Topics in Operator Algebras: Algebraic Conformal Field Theory

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0 Notations

$\mathbb{N} = \{0, 1, 2, \dots\}$. $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. $\overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \leqslant r\}$. $\mathbb{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$.

Unless otherwise stated, an **interval of $\mathbb{S}^1$** denotes a non-empty non-dense connected open subset of $\mathbb{S}^1$.

If X is a complex manifold, we let $\mathscr{O}(X)$ denote the set of holomorphic functions $f : X \to \mathbb{C}$.

Unless otherwise stated, an **unbounded operator** $T : \mathcal{H} \to \mathcal{K}$ (where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces) denotes a linear map from a dense linear subspace $\mathscr{D}(T) \subset \mathcal{H}$ to $\mathcal{H}$. $\mathscr{D}(T)$ is called the **domain** of T. We let T^* be the adjoint of T. In practice, we are also interested in T^* defined on a dense subspace of its domain $\mathscr{D}(T^*)$. We call its restriction a **formal adjoint** of T and denote it by $T^\dagger$.

Given a Hilbert space $\mathcal{H}$, its inner product is denoted by $(\xi, \eta) \in \mathcal{H}^2 \mapsto \langle \xi | \eta \rangle$. We assume that it is linear on the first variable and antilinear on the second one. (Namely, we are following mathematician's convention.)

Whenever we write $\langle \xi, \eta \rangle$, we understand that it is linear on both variables. E.g. $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual space.

If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, we let

$$\mathfrak{L}(\mathcal{H}, \mathcal{K}) = \{\text{Bounded linear maps } \mathcal{H} \to \mathcal{K}\} \qquad \mathfrak{L}(\mathcal{H}) = \mathfrak{L}(\mathcal{H}, \mathcal{H}) \tag{0.1}$$

If V, W are vector spaces, we let

$$\mathrm{Hom}(V, W) = \{\text{Linear maps } V \to W\} \qquad \mathrm{End}(V) = \mathrm{Hom}(V, V) \tag{0.2}$$

An unbounded operator $T : \mathcal{H} \to \mathcal{H}$ denotes a linear map $\mathscr{D}(T) \to \mathcal{H}$ where $\mathscr{D}(T)$ is a dense linear subspace of $\mathcal{H}$. We say that an unbounded operator T is **continuous** if it is continuous with respect to the norms on the domain and the codomain. Thus, "bounded" means continuous and $\mathscr{D}(T) = \mathcal{H}$.

If $z_\bullet = (z_1, \dots, z_k)$ are mutually commuting formal variables, for each $n_\bullet = (n_1, \dots, n_k) \in \mathbb{Z}^k$ we let

$$z_\bullet^{n_\bullet} = z_1^{n_1} \cdots z_k^{n_k}$$

For each vector space W, we let

$$W[[z_\bullet]] = \Big\{ \sum_{n_\bullet \in \mathbb{N}^k} w_{n_\bullet} z_\bullet^{n_\bullet} \Big\} \qquad W[[z_\bullet^{\pm 1}]] = \Big\{ \sum_{n_\bullet \in \mathbb{Z}^k} w_{n_\bullet} z_\bullet^{n_\bullet} \Big\}$$

$$W((z_\bullet)) = \Big\{ \sum_{n_\bullet \in \mathbb{Z}^k} w_{n_\bullet} z_\bullet^{n_\bullet} : w_{n_\bullet} = 0 \text{ when } n_1, \dots, n_k \ll 0 \Big\}$$

$$W[z_\bullet] = W((z_\bullet)) \cap W((z_\bullet^{-1})) = \text{polynomials of } z_\bullet \text{ with } W\text{-coefficients}$$

where $w_{n_\bullet} \in W$.

If X is a set, the n-fold **configuration space** $\mathrm{Conf}^n(X)$ is

$$\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X : x_i \neq x_j \text{ if } i \neq j\} \tag{0.3}$$

Definition 0.1. A map of complex vector spaces $T : V \to V'$ is called **antilinear** or **conjugate linear** if $T(a\xi + b\eta) = \overline{a}T\xi + \overline{b}T\eta$ for all $\xi, \eta \in V$ and $a, b \in \mathbb{C}$. If V and V' are (complex) inner product spaces, we say that T is **antiunitary** if it is am antiliear surjective and satisfies $\|T\xi\| = \|\xi\|$ for all $\xi \in V$, equivalently,

$$\langle T\xi|T\eta\rangle = \overline{\langle \xi|\eta\rangle} \equiv \langle \eta|\xi\rangle \tag{0.4}$$

for all $\xi, \eta \in V$.

For each $n \in \mathbb{Z}$, we let $\mathfrak{e}_n \in C^\infty(\mathbb{S}^1)$ be $\mathfrak{e}_n(z) = z^n$.

1 Introduction: PCT symmetry, Bisognano-Wichmann, Tomita-Takesaki

Algebraic quantum field theory (AQFT) is a mathematically rigorous approach to QFT using the language of functional analysis and operator algebras. The main subject of this course is 2d **algebraic conformal field theory (ACFT)**, namely, 2d CFT in the framework of AQFT.

1.1

Let $d \in \mathbb{Z}_+$. We first sketch the general picture of an $(1 + d)$ dimensional Poincaré invariant QFT in the spirit of **Wightman axioms**. We consider Bosonic theory for simplicity.

We let $\mathbb{R}^{1,d}$ be the $(1+d)$-dimensional **Minkowski space**. So it is $\mathbb{R}^{1+d}$ but with metric tensor

$$ds^2 = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^d)^2 \tag{1.1}$$

Here x^0 denotes the time coordinate, and $x^1, \ldots, x^d$ denote the spatial coordinates. The (restricted) **Poincaré group** is

$$\mathrm{P}^+(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}^+(1,d)$$

Here, $\mathbb{R}^{1,d}$ acts by translation on $\mathbb{R}^{1,d}$. $\mathrm{SO}^+(1,d)$ is the (restricted) **Lorentz group**, the identity component of the (full) Lorentz group $\mathrm{O}(1,d)$ whose elements are invertible linear maps on $\mathbb{R}^{1,d}$ preserving the Minkowski metric.

Remark 1.1. Any $g \in \mathrm{O}(1,d)$ must have determinent ± 1. One can show that $\mathrm{SO}^+(1,d)$ is precisely the elements $g \in \mathrm{O}(1,d)$ such that $\det g = 1$, and that g does not change the direction of time (i.e., if $\mathbf{v} = (v_0, \ldots, v_d) \in \mathbb{R}^{1,d}$ satisfies $v_0 > 0$, then the first component of $g\mathbf{v}$ is > 0). See [Haag, Sec. I.2.1].

Definition 1.2. We say that $\mathbf{x} = (x_0, \ldots, x_d), \mathbf{y} = (y_0, \ldots, y_d) \in \mathbb{R}^{1,d}$ are **spacelike (separated)** if their Minkowski distance is negative, i.e.,

$$(x_0 - y_0)^2 < (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$$

1.2

A Poincaré invariant QFT consists of the following data:

(1) We have a Hilbert space $\mathcal{H}$.

(2) There is a (strongly continuous) projective unitary representation U of $\mathrm{P}^+(1,d)$ on $\mathcal{H}$. In particular, its restriction to the translation on the k-th component (where $k = 0, 1, \ldots, d$) gives a one parameter unitary group $x^k \in \mathbb{R} \mapsto \exp(\mathbf{i} x^k P_k)$ where P_k is a self-adjoint operator on $\mathcal{H}$.

(3) (Positive energy) The following are positive operators:

$$P_0 \geqslant 0 \qquad (P_0)^2 - (P_1)^2 - \cdots - (P_d)^2 \geqslant 0$$

The operator P_0 is called the **Hamiltonian** or the **energy operator**. $P_1, \ldots, P_d$ are the momentum operators. $(P_0)^2 - (P_1)^2 - \cdots - (P_d)^2$ is the mass.

(4) We have a collection of **(quantum) fields** $\mathscr{Q}$, where each $\Phi \in \mathscr{Q}$ is an operator-valued function on $\mathbb{R}^{1,d}$. For each $\mathbf{x} \in \mathbb{R}^{1,d}$, $\Phi(x)$ is a "linear operator on $\mathcal{H}$".

(5) **(**Locality) If $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{1,d}$ are spacelike and $\Phi_1, \Phi_2 \in \mathscr{Q}$, then

$$[\Phi_1(x_1), \Phi_2(x_2)] = 0 \tag{1.2}$$

(6) (*-invariance) For each $\Phi \in \mathscr{Q}$, there exists $\Phi^\dagger \in \mathscr{Q}$ such that

$$\Phi(\mathbf{x})^\dagger = \Phi^\dagger(\mathbf{x}) \tag{1.3}$$

Moreover, $\Phi^{\dagger\dagger} = \Phi$.

(7) (Poincaré invariance) There is a distinguished unit vector[1] Ω, called the **vacuum vector**, such that

$$U(g)\Omega = \Omega \qquad \forall g \in \mathrm{P}^+(1,d)$$

Moreover, for each $g \in \mathrm{P}^+(1,d)$ and $\Phi \in \mathscr{Q}$, we have

$$U(g)\Phi(\mathbf{x})U(g)^{-1} = \Phi(g\mathbf{x}) \tag{1.4}$$

(8) (Cyclicity) Vectors of the form

$$\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega \tag{1.5}$$

(where $n \in \mathbb{N}$, $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$ are mutually spacelike, and $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$) span a dense subspace of $\mathcal{H}$.

Remark 1.3. In some QFT, there is a factor (a function of $\mathbf{x}$) before $\Phi(g\mathbf{x})$ in the Poincaré invariance relation (1.4). Similarly, there is a factor before $\Phi^\dagger(\mathbf{x})$ in the $*$-invariance formula (1.3). We will encounter these more general covariance property later. In this section, we content ourselves with the simplest case that the factors are 1.

Remark 1.4. By the Poincaré invariance and the cyclicity, the action of $\mathrm{P}^+(1,d)$ is uniquely determined by $\mathscr{Q}$ by

$$U(g)\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega = \Phi_1(g\mathbf{x}_1) \cdots \Phi_n(g\mathbf{x}_n)\Omega \tag{1.6}$$

[1] A unit vector denotes a vector with length 1

1.3

Technically speaking, $\Phi(\mathbf{x})$ can not be viewed as a linear operator on $\mathcal{H}$. It cannot be defined even on a sufficiently large subspace of $\mathcal{H}$. One should think about **smeared fields**

$$\Phi(f) = \int_{\mathbb{R}^{1,d}} \Phi(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{1.7}$$

where $f \in C_c^\infty(\mathbb{R}^{1,d})$. (In contrast, we call $\Phi(\mathbf{x})$ a **pointed field**.) Then $\Phi(f)$ is usually a closable unbounded operator on $\mathcal{H}$ with dense domain $\mathscr{D}(\Phi(f))$. Moreover, $\mathscr{D}(\Phi(f))$ is preserved by any smeared operator $\Psi(g)$. Therefore, for any $f_1, \ldots, f_n \in C_c^\infty(\mathbb{R}^{1,d})$ the following vector can be defined in $\mathcal{H}$:

$$\Phi_1(f_1) \cdots \Phi_n(f_n)\Omega \tag{1.8}$$

The precise meaning of cyclicity in Subsec. 1.2 means that vectors of the form (1.8) span a dense subspace of $\mathcal{H}$. Locality means that for $f_1, f_2 \in C_c^\infty(\mathbb{R}^{1,d})$ compactly supported in spacelike regions, on a reasonable dense subspace of $\mathcal{H}$ (e.g., the subspace spanned by (1.8)) we have

$$[\Phi_1(f_1), \Phi_2(f_2)] = 0 \tag{1.9}$$

The $*$-invariance means that

$$\langle \Phi(f)\xi | \eta \rangle = \langle \xi | \Phi^\dagger(f)\eta \rangle \tag{1.10}$$

for each ξ, η in the this good subspace.

1.4

In the remaining part of this section, if possible, we also understand $\Phi(\mathbf{x})$ as a smeared operator $\Phi(f)$ where $f \in C_c^\infty(\mathbb{R}^{1,d})$ satisfies $\int f = 1$ and is supported in a small region containing $\mathbf{x}$. Thus, $\Phi(\mathbf{x})$ can almost be viewed as a closable operator. Hence the expression (1.5) makes sense in $\mathcal{H}$.

We now explore the consequences of positive energy. As we will see, it implies that $\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega$, a function of $\mathbf{x}_\bullet$, can be analytically continued.

The fact that $P_0 \geqslant 0$ implies that when $t \leqslant 0$, e^{tP_0} is a bounded linear operator with operator norm $\leqslant 1$. Therefore, if τ belongs to

$$\mathfrak{I} = \{\mathrm{Im}\tau \geqslant 0\}$$

then $e^{\mathbf{i}\tau P_0} = e^{\mathbf{i}\mathrm{Re}\tau} \cdot e^{-\mathrm{Im}\tau}$ is bounded. Indeed, $\tau \in \mathfrak{I} \mapsto e^{\mathbf{i}\tau P_0}$ is continuous, and is holomorphic on $\mathrm{Int}\mathfrak{I}$.

Let $\mathbf{e}_0 = (1, 0, \ldots, 0)$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$ be distinct. By the Poincaré covariance, the relation

$$e^{\mathbf{i}\tau P_0}\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(\mathbf{x}_1 + \tau e_0)\cdots\Phi_n(\mathbf{x}_n + \tau e_0)\Omega \tag{1.11}$$

holds for all real τ. Moreover, the LHS is continuous on $\mathfrak{I}$ and holomorphic on $\mathrm{Int}\mathfrak{I}$. This suggests that the RHS of (1.11) can also be defined as an element of $\mathcal{H}$ when $\tau \in \mathfrak{I}$.

1.5

We shall further explore the question: for which $\mathbf{x}_i$ is in $\mathbb{C}^d$ can $\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega$ be reasonably defined as an element of $\mathcal{H}$?

Remark 1.5. We expect that the smeared fields should be defined on any P_0-**smooth vectors**, i.e., vectors in $\bigcap_{k\geqslant 0}\mathscr{D}(P_0^k)$. For each $r > 0$, since one can find $C_{k,r} \geqslant 0$ such that $\lambda^{2k} \leqslant C_{k,r}e^{2r\lambda}$ for all $\lambda \geqslant 0$, we conclude that

$$\mathrm{Rng}(e^{-rP_0}) \equiv \mathscr{D}(e^{rP_0}) \subset \bigcap_{k\geqslant 0}\mathscr{D}(P_0^k) \tag{1.12}$$

The above remark shows that $\Phi_1(\mathbf{x}_1)$, viewed as a smeared operator localized on a small neighborhood of $\mathbf{x}_1$, is definable on $e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\Omega = \Phi_2(\zeta_2\mathbf{e}_0 + \mathbf{x}_2)\Omega$ whenever $\mathrm{Im}\zeta_2 > 0$. Thus, heuristically, $(\zeta_1, \zeta_2) \mapsto e^{\mathbf{i}\zeta_1 P_0}\Phi_1(\mathbf{x}_1)e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\Omega$ should also be holomorphic on

$$\{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \mathrm{Im}\zeta_1, \mathrm{Im}\zeta_2 > 0\}$$

Repeating this procedure, we see that the holomorphicity holds for

$$e^{\mathbf{i}\zeta_1 P_0}\Phi_1(\mathbf{x}_1)e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\cdots e^{\mathbf{i}\zeta_n P_0}\Phi_n(\mathbf{x}_n)\Omega$$

when $\mathrm{Im}\zeta_i > 0$. By Poincaré covariance, the above expression equals

$$\Phi_1(\mathbf{x}_1 + \zeta_1\mathbf{e}_0)\Phi_2(\mathbf{x}_2 + (\zeta_1 + \zeta_2)\mathbf{e}_0)\cdots\Phi_n(\mathbf{x}_n + (\zeta_1 + \cdots + \zeta_n)\mathbf{e}_0)\Omega$$

Therefore,

$$(\zeta_1, \ldots, \zeta_n) \mapsto \Phi_1(\mathbf{x}_1 + \zeta_1\mathbf{e}_0)\cdots\Phi_n(\mathbf{x}_n + \zeta_n\mathbf{e}_0) \in \mathcal{H} \tag{1.13}$$

should be holomorphic on $\{\zeta_\bullet \in \mathbb{C}^n : 0 < \mathrm{Im}\zeta_1 < \cdots < \mathrm{Im}\zeta_n\}$.

By the locality axiom, the order of products of quantum fields can be exchanged. Thus, our expectation for a reasonable QFT includes the following condition:

Conclusion 1.6. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$. Then (1.13) is holomorphic on

$$\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \mathrm{Im}\zeta_i > 0, \text{ and } \mathrm{Im}\zeta_i \neq \mathrm{Im}\zeta_j \text{ if } i \neq j\} \tag{1.14a}$$

Moreover, since (1.13) is also definable and continuous on

$$\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n : \mathbf{x}_1 + \zeta_1\mathbf{e}_1, \ldots, \mathbf{x}_n + \zeta_n\mathbf{e}_0 \text{ are mutually spacelike}\} \tag{1.14b}$$

we expect that the function (1.13) is continuous on the union of (1.14a) and (1.14b).

1.6

We have (informally) derived some consequences from the positivity of P_0. Note that since $P_0 \geqslant 0$, we have $U(g)P_0U(g)^{-1} \geqslant 0$ for each $g \in \mathrm{SO}^+(1,d)$. Since P_0 is the generator of the flow $t \in \mathbb{R} \mapsto t\mathbf{e}_0 \in \mathbb{R}^{1,d} \subset \mathrm{P}^+(1,d)$, $U(g)P_0U(g)^{-1}$ is the generator of the flow

$$t \in \mathbb{R} \mapsto g(t\mathbf{e}_0)g^{-1} = t \cdot g\mathbf{e}_0 \tag{1.15}$$

Therefore, if $g\mathbf{e}_0 = (a_0, \dots, a_n)$, then

$$U(g)P_0U(g)^{-1} = a_0P_0 + \cdots + a_nP_n \tag{1.16}$$

Hence the RHS must be positive. But what are all the possible $g\mathbf{e}_0$?

Remark 1.7. One can show that the orbit of $\mathbf{e}_0 = (1, 0, \dots, 0)$ under $\mathrm{SO}^+(1,d)$ is the upper hyperbola with diameter 1, i.e., the set of all $(a_0, \dots, a_n) \in \mathbb{R}^{1,d}$ satisfying

$$a_0 > 0 \qquad (a_0)^2 - (a_1)^2 - \cdots - (a_n)^2 = 1 \tag{1.17}$$

Thus $\sum_i a_iP_i \geqslant 0$ for all such $a_\bullet$. What are the consequences of this positivity?

1.7

To simplify the following discussions, we set $d = 2$ and

$$t = x^0 \qquad x = x^1$$

We further set

$$u = t - x \qquad v = t + x \tag{1.18}$$

so that

$$t = \frac{u+v}{2} \qquad x = \frac{-u+v}{2} \tag{1.19}$$

The Minkowski metric becomes

$$\boxed{(dt)^2 - (dx)^2 = du \cdot dv} \tag{1.20}$$

Then

$$(u,v) \text{ is spacelike to } (u',v') \qquad \Longleftrightarrow \qquad (u-u')(v-v') < 0 \tag{1.21}$$

For each $\Phi \in \mathscr{Q}$, we write

$$\widetilde{\Phi}(u,v) := \Phi(t,x) = \Phi\Big(\frac{u+v}{2}, \frac{-u+v}{2}\Big) \tag{1.22}$$

We let H_0 and H_1 be the self-adjoint operators such that

$$H_0 = P_0 - P_1 \qquad H_1 = P_0 + P_1$$

so that they are the generators of the flow $t \mapsto (t,-t)$ and $t \mapsto (t,t)$.

Remark 1.8. Since $\mathbb{R}^{1,d}$ is an abelian group, we know that P_i commutes with P_j. Hence H_0 commutes with H_1.

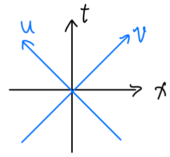

Figure 1.1. The coordinates u, v

1.8

The orbit of $\mathbf{e}_0$ under $\mathrm{SO}^+(1,1)$ is the unit upper hyperbola $(x^0)^2 - (x_1)^2 = 1, x^0 > 0$. Equivalently, it is $uv = 1, u > 0$. According to Subsec. 1.6, we conclude that $b_0 H_0 + b_1 H_1 \geqslant 0$ for each b_0, b_1 satisfying $b_0 b_1 = 1, b_0 > 0$ (equivalently, for each $b_0 > 0, b_1 > 0$). This implies

$$H_0 \geqslant 0 \qquad H_1 \geqslant 0 \tag{1.23}$$

Therefore, similar to the argument in Subsec. 1.5 (and specializing to the special case that $\mathbf{x}_1 = \cdots = \mathbf{x}_n = 0$), the holomorphicity of

$$(\zeta_\bullet, \gamma_\bullet) \mapsto e^{\mathbf{i}\zeta_1 H_0 + \mathbf{i}\gamma_1 H_1} \widetilde{\Phi}_1(0) e^{\mathbf{i}\zeta_2 H_0 + \mathbf{i}\gamma_2 H_1} \widetilde{\Phi}_2(0) \cdots e^{\mathbf{i}\zeta_n H_0 + \mathbf{i}\gamma_n H_1} \widetilde{\Phi}_n(0)\Omega$$

on the region $\mathrm{Im}\zeta_i > 0, \mathrm{Im}\gamma_i > 0$, together with locality, implies:

Conclusion 1.9. Let $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$. Then

$$(u_1, v_1, \ldots, u_n, v_n) \mapsto \widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}(u_n, v_n)\Omega \tag{1.24}$$

is holomorphic on

$$\{(u_\bullet, v_\bullet) \in \mathbb{C}^{2n} : \mathrm{Im}u_i > 0, \mathrm{Im}v_i > 0, \mathrm{Im}u_i \neq \mathrm{Im}u_j, \mathrm{Im}v_i \neq \mathrm{Im}v_j \text{ if } i \neq j\} \tag{1.25a}$$

and can be continuously extended to

$$\{(u_\bullet, v_\bullet) \in \mathbb{R}^{2n} : (u_i - u_j) \cdot (v_i - v_j) < 0 \text{ if } i \neq j\} \tag{1.25b}$$

Rigorously speaking, the above mentioned “continuity" of the extension should be understood in terms of distributions. Here, we ignore such subtlety and view pointed fields as smeared field in a small region.

1.9

We note that $\mathrm{diag}(-1, \pm 1)$ is not inside $\mathrm{SO}^+(1,1)$, since it reverses the time direction. Neither is $\mathrm{diag}(1,-1)$ in $\mathrm{SO}^+(1,1)$ because its determinant is negative. Consequently, the QFT is not necessarily symmetric under the following operations:

- **Time reversal** $t \mapsto -x$.
- **Parity transformation** $x \mapsto -x$.
- **PT transformation** $(t,x) \mapsto (-t,-x)$, the combination of time and parity inversions.

Mathematically, this means that the maps

$$\begin{aligned}
\Phi_1(t_1,x_1)\cdots\Phi_n(t_n,x_n)\Omega &\quad\mapsto\quad \Phi_1(-t_1,x_1)\cdots\Phi_n(-t_n,x_n)\Omega\\
\Phi_1(t_1,x_1)\cdots\Phi_n(t_n,x_n)\Omega &\quad\mapsto\quad \Phi_1(t_1,-x_1)\cdots\Phi_n(t_n,-x_n)\Omega\\
\Phi_1(t_1,x_1)\cdots\Phi_n(t_n,x_n)\Omega &\quad\mapsto\quad \Phi_1(-t_1,-x_1)\cdots\Phi_n(-t_n,-x_n)\Omega
\end{aligned}$$

(where $(t_1,x_1),\dots,(t_n,x_n)$ are mutually spacelike) are not necessarily unitary. (Compare Rem. 1.4.) Simiarly, the QFT is not necessarily symmetric under **Charge conjugation** $\Phi \mapsto \Phi^\dagger$, which means that the map

$$\begin{aligned}
\Phi_1(t_1,x_1)\cdots\Phi_n(t_n,x_n)\Omega \quad\mapsto\quad & \Phi_n(t_n,x_n)^\dagger\cdots\Phi_1(t_1,x_1)^\dagger\Omega\\
&=\Phi_1^\dagger(t_1,x_1)\cdots\Phi_n^\dagger(t_n,x_n)\Omega
\end{aligned}$$

is not necessarily (anti)unitary. However, as we shall explain, the combination of PCT transformations is actually unitary, and hence is a symmetry of the QFT. This is called the PCT theorem.

1.10

To prove the PCT theorem, we shall first prove that the PT transformation, though not implemented by a unitary operator, is actually implemented by the analytic continuation of a one parameter unitary group.

Definition 1.10. The one parameter group $s \mapsto \Lambda(s) \in \mathrm{SO}^+(1,1)$ defined by

$$\Lambda(s)(u,v) = (e^{-s}u, e^{s}v) \tag{1.26}$$

is called the **Lorentz boost**. Equivalently,

$$\Lambda(s)\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}\begin{bmatrix} t \\ x \end{bmatrix} \tag{1.27}$$

Define the (open) **right wedge** $\mathcal{W}$ and **left wedge** $-\mathcal{W}$ by

$$\mathcal{W} = \{(u,v) \in \mathbb{R}^2 : v > 0, u < 0\} = \{(t,x) \in \mathbb{R}^{1,1} : -x < t < x\} \tag{1.28}$$

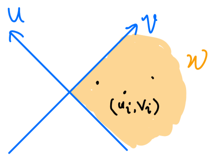

Figure 1.2.

Theorem 1.11 (PT theorem). *Let* $(u_1, v_1), \ldots, (u_n, v_n) \in \mathcal{W}$ *be mutually spacelike (i.e. satisfying* $(u_i - u_j)(v_i - v_j) < 0$ *if* $i \neq j$*), cf. Fig. 1.2. Let* $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$*. Let* K *be the self-adjoint generator of the Lorentz boost, i.e.,*

$$U(\Lambda(s)) = e^{\mathbf{i}sK}$$

Then $\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega$ *belongs to the domain of* $e^{-\pi K}$*, and*

$$e^{-\pi K}\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega = \Phi_1(-\mathbf{x}_1) \cdots \Phi_n(-\mathbf{x}_n)\Omega \tag{1.29}$$

Equivalently, $\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega$ belongs to the domain of $e^{-\pi K}$, and

$$e^{-\pi K}\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega = \widetilde{\Phi}_1(-u_1, -v_1) \cdots \widetilde{\Phi}_n(-u_n, -v_n)\Omega \tag{1.30}$$

Note that the requirement that $(u_1, v_1), \ldots, (u_n, v_n) \in \mathcal{W}$ are spacelike means, after relabeling the subscripts, that

$$0 < v_1 < \cdots < v_n \qquad 0 < -u_1 < \cdots < -u_n$$

Proof. This theorem relies on the following fact that we shall prove rigorously in the future:

- ⋆ Let $T \geqslant 0$ be a self-adjoint operator on $\mathcal{H}$ with $\mathrm{Ker}(T) = 0$. Let $r > 0$. Then $\xi \in \mathcal{H}$ belongs to $\mathscr{D}(T^r)$ iff the function $s \in \mathbb{R} \mapsto T^{\mathbf{i}s}\xi \in \mathcal{H}$ can be extended to a continuous function F on

$$\{z \in \mathbb{C} : -r \leqslant \mathrm{Im} z \leqslant 0\}$$

  and holomorphic on its interior. Moreover, for such ξ we have $F(-\mathbf{i}r) = T^r\xi$.

In fact, the function $F(z)$ is given by $z \mapsto T^z\xi$.

We shall apply this result to $T = e^{-K}$ and $r = \pi$. For that purpose, we must show that the $\mathcal{H}$-valued function of $s \in \mathbb{R}$ defined by

$$e^{\mathbf{i}\pi s}\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega = \widetilde{\Phi}_1(e^{-s}u_1, e^s v_1) \cdots \widetilde{\Phi}_n(e^{-s}u_n, e^s v_n)\Omega$$

can be extended to a continuous function on

$$\{z \in \mathbb{C} : 0 \leqslant \mathrm{Im} z \leqslant \pi\}$$

and holomorphic on its interior.

In fact, we can construct this $\mathcal{H}$-valued function, which is

$$z \mapsto \widetilde{\Phi}_1(e^{-z}u_1, e^z v_1) \cdots \widetilde{\Phi}_n(e^{-z}u_n, e^z v_n)\Omega$$

noting that the conditions in Conc. 1.9 are fulfilled. In particular, the condition $0 < \mathrm{Im} < \pi$ is used to ensure that, since $u_i < 0, v_i > 0$, we have $\mathrm{Im}(e^{-z}u_i) > 0$ and $\mathrm{Im}(e^z v_i) > 0$ as required by (1.25a). The value of this function at $z = \mathbf{i}\pi$ equals the RHS of (1.30). Therefore the theorem is proved. □

1.11

Theorem 1.12 (PCT theorem). *We have an antiunitary map $\Theta : \mathcal{H} \to \mathcal{H}$, called the **PCT operator**, such that*

$$\Theta \cdot \Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega = \Phi_1(-\mathbf{x}_1)^\dagger \cdots \Phi_n(-\mathbf{x}_n)^\dagger\Omega \tag{1.31}$$

for any $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$ and mutually spacelike $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Equivalently, Θ is defined by

$$\Theta \cdot \widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n) = \widetilde{\Phi}_1(-u_1, -v_1)^\dagger \cdots \widetilde{\Phi}_n(-u_n, -v_n)^\dagger\Omega \tag{1.32}$$

Proof. The existence of an antilinear isometry Θ satisfying (1.32) is equivalent to showing that (cf. (0.4))

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1) \cdots \widetilde{\Phi}_n(\mathbf{u}_n)\Omega|\widetilde{\Psi}_1(\mathbf{u}'_1) \cdots \widetilde{\Psi}_k(\mathbf{u}'_k)\Omega\rangle \\ =&\langle\widetilde{\Psi}_1(-\mathbf{u}'_1)^\dagger \cdots \widetilde{\Psi}_k(-\mathbf{u}'_k)^\dagger\Omega|\widetilde{\Phi}_1(-\mathbf{u}_1)^\dagger \cdots \widetilde{\Phi}_n(-\mathbf{u}_n)^\dagger\Omega\rangle\end{aligned} \tag{$\star$}$$

if $\mathbf{u}_1, \ldots \mathbf{u}_n$ are spacelike, and $\mathbf{u}'_1, \ldots \mathbf{u}'_k$ are spacelike. (We do not assume that, say, $\mathbf{u}_1$ and $\mathbf{u}'_1$ are spacelike.)

It suffices to prove this in the special case that $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}'_1, \ldots, \mathbf{u}'_k$ are mutually spacelike. Then the general case will follow that both sides of the above relation can be analytically continued to suitable regions as functions of $\mathbf{u}_1, \ldots, \mathbf{u}_n$. For example, the fact that $H_0, H_1 \geqslant 0$ implies that

$$e^{\mathbf{i}\zeta H_0 + \mathbf{i}\gamma H_1}\widetilde{\Phi}_1(\mathbf{u}_1) \cdots \widetilde{\Phi}_n(\mathbf{u}_n)\Omega = \widetilde{\Phi}_1(\mathbf{u}_1 + (\zeta, \gamma)) \cdots \widetilde{\Phi}_n(\mathbf{u}_n + (\zeta, \gamma))\Omega$$

is continuous on $\{(\zeta, \gamma) \in \mathbb{C}^2 : \mathrm{Im}\zeta \geqslant 0, \mathrm{Im}\gamma \geqslant 0\}$ and holomorphic on its interior.

Set $\Gamma_j = \Psi_j^\dagger$. Then ($\star$) is equivalent to

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1) \cdots \widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1) \cdots \widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle \\ =&\langle\widetilde{\Phi}_1(-\mathbf{u}_1) \cdots \widetilde{\Phi}_1(-\mathbf{u}_n)\widetilde{\Gamma}_1(-\mathbf{u}'_1) \cdots \widetilde{\Gamma}_k(-\mathbf{u}'_k)\Omega|\Omega\rangle\end{aligned}$$

By the PT Thm. 1.11, this relation is equivalent to

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1) \cdots \widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1) \cdots \widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle \\ =&\langle e^{-\pi K}\widetilde{\Phi}_1(\mathbf{u}_1) \cdots \widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1) \cdots \widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\end{aligned}$$

But this of course holds since $e^{-\pi K}\Omega = \Omega$ by Poincaré invariance. □

1.12

Combining the PT Thm. 1.11 with the PCT Thm. 1.12, we conclude that $e^{-\pi K}$ is an injective positive operator, Θ is antinitary, and

$$\Theta e^{-\pi K} A\Omega = A^\dagger \Omega \tag{1.33a}$$

where A is a product of spacelike separated field in $\mathcal{W}$. The rigorous statement should be that

$$A = \Phi_1(f_1)\cdots\Phi_n(f_n)$$

where $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$, and $f_i \in C_c^\infty(O_i)$ where $O_1, \ldots, O_n \subset \mathcal{W}$ are open and mutually spacelike. If we let $\mathscr{A}(\mathcal{W})$ be the $*$-algebra generated by all such A, then by the Poincaré invariance, for each $g \in \mathrm{P}^+(1,d)$ we have

$$U(g)\mathscr{A}(\mathcal{W})U(g)^{-1} = \mathscr{A}(g\mathcal{W})$$

In particular, since for the Lorentz boost Λ we have $\Lambda(s)\mathcal{W} = \mathcal{W}$, we therefore have

$$e^{\mathbf{i}sK}\mathscr{A}(\mathcal{W})e^{-\mathbf{i}sK} = \mathscr{A}(\mathcal{W}) \tag{1.33b}$$

for all $s \in \mathbb{R}$. Since the PT transformation sends $\mathcal{W}$ to $-\mathcal{W}$, the definition of Θ clearly also implies

$$\Theta\mathscr{A}(\mathcal{W})\Theta^{-1} = \mathscr{A}(-\mathcal{W}) \tag{1.33c}$$

Note that since $\mathcal{W}$ is local to $-\mathcal{W}$, we have $[\mathscr{A}(\mathcal{W}), \mathscr{A}(-\mathcal{W})] = 0$. Therefore, $\Theta\mathscr{A}(\mathcal{W})\Theta$ is a subset of the (in some sense) commutant of $\mathscr{A}(\mathcal{W})$.

1.13

The set of formulas (1.33) is reminiscent of the Tomita-Takesaki theory, one of the deepest theories in the area of operator algebras. The setting is as follows.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Namely, $\mathcal{M}$ is a $*$-subalgebra of $\mathfrak{L}(\mathcal{H})$ closed under the "strong operator topology". (We will formally introduce von Neumann algebras in a later section.) Let $\Omega \in \mathcal{H}$ be a unit vector. Assume that Ω is **cyclic** (i.e. $\mathcal{M}\Omega$ is dense) and **separating** (i.e., if $x \in \mathcal{M}$ and $x\Omega = 0$ then $x = 0$) under $\mathcal{M}$. Then the **Tomita-Takesaki theorem** says that the linear map

$$S : \mathcal{M}\Omega \to \mathcal{M}\Omega \qquad x\Omega \mapsto x^*\Omega$$

is antilinear and closable. Denote its closure also by S, and consider its polar decomposition $S = J\Delta^{\frac{1}{2}}$ where Δ is a positive closed operator, and J is an antiunitary map. Then Δ is injective, we have $J^{-1} = J^* = J$, and

$$\Delta^{\mathbf{i}s}\mathcal{M}\Delta^{-\mathbf{i}s} = \mathcal{M} \qquad J\mathcal{M}J = \mathcal{M}'$$

where $\mathcal{M}'$ is the commutant $\{y \in \mathfrak{L}(\mathcal{H}) : xy = yx \ (\forall x \in \mathcal{M})\}$.We call Δ and J respectively the **modular operator** and the **modular conjugation**.

1.14

To relate the Tomita-Takesaki theory to QFT, one takes $\mathcal{M}$ to be $\mathfrak{A}(\mathcal{W})$, the von Neumann algebra generated by $\mathscr{A}(\mathcal{W})$. Note that the elements of $\mathscr{A}(\mathcal{W})$ are typically unbounded operators, whereas those of $\mathfrak{A}(\mathcal{W})$ are bounded. Thus, the meaning of "the von Neumann algebra generated by a set of closed/closable operators" should be clarified. This is an important notion, and we will study it in a later section.

To apply the setting of Tomita-Takesaki, one should first show that the vacuum vector is cyclic and separating under $\mathfrak{A}(\mathcal{W})$. This is not an easy task, although it is relatively easier to show that Ω is cyclic and separating under $\mathscr{A}(\mathcal{W})$. Moreover, we have

Theorem 1.13 (Bisognano-Wichmann). *Let Δ and J be the modular operator and the modular conjugation of $(\mathfrak{A}(\mathcal{W}),\Omega)$. Then $J=\Theta$ and $\Delta^{\frac{1}{2}}=e^{-\pi K}$.*

Since (1.33c) easily implies $\Theta\mathfrak{A}(\mathcal{W})\Theta^{-1}=\mathfrak{A}(-\mathcal{W})$, together with $J\mathcal{M}J^{-1}=\mathcal{M}'$ we obtain

$$\mathfrak{A}(\mathcal{W})'=\mathfrak{A}(-\mathcal{W}) \tag{1.34}$$

a version of **Haag duality**.

One of the main goals of this course is to give a rigorous and self-contained proof of the PCT theorem, the Bisognano-Wichmann theorem, and the Haag duality for 2d chiral conformal field theories.

1.15

For a general odd number $d>0$, the above results should be modified as follows. Let K be the generator of the **Lorentz boost**

$$\Lambda(s)=\left(\begin{array}{cc|ccc} \cosh s & \sinh s & & 0 & \\ \sinh s & \cosh s & & & \\ \hline & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{array}\right)$$

Let $\Lambda(\mathbf{i}\pi)=\mathrm{diag}(-1,-1,1,\dots,1)$, which does not belong to $\mathrm{P}^+(1,d)$ since it reverses the time direction (although it has positive determinant). Then the PT Thm. 1.11 should be modified by replacing (1.29) with

$$e^{-\pi K}\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega=\Phi_1(\Lambda(\mathbf{i}\pi)\mathbf{x}_1)\cdots\Phi_n(\Lambda(\mathbf{i}\pi)\mathbf{x}_n)\Omega \tag{1.35}$$

Let $\rho=\mathrm{diag}(1,1,-1,\dots,-1)$, which has determinant 1 (since d is odd) and hence belongs to $\mathrm{SO}^+(1,d)$. Then the PCT Thm. 1.12 still holds verbatim. Let

$$\mathcal{W}=\{(a_0,\dots,a_n)\in\mathbb{R}^{1,d}:-a_1<a_0<a_1\} \tag{1.36}$$

Then the **Bisognano-Wichmann theorem** says that $e^{-\pi K}$ is the modular operator of $(\mathfrak{A}(\mathcal{W}), \Omega)$, and $\Theta U(\rho)$ is the modular conjugation.

We refer the readers to [Haag, Sec. V.4.1] and the reference therein for a detailed study.

2 2d conformal field theory

2.1

We look at a 2d **unitary full conformal field theory** (unitary full CFT) $\mathcal{Q}$ on the **space-compactified Minkowski space**

$$\mathbb{R}^{1,1}_{\mathbf{c}} = \mathbb{R} \times \mathbb{S}^1 \qquad \text{with metric tensor } (dt)^2 - (dx)^2 = dudv$$

The space $\mathbb{R}^{1,1}_{\mathrm{c}}$ describes the propagation of the closed string $\{0\} \times \mathbb{S}^1$. Here, as in Subsec. 1.7, we write a general element of $\mathbb{R}^{1,1}_{\mathrm{c}}$ as $\mathbf{x} = (t, x)$, and write

$$u = t - x \quad v = t + x \qquad \text{so that} \qquad t = \frac{u+v}{2} \quad x = \frac{-u+v}{2}$$

The field operators are of the form $\Phi(\mathbf{x}) = \Phi(t, x)$. Recall that

$$\widetilde{\Phi}(u, v) := \Phi(t, x) = \Phi\Big(\frac{u+v}{2}, \frac{-u+v}{2}\Big)$$

Identifying $\mathbb{R}/2\pi\mathbb{Z} = \mathbb{S}^1$ via $\exp$, a field Φ can be viewed as an "operator valued function" on $\mathbb{R}^{1,1}$ satisfying

$$\Phi(t, x + 2\pi) = \Phi(t, x) \qquad \text{equivalently} \qquad \widetilde{\Phi}(u, v) = \widetilde{\Phi}(u - 2\pi, v + 2\pi) \tag{2.1}$$

The field operators are "acting on" a Hilbert space $\mathcal{H}$ with vacuum vector Ω.

Compared to the axioms for Poincaré invariant QFT in Subsec. 1.2, some changes should be made to describe a CFT. We still have the locality (1.2). Instead of considering $\mathrm{P}^+(1,1)$ we must consider the group of orientation-preserving, time-direction preserving, and conformal (i.e. angle-preserving) transforms on $\mathbb{R}^{1,1}_{\mathrm{c}}$. "Conformal" means that the diffeomorphism $g : \mathbb{R}^{1,1}_{\mathrm{c}} \to \mathbb{R}^{1,1}_{\mathrm{c}}$ satisfies

$$g^*(dudv) = \lambda(u, v)dudv$$

for a smooth function $\lambda : \mathbb{R}^{1,1}_{\mathrm{c}} \to \mathbb{R}_{>0}$. Our next goal is to classify such g.

2.2

Definition 2.1. We let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the group of orientation-preserving diffeomorphisms of $\mathbb{S}^1$. Equivalently, it is the group of smooth functions $f : \mathbb{S}^1 \to \mathbb{S}^1$ whose lift $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ satisfies for all $x \in \mathbb{R}$ that

$$\widetilde{f}(x + 2\pi) = \widetilde{f}(x) + 2\pi \qquad \widetilde{f}'(x) > 0 \tag{2.2}$$

Note that by the basics of covering spaces, any element of $\mathrm{Diff}^+(\mathbb{S}^1)$ can be lifted to $\widetilde{f}$ satisfying (2.2). Conversely, if $\widetilde{f}$ satisfies (2.2), then $\widetilde{f}$ gives rise to an injective smooth map $f : \mathbb{S}^1 \to \mathbb{S}^1$. (Note that $\widetilde{f}' > 0$ implies that $\widetilde{f}$ is strictly increasing.) Since $\widetilde{f}'(x) > 0$, the function f is injective, and the inverse function theorem shows that the compact set $f(\mathbb{S}^1)$ is open, and hence equals $\mathbb{S}^1$. Thus $f \in \mathrm{Diff}^+(\mathbb{S}^1)$.

Remark 2.2. Note that f uniquely determines $\widetilde{f}$ up to an $2\pi\mathbb{Z}$-addition, i.e., both $\widetilde{f}$ and $\widetilde{f} + 2n\pi$ (where $n \in \mathbb{Z}$) correspond to f. Therefore, if we let $\widetilde{\mathbf{Diff}^+}(\mathbb{S}^1)$ be the topological group formed by all $\widetilde{f}$ satisfying (2.2),[2] then we have an exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \longrightarrow \mathrm{Diff}^+(\mathbb{S}^1) \longrightarrow 1 \tag{2.3}$$

where $\mathbb{Z}$ is freely generated by $x \in \mathbb{R} \mapsto x + 2\pi$.

Note that the map $(\widetilde{f}, t) \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times [0, 1] \mapsto \widetilde{f}_t \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ defined by

$$\widetilde{f}_t(x) = (1 - t)\widetilde{f}(x) + tx$$

shows that $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ is contractible (to the identity element) and hence simply-connected. (Therefore $\mathrm{Diff}^+(\mathbb{S}^1)$ is connected.) We conclude that $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ is the universal cover of $\mathrm{Diff}^+(\mathbb{S}^1)$. □

2.3

Theorem 2.3. *Under the coordinates (u, v), an orientation-preserving time-direction-preserving conformal transform g of $\mathbb{R}^{1,1}_{\mathrm{c}}$ is precisely of the form*

$$g(u, v) = (\alpha(u), \beta(v)) \tag{2.4}$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ belong to $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$.

Proof. Step 1. First, suppose that g is of the form (2.4). Then g gives a well-defined smooth map $\mathbb{R}^{1,1}_{\mathrm{c}} \to \mathbb{R}^{1,1}_{\mathrm{c}}$ because

$$g(u - 2\pi, v + 2\pi) = g(u, v) + (-2\pi, 2\pi) \tag{2.5}$$

One checks easily that g is a diffeomorphism (with inverse given by $(\alpha^{-1}(u), \beta^{-1}(v))$) preserving the orientation and the time direction. Since $g^*dudv = \alpha'(u)\beta'(v)dudv$, g is conformal.

[2]The topology is defined such that a net $\widetilde{f}_\alpha$ converges to $\widetilde{f}$ iff the n-th derivative $\widetilde{f}^{(n)}_\alpha$ converges uniformly to $\widetilde{f}^{(n)}$ for all $n \in \mathbb{N}$.

Step 2. Conversely, choose an orientation preserving conformal transform g. We lift g to a smooth conformal map $\mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ also denoted by $g = (\alpha, \beta)$. So $\alpha, \beta : \mathbb{R}^{1,1} \to \mathbb{R}$. Then, besides (2.5), g also satisfies:

$$\partial_u \alpha \partial_u \beta = 0 \qquad \partial_v \alpha \partial_v \beta = 0 \tag{a}$$
$$\partial_u \alpha \partial_v \beta + \partial_v \alpha \partial_u \beta > 0 \tag{b}$$
$$\partial_u \alpha \partial_v \beta - \partial_v \alpha \partial_u \beta > 0 \tag{c}$$

Here, (a) and (b) are due to the fact that

$$g^*(dudv) = (\partial_u \alpha du + \partial_v \alpha dv)(\partial_u \beta du + \partial_v \beta dv)$$

equals $\lambda(u, v)dudv$ for some smooth $\lambda : \mathbb{R}^{1,1} \to \mathbb{R}_{>0}$. (So λ is the LHS of (b).) Since g is orientation preserving, (c) follows from the computation

$$g^*(du \wedge dv) = (\partial_u \alpha \partial_v \beta - \partial_v \alpha \partial_u \beta) du \wedge dv$$

Step 3. By (a), at a given $p \in \mathbb{R}^{1,1}$, if $\partial_u \alpha \neq 0$, then $\partial_u \beta = 0$. Conversely, if at p we have $\partial_u \beta = 0$, then (b) shows that $\partial_u \alpha \partial_v \beta > 0$, and hence $\partial_u \alpha \neq 0$. Thus

$$\begin{aligned} \partial_u \alpha|_p \neq 0 \qquad &\Longleftrightarrow \qquad \partial_u \beta|_p = 0 \\ \partial_v \alpha|_p \neq 0 \qquad &\Longleftrightarrow \qquad \partial_v \beta|_p = 0 \end{aligned}$$

where the second equivalence follows from the same argument. Therefore, the set of p at which $\partial_v \alpha = 0$ is both open and closed, and hence must be either $\mathbb{R}^{1,1}$ or $\varnothing$. Similarly, either $\partial_u \beta = 0$ everywhere, or $\partial_u \beta \neq 0$ everywhere.

Let us prove that

$$\partial_v \alpha = 0 \qquad \partial_u \beta = 0$$

everywhere. Suppose the first is not true. Then by the previous paragraph, we have $\partial_v \alpha \neq 0$ and $\partial_v \beta = 0$ everywhere. Then (b) implies $\partial_v \alpha \partial_u \beta > 0$, and (c) implies $-\partial_v \alpha \partial_u \beta > 0$, impossible. So the first (and similarly the second) is true.

Step 4. Therefore, we can write $\alpha = \alpha(u)$ and $\beta = \beta(v)$, and we have $\alpha' \neq 0$ and $\beta' \neq 0$ everywhere. (b) implies that $\alpha'(u)\beta'(v) > 0$ for all u, v. Thus, either $\alpha' > 0$ and $\beta' > 0$ everywhere, or $\alpha' < 0$ and $\beta' < 0$ everywhere. The latter cannot happen, since g preserves the direction of time. Thus $\alpha' > 0$ and $\beta' > 0$ everywhere. Since g satisfies (2.5), we see that α satisfies (2.2). Similarly β satisfies (2.2). This finishes the proof. $\square$

2.4

We let $\mathbf{Cf^+(\mathbb{R}^{1,1}_c)}$ be the group of diffeomorphisms of $\mathbb{R}^{1,1}_c$ preserving the orientation and the time-direction. Then Thm. 2.3 says that any $g \in \mathrm{Cf}^+(\mathbb{R}^{1,1}_c)$ can be represented by some $(\alpha,\beta) \in \widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)^2$.

However, (α,β) is not uniquely determined by g. Indeed, in Step 2 of the proof of Thm. 2.3 we have lifted g to a smooth map on $\mathbb{R}^{1,1}$. This lift is unique up to addition by $(-2\pi,2\pi)\mathbb{Z}$ in the (u,v) coordinates (or $(0,2\pi)\mathbb{Z}$ in the (t,x) coordinates). Thus, (α,β) are unique up to addition by $(-2\pi,2\pi)\mathbb{Z}$. This non-uniqueness can be ignored once we pass to $(\breve{\alpha},\breve{\beta})$, the projection of (α,β) into $\mathrm{Diff}^+(\mathbb{S}^1)^2$. Thus, we have a well-defined (continuous) surjective group homomorphism $\Gamma : \mathrm{Cf}^+(\mathbb{R}^{1,1}_c) \to \mathrm{Diff}^+(\mathbb{S}^1) \times \mathrm{Diff}^+(\mathbb{S}^1)$ sending g to $(\breve{\alpha},\breve{\beta})$.

One checks easily that the kernel of this homomorphism is freely generated by $(2\pi,0)$ (equivalently, by $(0,2\pi)$) under the (u,v) coordinates, equivalently, by (π,π) under the (t,x) coordinates. Therefore, we have an exact sequence of groups

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cf}^+(\mathbb{R}^{1,1}_c) \xrightarrow{\Gamma} \mathrm{Diff}^+(\mathbb{S}^1)^2 \longrightarrow 1 \tag{2.6}$$

Since Γ is a covering map, we also have a covering map $\widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)^2 \twoheadrightarrow \mathrm{Cf}^+(\mathbb{R}^{1,1}_c)$ such that the following diagram commutes

$$\begin{tikzcd} & \widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)^2 \arrow[dl, two heads] \arrow[d, two heads] \\ \mathrm{Cf}^+(\mathbb{R}^{1,1}_c) \arrow[r, two heads, "\Gamma"'] & \mathrm{Diff}^+(\mathbb{S}^1)^2 \end{tikzcd} \tag{2.7}$$

2.5

Since we require that $\mathcal{Q}$ is a CFT with Hilbert space $\mathcal{H}$, we must have a **strongly continuous projective unitary representation** $\mathcal{U}$ of $\mathrm{Cf}^+(\mathbb{R}^{1,1}_c)$. Namely,

$$\mathcal{U} : \mathrm{Cf}^+(\mathbb{R}^{1,1}_c) \to \mathrm{PU}(\mathcal{H})$$

is a continuous group homomorphism. Here, $\mathrm{PU}(\mathcal{H})$ is the quotient group (with quotient topology) $U(\mathcal{H})/\sim$ where $U(\mathcal{H})$ is the group of unitary operators of $\mathcal{H}$ (equipped with the strong operator topology), and $U_1 \simeq U_2$ iff $U_1 = \lambda U_2$ for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. We suppress the adjectives "strongly continuous" when no confusion arises.

By (2.7), $\mathcal{U}$ can be lifted to a projective unitary representation of $\widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)^2$ on $\mathcal{H}$. Since $\widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)^2$ is simply connected, its projective unitary representations are (roughly) equivalent to the projective unitary representations of the Lie algebra of

$\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$, which is $\mathrm{Vec}(\mathbb{S}^1) \oplus \mathrm{Vec}(\mathbb{S}^1)$ where $\mathrm{Vec}(\mathbb{S}^1)$ is the Lie algebra of smooth real vector fields of $\mathbb{S}^1$.

The elements of $\mathrm{Vec}(\mathbb{S}^1)$ are of the form $f\partial_\theta$ where $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$ and ∂_θ is the unique vector field on $\mathbb{S}^1$ that is pulled back by $\exp(\mathbf{i}\cdot) : \mathbb{R} \to \mathbb{S}^1$ to $\partial_\theta \in \mathrm{Vec}(\mathbb{R}^1)$ where θ is the standard coordinate of $\mathbb{R}$ (sending x to x). The Lie bracket of $\mathrm{Vec}(\mathbb{S}^1)$ is the negative of the Lie derivative, i.e.

$$[f_1\partial_\theta, f_2\partial_\theta]_{\mathrm{Vec}(\mathbb{S}^1)} = (-f_1\partial_\theta f_2 + f_2\partial_\theta f_1)\partial_\theta$$

The negative choice is due to the fact that the group action of $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ on $h \in C^\infty(\mathbb{S}^1)$ is given by $h \circ g^{-1}$; however, the Lie derivative is defined by differentiating $g \mapsto h \circ g$.

2.6

The complexification $\mathrm{Vec}_\mathbb{C}(\mathbb{S}^1)$ of $\mathrm{Vec}(\mathbb{S}^1)$ is the Lie algebra of all $f\partial_\theta$ where $f \in C^\infty(\mathbb{S}^1) \equiv C^\infty(\mathbb{S}^1, \mathbb{C})$. Let $z = e^{\mathbf{i}\theta} \in C^\infty(\mathbb{S}^1)$, which is the inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{C}$. Then we can define $\partial_z \in \mathrm{Vec}_\mathbb{C}(\mathbb{S}^1)$ by

$$\partial_z = \frac{1}{\mathbf{i}z}\partial_\theta \qquad \text{so that} \qquad \partial_\theta = \mathbf{i}z\partial_z = \mathbf{i}e^{\mathbf{i}\theta}\partial_z \tag{2.8}$$

Then $\mathrm{Vec}_\mathbb{C}(\mathbb{S}^1)$ is a $*$-Lie algebra, i.e., a complex Lie algebra equipped with an involution $\dagger$. For $\mathrm{Vec}_\mathbb{C}(\mathbb{S}^1)$, the involution is defined by

$$(f\partial_\theta)^\dagger = -\overline{f}\partial_\theta$$

so that $\mathrm{Vec}(\mathbb{S}^1)$ is precisely the set of all $\mathfrak{x} \in \mathrm{Vec}_\mathbb{C}(\mathbb{S}^1)$ satisfying $\mathfrak{x}^\dagger = -\mathfrak{x}$. In particular, noting $\overline{z} = z^{-1}$ on $\mathbb{S}^1$, we have

$$(\partial_z)^\dagger = z^2\partial_z \tag{2.9}$$

$\mathrm{Vec}_\mathbb{C}(\mathbb{S}^1)$ contains a "sufficiently large" $*$-Lie subalgebra, the **Witt algebra** $\mathrm{Witt} = \mathrm{Span}_\mathbb{C}\{l_n : n \in \mathbb{Z}\}$, where

$$l_n = z^n\partial_z \tag{2.10}$$

One easily computes that $l_n^\dagger = l_{-n}$, and that

$$[l_m, l_n] = (m-n)l_{m+n}$$

Projective unitary representations of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ correspond to (honest) unitary representations of central extensions of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$, which (roughly) correspond to unitary representations of central extensions of Witt.

2.7

It can be shown that the central extensions of Witt are equivalent to the **Virasoro algebra Vir**. As a vector space, Vir has basis $\{C, L_n : n \in \mathbb{Z}\}$. These basis elements satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \qquad [L_n, C] = 0 \tag{2.11a}$$

$$L_n^\dagger = L_{-n} \qquad C^\dagger = C \tag{2.11b}$$

Thus, the projective unitary representation of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ on $\mathcal{H}$ can be described by a unitary representation of $\mathrm{Vir} \oplus \widehat{\mathrm{Vir}}$. Here, $\widehat{\mathrm{Vir}} = \{\widehat{C}, \widehat{L}_n : n \in \mathbb{Z}\}$ is isomorphic to Vir.

One can decompose a (unitary full) CFT $\mathcal{Q}$ into a "direct sum" of CFTs such that C and $\widehat{C}$ act as scalars $c, \widehat{c} \in \mathbb{R}$. (In fact, one can show that $c, \widehat{c} \geqslant 0$.) We call $(c, \widehat{c})$ the **central charge** of the CFT $\mathcal{Q}$. Since $L_0^\dagger = L_0$ and $\widehat{L}_0^\dagger = \widehat{L}_0$, one usually assume that $L_0, \widehat{L}_0$ act as self-adjoint operators on $\mathcal{H}$.

2.8

In a Poincaré invariant QFT, the vacuum vector is fixed by $\mathrm{P}^+(1,d)$. However, in our CFT $\mathcal{Q}$, the vacuum vector Ω is not fixed by $\mathrm{Cf}^+(\mathbb{R}^{1,1}_\mathfrak{c})$. In terms of Vir, then $L_n\Omega$ is not necessarily zero for all n. This phenomenon is related to the fact that an arbitrary one-parameter subgroup $t \in \mathbb{R} \mapsto g_t \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$, when each g_t acts on $\mathbb{S}^1$ and hence can be viewed as a map $g_t : \mathbb{S}^1 \to \mathbb{P}^1$, does not have a sufficiently large domain for the analytic continuation $z \mapsto g_z$.

On the other hand, we do have

$$L_n\Omega = 0 \qquad \text{if } n = -1, 0, 1 \tag{2.12}$$

(and similarly $\widehat{L}_0\Omega = \widehat{L}_{\pm 1}\Omega = 0$). These $L_0, L_{\pm 1}$ span a Lie $*$-subalgebra

$$\mathfrak{sl}(2, \mathbb{C}) = \mathrm{Span}_\mathbb{C}\{L_0, L_{\pm 1}\}$$

with skew-symmetric part

$$\mathfrak{su}(2) := \{\mathfrak{x} \in \mathfrak{sl}(2,\mathbb{C}) : \mathfrak{x}^\dagger = -\mathfrak{x}\} = \mathrm{Span}_\mathbb{R}\Big\{\mathbf{i}l_0, \frac{l_1 - l_{-1}}{2}, \frac{\mathbf{i}(l_1 + l_{-1})}{2}\Big\} \tag{2.13}$$

As we will see in the future, the one-parameter group generated by $(l_1 - l_{-1})/2$ is related to the PCT symmetry of the CFT.

The Lie subgroup of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ with Lie algebra $\mathfrak{su}(2)$ is $\widetilde{\mathbf{PSU}(1,1)}$, , the universal cover of the **Möbius group** $\mathrm{PSU}(1,1)$ whose elements are linear fractional transforms

$$z \in \mathbb{P}^1 \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad \text{where } \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1$$

The condition $|\alpha|^2 - |\beta|^2 = 1$ is to ensure that the transform sends $\mathbb{S}^1$ to $\mathbb{S}^1$. The exact sequence (2.3) restricts to

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{PSU}}(1,1) \longrightarrow \mathrm{PSU}(1,1) \longrightarrow 1 \tag{2.14}$$

where $\mathbb{Z}$ is freely generated by "the anticlockwise rotation by 2π". Thus, the projective action $\mathcal{U}(g)$ of any g in $\widetilde{\mathrm{PSU}}(1,1) \times \widetilde{\mathrm{PSU}}(1,1) \subset \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ fixes Ω up to $\mathbb{S}^1$-multiplications. Now we choose $\mathcal{U}(g)$ to be the unique one such that $\mathcal{U}(g)\Omega = \Omega$. Then $\mathcal{U}$ gives an (honest) strongly-continuous unitary representation of $\widetilde{\mathrm{PSU}}(1,1) \times \widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}$ fixing Ω.

2.9

A field $\Phi \in \mathcal{Q}$ is called **chiral** (resp. **antichiral**) if $\widetilde{\Phi}$ depends only on u (resp. v) but not on v (resp. u). We let $\boldsymbol{\mathcal{V}}$ resp. $\widehat{\boldsymbol{\mathcal{V}}}$ be the set of chiral resp. anti chiral fields. They can be viewed as algebraic structures. (We will say more about such structures in the future.)

Let $\mathcal{H}_0$ (resp. $\widehat{\mathcal{H}}_0$) be the closure of the subspace spanned by $\varphi(f_1)\cdots\varphi(f_n)\Omega$ where each $f_i \in C_c^\infty(\mathbb{R}_c^{1,1})$ and $\varphi_i \in \mathcal{V}$ (resp. $\varphi_i \in \widehat{\mathcal{V}}$). Then $\mathcal{H}_0$ can be viewed as a (unitary) representation of $\mathcal{V}$, called the **vacuum representation**. Clearly $\Omega \in \mathcal{H}_0 \cap \widehat{\mathcal{H}}_0$.

A basic assumption of unitary full CFT is the existence of orthogonal decomposition

$$\mathcal{H} = \bigoplus_{i \in \mathfrak{I}} \mathcal{H}_i \otimes \widehat{\mathcal{H}}_i \qquad \supset \mathcal{H}_0 \otimes \widehat{\mathcal{H}}_0 \tag{2.15}$$

where each $\mathcal{H}_i$ (resp. $\widehat{\mathcal{H}}_i$) is an irreducible unitary representation of $\mathcal{V}$ (resp. $\widehat{\mathcal{V}}$). Here, $\bigoplus$ could be a finite, or infinite discrete, or even continuous (i.e. a direct integral). A large class of important CFTs are called **rational CFTs** , which means that the direct sum is finite. Here, $\mathcal{H}_0$ is identified with $\mathcal{H}_0 \otimes \Omega$ so that it can thus be viewed as a subspace of $\mathcal{H}$; similarly $\widehat{\mathcal{H}}_0 \simeq \Omega \otimes \widehat{\mathcal{H}}_0$. Therefore, with respect to the decomposition (2.15), the vacuum vector $\Omega \in \mathcal{H}$ can be written as $\Omega \otimes \Omega$.

2.10

From now on, we slightly change our notation a bit:

Convention 2.4. An element of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ is not viewed as a function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, but rather a multivalued smooth function $f : \mathbb{S}^1 \to \mathbb{S}^1$ related to the original $\widetilde{f}$ by

$$f(e^{\mathbf{i}\theta}) = \widetilde{f}(\theta)$$

Following this convention, and similar to (2.8), we define

$$f'(e^{\mathbf{i}\theta}) \equiv \partial_z f(e^{\mathbf{i}\theta}) = \frac{\widetilde{f}'(\theta)}{\mathbf{i}e^{\mathbf{i}\theta}} \tag{2.16}$$

Similarly, for each $\Phi \in \mathcal{Q}$, we let

$$\mathring{\Phi}(e^{\mathbf{i}u}, e^{\mathbf{i}v}) \xlongequal{\text{def}} \widetilde{\Phi}(u,v) = \Phi\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) \tag{2.17}$$

viewing $\mathring{\Phi}$ as a multivalued function on $\mathbb{S}^1 \times \mathbb{S}^1$.

Although the projective unitary representation $\mathcal{U}$ of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$ does not fix Ω up to $\mathbb{S}^1$-multiplications, similar to (1.4), there is a large class of $\Phi \in \mathcal{Q}$, called **primary fields**, satisfying the **conformal covariance property**: For each such Φ, there exist $\delta, \widehat{\delta} \in \mathbb{R}_{\geqslant 0}$ (called the **conformal weights** of Φ) such that for all $(g,h) \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$ and

$$\mathcal{U}(g,h)\mathring{\Phi}(e^{\mathbf{i}u}, e^{\mathbf{i}v})\mathcal{U}(g,h)^{-1} = g'(e^{\mathbf{i}u})^{\delta} h'(e^{\mathbf{i}v})^{\widehat{\delta}} \cdot \mathring{\Phi}(g(e^{\mathbf{i}u}), h(e^{\mathbf{i}v})) \tag{2.18}$$

in the sense of smeared operators.

2.11

In the special case that $\varphi \in \mathcal{V}$, then (2.1) says that $\mathring{\varphi}$ is a single-valued function on $\mathbb{S}^1$, and hence has a Fourier series expansion

$$\mathring{\varphi}(z) = \sum_{n\in\mathbb{Z}} \mathring{\varphi}_n z^{-n-1} \tag{2.19}$$

So $\mathring{\varphi}_n = \mathrm{Res}_{z=0}\mathring{\varphi}(z)z^n dz$. The derivative $\mathring{\varphi}'(z) = \partial_z\mathring{\varphi}(z)$ is understood in the usual way, i.e.,

$$\mathring{\varphi}'(z) = \sum_n (-n-1)\mathring{\varphi}_n z^{-n-2}$$

Now, writing $\mathcal{U}(g,1)$ as $\mathcal{U}(g)$, then for primary chiral φ, (2.18) becomes

$$\mathcal{U}(g)\mathring{\varphi}(z)\mathcal{U}(g)^{-1} = g'(z)^{\delta} \cdot \mathring{\varphi}(g(z)) \tag{2.20}$$

We simply call δ the **conformal weight** of the chiral field φ. If (2.20) only holds for $g \in \widetilde{\mathrm{PSU}}(1,1)$, we say that the chiral field φ is **quasi-primary**.

Remark 2.5. For each primary (resp. quasi-primary) chiral φ, and for each $m \in \mathbb{Z}$ (resp. $m = 0, \pm 1$), we have

$$[L_m, \mathring{\varphi}(z)] = z^{m+1}\mathring{\varphi}'(z) + \delta\cdot(m+1)z^m\mathring{\varphi}(z) \tag{2.21a}$$

Equivalently, for each $n \in \mathbb{Z}$ we have

$$[L_m, \mathring{\varphi}_n] = -(m+n+1)\mathring{\varphi}_{m+n} + \delta\cdot(m+1)\mathring{\varphi}_{m+n} \tag{2.21b}$$

Heuristic proof. Let $t \mapsto g_t$ be the one-parameter group generated by $\mathfrak{x} = \sum_m a_m l_m$ (a finite sum) satisfying $\mathfrak{x}^\dagger = -\mathfrak{x}$, i.e., $\overline{a_m} = -a_{-m}$. So $g_0(z) = z$ and $\partial_t g_t(z)\big|_{t=0} = \sum_m a_m z^{m+1}$. Set $X = \sum_m a_m L_m$. Then, informally, we have

$$\frac{d}{dt}\mathcal{U}(g_t)\mathring{\varphi}(z)\mathcal{U}(g_t)^{-1}\big|_{t=0} = [X, \mathring{\varphi}(z)]$$

Also

$$\frac{d}{dt}\mathring{\varphi}(g_t(z))\big|_{t=0} = \mathring{\varphi}'(z) \cdot \partial_t g_t(z)\big|_{t=0} = \sum_m a_m z^{m+1}\mathring{\varphi}'(z)$$

Since $\delta \cdot g_0'(z)^{\delta-1} = \delta \cdot \big(\frac{d}{dz}(z)\big)^{\delta-1} = \delta$, we have

$$\frac{d}{dt}g_t'(z)^\delta\big|_{t=0} = \delta \cdot g_0'(z)^{\delta-1} \cdot \partial_t g_t'(z)\big|_{t=0} = \delta \sum_m (a_m z^{m+1})' = \delta \sum_m (m+1)a_m z^m$$

Combining the above three results with (2.20), we get (2.21a). $\square$

3 Local fields and chiral algebras

In this section, we introduce a rigorous approach to the algebra $\mathcal{V}$ of chiral fields. We will give an axiomatic description of (the modes of) the chiral fields acting on $\mathbb{V}$, the dense subspace of $\mathcal{H}_0$ with finite L_0-spectra. (So $\mathcal{H}_0$ is the Hilbert space completion of $\mathbb{V}$.) Some of the proofs will be sketched or even omitted. But details can be found in [Gui-V] (especially Sec. 7 and 8).

3.1

Unless otherwise stated, we fix a complex inner product space $\mathbb{V}$ together with a diagonalizable operator $L_0 \in \mathrm{End}(\mathbb{V})$ such that the eigenvalues of L_0 belong to $\mathbb{N}$. Thus, we have orthogonal decomposition $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ where $\mathbb{V}(n) = \{v \in \mathbb{V} : L_0 v = nv\}$. If $v \in \mathbb{V}$, we say that v is **homogeneous** if $v \in \mathbb{V}(n)$ for some n; in that case we write

$$\mathrm{wt}(v) = n$$

The Hilbert space completion of $\mathbb{V}$ is denoted by $\mathcal{H}_\mathbb{V}$. We assume that each $\mathbb{V}(n)$ is finite-dimensional so that $\mathbb{V}(n)^{**} = \mathbb{V}(n)$. Define

$$\mathbb{V}^{\mathrm{ac}} = \prod_{n\in\mathbb{N}} \mathbb{V}(n)$$

the **algebraic completion** of $\mathbb{V}$. Then clearly

$$\mathbb{V} \subset \mathcal{H}_\mathbb{V} \subset \mathbb{V}^{\mathrm{ac}}$$

Note that L_0 acts on $\mathbb{V}^{\mathrm{ac}}$ in a canonical way by acting on each $\mathbb{V}(n)$ as $n \cdot \mathrm{id}$. Similarly, for each $q \in \mathbb{C}^\times$, q^{L_0} acts on $\mathbb{V}^{\mathrm{ac}}$.

For each $n \in \mathbb{N}$, we define the projection onto the n-th component

$$P_n : \mathbb{V}^{\mathrm{ac}} \to \mathbb{V}(n) \tag{3.1}$$

Then for any $\xi \in \mathbb{V}^{\mathrm{ac}}$, it is clear that

$$\xi \in \mathcal{H}_\mathbb{V} \qquad \Longleftrightarrow \qquad \sum_{n\in\mathbb{N}} \|P_n\xi\|^2 < +\infty \tag{3.2}$$

Note that L_0 and q^{L_0} commute with P_n. We also let

$$P_{\leqslant n} = \sum_{k\in\mathbb{N},k\leqslant n} P_n \tag{3.3}$$

3.2

Definition 3.1. An **(homogeneous) field** on $\mathbb{V}$ is an element

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$$

(where each A_n is in $\mathrm{End}(\mathbb{V})$) satisfying

$$[L_0, A(z)] = \mathrm{wt}(A)\cdot A(z) + z\partial_z A(z) \tag{3.4a}$$

for some $\mathrm{wt}(A) \in \mathbb{N}$ (called the **(conformal) weight**) of $A(z)$; equivalently,

$$[L_0, A_n] = (\mathrm{wt}(A) - n - 1)A_n \tag{3.4b}$$

Remark 3.2. Note that by (3.4b), for each d, A_n restricts to

$$A_n : \mathbb{V}(d) \to \mathbb{V}(d + \mathrm{wt}(A) - n - 1) \tag{3.5}$$

Since no nonzero homogeneous vectors can have negative weights, we see that $A_n v = 0$ when $n \gg 0$, and that $\langle A_n \cdot |v\rangle = 0$ when $n \ll 0$. Thus

$$A(z)v \in \mathbb{V}((z)) \tag{3.6}$$

for each homogeneous $v \in \mathbb{V}$, and hence for all $v \in \mathbb{V}$. This is called the **lower truncation property**.

Note that A_n can be extended to $A_n^{\mathrm{tt}} : \mathbb{V}^{\mathrm{ac}} \to \mathbb{V}^{\mathrm{ac}}$. We abbreviate A_n^{tt} to A_n when no confusion arises.

Example 3.3. The field $\mathbf{1}(z) = \mathrm{id}_{\mathbb{V}}$ is called the **vacuum field**. By (3.4), we clearly have

$$\mathrm{wt}(\mathbf{1}) = 0$$

3.3

Let $A(z)$ be a homogeneous field. By (3.5), we have a well defined linear map $(A_n)^\dagger : \mathbb{V} \to \mathbb{V}$ being the formal adjoint of A_n, i.e.,

$$\langle A_n u|v\rangle = \langle u|(A_n)^\dagger v\rangle$$

This is because the restriction $A_n : \mathbb{V}(d) \to \mathbb{V}(d + \mathrm{wt}(A) - n - 1)$ has an adjoint due the the finite-dimensionality. Thus $(A_n)^\dagger$ restricts to

$$(A_n)^\dagger : \mathbb{V}(d) \to \mathbb{V}(d - \mathrm{wt}(A) + n + 1) \tag{3.7}$$

If z is a formal variable, we understand $\overline{z} \equiv z^\dagger$ as the formal conjugate of z. So $z, \overline{z}$ are mutually commuting formal variables.

Definition 3.4. Define the **quasi-primary contragredient $A^\theta(z)$** of $A(z)$ to be

$$A^\theta(z)=(-z^{-2})^{\mathrm{wt}(A)}A(\overline{z^{-1}})^\dagger=(-z^{-2})^{\mathrm{wt}(A)}\cdot\sum_{n\in\mathbb{Z}}(A_n)^\dagger z^{n+1} \tag{3.8}$$

One shows easily that

$$A^\theta_n=(-1)^{\mathrm{wt}(A)}\cdot(A_{-n-2+2\mathrm{wt}(A)})^\dagger \tag{3.9}$$

Comparing (3.9) with (3.7), we see that A^θ_n restricts to $\mathbb{V}(d)\to\mathbb{V}(d+\mathrm{wt}(A)-n-1)$. Hence A^θ is homogeneous with weight

$$\mathrm{wt}(A^\theta)=\mathrm{wt}(A) \tag{3.10}$$

One checks easily that $A^{\theta\theta}=A$.

The reason we need the extra term $(-z^{-2})^{\mathrm{wt}(A)}$ will be clear when studying PCT symmetry for chiral CFTs in the future (cf. Thm. 7.17). At present, we at least know that part of the reasons we need z^{-2} and its power $\mathrm{wt}(A)$ is because we want (3.10) to be true.

3.4

Remark 3.5. The field $A^\theta(z)$ can also be understood in the following way: For each $u,v\in\mathbb{V}$ we have

$$\langle A^\theta(z)u|v\rangle=(-z^{-2})^{\mathrm{wt}(A)}\langle u|A(\overline{z^{-1}})v\rangle \tag{3.11}$$

as elements of $\mathbb{C}[[z^{\pm1}]]$. By (3.6), the LHS resp. RHS is in $\mathbb{C}((z))$ resp. $\mathbb{C}((z^{-1}))$, we conclude that (3.11) is in $\mathbb{C}[z^{\pm1}]$. Similarly,

$$\langle A(z)u|v\rangle\in\mathbb{C}[z^{\pm1}]$$

Thus, $z\in\mathbb{C}^\times\to\langle A(z)u|v\rangle\in\mathbb{C}$ is a holomorphic function with finite poles at $0,\infty$, and (3.11) holds in $\mathscr{O}(\mathbb{C}^\times)$. It follows that for each $m,n\in\mathbb{V}$,

$$z\in\mathbb{C}^\times\mapsto P_mA(z)P_n$$

is an $\mathrm{Hom}(\mathbb{V}(n),\mathbb{V}(m))$-valued holomorphic function.

Proposition 3.6. *Let $u,v\in\mathbb{V}$. Let A be a homogeneous field. Then for each $z,q\in\mathbb{C}^\times$ we have*

$$\langle q^{L_0}A(z)q^{-L_0}u|v\rangle=q^{\mathrm{wt}(A)}\cdot\langle A(qz)u|v\rangle \tag{3.12}$$

In short, we have $q^{L_0}A(z)q^{-L_0}=q^{\mathrm{wt}(A)}A(qz)$ as linear maps $\mathbb{V}\to\mathbb{V}^{\mathrm{ac}}$. Compare this with Eq. (2.20).

Proof. For each fixed $q\in\mathbb{C}^\times$, by expanding both sides of (3.12) as Laurent series of z, we see that (3.12) is equivalent to

$$\langle q^{L_0}A_nq^{-L_0}u|v\rangle=q^{\mathrm{wt}(A)-n-1}\langle A_nu|v\rangle \tag{3.13}$$

By linearity, it suffices to assume that u,v are homogenous. In that case, this relation follows immediately from (3.5). □

3.5

Definition 3.7. Let $A(z), B(z)$ be homogeneous fields on $\mathbb{V}$. We say that $A(z), B(z)$ are mutually **local** if there exists $N \in \mathbb{N}$ (depending on A, B) such that the following relation holds in $\mathrm{End}(\mathbb{V})[[z^{\pm 1}, w^{\pm 1}]]$:

$$(z-w)^N [A(z), B(w)] = 0 \tag{3.14}$$

We call N an **order of pole between** A, B.

Remark 3.8. A field $A(z)$ is not necessarily local to itself. If $A(z)$ is local to $A(z)$, we say that $A(z)$ is **self-local**. A collection of fields $(A^i(z))_{i \in I}$ is called **mutually local** if $A^i(z)$ is local to $A^j(z)$ whenever $i, j \in I$ and $i \neq j$.

Eq. (3.14) needs explanation. Let R be a $\mathbb{C}$-algebra. Write $z_\bullet = (z_1, \ldots, z_k)$. Then $R[[z_\bullet^{\pm 1}]]$ is an $R[z_\bullet]$-module. However, this module is not necessarily torsion-free:

Example 3.9. Fix $N \in \mathbb{N}$. Let α, β be the expansions of the meromorphic function $(z_1 - z_2)^{-N}$ in $|z_1| < |z_2|$ and $|z_1| > |z_2|$, i.e.

$$\alpha = \sum_{j \in \mathbb{N}} \binom{-N}{j} z_1^j (-z_2)^{-N-j} \qquad \beta = \sum_{j \in \mathbb{N}} \binom{-N}{j} z_1^{-N-j} (-z_2)^j$$

Then $\alpha \in \mathbb{C}[[z_2^{\pm 1}]][z_1]$ and $\beta \in \mathbb{C}[[z_1^{\pm 1}]][z_2]$, and both belong to $\mathbb{C}[[z_\bullet^{\pm 1}]]$. So $\alpha \neq \beta$ as elements of $\mathbb{C}[[z_\bullet^{\pm 1}]]$. However, $(z_1 - z_2)^N \alpha = (z_1 - z_2)^N \beta = 1$. Thus $\alpha - \beta$ is an torsion element of the $\mathbb{C}[z_\bullet]$-module $\mathbb{C}[[z_\bullet^{\pm 1}]]$.

Then $R[[z_\bullet^{\pm 1}]]$ is not naturally a $\mathbb{C}$-algebra. In particular, not every two elements of $R[[z_\bullet^{\pm 1}]]$ can be multiplied. For example, the square of $\sum_{n \in \mathbb{Z}} z^n$ does not make sense. Moreover, the associativity of products does not necessarily hold even if the elements involved can be multiplied, as shown by Exp. 3.10.

Example 3.10. In Exp. 3.9, both $(\alpha \cdot (z_1 - z_2)^N) \cdot \beta$ and $\alpha \cdot ((z_1 - z_2)^N \cdot \beta)$ can be defined. However,

$$(\alpha \cdot (z_1 - z_2)^N) \cdot \beta = \beta \qquad \alpha \cdot ((z_1 - z_2)^N \cdot \beta) = \alpha$$

3.6

Assume that A, B are mutually local fields with order of pole N. Choose any $u, v \in \mathbb{V}$. By Rem. 3.5, we have $\langle A(z)B(w)u|v\rangle = (-z^{-2})^{\mathrm{wt}(A)} \langle B(w)|A^\theta(\overline{z^{-1}})v\rangle$, which belongs to $\mathbb{C}((z^{-1}, w))$ by the lower truncation property (3.6). Thus

$$\langle A(z)B(w)u|v\rangle \in \mathbb{C}((z^{-1}, w)) \qquad \langle B(w)A(z)u|v\rangle \in \mathbb{C}((z, w^{-1}))$$

Therefore, setting

$$g := (z-w)^N \langle A(z)B(w)u|v\rangle = (z-w)^N \langle B(w)A(z)u|v\rangle$$

we have that

$$g \in \mathbb{C}((z^{-1},w)) \cap \mathbb{C}((z,w^{-1})) = \mathbb{C}[z^{\pm 1},w^{\pm 1}]$$

Since the $\alpha(z,w),\beta(z,w)$ in Exp. 3.5 are respectively the inverses of $(z-w)^N$ in the $\mathbb{C}$-algebras $\mathbb{C}((z^{-1},w))$ and $\mathbb{C}((z,w^{-1}))$, we see that

$$\langle A(z)B(w)u|v\rangle = \beta g \qquad \langle B(w)A(z)u|v\rangle = \alpha g$$

Consequently, the series $\langle A(z)B(w)u|v\rangle$ of z,w converges **absolutely and locally uniformly (a.l.u)** on the region $\{(z,w)\in\mathbb{C} : 0<|w|<|z|\}$ in the sense that it converges uniformly on any compact subset of that open set. This is because the series βg converges a.l.u. on this domain.

When u,v are homogeneous, one sees easily that this a.l.u. convergence is equivalent to that of

$$\sum_{n\in\mathbb{N}} \langle A(z)P_nB(w)u|v\rangle$$

viewed as a series of functions of z,w on $\{0<|w|<|z|\}$. (This is because for each n, $\langle A(z)P_nB(w)u|v\rangle$ is a monomial of z,w.) Thus, by linearity, the a.l.u. convergence of this series of functions also holds for any $u,v\in\mathbb{V}$. Similarly,

$$\sum_{n\in\mathbb{N}} \langle B(w)P_nA(z)u|v\rangle$$

converges a.l.u. on $\{0<|z|<|w|\}$. Moreover, the limit functions of these two series can be analytically extended to the same holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times)$, namely, the rational function $(z_1-z_2)^{-N}g(z_1,z_2)$.

3.7

The results in the previous subsection can be generalized to the following theorem. The proof is similar, and hence will not be given here. See [Gui-V, Subsec. 8.2] for details.

Theorem 3.11. *Let $A^1,\dots,A^k$ be mutually local fields. Then for each $u,v\in\mathbb{V}$ and each permutation σ of $\{1,\dots,k\}$, the series of Laurent polynomials of $z_\bullet$*

$$\sum_{n_2,\dots,n_k\in\mathbb{N}} \langle A^{\sigma(1)}(z_{\sigma(1)})P_{n_2}A^{\sigma(2)}(z_{\sigma(2)})P_{n_3}\cdots P_{n_k}A^{\sigma(k)}(z_{\sigma(k)})u|v\rangle \tag{3.15}$$

converges a.l.u. on

$$\{z_\bullet \in \mathbb{C}^k : 0 < |z_{\sigma(k)}| < \cdots < |z_{\sigma(1)}|\} \tag{3.16}$$

and can be extended to some $f_{u,v} \in \mathscr{O}(\mathrm{Conf}^k(\mathbb{C}^\times))$ independent of σ. Indeed, $f_{u,v}$ is a rational function.

Remark 3.12. We say that u is **vacuum with respect to** $A(z)$ if $A(z)u \in \mathbb{V}[[z]]$, i.e., if $A_n u = 0$ if $n \geqslant 0$. If u is vacuum with respect to $A^1, \dots, A^k$, the same argument as in Subsec. 3.6 shows that $f_{u,v} \in \mathscr{O}(\mathrm{Conf}^k(\mathbb{C}))$. Thus (3.15) converges a.l.u. on

$$\{z_\bullet \in \mathbb{C}^k : |z_{\sigma(k)}| < \cdots < |z_{\sigma(1)}|\}$$

Definition 3.13. In the setting of Thm. 3.11, for each $u \in \mathbb{V}$ and $z_\bullet \in \mathrm{Conf}^k(\mathbb{C}^\times)$, define

$$A^1(z_1)\cdots A^k(z_k)u \in \mathbb{V}^{\mathrm{ac}} \tag{3.17}$$

to be the one whose inner product with any $v \in \mathbb{V}$ is $f_{u,v}(z)$. Thus $A^1(z_1)\cdots A^k(z_k)$ is a linear map $\mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$, and for each $u, v \in \mathbb{V}$ the function

$$z_\bullet \in \mathrm{Conf}^k(\mathbb{C}^\times) \mapsto \langle A^1(z_1)\cdots A^k(z_k)u|v\rangle \in \mathbb{C} \tag{3.18}$$

is holomorphic. When u is vacuum with respect to $A^1, \dots, A^k$, the same conclusion holds if we replace $\mathbb{C}^\times$ with $\mathbb{C}$.

It is clear that for each permutation σ of $\{1, \dots, k\}$ we have

$$A^{\sigma(1)}(z_{\sigma(1)})\cdots A^{\sigma(k)}(z_{\sigma(k)})u = A^1(z_1)\cdots A^k(z_k)u \tag{3.19}$$

Some of the results about single operators can be generalized to products of operators:

Proposition 3.14. *Let $A^1, \dots, A^k$ be mutually local fields. Then for each $z_\bullet \in \mathrm{Conf}^k(\mathbb{C}^\times)$ and $q \in \mathbb{C}^\times$, we have in $\mathrm{Hom}(\mathbb{V}, \mathbb{V}^{\mathrm{ac}})$ that*

$$q^{L_0}A^1(z_1)\cdots A^k(z_k) = q^{\mathrm{wt}(A^1)+\cdots+\mathrm{wt}(A^k)}A^1(qz_1)\cdots A^k(qz_k)q^{L_0} \tag{3.20}$$

Proof. Fix $q \in \mathbb{C}^\times$ and $u, v \in \mathbb{V}$. Let f, g denote the LHS and the RHS of (3.20) inserted in $\langle \cdot|v\rangle$. By Thm. 3.11, both f and g are holomorphic functions of $z_\bullet \in \mathrm{Conf}^k(\mathbb{C}^\times)$. Therefore, to prove $f = g$ on the connected region $\mathrm{Conf}^k(\mathbb{C}^\times)$ it suffices to prove it on a nonempty open subset, say $\{0 < |z_k| < \cdots < |z_1|\}$. In that case, the relation $f = g$ follows from the a.l.u. convergence in Thm. 3.11 and the fact that for all $n_2, \dots, n_k \in \mathbb{N}$ we have in $\mathrm{Hom}(\mathbb{V}, \mathbb{V}^{\mathrm{ac}})$ that

$$\begin{aligned}&q^{L_0}A^1(z_1)P_{n_2}A^2(z_2)P_{n_3}\cdots P_{n_k}A^k(z_k)\\ =&q^{\mathrm{wt}(A^1)+\cdots+\mathrm{wt}(A^k)}A^1(qz_1)P_{n_2}A^2(qz_2)P_{n_3}\cdots P_{n_k}A^k(qz_k)q^{L_0}\end{aligned}$$

The latter is due to Prop. 3.6 and the fact that q^{L_0} commutes with each P_{n_j}. □

Proposition 3.15. *Let $A^1, \dots, A^k$ be mutually local fields. Let $u, v \in \mathbb{V}$. Then for each $z_\bullet \in \mathrm{Conf}^k(\mathbb{C}^\times)$ we have*

$$\begin{aligned}&\langle A^1(z_1)\cdots A^k(z_k)u|v\rangle\\ =&(-z_1^{-2})^{\mathrm{wt}(A^1)}\cdots(-z_k^{-2})^{\mathrm{wt}(A^k)}\big\langle u\big|(A^k)^\theta(\overline{z_k^{-1}})\cdots(A^1)^\theta(\overline{z_1^{-1}})v\big\rangle\end{aligned} \tag{3.21}$$

Proof. Similar to Prop. 3.14, it suffices to prove (3.21) when $0 < |z_1| < \cdots < |z_k|$ (and hence $0 < |\overline{z_k^{-1}}| < \cdots < |\overline{z_1^{-1}}|$). This special case follows from the a.l.u. convergence Thm. 3.11 and Def. 3.4. □

3.8

We now discuss a further generalization (or variant) of the convergence Thm. 3.11. Its proof gives another application of the trick of analytic continuation (as in the proof of Prop. 3.14 and 3.15).

Theorem 3.16. *Assume that $A^1, \dots, A^m$ and $B^1, \dots, B^k$ are mutually local fields. Let*

$$O = \{(z_1, \dots, z_m, \zeta_1, \dots, \zeta_k) \in \mathrm{Conf}^{m+k}(\mathbb{C}^\times) : |z_i| > |\zeta_j| \text{ for all } i, j\}$$

Then for each $u, v \in \mathbb{V}$, the RHS of

$$\begin{aligned}&\langle A^1(z_1)\cdots A^m(z_m)B^1(\zeta_1)\cdots B^k(\zeta_k)u|v\rangle\\ =&\sum_{n\in\mathbb{N}}\langle A^1(z_1)\cdots A^m(z_m)P_nB^1(\zeta_1)\cdots B^k(\zeta_k)u|v\rangle\end{aligned} \tag{3.22}$$

converges a.l.u. on O to the LHS.

Proof. It suffices to prove the a.l.u. on

$$O_r = \{(z_1, \dots, z_m, \zeta_1, \dots, \zeta_k) \in \mathrm{Conf}^{m+k}(\mathbb{C}^\times) : |z_i| > r|\zeta_j| \text{ for all } i, j\}$$

for each $r > 1$. In fact, we shall show that the series of functions

$$\sum_{n\in\mathbb{N}}\langle A^1(z_1)\cdots A^m(z_m)q^{L_0}P_nB^1(\zeta_1)\cdots B^k(\zeta_k)u|v\rangle \tag{a}$$

converges a.l.u. on $(z_\bullet, \zeta_\star, q) \in O_r \times \mathbb{D}_r^\times$ to

$$q^\delta\langle A^1(z_1)\cdots A^m(z_m)B^1(q\zeta_1)\cdots B^k(q\zeta_k)q^{L_0}u|v\rangle \tag{b}$$

where $\delta = \mathrm{wt}(B^1) + \cdots + \mathrm{wt}(B^k)$. By Thm. 3.11 and Prop. 3.6, on

$$O_r' = \{(z_\bullet, \zeta_\star) : 0 < r|\zeta_k| < \cdots < r|\zeta_1| < |z_m| < \cdots < |z_1|\}$$

the series (a) is equivalent to

$$\begin{aligned}&\sum\langle A^1(z_1)P_{\nu_2}\cdots P_{\nu_m}A^m(z_m)q^{L_0}P_nB^1(\zeta_1)P_{n_2}\cdots P_{n_k}B^k(\zeta_k)u|v\rangle\\ =&\sum q^\delta\langle A^1(z_1)P_{\nu_2}\cdots P_{\nu_m}A^m(z_m)P_nB^1(q\zeta_1)P_{n_2}\cdots P_{n_k}B^k(q\zeta_k)q^{L_0}u|v\rangle\end{aligned}$$

and hence converges a.l.u. to (b). Therefore, if we let $\sum_\nu f_\nu q^\nu$ be the Laurent series expansion of (b) (where $f_\nu\in\mathscr{O}(O_r)$), then this series converges a.l.u. on $O_r\times\mathbb{D}_r^\times$, and this series equals the series (a) on $O_r'\times\mathbb{D}_r^\times$. Thus f_ν equals the coefficient before q^ν of (a) on O_r', and hence on O_r by the holomorphicity of the coefficients (as functions on O_r). Thus (a) converges a.l.u. on O_r to (b). □

The following theorem follows almost immediately from Thm. 3.16.

Theorem 3.17. *Let $A^1,\dots,A^k$ be homogeneous fields such that any two distinct members of $A^1,\dots,A^k,(A^1)^\theta,\dots,(A^k)^\theta$ are mutually local. Let $v\in\mathbb{V}$. Then we have a holomorhic function*

$$\mathrm{Conf}^k(\mathbb{D}_1^\times)\to\mathcal{H}_{\mathbb{V}}\qquad z_\bullet\mapsto A^1(z_1)\cdots A^k(z_k)v \tag{3.23}$$

If v is vacuum with respect to $A^1,\dots,A^k$, and if $\mathrm{Conf}^k(\mathbb{D}_1^\times)$ is replaced by $\mathrm{Conf}^k(\mathbb{D}_1)$, the function (3.23) is still holomorphic.

Proof. Step 1. By Prop. 3.15, we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\|P_nA^1(z_1)\cdots A^k(z_k)v\|^2\\ =&\sum_{n\in\mathbb{N}}(-\overline{z_1}^{-2})^{\mathrm{wt}(A^1)}\cdots(-\overline{z_k}^{-2})^{\mathrm{wt}(A^k)}\\ &\cdot\langle(A^k)^\theta(1/\overline{z_k})\cdots(A^1)^\theta(1/\overline{z_1})P_nA^1(z_1)\cdots A^k(z_k)v|v\rangle\end{aligned}$$

By Thm. 3.16, this series converges a.l.u. on $\mathrm{Conf}^k(\mathbb{D}_1^\times)$. Therefore, for each $z_\bullet\in\mathrm{Conf}^k(\mathbb{D}_1^\times)$ we have $A^1(z_1)\cdots A^k(z_k)v\in\mathcal{H}_{\mathbb{V}}$. Moreover, the above a.l.u. convergence implies the a.l.u. convergence of the series of $\mathcal{H}_{\mathbb{V}}$-valued functions

$$z_\bullet\in\mathrm{Conf}^k(\mathbb{D}_1^\times)\mapsto\sum_{n\in\mathbb{N}}P_nA^1(z_1)\cdots A^k(z_k)v$$

because the summands are mutually orthogonal for different n. Since the partial sums of this series are holomorphic, the limit of the above series (namely, $A^1(z_1)\cdots A^k(z_k)v$) is also holomorphic.

Step 2. We now address the case that v is vacuum. We want to show that for each open disk $U\subset\mathbb{D}_1^\times$ centered at 0, if we let

$$\Gamma=\mathrm{Conf}^{k-1}(\mathbb{D}_1\backslash U)$$

and define the holomorphic function $f:\Gamma\times U^\times\to\mathcal{H}_{\mathbb{V}}$ to be the restriction of (3.23) (where $U^\times=U\backslash\{0\}$), then f can be extended to a holomorphic function on $\Gamma\times U$. The proof will be completed by replacing z_k by any one of $z_1,\dots,z_k$.

It suffices to prove that the Laurent series expansion $f=\sum_{n\in\mathbb{Z}}f_n(z_1,\dots,z_{k-1})z_k^n$ (where $f_n\in\mathscr{O}(\Gamma)$) satisfies $f_n=0$ for all $n<0$; then $\sum_{n\in\mathbb{Z}}f_nz_k^n$ converges a.l.u. on $\Gamma\times U$ to a holomorphic function extending f, finishing the proof. Since Γ is connected, it suffices to prove $f_n=0$ on

$$\{(z_1,\dots,z_{k-1})\in\Gamma:|z_1|>\cdots>|z_{k-1}|\}$$

Choose any $(z_1,\dots,z_{k-1})$ in this set. Then for $z_k\in U$, and for each $w\in\mathbb{V}$, we have

$$\langle f(z_\bullet)|w\rangle=\sum_{n_2,\dots,n_k\in\mathbb{N}}\langle A^1(z_1)P_{n_2}\cdots P_{n_k}A^k(z_k)v|w\rangle$$

where $\mathrm{Res}_{z_k=0}(\mathrm{RHS})z_k^{-n-1}dz_k=0$ for $n<0$ (since v is A^k-vacuum), noting that $\mathrm{Res}_{z_k=0}$ commutes with $\sum$ due to the a.l.u. convergence of the RHS above over $z_k\in U^\times$. Thus $f_n(z_1,\dots,z_{k-1})=0$ when $n<0$. $\square$

3.9

A linear combination of mutually local fields is clearly local to the original fields. It turns out that there is a non-associative "product" A_kB (where $k\in\mathbb{Z}$) that is local to any field C whenever A,B,C are mutually local.

Definition 3.18. Let A,B be mutually local fields. Let $k\in\mathbb{Z}$. For each $z\in\mathbb{C}^\times$, define a linear map $(A_kB)(z):\mathbb{V}\to\mathbb{V}^{\mathrm{ac}}$ by

$$\langle(A_kB)(z)u|v\rangle=\oint_{\Gamma(z)}(\zeta-z)^k\langle A(\zeta)B(z)u|v\rangle\frac{d\zeta}{2\mathrm{i}\pi}\tag{3.24}$$

for each $u,v\in\mathbb{V}$. Here, $\Gamma(z)$ is an anticlockwise circle around z. Clearly (3.24) is holomorphic over $z\in\mathbb{C}^\times$. Let

$$(A_kB)_n:\mathbb{V}\to\mathbb{V}^{\mathrm{ac}}\qquad\langle(A_kB)_nu|v\rangle=\mathrm{Res}_{z=0}\,z^n\langle(A_kB)(z)u|v\rangle dz$$

So we have $(A_kB)(z)=\sum_{n\in\mathbb{Z}}(A_kB)_nz^{-n-1}$ in $\mathrm{Hom}(\mathbb{V},\mathbb{V}^{\mathrm{ac}})[[z]]$.

3.10

Let A,B be mutually local fields.

Theorem 3.19. *For each $n,k\in\mathbb{Z}$ we have*

$$(A_kB)_n=\sum_{l\in\mathbb{N}}(-1)^l\binom{k}{l}A_{k-l}B_{n+l}-\sum_{l\in\mathbb{N}}(-1)^{k+l}\binom{k}{l}B_{k+n-l}A_l\tag{3.25}$$

Note that by the lower truncation property (3.6), the RHS of (3.25) is a finite sum when acting on each $v \in \mathbb{V}$.

Proof. Fix $w \in \mathbb{C}^\times$. Then the function $f(z) = (z-w)^k \langle A(z)B(w)u|v\rangle$ is holomorphic on $z \in \mathbb{C}^\times \backslash \{w\}$. Let Γ_-, Γ_+ be circles around 0 with radii $< |w|$ and $> |w|$ respectively. Let $\Gamma(w)$ be a circle around w and between Γ_- and Γ_+. Then Cauchy's theorem implies that $\langle (A_k B)(w)u|v\rangle = \int_{\Gamma(w)} f(z)dz/2\mathbf{i}\pi$ equals $\int_{\Gamma_+ - \Gamma_-} f(z)dz/2\mathbf{i}\pi$.

We compute that $\int_{\Gamma_+} f(z)\frac{dz}{2\mathbf{i}\pi}$ equals

$$\int_{\Gamma_+} (z-w)^k \langle A(z)B(w)u|v\rangle \frac{dz}{2\mathbf{i}\pi} = \int_{\Gamma_+} \sum_{\nu\in\mathbb{N}} (z-w)^k \langle A(z)P_\nu B(w)u|v\rangle \frac{dz}{2\mathbf{i}\pi}$$

By Thm. 3.11, the series in the integral converges uniformly on $z \in \Gamma_+$. Thus $\int_{\Gamma_+}$ and $\sum_\nu$ can be exchanged. Therefore

$$\begin{aligned}
&\int_{\Gamma_+} f(z)\frac{dz}{2\mathbf{i}\pi} = \sum_{\nu\in\mathbb{N}} \int_{\Gamma_+} (z-w)^k \langle A(z)P_\nu B(w)u|v\rangle \frac{dz}{2\mathbf{i}\pi} \\
&= \sum_{\nu\in\mathbb{N}} \int_{\Gamma_+} \sum_{l\in\mathbb{N}} \binom{k}{l} z^{k-l}(-w)^l \langle A(z)P_\nu B(w)u|v\rangle \frac{dz}{2\mathbf{i}\pi} \\
&= \sum_{\nu\in\mathbb{N}} \sum_{l\in\mathbb{N}} \binom{k}{l} (-w)^l \langle A_{k-l}P_\nu B(w)u|v\rangle = \sum_{l\in\mathbb{N}} \binom{k}{l} (-w)^l \langle A_{k-l}B(w)u|v\rangle
\end{aligned}$$

Similarly, since $(z-w)^k = \sum_{l\in\mathbb{N}} \binom{k}{l} z^l (-w)^{k-l}$ when z is on Γ_-, we have

$$\int_{\Gamma_-} f(z)\frac{dz}{2\mathbf{i}\pi} = \sum_{\nu\in\mathbb{N}} \int_{\Gamma_-} (z-w)^k \langle B(w)P_\nu A(z)u|v\rangle \frac{dz}{2\mathbf{i}\pi} = \sum_{l\in\mathbb{N}} \binom{k}{l} (-w)^{k-l} \langle B(w)A_l u|v\rangle$$

To summarize, we have

$$\langle (A_k B)(w)u|v\rangle = \sum_{l\in\mathbb{N}} \binom{k}{l} (-w)^l \langle A_{k-l}B(w)u|v\rangle - \sum_{l\in\mathbb{N}} \binom{k}{l} (-w)^{k-l} \langle B(w)A_l u|v\rangle$$

Applying $\mathrm{Res}_{w=0} w^n(\cdots)dw$ to both sides, we get (3.25). □

Corollary 3.20. *Let $k \in \mathbb{Z}$. Then for each $n \in \mathbb{Z}$, the linear map $(A_k B)_n : \mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$ has range in $\mathbb{V}$. Moreover, $A_k B$ is a homogeneous field with weight*

$$\mathrm{wt}(A_k B) = \mathrm{wt}(A) + \mathrm{wt}(B) - k - 1 \tag{3.26}$$

Proof. Eq. (3.25) shows that $(A_k B)_n$ sends each $\mathbb{V}(d)$ to $\mathbb{V}(d')$ where

$$\begin{aligned}
d' &= d + (\mathrm{wt}(A) - k + l - 1) + (\mathrm{wt}(B) - n - l - 1) \\
&= d + (\mathrm{wt}(B) - k - n + l - 1) + (\mathrm{wt}(A) - l - 1)
\end{aligned}$$

which equals $d + \mathrm{wt}(A_k B) - n - 1$ if we let $\mathrm{wt}(A_k B)$ be the RHS of (3.26). □

3.11

With the help of $A_k B$, we obtain several equivalent descriptions of local fields:

Theorem 3.21. *Let A, B be homogeneous fields and $N \in \mathbb{N}$. Then the following are equivalent.*

(1) A, B are mutually local with pole of order N.

(2) For each $u, v \in \mathbb{V}$, the series

$$\sum_{n\in\mathbb{N}} \langle A(z)P_n B(w)u|v\rangle \qquad \textit{and} \qquad \sum_{n\in\mathbb{N}} \langle B(w)P_n A(z)u|v\rangle \tag{3.27}$$

converge a.l.u. on

$$\{(z,w)\in\mathbb{C}^2 : 0<|w|<|z|\} \qquad \textit{and} \qquad \{(z,w)\in\mathbb{C}^2 : 0<|z|<|w|\} \tag{3.28}$$

respectively, and can be extended to a common function $f_{u,v} \in \mathscr{O}(\mathrm{Conf}^2(\mathbb{C}^\times))$ such that $(z-w)^N f_{u,v}$ is holomorphic on $(\mathbb{C}^\times)^2$.

(3) For each $j = 0, 1, \dots, N-1$ there exists a sequence $(C^j_n)_{n\in\mathbb{Z}}$ in $\mathrm{End}(\mathbb{V})$ such that for all $m, k \in \mathbb{Z}$ we have

$$[A_m, B_k] = \sum_{l=0}^{N-1} \binom{m}{l} C^l_{m+k-l} \tag{3.29}$$

Moreover, if one of (1) and (2) is true, then $A_j B = 0$ for all $j \geqslant N$, and (3) holds if for each $0 \leqslant j \leqslant N$ we define $C^j(z) \equiv \sum_{n\in\mathbb{Z}} C^j_n z^{-n-1}$ to be

$$C^j(z) = (A_j B)(z)$$

Proof. (1)⇒(2) follows directly from Thm. 3.11.

(2)⇒(3): Note that $A_j B$ can be defined and satisfies Cor. 3.20 whenever (2) holds. Assume (2), and set $C^j(z) = (A_j B)(z)$. By Def. 3.18, if $j \geqslant N$ then

$$\langle (A_j B)(w)u|v\rangle = \mathrm{Res}_{z=w}(z-w)^j f_{u,v}(z,w)dz = 0$$

because $z \mapsto (z-w)^j f_{u,v}(z,w)$ is holomorphic on a neighborhood at w. So $C^j = 0$ for all $j \geqslant N$.

Fix $w \in \mathbb{C}^\times$ and $g(z) = z^m \langle A(z)B(w)u|v\rangle$. Let $\Gamma_\pm, \Gamma(w)$ be as in the proof of Thm. 3.19. Then $\int_{\Gamma(w)} g(z)\frac{dz}{2\mathbf{i}\pi} = \int_{\Gamma_+ - \Gamma_-} g(z)\frac{dz}{2\mathbf{i}\pi}$. Similar to the proof of Thm. 3.19, one computes that

$$\int_{\Gamma_+} g(z)\frac{dz}{2\mathbf{i}\pi} = \langle A_m B(w)u|v\rangle \qquad \int_{\Gamma_-} g(z)\frac{dz}{2\mathbf{i}\pi} = \langle B(w)A_m u|v\rangle$$

$$\int_{\Gamma(w)} g(z)\frac{dz}{2\mathbf{i}\pi} = \sum_{l\in\mathbb{N}}\binom{m}{l}w^{m-l}\langle (A_lB)(w)u|v\rangle$$

since $z^m = \sum_{l\in\mathbb{N}}\binom{m}{l}(z-w)^l w^{m-l}$ when $z\in\Gamma(w)$. Thus, for all $w\in\mathbb{C}^\times$ we have

$$\langle [A_m, B(w)]u|v\rangle = \sum_{l=0}^{N-1}\binom{m}{l}w^{m-l}\langle C^l(w)u|v\rangle$$

Applying $\mathrm{Res}_{w=0}w^k(\cdots)dw$ to both sides, we get (3.29).

(3)⇒(1): This is calculated by brutal force. Assume (3). Using $(z-w)^N = \sum_{j=0}^N\binom{N}{j}z^jw^{N-j}$, one computes that the coefficient before $z^{-m-1}w^{-n-1}$ of $(z-w)^N[A(z),B(w)]$ is

$$\mathrm{Res}_{z=0}\mathrm{Res}_{w=0}z^mw^n(z-w)^N[A(z),B(w)]dzdw = \sum_{l=0}^{N-1}\lambda_l C^l_{m+n+N-l}$$

where $\lambda_l = \sum_{j=0}^N\binom{N}{j}(-1)^{N-j}\binom{m+j}{l}$ is a number depending on N and $m\in\mathbb{Z}$. One shows that $p(z) := (1+z)^m z^N$ equals $\sum_{l\in\mathbb{N}}\lambda_l z^l$ by first writing $p(z)$ as a polynomial of $(1+z)$, and then expanding each power of $1+z$. So $\lambda_l = 0$ for $l<N$. This proves (1). See [Gui-V, Subset. 7.8] for details. □

3.12

Thm. 3.21 gives us useful methods of proving locality. In this subsection, we give applications of Thm. 3.21-(2). In the next subsection, we discuss applications of Thm. 3.21-(3).

The following theorem is called **Dong's lemma** or **Dong-Li's lemma**

Theorem 3.22. *Let A,B,C be mutually local fields. Then for each $k\in\mathbb{Z}$, A_kB is local to C.*

Proof. Choose any $u,v\in\mathbb{V}$. Define $g\in\mathscr{O}(\mathrm{Conf}^2(\mathbb{C}^\times))$ by

$$g(z_2,z_3) = \mathrm{Res}_{z_1=z_2}(z_1-z_2)^n\langle A(z_1)B(z_2)C(z_3)u|v\rangle$$

Using Def. 3.18 and Thm. 3.16, one shows that

$$\sum_{n\in\mathbb{N}}\langle (A_kB)(z_2)P_nC(z_3)u|v\rangle \qquad \text{resp.} \qquad \sum_{n\in\mathbb{N}}\langle C(z_3)P_n(A_kB)(z_2)u|v\rangle$$

converges a.l.u. on

$$\{(z_2,z_3)\in\mathbb{C}^2 : 0<|z_3|<|z_2|\} \qquad \text{resp.} \qquad \{(z_2,z_3)\in\mathbb{C}^2 : 0<|z_2|<|z_3|\}$$

to $g(z_2,z_3)$. Moreover, since $\langle A(z_1)B(z_2)C(z_3)u|v\rangle$ is a rational function of z_1,z_2,z_3, one checks easily that g has finite poles at $z_2-z_3=0$. Thus A_kB is local to C by Thm. 3.21-(2). See [Gui-V, Subsec. 8.7] for details. □

Corollary 3.23. *Let A, B be mutually local fields. Define $\partial A \equiv A'$ to be $\partial_z A(z) = \sum_{n\in\mathbb{Z}}(-n-1)A_n z^{-n-2}$, equivalently,*

$$(\partial A)_n = -nA_{n-1} \tag{3.30}$$

Then ∂A is homegeneous of weight

$$\mathrm{wt}(\partial A) = \mathrm{wt}(A) + 1 \tag{3.31}$$

Moreover, ∂A is local to B.

Proof. Eq. (3.31) is clear from (3.30). Using (3.25) and (3.30), one checks that

$$\partial A = (A_{-2}\mathbf{1}) \tag{3.32}$$

So the corollary follows from Thm. 3.22. □

Note that A^θ is not necessarily local to B even if A is local to B.

3.13

Example 3.24. Let $c \geqslant 0$. A field $T(z) = \sum_n L_n z^{-n-2}$ of weight 2 is called a **unitary Virasoro field** (or stress-energy field) of **central charge** c if L_0 coincides with the one in Subsec. 3.1, and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \qquad L_n^\dagger = L_{-n} \tag{3.33}$$

for all $m, n \in \mathbb{Z}$. Note that $L_n^\dagger = L_{-n}$ means $\langle L_n u|v\rangle = \langle u|L_{-n}v\rangle$. Thus $T_n = L_{n-1}$. The **Virasoro relation** (3.33) shows that $T(z)$ is self-local.

Proof of self-locality. Eq. (3.33) is equivalent to

$$[T_m, T_n] = (m-n)T_{m+n-1} + \frac{c}{2}\binom{m}{3}\delta_{m+n-2,0}$$

So $[T_m, T_n] = \sum_{l=0}^3 \binom{m}{l} C^l_{m+n-l}$ if we set

$$C_k^0 = -kT_{k-1} \qquad C_k^1 = 2T_k \qquad C_k^2 = 0 \qquad C_k^3 = \frac{c}{2}\delta_{k+1,0}$$

In other words,

$$C^0(z) = \partial_z T(z) \qquad C^1(z) = 2T(z) \qquad C^2(z) = 0 \qquad C^3(z) = \frac{c}{2}$$

Thus, by Thm. 3.21-(3), $T(z)$ is self-local. □

3.14

Definition 3.25. We say that $(\mathcal{V}, \mathbb{V})$ is a **(quasi-primary unitary) chiral algebra** if $\mathbb{V}$ is as in Subsec. 3.1, and $\mathcal{V}$ is a set of homogeneous fields satisfying the following conditions:

(1) Creation property: There is a distinguished vector $\Omega \in \mathbb{V}(0)$ such that $A(z)\Omega \in \mathbb{V}[[z]]$ (i.e., $A_n\Omega = 0$ if $n \geqslant 1$) for all $A \in \mathcal{V}$.

(2) Locality: Any two fields of $\mathcal{V}$ are mutually local. In particular, every field of $\mathcal{V}$ is self-local.

(3) Cyclicity: Vectors of the form $A^1_{n_1} \cdots A^k_{n_k}\Omega$ (where $k \in \mathbb{N}$, $A^1, \dots, A^k \in \mathcal{V}$, and $n_1, \dots, n_k \in \mathbb{Z}$) span $\mathbb{V}$

(4) Möbius covariance: The operator L_0 can be extended to $\{L_0, L_{\pm 1}\}$ satisfying for all $A \in \mathcal{V}$ and $m \in \{0, 1, -1\}$ that

$$[L_m, A(z)] = z^{m+1}\partial_z A(z) + \mathrm{wt}(A) \cdot (m+1)z^m A(z) \tag{3.34a}$$

in $\mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$. Equivalently, for all $n \in \mathbb{Z}$ we have

$$[L_m, A_n] = -(m+n+1)A_{m+n} + \mathrm{wt}(A) \cdot (m+1)A_{m+n} \tag{3.34b}$$

Moreover, we have

$$L_n\Omega = 0 \qquad \text{for all } n = 0, \pm 1 \tag{3.35}$$

(5) θ-invariance: If $A \in \mathcal{V}$, then $A^\theta \in \mathcal{V}$.

Remark 3.26. The adjective "quasi-primary" means that (3.34) holds for $A \in \mathcal{V}$. However, for $A, B \in \mathcal{V}$ and $k \in \mathbb{Z}$, the fields ∂A and $A_k B$ satisfy (3.34) only for $m = -1, 0$, but not necessarily for $m = 1$. In other words, ∂A and $A_k B$ are not necessarily quasi-primary. Non quasi-primary fields satisfy a more complicated Möbius covariance formula.

Definition 3.27. A chiral algebra $(\mathcal{V}, \mathbb{V})$ is called **conformal** if $L_0, L_{\pm 1}$ can be extended to a sequence $(L_n)_{n \in \mathbb{Z}}$ in $\mathrm{End}(\mathbb{V})$ such that the Virasoro relation (3.33) holds for some central charge c, and that $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies $\mathrm{wt}(T) = 2$ and belongs to $\mathcal{V}$.

If $(\mathcal{V}, \mathbb{V})$ is a conformal chiral algebra, we say that $A \in \mathcal{V}$ is a **primary field** if A satisfies (3.34) for all $m \in \mathbb{Z}$. □

Remark 3.28. Note that when $m = 0, \pm 1$, the Virasoro relation specializes to $[L_m, L_n] = (m-n)L_{m+n}$ (for all $n \in \mathbb{Z}$). Thus $T(z)$ automatically satisfies (3.34). However, if $c \neq 0$ and $m \neq 0, \pm 1$, then (3.34) does not hold for $T(z)$. Thus $T(z)$ is not primary.

Remark 3.29. When $(\mathcal{V}, \mathbb{V})$ is a conformal chiral algebra, then (3.35) is redundant, since the creation proerty for $T(z)$ implies that

$$L_n\Omega = 0 \qquad \text{for all } n = -1, 0, 1, 2, 3, \dots$$

3.15

Definition 3.30. A **unitary Lie algebra** is defined to be a complex Lie algebra $\mathfrak{g}$ together with an inner product (called **invariant inner product**) on $\mathfrak{g}$ and an **involution** $\dagger$ (i.e., an antilinear map $\dagger : \mathfrak{g} \to \mathfrak{g}$ satisfying $X^{\dagger\dagger} = X$ for all $X \in \mathfrak{g}$) satisfying the following properties for all $X, Y, Z \in \mathfrak{g}$:

(1) $\langle [X, Y] | Z \rangle = \langle Y | [X^\dagger, Z] \rangle$, i.e., the representation $X \mapsto [X, -]$ is unitary.

(2) $[X, Y]^\dagger = [Y^\dagger, X^\dagger]$.

(3) $\dagger : \mathfrak{g} \to \mathfrak{g}$ is antiunitary.

If W is an inner product space, we say that $\pi : \mathfrak{g} \to \mathrm{End}(W)$ is a **unitary representation** if $\pi([X, Y]) = [\pi(X), \pi(Y)]$ and $\pi(X)^\dagger = \pi(X^\dagger)$ (i.e. $\langle \pi(X)u|v\rangle = \pi(u|\pi(X^)\dagger v))$ for all $X, Y \in \mathfrak{g}$.

Remark 3.31. One can show that a finitely dimensional complex Lie algebra $\mathfrak{g}$ is unitary iff it is isomorphic (but not necessarily unitarily isomorphic) to $\mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ where $\mathfrak{z}$ is abelian (i.e. $\simeq \mathbb{C}^k$ for some $k \in \mathbb{N}$) and $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ are complex simple Lie algebras. There are no canonical choices of the invariant inner products on $\mathfrak{z}$, since any two inner products are unitarily equivalent. However, $\mathfrak{g}_i$ has a canonical choice of invariant inner product: the one under which the longest root has length $\sqrt{2}$. See [Was-10, Ch. II] for details.

Example 3.32. Let $\mathfrak{g}$ be a finite-dimensional unitary Lie algebra. Let $l \in \mathbb{R}_{>0}$. Suppose that $\mathcal{V}$ is a set of fields $X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}$ (where $X \in \mathfrak{g}$) such that

$$[X_m, Y_n] = [X, Y]_{m+n} + l \cdot m \langle X | Y^\dagger \rangle \delta_{m+n,0} \qquad (X_n)^\dagger = (X^\dagger)_{-n} \tag{3.36}$$

for all $X, Y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Using Thm. 3.21-(3), one checks that any two fields of $\mathcal{V}$ are mutually local. We call $X(z)$ a **current field**.

Now assume that the creation property and the cyclicity in Def. 3.25 holds for $(\mathcal{V}, \mathbb{V})$. Assume that $\mathfrak{g}$ is abelian resp. simple. Let $h^\vee$ be 0 resp. the dual Coxeter number of $\mathfrak{g}$. Define $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ via the **Sugawara construction**

$$T(z) = \frac{1}{2(l + h^\vee)} \sum_i \big((E_i^\dagger)_{-1} E_i \big)(z) \tag{3.37}$$

where (E_i) is an orthonormal basis of $\mathfrak{g}$. Then $T^\theta(z) = T(z)$ and $X^\theta(z) = -X^\dagger(z)$ (where $X \in \mathfrak{g}$), and $\mathcal{V} \cup \{T(z)\}$ is a conformal chiral algebra, and all $X(z)$ are primary with

$$\mathrm{wt}(X) = 1$$

We call $\mathcal{V}$ the **current algebra** of $\mathfrak{g}$ with **level** l. See [Gui-V, Sec. 6] for details.

If $\mathfrak{g}$ is abelian, all $l > 0$ are possible. In fact, in this case, $(\mathcal{V}, \mathbb{V})$ can be constructed from Bosonic Fock spaces. See [Gui-V, Subsec. 6.13]. If $\mathfrak{g}$ is simple and the invariant inner product is the canonical one (i.e., the one under which the longest root has length $\sqrt{2}$), one can show that all possible $l > 0$ form $\mathbb{Z}_+$. See [Was-10, Ch. III] for details. $\square$

4 Energy-bounded fields and their smeared fields

We assume the setting of Subsec. 3.1. Recall that $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ is graded by L_0. Therefore, L_0 is a symmetric operator on $\mathcal{H}$, and hence closable. Recall that $\oplus_n v_n \in \mathbb{V}^{\mathrm{ac}}$ belongs to $\mathcal{H}_{\mathbb{V}}$ iff $\sum_n \|v_n\|^2 < +\infty$. For each $n \in \mathbb{Z}$, define

$$\mathfrak{e}_n : \mathbb{S}^1 \to \mathbb{C} \qquad \mathfrak{e}_n(z) = z^n \tag{4.1}$$

For each $f \in C^\infty(\mathbb{S}^1)$ and $t \in \mathbb{R}$, define the t-th order Sobolev norm $|f|_t$ to be

$$|f|_t = \sum_{n\in\mathbb{Z}} (1 + |n|)^t |\widehat{f}(n)| \tag{4.2}$$

where $f = \sum_{n\in\mathbb{N}} \widehat{f}(n) \cdot \mathfrak{e}_n$ is the Fourier series. Note that $|f|_t < +\infty$ because $n \in \mathbb{Z} \mapsto \widehat{f}(n)$ is rapidly decreasing.

4.1

The following proposition implies, for example, that if $q \in \mathbb{C}^\times$ then the action of q^{L_0} on $\mathbb{V}^{\mathrm{ac}}$ described in Subsec. 3.1 is compatible with the Borel functional calculus $q^{\overline{L_0}}$ on $\mathcal{H}_{\mathbb{V}}$.

Proposition 4.1. *The closure $\overline{L_0}$ is a (closed self-adjoint) positive operator. Moreover, for each Borel function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{C}$ we have*

$$\mathscr{D}(f(\overline{L_0})) = \Big\{ \oplus_n v_n \in \mathcal{H}_{\mathbb{V}} : \sum_n |f(n)|^2 \|v_n\|^2 < +\infty \Big\} \tag{4.3a}$$

Moreover, $\mathbb{V}$ is a core for $f(\overline{L_0})$. For each $\oplus_n v_n \in \mathcal{H}_{\mathbb{V}}$ we have

$$f(\overline{L_0})(\oplus_n v_n) = \oplus_n f(n) v_n \tag{4.3b}$$

In particular, if $r \in \mathbb{R}$, we understand $(1 + \overline{L_0})^r$ as $f(\overline{L_0})$ where $f(x) = (1 + x)^r$.

Proof. By choosing an orthonormal basis for each $\mathbb{V}(n)$, we can view $\mathcal{H}_{\mathbb{V}}$ as a direct sum $\bigoplus_{j\in J} L^2(\mathbb{N}, \mu_j)$ where J is a countable set and μ_j is a Dirac measure on $\mathbb{N}$. Then clearly

$$\mathbb{V} = \bigcup_{n\in\mathbb{N}} M_{\chi_{[0,n]}} \mathcal{H}_{\mathbb{V}}$$

This shows that $\mathbb{V}$ is a core for the multiplication operator M_f (since $(M_{\chi_{[0,n]}})_{n\in\mathbb{N}}$ is a sequence of bounding projections for M_f, cf. [Gui-S, Sec. 8]).

Now let $x : \mathbb{N} \to \mathbb{C}, n \mapsto n$. Then $M_x|_{\mathbb{V}} = L_0$. Thus $\overline{L_0}$ is the positive operator M_x. Hence $f(\overline{L_0}) = f(M_x) = M_f$. Thus (4.3) follows from the fact that $\mathscr{D}(M_f)$ equals the RHS of (4.3a), and that the action of M_f on $\mathscr{D}(M_f)$ is described by the RHS of (4.3b). $\square$

Corollary 4.2. *Restrict each P_n to a projection on $\mathcal{H}_{\mathbb{V}}$. Then the von Neumann algebras $\{\overline{L_0}\}''$ and $\{P_n : n \in \mathbb{N}\}''$ are equal.*

Proof. By Prop. 4.1, we have $P_n = \chi_{\{n\}}(\overline{L_0})$, and hence $P_n \in \{\overline{L_0}\}''$. Thus $\{\overline{L_0}\}'' \supset \{P_n : n \in \mathbb{N}\}''$. If U is a unitary operator on $\mathcal{H}_{\mathbb{V}}$ commuting with each P_n, then U preserves each $\mathbb{V}(n)$. This implies $L_0 U = L_0 U$, and hence $U \in \{\overline{L_0}\}'$. Thus $\{\overline{L_0}\}'' \subset \{P_n : n \in \mathbb{N}\}''$. $\square$

4.2

Definition 4.3. Let $r \in \mathbb{R}$. The ***r*-th order Sobolev norm** is defined to be

$$\|\cdot\|_r : \mathbb{V}^{\mathrm{ac}} \to [0, +\infty] \qquad \|\xi\|_r^2 = \sum_{n\in\mathbb{N}} (1+n)^{2r} \|P_n\xi\|^2$$

Moreover, if $r \geqslant 0$, define the ***r*-th order Sobolev space** to be

$$\mathcal{H}_{\mathbb{V}}^r := \mathscr{D}((1+\overline{L_0})^r) \xlongequal{(4.3a)} \{\xi \in \mathbb{V}^{\mathrm{ac}} : \|\xi\|_r < +\infty\}$$

Then, on $\mathcal{H}_{\mathbb{V}}^r$, the norm $\|\cdot\|_r$ is induced by the ***r*-th order Sobolev inner product**

$$\langle \xi | \eta \rangle_r = \sum_{n\in\mathbb{N}} (1+n)^{2r} \langle P_n \xi | P_n \eta \rangle$$

Clearly for $r \leqslant r'$ we have for all $\xi \in \mathbb{V}^{\mathrm{ac}}$ that

$$\|\xi\|_r \leqslant \|\xi\|_{r'} \qquad \text{and hence} \mathcal{H}_{\mathbb{V}}^r \supset \mathcal{H}_{\mathbb{V}}^{r'}$$

We set

$$\mathcal{H}_{\mathbb{V}}^\infty = \bigcap_{r\geqslant 0} \mathcal{H}_{\mathbb{V}}^r$$

Vectors in $\boldsymbol{\mathcal{H}_{\mathbb{V}}^\infty}$ are called **smooth vectors**.

Remark 4.4. Note that for each $r \in \mathbb{R}$, the projection $P_n : (\mathcal{H}_{\mathbb{V}}^r, \langle\cdot,\cdot\rangle_r) \to (\mathcal{H}_{\mathbb{V}}^r, \langle\cdot,\cdot\rangle_r)$ onto $\mathbb{V}(n)$ is a projection in the sense that $P_n^2 = P_n$ and $\langle P_n\xi|\eta\rangle_r = \langle \xi | P_n \eta\rangle_r$ for all $\xi, \eta \in \mathcal{H}_{\mathbb{V}}^r$. Thus P_n has operator norm $\leqslant 1$ under $\langle\cdot,\cdot\rangle_r$. The same can be said about $P_{\leqslant n}$.

Remark 4.5. If T is a positive operator on a Hilbert space $\mathcal{H}$ satisfying $T \geqslant a$ for some $a > 0$, then $\mathscr{D}(T)$ is complete under the inner product $\langle \xi|\eta\rangle_T := \langle T\xi | T\eta\rangle$. It follows that for each $r \geqslant 0$, $\mathcal{H}_{\mathbb{V}}^r$ is complete under $\langle\cdot|\cdot\rangle_r$.

Proof. By (e.g.) spectral theory, $T^2 - a^2$ is positive. Thus $\|T\xi\|^2 \geqslant a^2\|\xi\|^2$. Thus, if (ξ_n) is a Cauchy sequence in $\mathscr{D}(T)$ under $\langle\cdot|\cdot\rangle_T$, then $T\xi_n$ and ξ_n both converge in $\mathcal{H}$. Let $\xi = \lim_n \xi_n$. Since T is closed, we see that $(\xi_n, T\xi_n)$ converges in $\mathcal{H} \times \mathcal{H}$ to some $(\xi, T\xi)$ in the graph of T. So $\lim_n \|\xi - \xi_n\|_T = 0$. $\square$

Definition 4.6. Let $r \geqslant 0$. We say that a field $A(z)$ satisfies ***r*-th order energy bounds** if there exist $M, t \geqslant 0$ such that for any $n \in \mathbb{Z}$ and $v \in \mathbb{V}$ we have

$$\|A_n v\| \leqslant M(1+|n|)^t \|v\|_r \tag{4.4}$$

In other words, the following linear map

$$A_n : (\mathbb{V}, \|\cdot\|_r) \to (\mathbb{V}, \|\cdot\|)$$

is bounded with operator norm $\leqslant M(1+|n|)^t$.

First order energy bounds are called **linear energy bounds**. A field satisfying r-th energy bounds for some $r \geqslant 0$ is called **(polynomial) energy-bounded**. $\square$

4.3

Proposition 4.7. *Let $A(z)$ be a field satisfying (4.4). Then for any $p \in \mathbb{R}$, there exists $M_p \geqslant 0$ such that for any $n \in \mathbb{Z}$ and $v \in \mathbb{V}$ we have*

$$\|A_n v\|_p \leqslant M_p (1+|n|)^{|p|+t} \|v\|_{p+r}$$

Proof. We want to prove

$$\|A_n v\|_p^2 \leqslant M_p^2 (1+|n|)^{2(|p|+t)} \|v\|_{p+r}^2 \tag{a}$$

For different $m \in \mathbb{N}$, $P_m v$ are mutually orthonormal under the $(p+r)$-th order Sobolev inner product. By Rem. 3.2, for different m, $A_n P_m v = P_{m+\mathrm{wt}(A)-n-1} A_n v$ are mutually orthonormal under the p-th order Sobolev inner product. Therefore, by replacing v with $P_m v$, it suffices to prove (a) under the assumption that v is homogeneous. We also assume that $A_n v \neq 0$. Then by Rem. 3.2, we have $\mathrm{wt}(A) + \mathrm{wt}(v) - 1 - n \geqslant 0$.

By (4.4), we have

$$\|A_n v\|^2 \leqslant M^2 (1+|n|)^{2t} (1+\mathrm{wt}(v))^{2r} \|v\|^2$$

Thus, using Rem. 3.2, we get

$$\begin{aligned}
&\|A_n v\|_p^2 = (\mathrm{wt}(A) + \mathrm{wt}(v) - n)^{2p} \|A_n v\|^2 \\
\leqslant&(\mathrm{wt}(A) + \mathrm{wt}(v) - n)^{2p} M^2 (1+|n|)^{2t} (1+\mathrm{wt}(v))^{2r} \|v\|^2 \\
=&\Big(\frac{\mathrm{wt}(A) - n + \mathrm{wt}(v)}{1+\mathrm{wt}(v)}\Big)^{2p} M^2 (1+|n|)^{2t} (1+\mathrm{wt}(v))^{2(p+r)} \|v\|^2 \\
=&\Big(\frac{\mathrm{wt}(A) - n + \mathrm{wt}(v)}{1+\mathrm{wt}(v)}\Big)^{2p} M^2 (1+|n|)^{2t} \|v\|_{p+r}^2
\end{aligned}$$

If $p \geqslant 0$, then we can choose $M_p = (1+\mathrm{wt}(A))^p M$, since

$$\Big(\frac{\mathrm{wt}(A) - n + \mathrm{wt}(v)}{1+\mathrm{wt}(v)}\Big)^{2p} \leqslant \Big(\frac{1+\mathrm{wt}(A) + |n| + \mathrm{wt}(v)}{1+\mathrm{wt}(v)}\Big)^{2p}$$

$$\leqslant (1+\mathrm{wt}(A)+|n|)^{2p} \leqslant (1+\mathrm{wt}(A))^{2p}(1+|n|)^{2p}$$

Now assume $p<0$. If $1 \leqslant \mathrm{wt}(A)-n$, then

$$\Big(\frac{\mathrm{wt}(A)-n+\mathrm{wt}(v)}{1+\mathrm{wt}(v)}\Big)^{2p} = \Big(\frac{1+\mathrm{wt}(v)}{\mathrm{wt}(A)-n+\mathrm{wt}(v)}\Big)^{2|p|} \leqslant 1 \leqslant (1+|2n|)^{2|p|}$$

If $1 \geqslant \mathrm{wt}(A)-n$, then since $\mathrm{wt}(A)-n+\mathrm{wt}(v)\geqslant 1$ (cf. the first paragraph),

$$\begin{aligned}&\Big(\frac{1+\mathrm{wt}(v)}{\mathrm{wt}(A)-n+\mathrm{wt}(v)}\Big)^{2|p|} = \Big(1+\frac{1+n-\mathrm{wt}(A)}{\mathrm{wt}(A)-n+\mathrm{wt}(v)}\Big)^{2|p|}\\ \leqslant&(2+n-\mathrm{wt}(A))^{2|p|} \leqslant (2+2n+2\mathrm{wt}(A))^{2|p|} \leqslant 2^{2|p|}(1+\mathrm{wt}(A))^{2|p|}(1+|n|)^{2|p|}\end{aligned}$$

Therefore, if $p<0$, we can choose $M_p = 2^{|p|}(1+\mathrm{wt}(A))^{2|p|}M$. □

Corollary 4.8. *Assume that $A(z)$ satisfies r-th order energy bounds where $r\geqslant 0$. Then the quasi-primary contragredient $A^\theta(z)$ (cf. Def. 3.4) also satisfies r-th order energy bounds.*

Proof. Assume that $A(z)$ satisfies (4.4). Then for each $u,v\in\mathbb{V}$, we use (3.9) to compute that

$$\begin{aligned}&\big|\langle A_n^\theta u|v\rangle\big| = \big|\langle u|A_{-n-2+\mathrm{wt}(A)}v\rangle\big| = \big|\langle (1+L_0)^r u|(1+L_0)^{-r}A_{-n-2+\mathrm{wt}(A)}v\rangle\big|\\ \leqslant&\|u\|_r\cdot\|A_{-n-2+\mathrm{wt}(A)}v\|_{-r}\end{aligned}$$

By Prop. 4.7, we have

$$\|A_{-n-2+\mathrm{wt}(A)}v\|_{-r} \leqslant M_{-r}(1+|n+2+\mathrm{wt}(A)|)^{r+t}\|v\| \leqslant C(1+|n|)^{r+t}\|v\|$$

where $C=M_{-r}(2+\mathrm{wt}(A))^{r+t}$. Thus, for any $u\in\mathbb{V}$, we have

$$\big|\langle A_n^\theta u|v\rangle\big| \leqslant C(1+|n|)^{r+t}\|u\|_r\cdot\|v\|$$

for all $v\in\mathbb{V}$, and hence $\|A_n^\theta u\|\leqslant C(1+|n|)^{r+t}\|u\|_r$. □

4.4

To prepare for the study of smeared fields we need:

Lemma 4.9. *Let $F:\mathbb{S}^1\to\mathrm{Hom}(\mathbb{V},\mathbb{V}^{\mathrm{ac}})$ satisfying the following properties:*

(a) For each $u,v\in\mathbb{V}$, the function $z\in\mathbb{S}^1\mapsto\langle F(z)u|v\rangle\in\mathbb{C}$ is continuous.

(b) For each $z\in\mathbb{S}^1$, there is a (necessarily unique) $F(z)^\dagger\in\mathrm{Hom}(\mathbb{V},\mathbb{V}^{\mathrm{ac}})$ such that

$$\langle F(z)u|v\rangle = \langle u|F(z)^\dagger v\rangle$$

for all $u,v\in\mathbb{V}$.

Then as elements of $\mathrm{Hom}(\mathbb{V}, \mathbb{V}^{\mathrm{ac}})$ *we have*

$$\left(\oint_{\mathbb{S}^1} F(z)\frac{dz}{2\mathbf{i}\pi}\right)^\dagger = \oint_{\mathbb{S}^1} z^{-2}\cdot F(z)^\dagger\frac{dz}{2\mathbf{i}\pi} \tag{4.5}$$

Namely, defining $S, T \in \mathrm{Hom}(\mathbb{V}, \mathbb{V}^{\mathrm{ac}})$ *by*

$$\langle Su|v\rangle = \oint_{\mathbb{S}^1}\langle F(z)u|v\rangle\frac{dz}{2\mathbf{i}\pi} \qquad \langle Tu|v\rangle = \oint_{\mathbb{S}^1}\langle z^{-2}F(z)^\dagger u|v\rangle\frac{dz}{2\mathbf{i}\pi}$$

(for all $u, v \in \mathbb{V}$*), then* $\langle Su|v\rangle = \langle u|Tv\rangle$.

For example, by Rem. 3.5, if $A(z)$ is a homogeneous field, then A satisfies the above properties, because for $z \in \mathbb{S}^1$ we have

$$A(z)^\dagger = (-z^2)^{\mathrm{wt}(A)}A^\theta(z) \tag{4.6}$$

Proof. We compute that

$$\langle Su|v\rangle = \int_{-\pi}^{\pi}\langle F(e^{\mathbf{i}\theta})u|v\rangle\cdot\frac{e^{\mathbf{i}\theta}d\theta}{2\pi} = \overline{\int_{-\pi}^{\pi}\langle v|F(e^{\mathbf{i}\theta})u\rangle\cdot\frac{e^{-\mathbf{i}\theta}d\theta}{2\pi}}$$
$$=\overline{\int_{-\pi}^{\pi}\langle F(e^{\mathbf{i}\theta})^\dagger v|u\rangle\cdot\frac{e^{-\mathbf{i}\theta}d\theta}{2\pi}} = \overline{\int_{-\pi}^{\pi}\langle e^{-2\mathbf{i}\theta}F(e^{\mathbf{i}\theta})^\dagger v|u\rangle\cdot\frac{e^{\mathbf{i}\theta}d\theta}{2\pi}} = \overline{\langle Tv|u\rangle}$$

□

Remark 4.10. There is a non-rigorous but heuristic way to prove (4.5) (and to help memorize (4.5)). Note that $\overline{z} = z^{-1}$ on $\mathbb{S}^1$. Therefore

$$\left(\oint_{\mathbb{S}^1} F(z)\frac{dz}{2\mathbf{i}\pi}\right)^\dagger = \oint_{\mathbb{S}^1} F(z)^\dagger\frac{dz^{-1}}{-2\mathbf{i}\pi} = \oint_{\mathbb{S}^1} z^{-2}F(z)^\dagger\frac{dz}{2\mathbf{i}\pi}$$

4.5

We now define smeared fields. Let $A(z)$ be a field satisfying r-th order energy bounds.

Definition 4.11. For each $f \in C^\infty(\mathbb{S}^1)$, define the **smeared field**

$$A(f) : \mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$$
$$\langle A(f)u|v\rangle = \oint_{\mathbb{S}^1}\langle A(z)u|v\rangle f(z)\frac{dz}{2\mathbf{i}\pi} \equiv \int_{-\pi}^{\pi}\langle A(e^{\mathbf{i}\theta})u|v\rangle f(e^{\mathbf{i}\theta})\cdot\frac{e^{\mathbf{i}\theta}d\theta}{2\pi}$$

for all $u, v \in \mathbb{V}$. (The domain of $A(f)$ will be slightly extended, see Conv. 4.13.)

Theorem 4.12. *Let $f \in C^\infty(\mathbb{S}^1)$. The following are true.*

(a) We have $A(f)\mathbb{V} \subset \mathcal{H}_{\mathbb{V}}$, and hence $A(f)$ is an unbounded operator on $\mathcal{H}_{\mathbb{V}}$ with dense domain $\mathbb{V}$. Moreover, writing $f(e^{\mathbf{i}\theta}) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{\mathbf{i}n\theta}$, then for each $u \in \mathbb{V}$, the RHS below converges in $\mathcal{H}_{\mathbb{V}}$ to the LHS:

$$A(f)u = \sum_{n\in\mathbb{Z}} \widehat{f}(n)A_n u \tag{4.7}$$

(b) We have

$$(-1)^{\mathrm{wt}(A)} A^\theta\big(\overline{\mathfrak{e}_{2-2\mathrm{wt}(A)} f}\big) \subset A(f)^* \tag{4.8}$$

Consequently, $A(f)$ is closable (because $\mathscr{D}(A(f)^*)$ is dense).

*(c) $\mathcal{H}_{\mathbb{V}}^\infty$ is an **invariant core** for $\overline{A(f)}$. (Namely, $\mathcal{H}_{\mathbb{V}}^\infty \subset \mathscr{D}(\overline{A(f)})$ is a core for $\overline{A(f)}$, and $\overline{A(f)}\mathcal{H}_{\mathbb{V}}^\infty \subset \mathcal{H}_{\mathbb{V}}^\infty$.) Moreover, assume that A satisfies (4.4). Let $p \geqslant 0$, and let M_p be as in Prop. 4.7. Then for each $\xi \in \mathcal{H}_{\mathbb{V}}^\infty$ we have*

$$\|\overline{A(f)}\xi\|_p \leqslant M_p|f|_{t+|p|} \cdot \|\xi\|_{p+r} \tag{4.9}$$

Note that (4.9) means that the linear map

$$\overline{A(f)}\big|_{\mathcal{H}_{\mathbb{V}}^\infty} : (\mathcal{H}_{\mathbb{V}}^\infty, \|\cdot\|_{p+r}) \to (\mathcal{H}_{\mathbb{V}}^\infty, \|\cdot\|_p) \tag{4.10}$$

is bounded with operator norm $\leqslant M_p \cdot |f|_{t+|p|}$.

Proof. (a): Let $u \in \mathbb{V}$ be homogemeous. Then for each $v \in \mathbb{V}$, since $\langle A(z)u|v\rangle = \sum_{n\in\mathbb{Z}}\langle A_n u|v\rangle z^{-n-1}$ (where the RHS is a finite sum), we have

$$\langle A(f)u|v\rangle = \int_{-\pi}^{\pi} \langle A(e^{\mathbf{i}\theta})u|v\rangle f(e^{\mathbf{i}\theta}) \cdot \frac{e^{\mathbf{i}\theta}d\theta}{2\pi} = \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}}\langle A_n u|v\rangle e^{-\mathbf{i}n\theta} \cdot f(e^{\mathbf{i}\theta}) \cdot \frac{d\theta}{2\pi}$$

$$= \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} \langle A_n u|v\rangle e^{-\mathbf{i}n\theta} \cdot f(e^{\mathbf{i}\theta}) \cdot \frac{d\theta}{2\pi} = \sum_{n\in\mathbb{Z}}\langle A_n u|v\rangle \widehat{f}(n)$$

where all sums $\sum_{n\in\mathbb{Z}}$ are indeed finite. So (4.7) holds when both sides are multiplied by P_m (for all $m \in \mathbb{N}$). In other words, (4.7) holds in $\mathbb{V}^{\mathrm{ac}}$.

Now assume that A satisfies (4.4). Then $A_n u$ are mutually orthogonal for different n (due to Rem. 3.2), and hence

$$\sum_{n\in\mathbb{Z}} \|\widehat{f}(n)A_n u\|^2 \leqslant \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \cdot M^2(1+|n|)^{2t} \cdot \|u\|_r^2$$

is finite because $n \mapsto \widehat{f}(n)$ is rapidly decreasing. Thus, the RHS of (4.7) converges in $\mathcal{H}_{\mathbb{V}}$. So (4.7) holds in $\mathcal{H}_{\mathbb{V}}$.

(b): Using Lem. 4.9 and (4.6), we compute

$$
\begin{aligned}
A(f)^\dagger &= \Big(\oint_{\mathbb{S}^1} f(z)A(z)\frac{dz}{2\mathbf{i}\pi}\Big)^\dagger = \oint_{\mathbb{S}^1} \overline{z^2 f(z)}A(z)^\dagger\frac{dz}{2\mathbf{i}\pi}\\
&= \oint_{\mathbb{S}^1} \overline{z^2 f(z)}\cdot(-z^2)^{\mathrm{wt}(A)}A^\theta(z)\frac{dz}{2\mathbf{i}\pi} = \oint_{\mathbb{S}^1}(-1)^{\mathrm{wt}(A)}\overline{z^{2-2\mathrm{wt}(A)}f(z)}A^\theta(z)\frac{dz}{2\mathbf{i}\pi}
\end{aligned}
$$

which equals $(-1)^{\mathrm{wt}(A)}A^\theta\big(\overline{\mathfrak{e}_{2-2\mathrm{wt}(A)}f}\big)$. This proves (4.8).

(c): Let $u\in\mathbb{V}$. For each $m\in\mathbb{N}$, by (4.7) and Prop. 4.7, we have

$$
\begin{aligned}
&\|P_{\leqslant m}A(f)u\|_p \leqslant \sum_{n\in\mathbb{N}}|\widehat{f}(n)|\cdot\|P_{\leqslant m}A_nu\|_p \leqslant \sum_{n\in\mathbb{N}}|\widehat{f}(n)|\cdot\|A_nu\|_p\\
\leqslant&\sum_{n\in\mathbb{N}}|\widehat{f}(n)|\cdot M_p(1+|n|)^{|p|+t}\|u\|_{p+t} = |f|_{|p|+t}M_p\|u\|_{p+t}
\end{aligned}
$$

noting that the second sum is finite, and also noting Rem. 4.4. Since m is arbitrary, we have thus proved (4.9) for $\xi=u\in\mathbb{V}$. In particular, $A(f)u\in\mathcal{H}_{\mathbb{V}}^p$ (for all $p\geqslant 0$).

Now we consider an arbitrary $\xi\in\mathcal{H}_{\mathbb{V}}^\infty$. Applying (4.9) to $(P_{\leqslant n}-P_{\leqslant m})\xi$ and $p=0$, we get

$$
\|\overline{A(f)}P_{\leqslant n}\xi-\overline{A(f)}P_{\leqslant m}\xi\|^2\leqslant M_0|f|_t\cdot\|(P_{\leqslant n}-P_{\leqslant m})\xi\|_r
$$

where the RHS converges to 0 as $m,n\to\infty$. Thus $\lim_n\overline{A(f)}P_{\leqslant n}\xi$ converges in $\mathcal{H}_{\mathbb{V}}$. Since $\lim_n P_{\leqslant n}\xi$ converges, by the closedness of $\overline{A(f)}$, we see that $\xi\in\mathscr{D}(\overline{A(f)})$ and

$$
\lim_{n\to\infty}\overline{A(f)}P_{\leqslant n}\xi=\overline{A(f)}\xi \tag{$\triangle$}
$$

In particular, we have proved that $\mathcal{H}_{\mathbb{V}}^\infty$ is contained in $\mathscr{D}(\overline{A(f)})$, and hence is a core for $\overline{A(f)}$ (since it contains $\mathbb{V}$).

Moreover, for each $m\in\mathbb{N}$, since $(1+\overline{L}_0)^pP_{\leqslant m}$ is bounded, we have

$$
\begin{aligned}
&\|P_{\leqslant m}\overline{A(f)}\xi\|_p=\|(1+\overline{L}_0)^pP_{\leqslant m}\overline{A(f)}\xi\|\xlongequal{(\triangle)}\lim_{n\to\infty}\|(1+\overline{L}_0)^pP_{\leqslant m}\overline{A(f)}P_{\leqslant n}\xi\|\\
=&\lim_{n\to\infty}\|P_{\leqslant m}\overline{A(f)}P_{\leqslant n}\xi\|_p\leqslant\limsup_{n\to\infty}\|\overline{A(f)}P_{\leqslant n}\xi\|_p\\
\leqslant&\limsup_{n\to\infty}M_p|f|_{t+|p|}\cdot\|P_{\leqslant n}\xi\|_{p+r}=M_p|f|_{t+|p|}\cdot\|\xi\|_{p+r}
\end{aligned}
$$

Since m is arbitrary, we conclude that $\overline{A(f)}\xi\in\mathcal{H}_{\mathbb{V}}^p$, and that (4.9) holds. Since p can be arbitrary, we conclude that $\overline{A(f)}\xi\in\mathcal{H}_{\mathbb{V}}^\infty$ (for all $\xi\in\mathcal{H}_{\mathbb{V}}^\infty$). $\square$

4.6

Convention 4.13. From now on, if A is an energy-bounded field and $f \in C^\infty(\mathbb{S}^1)$, we let

$$A(f) : \mathcal{H}_{\mathbb{V}}^\infty \to \mathcal{H}_{\mathbb{V}}^\infty$$

be the unique closable operator on $\mathcal{H}_{\mathbb{V}}$ with dense domain $\mathscr{D}(A(f)) = \mathcal{H}_{\mathbb{V}}^\infty$ such that $\mathbb{V}$ is a core for $A(f)$, and that $A(f)|_{\mathbb{V}}$ is the original map defined in Def. 4.11. Such $A(f)$ exists by Thm. 4.12.

By (4.8), we have $\mathcal{H}_{\mathbb{V}}^\infty \subset \mathscr{D}(A(f)^*)$ and $A(f)^*\mathcal{H}_{\mathbb{V}}^\infty \subset \mathcal{H}_{\mathbb{V}}^\infty$. Thus we define

$$\boldsymbol{A(f)^\dagger} := A(f)^*|_{\mathcal{H}_{\mathbb{V}}^\infty} : \mathcal{H}_{\mathbb{V}}^\infty \to \mathcal{H}_{\mathbb{V}}^\infty$$

viewed as an unbounded operator on $\mathcal{H}_{\mathbb{V}}$. Thus,

$$A(f)^\dagger = (-1)^{\mathrm{wt}(A)} A^\theta(\overline{e_{2-2\mathrm{wt}(A)} f}) \tag{4.11}$$

In general, if T is a closable operator on $\mathcal{H}_{\mathbb{V}}$ with invariant domain $\mathcal{H}_{\mathbb{V}}^\infty$, and if $T^*\mathcal{H}_{\mathbb{V}}^\infty \subset \mathcal{H}_{\mathbb{V}}^\infty$, we let

$$\boldsymbol{T^\dagger} := T^*|_{\mathcal{H}_{\mathbb{V}}^\infty} : \mathcal{H}_{\mathbb{V}}^\infty \to \mathcal{H}_{\mathbb{V}}^\infty$$

called the **formal adjoint** of T. □

Proposition 4.14. *Let $A^1, \dots, A^N$ be energy-bounded fields. Then there exist $t_1, \dots, t_N, r \geqslant 0$ such that for any $p \geqslant 0$, there exists $M_p \geqslant 0$ such that*

$$\|A^1(f_1)\cdots A^N(f_N)\xi\|_p \leqslant M_p |f_1|_{t_1+|p|} \cdots |f_N|_{t_N+|p|} \cdot \|\xi\|_{p+r} \tag{4.12}$$

for all $\xi \in \mathcal{H}_{\mathbb{V}}^\infty$ and $f_1, \dots, f_N \in C^\infty(\mathbb{S}^1)$. Moreover, $A^\bullet(f_\bullet) := A^1(f_1)\cdots A^N(f_N)$ is closable with core $\mathbb{V}$.

Note that by Conv. 4.13, $A^\bullet(f_\bullet)$ is an unbounded operator on $\mathcal{H}_{\mathbb{V}}^\infty$ with invariant domain $\mathcal{H}_{\mathbb{V}}^\infty$.

Proof. Eq. (4.12) follows easily from (4.20). Note that

$$A^\bullet(f_\bullet)^* \supset A^N(f_N)^* \cdots A^1(f_1)^* \supset A^N(f_N)^\dagger \cdots A^1(f_1)^\dagger \tag{4.13}$$

where the RHS has dense invariant domain $\mathcal{H}_{\mathbb{V}}^\infty$. So $A^\bullet(f_\bullet)$ is closable. If $\xi \in \mathcal{H}_{\mathbb{V}}^\infty$, then $\lim_n P_{\leqslant n}\xi = \xi$, and (similar to the proof of Thm. 4.12-(c)) Formula (4.12) implies that $(A^\bullet(f_\bullet)P_{\leqslant n}\xi)_{n\in\mathbb{N}}$ is a Cauchy sequence, which must converge to $\overline{A^\bullet(f_\bullet)}\xi$. So $\mathbb{V}$ is a core for $A^\bullet(f_\bullet)$. □

4.7

So far, we haven't given any nontrivial example of energy-bounded field. Here we introduce one.

Definition 4.15. For each field A we write

$$A(z)=\sum_{n\in\mathbb{Z}}A_{(n)}z^{-n-\mathrm{wt}(A)} \tag{4.14}$$

Thus, by Rem. 3.2, for each $d\in\mathbb{N}$ we have

$$[L_0,A_{(n)}]=-nA_{(n)} \tag{4.15}$$

Namely, $A_{(n)}$ increases the weight by $-n$.

For example, for the Virasoro field $T(z)$ in Exp. 3.24 we have $T_{(n)}=L_n$. For the current field $X(z)$ in Exp. 3.32 we have $X_{(n)}=X_n$.

Theorem 4.16. *Let $A(z)$ be a field. Assume that there exist $C,t,r\geqslant 0$ such that for each $n\in\mathbb{Z}$ and $v\in\mathbb{V}$ we have*

$$\big|\langle[A_{(n)},A_{(n)}^\dagger]v|v\rangle\big|\leqslant C^2(1+|n|)^{2t}\|v\|_r^2 \tag{4.16}$$

Then there exists $M\geqslant 0$ such that for each $v\in\mathbb{V}$ and each nonzero $n\in\mathbb{Z}$, we have

$$\|A_{(n)}v\|\leqslant M(1+|n|)^t\|v\|_{r+\frac{1}{2}} \tag{4.17}$$

Proof. We first assume that $n>0$ and prove the estimate for $A_{(-n)}$. For each $d\in\mathbb{N}$, let K_d be the operator norm of the restriction $A_{(-n)}:\mathbb{V}(d)\to\mathbb{V}(d+n)$, which is also the operator norm of $A_{(-n)}^\dagger:\mathbb{V}(d+n)\to\mathbb{V}(d)$. Choose any $v\in\mathbb{V}(d)$ with $\|v\|\leqslant 1$. Then

$$\begin{aligned}\|A_{(-n)}v\|^2&=\|A_{(-n)}^\dagger v\|^2+\langle[A_{(-n)}^\dagger,A_{(-n)}]v|v\rangle\leqslant K_{d-n}^2\|v\|^2+C^2(1+|n|)^{2t}\|v\|_r^2\\ &\leqslant K_{d-n}^2+C^2(1+|n|)^{2t}(1+d)^{2r}\end{aligned}$$

Therefore

$$\begin{aligned}K_d^2&\leqslant K_{d-n}^2+C^2(1+|n|)^{2t}(1+d)^{2r}\\ &\leqslant K_{d-2n}^2+C^2(1+|n|)^{2t}\big((1+d)^{2r}+(1+d-n)^{2r}\big)\leqslant\cdots\\ &\leqslant C^2(1+|n|)^{2t}\int_1^{2+2d}x^{2r}dx\leqslant\frac{2^{2r+1}C^2}{2r+1}(1+|n|)^{2t}(1+d)^{2r+1}\end{aligned}$$

Take $M=\sqrt{\frac{2^{2r+1}}{2r+1}}C$. Then since $K_d=\|A_{(-n)}|_{\mathbb{V}(d)}\|=\|A_{(-n)}^\dagger|_{\mathbb{V}(d+n)}\|$, we have

$$\|A_{(-n)}v\|\leqslant M(1+|n|)^t\|v\|_{r+\frac{1}{2}}\qquad\|A_{(-n)}^\dagger v\|\leqslant M(1+|n|)^t\|v\|_{r+\frac{1}{2}}$$

for each homogeneous v, and hence for all v (by the same reasoning as in the proof of Prop. 4.7).

Let $B(z)=\sum_{n\in\mathbb{Z}}B_{(n)}z^{-n-\mathrm{wt}A}$ where $B_{(n)}=A_{(-n)}^\dagger$. Then $[B_{(n)},B_{(n)}^\dagger]=-[A_{(-n)},A_{(-n)}^\dagger]$ satisfies a similar inequality to (4.16). Therefore, by the above paragraph, if $n>0$ then $B_{(-n)}^\dagger=A_{(n)}$ satisfies a similar estimate to (4.17). □

4.8

Let $(\mathcal{V}, \mathbb{V})$ be a chiral algebra.

Corollary 4.17. *The field $\sum_{n\in 0,\pm 1} L_n z^{-n-2}$ is linearly energy-bounded. If $(\mathcal{V}, \mathbb{V})$ is a conformal chiral algebra, then $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is linearly energy-bounded.*

Proof. By the Virasoro relation (3.33), there is $C \geqslant 0$ such that $\langle [L_n, L_n^\dagger]v|v\rangle \leqslant C(1+|n|)^3\|v\|_{\frac{1}{2}}^2$ for all $v \in \mathbb{V}$. Therefore, by Thm. 4.16, $\sum_{n\neq 0} L_n z^{-n-2}$ is linearly energy-bounded. So $\sum_n L_n z^{-n-2}$ is also linearly energy-bounded. □

Corollary 4.18. *Let $A \in \mathcal{V}$ and $\mathrm{wt}(A) > 0$. Assume that the field $\sum_{n\in\mathbb{Z}} [A_{(n)}, A_{(n)}^\dagger] z^{-n-1}$ satisfies r-th order energy bounds. Then $A(z)$ satisfies $(r+\frac{3}{2})$-th order energy bounds.*

Proof. Assume that there exist $M, t \geqslant 0$ such that $\|[A_{(n)}, A_{(n)}^\dagger]v\| \leqslant M(1+|n|)^t\|v\|_r$ for all n. Then

$$|\langle [A_{(n)}, A_{(n)}^\dagger]v|v\rangle| \leqslant \|[A_{(n)}, A_{(n)}^\dagger]v\| \cdot \|v\| \leqslant M(1+|n|)^t\|v\|_r^2$$

Thus, by Thm. 4.16, $\sum_{n\neq 0} A_{(n)} z^{-n-1}$ satisfies $(r+\frac{1}{2})$-th order energy bounds. Since $A_{(n)} = A_{n+\mathrm{wt}(A)-1}$, by (3.34b) we have $[L_{-1}, A_{(1)}] = -\mathrm{wt}(A)\cdot A_{(0)}$. By Cor. 4.17, L_{-1} satisfies 1-st order energy bounds. Since A_1 satisfies $(r+\frac{1}{2})$-th order energy bounds, by using Prop. 4.7 we see that $A_0 = -\mathrm{wt}(A)^{-1}[L_{-1}, A_1]$ satisfies $(r+\frac{3}{2})$-th order energy bounds. □

Corollary 4.19. *Assume that $(\mathcal{V}, \mathbb{V})$ is the current algebra associated to a finite-dimensional unitary Lie algebra $\mathfrak{g}$. Then for each $X \in \mathfrak{g}$, the field $X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}$ is linearly energy-bounded.*

Proof. It suffices to consider the case that $X^\dagger = X$. Then by (3.36) we have

$$[X_n, X_n^\dagger] = [X_n, X_{-n}] = l\cdot n\|X\|^2\cdot \mathbf{1}$$

Therefore, by Thm. 4.16, $\sum_{n\neq 0} X_n z^{-n-1}$ satisfies $\frac{1}{2}$-th order energy bounds.

It remains to study X_0. We have orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ is abelian and $\mathfrak{g}_2$ is a direct sum of simple Lie algebras. Then $[\mathfrak{g}_1, \mathfrak{g}] = 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2$. It suffices to assume that either $X \in \mathfrak{g}_1$ or $X \in \mathfrak{g}_2$. In the former case, by (3.36), we see that X_0 commutes with Y_n for every $Y \in \mathfrak{g}$ and $n \in \mathbb{Z}$. By the creating property in Def. 3.25, we have $X_0\Omega = 0$. By the cyclicity in Def. 3.25, we know that $X_0 v = 0$ for all $v \in \mathbb{V}$. Then X_0 is clearly linearly energy bounded.

In the later case, since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2$, it suffices to assume that $X = [Y, Z]$ for some $Y, Z \in \mathfrak{g}$. Then by (3.36) we have $X_0 = [Y_1, Z_{-1}] + l\langle Y|Z^*\rangle \cdot \mathbf{1}$. Since we have proved that Y_1 and Z_{-1} satisfy $\frac{1}{2}$-th order energy bounds, by Prop. 4.7, $[Y_1, Z_{-1}]$ is linearly energy-bounded. □

4.9

Proposition 4.20. *Let A, B be mutually local fields. Assume that A, B are both energy-bounded. Then for each $k \in \mathbb{Z}$, $A_k B$ is energy-bounded.*

Proof. As in the proof of Prop. 4.7, it suffices to establish the inequality for homogeneous vectors. Recall (3.25), i.e.

$$(A_k B)_n = \sum_{l\in\mathbb{N}} (-1)^l \binom{k}{l} A_{k-l} B_{n+l} - \sum_{l\in\mathbb{N}} (-1)^{k+l} \binom{k}{l} B_{k+n-l} A_l \tag{$\star$}$$

Note that $\limsup_{l\to\infty} |\binom{k}{l}| l^{-|k|} < +\infty$. Choose any homogeneous $v \in \mathbb{V}$. Then by the energy-boundedness of A, B, and by Prop. 4.7, there exist constants $C_\bullet, r_\bullet, t_\bullet$ independent of v and n (but possibly depending on k) such that

$$\begin{aligned}
&\Big\| \sum_{l\in\mathbb{N}} (-1)^{k+l} \binom{k}{l} B_{k+n-l} A_l v \Big\| \leqslant \sum_{l\geqslant 0} C_1 l^{|k|} \big\| B_{k+n-l} A_l v \big\| \\
\leqslant & \sum_{l\geqslant 0} C_2 l^{|k|} (1 + |k+n-l|)^{t_1} \|A_l v\|_{r_1} \\
\leqslant & \sum_{0\leqslant l\leqslant \mathrm{wt}(A)+\mathrm{wt}(v)-1} C_3 l^{|k|} (1+|n|)^{t_1} (1+l)^{t_1} \cdot (1+l)^{t_2} \|v\|_{r_2} \\
\leqslant & \sum_{0\leqslant l\leqslant \mathrm{wt}(A)+\mathrm{wt}(v)-1} C_3 (1+|n|)^{t_1} (1+l)^{|k|+t_1+t_2} \cdot \|v\|_{r_2} \\
\leqslant & C_4 (1+|n|)^{t_1} (1+\mathrm{wt}(v))^{1+|k|+t_1+t_2} \cdot \|v\|_{r_2} = C_4 (1+|n|)^{t_1} \|v\|_{1+|k|+t_1+t_2+r_2}
\end{aligned}$$

By a similar calculation, we obtain the desired bound for the first summand on the RHS of ($\star$) acting on v. □

4.10

Our last topic of this section is a relationship between products of smeared fields and "products" of pointed fields. We first expand the setting in Subsec. 4.4.

Definition 4.21. Let $I_1, \dots, I_N$ be real oriented smooth 1-submanifolds of $\mathbb{C}$. Let

$$I_\bullet = I_1 \times \cdots \times I_N$$

Suppose that $F : I_\bullet \to \mathrm{Hom}(\mathbb{V}, \mathbb{V}^{\mathrm{ac}})$ is continuous in the sense that for each $u, v \in \mathbb{V}$, the function $z_\bullet \in I_\bullet \to \langle F(z_\bullet)u|v\rangle \in \mathbb{C}$ is continuous. Define

$$\int_{I_\bullet} F(z_\bullet) đz_\bullet : \mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$$

such that for each $u, v \in \mathbb{V}$,

$$\begin{aligned}&\Big\langle \int_{I_\bullet} F(z_\bullet) \,\text{đ} z_\bullet \cdot u \Big| v \Big\rangle = \int_{I_N} \cdots \int_{I_1} \langle F(z_\bullet) u | v \rangle \frac{dz_1}{2\mathbf{i}\pi} \cdots \frac{dz_N}{2\mathbf{i}\pi} \\ &\equiv \int_{a_N}^{b_N} \cdots \int_{a_1}^{b_1} F(\gamma_1(\theta_1), \ldots, \gamma_N(\theta_N)) \gamma_1'(\theta_1) \cdots \gamma_N'(\theta_N) \frac{d\theta_1}{2\mathbf{i}\pi} \cdots \frac{d\theta_N}{2\mathbf{i}\pi}\end{aligned} \tag{4.18}$$

where each $\gamma_i : (a_i, b_i) \to I_i$ is a positive parametrization of I_i. (In Conv. 4.24, we will expand the domain of $\int_{I_\bullet} F(z_\bullet) \,\text{đ} z_\bullet$ to $\mathcal{H}_{\mathbb{V}}^\infty$ under reasonable conditions.)

A similar definition applies to any $F : I_\bullet \to \mathbb{V}^{\mathrm{ac}}$ whose evaluation with any $v \in \mathbb{V}$ is continuous over $I_\bullet$. □

Thus, in short we have

$$\text{đ} z_\bullet = \frac{dz_1}{2\mathbf{i}\pi} \cdots \frac{dz_N}{2\mathbf{i}\pi}$$

4.11

Theorem 4.22. *Let $A^1, \ldots, A^N$ be mutually commuting energy-bounded fields. Let $I_1, \ldots, I_N$ be mutually disjoint intervals of $\mathbb{S}^1$. Choose $f_i \in C_c^\infty(I_i)$ for each i. Then as linear maps $\mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$ we have*

$$A^1(f_1) \cdots A^N(f_N)\big|_{\mathbb{V}} = \int_{I_\bullet} f_1(z_1) \cdots f_N(z_N) A^1(z_1) \cdots A^N(z_N) \,\text{đ} z_\bullet \tag{4.19}$$

Recall Def. 3.13 for the continuity of (3.18), the function $z_\bullet \mapsto \langle A^1(z_1) \cdots A^N(z_N) u | v \rangle$.

Proof. Let $u, v \in \mathbb{V}$ be homogeneous. Let Δ be the set of all $(r_1, \ldots, r_N)$ satisfying $r_1, \ldots, r_N \in [1/2, 1)$ and $r_{i+1}/r_i \in [1/2, 1)$. For each $r_\bullet \in \Delta$, Let

$$F(r_\bullet) = \int_{I_\bullet} f(z_1) \cdots f_N(z_N) \langle A^1(r_1 z_1) \cdots A^N(r_N z_N) u | v \rangle \,\text{đ} z$$

Let $\mathbf{R}$ be the map on the RHS of (4.19). Then by the continuity of (3.18) we have

$$\lim_{r_1, \ldots, r_N \nearrow 1} F(r_\bullet) = \langle \mathbf{R} u | v \rangle$$

For each fixed $r_\bullet \in \Delta$, since $|r_1 z_1| > \cdots > |r_N z_N| > 0$ for all $z_\bullet \in I_\bullet$, by Prop. 3.6,

$$\begin{aligned}&\langle A^1(r_1 z_1) \cdots A^N(r_N z_N) u | v \rangle = \sum_{n_\bullet \in \mathbb{N}^N} \langle P_{n_1} A^1(r_1 z_1) P_{n_2} A^2(r_2 z_2) \cdots P_{n_N} A^N(r_N z_N) u | v \rangle \\ &= \sum_{n_\bullet \in \mathbb{N}^N} \langle P_{n_1} A^1(z_1) P_{n_2} A^2(z_2) \cdots P_{n_N} A^N(z_N) u | v \rangle \cdot \Delta(r_\bullet)\end{aligned}$$

which converges a.l.u. for $z_\bullet \in I_\bullet$, where

$$\Delta(r_\bullet) = r_1^{n_1}\left(\frac{r_2}{r_1}\right)^{n_2}\cdots\left(\frac{r_N}{r_{N-1}}\right)^{n_N}\cdot r_1^{-\mathrm{wt}(A^1)}\cdots r_N^{-\mathrm{wt}(A^N)}$$

Hence the sum commutes with the integrals, yielding

$$F(r_\bullet) = \sum_{n_\bullet\in\mathbb{N}^N}\int_{I_\bullet} f_1(z_1)\cdots f_N(z_N)\langle P_{n_1}A^1(z_1)P_{n_2}A^2(z_2)\cdots P_{n_N}A^N(z_N)u|v\rangle\cdot\Delta(r_\bullet)$$

$$= \sum_{n_\bullet\in\mathbb{N}^N}\langle P_{n_1}A^1(f_1)P_{n_2}A^2(f_2)\cdots P_{n_N}A^N(f_N)u|v\rangle\cdot\Delta(r_\bullet) \tag{$\Diamond$}$$

By (4.7), we have

$$\begin{aligned}&\sum_{n_\bullet\in\mathbb{N}^N}\big\|P_{n_1}A^1(f_1)P_{n_2}A^2(f_n)\cdots P_{n_N}A^N(f_N)u\big\|\\ =&\sum_{k_\bullet\in\mathbb{N}^N}\big\|A^1_{k_1}A^2_{k_2}\cdots A^N_{k_N}u\big\|\cdot|\widehat{f}_1(k_1)\cdots\widehat{f}_N(k_N)|\end{aligned} \tag{$\star$}$$

which is finite by Prop. 4.7 (and the fact that $k_i\mapsto\widehat{f}_i(k_i)$ is rapidly decreasing). Hence, the function

$$\sup_{r_\bullet\in\Delta}\Big|\big\langle P_{n_1}A^1(f_1)P_{n_2}A^2(f_n)\cdots P_{n_N}A^N(f_N)u|v\big\rangle\Big|\Delta(r_\bullet) < +\infty$$

Therefore, applying the dominated convergence theorem to ($\Diamond$), we get

$$\lim_{r_1,\dots,r_N\nearrow 1}F(r_\bullet) = \sum_{n_\bullet\in\mathbb{N}^N}\langle P_{n_1}A^1(f_1)P_{n_2}A^2(f_2)\cdots P_{n_N}A^N(f_N)u|v\rangle$$

$$=\langle A^1(f_1)\cdots A^N(f_N)u|v\rangle$$

where the last equality is due to Lem. 4.23. This proves (4.19). □

Lemma 4.23. *Let $T_1,\dots,T_N$ be closable operators on $\mathcal{H}_{\mathbb{V}}$ such that for each $1\leqslant i\leqslant N$, $\mathcal{H}^\infty_{\mathbb{V}}$ is contained in $\mathscr{D}(T_i)\cap\mathscr{D}(T_i^*)$ and invariant under both T_i and T_i^*. Then for each $\xi,\eta\in\mathcal{H}^\infty_{\mathbb{V}}$ we have*

$$\sum_{n_N\in\mathbb{N}}\cdots\sum_{n_1\in\mathbb{N}}\langle P_{n_1}T_1P_{n_2}T_2\cdots P_{n_N}T_N\xi|\eta\rangle = \langle T_1\cdots T_N\xi|\eta\rangle \tag{4.20}$$

Proof. The case $N=0$ is obvious. Assume case $N-1$ is true. Then for case N, the LHS of (4.20) equals

$$\sum_{n_N}\cdots\sum_{n_2}\sum_{n_1}\langle T_1P_{n_2}T_2\cdots P_{n_N}T_N\xi|P_{n_1}\eta\rangle = \sum_{n_N}\cdots\sum_{n_2}\langle T_1P_{n_2}T_2\cdots P_{n_N}T_N\xi|\eta\rangle$$

$$=\sum_{n_N}\cdots\sum_{n_2}\langle P_{n_2}T_2\cdots P_{n_N}T_N\xi|T_1^*\eta\rangle = \langle T_2\cdots T_N\xi|T_1^*\eta\rangle = \langle T_1T_2\cdots T_N\xi|\eta\rangle$$

□

4.12

Convention 4.24. Let $F : I_\bullet \to \mathrm{Hom}(\mathbb{V}, \mathbb{V}^{\mathrm{ac}})$ be as in Def. 4.21. Suppose that the linear map $\int_{I_\bullet} F(z_\bullet) đz_\bullet : \mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$ has range in $\mathcal{H}_{\mathbb{V}}$ and is closable. Suppose also that $\mathcal{H}_{\mathbb{V}}^\infty$ is contained in the domain of the closure of $\int_{I_\bullet} F(z_\bullet) đz_\bullet$. We then let

$$\int_{I_\bullet} F(z_\bullet) đz_\bullet : \mathcal{H}_{\mathbb{V}}^\infty \to \mathcal{H}_{\mathbb{V}}$$

denote the restriction to $\mathcal{H}_{\mathbb{V}}^\infty$ of the closure of the original $\int_{I_\bullet} F(z_\bullet) đz_\bullet$.

Example 4.25. Assume the setting of Thm. 4.22, and let $F(z_\bullet) = f_1(z_1) \cdots f(z_N) A^1(z_1) \cdots A^N(z_N)$ be the integrand on the RHS of (4.19). Then by Thm. 4.22, $\int_{I_\bullet} F(z_\bullet) đz_\bullet$ equals $A^\bullet(f_\bullet)|_{\mathbb{V}}$. By Prop. 4.14, $\mathbb{V}$ is a core for $A^\bullet(f_\bullet) : \mathcal{H}_{\mathbb{V}}^\infty \to \mathcal{H}_{\mathbb{V}}^\infty$, and hence $\mathscr{D}(\overline{A^\bullet(f_\bullet)|_{\mathbb{V}}}) \supset \mathcal{H}_{\mathbb{V}}^\infty$. Therefore Conv. 4.24 applies to $F(z_\bullet)$, and hence

$$A^1(f_1) \cdots A^N(f_N) = \int_{I_\bullet} f_1(z_1) \cdots f_N(z_N) A^1(z_1) \cdots A^N(z_N) đz_\bullet \tag{4.21}$$

holds as closable operators on $\mathcal{H}_{\mathbb{V}}$ with common domain $\mathcal{H}_{\mathbb{V}}^\infty$.

Corollary 4.26. *Let A, B be energy-bounded and mutually local fields. Let I, J be mutually disjoint intervals of $\mathbb{S}^1$. Then for each $f \in C_c^\infty(I)$ and $g \in C_c^\infty(J)$ we have (on $\mathcal{H}_{\mathbb{V}}^\infty$) that*

$$A(f)B(g) = B(g)A(f)$$

Proof. By Exp. 4.25, as unbounded operators with domain $\mathcal{H}_{\mathbb{V}}^\infty$ we have

$$A(f)B(g) = \int_{I \times J} f(z_1) g(z_2) A(z_1) B(z_2) đz_\bullet = B(g)A(f)$$

noting (3.19) that the $A(z_1)$ and $B(z_2)$ in the integral are interchangeable. $\square$

5 Methods of unbounded operators

In this section, we collect some useful properties about unbounded operators that will be used later.

Let $\mathcal{H}$ be a Hilbert space. Recall [Gui-S, Sec. 4] for the meaning of unbounded positive operators. Assume that T is self-adjoint. By spectral theorem, $T \geqslant 0$ iff $\langle T\xi|\xi\rangle \geqslant 0$ for all $\xi \in \mathscr{D}(T)$. Thus, if $\mathscr{D}_0 \subset \mathscr{D}(T)$ is a core for T, then $T \geqslant 0$ iff $\langle T\xi|\xi\rangle \geqslant 0$ for all $\xi \in \mathscr{D}_0$.

In this section, we fix a positive operator H on $\mathcal{H}$ satisfying $H - a \geqslant 0$ for some real number $a > 0$. Recall Rem. 4.5 that for each $p \geqslant 0$, $\mathscr{D}(H^p)$ is complete under the inner product

$$\langle\xi|\eta\rangle_p := \langle H^p\xi|H^p\eta\rangle \tag{5.1}$$

We always let $\|\cdot\|_p$ be the norm associated to this inner product.

Remark 5.1. Note that for each $r \geqslant 0$,

$$\mathcal{H}^\infty := \bigcap_{p\geqslant 0} \mathscr{D}(H^p)$$

is a core for H^r. This is because $\mathcal{H}^\infty$ contains $\bigcup_{\lambda\geqslant 0} \mathrm{Rng}(\chi_{[0,\lambda]}(H))$ where the latter is a core for $f(H)$.

5.1

Let $O \subset \mathbb{C}$ be open. Recall that a **holomorphic function** $f : O \to \mathcal{H}$ is a function whose derivative $f'(z)$ exists everywhere on Ω. More generally, if $O \subset \mathbb{C}^n$ is open, a **holomorphic function** $f : O \to \mathcal{H}$ is defined to be a continuous function which is holomorphic in each variable (when the other variables are fixed).

Proposition 5.2. *Let V be a dense linear subspace of $\mathcal{H}$. Let $O \subset \mathbb{C}$ be open and $f : O \to \mathcal{H}$ be a function. Then the following are equivalent:*

(1) f is holomorphic.

(2) f is continuous, and $z \in O \mapsto \langle f(z)|v\rangle$ is holomorhic for all $v \in V$.

(3) On each closed disk $D \subset O$ with center z_0 and radius r, $f(z)$ is the limit of a series $\sum_{n\in\mathbb{N}} \xi_n(z - z_0)^n$ where $\xi_n \in \mathcal{H}$ and $\sum_{n\in\mathbb{N}} \|\xi_n\| r^n < +\infty$.

Proof. (1)⇒(2) and (3)⇒(2) are obvious.

(2)⇒(1,3): Assume (2). Let C be an anticlockwise circle in O centered at z_0 with radius ρ slightly larger than r. Since f is continuous, for each $|z - z_0| < \rho$ we can define the Riemann integral

$$g(z) = \oint_C \frac{f(\zeta)}{\zeta - z} đ\zeta$$

By the residue theorem, for each $v \in V$ and each $z \in D$ we have $\langle f(z)|v\rangle = \langle g(z)|v\rangle$. So $f = g$ on D. Thus, whenever $|z - z_0| < r$ we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \oint_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} d\!\!\!{}^-\zeta$$

whose limit under $z \to z_0$ exists because the integrand on the RHS converges uniformly (over $\zeta \in C$) to $f(\zeta)(\zeta - z_0)^2$. This proves (1).

Since the RHS below converges uniformly on $(\xi, z) \in C \times D$ to the LHS

$$(\zeta - z)^{-1} = \sum_{n \in \mathbb{N}} (\zeta - z_0)^{-n-1}(z - z_0)^n$$

we conclude that $f(z) = \sum_{n\in\mathbb{N}} \xi_n (z - z_0)^n$ where

$$\xi_n = \oint_C (\zeta - z_0)^{-n-1} f(\zeta) d\!\!\!{}^-\zeta$$

Since $\|\xi_n\| \leqslant \rho^{-n-1}M$ where $M = \sup_{\zeta\in C} \|f(\zeta)\|$, we have $\sum_n \|\xi_n\| r^n < +\infty$. This proves (3). $\square$

5.2

We know that if A is self-adjoint, then $\mathscr{D}(A)$ is precisely the set of all ξ such that $t \in \mathbb{R} \mapsto e^{\mathbf{i}tA}\xi$ is differentiable at 0. In a similar spirit, the following theorem describes the domain of A^r.

Theorem 5.3. *Let A be a positive operator on $\mathcal{H}$ satisfying $\chi_{\{0\}}(A) = 0$ (equivalently, A is injective). Let $r > 0$ and*

$$\mathfrak{I} = \{z \in \mathbb{C} : -r \leqslant \mathrm{Im} z \leqslant 0\}$$

Let $\xi \in \mathcal{H}$. Then the following are equivalent.

(1) $\xi \in \mathscr{D}(A^r)$.

(2) The function $t \in \mathbb{R} \mapsto A^{\mathbf{i}t} \in \mathcal{H}$ can be extended to a (necessarily unique) continuous function $F : \mathfrak{I} \to \mathcal{H}$ and holomorphic on $\mathrm{Int}\mathfrak{I}$.

(3) There exist $\psi \in \mathcal{H}$ and a core $\mathscr{D}_0$ for A^r such that for each $\eta \in \mathscr{D}_0$, there is a function f_η continuous on $\mathfrak{I}$, holomorphic on $\mathrm{Int}\mathfrak{I}$, such that $f_\eta(t) = \langle A^{\mathbf{i}t}\xi|\eta\rangle$ for each $t \in \mathbb{R}$, and that $f_\eta(-\mathbf{i}r) = \langle\psi|\eta\rangle$.

Moreover, if the above statements are true, the function in (2) takes the form $F(z) = A^{\mathbf{i}z}\xi$. In particular, $F(-\mathbf{i}r) = A^r\xi$.

Proof. (1)⇒(2): The uniqueness follows from Lem. 5.4. As for the existence, one defines $F : z \in \mathfrak{I} \mapsto A^{\mathbf{i}z}\xi$, noting that condition (1) ensures that $\xi \in \mathscr{D}(A^{\mathbf{i}z})$ for all $z \in \mathfrak{I}$. The continuity of $(\tau, \mu) \in \mathfrak{I} \times \mathfrak{I} \mapsto \langle A^{\mathbf{i}\tau}\xi | A^{\mathbf{i}\mu}\xi\rangle$ (which follows from the spectral theory and the dominated convergence theorem) shows that F is continuous; the holomorphicity of $F|_{\mathrm{Int}\mathfrak{I}}$ follows e.g. from the spectral theory and Morera's theorem.

(2)⇒(3): Let $f_\eta(z) = \langle F(z)|\eta\rangle$ and $\psi = F(-\mathbf{i}r)$.

(3)⇒(1): Since $\eta \in \mathscr{D}_0$, the map $z \in \mathfrak{I} \mapsto A^{\mathbf{i}z}\eta \in \mathcal{H}$ is continuous on $\mathfrak{I}$ and holomorphic on $\mathrm{Int}\mathfrak{I}$. Therefore, we have a continuous function $\widetilde{f}_\eta : \mathfrak{I} \to \mathbb{C}$, holomorphic on $\mathrm{Int}\mathfrak{I}$, such that

$$\widetilde{f}_\eta(z) = \langle \xi | A^{-\mathbf{i}\overline{z}}\eta\rangle$$

Extend f_η to a continuous function $\mathfrak{I}$ holomorphic on $\mathrm{Int}\mathfrak{I}$. For each $t \in \mathbb{R}$ we have

$$\widetilde{f}_\eta(t) = \langle \xi | A^{-\mathbf{i}t}\eta\rangle = \langle A^{\mathbf{i}t}\xi|\eta\rangle = f_\eta(t)$$

Thus, by the following Lem. 5.4, we have $\widetilde{f}_\eta = f_\eta$ on $\mathfrak{I}$. Therefore

$$\langle\psi|\eta\rangle = f_\eta(-\mathbf{i}r) = \widetilde{f}_\eta(-\mathbf{i}r) = \langle\xi|A^r\eta\rangle$$

Since all such η form a core $\mathscr{D}_0$ for A^r, we see that $\xi \in \mathscr{D}((A^r|_{\mathscr{D}_0})^*) = \mathscr{D}(A^r)$ and $A^r\xi = \psi$. □

Lemma 5.4. *Let Ω be an open subset of $\{z \in \mathbb{C} : \mathrm{Im}z \geqslant 0\}$ whose interior $\mathrm{Int}_{\mathbb{C}}\Omega = \{z \in \Omega : \mathrm{Im}z > 0\}$ is connected. Suppose that I is a non-empty open interval contained in $\mathbb{R} \cap \Omega$. Let $F : \Omega \to \mathcal{H}$ be a continuous function holomorphic on $\mathrm{Int}_{\mathbb{C}}\Omega$. If $F|_I = 0$, then $F = 0$.*

Note that by applying a Möbious transform, the lemma also applies to the case that Ω is an open subset of the unit closed disk $\overline{\mathbb{D}}_1$ with connected interior, and I is an interval of $\mathbb{S}^1$.

Proof. By considering the function $z \in \Omega \mapsto \langle F(z)|\eta\rangle$ for all $\eta \in \mathcal{H}$, it suffices to assume that $\mathcal{H} = \mathbb{C}$. Let $\varepsilon > 0$ and $D = \{z \in \mathbb{C} : \mathrm{Re}z \in I, 0 \leqslant \mathrm{Im}z < \varepsilon\}$ such that $D \subset \Omega$. If we can show that $F|_{\mathrm{Int}D} = 0$, then $F|_{\mathrm{Int}\Omega} = 0$ because $\mathrm{Int}\Omega$ is connected, and hence $F = 0$ by the continuity.

By the Schwarz reflection principle, $F|_D$ can be extended to a holomorphic function on $\widetilde{D} = \{z \in \mathbb{C} : \mathrm{Re}z \in I, -\varepsilon \leqslant \mathrm{Im}z < \varepsilon\}$. (In fact, one defines $F(z) = \overline{F(\overline{z})}$ if $\overline{z} \in D$, and uses Morera's theorem to show that $F|_{\widetilde{D}}$ is holomorphic.) Now $\widetilde{D}$ is an open subset of $\mathbb{C}$. It is well-known that any holomorphic function on $\widetilde{D}$ vanishing on an interval must be vanishing everywhere. Thus $F|_I = 0$ implies $F|_{\widetilde{D}} = 0$. □

Corollary 5.5. *Let A be a positive operator on $\mathcal{H}$. Let $\mathfrak{I} = \{z \in \mathbb{C} : \mathrm{Im} z \geqslant 0\}$. Then for each $\xi \in \mathcal{H}$, the function $F : z \in \mathfrak{I} \mapsto e^{\mathbf{i}zA}\xi \in \mathcal{H}$ is continuous, and $F|_{\mathrm{Int}\mathfrak{I}}$ is holomorphic.*

Proof. Apply Thm. 5.3 to the operator $B = e^{-A}$, which is bounded (and hence $\mathcal{H} = \mathscr{D}(B^r)$ for all $r \geqslant 0$) because $A \geqslant 0$. Note that the fact $B^{\mathbf{i}z} = e^{-\mathbf{i}zA}$ is due to the composition rule for Borel functional calculus, i.e., if $f, g : \mathbb{C} \to \mathbb{C}$ are Borel functions then $(f \circ g)(A) = f(g(A))$, c.f. [Gui-S, Sec. 9]. $\square$

5.3

Proposition 5.6. *Let T be a symmetric unbounded operator on $\mathcal{H}$. The following are true.*

1. *$\overline{T}$ is self-adjoint iff $\mathrm{Rng}(\mathbf{i} + T)$ and $\mathrm{Rng}(\mathbf{i} - T)$ are dense in $\mathcal{H}$.*
2. *Assume that $\overline{T}$ is self-adjoint, and let $\mathscr{D}_0$ be a linear subspace of $\mathscr{D}(T)$. Then $\mathscr{D}_0$ is a core for T (equivalently, for $\overline{T}$) iff $(\mathbf{i} + T)\mathscr{D}_0$ is dense in $\mathcal{H}$.*

Proof. Let $\mathscr{G}(\overline{T})$ be the graph of T. Then

$$\Phi_\pm : \mathscr{G}(\overline{T}) \to \mathrm{Rng}(\mathbf{i} \pm \overline{T}) \qquad (\xi, \overline{T}\xi) \mapsto \mathbf{i}\xi \pm \overline{T}\xi$$

is unitary (because $\overline{T}$ is symmetric), restricting to a unitary map $\mathscr{G}(T|_{\mathscr{D}_0}) \to (\mathbf{i} \pm T)\mathscr{D}_0$. Thus $\mathscr{D}_0$ is a core for $\overline{T}$ iff $\mathscr{G}(T|_{\mathscr{D}_0})$ is dense in $\mathscr{G}(\overline{T})$, iff $(\mathbf{i} + T)\mathscr{D}_0$ is dense in $\mathrm{Rng}(\mathbf{i} + T)$, iff $(\mathbf{i} - T)\mathscr{D}_0$ is dense in $\mathrm{Rng}(\mathbf{i} - T)$. Therefore (2) follows from (1).

Since $\mathscr{D}(T)$ is a core for $\overline{T}$, by setting $\mathscr{D}_0 = \mathscr{D}(T)$, we see that $\mathrm{Rng}(\mathbf{i} \pm T)$ is dense in $\mathrm{Rng}(\mathbf{i} \pm \overline{T})$. Since $\overline{T}$ is self-adjoint iff $\mathrm{Rng}(\mathbf{i} + \overline{T}) = \mathrm{Rng}(\mathbf{i} - \overline{T}) = \mathcal{H}$ (cf. [Gui-S, Sec. 10]), (1) is proved. $\square$

5.4

We use Prop. 5.6 to give two criteria for cores.

Theorem 5.7. *Let A, B be self-adjoint closed operators on $\mathcal{H}$, and assume that B is affiliated with $\{A\}''$ (e.g. $B = f(A)$ for some Borel $f : \mathbb{R} \to \mathbb{R}$). Choose a dense linear subspace $\mathscr{D}_0 \subset \mathscr{D}(B)$ such that $e^{\mathbf{i}tA}\mathscr{D}_0 \subset \mathscr{D}_0$ for any $t \in \mathbb{R}$. Then $\mathscr{D}_0$ is a core for B.*

Proof. By Prop. 5.6, it suffices to prove that $(\mathbf{i} + B)\mathscr{D}_0$ is dense in $\mathcal{H}$. Choose any $\eta \in \mathcal{H}$ orthogonal to $(\mathbf{i} + B)\mathscr{D}_0$. Then for each $\xi \in \mathscr{D}_0$ and $t \in \mathbb{R}$, since $e^{\mathbf{i}tA}$ belongs to $\mathcal{M} := \{H\}''$ and hence commutes strongly with $\mathbf{i} + B$, we have $e^{-\mathbf{i}tA}(\mathbf{i} + B) \subset (\mathbf{i} + B)e^{-\mathbf{i}tA}$. Therefore, since $e^{-\mathbf{i}tA}\xi \in \mathscr{D}_0$, we have

$$\langle (\mathbf{i} + B)\xi | e^{\mathbf{i}tA}\eta \rangle = \langle (\mathbf{i} + B)e^{-\mathbf{i}tA}\xi | \eta \rangle = 0$$

Let $\mathscr{A} = \mathrm{Span}_{\mathbb{C}}\{e^{\mathbf{i}tA} : t \in \mathbb{R}\}$, which is a unital $*$-algebra generating $\mathcal{M}$. Then clearly

$$\langle(\mathbf{i} + B)\xi|x\eta\rangle = 0$$

for all $x \in \mathscr{A}$, and hence (by passing to the strong operator closure) for all $x \in \mathcal{M}$. Since $\{B\}'' \subset \mathcal{M}$, the projection $E_\lambda = \chi_{[-\lambda,\lambda]}(B)$ belongs to $\mathcal{M}$ for all $\lambda \geqslant 0$. Thus, noting that $(-\mathbf{i} + B)E_\lambda$ is bounded (cf. [Gui-S, Prop. 8.1]) and hence $E_\lambda\eta \in \mathscr{D}(-\mathbf{i} + B)$, we have

$$\langle\xi|(-\mathbf{i} + B)E_\lambda\eta\rangle = \langle(\mathbf{i} + B)\xi|E_\lambda\eta\rangle = 0$$

Since $\xi \in \mathscr{D}_0$ is arbitrary and $\mathscr{D}_0$ is dense, we have $(-\mathbf{i} + B)E_\lambda\eta = 0$. Thus $E_\lambda\eta = 0$ because $-\mathbf{i} + B$ is injective. Letting $\lambda \nearrow +\infty$, we get $\eta = 0$. □

5.5

Case (1) of the following theorem is [CKLW18, Lem.7.2]. Although Case (2) is motivated by the treatment of Case (1) in [CKLW18], and although Case (1) (as well as its proof) is interesting in its own right, we will not use Case (1) in this course. Thus, the proof of Case (1) below can be safely skipped.

Theorem 5.8. *Let A, B be self-adjoint closed operators on $\mathcal{H}$, and assume that B is affiliated with $\{A\}''$. Let $\mathscr{D}_0$ be a dense subspace of $\mathscr{D}(B)$. Suppose that there exist $\delta > 0$ and a dense subspace $\mathscr{D}_\delta \subset \mathscr{D}_0$ such that $e^{\mathbf{i}tA}\mathscr{D}_\delta \subset \mathscr{D}_0$ for all $-\delta < t < \delta$. Then $\mathscr{D}_0$ is a core for B provided that one of the following is true.*

(1) $B = A^k$ for some $k \in \mathbb{N}$.

(2) $A \geqslant 0$.

Proof. By Prop. 5.6, to show that $\mathscr{D}_0$ is a core for A, it suffices to choose any $\eta \in \mathcal{H}$ orthogonal to $(\mathbf{i} + B)\mathscr{D}_0$ and show that $\eta = 0$. Since $e^{-\mathbf{i}tA}$ commutes strongly with B, we have $e^{-\mathbf{i}tA}(\mathbf{i} + B) \subset (\mathbf{i} + B)e^{-\mathbf{i}tA}$. Therefore, for each $\xi \in \mathscr{D}_\delta$ and $|t| < \delta$, since $e^{-\mathbf{i}tA}\xi \in \mathscr{D}_0$, we have

$$\langle(\mathbf{i} + B)\xi|e^{\mathbf{i}tA}\eta\rangle = \langle(\mathbf{i} + B)e^{-\mathbf{i}tA}\xi|\eta\rangle = 0$$

Case (1). Let $B = A^k$. Choose $h \in C_c^\infty(-\delta, \delta)$ with $\int_{\mathbb{R}} h = 1$, and let $\widehat{h}$ be its Fourier transform. Recall that $\widehat{h}(A) = \int_{\mathbb{R}} h(t)e^{-\mathbf{i}tA}dt$, i.e., for each $\psi \in \mathcal{H}$ we have $\widehat{h}(A)\psi = \int_{\mathbb{R}} h(t)e^{-\mathbf{i}tA}\psi dt$ where the RHS is understood as the improper $\mathcal{H}$-valued Riemann integral. (See the end of [Gui-S, Sec. 10].) Therefore

$$\langle(\mathbf{i} + A^k)\xi|\widehat{h}(A)\eta\rangle = 0$$

By the basic properties of Borel functional calculus (cf. [Gui-S, Sec. 9]), if $\alpha,\beta:\mathbb{R}\rightarrow\mathbb{C}$ are Borel functions, then $(\alpha\beta)(A)=\overline{\alpha(A)\beta(A)}$ where the latter equals $\alpha(A)\beta(B)$ when β is bounded (cf. [Gui-S, Prop. 8.1]). Thus $A^k\widehat{h}(A)$ equals $(x^k\widehat{h})(A)=(-\mathbf{i})^k\widehat{h^{(k)}}(A)$, which is a bounded operator because $\widehat{h^{(k)}}$ is bounded. In particular $\widehat{h}(A)\eta\in\mathscr{D}(-\mathbf{i}+A^k)$. So

$$\langle\xi|(-\mathbf{i}+A^k)\widehat{h}(A)\eta\rangle=\langle(\mathbf{i}+A^k)\xi|\widehat{h}(A)\eta\rangle=0$$

Since $\xi\in\mathscr{D}_\delta$ is arbitrary, we get $(-\mathbf{i}+A^k)\widehat{h}(A)\eta=0$, and hence $\widehat{h}(A)\eta=0$.

Replacing h with $h_n(x)=nh(nx)$, we have $\widehat{h}_n(A)\eta=0$. Since

$$\lim_{n\rightarrow\infty}\widehat{h}_n(A)\eta=\lim_{n\rightarrow\infty}\int_{\mathbb{R}}nh(nx)e^{-\mathbf{i}tA}\eta dx=\eta$$

we get $\eta=0$.

Case (2). Let $\mathfrak{J}=\{z\in\mathbb{C}:\mathrm{Im}z\geqslant 0\}$ and $\xi\in\mathscr{D}_\delta$. By Cor. 5.5, the function

$$\mathfrak{J}\mapsto\mathcal{H}\qquad z\mapsto\langle(\mathbf{i}+B)\xi|e^{-\mathbf{i}\overline{z}A}\eta\rangle$$

is continuous, and is holomorphic on $\mathrm{Int}\mathfrak{J}$. Since we have shown that this function is zero on an interval of $\mathbb{R}$, by Lem. 5.4, it is zero for all $z\in\mathfrak{J}$ and, in particular, all $z=t\in\mathbb{R}$. Thus, as in the proof of Thm. 5.7, for each $\lambda\geqslant 0$ and $E_\lambda=\chi_{[-\lambda,\lambda]}(B)$ we have

$$\langle\xi|(-\mathbf{i}+B)E_\lambda\eta\rangle=\langle(\mathbf{i}+B)\xi|E_\lambda\eta\rangle=0$$

for all ξ in the dense space $\mathscr{D}_\delta$. Hence $(-\mathbf{i}+B)E_\lambda\eta=0$, and hence $E_\lambda\eta=0$ (for all λ), and hence $\eta=0$. □

5.6

As our last application of Prop. 5.6, we give an important criterion for self-adjointness originally due to A. Jaffe (cf. [GJ, Thm. 19.4.3]). Our presentation follows [FL74]. Recall our assumption on H at the beginning of this section.

Remark 5.9. Suppose that A is a self-adjoint operator on $\mathcal{H}$. Then any symmetric extension of A equals A, i.e., if $A\subset B$ where B is a symmetric operator, then $A=B$. Indeed, we have $A\subset B\subset B^*\subset A^*$ and $A=A^*$. Thus $A=B$.

Theorem 5.10. *Let T be a symmetric (i.e. $T\subset T^*$) closed operator on $\mathcal{H}$. Assume that $\mathscr{D}_0\subset\mathscr{D}(H)\cap\mathscr{D}(T)$ is a core for H, and there exists $C\geqslant 0$ such that for every $\xi,\eta\in\mathscr{D}_0$ we have*

$$\|T\xi\|\leqslant C\|H\xi\| \tag{5.2a}$$
$$\big|\langle T\xi|H\eta\rangle-\langle H\xi|T\eta\rangle\big|\leqslant C\|H\xi\|\cdot\|\eta\| \tag{5.2b}$$

Then $\mathscr{D}_0$ is a core for T, and $T=T^*$.

We are mainly interested in the case that $\mathscr{D}_0$ is invariant under T and H. Then (5.2b) reads that for all $\xi, \eta \in \mathscr{D}_0$,

$$\big|\langle [H,T]\xi|\eta\rangle\big| \leqslant C\|H\xi\| \cdot \|\eta\|$$

equivalently, for all $\xi \in \mathscr{D}_0$ we have

$$\big\|[H,T]\xi\big\| \leqslant C\|H\xi\|$$

Thus (5.2) holds, e.g., when $\mathscr{D}_0 = \mathcal{H}_{\mathbb{V}}^\infty$, $H = \overline{L_0}$, and T is the closure of a linearly energy-bounded field that is symmetric.

Proof. Step 1. By Rem. 5.9, if we can prove that $\overline{T|_{\mathscr{D}_0}}$ is self-adjoint, then $\overline{T|_{\mathscr{D}_0}} = T$. Thus $T = T^*$ and T has core $\mathscr{D}_0$. Therefore, by replacing T with $\overline{T|_{\mathscr{D}_0}}$, we may assume at the beginning that $\mathscr{D}_0$ is a core for T, and we only need to prove $T = T^*$.

We first show that $\mathscr{D}(H) \subset \mathscr{D}(T)$, and that (5.2) holds for all $\xi, \eta \in \mathscr{D}(H)$. Choose $\xi \in \mathscr{D}(H)$. Since $\mathscr{D}_0$ is an H-core, there exists a sequence (ξ_n) in $\mathscr{D}_0$ converging to ξ such that $H\xi_n$ converges to $H\xi$. By (5.2a), $(T\xi_n)$ is a Cauchy sequence. Since T is closed, we see that $\xi \in \mathscr{D}(T)$ and $T\xi = \lim_n T\xi_n$.

Let $\xi, \eta \in \mathscr{D}(H)$. By the above paragraph, we have sequences $(\xi_n), (\eta_\bullet)$ converging to ξ, η such that $\lim H\xi_n = H\xi$, $\lim T\xi_n = T\xi$, $\lim H\eta_n = H\eta$, $\lim T\eta_n = T\eta$. Since (5.2) holds when ξ, η are replaced by ξ_n, η_n, we see that (5.2) holds for ξ, η.

Step 2. Since $H \geqslant a > 0$, by spectral theorem (or by the relation between H and its inverse mentioned in [Gui-S, Sec. 4]), we have $\mathrm{Rng}(H^{-1}) = \mathscr{D}(H)$ and $HH^{-1} = \mathbf{1}_{\mathcal{H}}$. So

$$\mathrm{Rng}(H^{-1}) = \mathscr{D}(H) \subset \mathscr{D}(T)$$

Let us prove for each $\xi, \eta \in \mathcal{H}$ that

$$\big|\langle TH^{-1}\xi|\eta\rangle - \langle \xi|TH^{-1}\eta\rangle\big| \leqslant C\|\xi\| \cdot \|H^{-1}\eta\| \tag{5.3a}$$

$$\big|\langle TH^{-1}\xi|\eta\rangle - \langle \xi|TH^{-1}\eta\rangle\big| \leqslant C\|H^{-1}\xi\| \cdot \|\eta\| \tag{5.3b}$$

By (5.2b) (with ξ, η replaced by $H^{-1}\xi, H^{-1}\eta$, both are in $\mathscr{D}(H)$), we get (5.3a). Since $\langle TH^{-1}\xi|\eta\rangle - \langle \xi|TH^{-1}\eta\rangle$ has complex conjugate $-\langle TH^{-1}\eta|\xi\rangle + \langle \eta|TH^{-1}\xi\rangle$, (5.3b) follows by exchanging the ξ and η in (5.3a).

Step 3. Let us prove for each $\xi, \eta \in \mathcal{H}$ that

$$|\langle TH^{-2}\xi|\eta\rangle - \langle \xi|TH^{-2}\eta\rangle| \leqslant 2C\|H^{-1}\xi\| \cdot \|H^{-1}\eta\| \tag{5.4}$$

By (5.3a) (replacing ξ with $H^{-1}\xi$), we have

$$\big|\langle TH^{-2}\xi|\eta\rangle - \langle H^{-1}\xi|TH^{-1}\eta\rangle\big| \leqslant C\|H^{-1}\xi\| \cdot \|H^{-1}\eta\|$$

By (5.3b) (replacing η with $H^{-1}\eta$), we have

$$\big|\langle TH^{-1}\xi|H^{-1}\eta\rangle - \langle\xi|TH^{-2}\eta\rangle\big| \leqslant C\|H^{-1}\xi\|\cdot\|H^{-1}\eta\|$$

This proves (5.4), since $\langle H^{-1}\xi|TH^{-1}\eta\rangle = \langle TH^{-1}\xi|H^{-1}\eta\rangle$.

Step 4. For each $\xi\in\mathcal{H}$, we have

$$2\mathbf{i}\cdot\mathrm{Im}\langle TH^{-2}\xi|\xi\rangle = \langle TH^{-2}\xi|\xi\rangle - \langle\xi|TH^{-2}\xi\rangle$$

Thus (5.4) implies

$$|2\mathrm{Im}\langle TH^{-2}\xi|\xi\rangle| \leqslant 2C\|H^{-1}\xi\|^2 \tag{5.5}$$

Now, to prove that $T=T^*$, by Prop. 5.6, it suffices to find some $\lambda\in\mathbb{R}\backslash\{0\}$ such that $T+\lambda\mathbf{i}$ and $T-\lambda\mathbf{i}$ have dense ranges. Choose $\lambda\in\mathbb{R}\backslash\{0\}$ whose value will be determined shortly, let $\xi\in\mathcal{H}$ be orthogonal to the range of $T+\lambda\mathbf{i}$. Then

$$\mathrm{Im}\langle(T+\lambda\mathbf{i})H^{-2}\xi|\xi\rangle = 0$$

namely,

$$\lambda\langle H^{-2}\xi|\xi\rangle = -\mathrm{Im}\langle TH^{-2}\xi|\xi\rangle$$

Thus, by (5.5), we have $|\lambda|\cdot|\langle H^{-2}\xi|\xi\rangle| \leqslant C\|H^{-1}\xi\|^2$, i.e.

$$|\lambda|\cdot\|H^{-1}\xi\|^2 \leqslant C\|H^{-1}\xi\|^2$$

So $\xi=0$ whenever $|\lambda|>C$. Thus, taking $\lambda=C+1$, we see that $T+\lambda\mathbf{i}$ and $T-\lambda\mathbf{i}$ have dense ranges. □

5.7

Our next goal is to prove Thm. 5.20, an extremely important criterion for strong commutativity. It will imply that two energy-bounded fields smeared over disjointly supported smooth functions are strongly commuting if one of them is linearly energy-bounded and its smeared field is symmetric.

Let T be a closable operator on $\mathcal{H}$ with **invariant domain** $\mathcal{H}^\infty$ (i.e. $\mathscr{D}(T)=\mathcal{H}^\infty$ and $T\mathcal{H}^\infty\subset\mathcal{H}^\infty$). Note that $H^n\mathscr{D}(H^m)\subset\mathscr{D}(H^{n+m})$ shows that $\mathcal{H}^\infty$ is H^n-invariant for every $n\in\mathbb{Z}$.

Definition 5.11. We say that T satisfies ***H*-bounds of order** r (where $r\geqslant 0$) if for each $n\in\mathbb{N}$ there exists a constant $|T|_n\geqslant 0$ (the ***n*-th bounding constant**) such that for every $\xi\in\mathcal{H}^\infty$ we have

$$\|H^nT\xi\|\leqslant |T|_n\cdot\|H^{n+r}\xi\|. \tag{5.6}$$

In other words, if for each $s \in \mathbb{R}$ we define

$$\langle\xi|\eta\rangle_r = \langle H^s\xi|H^s\eta\rangle \qquad \text{for all } \xi,\eta \in \mathscr{D}(H^s)$$

then $T : (\mathcal{H}^\infty, \|\cdot\|_{n+r}) \to (\mathcal{H}^\infty, \|\cdot\|_n)$ is a bounded linear map. H-bounds of order 1 are called **linear H-bounds**.

Clearly, if $0 \leqslant r_1 \leqslant r_2$, then H-bounds of order r_1 implies H-bounds of order r_2.

Lemma 5.12. *Assume T is closable with invariant domain $\mathcal{H}^\infty$. Assume also that T satisfies linear H-bounds. Then for every $n \in \mathbb{Z}$ there exists a bounding constant $|T|_n \geqslant 0$ such that for every $\xi \in \mathcal{H}^\infty$ we have*

$$\|H^nT\xi\| \leqslant |T|_n \cdot \|H^{n+1}\xi\|$$

Proof. We know this is true when $n \geqslant 0$. Now assume $n < 0$ and let $m = -n$. Then for every $\xi,\eta \in \mathcal{H}^\infty$,

$$\begin{aligned}&\big|\langle H^{-m}T\xi|H^m\eta\rangle\big| = \big|\langle\xi|T\eta\rangle\big| = \big|\langle H^{-m+1}\xi|H^{m-1}T\eta\rangle\big|\\ \leqslant&\|H^{-m+1}\xi\|\cdot|T|_{m-1}\|H^m\eta\|.\end{aligned}$$

Since $H^m\mathcal{H}^\infty$ is dense (because $\mathcal{H}^\infty$ is a core for H^m, and $\mathrm{Rng}(H^m)$ is dense), we conclude that $\|H^{-m}T\xi\| \leqslant |T|_{m-1}\|H^{-m+1}\xi\|$. □

5.8

The following theorem is [Tol99, Prop. 2.1].

Theorem 5.13. *Assume T is a symmetric operator on $\mathcal{H}$ with invariant domain $\mathcal{H}^\infty$. Assume that both T and $[H,T]$ satisfy linear H-bounds. Then $\overline{T}$ is self-adjoint. Moreover, for any $n \in \mathbb{N}$, $\mathscr{D}(H^n)$ is $e^{\mathbf{i}t\overline{T}}$-invariant, and there exists a constant $C_n \geqslant 0$ such that for every $\xi \in \mathscr{D}(H^n)$ and $t \in \mathbb{R}$ we have*

$$\|H^ne^{\mathbf{i}t\overline{T}}\xi\| \leqslant e^{C_n|t|} \cdot \|H^n\xi\| \tag{5.7}$$

In short, our conclusion is that $\mathcal{H}^\infty$ is $e^{\mathbf{i}t\overline{T}}$-invariant, and $e^{\mathbf{i}t\overline{T}} : (\mathcal{H}^\infty, \langle\cdot|\cdot\rangle_n) \to (\mathcal{H}^\infty, \langle\cdot|\cdot\rangle_n)$ is bounded with operator norm $\leqslant e^{C_n|t|}$.

Idea of the proof. We know from Thm. 5.10 that $\overline{T}$ is self-adjoint. Take $N = H^{2n}$. Using the fact that $[H,T]$ satisfies linear H-bounds, it is not hard to check that $[N,T]$ satisfies H-bounds of order $2n$, and hence satisfies linear N-bounds. From this, one shows that $\mathbf{i}[N,T] \leqslant cN$ for $c > 0$, namely, $\mathbf{i}\langle[N,T]\xi|\xi\rangle \leqslant c\langle N\xi|\xi\rangle$ where $\xi \in \mathcal{H}^\infty$.

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