal{E} \otimes_{\mathcal{B}} J \to \mathcal{E} \otimes_{\mathcal{B}} \mathcal{B}$ is injective. The natural idea is to tensor it by $\mathcal{B}/\mathfrak{n}^k$, where we choose $k \geqslant k_0$. But this is not a good choice, since $J \to \mathcal{B}$ tensored by $\mathcal{B}/\mathfrak{n}^k$ is not even injective. Indeed, by Exp. 5.3.10, if we tensor $\mathcal{B}/\mathfrak{n}^k$ with the short exact sequence $0 \to J \to \mathcal{B} \to \mathcal{B}/J \to 0$, we get a long one

$$0 \to \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{B}/\mathfrak{n}^k, \mathcal{B}/J) \to J/\mathfrak{n}^k J \to \mathcal{B}/\mathfrak{n}^k \to \mathcal{B}/(\mathfrak{n}^k + J) \to 0 \tag{5.4.1}$$

But we clearly have an exact sequence

$$0 \to J/(\mathfrak{n}^k \cap J) \to \mathcal{B}/\mathfrak{n}^k \to \mathcal{B}/(\mathfrak{n}^k + J) \to 0 \tag{5.4.2}$$

where $J/(\mathfrak{n}^k\cap J)=(\mathfrak{n}^k+J)/\mathfrak{n}^k$ is the kernel of the subsequent morphism.

One may tensor $\mathcal{E}$ with (5.4.2). But since we know that $\mathcal{E}\otimes_{\mathcal{B}}(\mathcal{B}/\mathfrak{n}^k)$ is $(\mathcal{B}/\mathfrak{n}^k)$-flat, namely, $\mathrm{Tor}_1^{\mathcal{B}/\mathfrak{n}^k}(\mathcal{E}\otimes_{\mathcal{B}}(\mathcal{B}/\mathfrak{n}^k),-)$ vanishes, and since (5.4.2) is clearly an exact sequence in $\mathrm{Mod}(\mathcal{B}/\mathfrak{n}^k)$, we $(\mathcal{B}/\mathfrak{n}^k)$-tensor $\mathcal{E}\otimes_{\mathcal{B}}(\mathcal{B}/\mathfrak{n}^k)$ with (5.4.2) to get an exact sequence as the second line of the following diagram:

$$\begin{tikzcd}
 & \mathcal{E}\otimes_{\mathcal{B}}J \arrow[r] \arrow[d, "\phi"'] & \mathcal{E}\otimes_{\mathcal{B}}\mathcal{B} \arrow[r] \arrow[d] & \mathcal{E}\otimes_{\mathcal{B}}(\mathcal{B}/J) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}\otimes_{\mathcal{B}}\big(J/(\mathfrak{n}^k\cap J)\big) \arrow[r] & \mathcal{E}\otimes_{\mathcal{B}}(\mathcal{B}/\mathfrak{n}^k) \arrow[r] & \mathcal{E}\otimes_{\mathcal{B}}\big(\mathcal{B}/(\mathfrak{n}^k+J)\big) \arrow[r] & 0
\end{tikzcd} \tag{5.4.3}$$

The first row is clearly exact, and we can easily find canonical morphisms as the vertical arrows making the above diagram commutes.

Choose any ξ in the kernel of $\mathcal{E}\otimes_{\mathcal{B}}J\to\mathcal{E}\otimes_{\mathcal{B}}\mathcal{B}$. Our goal is to show that $\xi=0$. By the commutativity of the first cell in (5.4.3), ξ is sent by ϕ to 0 in $\mathcal{E}\otimes_{\mathcal{B}}\big(J/(\mathfrak{n}^k\cap J)\big)$ for each $k\in\mathbb{Z}_+$. Now, by Artin-Rees lemma 1.4.5, there exists $s\in\mathbb{Z}_+$ such that

$$\mathfrak{n}^t(\mathfrak{n}^s\cap J)=\mathfrak{n}^{t+s}\cap J$$

for all $t\in\mathbb{N}$. So $\mathfrak{n}^{t+s}\cap J\subset\mathfrak{n}^tJ$. Assume for simplicity that $s\geqslant k_0$. Then for each $k\geqslant s$ we have

$$\mathfrak{n}^k\cap J\subset\mathfrak{n}^{k-s}J.$$

Therefore we have a surjection $\mathcal{E}\otimes_{\mathcal{B}}\big(J/(\mathfrak{n}^k\cap J)\big)\to\mathcal{E}\otimes_{\mathcal{B}}(J/\mathfrak{n}^{k-s}J)$. (To summarize, we are using Artin-Rees lemma to replace the $J/(\mathfrak{n}^k\cap J)$ in (5.4.2) with the $J/\mathfrak{n}^kJ$ in (5.4.1)!) Compose this morphism with ϕ, and we see that ξ is sent to 0 in

$$\mathcal{E}\otimes_{\mathcal{B}}(J/\mathfrak{n}^{k-s}J)\simeq\frac{\mathcal{E}\otimes_{\mathcal{B}}J}{\mathfrak{n}^{k-s}(\mathcal{E}\otimes_{\mathcal{B}}J)}$$

for all $k\geqslant s$. So ξ belongs to $\mathfrak{n}^t(\mathcal{E}\otimes_{\mathcal{B}}J)$ for all $t\in\mathbb{N}$.

Consider $\mathcal{E}\otimes_{\mathcal{B}}J$ as an $\mathcal{A}$-module. Then it is clearly finitely-generated. Since $\mathfrak{n}^t(\mathcal{E}\otimes_{\mathcal{B}}J)\subset\mathfrak{m}^t(\mathcal{E}\otimes_{\mathcal{B}}J)$, we have $\xi\in\bigcap_{t\in\mathbb{N}}\mathfrak{m}^t(\mathcal{E}\otimes_{\mathcal{B}}J)=0$ by Krull's intersection Thm. 1.4.4. $\square$

Let us phrase Thm. 5.4.7 in the language of complex analytic geometry.

Theorem 5.4.8. *Let $\varphi:X\to Y$ be a holomorphic map, and let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Let $x\in X$, $y=\varphi(x)$. Let $Y_k=\mathrm{Specan}(\mathscr{O}_Y/\mathfrak{m}_{Y,y}^k)$ where $\mathfrak{m}_{Y,y}$ is considered as the ideal of all $g\in\mathscr{O}_Y$ vanishing at y. Suppose that there exists $k_0\in\mathbb{N}$ such that the $\mathscr{O}_{\varphi^{-1}(Y_k)}$-module $\mathscr{E}|_{\varphi^{-1}(Y_k)}$ is flat over Y_k at x for all $k\geqslant k_0$. Then $\mathscr{E}$ is flat over Y at x.*

Recall that by Rem. 1.12.3, there is a canonical $\mathscr{O}_X$-module isomorphism

$$\mathscr{E}|_{\varphi^{-1}(Y_k)} \simeq \mathscr{E} \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathfrak{m}_{Y,y}^k)$$

Theorem 5.4.9. *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps, and let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module. Consider the Cartesian product*

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"] \\ S & Y \arrow[l, "\psi"'] \end{tikzcd}$$

Choose $x \in X, y \in Y$ such that $t = \varphi(x)$ equals $\psi(y)$. Assume that $\mathscr{E}$ is flat over S at x. Then $\mathrm{pr}_X^*\mathscr{E}$ is flat over Y at (x, y).

Proof. We first consider the special case that ψ is finite. By Thm. 2.7.2, we can shrink X, S to neighborhoods of x, t respectively and replace Y by $\psi^{-1}(S)$, so that x is the single point of the set $\varphi^{-1}(t)$. Then (x, y) is the single point of the set $\mathrm{pr}_Y^{-1}(y)$. By Prop. 2.4.5, we have isomorphisms of $\mathscr{O}_{S,t}$-modules and of $\mathscr{O}_{Y,y}$-modules

$$(\varphi_*\mathscr{E})_t \simeq \mathscr{E}_x \qquad (\mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{E})_y \simeq (\mathrm{pr}_X^*\mathscr{E})_{x\times y}$$

Thus, by Thm. 2.8.2 (and Rem. 2.8.1), we have an $\mathscr{O}_{Y,y}$-module isomorphism

$$(\mathrm{pr}_X^*\mathscr{E})_{x\times y} \simeq \mathscr{E}_x \otimes_{\mathscr{O}_{S,t}} \mathscr{O}_{Y,y}$$

Since $\mathscr{E}_x$ is $\mathscr{O}_{S,t}$-flat, $\mathscr{E}_x \otimes_{\mathscr{O}_{S,t}} \mathscr{O}_{Y,y}$ is clearly $\mathscr{O}_{Y,y}$-flat. Therefore $(\mathrm{pr}_X^*\mathscr{E})_{x\times y}$ is $\mathscr{O}_{Y,y}$-flat.

Now consider the general case. For each $k \in \mathbb{Z}_+$, let $Y_k = \mathrm{Specan}(\mathscr{O}_Y/\mathfrak{m}_{Y,y}^k)$. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"] & \mathrm{pr}_Y^{-1}(Y_k) \arrow[l, hook', "\widetilde{\iota}"'] \arrow[d] \\ S & Y \arrow[l, "\psi"'] & Y_k \arrow[l, hook', "\iota"'] \end{tikzcd}$$

where the two cells are Cartesian squares. So the largest rectangle is also Cartesian. Since $Y_k \to S$ is clearly finite, by the first paragraph, $\widetilde{\iota}^*\mathrm{pr}_X^*\mathscr{E} = (\mathrm{pr}_X^*\mathscr{E})|_{\mathrm{pr}_Y^{-1}(Y_k)}$ is flat over Y_k at (x, y) for all $k \in \mathbb{Z}_+$. Therefore, by Thm. 5.4.8, $\mathrm{pr}_X^*\mathscr{E}$ is flat over Y at (x, y). $\square$

Example 5.4.10. Let $\mathrm{pr}_Y : X \times Y \to Y$ be the projection onto the Y-component. Then pr_Y is flat, because it is the pullback of $X \to \{0\}$ (which is clearly flat) along $Y \to \{0\}$.

Example 5.4.11. Any holomorphic submersion of complex manifolds is flat because it is locally equivalent to $X \times Y \to Y$ where X, Y are open subsets of number spaces.

5.5 Slicing criterion for flatness

In this section, we give more useful criteria on flatness.

Lemma 5.5.1. *Fix a ring morphism $\mathcal{B} \to \mathcal{A}$. Let*

$$0 \to \mathcal{E}'' \to \mathcal{E}' \to \mathcal{E} \to 0$$

be an exact sequence in $\mathrm{Mod}(\mathcal{A})$*. Then for every $\mathcal{M} \in \mathrm{Mod}(\mathcal{B})$, we have an isomorphism of $\mathcal{B}$-modules*

$$\begin{aligned} &\mathrm{Coker}\Big(\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{E}') \to \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{E})\Big) \\ \simeq &\mathrm{Coker}\Big(\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{A},\mathcal{E}') \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{A},\mathcal{E})\Big) \end{aligned} \tag{5.5.1}$$

Proof. Apply $\mathcal{M}\otimes_{\mathcal{B}} -$ to the above short exact sequence to get a long one

$$\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{E}') \to \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{E}) \to \mathcal{M}\otimes_{\mathcal{B}}\mathcal{E}'' \to \mathcal{M}\otimes_{\mathcal{B}}\mathcal{E}'$$

Applying $\mathcal{M}\otimes_{\mathcal{B}}\mathcal{A}\otimes_{\mathcal{A}} -$ instead, we get a long exact sequence

$$\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{A},\mathcal{E}') \to \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{A},\mathcal{E}) \to \mathcal{M}\otimes_{\mathcal{B}}\mathcal{E}'' \to \mathcal{M}\otimes_{\mathcal{B}}\mathcal{E}'$$

These two exact sequences imply (5.5.1). □

We consider two important special cases of Lemma 5.5.1:

Proposition 5.5.2. *Fix a ring morphism $\mathcal{B} \to \mathcal{A}$. Choose $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$ and $\mathcal{M} \in \mathrm{Mod}(\mathcal{B})$. Then*

$$\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{E}) = 0 \qquad \Longrightarrow \qquad \mathrm{Tor}_1^{\mathcal{A}}(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{A},\mathcal{E}) = 0 \tag{5.5.2}$$

Proof. Choose an epimorphism $\mathcal{E}' \twoheadrightarrow \mathcal{E}$ where $\mathcal{E}'$ is a free $\mathcal{A}$-module. Then $\mathrm{Tor}_1^{\mathcal{A}}(-,\mathcal{E}') = 0$ since free modules are clearly flat. The proposition follows immediately from (5.5.1). □

Proposition 5.5.3. *Let $\mathcal{M}$ be a $\mathcal{B}$-module, and let $\mathcal{F}$ be a $(\mathcal{B}/\tau\mathcal{B})$-module where $\tau \in \mathcal{B}$. Assume that τ is a non zero-divisor of both $\mathcal{B}$ and $\mathcal{M}$. Then*

$$\mathrm{Tor}_1^{\mathcal{B}/\tau\mathcal{B}}\big(\mathcal{M}\otimes_{\mathcal{B}}(\mathcal{B}/\tau\mathcal{B}),\mathcal{F}\big) \simeq \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{F}) \tag{5.5.3}$$

Proof. Again we choose an epimorphism $\mathcal{F}' \to \mathcal{F} \to 0$ where $\mathcal{F}'$ is a free $(\mathcal{B}/\tau\mathcal{B})$-module. Then $\mathrm{Tor}_1^{\mathcal{B}/\tau\mathcal{B}}(-,\mathcal{F}') = 0$. So (5.5.3) follows immediately from (5.5.1) if we can show that $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{F}') = 0$. Since $\mathcal{F}'$ is a direct sum of $\mathcal{B}/\tau\mathcal{B}$, by Prop. 5.3.9, it suffices to show $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M},\mathcal{B}/\tau\mathcal{B}) = 0$. This follows from the next result. □

Lemma 5.5.4. *Let $\mathcal{B}$ be a ring and let $\tau \in \mathcal{B}$ be a non zero-divisor of $\mathcal{B}$. Let $\mathcal{M}$ be a $\mathcal{B}$-module. Then the following are equivalent.*

(1) τ is a non zero-divisor of $\mathcal{M}$.

(2) $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M}, \mathcal{B}/\tau\mathcal{B}) = 0$.

Proof. Since τ is a non zero-divisor of $\mathcal{B}$, we have a short exact sequence

$$0 \to \mathcal{B} \xrightarrow{\times\tau} \mathcal{B} \to \mathcal{B}/\tau\mathcal{B} \to 0$$

which, tensored by $\mathcal{M}$, gives a commutative diagram where the horizontal lines are exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M}, \mathcal{B}/\tau\mathcal{B}) \arrow[r] & \mathcal{M} \otimes_{\mathcal{B}} \mathcal{B} \arrow[r] \arrow[d, "\simeq"'] & \mathcal{M} \otimes_{\mathcal{B}} \mathcal{B} \arrow[r] \arrow[d, "\simeq"'] & \mathcal{M} \otimes_{\mathcal{B}} (\mathcal{B}/\tau\mathcal{B}) \arrow[r] \arrow[d, "\simeq"'] & 0 \\
& & \mathcal{M} \arrow[r, "\times\tau"] & \mathcal{M} \arrow[r] & \mathcal{M}/\tau\mathcal{M} &
\end{tikzcd}$$

This shows

$$\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{M}, \mathcal{B}/\tau\mathcal{B}) \simeq \mathrm{Ker}(\mathcal{M} \xrightarrow{\times\tau} \mathcal{M}) \tag{5.5.4}$$

The equivalence of (1) and (2) follows immediately. □

As an application of Prop. 5.5.2, we prove a variant of Thm. 5.4.7

Theorem 5.5.5. *Let $(\mathcal{B}, \mathfrak{n}) \to (\mathcal{A}, \mathfrak{m})$ be a morphism of Noetherian local rings. Let $\mathcal{E}$ be a finitely-generated $\mathcal{A}$-module. The following are equivalent.*

(1) $\mathcal{E}$ is $\mathcal{B}$-flat.

(2) $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{B}/\mathfrak{n}) = 0$.

Proof. Clearly (1) implies (2). Assume (2). To prove that $\mathcal{E}$ is $\mathcal{B}$-flat, by Thm. 5.4.7, it suffices to prove that

$$\mathrm{Tor}_1^{\mathcal{B}/\mathfrak{n}^k}(\mathcal{E} \otimes_{\mathcal{B}} (\mathcal{B}/\mathfrak{n}^k), \mathcal{N}) = 0$$

for any $k \in \mathbb{Z}_+$ and any $(\mathcal{B}/\mathfrak{n}^k)$-module $\mathcal{N}$ (equivalently, any $\mathcal{B}$-module $\mathcal{N}$ such that $\mathfrak{n}^k\mathcal{N} = 0$). By Prop. 5.5.2, it suffices to prove $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{N}) = 0$ whenever $\mathfrak{n}^k\mathcal{N} = 0$ for some $k > 0$.

If $\mathfrak{n}\mathcal{N} = 0$, then $\mathcal{N} = \mathcal{N}/\mathfrak{n}\mathcal{N} \simeq \mathcal{N} \otimes_{\mathcal{B}} (\mathcal{B}/\mathfrak{n})$ is a vector space over the field $\mathcal{B}/\mathfrak{n}$. Thus $\mathcal{N}$ is a direct sum of $\mathcal{B}/\mathfrak{n}$. So clearly $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{N}) = 0$ by assumption (2) and Prop. 5.3.9. The general case follows by induction on k and the exact sequence

$$\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathfrak{n}\mathcal{N}) \to \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{N}) \to \mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{N}/\mathfrak{n}\mathcal{N})$$

where the last term is 0 because $\mathcal{N}/\mathfrak{n}\mathcal{N}$ is annihilated by $\mathfrak{n}$. □

Theorem 5.5.6 (Slicing criterion). *Let $(\mathcal{B}, \mathfrak{n}) \to (\mathcal{A}, \mathfrak{m})$ be a morphism of Noetherian local rings. Let $\mathcal{E}$ be a finitely-generated $\mathcal{A}$-module. Let $\tau \in \mathfrak{n}$ be a non zero-divisor of $\mathcal{B}$. The following are equivalent.*

(1) $\mathcal{E}$ is $\mathcal{B}$-flat.

(2) τ is a non zero-divisor of $\mathcal{E}$, and $\mathcal{E} \otimes_{\mathcal{B}} (\mathcal{B}/\tau\mathcal{B})$ is $(\mathcal{B}/\tau\mathcal{B})$-flat.

Proof. Assume (1). Then $\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{B}/\tau\mathcal{B}) = 0$ implies that τ is a non zero-divisor of $\mathcal{E}$ (by Lemma 5.5.4). And $\mathrm{Tor}_1^{\mathcal{B}/\tau\mathcal{B}}(\mathcal{E} \otimes_{\mathcal{B}} (\mathcal{B}/\tau\mathcal{B}), -)$ is trivial by Lemma 5.5.2 or Prop. 5.5.3. Conversely, assume (2). Then by Prop. 5.5.3,

$$\mathrm{Tor}_1^{\mathcal{B}}(\mathcal{E}, \mathcal{B}/\mathfrak{n}) \simeq \mathrm{Tor}_1^{\mathcal{B}/\tau\mathcal{B}}(\mathcal{E} \otimes_{\mathcal{B}} (\mathcal{B}/\tau\mathcal{B}), \mathcal{B}/\mathfrak{n}) = 0$$

So $\mathcal{E}$ is $\mathcal{B}$-flat by Thm. 5.5.5. □

Let us phrase Thm. 5.5.6 in the language of complex geometry.

Theorem 5.5.7 (Slicing criterion). *Let $\varphi : X \to Y$ be holomorphic and let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Choose $\tau \in \mathscr{O}(Y)$. Let $T = \mathrm{Specan}(\mathscr{O}_Y/\tau\mathscr{O}_Y)$. Choose $y \in T$ and $x \in \varphi^{-1}(T)$, and assume that τ is a non zero-divisor of $\mathscr{O}_{Y,y}$. Then the following are equivalent.*

(1) $\mathscr{E}$ is flat over Y at x.

(2) τ is a non zero-divisor of $\mathscr{E}_x$, and $\mathscr{E}|_{\varphi^{-1}(T)}$ is flat over T at x.

Corollary 5.5.8. *Let $f \in \mathscr{O}(X)$ and $x \in X$. Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Let z be the standard coordinate of $\mathbb{C}$. Then the following are equivalent.*

(1) $\mathscr{E}$ is f-flat at x.

(2) $f - f(x)$ is a non zero-divisor of $\mathscr{E}_x$.

Proof. In this case, $T = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}}/z\mathscr{O}_{\mathbb{C}})$ is a single reduced point. So $\mathscr{E}|_{f^{-1}(T)}$ is clearly flat over T. □

We now give a new

Proof of Prop. 3.7.2. It suffices to show that if $f \in \mathscr{O}(X)$ is a non zero-divisor of $\mathscr{O}_{X,x}$ then $f \otimes 1$ is a non zero-divisor of $\mathscr{O}_{X\times Y, x\times y}$. This is clearly true when $f(x) \neq 0$. So let us assume $f(x) = 0$. By Cor. 5.5.8, $f : X \to \mathbb{C}$ is flat at x. Since $f\otimes 1 : X \times Y \to \mathbb{C}$ is the composition of the projection $X \times Y \to X$ (which is flat by Exp. 5.4.10) and f, $f\otimes 1$ is flat at $x \times 1$. By Cor. 5.5.8, $f\otimes 1$ is a non zero-divisor. □

Cor. 5.5.8 can be easily generalized to a criterion on the flatness of a holomorphic map from a complex space to a complex manifold (Cor. 5.5.10), as shown below.

Definition 5.5.9. Let $\mathcal{E}$ be an $\mathcal{A}$-module. A finite sequence $a_1, \ldots, a_n \in \mathcal{A}$ is called an $\mathcal{E}$**-regular sequence** if the following are satisfied:

(1) For each $1 \leqslant i \leqslant n$, a_i is a non zero-divisor of $\dfrac{\mathcal{E}}{\sum_{j<i} a_j\mathcal{E}}$. In particular, a_1 is a non zero-divisor of $\mathcal{E}$.

(2) $\sum_{i=1}^n a_i\mathcal{E} \neq \mathcal{E}$.

We are mainly interested in the case that $(\mathcal{A}, \mathfrak{m})$ is a Noetherian local ring and $a_1, \ldots, a_n \in \mathfrak{m}$. In this case, if $\mathcal{E}$ is a non-zero finitely-generated $\mathcal{A}$-module, then $\mathfrak{m}\mathcal{E} \neq \mathcal{E}$ by Nakayama's lemma 1.2.15. Then condition (2) is redundant.

Corollary 5.5.10. *Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module, and let $f_1, \ldots, f_n \in \mathscr{O}(X)$. Let $x \in X$. Set $F = (f_1, \ldots, f_n) : X \to \mathbb{C}^n$. Then $\mathscr{E}$ is F-flat at x if and only if the germs at x of $f_1 - f_1(x), \ldots, f_n - f_n(x)$ form an $\mathscr{E}_x$-regular sequence.*

Proof. By induction on n. The case $n = 0$ is obvious. Assume case $n-1$ holds where $n \in \mathbb{Z}_+$. Now consider case n. Assume for simplicity that $f_1(x) = \cdots = f_n(x) = 0$. Let $(z_1, \ldots, z_n)$ be the standard coordinates of $\mathbb{C}^n$. Then $\mathbb{C}^{n-1} \simeq 0 \times \mathbb{C}^{n-1}$ is $\mathrm{Specan}(\mathscr{O}_{\mathbb{C}^n}/z_1\mathscr{O}_{\mathbb{C}^n})$. Note that that z_1 is a non zero-divisor of $\mathscr{E}_x$ is the same as that f_1 is a non zero-divisor of $\mathscr{E}_x$. Thus, by Slicing criterion (5.5.7), $\mathscr{E}$ is F-flat at x if and only if f_1 is a non zero-divisor of $\mathscr{E}_x$ and $\mathscr{E}|_{F^{-1}(\mathbb{C}^{n-1})} \simeq \mathscr{E}/z_1\mathscr{E}$ is flat over $\mathbb{C}^{n-1}$ at x. By induction, the second condition is equivalent to that $f_2, \ldots, f_n$ is an $(\mathscr{E}_x/z_1\mathscr{E}_x)$-regular sequence. □

We close this section with another interesting application of Thm. 5.5.5 which was used in the proof of Lemma 3.8.2. In this application, the $(\mathcal{B}, \mathfrak{n})$ and $(\mathcal{A}, \mathfrak{m})$ in Thm. 5.5.5 are set to be equal.

Theorem 5.5.11. *Let $(\mathcal{A}, \mathfrak{m})$ be a Noetherian local ring. Let $\mathcal{E}$ be a finitely-generated $\mathcal{A}$-module. Then the following are equivalent.*

(1) $\mathcal{E}$ is a free $\mathcal{A}$-module of finite rank.

(2) $\mathcal{E}$ is a projective $\mathcal{A}$-module.

(3) $\mathcal{E}$ is a flat $\mathcal{A}$-module.

Proof. Clearly (1)⇒(2). By Thm. 5.2.2, if $\mathcal{E}$ is projective then $\mathrm{Tor}^{\mathcal{A}}_{>0}(\mathcal{E}, -) = 0$. Therefore (2)⇒(3).

Assume (3). Choose $x_1, \ldots, x_n \in \mathcal{E}$ forming a basis of the $(\mathcal{A}/\mathfrak{m})$-vector space $\mathcal{E} \otimes_{\mathcal{A}} (\mathcal{A}/\mathfrak{m})$. By Nakayama's lemma 1.2.15, the morphism $\mathcal{A}^n \to \mathcal{M}$ sending the j-th basis of $\mathcal{A}^n$ to x_j is surjective. Let $\mathcal{N}$ be the kernel of this morphism. Then the short exact sequence $0 \to \mathcal{N} \to \mathcal{A}^n \to \mathcal{E} \to 0$ gives a long one

$$0 \to \mathrm{Tor}^{\mathcal{A}}_1(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \to \mathcal{N} \otimes_{\mathcal{A}} (\mathcal{A}/\mathfrak{m}) \to (\mathcal{A}/\mathfrak{m})^n \xrightarrow{\simeq} \mathcal{E} \otimes_{\mathcal{A}} (\mathcal{A}/\mathfrak{m}) \to 0$$

That the second last morphism is an isomorphism is due to the fact that $x_1, \dots, x_n$ are a basis. Since $\mathcal{E}$ is flat, $\mathrm{Tor}_1^{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) = 0$. So $\mathcal{N} \otimes_{\mathcal{A}} (\mathcal{A}/\mathfrak{m}) = 0$. Hence $\mathcal{N} = 0$ by Nakayama's lemma. Therefore $\mathcal{A}^n \to \mathcal{M}$ is an isomorphism. This proves (1). □

5.6 Flatness, openness, and dimensions of fibers I

In this section, X, Y denote complex spaces.

Theorem 5.6.1. *Let $\varphi : X \to Y$ be a holomorphic map, and let $x \in X$. Suppose that φ is flat at $x \in X$. Then the following Dimension Formula holds*

$$\dim_x X_{\varphi(x)} = \dim_x X - \dim_{\varphi(x)} Y \tag{5.6.1}$$

and φ is open at x.

Recall that X_y (where $y \in Y$) means the inverse image $\varphi^{-1}(y)$ of the reduced point $\{y\}$.

Proof-Step 1. Let $y = \varphi(x)$. We prove (5.6.1) by induction on $\dim_y Y$. The case $\dim_y Y = 0$ is obvious. Assume the theorem is proved when $\dim Y_y \leqslant n-1$, where $n \in \mathbb{Z}_+$. Assume $\dim_y Y = n$. Note that if we let Y_0 be the reduction $\mathrm{red}(Y)$, and let $X_0 = \varphi^{-1}(Y_0)$, then $\varphi : X_0 \to Y_0$ is flat at x by Thm. 5.4.9, and it suffices to prove (5.6.1) where X, Y are replaced by X_0, Y_0 (since dimensions are invariant under reductions). Therefore, by replacing X, Y with X_0, Y_0, we may well assume that Y is reduced.

By Rem. 3.10.4, we may shrink Y to a neighborhood of y and shrink X to $\varphi^{-1}(Y)$ so that there exists $\tau \in \mathscr{O}_Y$ vanishing at y which is a non zero-divisor of $\mathscr{O}_{Y,y}$. Let $Y' = \mathrm{Specan}(\mathscr{O}_Y/\tau\mathscr{O}_Y)$ and $X' = \varphi^{-1}(Y') = \mathrm{Specan}(\mathscr{O}_X/\tau\mathscr{O}_X)$. By Active lemma,

$$\dim_y Y' = \dim_y Y - 1.$$

Clearly $X_y = X'_y$. By Thm. 5.4.9, the restriction $\varphi : X' \to Y'$ is flat at x. Therefore, by case $n-1$,

$$\dim_x X_y = \dim_x X' - \dim_y Y'.$$

Since φ is flat at x, by Slicing criterion 5.5.7 (indeed, here we do not use the full power of Slicing criterion), τ is a non zero-divisor of $\mathscr{O}_{X,x}$. So by Active lemma,

$$\dim_x X' = \dim_x X - 1.$$

Dimension Formula (5.6.1) follows. □

Proof-Step 2. We now prove that φ is open at x. As in Step 1, we may well assume that Y is reduced. If Y is locally irreducible, then the openness of φ at x follows immediately from Dimension Formula (5.6.1) and Thm. 3.13.1. In the general case, we let $\nu : \widehat{Y} \to Y$ be the normalization of Y, and consider the Cartesian square

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & Z \arrow[l, "\pi"'] \arrow[d, "\psi"] \\ Y & \widehat{Y} \arrow[l, "\nu"'] \end{tikzcd}$$

Choose any neighborhood $U \subset X$ of x. The commutativity of the above diagram implies

$$\psi(\pi^{-1}(U)) \subset \nu^{-1}(\nu \circ \psi(\pi^{-1}(U))) = \nu^{-1}(\varphi \circ \pi(\pi^{-1}(U))) \subset \nu^{-1}(\varphi(U))$$

But note that as a set, Z is $\{(x, \widehat{y}) \in X \times \widehat{Y} : \varphi(x) = \nu(\widehat{y})\}$. From this observation, we see

$$\psi(\pi^{-1}(U)) = \nu^{-1}(\varphi(U)). \tag{5.6.2}$$

Note that π is finite (Prop. 2.4.11). By Thm. 5.4.9, ψ is flat at any point of the finite set $\pi^{-1}(x)$. So ψ is open at any point of $\pi^{-1}(x)$ because $\widehat{Y}$ is locally irreducible (Prop. 4.8.1). Note that $\psi(\pi^{-1}(x)) = \nu^{-1}(y)$. So $\psi(\pi^{-1}(U))$ contains a neighborhood of $\nu^{-1}(y)$, which (due to Prop. 2.4.1) can be of the form $\nu^{-1}(V)$ where $V \subset Y$ is a neighborhood of y. By (5.6.2), $\nu^{-1}(V) \subset \nu^{-1}(\varphi(U))$. Hence $V \subset \varphi(U)$ because ν is surjective. So $\varphi(U)$ contains a neighborhood of y. $\square$

We shall give a converse of Thm. 5.6.1. We first need a preparatory result on reducedness.

Proposition 5.6.2. *Choose $f \in \mathscr{O}(X)$ and $x \in X$ such that $f(x) = 0$. Assume that f is active at x, and that* $\mathrm{Specan}(\mathscr{O}_X/f\mathscr{O}_X)$ *is reduced at x. Then X is reduced at x, and hence f is a non zero-divisor of $\mathscr{O}_{X,x}$.*

Proof. By Prop. 3.4.1, we may shrink X to a neighborhood of x so that $N(f)$ is nowhere dense in X. We claim that $\sqrt{0_{X,x}} \subset f_x\sqrt{0_{X,x}}$. Then $\sqrt{0_{X,x}} = \mathfrak{m}_{X,x}\sqrt{0_{X,x}}$ and hence $\sqrt{0_{X,x}} = 0$ by Nakayama's lemma 1.2.15. Hence X is reduced at x.

To prove the claim, choose any $g \in \sqrt{0_{X,x}}$. By shrinking X further, we have $g \in \mathscr{O}(X)$ and g takes value 0 at any point of X. In particular, g vanishes on $N(f)$. So $g_x \in f_x\mathscr{O}_{X,x}$ because $\mathscr{O}_{X,x}/f_x\mathscr{O}_{X,x}$ is reduced by assumption. Shrink X so that $g = fh$ where $h \in \mathscr{O}(X)$. Since $N(f)$ is nowhere dense in X and g vanishes at every point, h also vanishes at every point. Thus $h_x \in \sqrt{0_{X,x}}$. $\square$

Theorem 5.6.3. *Let $\varphi : X \to Y$ be a holomorphic map where Y is a complex manifold. Assume that one of the following equivalent conditions holds:*

(1) Dimension Formula (5.6.1) holds for all $x \in X$.

(2) φ is open.

Choose $x \in X$, let $y = \varphi(x)$, and assume that $X_y = \varphi^{-1}(y)$ is reduced at x. Then X is reduced at x, and φ is flat at x.

Recall that the equivalence of (1) and (2) is due to Cor. 3.12.3.

Proof. By shrinking Y to a neighborhood of y and shrinking X to $\varphi^{-1}(X)$, we assume that Y is an open subset of $\mathbb{C}^n$. We prove the theorem by induction on n. The case $n = 0$ is obvious. Assume case $n-1$ is proved. Assume for simplicity that $y = 0$. Let $(z_1, \dots, z_n)$ be the standard coordinates of $\mathbb{C}^n$. Let $Y' = \mathrm{Specan}(\mathscr{O}_Y/z_1\mathscr{O}_Y)$, namely, Y' is the intersection of Y and $\mathbb{C}^{n-1} \simeq 0 \times \mathbb{C}^{n-1}$. Let $X' = \varphi^{-1}(Y') = \mathrm{Specan}(\mathscr{O}_X/z_1\mathscr{O}_X)$. Then by case $n-1$, X' is reduced at x, and $\mathscr{O}_{X',x}$ is $\mathscr{O}_{Y',0}$-flat. If we can show that z_1 is active in $\mathscr{O}_{X,x}$, then by Prop. 5.6.2, X is reduced at x, and hence z_1 is a non zero-divisor of $\mathscr{O}_{X,x}$. Therefore φ is flat at x due to Slicing criterion 5.5.7.

Let us show that z_1 is active in $\mathscr{O}_{X,x}$. In other words, we need to show that $z_1 \circ \varphi$ is active at x. But this follows immediately from Prop. 3.13.5 and the fact that $z_1 \circ \varphi : X \to \mathbb{C}$ is open. $\square$

The assumption on the reducedness of fibers is sometimes too strong for applications. For instance, all Weierstrass maps are flat, but their fibers are not necessarily reduced even when the base spaces are smooth. We will give a different criterion on flatness later (cf. Thm. 5.9.7), in which the reducedness condition in Thm. 5.6.3 is replaced by assuming that $\mathscr{O}_{X,x}$ is Cohen-Macaulay. Indeed, the rest of this chapter is devoted to proving and understanding Thm. 5.9.7.

5.7 Associated primes

We fix a Noetherian ring $\mathcal{A}$.

Definition 5.7.1. Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$. An **associated prime** of $\mathcal{E}$ is a prime ideal $\mathfrak{p}$ of the form $\mathrm{Ann}_{\mathcal{A}}(\xi)$ where $\xi \in \mathcal{E}$. The set of associated primes is denoted by $\mathrm{Ass}(\mathcal{E})$.

For instance, if X is a complex space and $\mathscr{E}$ is a coherent $\mathscr{O}_X$-module, then by Def. 2.3.3, for each $\sigma \in \mathscr{E}$ we have $\mathrm{Supp}(\mathscr{O}_X\sigma) = \mathrm{Specan}(\mathscr{O}_X/\mathscr{A}nn_{\mathscr{O}_X}(\sigma))$. So the complex subspace $\mathrm{Supp}(\mathscr{O}_X\sigma)$ is irreducible at x iff $\mathrm{Ann}_{\mathscr{O}_{X,x}}(\sigma_x)$ is prime.

Example 5.7.2. By Prop. 3.1.7, we know that if $\mathcal{A}$ is reduced, then the prime ideals in the prime decomposition of $0 \subset \mathcal{A}$ form the set $\mathrm{Ass}(\mathcal{A})$. More generally, if I is an ideal of $\mathcal{A}$ such that $I = \sqrt{I}$, and if we let $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$ be the prime decomposition of $I \subset \mathcal{A}$, then $\mathfrak{p}_1, \ldots, \mathfrak{p}_N$ are precisely the associated primes of the $\mathcal{A}$-module $\mathcal{A}/I$.

Recall that prime ideals are assumed to be not $\mathcal{A}$. And $\mathrm{Ann}_{\mathcal{A}}(\xi) = \mathcal{A}$ iff $\xi = 0$. There is a simple criterion on whether the ideal $\mathrm{Ann}_{\mathcal{A}}(\xi)$ is prime.

Lemma 5.7.3. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$ and assume that $\xi \in \mathcal{E}$ is non-zero. Then the following are equivalent.*

(1) $\mathrm{Ann}_{\mathcal{A}}(\xi)$ *is prime.*

(2) For each $a \in \mathcal{A}$, if $a\xi \neq 0$ then $\mathrm{Ann}_{\mathcal{A}}(a\xi) = \mathrm{Ann}_{\mathcal{A}}(\xi)$.

Proof. Clearly $\mathrm{Ann}_{\mathcal{A}}(a\xi) \supset \mathrm{Ann}_{\mathcal{A}}(\xi)$ in general. Suppose that $\mathrm{Ann}_{\mathcal{A}}(\xi)$ is prime. If $b \in \mathrm{Ann}_{\mathcal{A}}(a\xi)$, then $ab\xi = 0$, and hence ab belongs to the prime ideal $\mathrm{Ann}_{\mathcal{A}}(\xi)$. Since $a\xi \neq 0$, $a \notin \mathrm{Ann}_{\mathcal{A}}(\xi)$. So $b \in \mathrm{Ann}_{\mathcal{A}}(\xi)$. This proves (2).

Conversely, if $\mathrm{Ann}_{\mathcal{A}}(\xi)$ is not prime, then there is $a, b \in \mathcal{A}$ not in $\mathrm{Ann}_{\mathcal{A}}(\xi)$ such that $ab \in \mathrm{Ann}_{\mathcal{A}}(\xi)$. Namely, $a\xi \neq 0$, $b\xi \neq 0$, and $ab\xi = 0$. Then b belongs to $\mathrm{Ann}_{\mathcal{A}}(a\xi)$ but not to $\mathrm{Ann}_{\mathcal{A}}(\xi)$. □

Proposition 5.7.4. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$. Then for any non-zero $\xi \in \mathcal{E}$, there exists $a \in \mathcal{A}$ such that ($a\xi \neq 0$ and) $\mathrm{Ann}_{\mathcal{A}}(a\xi)$ is prime. In particular, if $\mathcal{E}$ is non-zero then $\mathrm{Ass}(\mathcal{E})$ is non-empty.*

Proof. The Noether property of $\mathcal{A}$ implies that any chain inside the partially ordered set $\{\mathrm{Ann}_{\mathcal{A}}(a\xi) : a \in \mathcal{A}, a\xi \neq 0\}$ must be stationary, and hence has an upper bound. By Zorn's lemma, this set contains a maximal element, which we denote by $\mathrm{Ann}_{\mathcal{A}}(a\xi)$. If $b \in \mathcal{A}$ and $ab\xi \neq 0$, then the maximality shows that $\mathrm{Ann}_{\mathcal{A}}(ab\xi) = \mathrm{Ann}_{\mathcal{A}}(\xi)$. Therefore $\mathrm{Ann}_{\mathcal{A}}(a\xi)$ is prime by Lemma 5.7.3. □

Corollary 5.7.5. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$. Let $x \in \mathcal{A}$. Then the following are equivalent.*

(1) x is a non zero-divisor of $\mathcal{E}$.

(2) x does not belong to any associated prime of $\mathcal{E}$.

Note that this property generalizes Cor. 3.1.6.

Proof. If x belongs to an associated prime $\mathrm{Ann}_{\mathcal{A}}(\xi)$ then $\xi \neq 0$ and $x\xi = 0$. So x is a zero-divisor of $\mathcal{E}$. Conversely, if x is a zero-divisor of $\mathcal{E}$, then $a\xi = 0$ for some non-zero $\xi \in \mathcal{E}$. By Prop. 5.7.4, we may find $a \in \mathcal{A}$ so that $\mathrm{Ass}_{\mathcal{A}}(a\xi)$ is prime, which clearly contains ξ. □

According to the above corollary, associated primes are useful for studying the zero-divisors of modules. But why are we interested in those $\mathrm{Ann}_{\mathcal{A}}(\xi)$ that are prime? A main reason is that any finitely generated $\mathcal{A}$-module has finitely many associated primes. To prove this important fact, we first need a lemma.

Lemma 5.7.6. *Let* $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ *be an exact sequence in* $\mathrm{Mod}(\mathcal{A})$. *Then* $\mathrm{Ass}(\mathcal{E}) \subset \mathrm{Ass}(\mathcal{E}') \cup \mathrm{Ass}(\mathcal{E}'')$.

Proof. Choose $\xi \in \mathcal{E}$ such that $\mathrm{Ann}_{\mathcal{A}}(\xi)$ is an associated prime of $\mathcal{E}$. We view $\mathcal{E}''$ as $\mathcal{E}/\mathcal{E}'$, and let $[\xi]$ be the residue class of ξ in $\mathcal{E}/\mathcal{E}''$. Then $\mathrm{Ann}_{\mathcal{A}}([\xi]) = \{a \in \mathcal{A} : a\xi \in \mathcal{E}'\}$. Clearly $\mathrm{Ann}_{\mathcal{A}}(\xi) \subset \mathrm{Ann}_{\mathcal{A}}([\xi])$. Assume that $\mathrm{Ann}_{\mathcal{A}}(\xi)$ does not belong to $\mathrm{Ass}(\mathcal{E}'')$. Then $\mathrm{Ann}_{\mathcal{A}}(\xi) \neq \mathrm{Ann}_{\mathcal{A}}([\xi])$, otherwise $\mathrm{Ann}_{\mathcal{A}}([\xi])$ would be an associated prime of $\mathcal{E}''$.

Pick $a \in \mathcal{A}$ belonging to $\mathrm{Ann}_{\mathcal{A}}([\xi])$ but not $\mathrm{Ann}_{\mathcal{A}}(\xi)$. So $a\xi$ is a non-zero element of $\mathcal{E}'$. Since $\mathrm{Ann}_{\mathcal{A}}(\xi)$ is prime, by Lemma 5.7.3, $\mathrm{Ann}_{\mathcal{A}}(\xi)$ equals $\mathrm{Ann}_{\mathcal{A}}(a\xi)$, and hence is an associated prime of $\mathcal{E}'$. □

Theorem 5.7.7. *Let* $\mathcal{E}$ *be a finitely-generated* $\mathcal{A}$*-module. Then* $\mathcal{E}$ *has finitely many associated primes.*

Proof. If $\mathcal{N}$ is a submodule of $\mathcal{E}$ and $\mathcal{N} \neq \mathcal{E}$, then by the fact that $\mathcal{E}/\mathcal{N}$ has at least one associated prime, we can find $\xi \in \mathcal{E}\backslash\mathcal{N}$ such that $\mathrm{Ann}_{\mathcal{A}}([\xi])$ is prime. Here $[\xi]$ denotes the residue class of ξ in $\mathcal{E}/\mathcal{N}$. Let $\mathcal{N}_1$ be generated by $\mathcal{N}$ and ξ. Then $\mathcal{N}_1/\mathcal{N} \simeq \mathcal{A}/\mathrm{Ann}_{\mathcal{A}}([\xi])$ has $\mathrm{Ann}_{\mathcal{A}}([\xi])$ as its only associated prime (cf. Exp. 5.7.2).

The above discussion shows that we can find a chain of submodules $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \cdots$ of $\mathcal{E}$ such that each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is equivalent to $\mathcal{A}/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Since $\mathcal{E}$ is finitely-generated and $\mathcal{A}$ is Noetherian, this chain must have finite length. So it is of the form $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subset \cdots \subsetneq \mathcal{E}_n = \mathcal{E}$. By Exp. 5.7.2, $\mathfrak{p}_i$ is the only associated prime of $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{A}/\mathfrak{p}_i$. Thus, by Lemma 5.7.6 we have

$$\mathrm{Ass}(\mathcal{E}) \subset \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\} \tag{5.7.1}$$

So $\mathcal{E}$ has finitely many associated primes. □

5.8 Depth

In this section, we fix a Noetherian local ring $\mathcal{A}$ with maximal ideal $\mathfrak{m}$.

Definition 5.8.1. For any $\mathcal{A}$-module $\mathcal{E}$, the **depth of** $\mathcal{E}$, written as $\mathrm{depth}_{\mathcal{A}}(\mathcal{E})$ or simply $\mathrm{depth}(\mathcal{E})$, is

$$\mathrm{depth}(\mathcal{E}) = \sup\{n : \text{there exists an } \mathcal{E}\text{-regular sequence } a_1, \ldots, a_n \in \mathfrak{m}\}$$

In particular, $\mathcal{E}$ has a non zero-divisor in $\mathfrak{m}$ iff $\mathrm{depth}(\mathcal{E}) > 0$.

Our starting point of analysis is the following application of associated primes. Interestingly, the statement of this lemma does not involve associated primes, but the proof actually does.

Lemma 5.8.2. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$ be finitely generated. Then the following are equivalent.*

(1) $\mathrm{depth}(\mathcal{E}) > 0$*, i.e.,* $\mathcal{E}$ *has a non zero-divisor in* $\mathfrak{m}$*.*

(2) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) = 0$.

Proof. We first observe that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E})$ is equivalent to $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{E}) \to \mathrm{Hom}_{\mathcal{A}}(\mathfrak{m}, \mathcal{E}))$. In particular, if we identify $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{E})$ naturally with $\mathcal{E}$, then

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \simeq \{\xi \in \mathcal{E} : \mathfrak{m}\xi = 0\}$$

It follows that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E})$ is non-zero iff $\mathfrak{m}$ is an associated prime of $\mathcal{E}$.

Assume that (2) does not hold. Then $\mathfrak{m}$ is an associated prime. Write $\mathfrak{m} = \mathrm{Ann}_{\mathcal{A}}(\xi)$ where $\xi \neq 0$. Then every element of $\mathfrak{m}$ annihilates ξ. So every element of $\mathfrak{m}$ is a zero-divisor of $\mathcal{E}$. So (1) does hot hold.

Assume that (2) holds. Then $\mathfrak{m}$ is not an associated prime. By Thm. 5.7.7, $\mathcal{E}$ has finitely may associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_N$, which are all proper subspaces of $\mathfrak{m}$. In the special case that $\mathcal{A}$ is a local $\mathbb{C}$-algebra (which is indeed the only situation we will encounter in out notes), $\mathfrak{m}$ does not equal any finite union of proper subspaces. So we can find $a \in \mathfrak{m}$ not inside $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_N$. Then by Cor. 5.7.5, a is a non zero-divisor of $\mathcal{E}$. Therefore (1) holds. In the general case, that $\mathfrak{m} \neq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_N$ follows from the prime avoidance property. We refer the readers to (for instance) [Vak17, Sec. 11.2] for details. □

Starting from Lemma 5.8.2, we can establish a cohomological characterization of depth. This is achieved with the help of the following fact:

Proposition 5.8.3. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$, and let $a \in \mathfrak{m}$ be a non zero-divisor of $\mathcal{E}$. Then for each $n \in \mathbb{N}$,*

$$\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/a\mathcal{E}) \simeq \mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \tag{5.8.1}$$

$$\mathrm{Tor}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/a\mathcal{E}) \simeq \mathrm{Tor}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \tag{5.8.2}$$

Proof. The short exact sequence $0 \to \mathcal{E} \xrightarrow{\times a} \mathcal{E} \to \mathcal{E}/a\mathcal{E} \to 0$ yields a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \to \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \to \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/a\mathcal{E}) \to \mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E})$$
$$\to \mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \to \mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \to \cdots$$

By Rem. 5.2.7, the Ext functor preserves multiplications on both components since Hom clearly does. This implies that the morphism $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \to \mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E})$ in the above long exact sequence, which is $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \times a)$, is simply the action of a on the $\mathcal{A}$-module $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E})$. It also equals $\mathrm{Ext}^n_{\mathcal{A}}(\times a, \mathcal{E})$, which is indeed zero since $\mathcal{A}/\mathfrak{m} \xrightarrow{\times a} \mathcal{A}/\mathfrak{m}$ is zero. So $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \times a)$ is zero, and similarly

$\mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \times a)$ is zero. Therefore, by the exactness of the above sequence, the connecting map $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/a\mathcal{E}) \to \mathrm{Ext}^{n+1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E})$ must be an isomorphism. (5.8.1) is proved. (5.8.2) can be proved in a similar way. □

Now we can generalize Lemma 5.8.2 as follows.

Lemma 5.8.4. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$ be finitely generated. Let $k \in \mathbb{N}$, and let $a_1, \dots, a_k \in \mathfrak{m}$ be an $\mathcal{E}$-regular sequence. Then the following are equivalent.*

(1) $a_1, \dots, a_k$ can be extended to an $\mathcal{E}$-regular sequence $a_1, \dots, a_k, a_{k+1} \in \mathfrak{m}$.

(2) $\mathrm{Ext}^k_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) = 0$.

Proof. By Prop. 5.8.3,

$$\begin{aligned} &\mathrm{Ext}^k_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) \simeq \mathrm{Ext}^{k-1}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/a_1\mathcal{E}) \simeq \mathrm{Ext}^{k-2}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/(a_1\mathcal{E} + a_2\mathcal{E}))\\ \simeq{}& \cdots \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}/(a_1\mathcal{E} + \cdots + a_k\mathcal{E})) \end{aligned}$$

By Lemma 5.8.2, the last term is zero if and only if $\mathcal{E}/(a_1\mathcal{E} + \cdots + a_k\mathcal{E})$ has a non zero-divisor in $\mathfrak{m}$. This is equivalent to saying that (1) holds. □

Theorem 5.8.5. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$ be finitely generated. Then $\mathrm{depth}(\mathcal{E})$ is the smallest $n \in \mathbb{N}$ such that $\mathrm{Ext}^k_{\mathcal{A}}(\mathcal{A}/\mathfrak{m}, \mathcal{E}) = 0$ for all $k < n$.*

Proof. Apply Lemma 5.8.4 to any longest $\mathcal{E}$-regular sequence in $\mathfrak{m}$. □

Corollary 5.8.6. *Let $\mathcal{E} \in \mathrm{Mod}(\mathcal{A})$ be finitely generated, and let $a_1, \dots, a_k \in \mathfrak{m}$ be an $\mathcal{E}$-regular sequence. Then*

$$\mathrm{depth}(\mathcal{E}) = \mathrm{depth}\big(\mathcal{E}/(a_1\mathcal{E} + \cdots + a_k\mathcal{E})\big) + k$$

Namely, $a_1, \dots, a_k$ can be extended to an $\mathcal{E}$-regular sequence in $\mathfrak{m}$ of length $\mathrm{depth}(\mathcal{E})$.

Proof. By induction, it suffices to prove the case $k = 1$. Then the formula follows from Prop. 5.8.3 and Thm. 5.8.5. □

Theorem 5.8.7. *Let X be a complex space and let $x \in X$. Let $\mathscr{E}, \mathscr{M}$ be coherent $\mathscr{O}_X$-modules. Assume that*

$$\mathrm{depth}_{\mathscr{O}_{X,x}}(\mathscr{E}_x) = n, \qquad \dim_x \mathrm{Supp}(\mathscr{M}) = m.$$

Then we have

$$\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x) = 0 \qquad \forall k < n - m. \tag{5.8.3}$$

This theorem (as well as the subsequent corollary) also holds for any finitely-generated modules of Noetherian local rings. And the proof for the general case is similar to the one below. (See [Vak17, Sec. 26.1].) Since we have established dimension theory only for analytic local $\mathbb{C}$-algebras, we shall focus on this special case, which is clearly sufficient for our applications.

Proof. First note that if we have an exact sequence of morphisms of coherent $\mathscr{O}_X$-modules $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$, then we clearly have

$$\mathrm{Supp}(\mathscr{M}) = \mathrm{Supp}(\mathscr{M}') \cup \mathrm{Supp}(\mathscr{M}'')$$

as analytic subsets of X. (Namely, for each $p \in X$, $\mathscr{M}_p = 0$ iff $\mathscr{M}'_p = \mathscr{M}''_p = 0$.) Therefore, by Prop. 3.10.7, we have

$$\dim_x \mathrm{Supp}(\mathscr{M}) = \max\left\{ \dim_x \mathrm{Supp}(\mathscr{M}'), \dim_x \mathrm{Supp}(\mathscr{M}'')\right\} \tag{5.8.4}$$

Recall that the germs of coherent $\mathscr{O}_X$-modules at x are equivalent to finitely-generated $\mathscr{O}_{X,x}$-modules (Thm. 2.2.2). We now prove the theorem by induction on $\dim_x \mathrm{Supp}(\mathscr{M})$. (5.8.3) clearly holds whenever $\dim_x \mathrm{Supp}(\mathscr{M}) = -\infty$, i.e. when $\mathscr{M}_x = 0$. It also holds when $\mathscr{M}_x = \mathbb{C} = \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$ due to Thm. 5.8.5. Thus, if $\dim_x \mathrm{Supp}(\mathscr{M}) = 0$, then x is a single point of $\mathrm{Supp}(\mathscr{M})$. Hence $\mathfrak{m}_{X,x}^l\mathscr{M}_x = 0$ for some $l \in \mathbb{Z}_+$ by Nullstellensatz (Rem. 2.10.4-3). Then an induction on l and the exact sequence

$$\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathfrak{m}_{X,x}\mathscr{M}_x, \mathscr{E}_x) \to \mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x) \to \mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x \otimes_{\mathscr{O}_{X,x}} \mathbb{C}, \mathscr{E}_x)$$

proves (5.8.3) whenever $\dim_x \mathrm{Supp}(\mathscr{M}) = 0$.

Now suppose that the theorem holds whenever $\dim_x \mathrm{Supp}(\mathscr{M}) < m$. Assume $\dim_x \mathrm{Supp}(\mathscr{M}) = m$. As in the proof of Prop. 5.7.7, we may shrink X to a neighborhood of x and find a chain of coherent $\mathscr{O}_X$-modules $0 = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \cdots \subset \mathscr{M}_l = \mathscr{M}$ such that for each $1 \leqslant i \leqslant l$, $\mathscr{M}_i/\mathscr{M}_{i-1}$ is equivalent to $\mathscr{O}_X/\mathscr{P}_i$ where $\mathscr{P}_i$ is a coherent ideal of $\mathscr{O}_X$ such that $\mathscr{P}_{i,x}$ is prime. Therefore, by (5.8.4),

$$\dim_x \mathrm{Supp}(\mathscr{M}) = \sup_i \dim_x N(\mathscr{P}_i).$$

So $\dim_x N(\mathscr{P}_i) \leqslant m$ for all i. Since we have an exact sequence

$$\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_{i-1,x}, \mathscr{E}_x) \to \mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_{i,x}, \mathscr{E}_x) \to \mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/\mathscr{P}_{i,x}, \mathscr{E}_x)$$

if we can show that $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/\mathscr{P}_{i,x}, \mathscr{E}_x) = 0$ for all i, then by induction on i, we obtain $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x) = 0$.

Therefore, it suffices to prove (5.8.3) in the special case that $\mathscr{M} = \mathscr{O}_X/\mathscr{P}$ where $\mathscr{P}$ is a coherent ideal of $\mathscr{O}_X$, $\mathscr{P}_x$ is prime, and $\dim_x N(\mathscr{P}) = m \geqslant 1$. Shrink

X further so that we can choose $a \in \mathscr{O}(X)$ with $a(x) = 0$ such that the germ $a_x \in \mathfrak{m}_{X,x}$ is not in $\mathscr{P}_x$. So a_x is a non zero-divisor of $\mathscr{M}_x$. Thus we have a short exact sequence

$$0 \to \mathscr{M}_x \xrightarrow{\times a_x} \mathscr{M}_x \to \mathscr{M}_x/a_x\mathscr{M}_x \to 0$$

which gives rise to a long one

$$\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x) \to \mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x) \to \mathrm{Ext}^{k+1}_{\mathscr{O}_{X,x}}(\mathscr{M}_x/a_x\mathscr{M}_x, \mathscr{E}_x)$$

The support of $\mathscr{M}_x/a_x\mathscr{M}_x$ has dimension $m-1$ at x by Active lemma 3.10.3. Assume $k < n-m$. Then by case $m-1$, $\mathrm{Ext}^{k+1}_{\mathscr{O}_{X,x}}(\mathscr{M}_x/a_x\mathscr{M}_x, \mathscr{E}_x)$ is zero. So the endomorphism $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\times a_x, \mathscr{E}_x)$ on $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x)$ is surjective. By Rem. 5.2.7, this endomorphism is the multiplication of a_x on the $\mathscr{O}_{X,x}$-module $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x)$. Since $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x)$ is finitely generated (because we can choose a finite-rank free resolution of $\mathscr{M}_x$), and since $a_x \in \mathfrak{m}_{X,x}$, $\mathrm{Ext}^k_{\mathscr{O}_{X,x}}(\mathscr{M}_x, \mathscr{E}_x)$ must be zero by Nakayama's lemma 1.2.15. □

We shall only use the following very special case of Thm. 5.8.7:

Corollary 5.8.8. *Let X be a complex space and $x \in X$. Let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module. Then*

$$\mathrm{depth}(\mathscr{E}_x) \leqslant \inf \left\{ \dim \mathscr{O}_{X,x}/\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}(\mathscr{E}_x) \right\}$$

In particular, $\mathrm{depth}(\mathscr{E}_x) \leqslant \dim_x X$.

Proof. Choose any associated prime $\mathfrak{p}$ of $\mathscr{E}_x$. Let $m = \dim \mathscr{O}_{X,x}/\mathfrak{p}$ and $n = \mathrm{depth}(\mathscr{E}_x)$. If $n > m$, then $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/\mathfrak{p}, \mathscr{E}_x) = 0$ by Thm. 5.8.7. But

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/\mathfrak{p}, \mathscr{E}_x) \simeq \mathrm{Ker}\big(\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}, \mathscr{E}_x) \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathfrak{p}, \mathscr{E}_x)\big) \\ \simeq &\{\xi \in \mathscr{E}_x : \mathfrak{p}\xi = 0\} \end{aligned}$$

and the set $\{\xi \in \mathscr{E}_x : \mathfrak{p}\xi = 0\}$ is non-zero by the very definition of associate primes. This is impossible. So $n \leqslant m$. □

5.9 Flatness, openness, and dimensions of fibers II: Cohen-Macaulay

In this section, X is a complex space. Note that if $x \in X$, then $\mathrm{depth}(\mathscr{O}_{X,x}) \leqslant \dim_x X$ by Cor. 5.8.8.

Definition 5.9.1. Let $x \in X$. We say that X is **Cohen-Macaulay at** x or that $\mathscr{O}_{X,x}$ is a **Cohen-Macaulay ring**, if

$$\operatorname{depth}(\mathscr{O}_{X,x}) = \dim_x X.$$

If X is Cohen-Macaulay at every $x \in X$, we say that X is a **Cohen-Macaulay complex space**.

Example 5.9.2. If X is smooth of dimension n at x, then there clearly exists an $\mathscr{O}_{X,x}$-regular sequence in $\mathfrak{m}_{X,x}$ of length n. (Take the coordinate functions.) Therefore, complex manifolds are Cohen-Macaulay.

Example 5.9.3. If $\dim_x X = 0$ then X is clearly Cohen-Macaulay at x.

Proposition 5.9.4. *Choose $f_1, \dots, f_n \in \mathscr{O}(X)$ vanishing at x, and assume that their germs $f_{1,x}, \dots, f_{n,x}$ form an $\mathscr{O}_{X,x}$-regular sequence. Then the following are equivalent.*

(1) X is Cohen-Macaulay at x.

(2) $Y = \operatorname{Specan}\big(\mathscr{O}_X/(f_1\mathscr{O}_X + \cdots + f_n\mathscr{O}_X)\big)$ is Cohen-Macaulay at x.

Proof. By Active lemma 3.10.3, $\dim_x X = \dim_x Y + n$. Then the equivalence of the two statements follows immediately from Cor. 5.8.6. □

Proposition 5.9.5. *Suppose that X is Cohen-Macaulay at x. For each associated prime $\mathfrak{p}$ of $\mathscr{O}_{X,x}$, we have $\dim \mathscr{O}_{X,x}/\mathfrak{p} = \dim_x X$.*

Proof. Clearly, in general we have $\dim \mathscr{O}_{X,x}/\mathfrak{p} = \dim_x N(\mathfrak{p}) \leqslant \dim_x X$. That we have " $\geqslant$ " when $\mathscr{O}_{X,x}$ is Cohen-Macaulay is due to Cor. 5.8.8. □

The miracle of Cohen-Macaulayness lies in the following fact:

Theorem 5.9.6. *Let $f \in \mathscr{O}(X)$ vanish at x. Let $Z = \operatorname{Specan}(\mathscr{O}_X/f\mathscr{O}_X)$. Suppose that X is Cohen-Macaulay at x. Then the following are equivalent.*

(1) f is a non zero-divisor of $\mathscr{O}_{X,x}$.

(2) $\dim_x X = \dim_x Z + 1$.

If one of them holds, then Z is Cohen-Macaulay at x.

Proof. (1)⇒(2): By Active lemma 3.10.3. Then by Prop. 5.9.4, Z is Cohen-Macaulay at x.

Not (1) ⇒ Not (2): Assume that the germ f_x is a zero-divisor of $\mathscr{O}_{X,x}$. Then by Cor. 5.7.5, f_x belongs to some associated prime $\mathfrak{p}$ of $\mathscr{O}_{X,x}$. We shrink X to a neighborhood of x so that $\mathfrak{p} = \mathscr{P}_x$ for a coherent ideal $\mathscr{P}$ of $\mathscr{O}_X$, and that $f \in \mathscr{P}(X)$. Let $n = \dim_x X$. Then by Prop. 5.9.5, $\dim_x N(\mathscr{P}) = n$. Hence

$$\dim_x Z = \dim_x N(f) \geqslant \dim_x N(\mathscr{P}) = n.$$

So (2) does not hold. □

We are now able to prove the following remarkable result which is parallel to Thm. 5.6.3.

Theorem 5.9.7 (Miracle flatness theorem). *Let $\varphi : X \to Y$ be a holomorphic map where Y is a complex manifold. Assume that X is Cohen-Macaulay at $x \in X$. Then the following are equivalent.*

(1) The following Dimension Formula holds

$$\dim_x X_{\varphi(x)} = \dim_x X - \dim_{\varphi(x)} Y \tag{5.9.1}$$

(2) φ is flat at x.

Moreover, if any of them holds, then $X_{\varphi(x)} = \varphi^{-1}(\varphi(x))$ is Cohen-Macaulay at x.

Proof. That (2)⇒(1) is due to Thm. 5.6.1. To prove (1)⇒(2), we may assume that Y is an open subset of $\mathbb{C}^n$, and that $\varphi(x) = 0$. Then φ is represented by $(f_1, \dots, f_n) \in \mathscr{O}(X)^n$. For each $1 \leqslant k \leqslant n$, let

$$Z^k = \mathrm{Specan}(\mathscr{O}_X/(f_1\mathscr{O}_X + \cdots + f_k\mathscr{O}_X)).$$

Set $Z^0 = X$. Then $Z^k = \mathrm{Specan}(\mathscr{O}_{Z^{k-1}}/f_k\mathscr{O}_{Z^{k-1}})$. So by the definition of dimensions (Def. 3.9.1),

$$\dim_x Z^k + 1 \geqslant \dim_x Z^{k-1}.$$

Thus we have (noting that $Z^n = \varphi^{-1}(0) = X_{\varphi(x)}$)

$$\dim_x Z^n + n \geqslant \dim_x Z^0 \qquad \text{equivalently} \qquad \dim_x X_{\varphi(x)} + n \geqslant \dim_x X$$

and "$\geqslant$" becomes "$=$" whenever

$$\dim_x Z^k + 1 = \dim_x Z^{k-1} \qquad \forall 1 \leqslant k \leqslant n \tag{5.9.2}$$

Since we assume (1) is true, we have (5.9.2). By assumption, Z^0 is Cohen-Macaulay at x. Suppose we have proved that Z^{k-1} is Cohen-Macaulay at x where $1 \leqslant k \leqslant n$, then by (5.9.2) and Thm. 5.9.6, f_k is a non zero-divisor of $\mathscr{O}_{Z^{k-1},x}$ and Z^k is Cohen-Macaulay at x. Therefore, by induction, we see that the germs $f_{1,x}, \dots, f_{n,x}$ form an $\mathscr{O}_{X,x}$-regular sequence, and $Z^n = X_{\varphi(x)}$ is Cohen-Macaulay at x. Hence φ is flat at x by Cor. 5.5.10. □

Corollary 5.9.8. *Assume that X is a Cohen-Macaulay complex space (e.g. a complex manifold) and Y is a complex manifold. Let $\varphi : X \to Y$ be a holomorphic map. Then the following are equivalent:*

(1) Dimension Formula (5.9.1) holds for all $x \in X$.

(2) φ is open.

(3) φ is flat.

Proof. By Cor. 3.12.3 and Thm. 5.9.7. □

Chapter 6

Cohomology and base change

6.1 Sheaf cohomology and higher direct images

Let X be a ringed space with structure sheaf $\mathscr{O}_X$. Let $\mathrm{Mod}(\mathscr{O}_X)$ and $\mathrm{Coh}(\mathscr{O}_X)$ (or simply $\mathrm{Mod}(X)$ and $\mathrm{Coh}(X)$) be respectively the category of $\mathscr{O}_X$-modules and the category of coherent $\mathscr{O}_X$-modules. Note that if $\mathscr{O}_X$ equals $\mathbb{Z}$, the sheaf of locally constant $\mathbb{Z}$-valued functions, then $\mathrm{Mod}(\mathscr{O}_X)$ is the category of sheaves (of abelian groups) on the topological space X.

6.1.1 $\mathrm{Mod}(\mathscr{O}_X)$ has enough injectives

Our aim of this section is to construct various derived functors from $\mathrm{Mod}(\mathscr{O}_X)$. The first step is to show that $\mathrm{Mod}(\mathscr{O}_X)$ has enough injectives. (In general, $\mathrm{Mod}(\mathscr{O}_X)$ need not have enough projectives.) We first note the following elementary fact:

Remark 6.1.1. If $(\mathscr{E}_\alpha)_\alpha$ is a family of $\mathscr{O}_X$-modules, we can define the **direct product** $\prod_\alpha \mathscr{E}_\alpha$ in a natural way, i.e. whose space of sections on each open subset $U \subset X$ is $\prod_\alpha \mathscr{E}_\alpha(U)$. (It is in general not true that the stalk of the direct product equals the direct product of stalks.)

If $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X)$, then a morphism $\varphi : \mathscr{F} \to \prod_\alpha \mathscr{E}_\alpha$ is equivalently a collection of morphism $\mathscr{F} \to \mathscr{E}_\alpha$ for all α. Namely, we have a natural equivalence

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \prod_\alpha \mathscr{E}_\alpha) \simeq \prod_\alpha \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}_\alpha) \tag{6.1.1}$$

From (6.1.1), it is clear that if each $\mathscr{E}_\alpha$ is injective, then $\prod_\alpha \mathscr{E}_\alpha$ is injective. $\square$

Definition 6.1.2. Let $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$. We view each stalk $\mathscr{E}_x$ as an $\mathscr{O}_X$-module: if $U \subset X$ is open and $f \in \mathscr{O}_X(U)$, and if $s \in \mathscr{E}_x$, then $fs = 0$ if $x \neq U$ and $fs = f_x s$ if

$x \in U$. Then the $\mathscr{O}_X$-module

$$\mathrm{Gode}(\mathscr{E}) = \prod_{x \in X} \mathscr{E}_x \tag{6.1.2}$$

is called the **Godement sheaf** of $\mathscr{E}$. More explicitly, if $U \subset X$ is open, then

$$\mathrm{Gode}(\mathscr{E})(U) = \prod_{x \in U} \mathscr{E}_x \tag{6.1.3}$$

namely, a section $s \in \mathrm{Gode}(\mathscr{E})(U)$ is equivalently a function on U whose value at each $x \in X$ is an element of $\mathscr{E}$.

We have an obvious monomorphism $\mathscr{E} \hookrightarrow \mathrm{Gode}(\mathscr{E})$ sending each $s \in \mathscr{E}(U)$ to the function on U whose value at each $x \in U$ is the stalk s_x. □

Proposition 6.1.3. *$\mathrm{Mod}(\mathscr{O}_X)$ has enough injectives, namely, for each $\mathscr{O}_X$-module $\mathscr{E}$, there is an injective object $\mathscr{E}^0 \in \mathrm{Mod}(\mathscr{O}_X)$ and a monomorphism $\mathscr{E} \hookrightarrow \mathscr{E}^0$.*

Proof. For each $x \in X$, we choose a monomorphism $\mathscr{E}_x \hookrightarrow \mathscr{I}_x$ of $\mathscr{O}_{X,x}$-modules such that $\mathscr{I}_x$ is injective, which exists due to Prop. 5.3.3. Set

$$\mathscr{E}^0 = \prod_{x \in X} \mathscr{I}_x$$

and define the monomorphism $\mathscr{E} \hookrightarrow \mathscr{E}^0$ to be the composition

$$\mathscr{E} \hookrightarrow \prod_{x \in X} \mathscr{E}_x \hookrightarrow \prod_{x \in X} \mathscr{I}_x.$$

To show that $\mathscr{E}^0$ is injective, by Rem. 6.1.1, it suffices to show that each $\mathscr{I}_x$ is an injective $\mathscr{O}_X$-module. But this is clear because $\mathscr{I}_x$ is injective in $\mathrm{Mod}(\mathscr{O}_{X,x})$, and

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}_x) \simeq \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{I}_x)$$

for each $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X)$. □

6.1.2 $H^q(X, \mathscr{E})$ and $R^q\varphi_*(\mathscr{E})$

Thanks to Prop. 6.1.3, we can make the following definition:

Definition 6.1.4. Let $H^\bullet(X, -)$ be the right derived functor (cf. Thm. 5.2.2) of the functor $H^0(X, -)$ from $\mathrm{Mod}(\mathscr{O}_X)$ to $\mathrm{Mod}(\mathscr{O}(X))$, sending $\mathscr{E}$ to $H^0(X, \mathscr{E}) = \mathscr{E}(X)$. For each $q \in \mathbb{N}$, $H^q(X, \mathscr{E})$ is called the q**-th cohomology group** of X with coefficients in $\mathscr{E}$. As usual, we set $H^q(X, \mathscr{E}) = 0$ if $q < 0$.

Note that if $\mathscr{O}_X$ is a sheaf of $\mathbb{F}$-algebras where $\mathbb{F}$ is a field, then $H^q(X, \mathscr{E})$ is a vector space over $\mathbb{F}$. □

Remark 6.1.5. If $U \subset X$ is open and $V \subset U$ is open, then as $H^\bullet(U, -)$ is a universal δ-functor, the natural morphism $H^0(U, -) \to H^0(V, -)$ defined by restriction extends uniquely to a morphism of δ-functors $H^\bullet(U, -) \to H^\bullet(V, -)$. (See Def. 5.1.5.)

Namely (Def. 5.1.4), we have a unique collection of morphisms $H^q(U, \mathscr{E}) \to H^q(V, \mathscr{E})$ for all $q \in \mathbb{N}$ and $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$ satisfying the following: If $\mathscr{E} \to \mathscr{F}$ is a morphism in $\mathrm{Mod}(\mathscr{O}_X)$, the following diagram commutes

$$\begin{array}{ccc} H^q(U,\mathscr{E}) & \longrightarrow & H^q(U,\mathscr{F}) \\ \downarrow & & \downarrow \\ H^q(V,\mathscr{E}) & \longrightarrow & H^q(V,\mathscr{F}) \end{array} \tag{6.1.4}$$

If $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ is a short exact sequence in $\mathrm{Mod}(\mathscr{O}_X)$, the following diagram commutes

$$\begin{array}{ccc} H^q(U,\mathscr{G}) & \xrightarrow{\delta} & H^{q+1}(U,\mathscr{E}) \\ \downarrow & & \downarrow \\ H^q(V,\mathscr{G}) & \xrightarrow{\delta} & H^{q+1}(V,\mathscr{E}) \end{array} \tag{6.1.5}$$

□

Definition 6.1.6. Let $\varphi : X \to Y$ be a morphism of ringed spaces. Note that the direct image functor $\varphi_* : \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_Y)$ sending $\mathscr{E}$ to $\varphi_*\mathscr{E}$ is left exact. Note that for each open $V \subset Y$, we have

$$\varphi_*\mathscr{E}(V) = \mathscr{E}(\varphi^{-1}(V)) = H^0(\varphi^{-1}(V), \mathscr{E})$$

by our notations. For each $q \in \mathbb{N}$, the sheafification of the presheaf of $\mathscr{O}_Y$-modules associating to each open set $V \subset Y$ the $\mathscr{O}_Y(V)$-module

$$(R^q\varphi_*(\mathscr{E}))^{\mathrm{pre}}(V) = H^q(\varphi^{-1}(V), \mathscr{E})$$

is denoted by $R^q\varphi_*(\mathscr{E})$ and called the **q-th higher direct image** of φ.

Clearly $R^0\varphi_*(\mathscr{E}) = \varphi_*\mathscr{E}$, and the stalk of $R^q\varphi_*(\mathscr{E})$ at each $y \in Y$ is

$$R^q\varphi_*(\mathscr{E})_y = \varinjlim_{V \ni y} H^q(\varphi^{-1}(V), \mathscr{E}) \tag{6.1.6}$$

where the direct limit is over all neighborhoods of y. □

Since $H^\bullet(\varphi^{-1}(V), -)$ is a δ-functor, for each short exact sequence $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ in $\mathrm{Mod}(\mathscr{O}_X)$ we have a long exact sequence

$$0 \to H^0(\varphi^{-1}(V), \mathscr{E}) \to H^0(\varphi^{-1}(V), \mathscr{F}) \to H^0(\varphi^{-1}(V), \mathscr{G})$$

$$\begin{aligned}&\xrightarrow{\delta} H^1(\varphi^{-1}(V),\mathscr{E}) \to H^1(\varphi^{-1}(V),\mathscr{F}) \to H^1(\varphi^{-1}(V),\mathscr{G})\\&\xrightarrow{\delta} H^2(\varphi^{-1}(V),\mathscr{E}) \to \cdots\end{aligned} \tag{6.1.7}$$

By Rem. 6.1.5, if $W \subset V$ is open, we have a morphism of exact sequences from (6.1.7) to a similar one about $\varphi^{-1}(W)$. Therefore, since direct limit preserves exactness, we obtain an exact sequence in $\mathrm{Mod}(\mathscr{O}_Y)$

$$0 \to \varphi_*\mathscr{E} \to \varphi_*\mathscr{F} \to \varphi_*\mathscr{G} \xrightarrow{\delta} R^1\varphi_*\mathscr{E} \to R^1\varphi_*\mathscr{F} \to R^1\varphi_*\mathscr{G} \xrightarrow{\delta} R^2\varphi_*\mathscr{E} \to \cdots \tag{6.1.8}$$

since its stalk at each $y \in Y$ is the direct limit of (6.1.7) over all neighborhoods V of y.

Proposition 6.1.7. *$(R^\bullet\varphi_*,\delta)$ is the universal δ-functor (cf. Def. 5.1.5) from $\mathrm{Mod}(\mathscr{O}_X)$ to $\mathrm{Mod}(\mathscr{O}_Y)$ extending φ_*. Therefore, it is the right derived functor of φ_*.*

Proof. Since $H^\bullet(\varphi^{-1}(V),-)$ is a δ-functor, a morphism of short exact squences in $\mathrm{Mod}(\mathscr{O}_X)$

$$\begin{array}{ccccccccc}0 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{E}' & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{G}' & \longrightarrow & 0\end{array} \tag{6.1.9}$$

gives rise to a commutative diagram

$$\begin{array}{ccc}H^q(\varphi^{-1}(V),\mathscr{G}) & \xrightarrow{\delta} & H^{q+1}(\varphi^{-1}(V),\mathscr{E})\\ \downarrow & & \downarrow\\ H^q(\varphi^{-1}(V),\mathscr{G}') & \xrightarrow{\delta} & H^{q+1}(\varphi^{-1}(V),\mathscr{E}')\end{array}$$

for each $q \in \mathbb{N}$ and open $V \subset Y$. By passing to direct limits, we obtain a commutative diagram

$$\begin{array}{ccc}R^q\varphi_*(\mathscr{G}) & \xrightarrow{\delta} & R^{q+1}\varphi_*(\mathscr{E})\\ \downarrow & & \downarrow\\ R^q\varphi_*(\mathscr{G}') & \xrightarrow{\delta} & R^{q+1}\varphi_*(\mathscr{E}')\end{array}$$

This verifies that $(R^\bullet\varphi_*,\delta)$ is a δ-functor.

To show that it is universal, by Thm. 5.1.6 and that $\mathrm{Mod}(\mathscr{O}_X)$ has enough injectives, it suffices to show that $R^q\varphi_*(\mathscr{I})$ is zero whenever $q > 0$ and $\mathscr{I}$ is an injective object in $\mathrm{Mod}(\mathscr{O}_X)$. But this is obvious since $H^{>0}$ vanishes on injective objects (as right derived functors do, cf. Thm. 5.2.2), which shows that $H^{>0}(\varphi^{-1}(V),\mathscr{I}) = 0$ for every open $V \subset Y$. $\square$

6.2 Čech cohomology

Fix a ringed topological space $(X, \mathscr{O}_X)$. In this section, we introduce Čech cohomology as an easier way to compute sheaf cohomology. We follow mainly the approach of [Dem]. Čech cohomology is equivalent to sheaf cohomology in most cases. Čech cohomology is easier to compute, while sheaf cohomology is more functorial and can easily explain why other cohomology theories (e.g. de Rham cohomology, Dolbeault cohomology) agree with Čech cohomolgy in explicit situations.

Most part of this section is self-contained in the sense that it does not assume the knowledge of sheaf cohomology or derived functors. Indeed, it is recommended that the readers read this section before they read the more abstract approach of sheaf cohomology.

6.2.1 Čech cohomology $\check{H}^q(\mathfrak{U}, \mathscr{E})$

Fix an open cover $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ of X. For each $\alpha_0, \alpha_1, \ldots, \alpha_q \in I$, set

$$U_{\alpha_0\alpha_1\cdots\alpha_q} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}$$

Definition 6.2.1. For each $q \in \mathbb{N}$, the **(alternate) Čech q-cochain** is the $\mathscr{O}(X)$-module

$$C^q(\mathfrak{U}, \mathscr{E}) = \Big\{(c_{\alpha_0\alpha_1\cdots\alpha_q}) \in \prod_{(\alpha_0,\ldots,\alpha_q)\in I^{q+1}} \mathscr{E}(U_{\alpha_0\alpha_1\cdots\alpha_q}) : \\ c_{\alpha_0\cdots\alpha_q} = \mathrm{sgn}(\sigma)\cdot c_{\alpha_{\sigma(0)}\cdots\alpha_{\sigma(q)}} \text{ for all } \sigma \in \mathrm{Aut}\{0,1,\ldots,q\}\Big\}$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.[1] The q-th **coboundary operator** $\delta = \delta^q : C^q(\mathfrak{U}, \mathscr{E}) \to C^{q+1}(\mathfrak{U}, \mathscr{E})$ is an $\mathscr{O}(X)$-module morphism defined by

$$(\delta^q c)_{\alpha_0\cdots\alpha_q} = \sum_{0\leqslant j\leqslant q+1} (-1)^j c_{\alpha_0\cdots\widehat{\alpha_j}\cdots\alpha_{q+1}}\big|_{U_{\alpha_0\cdots\alpha_q}} \tag{6.2.1}$$

We set $C^q(\mathfrak{A}, \mathscr{E}) = 0$ if $q < 0$.

It is not hard to check that $d^{q+1}d^q = 0$. So $(C^\bullet(\mathfrak{U}, \mathscr{E}), \delta)$ is a complex. Its cohomology

$$H^\star(\mathfrak{U}, \mathscr{E}) := \mathcal{H}^\star(C^\bullet(\mathfrak{U}, \mathscr{E}))$$

is called the **Čech cohomology groups** of $\mathfrak{U}$ with coefficients in $\mathscr{E}$. □

[1] One can also define Čech cohomology without assuming the alternate condition. The cohomology theory one gets is equivalent to the alternate one. See [Dem] Sec. IV.5.D.

Remark 6.2.2. The boundary operators can be defined in a similar way as the differential operators of differential forms. Choose any $s \in \mathscr{E}(U_{\alpha_0\cdots\alpha_q})$. We define

$$s \cdot d\alpha_0 d\alpha_1 \cdots d\alpha_q \in C^q(\mathfrak{U}, \mathscr{E})$$

to be

$$\begin{aligned} s \cdot d\alpha_0 \cdots d\alpha_q|_{\alpha_{\sigma(0)}\cdots\alpha_{\sigma(q)}} &= \mathrm{sgn}(\sigma) \cdot s \qquad \in \mathscr{E}(U_{\sigma(0)} \cdots \alpha_{\sigma(q)}) \\ s \cdot d\alpha_0 \cdots d\alpha_q|_{\beta_0\cdots\beta_q} &= 0 \qquad (\beta_0, \dots, \beta_q \text{ is not a permutation of } \alpha_0, \dots, \alpha_q) \end{aligned} \tag{6.2.2}$$

It is helpful to view $s \cdot d\alpha_0 d\alpha_1 \cdots d\alpha_q$ as the multiplication of $s \in \mathscr{E}(U_{\alpha_0\cdots\alpha_q})$ and $d\alpha_0 \cdots d\alpha_q \in C^q(\mathfrak{U}, \mathscr{O}_X)$.

Clearly, for each permutation $\sigma \in \mathrm{Aut}\{0, 1, \dots, q\}$,

$$s \cdot d\alpha_0 \cdots d\alpha_q = \mathrm{sgn}(\sigma) s \cdot d\alpha_{\sigma(0)} \cdots d\alpha_{\sigma(q)}.$$

And $\delta^q : C^q(\mathfrak{U}, \mathscr{E}) \to C^{q+1}(\mathfrak{U}, \mathscr{E})$ can be defined to be the $\mathscr{O}(X)$-module morphism determined by

$$\delta(s \cdot d\alpha_0 \cdots d\alpha_q) = \sum_{\beta \in I} s \cdot d\beta d\alpha_0 \cdots d\alpha_q. \tag{6.2.3}$$

It is well defined, namely, it is compatible with the expression of $\delta(s \cdot d\alpha_{\sigma(0)} \cdots d\alpha_{\sigma(q)})$ for any permutation σ of $\{0, 1, \dots, q\}$. □

6.2.2 Partition of unity and vanishing of $\check{H}^{>0}(\mathfrak{U}, \mathscr{E})$

Fix an open cover $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ of X. An important feature of Čech cohomology is that $\check{H}^{>0}(\mathfrak{U}, -)$ vanishes when one can construct partition of unity. More precisely:

Definition 6.2.3. A **partition of unity in $\mathscr{O}_X$ subordinated to** $\mathfrak{U}$ is a collection $(\psi_\alpha)_{\alpha \in I}$ where each $\psi_\alpha \in \mathscr{O}(X)$, satisfying the following conditions:

- $\mathrm{Supp}(\psi_\alpha) \subset U_\alpha$ for each $\alpha \in I$.
- The family of subset $(\mathrm{Supp}(\psi_\alpha))_{\alpha \in I}$ is **locally finite**, namely, each $x \in X$ is contained in a neighborhood which intersects only finitely many members of $(\mathrm{Supp}(\psi_\alpha))_{\alpha \in I}$.
- $\sum_{\alpha \in I} \psi_\alpha(x) = 1$ for each $x \in X$.

Proposition 6.2.4. *Suppose that $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$ is also an $\mathscr{R}_X$-module where $\mathscr{R}_X$ is a sheaf of rings on X (possibly different from $\mathscr{O}_X$). Suppose that there is a partition of unity in $\mathscr{R}_X$ subordinate to $\mathfrak{U}$. Then $\check{H}^{>0}(\mathfrak{U}, \mathscr{E}) = 0$.*

Proof. Let $(\psi_\alpha)_{\alpha\in I}$ be a partition in $\mathscr{R}_X$ subordinate to $\mathfrak{U}$. For each $q \in \mathbb{Z}_+$, define an $\mathscr{R}_X(X)$-module morphism

$$w^q : C^q(\mathfrak{U}, \mathscr{E}) \to C^{q-1}(\mathfrak{U}, \mathscr{E})$$
$$w^q(s \cdot d\alpha_0 \cdots d\alpha_q) = \sum_{j=0}^{q} (-1)^j \psi_{\alpha_j} s \cdot d\alpha_0 \cdots \widehat{d\alpha_j} \cdots d\alpha_q \tag{6.2.4}$$

where each $\psi_{\alpha_j} s$, a priori an element of $\mathscr{E}(U_{\alpha_0\cdots\alpha_q})$, is extended by zero to an element of $\mathscr{E}(U_{\alpha_0\cdots\widehat{\alpha_j}\cdots\alpha_q})$. (Also, (6.2.4) is well-defined, i.e. is invariant under a permutation of $\{0, 1, \dots, q\}$.)

For each $q > 0$, it is a routine check that

$$\delta^{q-1} w^q + w^{q+1} \delta^q = \mathbf{1}.$$

Therefore, the identity map on $C^q(\mathfrak{U}, \mathscr{E})$ is homotopic to 0. Thus $\mathcal{H}^q(C^\bullet(\mathfrak{U}, \mathscr{E}))$ vanishes when $q > 0$. □

The analog of Prop. 6.2.4 for sheaf cohomology (namely, the degree> 0 sheaf cohomology groups of fine sheaves are zero) is also true. See [Voi, Prop. 4.36].

Using a similar idea, we prove:

Lemma 6.2.5. *For each $x \in X$, choose an $\mathscr{O}_{X,x}$-module $\mathcal{F}_x$, and view it as an $\mathscr{O}_X$-module. Let*

$$\mathscr{F} = \prod_{x \in X} \mathcal{F}_x$$

Then $\check{H}^{>0}(\mathfrak{U}, \mathscr{F})$.

In particular, for each $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$, if we let $\mathscr{E}^0 \in \mathrm{Mod}(\mathscr{O}_X)$ be the Godement sheaf $\mathrm{Gode}(\mathscr{E})$, *then $\check{H}^{>0}(\mathfrak{U}, \mathscr{E}^0) = 0$.*

Proof. $\mathscr{F}$ is clearly an $\mathscr{R}_X$-module, where

$$\mathscr{R}_X := \mathrm{Gode}(\mathscr{O}_X) = \prod_{x \in X} \mathscr{O}_{X,x}.$$

An easy application of Zorn's lemma shows that we have a disjoint union $X = \coprod_{\alpha \in I} E_\alpha$ (over the same index set I as that of $\mathfrak{U}$) such that $E_\alpha \subset U_\alpha$ for each $\alpha \in I$. For each $\alpha \in I$, define $\psi_\alpha \in \mathscr{R}_X(X)$ to be the characteristic function of U_α, namely, $\psi_\alpha = (\psi_\alpha(x))_{x\in X}$ where $\psi_\alpha(x) = 1$ if $x \in U_\alpha$ and $\psi_\alpha(x) = 0$ if $x \in X \backslash U_\alpha$. Though the support of ψ_α (which is U_α^{cl}) is not contained in U_α, we still have that for each open $V \subset X$ and $s \in \mathscr{E}(V \cap U_\alpha)$, $\psi_\alpha s$ extends by zero to an element of V. Thus, for each $q > 0$ we can define an $\mathscr{R}_X(X)$-module morphism $w^q : C^q(\mathfrak{U}, \mathscr{F}) \to C^{q-1}(\mathfrak{U}, \mathscr{F})$ by (6.2.4) and show again that $\delta^{q-1} w^q + w^{q+1} \delta^q = \mathbf{1}$. □

Definition 6.2.6. We say that $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$ is a **fine sheaf** if $\mathscr{E}$ is over a sheaf $\mathscr{R}_X$ of rings on X, where $\mathscr{R}_X$ satisfies that for every open cover $\mathfrak{U}$ of X there is a partition of unity in $\mathscr{R}_X$ subordinate to $\mathfrak{U}$.

For instance, if X is a smooth manifold and $\mathscr{E}$ is over the sheaf $\mathscr{C}^\infty_{X,\mathbb{R}}$ of real valued smooth functions, then $\mathscr{E}$ is fine.

By Prop. 6.2.4, if $\mathscr{E}$ is a fine sheaf, then for every open cover $\mathfrak{U}$ of X, $\check{H}^{>0}(\mathfrak{U}, \mathscr{E}) = 0$.

6.2.3 Long exact sequences for Čech cohomology

A short exact sequence $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ in $\mathrm{Mod}(\mathscr{O}_X)$ gives an exact sequence of complexes

$$0 \to C^\bullet(\mathfrak{U}, \mathscr{E}) \to C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}) \to 0$$

where

$$C^q_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}) = \mathrm{Im}\big(C^q(\mathfrak{U}, \mathscr{F}) \to C^q(\mathfrak{U}, \mathscr{G})\big) \tag{6.2.5}$$

This gives a long exact sequence of their cohomologies

$$\cdots \to \check{H}^{q-1}_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}) \xrightarrow{\delta} \check{H}^q(\mathfrak{U}, \mathscr{E}) \to \check{H}^q(\mathfrak{U}, \mathscr{F}) \to \check{H}^q_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}) \xrightarrow{\delta} \check{H}^{q+1}(\mathfrak{U}, \mathscr{E}) \to \cdots \tag{6.2.6}$$

where we set

$$\check{H}^q_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}) = \mathcal{H}^q(C^\bullet_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}))$$

and recall that $\check{H}^q(\mathfrak{U}, -) = \mathcal{H}^q(C^\bullet(\mathfrak{U}, -))$. Clearly, a morphism of short exact sequences (6.1.9) gives a morphism of long exact sequences from (6.2.6) to a similar one for $\mathscr{E}', \mathscr{F}', \mathscr{G}'$.

Remark 6.2.7. According to the definition (6.2.5), an element of $C^q_{\mathscr{F}}(\mathfrak{U}, \mathscr{G})$ is precisely an element $g = (g_{\alpha_0\cdots\alpha_q})$ of $C^q(\mathfrak{U}, \mathscr{G})$ such that each $g_{\alpha_0\cdots\alpha_q} \in \mathscr{G}(U_{\alpha_0\cdots\alpha_q})$ can be lifted to an element $f_{\alpha_0\cdots\alpha_q} \in \mathscr{F}(U_{\alpha_0\cdots\alpha_q})$, and that the lifting can be chosen in such a way that the skew-symmetry condition

$$f_{\alpha_0\cdots\alpha_q} = \mathrm{sgn}(\sigma) f_{\alpha_{\sigma(0)}\cdots\alpha_{\sigma(q)}} \tag{6.2.7}$$

is satisfied for each $(\alpha_0, \ldots, \alpha_q) \in I^{q+1}$ and each permutation σ of $\{0, 1, \ldots, q\}$.

The skew-symmetry condition is redundant. For if a lift $f_{\alpha_0\cdots\alpha_q}$ is chosen for each ordered tuple $(\alpha_0, \ldots, \alpha_q) \in I^{q+1}$, then the alternating sum

$$\widetilde{f}_{\alpha_0\cdots\alpha_q} = \frac{1}{(q+1)!} \sum_{\sigma \in \mathrm{Aut}\{0,\ldots,q\}} \mathrm{sgn}(\sigma) f_{\alpha_{\sigma(0)}\cdots\alpha_{\sigma(q)}}$$

is also a lift of $g_{\alpha_0\cdots\alpha_q}$, and the skew-symmetry condition is satisfied: $\widetilde{f}_{\alpha_0\cdots\alpha_q} = \mathrm{sgn}(\sigma)\widetilde{f}_{\alpha_{\sigma(0)}\cdots\alpha_{\sigma(q)}}$. □

Remark 6.2.8. The connecting morphism $\delta^q : \check{H}^q_{\mathscr{F}}(\mathfrak{U},\mathscr{G}) \to \check{H}^{q+1}(\mathfrak{U},\mathscr{E})$ is described as follows.

1. Choose any element $[g]$ of $\check{H}^q_{\mathscr{F}}(\mathfrak{U},\mathscr{G})$, represented by some $g = (g_{\alpha_0\cdots\alpha_q}) \in C^q_{\mathscr{F}}(\mathfrak{U},\mathscr{G})$ which is a cocycle, i.e. satisfies $\delta g = 0$. (Here δ is defined by (6.2.3).) According to the definition (6.2.5) of $C^q_{\mathscr{F}}(\mathfrak{U},\mathscr{G})$, we can lift g to an element $f = (f_{\alpha_0\cdots\alpha_q})$ of $C^q(\mathfrak{U},\mathscr{F})$.

2. That $\delta g = 0$ implies that $\delta^q f \in C^{q+1}(\mathfrak{U},\mathscr{F})$ is sent to zero by $C^{q+1}(\mathfrak{U},\mathscr{F}) \to C^{q+1}(\mathfrak{U},\mathscr{G})$. So $\delta^q f \in C^{q+1}(\mathfrak{U},\mathscr{F})$ belongs to the image of $C^{q+1}(\mathfrak{U},\mathscr{E}) \to C^{q+1}(\mathfrak{U},\mathscr{F})$.

3. Choose an arbitrary $e \in C^{q+1}(\mathfrak{U},\mathscr{E})$ which is sent to δf. Then $\delta \circ \delta f = 0$ implies $\delta e = 0$. Then $[e] \in \check{H}^{q+1}(\mathfrak{U},\mathscr{E})$ is $\delta[g]$.

□

Unfortunately, $\check{H}^\bullet(\mathfrak{U},\mathscr{G})$ and $\check{H}^\bullet_{\mathscr{F}}(\mathfrak{U},\mathscr{G})$ are not equal in general. So (6.2.6) does not give a δ-functor. There are two ways to overcome this difficulty:

(a) Take direct limit of $\check{H}(\mathfrak{U},-)$ and $\check{H}_{\mathscr{F}}(\mathfrak{U},-)$ over all open covers $\mathfrak{U}$. Then these two spaces agree when X is paracompact.

(b) For many important examples of sheaves, one can choose a nice cover $\mathfrak{U}$ such that $\check{H}(\mathfrak{U},-)$ and $\check{H}_{\mathscr{F}}(\mathfrak{U},-)$ are equal. Then there is a chance that $\check{H}(\mathfrak{U},-)$ equals the sheaf cohomology.

We first explain approach (a) in the next section.

6.3 Čech cohomology on paracompact spaces

Let X be a ringed space.

6.3.1 Čech cohomology $\check{H}^q(X,\mathscr{E})$

Suppose that $\mathfrak{V} = (V_\beta)_{\beta\in J}$ is another open cover of X which is a **refinement** of $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$. This means that there is a map $\rho : J \to I$ satisfying $V_\beta \subset U_{\rho(\beta)}$ for all $\beta \in J$. Then we obtain an $\mathscr{O}(X)$-module morphism

$$\rho^q : C^q(\mathfrak{U},\mathscr{E}) \to C^q(\mathfrak{V},\mathscr{E})$$
$$(\rho^q c)_{\beta_0\cdots\beta_q} = c_{\rho(\beta_0)\cdots\rho(\beta_q)}|_{V_{\beta_0\cdots\beta_q}} \tag{6.3.1}$$

Proposition 6.3.1. *Assume that $\widetilde{\rho}: J \to I$ satisfies $V_\beta \subset U_{\widetilde{\rho}(\beta)}$ for all $\beta \in J$. Then $\rho^\bullet$ and $\widetilde{\rho}^\bullet$ induce the same $\mathscr{O}(X)$-module morphism (called restriction map)*

$$\mathcal{H}^q(\rho^\bullet) = \mathcal{H}^q(\widetilde{\rho}^\bullet) : \check{H}^q(\mathfrak{U}, \mathscr{E}) \to \check{H}^q(\mathfrak{V}, \mathscr{E}). \tag{6.3.2}$$

Proof. (6.3.2) is obvious when $q = 0$. Let $q > 0$. For each $c \in C^q(\mathfrak{U}, \mathscr{E})$, define $w^q c \in \prod_{(\beta_0,\dots,\beta_{q-1})} \mathscr{E}(V_{\beta_0\cdots\beta_{q-1}})$ by

$$(w^q c)_{\beta_0\cdots\beta_{q-1}} = \sum_{0\leqslant j\leqslant q-1} (-1)^j c_{\rho(\beta_0)\cdots\rho(\beta_j)\widetilde{\rho}(\beta_j)\cdots\widetilde{\rho}(\beta_{q-1})}|_{V_{\beta_0\cdots\beta_{q-1}}}$$

Using the definition of δ in (6.2.1), one checks that $\delta^{q-1}w^q + w^{q+1}\delta^q$ equals $\widetilde{\rho}^q - \rho^q$ on $C^q(\mathfrak{U}, \mathscr{E})$. (See [Kod] Sec. 3.3 Lemma 3.2 for the details of computation.)

Thus, if $\delta^q c = 0$, then $\widetilde{\rho}^q c - \rho^q c$ equals $\delta^{q-1} w^q c$. Let b be the alternating sum of $w^q c$, namely

$$b_{\beta_0\cdots\beta_q} = \frac{1}{q!} \sum_{\sigma\in\mathrm{Aut}\{0,\dots,q-1\}} \mathrm{sgn}(\sigma) b_{\beta_{\sigma(0)}\cdots\beta_{\sigma(q-1)}}.$$

Then $b \in C^{q-1}(\mathfrak{V}, \mathscr{E})$, and $\widetilde{\rho}^q c - \rho^q c = \delta^{q-1} b$. This finishes the proof. □

Thanks to Prop. 6.3.1, we can make the following

Definition 6.3.2. For each $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$ and $q \in \mathbb{N}$,

$$\check{H}^q(X, \mathscr{E}) = \varinjlim_{\mathfrak{U} \text{ is an open cover of } X} \check{H}^q(\mathfrak{U}, \mathscr{E})$$

is called the **q-th Čech cohomology group of X with coefficients in $\mathscr{E}$**. Clearly

$$\check{H}^0(X, \mathscr{E}) = \check{H}^0(\mathfrak{U}, \mathscr{E}) = \mathscr{E}(X).$$

Any morphism of $\mathscr{O}_X$-modules $\mathscr{E} \to \mathscr{F}$ gives a canonical morphism $\check{H}^q(X, \mathscr{E}) \to \check{H}^q(X, \mathscr{F})$.

Let $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ be a short exact sequence in $\mathrm{Mod}(\mathscr{O}_X)$. The proof of Prop. 6.3.1 (together with Rem. 6.2.7) implies that

$$\mathcal{H}^q_{\mathscr{F}}(\rho^\bullet) = \mathcal{H}^q_{\mathscr{F}}(\widetilde{\rho}^\bullet) : \check{H}^q_{\mathscr{F}}(\mathfrak{U}, \mathscr{G}) \to \check{H}^q_{\mathscr{F}}(\mathfrak{V}, \mathscr{G}). \tag{6.3.3}$$

Thus, we can also define

$$\check{H}^q_{\mathscr{F}}(X, \mathscr{G}) = \varinjlim_{\mathfrak{U} \text{ is an open cover of } X} \check{H}^q_{\mathscr{F}}(\mathfrak{U}, \mathscr{G})$$

Then the direct limit of (6.2.6) over all $\mathfrak{U}$ gives an exact sequence (note that direct limit is an exact functor)

$$\cdots \to \check{H}^{q-1}_{\mathscr{F}}(X,\mathscr{G}) \xrightarrow{\delta} \check{H}^q(X,\mathscr{E}) \to \check{H}^q(X,\mathscr{F}) \to \check{H}^q_{\mathscr{F}}(X,\mathscr{G}) \xrightarrow{\delta} \check{H}^{q+1}(X,\mathscr{E}) \to \cdots \tag{6.3.4}$$

which is functorial in the sense that a morphism of short exact sequences (6.1.9) gives a morphism of long exact sequences from (6.3.4) to a similar one for $\mathscr{E}',\mathscr{F}',\mathscr{G}'$.

The monomorphism of complexes $C^\bullet_{\mathscr{F}}(\mathfrak{U},\mathscr{G}) \to C^\bullet(\mathfrak{U},\mathscr{G})$ gives a natural morphism of their cohomology groups

$$\check{H}^q_{\mathscr{F}}(\mathfrak{U},\mathscr{G}) \to \check{H}^q(\mathfrak{U},\mathscr{G})$$

which is compatible with restricting to a finer open cover $\mathfrak{V}$. Thus, passing to the direct limit over all open cover $\mathfrak{U}$, we obtain a natural morphism

$$\check{H}^q_{\mathscr{F}}(X,\mathscr{G}) \to \check{H}^q(X,\mathscr{G}) \tag{6.3.5}$$

Theorem 6.3.3. *Assume that X is paracompact. Then (6.3.5) is an isomorphism. Therefore, by (6.3.4), $(\check{H}^\bullet(X,-),\delta)$ is a δ-functor, which is indeed universal, and hence is isomorphic to the sheaf cohomology $(H^\bullet(X,-),\delta)$*

Recall that a paracompact space is a Hausdorff space satisfying that any open cover has a refinement which is locally finite. For instance, every second-countable locally compact Hausdorff space is paracompact. Therefore, every second-countable complex space is paracompact.

Proof. Step 1. Once we have proved that (6.3.5) is an isomorphism, then by Thm. 5.1.6, the δ-functor $(\check{H}^\bullet(X,-),\delta)$ is universal because that each $\mathscr{E}\in\mathrm{Mod}(\mathscr{O}_X)$ has a monomorphism into its Godement sheaf $\mathscr{E}^0=\mathrm{Gode}(\mathscr{E})$, and that $\check{H}^{>0}(X,\mathscr{E}^0)=0$ by Lemma 6.2.5.

To show that (6.3.5) is an isomorphism, it suffices to show that for each open cover $\mathfrak{U}$ of X and each $c=(c_{\alpha_0\cdots\alpha_q})\in C^q(\mathfrak{U},\mathscr{G})$, there is a refinement $\mathfrak{V}=(V_\beta)_{\beta\in J}$ together with a map $\rho:J\to I$ satisfying $V_\beta\subset U_{\rho(\beta)}$ for all $\beta\in J$, such that $\rho^q c\in C^q(\mathfrak{V},\mathscr{G})$ (as defined in (6.3.1)) belongs to $C^q_{\mathscr{F}}(\mathfrak{V},\mathscr{G})$. By Rem. 6.2.7, the last sentence is equivalent to that $\rho^q c$ is liftable in $\mathscr{F}$, namely, for each $\beta_0,\dots,\beta_q\in J$,

$$(\rho^q c)_{\beta_0\cdots\beta_q}=c_{\rho(\beta_0)\cdots\rho(\beta_q)}|_{V_{\beta_0\cdots\beta_q}}$$

lifts to an element of $\mathscr{F}(V_{\beta_0\cdots\beta_q})$.

Step 2. Since X is paracompact, by replacing $\mathfrak{U}$ with a refinement, we may assume that $\mathfrak{U}=(U_\alpha)_{\alpha\in I}$ itself is locally finite. For each $x\in X$, we choose a neighborhood V_x such that

(a) If $x \in X$ and $\alpha \in I$ is such that $x \in U_\alpha$, then $V_x \subset U_\alpha$.

(b) If $x \in U_{\alpha_0\cdots\alpha_q}$, (note that $V_x \subset U_{\alpha_0\cdots\alpha_q}$) then $c_{\alpha_0\cdots\alpha_q}|_{V_x}$ (which belongs to $\mathscr{G}(V_x)$) lifts to an element of $\mathscr{F}(V_x)$.

Let $\mathfrak{V} = (V_x)_{x\in X}$. For each $x \in X$, choose $\rho(x) \in I$ such that $V_x \subset U_{\rho(x)}$. By the end of Step 1, it suffices to show that for each $x_0, \dots, x_q \in X$, $c_{\rho(x_0)\cdots\rho(x_q)}|_{V_{x_0\cdots x_q}}$ lifts to an element of $\mathscr{F}(V_{x_0\cdots x_q})$. It suffices to prove that

$$x_0 \in U_{\rho(x_0)\cdots\rho(x_q)} \qquad (\forall x_0, \dots, x_q \in X \text{ such that } V_{x_0\cdots x_q} \neq \varnothing)$$

because it would then imply (by (a)) that $V_{x_0} \subset U_{\rho(x_0)\cdots\rho(x_q)}$, and (by (b)) that $c_{\rho(x_0)\cdots\rho(x_q)}|_{V_{x_0}}$ lifts to an element of $\mathscr{F}(V_{x_0})$. Therefore, it suffices to prove that for each $x, y \in X$,

$$V_x \cap V_y \neq \varnothing \qquad \Longrightarrow \qquad x \in U_{\rho(y)} \tag{6.3.6}$$

To show (6.3.6), we need to shrink each V_x further. Since X is paracompact, we may choose an open subcover $\mathfrak{W} = (W_\alpha)_{\alpha\in I}$ of $\mathfrak{U}$ with the same index set I such that W_α^{cl} belongs to U_α for each $\alpha \in I$. Clearly $\mathfrak{W}$ is also locally finite. Therefore, we may shrink each V_y further so that

(c) For each $y \in X$ there exists $\alpha \in I$ such that $V_y \subset W_\alpha$.

Thus for each $x, y \in X$, choose $\alpha \in I$ such that $V_y \subset W_\alpha$. If $V_x \cap V_y \neq \varnothing$, then $V_x \cap W_\alpha \neq \varnothing$. Then (6.3.6) follows if we can show $x \in V_\alpha$. To summarize, it remains to prove that for each $x \in X, \alpha \in I$,

$$V_x \cap W_\alpha \neq \varnothing \qquad \Longrightarrow \qquad x \in U_\alpha \tag{6.3.7}$$

This is certainly true if for each $x \in X$ we shrink V_x further: We first shrink V_x so that V_x intersects finitely many members of $\mathfrak{U}$, say $U_{\alpha_1}, \dots, U_{\alpha_k}$. If $1 \leqslant j \leqslant k$ is such that $x \neq U_{\alpha_j}$, since $W_{\alpha_j}^{\mathrm{cl}} \subset U_{\alpha_j}$, we may shrink V_x so that $V_x \cap W_{\alpha_j} = \varnothing$. Then (6.3.7) is clearly fulfilled. $\square$

Definition 6.3.4. Let $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$. A resolution $0 \to \mathscr{E} \to \mathscr{E}^\bullet$ is called a **fine resolution** of $\mathscr{E}$ if each $\mathscr{E}^q$ is a fine sheaf.

Example 6.3.5. Let X be a paracompact (e.g. second countable) topological (resp. smooth) manifold. For every open cover $\mathfrak{U}$ of X there is a continuous (resp. smooth) partition of unity of X subordinated to $\mathfrak{U}$. Therefore, every X-sheaf over the sheaf of germs of continuous (resp. smooth) functions on X is a fine sheaf.

Corollary 6.3.6. *Assume that X is paracompact. Let $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$, and let $0 \to \mathscr{E} \to \mathscr{E}^\bullet$ be a fine resolution of $\mathscr{E}$. Set $\mathscr{E}^q = 0$ if $q < 0$. Then there are $\mathscr{O}(X)$-module isomorphisms*

$$\check{H}^q(X, \mathscr{E}) \simeq H^q(X, \mathscr{E}) \simeq \mathcal{H}^q(\mathscr{E}^\bullet(X))$$

Proof. The first equivalence is due to Thm. 6.3.3. Since $\check{H}^{>0}(X,\mathscr{E}^\bullet)=0$ by Prop. 6.2.4, the second equivalence holds (cf. Subsec. 5.2.2 and especially (5.2.7)). $\square$

Example 6.3.7. Let X be a paracompact smooth manifold. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Then we have a resolution of the constant sheaf $\mathbb{F}$ called **de Rham resolution**:

$$0\to\mathbb{F}\xrightarrow{d}\bigwedge\nolimits_{\mathbb{F}}^{0}X\xrightarrow{d}\bigwedge\nolimits_{\mathbb{F}}^{1}X\xrightarrow{d}\bigwedge\nolimits_{\mathbb{F}}^{2}X\xrightarrow{d}\cdots \tag{6.3.8}$$

where $\bigwedge_{\mathbb{F}}^{q}X$ is the sheaf of germs of $\mathbb{F}$-valued differential q-forms, and d is the differential operator. Let $\bigwedge_{\mathbb{F}}^{q}X=0$ if $q<0$. The exactness of (6.3.8) is due to Poincaré's lemma. Then by Exp. 6.3.5, $\bigwedge_{\mathbb{F}}^{q}X$ is a fine sheaf since it is over the sheaf of $\mathbb{F}$-valued smooth functions $\mathscr{C}^\infty_{X,\mathbb{F}}$. Define the **de Rham cohomology group**

$$H^q_{\mathrm{dR}}(X,\mathbb{F}):=\mathcal{H}^q\Big(H^0\big(X,\bigwedge\nolimits_{\mathbb{F}}^{\bullet}X\big)\Big). \tag{6.3.9}$$

Then by Cor. 6.3.6, we have isomorphisms of $\mathbb{F}$-vector spaces

$$\check{H}^q(X,\mathbb{F})\simeq H^q(X,\mathbb{F})\simeq H^q_{\mathrm{dR}}(X,\mathbb{F}).$$

Example 6.3.8. Let X be a paracompact complex manifold. Let $\mathscr{C}^\infty_{X,\mathbb{C}}$ be the sheaf of germs of complex smooth functions on X. Let $\bigwedge^{p,q}X$ be the sheaf of germs of complex differential forms on X of degree (p,q). Locally, by choosing a set of coordinates $(z_1,\dots,z_n)$ of X, a section of $\Omega_X^{p,q}$ is a sum of those of the form $fdz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}$ where $f\in\mathscr{C}^\infty_{X,\mathbb{C}}$. Then we have a resolution

$$0\to\Omega_X^p\to\bigwedge\nolimits^{p,0}X\xrightarrow{\overline{\partial}^0}\bigwedge\nolimits^{p,1}X\xrightarrow{\overline{\partial}^1}\bigwedge\nolimits^{p,2}X\xrightarrow{\overline{\partial}^2}\cdots \tag{6.3.10}$$

called the **Dolbeault resolution** of Ω_X^p, where Ω_X^p is the sheaf of germs of holomorphic p-forms (which are locally a sum of those of the form $fdz_{i_1}\wedge\cdots\wedge dz_{i_p}$ where $f\in\mathscr{O}_X$), and $\overline{\partial}^q$ is a $\mathbb{C}$-linear sheaf map determined by

$$\begin{aligned}&\overline{\partial}(fdz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q})\\ =&\sum_k\frac{\partial f}{\partial\overline{z}_k}d\overline{z}_k\wedge dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}\end{aligned}$$

The exactness of (6.3.10) is due to Dolbeault lemma.

In (6.3.10), set $p=0$. Then $\Omega_X^0=\mathscr{O}_X$. Choose any locally free $\mathscr{O}_X$-module (i.e. holomorphic vector bundle) $\mathscr{E}$ and tensor it with (6.3.10). Since $\mathscr{E}$ is $\mathscr{O}_X$-flat, we obtain an exact sequence

$$0\to\mathscr{E}\to\mathscr{E}\otimes_{\mathscr{O}_X}\bigwedge\nolimits^{0,0}X\xrightarrow{\mathbf{1}\otimes\overline{\partial}}\mathscr{E}\otimes_{\mathscr{O}_X}\bigwedge\nolimits^{0,1}X\xrightarrow{\mathbf{1}\otimes\overline{\partial}}\mathscr{E}\otimes_{\mathscr{O}_X}\bigwedge\nolimits^{0,2}X\xrightarrow{\mathbf{1}\otimes\overline{\partial}}\cdots \tag{6.3.11}$$

called the **Dolbeault resolution** of $\mathscr{E}$. This is a fine resolution since each $\mathscr{E}\otimes_{\mathscr{O}_X}\bigwedge^{0,1}X$ is over $\mathscr{C}^\infty_{X,\mathbb{C}}$ (cf. Exp. 6.3.5). Define the **Dolbeault cohomology group**

$$H^q_{\overline{\partial}}(X,\mathscr{E}):=\mathcal{H}^q\Big(H^0\big(X,\mathscr{E}\otimes_{\mathscr{O}_X}\bigwedge\nolimits^{0,\bullet}X\big)\Big)$$

Then by Cor. 6.3.6, we have isomorphisms of $\mathbb{C}$-vector spaces

$$\check{H}^q(X,\mathscr{E})\simeq H^q(X,\mathscr{E})\simeq H^q_{\overline{\partial}}(X,\mathscr{E}) \tag{6.3.12}$$

6.4 Leray's theorem; Stein spaces

In this section, we explain approach (b) at the end of Subsec. 6.2.3. Again, we assume X is a ringed space. Let $\mathfrak{U}=(U_\alpha)_{\alpha\in I}$ be an open cover of X.

6.4.1 Leray's theorem

Theorem 6.4.1 (Leray's theorem)**.** *Assume that X is paracompact. Let $\mathscr{E}\in\mathrm{Mod}(\mathscr{O}_X)$ and assume that*

$$\check{H}^{>0}(U_{\alpha_0\cdots\alpha_n},\mathscr{E})=0 \tag{6.4.1}$$

for all $n\in\mathbb{N}$ and $\alpha_0,\dots,\alpha_n\in I$. Then for each $q\in\mathbb{N}$, the natural $\mathscr{O}(X)$-module morphism

$$\check{H}^q(\mathfrak{U},\mathscr{E})\to\check{H}^q(X,\mathscr{E}) \tag{6.4.2}$$

(defined by passing to direct limit over all open covers) is an isomorphism.

Proof. (6.4.2) is clearly an isomorphism when $q=0$, since they are both identified with $\mathscr{E}(X)$. We now prove Leray's theorem by induction on q. We know it holds when $q=0$. Assume it holds for case q where $q\in\mathbb{N}$. Let us prove it for case $q+1$. Let $\mathscr{F}=\mathrm{Gode}(\mathscr{E})$ so that we have a monomorphism $\mathscr{E}\hookrightarrow\mathscr{F}$ where $\mathscr{F}$ is killed by $\check{H}^{>0}(\mathfrak{U},-)$ and by $\check{H}^{>0}(X,-)$ (cf. Lemma 6.2.5). Let $\mathscr{G}=\mathscr{F}/\mathscr{E}$. Then we have a short exact sequence $0\to\mathscr{E}\to\mathscr{F}\to\mathscr{G}\to 0$.

Since $\check{H}^1(U_{\alpha_0\cdots\alpha_n},\mathscr{E})=0$, we have an exact sequence

$$0\to\mathscr{E}(U_{\alpha_0\cdots\alpha_n})\to\mathscr{F}(U_{\alpha_0\cdots\alpha_n})\to\mathscr{G}(U_{\alpha_0\cdots\alpha_n})\to 0$$

showing $C^\bullet_{\mathscr{F}}(\mathfrak{U},\mathscr{G})=C^\bullet(\mathfrak{U},\mathscr{G})$ (Rem. 6.2.7) and hence $\check{H}^\bullet_{\mathscr{F}}(X,\mathscr{G})=\check{H}^\bullet(X,\mathscr{G})$. Therefore, by (6.3.4) and its functoriality, we have a morphism of exact sequences

$$\begin{tikzcd}
\check{H}^q(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d, "\simeq"'] & \check{H}^q(\mathfrak{U},\mathscr{G}) \arrow[r] \arrow[d] & \check{H}^{q+1}(\mathfrak{U},\mathscr{E}) \arrow[r] \arrow[d] & 0 \\
\check{H}^q(X,\mathscr{F}) \arrow[r] & \check{H}^q(X,\mathscr{G}) \arrow[r] & \check{H}^{q+1}(X,\mathscr{E}) \arrow[r] & 0
\end{tikzcd} \tag{6.4.3}$$

The first vertical arrow is an isomorphism: It is clearly so if $q = 0$, and it is so when $q > 0$ because the domain and the codomain are both 0. Clearly, for each $p \in \mathbb{N}$, we have an exact sequence

$$\check{H}^p(U_{\alpha_0\cdots\alpha_n}, \mathscr{F}) \to \check{H}^p(U_{\alpha_0\cdots\alpha_n}, \mathscr{G}) \to \check{H}^{p+1}(U_{\alpha_0\cdots\alpha_n}, \mathscr{E})$$

where the third term is zero by assumption (6.4.1) and the first term is zero when $p > 0$ by Lemma 6.2.5. Therefore $\check{H}^{>0}(U_{\alpha_0\cdots\alpha_n}, \mathscr{G}) = 0$. So by case q of Leray's theorem (applied to $\mathscr{G}$), the middle vertical arrow of (6.4.3) is an isomorphism. So the third vertical arrow is also an isomorphism due to Five lemma. This proves case $q + 1$ for $\mathscr{E}$. $\square$

Without assuming that X is paracompact, Thm. 6.4.1 still holds if we replace $\check{H}^{>0}(U_{\alpha_0\cdots\alpha_n}, \mathscr{E})$ with $H^{>0}(U_{\alpha_0\cdots\alpha_n}, \mathscr{E})$, replace $\check{H}^q(X, \mathscr{E})$ with $H^q(X, \mathscr{E})$, and define the map (6.4.2) in an appropriate way. Indeed, this sheaf cohomology version is the common one that people refer to when talking about Leray's theorem. This version is especially useful in algebraic geometry, since schemes are not even Hausdorff. It also allows us to compute the sheaf cohomology of coherent sheaves over a complex space X which is non-necessarily paracompact by computing the Čech cohomology of a Stein open cover of X. We present this version in the following subsection.

6.4.2 Sheaves of Čech cochains $\mathfrak{C}^q(\mathfrak{U}, \mathscr{E})$ and Čech resolution

Let us consider the **sheaf of Čech q-cochains** $\mathfrak{C}^q(\mathfrak{U}, \mathscr{E})$, which is an $\mathscr{O}_X$-module associating to each open $W \subset X$ the $\mathscr{O}_X(W)$-module

$$\mathfrak{C}^q(\mathfrak{U}, \mathscr{E})(W) = C^q(W \cap \mathfrak{U}, \mathscr{E}) \tag{6.4.4}$$

$$(\text{where } W \cap \mathfrak{U} = \{W \cap U_\alpha\}_{\alpha \in I})$$

Then $(\mathfrak{C}^q(\mathfrak{U}, \mathscr{E}), \delta)$ is a cochain complex of $\mathscr{O}_X$-modules.

Note that we have an obvious inclusion $H^0(W, \mathscr{E}) \hookrightarrow \mathfrak{C}^0(\mathfrak{U}, \mathscr{E})(W)$, which gives rise to a monomorphism $\mathscr{E} \hookrightarrow \mathfrak{C}^0(\mathfrak{U}, \mathscr{E})$. Moreover, we have:

Proposition 6.4.2. *The following is an exact sequence*

$$0 \to \mathscr{E} \to \mathfrak{C}^0(\mathfrak{U}, \mathscr{E}) \xrightarrow{\delta^0} \mathfrak{C}^1(\mathfrak{U}, \mathscr{E}) \xrightarrow{\delta^1} \mathfrak{C}^2(\mathfrak{U}, \mathscr{E}) \xrightarrow{\delta^2} \cdots \tag{6.4.5}$$

In other words, $0 \to \mathscr{E} \to \mathfrak{C}^{\geqslant 0}(\mathfrak{U}, \mathscr{E})$ is a right resolution of $\mathscr{E}$.

Moreover, if $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X)$, then a morphism $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$ gives rise to a natural morphism of cochain complexes $C^\bullet(\mathfrak{U}, \mathscr{E}) \to C^\bullet(\mathfrak{U}, \mathscr{F})$. Therefore, $\mathfrak{C}^\bullet(\mathfrak{U}, -)$ is an (additive) functor from $\mathrm{Mod}(\mathscr{O}_X)$ to the category of complexes of $\mathscr{O}_X$-modules.

To summarize, we have a right resolution $\mathscr{E} \hookrightarrow \mathfrak{C}^{\geqslant 0}(\mathfrak{U}, \mathscr{E})$ for all $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$ which is functorial. This is called the **Čech resolution** of $\mathscr{E}$ with respect to $\mathfrak{U}$.

Proof. We only prove (6.4.5). The remaining part of the proposition is obvious. Choose any $x \in X$, and choose $\beta \in I$ such that $x \in U_\beta$. Then for each neighborhood W of x contained inside U_β, there is a partition of unity in $\mathscr{O}_W$ subordinate to $W \cap \mathfrak{U} = (W \cap U_\alpha)_{\alpha \in I}$: since $W \cap U_\beta = W$, one can take $\psi_\beta = 1$, and take $\psi_\alpha = 0$ if $\alpha \neq \beta$. Therefore, by Prop. 6.2.4, $\check{H}^{>0}(W \cap \mathfrak{U}, \mathscr{E})$ is zero, namely, the sequence

$$C^{q-1}(W \cap \mathfrak{U}, \mathscr{E}) \xrightarrow{\delta^{q-1}} C^q(W \cap \mathfrak{U}, \mathscr{E}) \xrightarrow{\delta^q} C^{q+1}(W \cap \mathfrak{U}, \mathscr{E})$$

is exact if $q > 0$. By taking direct limit over all such W, we see that the stalk of (6.4.5) at x is exact. $\square$

Theorem 6.4.3 (Leray's theorem)**.** *Let $\mathscr{E} \in \mathrm{Mod}(\mathscr{O}_X)$ and assume that*

$$H^{>0}(U_{\alpha_0 \cdots \alpha_n}, \mathscr{E}) = 0 \tag{6.4.6}$$

for all $n \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_n \in I$. Then for each $q \in \mathbb{N}$, there is a natural $\mathscr{O}(X)$-module isomorphism

$$\check{H}^q(\mathfrak{U}, \mathscr{E}) \simeq H^q(X, \mathscr{E}) \tag{6.4.7}$$

Proof. (6.4.6) implies that (6.4.5) is an **acyclic resolution** of $\mathscr{E}$, namely,

$$H^{>0}(X, \mathfrak{C}^p(\mathfrak{U}, \mathscr{E})) = 0 \tag{6.4.8}$$

for all $p \geqslant 0$. Therefore, by (5.2.7), we have an isomorphism (6.4.7) which is natural by Exe. 5.2.6. $\square$

6.4.3 Stein spaces and Cartan's theorems

In this subsection, we assume X is a complex space.

An important situation to which Leray's theorem can be applied is when $\mathscr{E}$ is $\mathscr{O}_X$-coherent and each U_α is a **Stein space**. The definition of Stein spaces is quite technical and will not be used in our notes. Instead, we use the following important fact about Stein spaces. Indeed, every complex space satisfying Cartan's theorem B is a Stein space.

Theorem 6.4.4 (Cartan's theorem B)**.** *Suppose that X is a Stein space. Then for each $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ we have $H^{>0}(X, \mathscr{E}) = 0$.*

An immediate consequence of Thm. B is:

Theorem 6.4.5 (Cartan's theorem A). *Suppose that X is a Stein space. Then for each $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ and each $x \in X$, the germs at x of the elements of $\mathscr{E}(X)$ generate the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$.*

Proof. By Nakayama's lemma, it suffices to show that $\mathscr{E}(X)$ spans the fiber $\mathscr{E}|x = \mathscr{E}_x/\mathfrak{m}_{X,x}\mathscr{E}_x$. We view $\mathfrak{m}_{X,x}$ is the ideal sheaf of all sections of $\mathscr{O}_X$ vanishing at x. Then $\mathscr{E}|x$ as a coherent $\mathscr{O}_X$-module. We have a short exact sequence

$$0 \to \mathfrak{m}_{X,x}\mathscr{E} \to \mathscr{E} \to \mathscr{E}|x \to 0$$

showing that $\mathfrak{m}_{X,x}\mathscr{E}$ is coherent and hence $H^1(X, \mathfrak{m}_{X,x}\mathscr{E}) = 0$ by Cartan's theorem B. Therefore, $H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E}|x)$ is surjective. □

We refer the readers to [Tay, Chapter 10, 11] for a proof of Cartan's theorem B. See also [GR-a] for a comprehensive account of the theory of Stein spaces.

Example 6.4.6. The following are some important examples of Stein spaces. We let X, Y, S denote complex spaces. Cf. [GR-a, Sec. V.1].

(a) Every connected non-compact Riemann surface is Stein.

(b) If X, Y are Stein, then $X \times Y$ is Stein.

(c) If Y is Stein and $\varphi : X \to Y$ is a *finite* holomorphic map (in particular, if X is a closed complex subspace of Y), then X is Stein.

From (a) and (b) we see that every polydisc is Stein. Therefore, for every complex space X, the set of Stein open subsets of X form a base of topology of X.

We also have:

(d) If $\varphi : X \to S$ and $\psi : Y \to S$ are holomorphic maps, and if X, Y are Stein, then $X \times_S Y$ is Stein.

(e) If $U_1, \dots, U_n$ are Stein open subsets of X, then $U_1 \cap \cdots \cap U_n$ is Stein.

(f) If $\varphi : X \to Y$ is a holomorphic map, $U \subset X$ and $W \subset Y$ are open subsets which are Stein, then $U \cap \varphi^{-1}(W)$ is Stein.

Indeed, (d) is due to that $X \times Y$ is Stein and that $X \times_S Y$ is a closed subspace of $X \times Y$ (by Prop. 1.13.10). (f) follows from (d) because $U \cap \varphi^{-1}(W)$ is the fiber product of $\varphi \circ \iota_{U,X} : U \to Y$ and $\iota_{W,Y} : W \to Y$. To show (e), it suffices to assume $n = 2$. Then (e) is a special case of (f). Furthermore, we have (cf. [GR-a, Sec. V.4.3])

(g) X is Stein if and only if its reduction $\mathrm{red}(X)$ is Stein.

□

Definition 6.4.7. An open cover $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ of a complex space X is called a **Stein cover** if each U_α is Stein.

If $\mathfrak{U}$ is a Stein cover, then by Exp. 6.4.6-(e), each intersection $U_{\alpha_0\cdots\alpha_n}$ is a Stein open subset of X. Therefore, by Cartan's theorem B and Leray's Thm. 6.4.3, we have:

Corollary 6.4.8. *Suppose that X is a complex space and $\mathfrak{U}$ is a Stein open cover of X. Then there is a natural equivalence of $\mathscr{O}(X)$-modules for each $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$:*

$$\check{H}^q(\mathfrak{U}, \mathscr{E}) \simeq H^q(X, \mathscr{E}).$$

6.5 Higher direct images and formal completion

Beginning with this section, **all complex spaces are assumed to be paracompact**. Thus, for complex spaces, we identify sheaf cohomology and Čech cohomology.

Let X and Y be complex spaces. Let $\varphi : X \to S$ be a holomorphic map. The following deep result is due to Grauert. See [GR-b, Chapter 10], [BS, Sec. 3.2], or [Dem, Sec. IX.5] for proofs. This theorem will be implicitly used in the remaining part of our notes. Especially, all major results of this chapter rely in an essential way on this theorem.

Theorem 6.5.1 (Grauert direct image theorem). *Let $\varphi : X \to S$ be a proper holomorphic map. For every $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ and every $q \in \mathbb{Z}$, $R^q\varphi_*(\mathscr{E})$ is a coherent $\mathscr{O}_S$-module.*

In the special case that S is a reduced point, the direct image theorem says:

Corollary 6.5.2 (Cartan-Serre theorem). *Suppose that X is compact and $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$, then $\dim_{\mathbb{C}} H^q(\mathscr{E}) < +\infty$.*

6.5.1 Higher direct images and tensor product

Let $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ and $\mathscr{M} \in \mathrm{Coh}(\mathscr{O}_S)$. Then there is a natural morphism of $\mathscr{O}_S$-modules

$$\boxed{R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{M} \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})} \tag{6.5.1}$$

defined as follows. If V is an open subset of S and $U \subset \varphi^{-1}(V)$ is open, we have a natural $\mathscr{O}_S(V)$-module morphism

$$\mathscr{E}(U) \otimes_{\mathscr{O}_{S(V)}} \mathscr{M}(V) \to (\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})(U).$$

Thus, if $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ is an open cover of $\varphi^{-1}(V)$, we have a canonical $\mathscr{O}_S(V)$-module morphism

$$C^q(\mathfrak{U},\mathscr{E})\otimes_{\mathscr{O}_S(V)}\mathscr{M}(V)\to C^q(\mathfrak{U},\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M})$$

which gives canonical morphisms

$$\begin{aligned}&\mathcal{H}^q(C^\bullet(\mathfrak{U},\mathscr{E}))\otimes_{\mathscr{O}_S(V)}\mathscr{M}(V)\to\mathcal{H}^q(C^\bullet(\mathfrak{U},\mathscr{E})\otimes_{\mathscr{O}_S(V)}\mathscr{M}(V))\\ \to&\mathcal{H}^q(C^\bullet(\mathfrak{U},\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M})).\end{aligned}$$

Passing to the direct limit over all open covers $\mathfrak{U}$, we get a canonical $\mathscr{O}_S(V)$-module morphism

$$H^q(\varphi^{-1}(V),\mathscr{E})\otimes_{\mathscr{O}_S(V)}\mathscr{M}(V)\to H^q(\varphi^{-1}(V),\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M})$$

Sheafifying this map gives (6.5.1).

A fundamental question about higher direct images is when the map (6.5.1) is an isomorphism. It is a main goal of this chapter to give a satisfying answer to this question. This question has important geometric implications. Choose $t\in S$ and take $\mathscr{M}=\mathscr{O}_S/\mathfrak{m}_{S,t}$ where, as usual, $\mathfrak{m}_{S,t}$ is understood as the $\mathscr{O}_S$-ideal of all sections vanishing at t. Then (6.5.1) reads

$$R^q\varphi_*(\mathscr{E})|_t\to H^q(X_t,\mathscr{E}|_{X_t}) \tag{6.5.2}$$

where we set

$$X_t=\varphi^{-1}(t)$$

as usual. That (6.5.2) is surjective means, in the case $q=0$, that any global sections of $\mathscr{E}|_{X_t}$ on X_t can be extended holomorphically to sections on the nearby fibers of X_t.

In the next section, we will prove a deep result saying that if $\mathscr{E}$ is φ-flat and (6.5.2) is surjective, then (6.5.2) must be an isomorphism, and more generally, the stalk map of (6.5.1) at t is an isomorphism for every $\mathscr{M}\in\mathrm{Coh}(\mathscr{O}_S)$. To prove this fact, we need to show that a formal version of (6.5.1) is an isomorphism. This is the task of this section.

6.5.2 Higher direct images and formal completion

Fix $t\in S$ and write $\mathfrak{m}_{S,t}$ as $\mathfrak{m}_t$ for simplicity. Let the $\mathscr{M}$ in (6.5.1) be $\mathscr{O}_{S,t}/\mathfrak{m}_t^k$ and $\mathscr{O}_{S,t}/\mathfrak{m}_t^l$ where $l\geqslant k$. Then we have a commutative diagram

$$\begin{tikzcd}
R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^l R^q\varphi_*(\mathscr{E})_t \arrow[r] \arrow[d] & R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t \arrow[d] \\
R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t \arrow[r] & R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t
\end{tikzcd}$$

where the horizontal maps are defined by (6.5.1). In other words, we get a morphism of inverse systems $R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^\bullet R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^\bullet\mathscr{E})_t$. Passing to the inverse limit gives an $\mathscr{O}_{S,t}$-module morphism

$$\boxed{\varprojlim_{k\in\mathbb{N}} R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t \longrightarrow \varprojlim_{k\in\mathbb{N}} R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t} \tag{6.5.3}$$

It is a deep result that this morphism is indeed an isomorphism. To show this fact, we need to show:

Lemma 6.5.3. *Assume that $\varphi: X\to S$ is a proper holomorphic map and $\mathscr{E}\in\mathrm{Coh}(\mathscr{O}_X)$. Then for each $k\in\mathbb{N}$ there exists $l\in\mathbb{N}$ such that*

$$\mathrm{Ker}\big(R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t\big) \subset \mathfrak{m}_t^k\cdot R^q\varphi_*(\mathscr{E})_t \tag{6.5.4}$$

Note that if (6.5.4) holds, then it holds if l is replaced by any $\widetilde{l}\geqslant l$. This is because $\mathfrak{m}_t^{\widetilde{l}}\subset\mathfrak{m}_t^l$ and hence we have a commutative diagram

$$\begin{tikzcd} R^q\varphi_*(\mathscr{E})_t \arrow[rr] \arrow[dr] & & R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t \\ & R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^{\widetilde{l}}\mathscr{E})_t \arrow[ur] & \end{tikzcd}$$

Theorem 6.5.4 (Grauert comparison theorem). *Assume that $\varphi: X\to S$ is a proper holomorphic map and $\mathscr{E}\in\mathrm{Coh}(\mathscr{O}_X)$. Then (6.5.3) is an $\mathscr{O}_{S,t}$-module isomorphism.*

Indeed, we will only use the injectivity of (6.5.3). (See Thm. 6.6.2 (e)⇒(d) and (c)⇒(f).) So we postpone the proof of surjectivity to the end of this section.

Proof that* (6.5.3) *is injective. Choose $(\sigma_k)_{k\in\mathbb{N}}$ in $\varprojlim_{k\in\mathbb{N}} R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t$. Namely, each σ_k belongs to $R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t$, and σ_l is sent to σ_k if $l\geqslant k$.

Suppose that $(\sigma_k)_{k\in\mathbb{N}}$ is sent to 0 by the map (6.5.3). Fix any $k\in\mathbb{N}$, and let l be as in Lemma 6.5.3. Since we can safely make l larger, we assume $l\geqslant k$. Then σ_l (which is in $R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^l R^q\varphi_*(\mathscr{E})_t$) is sent to 0 in $R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t$. Lift σ_l to an element $\varsigma_l\in R^q\varphi_*(\mathscr{E})_t$. Then ς_l is sent to 0 by the map

$$R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t$$

By Lemma 6.5.3, ς_l belongs to $\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t$. But ς_l is clearly sent to σ_k by the map $R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t$. So $\sigma_k=0$. □

6.5.3 Proof of Lemma 6.5.3

To prove Lemma 6.5.3 we first need a lemma.

Lemma 6.5.5. *Let X be a complex space and $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$. Let $f \in \mathscr{O}(X)$. Then for each precompact open subset $U \subset X$ there exists $d \in \mathbb{Z}_+$ such that the multiplication of f is injective on $f^d\mathscr{E}|_U$, namely, the map*

$$f^d\mathscr{E}|_U \xrightarrow{\times f} f^{d+1}\mathscr{E}|_U$$

is injective.

Proof. $\mathscr{F}_n = \mathrm{Ker}(\mathscr{E} \xrightarrow{\times f^n} \mathscr{E})$ is an ascending chain of coherent $\mathscr{O}_X$-submodules of $\mathscr{E}$ (as n increases). Therefore, by the Noether property of coherent sheaves (Thm. 3.11.5), this chain is stationary at some $n = d$ when restricted to U. So $\mathscr{F}_d|_U = \mathscr{F}_{d+1}|_U$. If $s \in \mathscr{E}|_U$ and $f^d s$ is such that $f \times f^d s = 0$, then $s \in \mathscr{F}_{d+1}|_U = \mathscr{F}_d|_U$, and hence $f^d s = 0$. So $\times f$ is injective on $f^d\mathscr{E}|_U$. □

***Proof of Lemma 6.5.3**.* Step 1. Shrink S to a neighborhood of $t \in S$ and choose $g_1, \dots, g_n \in \mathscr{O}(S)$ generating the ideal $\mathfrak{m}_t$.

Claim: It suffices to show that for each $k \in \mathbb{N}$ there exists $\lambda_1, \dots, \lambda_n \in \mathbb{N}$ such that

$$\mathrm{Ker}\Big(R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*\big(\mathscr{E}/\sum_{i=1}^n g_i^{\lambda_i}\mathscr{E}\big)_t\Big) \subset \sum_{i=1}^n g_i^k \cdot R^q\varphi_*(\mathscr{E})_t$$

Suppose the claim is proved. For each $k \in \mathbb{N}$, choose the corresponding $\lambda_1, \dots, \lambda_n$, and let $l = \lambda_1 + \cdots + \lambda_n$. Then

$$\mathfrak{m}_t^l\mathscr{E} \subset \sum_{i=1}^n g_i^{\lambda_i}\mathscr{E}$$

and hence we have a commutative diagram

$$\begin{tikzcd}
R^q\varphi_*(\mathscr{E})_t \arrow[rr] \arrow[dr] & & R^q\varphi_*(\mathscr{E}/\sum_{i=1}^n g_i^{\lambda_i}\mathscr{E})_t \\
& R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t \arrow[ur] &
\end{tikzcd}$$

Thus (6.5.4) follows from the claim.

Step 2. Fix $k \in \mathbb{N}$. For $\nu = 1, \dots, n$, we construct inductively λ_ν such that

$$\mathrm{Ker}\Big(R^q\varphi_*(\mathscr{E}) \to R^q\varphi_*\big(\mathscr{E}/\sum_{i=1}^\nu g_i^{\lambda_i}\mathscr{E}\big)\Big)_t \subset \sum_{i=1}^\nu g_i^k \cdot R^q\varphi_*(\mathscr{E})_t \tag{6.5.5}$$

In this step, we do this for $\nu = 1$. Namely, after shrinking S to a precompact neighborhood of t (and shrinking X correspondingly to $\varphi^{-1}(S)$), we find $\lambda_1 \in \mathbb{N}$ such that

$$\operatorname{Ker}\Big(R^q\varphi_*(\mathscr{E}) \to R^q\varphi_*\big(\mathscr{E}/g_1^{\lambda_1}\mathscr{E}\big)\Big) \subset g_1^k \cdot R^q\varphi_*(\mathscr{E}) \tag{6.5.6}$$

By Lemma 6.5.5, we can shrink S and find $b_1 \in \mathbb{N}$ so that $\times g_1$ is injective on $g_1^{b_1}\mathscr{E}$. So $\times g_1^k$ is also injective on $g_1^{b_1}\mathscr{E}$. Therefore, we have a short exact sequence

$$0 \to g_1^{b_1}\mathscr{E} \xrightarrow{\times g_1^k} \mathscr{E} \to \mathscr{E}/g_1^{b_1+k}\mathscr{E} \to 0 \tag{6.5.7}$$

and hence an exact sequence

$$R^q\varphi_*(g_1^{b_1}\mathscr{E}) \xrightarrow{R^q\varphi_*(\times g_1^k)} R^q\varphi_*(\mathscr{E}) \to R^q\varphi_*(\mathscr{E}/g_1^{b_1+k}\mathscr{E})$$

Set $\lambda_1 = b_1+k$. Then the LHS of (6.5.6) equals the image of $R^q\varphi_*(g_1^{b_1}\mathscr{E}) \to R^q\varphi_*(\mathscr{E})$, and hence is a subsheaf of $g_1^k \cdot R^q\varphi_*(\mathscr{E})$ since the following diagram commutes

$$\begin{tikzcd}
R^q\varphi_*(g_1^{b_1}\mathscr{E}) \arrow[rr, "R^q\varphi_*(\times g_1^k)"] \arrow[dr, "R^q\varphi_*(\iota)"'] & & R^q\varphi_*(\mathscr{E}) \\
& R^q\varphi_*(\mathscr{E}) \arrow[ur, "\times g_1^k"'] &
\end{tikzcd}$$

where $\iota : g_1^{b_1}\mathscr{E} \hookrightarrow \mathscr{E}$ is the inclusion.

Step 3. Let $\nu \in \{2, \ldots, n\}$, and assume that $\lambda_1, \ldots, \lambda_{\nu-1}$ are chosen such that

$$\operatorname{Ker}\Big(R^q\varphi_*(\mathscr{E}) \xrightarrow{\pi} R^q\varphi_*\big(\mathscr{E}/\mathscr{G}\big)\Big) \subset \sum_{i=1}^{\nu-1} g_i^k \cdot R^q\varphi_*(\mathscr{E}) \tag{6.5.8}$$

where we write

$$\mathscr{G} = \sum_{i=1}^{\nu-1} g_i^{\lambda_i}\mathscr{E}.$$

(Namely, we assume (6.5.5) holds for $\nu - 1$ instead of ν.) By Lemma 6.5.5, after shrinking S, there is b_ν such that $\times g_\nu$ is injective on $g_\nu^{b_\nu} \cdot (\mathscr{E}/\mathscr{G})$. Similar to (6.5.7), for each $d_\nu \in \mathbb{N}$ (to be determined later) we have a short exact sequence

$$0 \to g_\nu^{b_\nu}(\mathscr{E}/\mathscr{G}) \xrightarrow{\times g_\nu^{d_\nu+k}} \mathscr{E}/\mathscr{G} \to \mathscr{E}/(\mathscr{G} + g_\nu^{b_\nu+d_\nu+k}\mathscr{E}) \to 0$$

Since we also have short exact sequences $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{E}/\mathscr{G} \to 0$, we obtain a diagram where the row and the column are exact:

$$\begin{tikzcd}
& R^q\varphi_*(\mathscr{E}) \arrow[d, "\pi"'] & \\
R^q\varphi_*\big(g_\nu^{b_\nu}(\mathscr{E}/\mathscr{G})\big) \arrow[r] & R^q\varphi_*(\mathscr{E}/\mathscr{G}) \arrow[r, "\rho"] \arrow[d, "\delta"'] & R^q\varphi_*\big(\mathscr{E}/(\mathscr{G}+g_\nu^{b_\nu+d_\nu+k}\mathscr{E})\big) \\
& R^{q+1}\varphi_*(\mathscr{G}) &
\end{tikzcd}$$

Set $\lambda_\nu = b_\nu + d_\nu + k$. Then $\mathrm{Ker}(\rho \circ \pi)_t$ is the LHS of (6.5.5). We want to show that it is a subset of the RHS of (6.5.5). Choose any

$$\varsigma \in \mathrm{Ker}(\rho \circ \pi)_t$$

Then $\pi(\varsigma) \in R^q\varphi_*(\mathscr{E}/\mathscr{G})_t$ belongs to $\mathrm{Ker}(\rho)_t$. Thus, as argued in Step 2, we have

$$\pi(\varsigma) \in g_\nu^{d_\nu+k} R^q\varphi_*(\mathscr{E}/\mathscr{G})_t$$

Choose $\sigma \in R^q\varphi_*(\mathscr{E}/\mathscr{G})_t$ such that

$$\pi(\varsigma) = g_\nu^{d_\nu+k}\sigma$$

Then $g_\nu^{d_\nu+k} \cdot \delta(\sigma) = \delta(g_\nu^{d_\nu+k}\sigma) = \delta \circ \pi(\varsigma) = 0$.

By Lemma 6.5.5, we can find d_ν such that $\times g_\nu$ is injective on $g_\nu^{d_\nu} R^{q+1}\varphi_*(\mathscr{G})$. (Note that the coherence of $R^{q+1}\varphi_*(\mathscr{G})$ is due to Grauert direct image Thm. 6.5.1.) Therefore, from $g_\nu^{d_\nu+k} \cdot \delta(\sigma) = 0$ we conclude

$$\delta(g_\nu^{d_\nu}\sigma) = g_\nu^{d_\nu} \cdot \delta(\sigma) = 0.$$

Thus, there exists $\varsigma' \in R^q\varphi_*(\mathscr{E})_t$ such that $\pi(\varsigma') = g_\nu^{d_\nu}\sigma$, and hence

$$\pi(g_\nu^k\varsigma') = g_\nu^{d_\nu+k}\sigma = \pi(\varsigma)$$

So $\varsigma - g_\nu^k\varsigma' \in \mathrm{Ker}(\pi)_t$. Thus, by (6.5.8),

$$\varsigma \in \mathrm{Ker}(\pi)_t + g_\nu^k\varsigma' \subset \sum_{i=1}^{\nu-1} g_i^k \cdot R^q\varphi_*(\mathscr{E})_t + g_\nu^k\varsigma' \subset \sum_{i=1}^{\nu} g_i^k \cdot R^q\varphi_*(\mathscr{E})_t$$

This proves (6.5.5). $\square$

6.5.4 Inverse limit and exactness

In general, the inverse limit functor is only left exact. It preserves exactness when certain "Mittag-Leffler condition" is satisfied. We do not need this general version of exactness result. We are satisfied with the following version which will be used, together with Grauert comparison theorem, to prove the base change theorem in the next section.

Proposition 6.5.6. *Let $\mathcal{A}$ be a ring and let*

$$0 \to (\mathcal{M}'_n)_{n\in\mathbb{N}} \to (\mathcal{M}_n)_{n\in\mathbb{N}} \to (\mathcal{M}''_n)_{n\in\mathbb{N}} \to 0$$

be an exact sequence of inverse systems of $\mathcal{A}$-modules, indexed by $\mathbb{N}$. Assume that each $\mathcal{M}'_n$ is Artinian. Then the following sequence is exact:

$$0 \to \varprojlim_{n\in\mathbb{N}} \mathcal{M}'_n \to \varprojlim_{n\in\mathbb{N}} \mathcal{M}_n \to \varprojlim_{n\in\mathbb{N}} \mathcal{M}''_n \to 0$$

We are mainly interested in the case that $\mathcal{A} = \mathscr{O}_{X,x}$ where $\dim_x X = 0$ (so that $\dim_{\mathbb{C}} \mathscr{O}_{X,x} < +\infty$) and each $\mathcal{M}'_n$ is a finitely-generated $\mathscr{O}_{X,x}$-module. Then $\mathcal{M}'_n$ is Artinian because it is a finite-dimensional $\mathbb{C}$-vector space.

Proof. The only non-trivial part is the surjectivity of $\varprojlim \mathcal{M}_n \to \varprojlim \mathcal{M}''_n$. Since each $\mathcal{M}_n$ is Artinian, there is $k \geqslant n$ such that for all $l \geqslant k$ the image of $\mathcal{M}'_l \to \mathcal{M}'_n$ agrees with that of $\mathcal{M}'_k \to \mathcal{M}'_n$. Thus, we may find a subsequence $(\mathcal{M}'_{n_k})_{k\in\mathbb{N}}$ of $(\mathcal{M}_n)_{n\in\mathbb{N}}$ such that for each k and each $l, \widetilde{l} > k$, the images of $\mathcal{M}'_{n_l} \to \mathcal{M}'_{n_k}$ and $\mathcal{M}'_{n_{\widetilde{l}}} \to \mathcal{M}'_{n_k}$ are equal. It suffices to show that $\varprojlim_k \mathcal{M}_{n_k} \to \varprojlim_k \mathcal{M}''_{n_k}$ is surjective. Thus, by replacing the original sequence by the subsequence, we assume that for each $n \in \mathbb{N}$ and each $m > n$,

$$\mathrm{Im}(\mathcal{M}'_m \to \mathcal{M}'_n) = \mathrm{Im}(\mathcal{M}'_{n+1} \to \mathcal{M}'_n)$$

We assume for simplicity that $\mathcal{M}'_n$ is a submodule of $\mathcal{M}_n$ and $\mathcal{M}''_n = \mathcal{M}_n/\mathcal{M}'_n$. If $m \in \mathcal{M}''_n$, we denote its residue class in $\mathcal{M}_n/\mathcal{M}'_n$ by $[m]$.

Choose $(m''_k)_{k\in\mathbb{N}} \in \varprojlim \mathcal{M}''_k$. It suffices to prove, by induction on $n \in \mathbb{N}$, that if $m_{n+1} \in \mathcal{M}_{n+1}$ is chosen such that $[m_{n+1}] = m''_{n+1}$ and if m_n is the image of m_{n+1} under $\mathcal{M}_{n+1} \to \mathcal{M}_n$, then there exists $m_{n+2} \in \mathcal{M}_{n+2}$ such that $[m_{n+2}] = m''_{n+2}$, and that m_{n+2} is sent to m_n by the map $\mathcal{M}_{n+2} \to \mathcal{M}_n$. We prove this for the case $n = 0$, since the general case can be proved in the same way.

So we are given $m''_\bullet \in \varprojlim_k \mathcal{M}''_k$ and $m_1 \in \mathcal{M}_1$ such that $[m_1] = m''_1$. And we let m_0 be the image of m_1 under $\mathcal{M}_1 \to \mathcal{M}_0$. Choose $\alpha_2 \in \mathcal{M}_2$ such that $[\alpha_2] = m''_2$, and let $\alpha_1 \in \mathcal{M}_1$ be the image of α_2 under $\mathcal{M}_2 \to \mathcal{M}_1$. Then $[\alpha_1] = m''_1$. So $[m_1 - \alpha_1] = 0$, namely $m_1 - \alpha_1 \in \mathcal{M}'_1$.

Let α_0 be the image of α_1 under $\mathcal{M}_1 \to \mathcal{M}_0$. Since $\mathrm{Im}(\mathcal{M}'_1 \to \mathcal{M}'_0) = \mathrm{Im}(\mathcal{M}'_2 \to \mathcal{M}'_0)$, $m_0 - \alpha_0$ (which is the image of $m_1 - \alpha_1$ under $\mathcal{M}'_1 \to \mathcal{M}'_0$) can be lifted to an element $\beta_2 \in \mathcal{M}_2'$. Then $m_2 = \alpha_2 + \beta_2$ satisfies that $[m_2] = m_2''$ and that its image under $\mathcal{M}_2 \to \mathcal{M}_0$ is m_0. See the following diagrams.

$$
\begin{tikzcd}
\alpha_2 \arrow[rr] \arrow[d] & & m_2'' \arrow[d] \\
\alpha_1 \arrow[rr, bend left] \arrow[d] & m_1 \arrow[r] \arrow[d] & m_1'' \arrow[d] \\
\alpha_0 & m_0 \arrow[r] & m_0''
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta_2 \arrow[ddr, bend right] & \\
& m_1 - \alpha_1 \arrow[d] \\
& m_0 - \alpha_0
\end{tikzcd}
$$

□

We are now ready to prove the second half of Grauert comparison theorem.

Proof that* (6.5.3) *is surjective. Let α_k denote the map

$$\alpha_k : R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t$$

Then we have a short exact sequence of inverse systems of $\mathscr{O}_{S,t}$-modules

$$0 \to \mathrm{Ker}(\alpha_\bullet) \to R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^\bullet R^q\varphi_*(\mathscr{E})_t \to \mathrm{Im}(\alpha_\bullet) \to 0$$

Since each $R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t$ is a coherent $\mathscr{O}_{S,t}/\mathfrak{m}_t^k$-module, by Cor. 2.7.4, it is $\mathbb{C}$-finite dimensional. So $\mathrm{Ker}(\alpha_k)$ is a finite-dimensional $\mathbb{C}$-vector space, and hence Artinian. Therefore, by Prop. 6.5.6, we obtain a short exact sequence

$$0 \to \varprojlim_k \mathrm{Ker}(\alpha_k) \to \varprojlim_k R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t^k R^q\varphi_*(\mathscr{E})_t \to \varprojlim_k \mathrm{Im}(\alpha_k) \to 0$$

Since each $\mathrm{Im}(\alpha_k)$ is a submodule of $R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t$, $\varprojlim_k \mathrm{Im}(\alpha_k)$ is a submodule of $\varprojlim_k R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t$. Therefore, to prove that (6.5.3) is surjective, it suffices to prove that for each element

$$(\sigma_k)_{k\in\mathbb{N}} \in \varprojlim_k R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t$$

we have $\sigma_k \in \mathrm{Im}(\alpha_k)$ for all k.

Choose any k. Note that $\mathrm{Im}(\alpha_k)$ equals the image of the map $R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t$. So $\mathrm{Im}(\alpha_k) = \mathrm{Ker}(\delta_k)$ where δ_k is the connecting map in the following commutative diagram

$$
\begin{tikzcd}
R^q\varphi_*(\mathscr{E})_t \arrow[r, "="] \arrow[d] & R^q\varphi_*(\mathscr{E})_t \arrow[d] & \\
R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t \arrow[r] \arrow[d, "\delta_l"'] & R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t^k\mathscr{E})_t \arrow[d, "\delta_k"'] & \\
R^{q+1}\varphi_*(\mathfrak{m}_t^l\mathscr{E})_t \arrow[r, "\mu"] & R^{q+1}\varphi_*(\mathfrak{m}_t^k\mathscr{E})_t \arrow[r, "\eta"] & R^{q+1}\varphi_*(\mathfrak{m}_t^k\mathscr{E}/\mathfrak{m}_t^l\mathscr{E})_t
\end{tikzcd}
$$

where $l \geqslant k$. Apply Lemma 6.5.3 to the sheaf $\mathfrak{m}_t^k\mathscr{E}$. We see that for each $r \in \mathbb{N}$, there exists $l \geqslant k$ such that $\mathrm{Im}(\mu) = \mathrm{Ker}(\eta)$ is a subset of $\mathfrak{m}_t^r R^{q+1}\varphi_*(\mathfrak{m}_t^k\mathscr{E})_t$. For the element $\sigma_\bullet$ chosen above, we have $\mu\delta_l(\sigma_l) = \delta_k(\sigma_k)$. So $\eta\delta_k(\sigma_k) = \eta\mu\delta_l(\sigma_l) = 0$. Therefore $\delta_k(\sigma_k)$ belongs to $\mathrm{Ker}(\eta)$, and hence belongs to $\mathfrak{m}_t^r R^{q+1}\varphi_*(\mathfrak{m}_t^k\mathscr{E})_t$. Since this is true for all $r \in \mathbb{N}$, by Krull's intersection Thm. 1.4.4, we obtain $\delta_k(\sigma_k) = 0$ and hence $\sigma_k \in \mathrm{Ker}(\delta_k) = \mathrm{Im}(\alpha_k)$, finishing the proof. □

6.6 Base change theorem

Let X, S be complex spaces and $\varphi : X \to S$ be a holomorphic map. The main reference of this section is [BS, Sec. III.3].

Notice that if $\mathscr{M} \in \mathrm{Mod}(\mathscr{O}_X)$, we have a **pullback map**

$$\varphi^* : H^q(S, \mathscr{M}) \to H^q(X, \varphi^*\mathscr{M}) \tag{6.6.1}$$

which is a natural $\mathscr{O}(S)$-module morphism described as follows. If $W \subset S$ is open, then we have a pullback morphism (cf. Def. 1.10.2)

$$\varphi^* = \mathscr{M}(W) \to (\varphi^*\mathscr{M})(\varphi^{-1}(W))$$

Thus, if $\mathfrak{W} = (W_\alpha)_{\alpha\in I}$ is an open cover of S, then $\varphi^*\mathfrak{W} = (\varphi^{-1}(W_\alpha))_{\alpha\in I}$ is an open cover of X, and the above map yields a morphism of complexes $\mathfrak{C}^\bullet(\mathfrak{W}, \mathscr{M}) \to \mathfrak{C}^\bullet(\varphi^*\mathfrak{W}, \varphi^*\mathscr{M})$. Taking cohomology and passing to the direct limit over all open covers $\mathfrak{W}$, we obtain (6.6.1).

6.6.1 Base change maps

Recall that for each $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ and $\mathscr{M} \in \mathrm{Coh}(\mathscr{O}_S)$ we have an $\mathscr{O}_S$-module morphism

$$\boxed{R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{M} \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})} \tag{6.6.2}$$

defined in Subsec. 6.5.1. Let us call it the (algebraic) **base change map**. The goal of this section and next one is to give useful criteria about whether this map is an isomorphism.

There is a geometric version of base change map. Let $\psi : Y \to S$ be a holomorphic map of complex spaces. Let $Z = X \times_S Y$ be the fiber product with Cartesian square

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & Z \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\varphi}"] \\ S & Y \arrow[l, "\psi"'] \end{tikzcd} \tag{6.6.3}$$

Then we have the (geometric) **base change map**

$$\boxed{\psi^*(R^q\varphi_*\mathscr{E}) \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})} \tag{6.6.4}$$

equivalently

$$R^q\varphi_*\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_Y \to R^q\widetilde{\varphi}_*(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z)$$

which is a natural $\mathscr{O}_Y$-module morphism defined as follows. Choose any open $W \subset S$. Then by (6.6.1), we have pullback map

$$\widetilde{\psi}^* : H^q(\varphi^{-1}(W), \mathscr{E}) \to H^q(\widetilde{\varphi}^{-1}\psi^{-1}(W), \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z)$$

Sheafifying this map gives an $\mathscr{O}_S$-module morphism

$$R^q\varphi_*(\mathscr{E}) \to \psi_*(R^q\widetilde{\varphi}_*(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z))$$

which is equivalent to (6.6.4) because ψ^* is the left adjoint of ψ_* (Prop. 1.10.3).

Remark 6.6.1. Suppose that the following two cells are Cartesian squares of complex spaces

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & Z \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\varphi}"] & Z' \arrow[l, "\alpha"'] \arrow[d, "\beta"] \\ S & Y \arrow[l, "\psi"'] & Y' \arrow[l, "\gamma"'] \end{tikzcd} \tag{6.6.5}$$

Again we choose $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$. Consider the base change maps

$$\Phi : \psi^*(R^q\varphi_*\mathscr{E}) \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})$$
$$\Psi : \gamma^*(R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})) \to R^q\beta_*(\alpha^*\widetilde{\psi}^*\mathscr{E})$$

The second one is the base change map for $\widetilde{\psi}^*\mathscr{E}$ and the second Cartesian square. It is not hard to check that the pullback of Φ

$$\gamma^*\Phi : \gamma^*\psi^*(R^q\varphi_*\mathscr{E}) \to \gamma^*R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})$$

composed with Ψ gives the base change map for $\mathscr{E}$ and the largest Cartesian square

$$\Psi \circ \gamma^*\Phi : \gamma^*\psi^*(R^q\varphi_*\mathscr{E}) \to R^q\beta_*(\alpha^*\widetilde{\psi}^*\mathscr{E})$$

6.6.2 Base change theorem

The main result of this section is the following theorem. For any Noetherian ring $\mathcal{A}$, we let $\mathrm{Mod}^{\mathrm{f}}(\mathcal{A})$ be the abelian category of finitely-generated $\mathcal{A}$-modules. We write $\mathfrak{m}_{S,t}$ as $\mathfrak{m}_t$.

Theorem 6.6.2 (Base change theorem). *Let $\varphi : X \to S$ be a proper holomorphic map. Let $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$, and assume that $\mathscr{E}$ is φ-flat. Let $q \in \mathbb{Z}$. Choose $t \in S$. Then the following are equivalent.*

(a) The functor $\mathcal{M} \mapsto R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M})$ on $\mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$ is right exact.

(b) The functor $\mathcal{M} \mapsto R^{q+1}\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M})$ on $\mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$ is left exact.

(c) For each $\mathcal{M}$, the base change map

$$R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathcal{M} \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}) \tag{6.6.6}$$

is an isomorphism.

(d) For each $\mathcal{M}$, the base change map (6.6.6) is an epimorphism.

(e) The canonical map $R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t\mathscr{E})_t$ is surjective.

(f) For any holomorphic map of complex spaces $\psi : Y \to S$, if we let (6.6.3) denote the Cartesian square, then for each $y \in \psi^{-1}(t)$, the base change map

$$\psi^*(R^q\varphi_*\mathscr{E})_y \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})_y \tag{6.6.7}$$

is an isomorphism.

The notations in (6.6.6) are understood as follows. Recall that finitely-generated $\mathscr{O}_{S,t}$-modules are equivalently germs at t of coherent $\mathscr{O}_S$-modules (cf. Thm. 2.2.2). After shrinking S to a neighborhood of t, there is $\mathscr{M} \in \mathrm{Coh}(\mathscr{O}_S)$ such that $\mathscr{M}_t = \mathcal{M}$, Then we set

$$R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathcal{M} := \big(R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{M}\big)_t \simeq R^q\varphi_*(\mathscr{E})_t \otimes_{\mathscr{O}_{S,t}} \mathcal{M}$$
$$R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}) := R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_t$$

Definition 6.6.3. If any of the equivalent statements in Thm. 6.6.2 holds, we say that $\mathscr{E}$ **satisfies base change property** in order q at t.

Remark 6.6.4. Suppose that $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ satisfies base change property in order q at t. Notice that $\widetilde{\psi}^*\mathscr{E}$ is $\mathscr{O}_Z$-coherent, and that $\widetilde{\psi}^*\mathscr{E}$ is $\widetilde{\varphi}$-flat by Thm. 5.4.9. Then $\widetilde{\psi}^*\mathscr{E}$ satisfies base change property in order q at any $y \in \psi^{-1}(t)$.

Indeed, consider (6.6.5) where the two cells are Cartesian squares. We use the notations of Rem. 6.6.1. Then by the equivalence condition (f) of Thm. 6.6.2, Φ and $\Psi \circ \gamma^*\Phi$ (both are base change maps) are isomorphisms. Therefore Ψ is an isomorphism. □

6.6.3 Preliminary discussions

We recall the following basic fact which can be proved by diagram chasing:

Lemma 6.6.5 (Four lemma). *Suppose we have a commutative diagram in an Abelian category*

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}
$$

Suppose that the rows are exact.

1. *If α, γ are epimorphisms and δ is a monomorphism, then β is an epimorphism.*
2. *If β, δ are monomorphisms and α is an epimorphism, then γ is a monomorphism.*

Assume the setting of Thm. 6.6.2. Since $\mathcal{M} \in \mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$, we have a short exact sequence in $\mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$:

$$
0 \to \mathcal{N} \to \mathscr{O}_{S,t}^n \to \mathcal{M} \to 0 \tag{6.6.8}
$$

Since $\mathscr{E}$ is φ-flat, we have a short exact sequence

$$
0 \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{N} \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_{S,t}^n \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M} \to 0
$$

in $\mathrm{Coh}(\varphi^{-1}(W))$ for some neighborhood $W \subset S$ of t. Thus we have a commutative diagram in $\mathrm{Coh}(\varphi^{-1}(W))$:

$$
\begin{tikzcd}
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathcal{N} \arrow[r] \arrow[d] & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{N}) \arrow[d] \\
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_{S,t}^n \arrow[r, "\simeq"] \arrow[d] & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_{S,t}^n) \arrow[d, "\Gamma"] \\
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathcal{M} \arrow[r, "\Phi"] \arrow[d] & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}) \\
0 &
\end{tikzcd} \tag{6.6.9}
$$

where the columns are exact.

Observation 6.6.6. By Four lemma (or by easy diagram chasing), Φ is surjective if and only if Γ is surjective.

6.6.4 Proof of Thm. 6.6.2

Proof of (a)⇔(b). Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be a short exact sequence in $\mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$. Then since $\mathscr{E}$ is φ-flat, we have a short exact sequence $0 \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}' \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M} \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}'' \to 0$ and hence an exact sequence

$$\begin{gathered}R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}') \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}) \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}'')\\ \xrightarrow{\delta^q} R^{q+1}\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}') \to R^{q+1}\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}) \to R^{q+1}\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}'')\end{gathered}$$

So the second map is surjective iff δ^p is zero iff the fourth map is injective. This proves that (a) and (b) are equivalent. □

Proof of (a)⇒(d). Choose a short exact sequence (6.6.8). By (a), the map Γ in (6.6.9) is surjective. So Φ is surjective. □

Proof of (d)⇒(c). Again, we choose a short exact sequence (6.6.8). By (d), we know that in the diagram (6.6.9), the map Φ is surjective. Since, similarly, the first row is also surjective, by Four lemma, Φ is injective. This proves (c). □

Proof of (c)⇒(a). The functor $\mathcal{M} \mapsto R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathcal{M}$ is right exact. □

We have finished proving the equivalence of (a,b,c,d).

Proof of (d)⇒(e). Set $\mathcal{M} = \mathscr{O}_{S,t}/\mathfrak{m}_t$. Then (d) says that $R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t R^q\varphi_*(\mathscr{E})_t \to R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t\mathscr{E})_t$ is surjective. This proves (e). □

Proof of (e)⇒(d). Step 1. We first prove that the base change map (6.6.6) is surjective when $\mathfrak{m}_t^k\mathcal{M} = 0$ for some $k \in \mathbb{Z}_+$. We prove it by induction on k. First consider the case $k = 1$. Then $\mathfrak{m}_t\mathcal{M} = 0$. So $\mathcal{M}$ as an $\mathscr{O}_{S,t}$-module is equivalently a module over $\mathscr{O}_{S,t}/\mathfrak{m}_t = \mathbb{C}$, which is finite direct sum of $\mathscr{O}_{S,t}/\mathfrak{m}_t$. So we may assume without loss of generality that $\mathcal{M} = \mathscr{O}_{S,t}/\mathfrak{m}_t$. Then (e) clearly implies that (6.6.6) is surjective.

Assume that (6.6.6) is surjective whenever $\mathfrak{m}_t^k\mathcal{M} = 0$. Now consider case $k+1$, namely, assume $\mathcal{M}$ is such that $\mathfrak{m}_t^{k+1}\mathcal{M} = 0$. Since $\mathscr{E}$ is φ-flat, we have a short exact sequence

$$0 \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathfrak{m}_t\mathcal{M} \to \mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M} \to \mathscr{E} \otimes_{\mathscr{O}_S} (\mathcal{M}/\mathfrak{m}_t\mathcal{M}) \to 0$$

Therefore, similar to (6.6.9), we have a commutative diagram

$$\begin{array}{ccc}
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathfrak{m}_t\mathcal{M} & \longrightarrow & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathfrak{m}_t\mathcal{M}) \\
\downarrow & & \downarrow \\
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathcal{M} & \longrightarrow & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M}) \\
\downarrow & & \downarrow \\
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} (\mathcal{M}/\mathfrak{m}_t\mathcal{M}) & \longrightarrow & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} (\mathcal{M}/\mathfrak{m}_t\mathcal{M})) \\
\downarrow & & \\
0 & &
\end{array}$$

where the columns are exact. By case k, the first and the third rows are surjective. Therefore, by Four lemma, the second row is surjective.

Step 2. We consider the general case. By Step 1, for each k, the map

$$R^q\varphi_*(\mathscr{E}) \otimes \mathcal{M} \otimes \mathscr{O}_{S,t}/\mathfrak{m}_t^k \to R^q\varphi_*(\mathscr{E} \otimes \mathcal{M} \otimes \mathscr{O}_{S,t}/\mathfrak{m}_t^k)$$

is surjective. Since its kernel is a finitely-generated $\mathscr{O}_{S,t}/\mathfrak{m}_t^k$-module and hence has finite $\mathbb{C}$-dimension (Cor. 2.7.4), by Prop. 6.5.6, the inverse limit over $k \in \mathbb{N}$ of the above map is still surjective. Therefore, we have a commutative diagram

$$\begin{tikzcd}
\varprojlim_{k\in\mathbb{N}} R^q\varphi_*(\mathscr{E}) \otimes \mathcal{M} \otimes \mathscr{O}_{S,t}/\mathfrak{m}_t^k \arrow[rr] \arrow[dr, two heads] & & \varprojlim_{k\in\mathbb{N}} R^q\varphi_*(\mathscr{E} \otimes \mathcal{M}) \otimes \mathscr{O}_{S,t}/\mathfrak{m}_t^k \arrow[dl] \\
& \varprojlim_{k\in\mathbb{N}} R^q\varphi_*(\mathscr{E} \otimes \mathcal{M} \otimes \mathscr{O}_{S,t}/\mathfrak{m}_t^k) &
\end{tikzcd}$$

where the lower left arrow is surjective. By Grauert comparison Thm. 6.5.4, the lower right arrow is injective. Therefore the horizontal arrow is surjective. Thus, the base change map $R^q\varphi_*(\mathscr{E}) \otimes \mathcal{M} \to R^q\varphi_*(\mathscr{E} \otimes \mathcal{M})$ is surjective by Lemma 6.6.7. □

Lemma 6.6.7. *Let $(\mathcal{A}, \mathfrak{m})$ be a Noetherian local ring. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a morphism of finitely-generated $\mathcal{A}$-modules. If the induced map*

$$\widehat{\varphi} : \varprojlim_{k\in\mathbb{N}} \mathcal{M}/\mathfrak{m}^k\mathcal{M} \to \varprojlim_{k\in\mathbb{N}} \mathcal{N}/\mathfrak{m}^k\mathcal{N}$$

is surjective, then φ is surjective. If $\widehat{\varphi}$ is injective, then φ is injective.

Proof. Suppose that $\widehat{\varphi}$ is surjective. Since we have a commutative diagram

$$\begin{tikzcd}
\varprojlim_{k\in\mathbb{N}} \mathcal{M}/\mathfrak{m}^k\mathcal{M} \arrow[r, "\widehat{\varphi}"] \arrow[d] & \varprojlim_{k\in\mathbb{N}} \mathcal{N}/\mathfrak{m}^k\mathcal{N} \arrow[d, two heads] \\
\mathcal{M}/\mathfrak{m}\mathcal{M} \arrow[r] & \mathcal{N}/\mathfrak{m}\mathcal{N}
\end{tikzcd}$$

where the vertical arrows are surjective, $\mathcal{M}/\mathfrak{m}\mathcal{M} \to \mathcal{N}/\mathfrak{m}\mathcal{N}$ is surjective. So φ is surjective by Nakayama's lemma.

Suppose that $\widehat{\varphi}$ is injective. Choose $\xi \in \mathrm{Ker}(\varphi)$. Then ξ corresponds to the zero element of $\varprojlim_k \mathcal{M}/\mathfrak{m}^k\mathcal{M}$. So $\xi = 0$ by Krull's intersection Thm. 1.4.4. □

Proof of (f)⇒(e). Let $Y = \mathrm{Specan}(\mathscr{O}_S/\mathfrak{m}_t)$ and let $\psi : Y \to S$ be the inclusion map. Let $y = t$. Then the geometric base change map (6.6.7) becomes

$$R^q\varphi_*\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_{S,t}/\mathfrak{m}_t \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_{S,t}/\mathfrak{m}_t)$$

whose surjectivity clearly implies (e). □

Proof of (c)⇒(f). Step 1. Consider the Cartesian square

$$\begin{tikzcd} X & Z \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\varphi}"] \\ S \arrow[from=u, "\varphi"'] & Y \arrow[l, "\psi"] \end{tikzcd} \tag{6.6.10}$$

We consider the case that Y and S are (non-necessarily reduced) single points. Then by Cor. 2.8.3, we have a natural equivalence of $\mathscr{O}_X$-algebras

$$\widetilde{\varphi}^*\psi_*\mathscr{O}_Y \simeq \widetilde{\psi}_*\mathscr{O}_Z. \tag{6.6.11}$$

We identify Y and S as topological spaces through the map ψ. Then Z can be identified with X as topological spaces through $\widetilde{\psi}$. Now there are two different sheaves of local $\mathbb{C}$-algebras on $X = Z$, namely $\mathscr{O}_X$ and $\mathscr{O}_Z$. And we have a morphism $\mathscr{O}_X \to \mathscr{O}_Z$ so that $\mathscr{O}_Z$ is an $\mathscr{O}_X$-algebra. Similar things can be said about S and Y.

Now (6.6.11) reads $\varphi^*\mathscr{O}_Y \simeq \mathscr{O}_Z$ (as an equivalence of $\mathscr{O}_X$-algebras). By replacing $\mathscr{O}_Z$ with $\varphi^*\mathscr{O}_Y$ (namely, defining $\varphi^*\mathscr{O}_Y$ to be the new structure sheaf of Z), we may assume $\mathscr{O}_Z = \varphi^*\mathscr{O}_Y$.

Thus, for $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$,

$$R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E}) = R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z) = R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_X} \varphi^*\mathscr{O}_Y) = R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_Y)$$

Thus the geometric base change map $\psi^*(R^q\varphi_*\mathscr{E})_y \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})_y$ is equal to the algebraic one

$$R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_Y \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_Y)$$

which is an isomorphism by (c). This proves (f) in this special case.

Step 2. We consider the case that Y is a single point but S is not necessarily so. Write $Y = \{y\}$. Let $T = \mathrm{Supp}(\psi)$ (recall Def. 2.3.3). So T equals $\{t\}$ (where $t = \psi(y)$) as a set, and $\mathscr{O}_T = \mathscr{O}_S/\mathscr{J}$ where $\mathscr{J} = \mathscr{A}nn_{\mathscr{O}_S}(\psi_*\mathscr{O}_Y)$. So T is a closed complex subspace of S, and $\psi : Y \to S$ equals $\iota\alpha$ where $\alpha : Y \to T$ is the restriction of ψ (cf. Thm. 1.4.8) and $\iota : T \to S$ is the inclusion map.

Thus, we have commutative diagrams

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & \varphi^{-1}(T) \arrow[l, hook'] \arrow[d] & Z \arrow[l] \arrow[d] \\ S & T \arrow[l, hook', "\iota"'] & Y \arrow[l, "\alpha"'] \end{tikzcd}$$

where the two cells are Cartesian squares, and the largest rectangle is equal to (6.6.10). The (geometric) base change map for $\mathscr{E}$ and the first cell is

$$R^q\varphi_*(\mathscr{E})\otimes_{\mathscr{O}_S}\mathscr{O}_T \to R^q\varphi_*(\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{O}_T)$$

which, by (c), is an isomorphism if we shrink S to a neighborhood of t. We claim that the base change map for $\mathscr{E}|_{\varphi^{-1}(T)} = \mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{O}_T$ and the second cell is an isomorphism. Then the base change map for $\mathscr{E}$ and the largest rectangle (namely (6.6.10)) is an isomorphism by Rem. 6.6.1, which finishes the proof of (f) in this case.

To prove the claim, notice the commutative diagram

$$\begin{tikzcd}[column sep=tiny] R^q\varphi_*(\mathscr{E})\otimes_{\mathscr{O}_S}\mathscr{O}_T\otimes_{\mathscr{O}_T}\mathscr{O}_T/\mathfrak{m}_{T,t} \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & R^q\varphi_*(\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{O}_T)\otimes_{\mathscr{O}_T}\mathscr{O}_T/\mathfrak{m}_{T,t} \arrow[dl] \\ & R^q\varphi_*(\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{O}_T\otimes_{\mathscr{O}_T}\mathscr{O}_T/\mathfrak{m}_{T,t}) & \end{tikzcd}$$

where the horizontal and the lower left arrows are isomorphisms by (c) (after shrinking S). Thus the lower right arrow is an isomorphism. Thus $\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{O}_T$ satisfies base change property-condition (e) in order q at t. Therefore, by Step 1, it satisfies base change property-condition (f). This proves the claim.

Step 3. Choose any $k\in\mathbb{Z}_+$, let $Y' = \mathrm{Specan}(\mathscr{O}_Y/\mathfrak{m}_{Y,y}^k)$, and let $\gamma : Y' \to Y$ be the inclusion map. We consider the Cartesian squares

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & Z \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\varphi}"] & Z' \arrow[l, hook', "\alpha"'] \arrow[d, "\beta"] \\ S & Y \arrow[l, "\psi"'] & Y' \arrow[l, hook', "\gamma"'] \end{tikzcd}$$

and let Φ and Ψ be as in Rem. 6.6.1. By Step 2, $\Psi\circ\gamma^*\Phi$, the base change map for $\mathscr{E}$ and the largest Cartesian square, is an isomorphism. We have

$$\gamma^*\Phi = \Phi\otimes\mathbf{1} : \psi^*(R^q\varphi_*\mathscr{E})_y\otimes\mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^k \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})_y\otimes\mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^k$$
$$\Psi : R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})_y\otimes\mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^k \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E}/\mathfrak{m}_{Y,y}^k\widetilde{\psi}^*\mathscr{E})_y$$

Taking inverse limit over all k, we have a commutative diagram

$$\begin{tikzcd}
\varprojlim_k \psi^*(R^q\varphi_*\mathscr{E})_y \otimes \mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^k \arrow[rr] \arrow[dr, "\simeq"] & & \varprojlim_k R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})_y \otimes \mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^k \arrow[dl] \\
& \varprojlim_k R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E}/\mathfrak{m}_{Y,y}^k\widetilde{\psi}^*\mathscr{E})_y &
\end{tikzcd}$$

where the lower left arrow is an isomorphism. By Grauert comparison Thm. 6.5.4, the lower right arrow is injective. Therefore the horizontal arrow is an isomorphism. Therefore, by Lemma 6.6.7, the stalk map

$$\Phi_y : \psi^*(R^q\varphi_*\mathscr{E})_y \to R^q\widetilde{\varphi}_*(\widetilde{\psi}^*\mathscr{E})_y$$

is an isomorphism. This finishes the proof of (f). □

6.6.5 Applications

Let $\varphi : X \to S$ be a proper holomorphic map. Let $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$, and assume that $\mathscr{E}$ is φ-flat. Let $q \in \mathbb{Z}$ and $t \in S$.

Definition 6.6.8. We say that $\mathscr{E}$ is **cohomologically flat** in order q at t, if $\mathscr{E}$ satisfies base change property in orders q and $q-1$ at t.

Proposition 6.6.9. *The following are equivalent.*

(a) $\mathscr{E}$ is cohomologically flat in order q at t.

(b) The functor $\mathcal{M} \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathcal{M})$ on $\mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$ is exact.

(c) $\mathscr{E}$ satisfies base change property in order q at t, and $R^q\varphi_*(\mathscr{E})_t$ is a (finite-rank) free $\mathscr{O}_{S,t}$-module.

Proof. (a)⇔(b) is obvious.

(c)⇒(b): Since $R^q\varphi_*(\mathscr{E})_t$ is $\mathscr{O}_{S,t}$-free, the functor $\mathcal{M} \to R^q\varphi_*(\mathscr{E})_t \otimes \mathcal{M}$ is exact. Since $\mathscr{E}$ satisfies base change property in order q, $R^q\varphi_*(\mathscr{E} \otimes \mathcal{M})$ is naturally equivalent to $R^q\varphi_*(\mathscr{E})_t \otimes \mathcal{M}$, and hence is exact over $\mathcal{M}$. This proves (b).

(a,b)⇒(c): $R^q\varphi_*(\mathscr{E})_t \otimes \mathcal{M}$ is naturally equivalent to $R^q\varphi_*(\mathscr{E} \otimes \mathcal{M})$, and the latter is exact over $\mathcal{M}$. So $R^q\varphi_*(\mathscr{E}) \otimes -$ is exact on $\mathrm{Mod}^{\mathrm{f}}(\mathscr{O}_{S,t})$. By Exp. 5.3.11, we have $\mathrm{Tor}_1^{\mathscr{O}_{S,t}}(R^q\varphi_*(\mathscr{E})_t, \mathscr{O}_{S,t}/I) = 0$ for each ideal $I \subset \mathscr{O}_{S,t}$. Therefore, by Prop. 5.4.1, $R^q\varphi_*(\mathscr{E})_t$ is a flat $\mathscr{O}_{S,t}$-module. So it is free by Thm. 5.5.11. □

Recall that $X_t = \varphi^{-1}(t) = \mathrm{Specan}(\mathscr{O}_X/\mathfrak{m}_{S,t}\mathscr{O}_X)$. We now give an extremely useful criterion on cohomological flatness.

Theorem 6.6.10. *Suppose that $H^q(X_t, \mathscr{E}|_{X_t}) = 0$. Then $R^q\varphi_*(\mathscr{E})_t = 0$, and $\mathscr{E}$ is cohomologically flat in order q at t.*

The semicontinuity theorem 6.7.4, to be proved in the next section, implies that if $H^q(X_t, \mathscr{E}|_{X_t}) = 0$, then $H^q(X_s, \mathscr{E}|_{X_s}) = 0$ for each s in a neighborhood of t. Then this theorem implies that $\mathscr{E}$ satisfies base change property in order $q-1$ on that neighborhood.

Proof. The canonical map from $R^q\varphi_*(\mathscr{E})_t$ to $R^q\varphi_*(\mathscr{E}/\mathfrak{m}_t\mathscr{E})_t = H^q(X_t, \mathscr{E}|_{X_t})$ is surjective since the latter is 0. So $\mathscr{E}$ satisfies condition (e) of Base change Thm. 6.6.2. Thus $\mathscr{E}$ satisfies base change property in order q at t. By (c) of Thm. 6.6.2, the map

$$R^q\varphi_*(\mathscr{E})_t \otimes_{\mathscr{O}_{S,t}} \mathscr{O}_{S,t}/\mathfrak{m}_t \to R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_S/\mathfrak{m}_t)_t = 0$$

is bijective. Hence $R^q\varphi_*(\mathscr{E})_t/\mathfrak{m}_t R^q\varphi_*(\mathscr{E})_t = 0$. So $R^q\varphi_*(\mathscr{E})_t = 0$ by Nakayama's lemma. Therefore $\mathscr{E}$ is cohomologically flat in order q at t by Prop. 6.6.9-(c). $\square$

Corollary 6.6.11. *Assume that $H^{q+1}(X_t, \mathscr{E}|_t) = H^{q-1}(X_t, \mathscr{E}|_{X_t}) = 0$. Then $R^q\varphi_*(\mathscr{E})$ is a finite-rank free $\mathscr{O}_{S,t}$-module, and the canonical map*

$$R^q\varphi_*(\mathscr{E})|_t \to H^q(X_t, \mathscr{E}|_{X_t})$$

is an isomorphism of $\mathbb{C}$-vector spaces.

Proof. By Thm. 6.6.10, $\mathscr{E}$ satisfies base change property in orders $q-1, q, q+1$ at t. So it is cohomologically flat in order q at t. $\square$

6.7 Semicontinuity and base change theorem

In this section, we let $\varphi : X \to S$ be a proper holomorphic map of complex spaces, and let $\mathscr{E} \in \mathrm{Coh}(\mathscr{O}_X)$ be a φ-flat.

The material of this section is adapted from [GR-b, Sec. 10.5].

Remark 6.7.1. Choose any precompact open subset $W \subset S$. Then there exists $q_0 \in \mathbb{N}$ such that for every $q > q_0$, $H^q(\varphi^{-1}(V), \mathscr{E}) = 0$ and $H^q(X_t, \mathscr{E}|_{X_t}) = 0$ whenever V is a Stein open subset of W and $t \in W$. In particular, for each $t \in W$, $R^q\varphi_*(\mathscr{E})_t = 0$.

Proof. Since φ is proper, we can cover the compact set $\varphi^{-1}(\overline{W})$ by finitely many Stein open subsets of X, denoted by $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$. Let $q_0 + 1$ be the cardinality of I. Then for each Stein open subset $V \subset W$, $\varphi^{-1}(V) \cap \mathfrak{U} = (\varphi^{-1}(V) \cap U_\alpha)_{\alpha\in I}$ is a Stein cover of $\varphi^{-1}(V)$. (Recall Exp. 6.4.6). Therefore, by Leray's Thm. 6.4.1 and Cartan's Thm. B (Thm. 6.4.4), we have $H^\bullet(\varphi^{-1}(V), \mathscr{E}) = H^\bullet(\varphi^{-1}(V) \cap \mathfrak{U}, \mathscr{E})$. By the alternate condition of Čech cochains, if $q > q_0$ then $C^q(\varphi^{-1}(V) \cap \mathfrak{U}, \mathscr{E}) = 0$. Hence $H^q(\varphi^{-1}(V), \mathscr{E}) = 0$. Likewise, for $q > q_0$ and $t \in W$ we have $H^q(X_t, \mathscr{E}|_{X_t}) = 0$. $\square$

Thanks to the above observation, for each $t \in S$ we can define the **character of the sheaf** $\mathscr{E}|_{X_t}$ to be

$$\chi(X_t, \mathscr{E}|_{X_t}) = \sum_{q \in \mathbb{N}} (-1)^q \dim_{\mathbb{C}} H^q(X_t, \mathscr{E}|_{X_t})$$

which is a finite sum on each precompact Stein open subset of S.

Theorem 6.7.2 (Invariance of characters). *The character function*

$$t \in S \mapsto \chi(X_t, \mathscr{E}|_{X_t})$$

is locally constant.

We will not use this theorem in this chapter, and we refer the readers to [BS, Sec. III.4] for the proof. In the next chapter, we will prove this result under the assumption that φ is projective.

Definition 6.7.3. We say that $\mathscr{E}$ satisfies **base change property** (resp. is **cohomologically flat**) in order q, if it does so at every $t \in S$.

6.7.1 Semicontinuity theorem

Theorem 6.7.4 (Semicontinuity theorem). *For each $q \in \mathbb{Z}$, the dimension function*

$$\mathbf{d} : t \in S \mapsto \dim_{\mathbb{C}} H^q(X_t, \mathscr{E}|_{X_t}) \tag{6.7.1}$$

is upper-semicontinuous.

We divide the proof into several steps.

Lemma 6.7.5. *Let $p \in \mathbb{Z}$, and assume that $R^q\varphi_*(\mathscr{E})$ is locally free for every $q \geqslant p$. Then for every $q \geqslant p$, $\mathscr{E}$ is cohomologically flat in order q, and the dimension function $t \in S \mapsto \dim H^q(X_t, \mathscr{E}|_{X_t})$ is locally constant.*

Proof. Shrink S to any precompact Stein open subset. Then by Rem. 6.7.1, this lemma is clearly true for sufficiently large q. Now choose $q \geqslant p$ and assume that $\mathscr{E}$ is cohomologically flat in order $q+1$. Then $\mathscr{E}$ satisfies base change property in order q and, as $R^q\varphi_*(\mathscr{E})$ is locally free, $\mathscr{E}$ is cohomologically flat in order q by Prop. 6.6.9.

Since $R^q\varphi_*(\mathscr{E})$ is locally free, its fibers have locally constant dimensions. By base change property, $H^q(X_t, \mathscr{E}|_{X_t}) \simeq R^q\varphi_*(\mathscr{E})|_t$. So $H^q(X_t, \mathscr{E}|_{X_t})$ is locally constant with respect to t. $\square$

Lemma 6.7.6. *Suppose that $\psi : Y \to S$ is a surjective finite holomorphic map of (non-necessarily reduced) complex spaces. Let*

$$\begin{tikzcd} X & Z \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\varphi}"] \\ S \arrow[from=u, "\varphi"'] & Y \arrow[l, "\psi"'] \end{tikzcd}$$

be the Cartesian square. Suppose that the semicontinuity theorem holds for $\widetilde{\psi}^*\mathscr{E}$, namely, $y \in Y \mapsto \dim H^q(Z_y, \widetilde{\psi}^*\mathscr{E}|_{Z_y})$ is upper-semicontinuous. Then the semicontinuity theorem holds for $\mathscr{E}$.

Proof. For each $y \in Y$, since we have Cartesian squares

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & Z \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\varphi}"] & Z_y \arrow[l, hook'] \arrow[d] \\ S & Y \arrow[l, "\psi"'] & y \arrow[l, hook'] \end{tikzcd}$$

where the largest Cartesian square is also equivalent to

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & X_{\psi(y)} \arrow[l, hook'] \arrow[d] \\ S & \psi(y) \arrow[l, hook'] \end{tikzcd}$$

we have equivalences of vector spaces

$$H^q(Z_y, \widetilde{\psi}^*\mathscr{E}|_{Z_y}) \simeq H^q(X_{\psi(y)}, \mathscr{E}|_{X_{\psi(y)}}) \tag{6.7.2}$$

Choose any $t \in S$, and let $r = \dim H^q(X_t, \mathscr{E}|_{X_t})$. Then $\psi^{-1}(t)$ is a non-empty finite set. By (6.7.2), for each $y \in \psi^{-1}(t)$, we have $\dim H^q(Z_y, \widetilde{\psi}^*\mathscr{E}|_{Z_y}) = r$. Since $\widetilde{\psi}^*\mathscr{E}$ satisfies the semicontinuity theorem, there is a neighborhood $V \subset Y$ of $\psi^{-1}(t)$ such that for each $y' \in V$ we have $\dim H^q(Z_{y'}, \widetilde{\psi}^*\mathscr{E}|_{Z_{y'}}) \leqslant r$. By Prop. 2.4.1, there is a neighborhood $W \subset S$ of t such that $\psi^{-1}(W) \subset V$. Then by (6.7.2), for each $t' \in W$ we have $\dim H^q(X_{t'}, \mathscr{E}|_{X_{t'}}) \leqslant r$. This proves that $\mathscr{E}$ satisfies the semicontinuity theorem. □

The starting point of the proof of Semicontinuity theorem is the following special case:

Lemma 6.7.7. *The Semicontinuity Thm. 6.7.4 holds whenever S is a smooth 1-dimensional complex manifold, i.e. a Riemann surface.*

Proof. We may well assume that S is an open subset of $\mathbb{C}$. Let $z \in \mathscr{O}(\mathbb{C})$ be the standard coordinate function of $\mathbb{C}$. We note that for each $t \in S$, the map

$$\Phi_t^q : R^q\varphi_*(\mathscr{E})|_t \to H^q(X_t, \mathscr{E}|_{X_t}) \tag{6.7.3}$$

is injective. Indeed, consider the short exact sequence

$$0 \to \mathscr{O}_S \xrightarrow{\times(z-t)} \mathscr{O}_S \to \mathscr{O}_S/\mathfrak{m}_{S,t} \to 0$$

as a special case of (6.6.8). Then (6.6.9) becomes the commutative diagram

$$\begin{tikzcd}
R^q\varphi_*(\mathscr{E}) \arrow[rr, "="] \arrow[d, "\times(z-t)"] & & R^q\varphi_*(\mathscr{E}) \arrow[d, "\times(z-t)"] \\
R^q\varphi_*(\mathscr{E}) \arrow[rr, "="] \arrow[d] & & R^q\varphi_*(\mathscr{E}) \arrow[d, "\Gamma_t^q"] \\
R^q\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_S/\mathfrak{m}_{S,t} \arrow[rr, "\Phi_t^q"] \arrow[d] & & R^q\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_S/\mathfrak{m}_{S,t}) \\
0 & &
\end{tikzcd} \tag{6.7.4}$$

where the columns are exact. The map Φ_t^q in the above diagram is precisely the map Φ_t^q in (6.7.3). And Φ_t^q is injective by Four Lemma 6.6.5.

By Rem. 6.7.1, for sufficiently large q we have $R^q\varphi_*(\mathscr{E}) = 0$ and $H^q(X_t, \mathscr{E}|_{X_t}) = 0$ for all $t \in S$. Thus, by Thm. 3.8.3, there is a nowhere dense analytic subset A of S such that $R^q\varphi_*(\mathscr{E})$ is locally free outside A for all q. By Ritt's Lemma 3.10.6, $\dim A = 0$. So A is an isolated Hausdorff space, and hence is a discrete subset of S.

By Lemma 6.7.5, the dimension function

$$\mathbf{d} : t \in S \mapsto \dim H^q(X_t, \mathscr{E}|_{X_t})$$

is locally constant on $S \backslash A$. It is clear that the function

$$\mathbf{r} : t \in S \mapsto \dim R^q\varphi_*(\mathscr{E})|_t$$

is upper-semicontinuous (cf. Cor. 1.2.19). For each $t \in A$, choose a neighborhood $W \subset S$ of t such that $W \cap A = \{t\}$ and that $\mathbf{r}|_W \leqslant \mathbf{r}(t)$. Since the map Φ_t^q in (6.7.3) is injective, we have $\mathbf{r}(t) \leqslant \mathbf{d}(t)$. By Lemma 6.7.5, $\mathscr{E}$ satisfies base change property in all orders outside $S \backslash A$. So $\mathbf{d} = \mathbf{r}$ outside A. Therefore $\mathbf{d}|_W \leqslant \mathbf{d}(t)$. This proves that d is upper-seminicontinuous at every point of A, and hence everywhere on S. □

Let us make some comments on the above proof, which will be helpful for the following proof of base change theorem.

Remark 6.7.8. As noticed in Obs. 6.6.6, by Four Lemma, the map Φ_t^q in (6.7.3) is surjective iff the map Γ_t^q in (6.7.4) is surjective. Since we have a long exact sequence

$$R^q\varphi_*(\mathscr{E})_t \xrightarrow{\Gamma_t^q} R^q\varphi_*(\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{O}_S/\mathfrak{m}_{S,t})_t \to R^{q+1}\varphi_*(\mathscr{E})_t \xrightarrow{\times(z-t)} R^{q+1}\varphi_*(\mathscr{E})_t$$

induced by $0 \to \mathscr{E}\otimes\mathscr{O}_S \xrightarrow{\times(z-t)} \mathscr{E}\otimes\mathscr{O}_S \to \mathscr{E}\otimes\mathscr{O}_S/\mathfrak{m}_{S,t} \to 0$, we see that Γ_t^q is surjective iff $(z-t)$ is a non zero-divisor of $R^{q+1}\varphi_*(\mathscr{E})_t$. And the latter is equivalent to that $R^{q+1}\varphi_*(\mathscr{E})_t$ is $\mathscr{O}_{S,t}$-flat (by Cor. 5.5.8 or by Slicing Criterion 5.5.7), and is equivalent to that $R^{q+1}\varphi_*(\mathscr{E})_t$ is free of finite-rank (Thm. 5.5.11). We conclude that

$$\Phi_t^q \text{ is surjective (and hence bijective)} \quad\Longleftrightarrow\quad R^{q+1}\varphi_*(\mathscr{E})_t \text{ is } \mathscr{O}_{S,t}\text{-free} \tag{6.7.5}$$

Remark 6.7.9. Suppose that the dimension function $\mathbf{d}: t\in S\mapsto \dim H^q(X_t,\mathscr{E}|_{X_t})$ is locally constant. Using the notations in the last paragraph of the proof of Lemma 6.7.7, for each $t\in A$, the neighborhood W can be chosen such that $\mathbf{d}$ is constant on W. So for each $s\in W\backslash\{t\}$ we have $\mathbf{d}(t)=\mathbf{d}(s)=\mathbf{r}(s)\leqslant\mathbf{r}(t)$. Since $\mathbf{r}\leqslant\mathbf{d}$, we conclude $\mathbf{d}(t)=\mathbf{r}(t)$. So $\mathbf{d}=\mathbf{r}$ on A, and hence on S. So Φ_t^q is bijective for all $t\in S$. It follows that $\mathscr{E}$ satisfies base change in order q. We conclude that

$$\mathbf{d} \text{ is locally constant} \quad\Longrightarrow\quad \Phi_t^q \text{ is bijective for all } t\in S \tag{6.7.6}$$

Proof of Semicontinuity Thm. 6.7.4. It suffices to assume $\dim S<+\infty$. We prove the theorem by induction on $n=\dim S$. Since $\mathrm{red}(S)\to S$ is finite and surjective, by Lemma 6.7.6, it suffices to assume that S is reduced. Similarly, since the normalization $\widehat{S}\to S$ is finite and surjective, by Lemma 6.7.6 again, it suffices to assume that S is (reduced and) normal. By Prop. 4.9.2, S is then a disjoint union of connected (equivalently, irreducible) normal open subspaces. So we may well assume that S is normal and connected.

Assume $\dim S\leqslant 1$. If we assume that S is normal and connected, then by Thm. 4.8.2, S is either a single reduced point or a connected Riemann surface. Then Thm. 6.7.4 follows from Lemma 6.7.7.

Choose $n\in\mathbb{Z}_+$ and assume that Thm. 6.7.4 holds whenever $\dim S\leqslant n$. Now assume that $\dim S=n+1$ and S is normal and connected. Note that by Prop. 4.8.1, S has pure dimension $n+1$. By Thm. 3.8.3 and Lemma 6.7.5, there is a nowhere dense analytic subset A of S such that $R^q\varphi_*(\mathscr{E})$ is locally free outside A for all q. By Lemma 6.7.5, the dimension function $\mathbf{d}$ in (6.7.1) is locally constant on $S\backslash A$. By Prop. 4.9.1, $S\backslash A$ is connected. So

$$\mathbf{d} \text{ is a constant on } S\backslash A.$$

Choose any $t \in A$. It remains to show that $\mathbf{d}|_W \leqslant d(t)$ for some neighborhood W of t. By Ritt's Lemma 3.10.6, $\dim A \leqslant n$. Thus, by induction on case n, we can find a neighborhood W such that

$$\mathbf{d}|_{W \cap A} \leqslant \mathbf{d}(t).$$

We claim that we can shrink W to a smaller neighborhood of t such that there exists $f \in \mathscr{O}_S(W)$ satisfying that the germ $(N(f), t)$ is not inside the germ (A, t), and that the germ f_t is a non-zero element of $\mathscr{O}_{S,t}$. (Recall that $N(f)$ is the zero set of f.)

Suppose this claim is true. Then f_t is a non zero-divisor of $\mathscr{O}_{S,t}$ because $\mathscr{O}_{S,t}$ is normal and hence an integral domain (Prop. 4.8.1). Then by Active Lemma 3.10.3, we have $\dim_t N(f) = \dim_t S - 1 = n$. By Cor. 3.9.4, we may shrink W further so that $\dim N(f) = n$. By assumption on case n, we may shrink W so that $\mathbf{d}|_{W \cap N(f)} \leqslant \mathbf{d}(t)$. Since $W \cap N(f) \not\subset A \cap N(f)$, there exists $p \in (W \cap N(f)) \backslash A$. Then since $\mathbf{d}|_{S \backslash A}$ is constant, we have $\mathbf{d}|_{S \backslash A} = \mathbf{d}(p) \leqslant \mathbf{d}(t)$. This, together with $\mathbf{d}|_{W \cap A} \leqslant \mathbf{d}(t)$, shows $\mathbf{d}|_W \leqslant \mathbf{d}(t)$. The proof is then finished.

Let us prove the claim. Suppose that $\dim_t A > 0$. Then by Rem. 3.10.4, $\mathscr{O}_{A,t} = \mathscr{O}_{S,t}/\mathscr{I}_{A,t}$ contains a non zero-divisor g. We lift it to an element $f \in \mathscr{O}_{S,t}$. So $N(g) = N(f) \cap A$. By Active Lemma, we have $\dim_t N(f) \cap A = \dim_t A - 1 \leqslant n - 1$. So $(N(f), t)$ is not inside (A, t), otherwise we have

$$\dim_t N(f) = \dim_t N(f) \cap A \leqslant n - 1 = \dim_t S - 2$$

which is impossible since, according to the definition of dimensions (Def. 3.9.1), we must have $\dim_t N(f) \geqslant \dim_t S - 1$.

Suppose $\dim_t A = 0$. We shrink W and choose $f \in \mathscr{O}_S(W)$ such that $f_t \neq 0$. Then since $\dim_t N(f) \geqslant n > 0 = \dim_t A$, $(N(f), t)$ cannot be inside (A, t). Thus, in both cases we have proved the claim. $\square$

6.7.2 Base change theorem

Theorem 6.7.10 (Grauert base change theorem). *Let $q \in \mathbb{Z}$. Consider the following statements:*

(a) $\mathscr{E}$ is cohomologically flat in order q.

(b) The dimension function

$$\mathbf{d} : t \in S \mapsto \dim_{\mathbb{C}} H^q(X_t, \mathscr{E}|_{X_t}) \tag{6.7.7}$$

is locally constant.

Then (a)⇒(b). If S is reduced, then (b)⇒(a).

Proof of (a)⇒(b). By Prop. 6.6.9, $R^q\varphi_*(\mathscr{E})$ is locally free, and its fiber at t is isomorphic to $H^q(X_t,\mathscr{E}|_{X_t})$ since $\mathscr{E}$ satisfies base change property in order q. So (b) follows. □

We shall only prove (b)⇒(a) in the case that S is smooth. See [BS, Sec. III.4] for the proof of the general case.

Lemma 6.7.11. *Assume that S is smooth and* $\mathbf{d}$ *is locally constant. Then $R^{q+1}\varphi_*(\mathscr{E})$ is a torsion free $\mathscr{O}_S$-module.*

Proof. Recall that the torsion sheaf of any coherent sheaf is coherent by Cor. 3.14.5. Assume that the complex manifold S has pure dimension. We prove the lemma by induction on $\dim S$. If $\dim S=0$, then the lemma is trivial. If $\dim S=1$ then $R^{q+1}\varphi_*(\mathscr{E})$ is locally free by (6.7.5) and (6.7.6). So it is torsion free.

Now assume that the lemma holds whenever $\dim S\leqslant n$ ($n\in\mathbb{Z}_+$). Assume that $\dim S=n+1$. Let $\mathscr{T}$ be the torsion sheaf of $\mathscr{M}:=R^{q+1}\varphi_*(\mathscr{E})$. Suppose that $\mathscr{T}$ is non-zero. We assume for simplicity that S is an open subset of $\mathbb{C}^{n+1}$ and $\mathscr{T}_0\neq 0$, and we shall find a contradiction.

Step 1. Since $\mathscr{M}$ is locally free outside a nowhere dense analytic subset of S (Thm. 3.8.3), the support $A=\mathrm{Supp}(\mathscr{T})$ must be a nowhere dense analytic subset of S. By our assumption, $0\in A$. By shrinking S, we assume that S is an open ball centered at 0. Then there must be a hyperplane H of S passing through 0 whose intersection with A is nowhere dense: otherwise, by Prop. 4.10.2, for each H we have $A\cap H=H$, and hence $A=S$ (here we use the fact $n+1>1$), impossible. By an invertible linear transform, we assume that $H=N(z_1)=\{t\in S:z_1(t)=0\}$ where $(z_1,\dots,z_{n+1})$ are the standard coordinates of $\mathbb{C}^{n+1}$.

So $A\cap N(z_1)$ is a nowhere dense analytic subset of $N(z_1)$. Note that

$$\mathrm{Supp}(\mathscr{T}/z_1\mathscr{M}\cap\mathscr{T})\subset\mathrm{Supp}(\mathscr{T})\cap N(z_1)=A\cap N(z_1)$$

We consider $\mathscr{T}/z_1\mathscr{M}\cap\mathscr{T}$ as a coherent sheaf on $H=N(z_1)$. Then by Prop. 3.14.3, $\mathscr{T}/z_1\mathscr{M}\cap\mathscr{T}$ is the $\mathscr{O}_H$-torsion sheaf of itself. In other words, every element of $\mathscr{T}/z_1\mathscr{M}\cap\mathscr{T}$ is a torsion element of $\mathscr{O}_H$.

$\mathscr{T}_0/z_1\mathscr{M}_0\cap\mathscr{T}_0$, the stalk of $\mathscr{T}/z_1\mathscr{M}\cap\mathscr{T}$ at 0, is non-zero. Otherwise, we have $\mathscr{T}_0\subset z_1\mathscr{M}_0$. Every $s\in\mathscr{T}_0$ can be written as $z_1\gamma$ for some $\gamma\in\mathscr{M}_0$, and since s kills a non-zero element of $\mathscr{O}_{S,0}$, so does γ. This implies $\mathscr{T}_0\subset z_1\mathscr{T}_0$, and hence $\mathscr{T}_0=0$ by Nakayama's lemma, which is impossible.

Step 2. To summarize, we have shown that $\mathscr{T}_0/z_1\mathscr{M}_0\cap\mathscr{T}_0$ is a non-zero torsion $\mathscr{O}_{H,0}$-module. It is clearly a submodule of $\mathscr{M}_0/z_1\mathscr{M}_0$, the stalk at 0 of

$$\mathscr{M}\otimes_{\mathscr{O}_S}\mathscr{O}_S/z_1\mathscr{O}_S=R^{q+1}\varphi_*(\mathscr{E})\otimes_{\mathscr{O}_S}\mathscr{O}_S/z_1\mathscr{O}_S$$

We claim that the canonical map

$$\Phi : R^{q+1}\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_S/z_1\mathscr{O}_S \to R^{q+1}\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_S/z_1\mathscr{O}_S) = R^{q+1}\varphi_*(\mathscr{E}|_{\varphi^{-1}(H)})$$

is a monomorphism. Then the stalk at 0 of $R^{q+1}\varphi_*(\mathscr{E}|_{\varphi^{-1}(H)})$ has non-zero $\mathscr{O}_{H,0}$-torsion elements. So the $\mathscr{O}_H$-module $R^{q+1}\varphi_*(\mathscr{E}|_{\varphi^{-1}(H)})$ is not torsion free. But the function $t \in H \mapsto \dim H^q(X_t, \mathscr{E}|_{X_t})$ is locally constant. So by assumption on case n, $R^{q+1}\varphi_*(\mathscr{E}|_{\varphi^{-1}(H)})$ is $\mathscr{O}_H$-torsion free. This gives a contradiction.

Step 3. The argument that Φ is injective is similar to that in the proof of Lemma 6.7.7: the short exact sequence

$$0 \to \mathscr{O}_S \xrightarrow{\times z_1} \mathscr{O}_S \to \mathscr{O}_S/z_1\mathscr{O}_S \to 0 \tag{6.7.8}$$

gives, by (6.6.9), a commutative diagram where the columns are exact

$$\begin{tikzcd}
R^p\varphi_*(\mathscr{E}) \arrow[r, "="] \arrow[d, "\times z_1"] & R^p\varphi_*(\mathscr{E}) \arrow[d, "\times z_1"] \\
R^p\varphi_*(\mathscr{E}) \arrow[r, "="] \arrow[d] & R^p\varphi_*(\mathscr{E}) \arrow[d, "\Gamma"] \\
R^p\varphi_*(\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_S/z_1\mathscr{O}_S \arrow[r, "\Phi"] \arrow[d] & R^p\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_S/z_1\mathscr{O}_S) \\
0 &
\end{tikzcd} \tag{6.7.9}$$

and $p = q+1$. So by Four lemma, Φ is injective. □

Proof of Thm. 6.7.10 (b)⇒(a) when S is smooth. We assume without loss of generality that S is an open subset of $\mathbb{C}^n$. Assume that (b) holds. The short exact sequence (6.7.8) gives a commutative diagram (6.7.9) with exact columns, where we choose $p = q$. We also have a long exact sequence

$$R^q\varphi_*(\mathscr{E}) \xrightarrow{\Gamma} R^p\varphi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{O}_S/z_1\mathscr{O}_S) \to R^{q+1}\varphi_*(\mathscr{E}) \xrightarrow{\times z_1} R^{q+1}\varphi_*(\mathscr{E})$$

where the endomorphism $\times z_1$ on $R^{q+1}\varphi_*(\mathscr{E})$ is injective because $R^{q+1}\varphi_*(\mathscr{E})$ is torsion free by Lemma 6.7.11. Therefore, Γ is surjective. So by Five lemma, the map Φ in (6.7.9) is an isomorphism. Thus, noting that $N(z_1) = 0 \times \mathbb{C}^{n-1}$, we have a canonical isomorphism

$$R^q\varphi_*(\mathscr{E})|_{0\times\mathbb{C}^{n-1}} \simeq R^q\varphi_*(\mathscr{E}|_{\varphi^{-1}(0\times\mathbb{C}^{n-1})})$$

Thus, an easy induction on n shows that the canonical map $R^q\varphi_*(\mathscr{E})|_0 \to R^q\varphi_*(\mathscr{E}|_{X_0}) = H^q(X_0, \mathscr{E}|_{X_0})$ is an isomorphism. The same argument shows that for each $t \in S$, the canonical map

$$R^q\varphi_*(\mathscr{E})|_t \to H^q(X_t, \mathscr{E}|_{X_t}) \tag{6.7.10}$$

is an isomorphism. This shows that $\mathscr{E}$ satisfies base change property in order q and that, since $\mathbf{d}$ is locally constant, the function $t \in S \mapsto \dim R^q\varphi_*(\mathscr{E})|_t$ is locally constant. Thus $R^q\varphi_*(\mathscr{E})$ is locally free by Prop. 1.3.14. Therefore, by Prop. 6.6.9, $\mathscr{E}$ is cohomologically flat in order q. □

Thm. 6.7.10 is often used in the following form:

Corollary 6.7.12. *Let $q \in \mathbb{Z}$. Assume that S is smooth and the dimension function $\mathbf{d} : t \in S \mapsto \dim H^q(X_t, \mathscr{E}|_{X_t})$ is locally constant. Then $R^q\varphi_*(\mathscr{E})$ is locally free, and the canonical map* (6.7.10) *is an isomorphism of $\mathbb{C}$-vector spaces for all $t \in S$.*

In other words, the conclusion of the above corollary is that all $H^q(X_t, \mathscr{E}|_{X_t})$ (where $t \in S$) form a holomorphic vector bundle over S, namely $R^q\varphi_*(\mathscr{E})$.

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