Qiuzhen Lectures on Functional Analysis

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1 Preliminaries

1.1 Notation

In this monograph, unless otherwise stated, we understand the field $\mathbb{F}$ as either $\mathbb{R}$ or $\mathbb{C}$.

We use frequently the abbreviations:

iff=if and only if

LHS=left hand side   RHS=right hand side

$\exists$=there exists $\forall$=for all

i.e.=id est=that is=namely   e.g.=for example

cf.=compare/check/see/you are referred to

resp.=respectively   WLOG=without loss of generality

LCH=locally compact Hausdorff

MCT=monotone convergence theorem

DCT=dominated convergence theorem

When we write $A := B$ or $A \xlongequal{\text{def}} B$, we mean that A is defined by the expression B. When we write $A \equiv B$, we mean that A are B are different symbols of the same object.

Unless otherwise stated, an inner product space V denotes a complex inner product space, and its sesquilinear form $\langle\cdot|\cdot\rangle$ is linear on the right argument $|\cdot\rangle$ and antilinear on the left argument $\langle\cdot|$. Note that this convention is different from that of [Gui-A], where the right variable is antilinear.

If V is an $\mathbb{F}$-vector space, then for each $v \in V$ and each linear map $\varphi : V \to \mathbb{F}$, we write

$$\langle v, \varphi\rangle = \langle\varphi, v\rangle := \varphi(v)$$

We assume $a\cdot(+\infty) = (+\infty)\cdot a = +\infty$ if $a \in (0, +\infty]$, and $0\cdot(+\infty) = (+\infty)\cdot 0 = 0$. An increasing function/sequence/net means a non-decreasing one.

- Unless otherwise specified, completeness of a metric space or normed vector space refers to Cauchy completeness.
- $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{1, 2, \dots\}$.
- $\mathbb{R}_{\geqslant 0} = [0, +\infty)$, $\overline{\mathbb{R}}_{\geqslant 0} = [0, +\infty]$, $\overline{\mathbb{R}} = [-\infty, +\infty]$. We equip $\overline{\mathbb{R}}$ with the topology generated by all $(a, b), (a, +\infty], [-\infty, b)$ where $a, b \in \mathbb{R}$. Clearly, any strictly increasing function $[0, 1] \to \overline{\mathbb{R}}$ is a homeomoephism. Therefore, $\overline{\mathbb{R}}$ is a compact Hausdorff space.

- An **interval** denotes a connected subset of $\overline{\mathbb{R}}$. A **proper interval** denotes an interval with non-zero Lebesgue measure.

- Y^X is the set of functions with domain X and codomain Y.

- 2^X is the set of subsets of X.

- $\mathrm{fin}(2^X)$ is the set of finite subsets of X.

- For each vector space V, we let $V[x_1,\dots,x_k]$ be the space of polynomials of the (mutually commuting) abstract variables $x_1,\dots,x_k$ with coefficients in V. Therefore, its elements are of the form
$$\sum_{n_1,\dots,n_k\in E} v_{n_1,\dots,n_k} x_1^{n_1}\cdots x_k^{n_k} \qquad \text{where } E\in \mathrm{fin}(2^{\mathbb{N}}) \text{ and } v_{n_1,\dots,n_k}\in V$$

- If $f:X\to Y$ is a map, then
$$\mathrm{Rng}(f)=f(X)$$
If X,Y are vector spaces and f is linear, then
$$\mathrm{Ker}(f)=f^{-1}(0)$$

- If V is a vector space and X is a set, then V^X is viewed as a vector space whose linear structure is defined by
$$(af+bg)(x)=af(x)+bg(x) \qquad \text{for all } f,g\in V^X \text{ and } a,b\in\mathbb{F}$$

- If X is a set and $A\subset X$, the **characteristic function** is
$$\chi_A: X\to\{0,1\} \qquad x\mapsto \begin{cases} 1 & \text{if } x\in A\\ 0 & \text{if } x\in X\backslash A\end{cases}$$

- $\mathrm{Cl}_X(A)$, also denoted by $\mathrm{Cl}(A)$ or $\overline{A}$, is the closure of $A\subset X$ with respect to the topological space X.

- If X is a metric space and $p\in X, r\in[0,+\infty]$, we let
$$B_X(p,r)=\{x\in X: d(x,p)<r\} \qquad \overline{B}_X(p,r)=\{x\in X: d(x,p)\leqslant r\}$$
For each $E\subset X$, we define the **diameter**
$$\mathrm{diam}(E)=\sup\{d(x,y): x,y\in E\}$$

- If X is a topological space, then $\mathcal{T}_X$ denotes the topology of X, i.e.,

$$\mathcal{T}_X = \{\text{open subsets of } X\}$$

If $x \in X$, a **neighborhood** of x denotes an *open* subset of X containing x. We let

$$\mathrm{Nbh}_X(x) \equiv \mathrm{Nbh}(x) := \{\text{neighborhoods of } x \text{ in } X\}$$

- If X, Y are topological spaces, then

$$C(X,Y) = \{f \in Y^X : f \text{ is continuous}\}$$
$$\mathfrak{B}_X = \text{the Borel } \sigma\text{-algebra of } X$$
$$\mathscr{Bor}(X,Y) = \{f \in Y^X : f \text{ is Borel}\}$$

- m^n, as a measure, denotes the Lebesgue measure on $\mathbb{R}^n$, and is abbreviated to m when no confusion arises.

- $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} \simeq \mathbb{R}/2\pi\mathbb{Z}$. If f is a function on $\mathbb{S}^1$, equivalently, a 2π-periodic function on $\mathbb{R}$, then

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathbf{i}nx}dm(x)$$

is its n-th Fourier coefficient (whenever the integral can be defined).

- $(X, \mathfrak{M}, \mu)$, often abbreviated to (X, μ), denotes a measure space where $\mathfrak{M}$ is the σ-algebra and $\mu : \mathfrak{M} \to \overline{\mathbb{R}}_{\geqslant 0}$ is the measure.

- Let V be a normed vector space. Let X is either a set or a topological space, depending on the context. Let $1 \leqslant p < +\infty$. For each $f \in V^X$,

$$\mathrm{Supp}_X(f) \equiv \mathrm{Supp}(f) = \mathrm{Cl}_X(\{x \in X : f(x) \neq 0\})$$
$$\|f\|_{l^\infty(X,V)} = \|f\|_{l^\infty} = \sup_{x\in X}\|f(x)\|$$
$$\|f\|_{l^p(X,V)} = \|f\|_{l^p} = \Big(\sum_{x\in X}\|f(x)\|^p\Big)^{\frac{1}{p}}$$
$$|f| \text{ is the function } X \to \mathbb{R}_{\geqslant 0} \text{ such that } |f|(x) = \|f(x)\|$$

We call $|f|$ the **absolute value function** of f. For each $E \subset V$, we let

$$C_c(X,E) = \{f \in C(X,E) : \mathrm{Supp}(f) \text{ is compact in } X\}$$
$$l^\infty(X,V) = \{f \in V^X : \|f\|_\infty < +\infty\}$$
$$l^p(X,V) = \{f \in V^X : \|f\|_p < +\infty\}$$
$$\mathscr{Bor}^{\mathrm{b}}(X,V) = \mathscr{Bor}(X,V) \cap l^\infty(X,V) = \{f \in V^X : f \text{ is Borel and bounded}\}$$

We are particularly interested in the case that $E = V$, $E = [0,1]$, and $E = \mathbb{R}_{\geqslant 0}$.

- Let V be a normed vector space. Let X be a set. We say that a family $(f_\alpha)_{\alpha\in\mathscr{A}}$ in V^X is **uniformly bounded** if $\sup_{\alpha\in\mathscr{A}}\|f_\alpha\|_{l^\infty(X,V)}<+\infty$.

- If X is LCH and V is a normed $\mathbb{F}$-vector space, we understand $C_c(X,V)$ as a normed $\mathbb{F}$-vector space whose linear structure inherits from that of V^X, and whose norm is chosen to be the l^∞-norm.

- If $(X,\mathfrak{M})$ and $(Y,\mathfrak{N})$ are measurable spaces, then

$$\mathcal{L}(X,Y)=\{\text{measurable functions } X\rightarrow Y\}$$

If V is a normed vector space, for each $f\in\mathcal{L}(X,V)$ and $1\leqslant p<+\infty$, we let

$$\|f\|_{L^p(X,\mu)}=\|f\|_{L^p}=\Big(\int_X|f|^pd\mu\Big)^{\frac{1}{p}}$$
$$\|f\|_{L^\infty(X,\mu)}=\|f\|_{L^\infty}=\inf\{\lambda\in\overline{\mathbb{R}}_{\geqslant 0}:\mu\{x\in X:\|f(x)\|>a\}=0\}$$

which are potentially infinite.

- In the notation of function spaces, the codomain is understood to be $\mathbb{C}$ when it is suppressed. For example,

$$C_c(X)=C_c(X,\mathbb{C})\qquad \mathscr{Bor}(X)=\mathscr{Bor}(X,\mathbb{C})\qquad L^p(X,\mu)=L^p(X,\mu,\mathbb{C})$$

However, this convention does not apply to $\mathfrak{L}(V)$: If V is a normed vector space, then $\mathfrak{L}(V)$ denotes $\mathfrak{L}(V,V)$, the space of bounded linear operators on V.

1.2 Nets

In this section, we present results on nets that are essential for the topics covered in this course. For more details, see [Gui-A].

1.2.1 Basic definitions

Definition 1.2.1. A relation $\leqslant$ on a set I is called a **preorder** if for all $\alpha,\beta,\gamma\in I$, the following are satisfied:

- (Reflexivity) $\alpha\leqslant\alpha$.

- (Transitivity) If $\alpha\leqslant\beta$ and $\beta\leqslant\gamma$ then $a\leqslant\gamma$.

The pair $(I,\leqslant)$ (or simply I) is called a **preordered set**. For each $\beta\in I$, we write

$$I_{\geqslant\beta}=\{\alpha\in I:\alpha\geqslant\beta\}\tag{1.1}$$

Definition 1.2.2. A preordered set $(I, \leqslant)$ is called a **directed set** if

$$\forall \alpha, \beta \in I \quad \exists \gamma \in I \quad \text{such that } \alpha \leqslant \gamma, \beta \leqslant \gamma \tag{1.2}$$

If I is a directed set and X is a set, then a function $x : I \to X$ is called a **net** with directed set/index set I. We often write $x(\alpha)$ as x_α if $\alpha \in I$, and write x as $(x_\alpha)_{\alpha \in I}$.

Unless otherwise stated, for any net $(x_\alpha)_{\alpha \in I}$ we assume that $I \neq \emptyset$. □

Example 1.2.3. $(\mathbb{Z}_+, \leqslant)$ is a directed set. A net with index set $\mathbb{Z}_+$ in a set X is precisely a sequence in X.

Example 1.2.4. Let X be a topological space and $x \in X$. Then $\mathrm{Nbh}_X(x)$, together with the preorder $\supset$ (that is, $U \leqslant V$ iff $U \supset V$), is a directed set. Unless otherwise stated, the preorder on $\mathrm{Nbh}_X(x)$ is always chosen to be $\supset$.

Definition 1.2.5. Suppose that $(I, \leqslant_I)$ and $(J, \leqslant_J)$ are preordered set (resp. directed set), then the **product** $(I \times J, \leqslant)$ is a preordered set (resp. directed set) if for every $\alpha, \alpha' \in I, \beta, \beta' \in J$ we define

$$(\alpha, \beta) \leqslant (\alpha', \beta') \qquad \Longleftrightarrow \qquad \alpha \leqslant_I \alpha' \text{ and } \beta \leqslant_J \beta' \tag{1.3}$$

Unless otherwise stated, the preorder on $I \times J$ is assumed to be defined by (1.3).

Definition 1.2.6. If X is a set, then $(2^X, \subset)$ and $(\mathrm{fin}(2^X), \subset)$ are directed sets where

$$\mathrm{fin}(2^X) = \{A \subset X : A \text{ is a finite set}\} \tag{1.4}$$

We will use nets with index set $\mathrm{fin}(2^X)$ to study infinite sums.

Definition 1.2.7. Let P be a property about elements of a set X, i.e., P is a function $X \to \{\text{true}, \text{false}\}$. Let $(x_\alpha)_{\alpha \in I}$ be a net in X.

We say that x_α **eventually** satisfies P (equivalently, we say that x_α satisfies P for **sufficiently large** α) if:

- There exists $\beta \in I$ such that for every $\alpha \in I_{\geqslant \beta}$, the element x_α satisfies P.

"Sufficiently large" is also called "**large enough**".

We say that x_α **frequently** satisfies P if:

- For every $\beta \in I$ there exists $\alpha \in I_{\geqslant \beta}$ such that x_α satisfies P.

□

Remark 1.2.8. Let P and Q be two properties about elements of X. Then

$$\neg(x_\alpha \text{ eventually satisfies } P) \quad = \quad (x_\alpha \text{ frequently satisfies } \neg P)$$

By the crucial condition (1.2) for directed sets, we have

$$\begin{gathered}(x_\alpha \text{ eventually satisfies } P) \wedge (x_\alpha \text{ eventually satisfies } Q)\\ \Downarrow \\ x_\alpha \text{ eventually satisfies } P \wedge Q\end{gathered} \tag{1.5a}$$

We will use (1.5a) very frequently without explicitly mentioning it. Clearly, we also have

$$\begin{gathered}(x_\alpha \text{ eventually satisfies } P) \wedge (x_\alpha \text{ frequently satisfies } Q)\\ \Downarrow \\ x_\alpha \text{ frequently satisfies } P \wedge Q\end{gathered} \tag{1.5b}$$

1.2.2 Nets and topological spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Definition 1.2.9. Let $(x_\alpha)_{\alpha\in I}$ be a net in X. Let $x \in X$. We say that (x_α) **converges to** x and write

$$\lim_{\alpha\in I} x_\alpha \equiv \lim_{\alpha} x_\alpha = x$$

or simply $x_\alpha \to x$ if the following statement holds:

- For every $U \in \mathrm{Nbh}_X(x)$, the net (x_α) is eventually in U.

Convention 1.2.10. If I is a directed set, and if $(x_{\alpha_1,\dots,\alpha_n})_{(\alpha_1,\dots,\alpha_n)\in I^n}$ is a net in X (where I^n is equipped with the product preorder, cf. Def. 1.2.5), we

$$\text{abbreviate} \qquad \lim_{(\alpha_1,\dots,\alpha_n)\in I^n} x_{\alpha_1,\dots,\alpha_n} \qquad \text{to} \qquad \lim_{\alpha_1,\dots,\alpha_n\in I} x_{\alpha_1,\dots,\alpha_n}$$

Example 1.2.11. Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in a metric space. Then $(x_n)_{n\in\mathbb{Z}_+}$ is a Cauchy sequence iff $\lim_{m,n\in\mathbb{Z}_+} d(x_m, x_n) = 0$.

Proposition 1.2.12. *Let $(x_\alpha)_{\alpha\in I}$ be an increasing (resp. decreasing) net in $\overline{\mathbb{R}}$. Let x be the supremum (resp. infimum) of $\{x_\alpha : \alpha \in I\}$. Then $\lim_\alpha x_\alpha = x$.*

Proof. We address the case that (x_α) is increasing; the other case is similar. Let $S = \{x_\alpha : \alpha \in I\}$. Since $x = \sup S$, for each interval U open in $\overline{\mathbb{R}}$ and containing x, we have $S \cap U \neq \emptyset$, and hence there exists $\alpha \in I$ such that $x_\alpha \in U$. Since the net is increasing, we have $x_\alpha \leqslant x_\beta \leqslant x$ for each $\beta \geqslant \alpha$. Since U is an interval containing x_α, x, it must contain x_β for all $\beta \geqslant \alpha$. This proves that $\lim_\alpha x_\alpha = x$.. □

Definition 1.2.13. A net $(x_\alpha)_{\alpha\in I}$ in a metric space X is called a **Cauchy net** if $\lim_{\alpha,\beta\in I} d(x_\alpha, x_\beta) = 0$.

Proposition 1.2.14. *Let (x_α) be a convergent net in a metric space X. Then (x_α) is a Cauchy net.*

Proof. Let x be the limit of (x_α). Then for each $\varepsilon > 0$, there exists $\gamma \in I$ such that $d(x, x_\alpha) < \varepsilon$ for all $\alpha \geqslant \gamma$. Therefore, for all $\alpha, \beta \geqslant \gamma$ we have $d(x_\alpha, x_\beta) \leqslant d(x_\alpha, x) + d(x, x_\beta) < 2\varepsilon$. $\square$

Recall that a metric space X is called **(Cauchy) complete** if each Cauchy sequence in X converges.

Theorem 1.2.15. *Suppose that X is a complete metric space. Then every Cauchy net in X converges.*

Proof. Let $(x_\alpha)_{\alpha \in I}$ be a Cauchy net in X. Then for each $n \in \mathbb{Z}_+$ there exists $\gamma_n \in I$ such that $d(x_\alpha, x_\beta) < 1/n$ for all $\alpha, \beta \geqslant \gamma_n$. Since I is directed, by successively replacing $\gamma_1, \gamma_2, \dots$ with larger ones, we may assume that $(\gamma_n)_{n \in \mathbb{Z}_+}$ is increasing. Thus $d(x_{\gamma_m}, x_{\gamma_n}) < 1/m$ whenever $m \leqslant n$, and hence $(x_{\gamma_n})_{n \in \mathbb{Z}_+}$ is a Cauchy sequence in X. Since X is complete, $(x_{\gamma_n})_{n \in \mathbb{Z}_+}$ converges to some $x \in X$.

Now, for each n, and for each $\alpha \geqslant \gamma_n$ and $k \geqslant n$, we have $d(x_\alpha, x_{\gamma_k}) < 1/n$. Since $\lim_k d(x_{\gamma_k}, x) = 0$, we may find $k \geqslant n$ such that $d(x_{\gamma_k}, x) < 1/n$. Therefore $d(x_\alpha, x) < 2/n$ for all $\alpha \geqslant \gamma_n$. This proves $\lim_\alpha x_\alpha = x$. $\square$

Proposition 1.2.16. *X is Hausdorff iff every net in X has at most one limit.*

Proof. First, assume that X is not Hausdorff. Then there exist $x \neq y$ in X such that every neighborhood of x intersects every neighborhood of y. Let $I = \mathrm{Nbh}_X(x) \times \mathrm{Nbh}_X(y)$. For each $\alpha = (U, V) \in I$, by assumption, there exists $x_\alpha \in U \cap V$. Then $(x_\alpha)_{\alpha \in I}$ is a net in X converging to both x and y.

Conversely, assume that X has a net (x_α) converging to distinct points $x, y \in X$. Then for each $U \in \mathrm{Nbh}_X(x)$, (x_α) is eventually in X. Similarly, for each $V \in \mathrm{Nbh}_X(y)$, (x_α) is eventually in Y. By Rem. 1.2.8, (x_α) is eventually in $U \cap V$. In particular, there exists α such that $x_\alpha \in U \cap V$. So $U \cap V \neq \emptyset$. Thus, X is not Hausdorff. $\square$

Proposition 1.2.17. *Let $A \subset X$ and $x \in X$. Then $x \in \mathrm{Cl}_X(A)$ iff there is a net (x_α) in A whose limit is x.*

Proof. Suppose that $x \in \overline{A}$. Then each $U \in \mathrm{Nbh}_X(x)$ intersects A, and hence there exists $x_U \in U \cap A$. Then $(x_U)_{U \in \mathrm{Nbh}_X(x)}$ is a net in A converging to x.

Conversely, assume that $x \notin \overline{A}$. Then there exists $U \in \mathrm{Nbh}_X(x)$ disjoint from A. Therefore, any net (x_α) in A is never in U, and hence does not converge to x. $\square$

Theorem 1.2.18. *Let $f : X \to Y$ be a map. Let $x \in X$. Then the following are equivalent:*

(1) *f is continuous at x, that is, for each $V \in \mathrm{Nbh}_Y(f(x))$, there exists $U \in \mathrm{Nbh}_X(x)$ that is contained in $f^{-1}(V)$.*

(2) For each net (x_α) in X converging to x, the net $f(x_\alpha)$ converges to $f(x)$.

(3) For each net (x_α) in $X\backslash\{x\}$ converging to x, the net $f(x_\alpha)$ converges to $f(x)$.

Proof. (1)⇒(2): Assume (1). Let (x_α) be a net in X converging to x. Then for any $V \in \mathrm{Nbh}(f(x))$, by choosing $U \in \mathrm{Nbh}(x)$ contained in $f^{-1}(V)$, we have that (x_α) is eventually in U, and hence $(f(x_\alpha))$ is eventually in V. Therefore $f(x_\alpha) \to f(x)$. This proves (2).

(2)⇒(3): Obvious.

(1)⇒(3): Note that f is continuous at any discrete point, i.e., any point x such that $\{x\}$ is open in X. Therefore, it suffices to assume that $\{x\}$ is not open. Let us assume that (1) is not true, and prove that (3) is not true.

Since (1) is not true, there exists $V \in \mathrm{Nbh}(f(x))$ such that any $U \in \mathrm{Nbh}(x)$ is not contained in $f^{-1}(V)$, and hence $U\backslash\{x\}$ is not contained in $f^{-1}(V)$. Since $\{x\}$ is not open, the set $U\backslash\{x\}$ is not empty. Therefore, we can choose $x_U \in U\backslash\{x\}$ such that $f(x_U) \notin V$. It follows that $(x_U)_{U\in\mathrm{Nbh}_X(x)}$ is a net in $X\backslash\{x\}$ converging to x that is always outside V. Thus $f(x)$ is not a limit of $(f(x_U))_{U\in\mathrm{Nbh}_X(x)}$. Therefore, (3) is not true. □

Corollary 1.2.19. *Let $f : X \to Y$ be a bijection. Then the following are equivalent.*

(1) f is a homeomorphism.

(2) For each $x \in X$ and each net (x_α) in X, we have $\lim_\alpha x_\alpha = x$ iff $\lim_\alpha f(x_\alpha) = f(x)$.

Proof. This follows immediately from Thm. 1.2.18. □

Corollary 1.2.20. *Let $\mathcal{T}_X$ and $\mathcal{T}'_X$ be two topologies on X. Then the following are equivalent.*

(1) $\mathcal{T}_X = \mathcal{T}'_X$.

(2) For each $x \in X$ and each net (x_α) in X, the net (x_α) converges to x under $\mathcal{T}_X$ iff (x_α) converges to x under $\mathcal{T}'_X$.

In other words, topologies are determined by net convergence.

Proof. Apply Cor. 1.2.19 to the identity map of X. □

Definition 1.2.21. Recall that a topological space X is called **first-countable** if for each $x \in X$, there exists a sequence $(U_n)_{n\in\mathbb{Z}_+}$ in $\mathrm{Nbh}_X(x)$ such that $\{U_n : n \in \mathbb{Z}_+\}$ is cofinal in $\mathrm{Nbh}_X(x)$ (i.e., for each $V \in \mathrm{Nbh}_X(x)$ there exists n such that $U_n \subset V$). Moreover, once we have such $(U_n)_{n\in\mathbb{Z}_+}$, by replacing U_n with $U_1 \cap \cdots \cap U_n$, we may assume that $U_1 \supset U_2 \supset \cdots$. Therefore, X being first-countable means that for each $x \in X$, the net $(U)_{U\in\mathrm{Nbh}_X(x)}$ has a subnet that is also a sequence.

Remark 1.2.22. In Prop. 1.2.16, Prop. 1.2.17, Thm. 1.2.18, Cor. 1.2.19, and Cor. 1.2.20, if all the topologies involved are assumed to be first-countable, then the statements remain valid when nets are replaced by sequences. We leave the proof to the readers.

For example, Cor. 1.2.20 can be modified as follows: If $\mathcal{T}_X$ and $\mathcal{T}'_X$ are two first-countable topologies on X, then $\mathcal{T}_X = \mathcal{T}'_X$ iff any sequence (x_n) converges to x under $\mathcal{T}_X$ iff (x_n) converges to x under $\mathcal{T}'_X$.

Note that in Thm. 1.2.18, only the domain X needs to be first-countable; the codomain Y does not. □

1.2.3 Subnets

Definition 1.2.23. A subset E of a directed set I is called **cofinal** if:

$$\forall \alpha \in I \quad \exists \beta \in E \quad \text{such that } \alpha \leqslant \beta$$

By the transitivity in Def. 2.3.6 and property (2.24), we clearly have

$$\forall \alpha_1, \dots, \alpha_n \in I \quad \exists \beta \in E \quad \text{such that } \alpha_1 \leqslant \beta, \dots, \alpha_n \leqslant \beta$$

Definition 1.2.24. Let $(x_\alpha)_{\alpha\in I}$ be a net in a set X. A **subnet** of $(x_\alpha)_{\alpha\in I}$ is, by definition, of the form $(x_{\alpha_s})_{s\in S}$ where S is a directed set, and

$$(\alpha_s)_{s\in S} : S \to I \qquad s \mapsto \alpha_s$$

is an increasing function satisfying one of the following (clearly) equivalent conditions:

(a) The range $\{\alpha_s : s \in S\}$ is cofinal in I.

(b) For each $\beta \in I$, the net $(\alpha_s)_{s\in S}$ is eventually $\geqslant \beta$.

Example 1.2.25. A subsequence of a sequence is a subnet of that sequence.

Proposition 1.2.26. *Let $(x_\alpha)_{\alpha\in I}$ be a net in a topological space X converging to $x \in X$. Then every subnet $(x_{\alpha_s})_{s\in S}$ converges to x.*

Proof. Choose any $U \in \mathrm{Nbh}(x)$. Since $x_\alpha \to x$, there exists $\beta \in I$ such that for all $\alpha \geqslant \beta$ we have $x_\alpha \in U$. Since α_t is eventually $\geqslant \beta$, we see that x_{α_t} is eventually in U. □

Note that a subnet does not necessarily have the same index set as the original net. This definition of subnets is motivated largely by the following important property, which will play a crucial role in Sec. 1.3.1.

Theorem 1.2.27. *Let $(x_\alpha)_{\alpha\in I}$ be a net in a topological space X. Let $x \in X$. Then the following are equivalent.*

(1) $(x_\alpha)_{\alpha\in I}$ has a subnet converging to x.

(2) For every $U \in \mathrm{Nbh}_X(x)$, the net (x_α) is frequently in U.

(3) x belongs to $\bigcap_{\alpha\in I} \overline{\{x_\beta : \beta \geqslant \alpha\}}$.

*Any $x \in X$ satisfying one of these three conditions is called a **cluster point** of $(x_\alpha)_{\alpha\in I}$.*

Proof. (2)⇔(3): (3) holds iff x belongs to $\overline{\{x_\beta : \beta \geqslant \alpha\}}$ for each α, iff each $U \in \mathrm{Nbh}(x)$ intersects $\{x_\beta : \beta \geqslant \alpha\}$ for each α, iff for each $U \in \mathrm{Nbh}(x)$ and each α, there exists $\beta \geqslant \alpha$ such that $x_\beta \in U$, iff (2) holds.

(1)⇒(2): Let (x_{α_s}) be a subnet converging to x. Then for each $U \in \mathrm{Nbh}_X(x)$ and $\beta \in I$, the net (x_{α_s}) is eventually in U, and α_s is eventually $\geqslant \beta$. Therefore, by Rem. 1.2.8, it is eventually true that x_{α_s} is in U and, simultaneously, $\alpha_s \geqslant \beta$. In particular, there exists s such that $x_{\alpha_s} \in U$ and $\alpha_s \geqslant \beta$. This proves that (x_α) is frequently in U.

(2)⇒(1): Assume (2). Define a preordered set $(J, \leqslant)$ by

$$\begin{gathered} J = \{(\alpha, U) \in I \times \mathrm{Nbh}_X(x) : x_\alpha \in U\} \\ (\alpha, U) \leqslant (\alpha', U') \qquad \Longleftrightarrow \qquad \alpha \leqslant \alpha' \text{ and } U \supset U' \end{gathered} \tag{1.6}$$

Let us prove that J is directed: Suppose that (α_1, U_1) and (α_2, U_2) belong to J. Since $(\alpha)_{\alpha\in I}$ is eventually $\geqslant \alpha_1$ and eventually $\geqslant \alpha_2$, and since (by (2)) (x_α) is frequently in U, by Rem. 1.2.8, it is frequently true that $\alpha \geqslant \alpha_1, \alpha_2$ and $x_\alpha \in U_1 \cap U_2$. Choose $\alpha \in I_{\geqslant \alpha_1} \cap I_{\geqslant \alpha_2}$ such that $x_\alpha \in U_1 \cap U_2$. Then $(\alpha, U_1 \cap U_2)$ belongs to J and is $\geqslant (\alpha_1, U_1)$ and $\geqslant (\alpha_2, U_2)$. This proves that J is directed.

The map $(\alpha, U) \in J \mapsto \alpha \in I$ is clearly increasing; its range is cofinal, since $(\alpha, X) \in J$ for each $\alpha \in I$. Therefore, $(x_{\alpha,U})_{(\alpha,U)\in J}$ is a subnet of (x_α), which clearly converges to x. This proves (1). □

Theorem 1.2.28. *Assume that X is first-countable, and let (x_n) be a sequence in X. Let $x \in X$. Then x is a cluster point of (x_n) iff (x_n) has a subsequence converging to x.*

Proof. We leave the proof to the reader as an exercise. □

1.2.4 First-countable nets

Definition 1.2.29. Let $(I, \leqslant)$ be a directed set. Let ∞_I (often abbreviated to ∞) be a new symbol not in I. Then

$$I^* = I \cup \{\infty_I\}$$

is also a directed set if we extend the preorder $\leqslant$ of I to I^* by setting

$$\alpha \leqslant \infty_I \qquad (\forall \alpha \in I^*)$$

For each $\alpha \in I$, let

$$I^*_{\geqslant \alpha} = \{\beta \in I^* : \beta \geqslant \alpha\}$$

The **standard topology** on I^* is defined to be the one induced by the basis

$$\mathcal{B} = \big\{\{\alpha\} : \alpha \in I\big\} \cup \big\{I^*_{\geqslant \alpha} : \alpha \in I\big\} \tag{1.7}$$

Remark 1.2.30. Suppose that $(x_\alpha)_{\alpha \in I}$ is a net in a topology space X. Let $x_\infty \in X$. Extend (x_α) to a function

$$x : I^* \to X \qquad \alpha \mapsto x_\alpha, \quad \infty \mapsto x_\infty$$

The following facts are easy to check:

1. x is continuous at every point of I.

2. x is continuous at ∞ iff the net $(x_\alpha)_{\alpha \in I}$ converges to x_∞.

In particular, $(x_\alpha)_{\alpha \in I}$ converges to x_∞ iff the function $x : I^* \to X$ is continuous.

Definition 1.2.31. A directed set I is called **first countable** if one of the following equivalent conditions holds:

(1) The standard topology on I^* is first-countable.

(2) I has a countable cofinal subset.

(3) The net $(\alpha)_{\alpha \in I}$ has a subnet which is also a sequence. (In other words, there is an increasing sequence in I converging to ∞.)

If $(x_\alpha)_{\alpha \in I}$ is a net in a set X such that the index set I is first-countable, we also say that $(x_\alpha)_{\alpha \in I}$ is a **first-countable net**.

Proof of equivalence. (1)$\Rightarrow$(2): Assume (1). Then $\mathrm{Nbh}_{I^*}(\infty)$ has a countable cofinal subset, which (due to (1.7)) can be chosen to be of the form $I^*_{\geqslant \alpha_1}, I^*_{\geqslant \alpha_2}, \dots$ where $(\alpha_n)_{n \in \mathbb{Z}_+}$ is a sequence in I. Since $(I^*_{\geqslant \alpha_n})_{n \in \mathbb{Z}_+}$ is cofinal in $\mathrm{Nbh}_{I^*}(\infty)$, for each $\beta \in I$ there exists n such that $I^*_{\geqslant \alpha_n} \subset I^*_{\geqslant \beta}$, and hence $\alpha_n \geqslant \beta$. This proves that $\{\alpha_n : n \in \mathbb{Z}_+\}$ is a countable cofinite subset of I. Hence (2) is proved.

(2)$\Rightarrow$(1): Assume (2). For every $\alpha \in I$, the direct set $\mathrm{Nbh}_{I^*}(\alpha)$ clearly has a countable cofinal subset, namely $\{\{\alpha\}\}$. Let (α_n) be a countable cofinal sequence in I. One checks easily that $(I^*_{\geqslant \alpha_n})_{n \in \mathbb{Z}_+}$ is a cofinal sequence in $\mathrm{Nbh}_{I^*}(\infty)$. This proves (1).

(3)$\Rightarrow$(2): Obvious.

(2)$\Rightarrow$(3): Let $(\alpha_n)_{n \in \mathbb{Z}_+}$ be a cofinal sequence in I. Since I is directed, by increasing $\alpha_2, \alpha_3, \dots$ successively, we can assume that $(\alpha_n)_{n \in \mathbb{Z}_+}$ is increasing. Clearly (α_n) converges to ∞. $\square$

Example 1.2.32. If X is a topological space, then X is first-countable iff $\mathrm{Nbh}_X(x)$ is a first-countable directed set for each $x \in X$.

Example 1.2.33. If I, J are first-countable directed sets, then the directed set $I \times J$ is also first-countable. In particular, double sequences (i.e., nets of the form $(x_{m,n})_{m,n\in\mathbb{Z}_+}$) are first-countable nets.

Theorem 1.2.34. *Let $(x_\alpha)_{\alpha\in I}$ be a first-countable net in a topological space X. Let $x_\infty \in X$. Then the following are equivalent.*

(1) (x_α) converges to x_∞.

(2) For each sequence $(\alpha_n)_{n\in\mathbb{Z}_+}$ in I converging to ∞, the sequence $(x_{\alpha_n})_{n\in\mathbb{Z}_+}$ converges to x_∞.

Proof. This follows immediately from Thm. 1.2.18 and Rem. 1.2.30. □

With the help of Thm. 1.2.34, one can easily generalize the MCT, the DCT, and Fatou's lemma from sequences to first-countable nets of functions. We leave the details of the proofs to the reader.

Theorem 1.2.35 (Monotone convergence theorem (MCT)). *Let (X, μ) be a measure space. Let (f_α) be an increasing first-countable net of measurable functions $X \to \overline{\mathbb{R}}_{\geqslant 0}$. Let $f : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be the pointwise limit of (f_α). Then f is measurable, and*

$$\lim_\alpha \int_X f_\alpha d\mu = \int_X f d\mu$$

Theorem 1.2.36 (Dominated convergence theorem (DCT)). *Let (X, μ) be a measure space. Let (f_α) be an increasing first-countable net of measurable functions $X \to \mathbb{C}$. Assume that (f_α) converges pointwise to a measurable function $f : X \to \mathbb{C}$. Suppose that there exists an integrable function $g : X \to \overline{\mathbb{R}}_{\geqslant 0}$ such that $|f_\alpha| \leqslant g$ for all α. Then f_α, f are integrable, and*

$$\lim_\alpha \int_X f_\alpha d\mu = \int_X f d\mu$$

Note that without assuming the monotonicity of (f_α), the measurability of the limit function f must be included as an assumption.

Theorem 1.2.37 (Fatou's lemma). *Let (X, μ) be a measure space. Let (f_α) be a first-countable net of measurable functions $X \to \overline{\mathbb{R}}_{\geqslant 0}$. Assume that (f_α) converges pointwise to a measurable function $f : X \to \overline{\mathbb{R}}_{\geqslant 0}$. Then*

$$\liminf_\alpha \int_X f_\alpha d\mu \geqslant \int_X f d\mu$$

See Def. 1.3.10 for the definition of $\liminf$.

1.2.5 Unordered sum

In this subsection, we fix a normed vector space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Fix a set X. Recall that $\mathrm{fin}(2^X)$ is a directed set, ordered by the inclusion ($A \leqslant B$ meaning $A \subset B$).

Definition 1.2.38. Let $f : X \to V$ be a map. The expression

$$\sum_{x \in X} f(x)$$

(or simply $\sum_X f$) is called an **unordered sum**. If $v \in V$, we say that $\sum_{x\in X} f(x)$ equals (or converges to) v, if

$$\lim_{A \in \mathrm{fin}(2^X)} \sum_{x \in A} f(x) = v \tag{1.8}$$

In this case, we write

$$\sum_{x \in X} f(x) = v \tag{1.9}$$

Unwinding the definition of net convergence, (1.8) says that for every $\varepsilon > 0$, there exists a finite set $B \subset X$ such that for every finite set A satisfying $B \subset A \subset X$, we have $\|v - \sum_{x\in A} f(x)\| < \varepsilon$.

Remark 1.2.39. If V is complete, then $\sum_X f$ converges precisely when the associated net $(\sum_A f)_{A \in \mathrm{fin}(2^X)}$ satisfies the Cauchy condition. Let us spell out what this Cauchy condition means:

(1) For every $\varepsilon > 0$, there exists a finite set $B \subset X$ such that for any finite sets A_1, A_2 satisfying $B \subset A_1 \subset X, B \subset A_2 \subset X$, we have

$$\Big\| \sum_{A_1 \backslash A_2} f - \sum_{A_2 \backslash A_1} f \Big\| < \varepsilon$$

Note that the term inside the norm is $\sum_{A_1} f - \sum_{A_2} f$. This is also equivalent to:

(2) For every $\varepsilon > 0$, there exists a finite set $B \subset X$ such that for any finite set $E \subset X \backslash B$, we have

$$\Big\| \sum_{E} f \Big\| < \varepsilon$$

In practice, we will mainly use (2) as the Cauchy criterion for the convergence of $\sum_X f$.

Proof of the equivalence. (2) follows from (1) by taking $A_1 = B$ and $A_2 = B \cup E$. (1) follows from (2) by taking $E_1 = A_1 \backslash A_2$ and $E_2 = A_2 \backslash A_1$ and then concluding $\|\sum_{E_1} f - \sum_{E_2} f\| < 2\varepsilon$. □

Definition 1.2.40. Let $g : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be a map. Note that the net $(\sum_A g)_{A \in \mathrm{fin}(2^X)}$ is increasing. Hence, its limit exists in $\overline{\mathbb{R}}$ and equals $\sup_{A \in \mathrm{fin}(2^X)} \sum_A g$ (by Prop. 1.2.12). We write this as $\sum_X g$, or more precisely:

$$\sum_X g \equiv \sum_{x \in X} g(x) \xlongequal{\mathrm{def}} \lim_{A \in \mathrm{fin}(2^X)} \sum_A g = \sup_{A \in \mathrm{fin}(2^X)} \sum_A g \tag{1.10}$$

We say that $\sum_X g$ **converges** or **converges absolutely**, if $\sum_X g < +\infty$.

It is clear that $\sum_X g < +\infty$ iff there exists $C \in \mathbb{R}_{\geqslant 0}$ such that $\sum_A g < C$ for all $A \in \mathrm{fin}(2^X)$.

Remark 1.2.41. Note that when $g : X \to \mathbb{R}_{\geqslant 0}$, the convergence in Def. 1.2.40 agrees with that in Def. 1.2.38. Therefore, Rem. 1.2.39 still gives a Cauchy criterion for convergence.

Definition 1.2.42. Let $f : X \to V$. We say that $\sum_X f$ **converges absolutely** if

$$\sum_{x \in X} \|f(x)\| < +\infty$$

Proposition 1.2.43. *Let $f : X \to V$, and assume that $\sum_X f$ converges absolutely. Then $\mathrm{Supp}(f) := \{x \in X : f(x) \neq 0\}$ is a countable set.*

Proof. For each $\varepsilon > 0$, let $A_\varepsilon = \{x \in X : |f(x)| \geqslant \varepsilon\}$. Then

$$\sum_X |f| \geqslant \sum_{A_\varepsilon} |f| \geqslant \varepsilon \sum_{A_\varepsilon} 1$$

So $\sum_{A_\varepsilon} 1 < +\infty$, and hence A_ε is a finite set. Since $\mathrm{Supp}(f) = \bigcup_{n \in \mathbb{Z}_+} A_{1/n}$, the set $\mathrm{Supp}(f)$ must be countable. □

Proposition 1.2.44. *Assume that V is complete. Let $f : X \to V$. If $\sum_X f$ converges absolutely, then it converges, and*

$$\Big\| \sum_{x \in X} f(x) \Big\| \leqslant \sum_{x \in X} \|f(x)\| \tag{1.11}$$

We write this simply as $\|\sum_X f\| \leqslant \sum_X |f|$.

Proof. (1.11) clearly holds when X is finite. In the general case, assume that $\sum_X f$ converges absolutely. Then by the Cauchy criterion, for every $\varepsilon > 0$ there is $A \in$

$\mathrm{fin}(2^X)$ such that for each finite $E \subset X\backslash A$ we have $\sum_E |f| < \varepsilon$, and hence $\|\sum_E f\| < \varepsilon$. Therefore $\sum_X f$ converges by Cauchy criterion again.

By the continuity of the norm function $v \in V \mapsto \|v\| \in \mathbb{R}_{\geqslant 0}$, we have

$$\Big\|\sum_X f\Big\| = \Big\|\lim_A \sum_A f\Big\| = \lim_A \Big\|\sum_A f\Big\|$$

Since $\|\sum_A f\| \leqslant \sum_A |f|$, by by Prop. 1.2.12, the above expression is no less than

$$\lim_A \sum_A |f| = \sum_X |f|$$

□

Example 1.2.45. Let $f : \mathbb{Z}_+ \to V$, and assume that $\sum_{\mathbb{Z}_+} f$ converges. Then

$$\sum_{\mathbb{Z}_+} f = \lim_{n\to\infty} (f(1) + \cdots + f(n))$$

Proof. By assumption, the net $(\sum_A f)_{A\in\mathrm{fin}(2^{\mathbb{Z}_+})}$ converges to $v := \sum_{\mathbb{Z}_+} f$. By Prop. 1.2.26, the subnet $(\sum_{\{1,\dots,n\}} f)_{n\in\mathbb{Z}_+}$ also converges to v. □

1.3 Nets and compactness

1.3.1 Compactness and cluster points

Proposition 1.3.1. *Let X be a topological space. Then the following are equivalent.*

(1) X is compact.

*(2) (**Increasing chain property**) If $(U_\mu)_{\mu\in I}$ is an increasing net of open subsets of X satisfying $\bigcup_{\mu\in I} U_\mu = X$, then $U_\mu = X$ for some μ.*

*(3) (**Decreasing chain property**) If $(E_\mu)_{\mu\in I}$ is a decreasing net of nonempty closed subsets of X, then $\bigcap_{\mu\in I} E_\mu \neq \emptyset$.*

Here, "increasing net" means $U_\mu \subset U_\nu$ if $\mu \leqslant \nu$, and "decreasing net" means the opposite.

Proof. (1)⇒(2): Assume (1). Then $X = \bigcup_\mu U_\mu$ is an open cover of X. So, by the compactness of X, we have $X = U_{\mu_1} \cup \cdots \cup U_{\mu_n}$ for some $\mu_1, \dots, \mu_n \in I$. Choose $\mu \in I$ which is $\geqslant \mu_1, \dots, \mu_n$. Then $X = U_\mu$.

(2)⇒(1): Assume (2). Let $X = \bigcup_{\alpha\in\mathscr{A}} W_\alpha$ be an open cover of X. Let $I = \mathrm{fin}(2^{\mathscr{A}})$. For each $\mu = \{\alpha_1, \dots, \alpha_n\} \in I$, let $U_\mu = W_{\alpha_1} \cup \cdots \cup W_{\alpha_n}$. Then $(U_\mu)_{\mu\in I}$ is an increasing net of open sets covering X. Thus, by (2), we have $U_\mu = X$ for some μ. This proves (1).

(2)⇔(3): If we let $E_\mu = X\backslash U_\mu$, then (2) says that if (E_μ) is a decreasing net of closed sets whose intersection is $\emptyset$, then $E_\mu = \emptyset$ for some μ. This is the contraposition of (3). □

Theorem 1.3.2. *Let X be a topological space. Then X is compact iff every net in X has at least one cluster point.*

Proof. Assume that X is compact. Let $(x_\alpha)_{\alpha\in I}$ be a net in X. Define F_α by

$$F_\alpha = \{x_\beta : \beta \in I, \beta \geqslant \alpha\} \tag{1.12}$$

Then $(\overline{F}_\alpha)_{\alpha\in I}$ is a decreasing net of nonempty closed subsets. So $\bigcap_\alpha \overline{F}_\alpha$ is nonempty by the decreasing chain property (cf. Prop. 1.3.1). By Thm. 1.2.27, $(\overline{F}_\alpha)_{\alpha\in I}$ is the set of cluster points of (x_α). Therefore, (x_α) has a cluster point.

Conversely, assume that every net of X has a cluster point. By Prop. 1.3.1, to prove that X is compact, it suffices to prove that X satisfies the decreasing chain property. Let $(E_\alpha)_{\alpha\in I}$ be a decreasing net of nonempty closed subsets of X. For each α we choose $x_\alpha \in E_\alpha$, which gives a net $(x_\alpha)_{\alpha\in I}$ in X. The fact that (E_α) is decreasing implies that $F_\alpha \subset E_\alpha$ if we let $F_\alpha =$ (1.12). Thus, the closure $\overline{F}_\alpha$ is a subset of E_α since E_α is closed.

By Thm. 1.2.27, $\bigcap_{\alpha\in I} \overline{F}_\alpha$ is the set of cluster points of (x_α), which is non-empty by assumption. Therefore, $\bigcap_\alpha E_\alpha$ is nonempty. □

Definition 1.3.3. Recall that X is called

- **separable** if X has a countable dense subset.
- **second countable** if the topology $\mathcal{T}_X$ has a countable basis.
- **Lindelöf** if every open cover of X has a countable subcover.

Example 1.3.4. Any subspace of a second countable space is second countable.

Proposition 1.3.5. *We have*

$$\textit{separable} \quad \Longleftarrow \quad \textit{second countable} \quad \Longrightarrow \quad \textit{Lindelöf}$$

Moreover, for first countable spaces, we have

$$\textit{separable} \quad \Longleftrightarrow \quad \textit{second countable}$$

Proof. Step 1. Suppose that X is second countable. Let $\{U_1, U_2, \dots\}$ be a countable basis of $\mathcal{T}_X$. We may assume WLOG that each U_n is nonempty. Choose $x_n \in U_n$. Then $\{x_n : n \in \mathbb{Z}_+\}$ is a countable dense subset of X. Therefore, X is separable. Let $\mathcal{W}$ be an open cover of X. For each n, let W_n be any member of $\mathcal{W}$ containing U_n; if no such W_n exists then we set $W_n = \emptyset$. By the fact that $\{U_1, U_2, \dots\}$ is a basis of $\mathcal{T}_X$, one easily shows that $\bigcup_n W_n = X$. Therefore, (W_n) is a countable subcover of $\mathcal{W}$. This proves that X is Lindelöf.

Step 2. Suppose that X is first countable and separable. Let $\{x_1, x_2, \dot{\}}$ be a dense subset of X. For each n, let $U_{n,1}, U_{n,2}, \dots$ be a cofinal sequence in $\mathrm{Nbh}_X(x_n)$. Then $(U_{n,k})_{n,k\in\mathbb{Z}_+}$ is a countable basis of $\mathcal{T}_X$. This proves that X is second countable. □

Definition 1.3.6. Recall that a topological space X is called **sequentially compact** if every sequence in X has a convergent subsequence. By Thm. 1.2.28, if X is first-countable (in particular, if X metrizable or second-countable), then X is sequentially compact iff every sequence in X has a cluster point.

Theorem 1.3.7. *Let X be a second-countable topological space. Then X is compact iff X is sequentially compact.*

Proof. Similar to the proof of Thm. 1.3.2, one shows that every sequence (x_n) has a cluster point, (i.e., $\bigcap_n \overline{x_k : k \geqslant n}$ is nonempty, cf. Thm. 1.2.27) iff the intersection of a decreasing *sequence* of non-empty closed subsets of X is non-empty. The latter condition is equivalent to that X is **countably compact**, that is, every countable open cover of X has a finite subcover. Since X is Lindelöf (Prop. 1.3.5), compactness and countable compactness are equivalent. □

Corollary 1.3.8. *Let X be a metric space. Then X is compact iff X is sequentially compact.*

Proof. By Thm. 1.3.7 and Prop. 1.3.5, it suffices to prove that X is separable. Note that for each $\varepsilon > 0$, there exists a finite set $E_\varepsilon \subset X$ such that the distance from any point of X to E_ε is $\leqslant \varepsilon$. (Otherwise, one can find a sequence $(x_n)_{n\in\mathbb{Z}_+}$ such that for each n, the distance from x_{n+1} to $\{x_1, \dots, x_n\}$ is $> 1/\varepsilon$. Then any subsequence of (x_n) is not a Cauchy sequence, and hence does not converge.) Then $\bigcup_{n\in\mathbb{Z}_+} E_{1/n}$ is a countable dense subset of X. □

1.3.2 lim inf and lim sup

Theorem 1.3.9. *Let (x_α) be a net in a compact Hausdorff space X, and let $x \in X$. The following are equivalent.*

(1) (x_α) converges to x.

(2) x is the only cluster point of (x_α).

In other words, (x_α) converges to x iff any convergent subnet converges to x.

Proof. (1)⇒(2): This is obvious even without assuming that the Hausdorff space X is compact.

¬(1)⇒¬(2): Assume that (1) is not true. Since (x_α) does not converge to x, there exists $U \in \mathrm{Nbh}_X(x)$ such that (x_α) is frequently not in U. Therefore, $J = \{\alpha \in I : x_\alpha \notin U\}$ is a cofinal subset of I. It follows that $(x_\alpha)_{\alpha\in J}$ is a subnet of $(x_\alpha)_{\alpha\in I}$ that is always outside U.

Since X is compact, by Thm. 1.3.2, $(x_\alpha)_{\alpha\in J}$ has a subnet $(x_\mu)_{\mu\in K}$ converging to some $y \in X$. Then $y \notin U$, since (x_μ) is always not in U. Therefore, $x \neq y$, and y is a cluster point of (x_α). Hence (2) is not true. □

Definition 1.3.10. Let $(x_\alpha)_{\alpha\in I}$ be a net in the compact Hausdorff space $\overline{\mathbb{R}}$. Let S be the set of cluster points of (x_α) in $\overline{\mathbb{R}}$. Recall that S is a closed subset by Thm. 1.2.27, and is non-empty by Thm. 1.3.2. Define

$$\liminf_{\alpha\in I} x_\alpha = \inf S \qquad \limsup_{\alpha\in I} x_\beta = \sup S \tag{1.13}$$

Since S is closed, $\liminf_{\alpha\in I} x_\alpha$ and $\limsup_{\alpha\in I} x_\beta$ are both cluster points of (x_α). Therefore, they are the smallest and the largest cluster points of (x_α), respectively.

Remark 1.3.11. Let (x_α) be a net in $\overline{\mathbb{R}}$. Clearly $\liminf_\alpha x_\alpha \leqslant \limsup_\alpha x_\alpha$. By Thm. 1.3.9, if $x \in \overline{\mathbb{R}}$, then

$$\lim_\alpha x_\alpha = x \qquad \Longleftrightarrow \qquad \liminf_\alpha x_\alpha = x = \limsup_\alpha x_\alpha \tag{1.14}$$

Proposition 1.3.12. *Let (x_α) be a net in $\overline{\mathbb{R}}$. For each $\alpha \in I$, define*

$$A_\alpha = \inf\{x_\beta : \beta \geqslant \alpha\} \qquad B_\alpha = \sup\{x_\beta : \beta \geqslant \alpha\} \tag{1.15}$$

Then (A_α) is increasing and (B_α) is decreasing; in particular, they converge in $\overline{\mathbb{R}}$. Moreover, we have

$$\lim_{\alpha\in I} A_\alpha = \liminf_{\alpha\in I} x_\alpha \qquad \lim_{\alpha\in I} B_\alpha = \limsup_{\alpha\in I} x_\alpha \tag{1.16}$$

Proof. The monotonicities of (A_α) and (B_α) are obvious. Let $E_\alpha = \{x_\beta : \beta \geqslant \alpha\}$. Then $B_\alpha = \sup E_\alpha = \sup \overline{E_\alpha}$. By Thm. 1.2.27, we have $\limsup_\alpha x_\alpha = \sup \bigcap_\alpha \overline{E_\alpha}$. Since $\bigcap_\alpha \overline{E_\alpha} \subset \overline{E_\beta}$, we have $\limsup_\alpha x_\alpha \leqslant \sup \overline{E_\beta} = B_\beta$. Therefore $\limsup x_\alpha \leqslant \lim B_\alpha$.

For each open interval U in $\overline{\mathbb{R}}$ containing $\lim B_\alpha$, the net $(B_\alpha)_{\alpha\in I} = (\sup E_\alpha)_{\alpha\in I}$ must be eventually in U. From the definition of E_α, we see that (x_α) is frequently in U. It follows from Thm. 1.2.27 that $\lim B_\alpha$ is a cluster point of (x_α). Therefore $\limsup x_\alpha \geqslant \lim B_\alpha$. This proves one of the two relations in (1.16); the other one can be proved in the same way. □

1.4 Review of important facts in point-set topology

Fix a normed vector space $\mathcal{V}$.

1.4.1 Miscellaneous definitions and properties

Definition 1.4.1. If X, Y are metric spaces and $f : X \to Y$ is map, we say that $C \in \mathbb{R}_{\geqslant 0}$ is a **Lipschitz constant** of f if

$$d(f(x_1), f(x_2)) \leqslant C d(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X$$

If f has a Lipschitz constant, we say that f is **Lipschitz continuous**.

Definition 1.4.2. If d and d' are two metrics on a set X, we say that d and d' are **equivalent** if there exists $\alpha, \beta \in \mathbb{R}_{>0}$ such that

$$d(x,y) \leqslant \alpha d'(x,y) \qquad d'(x,y) \leqslant \beta d(x,y) \qquad \text{for all } x,y \in X$$

Definition 1.4.3. Let $X_1, \dots, X_N$ be metric spaces. For each $1 \leqslant p < +\infty$, the ***l*$^\infty$-product metric** d_∞ and the ***l*^{p}-product metric** d_p are the metrics on $X_1 \times \cdots \times X_N$ defined by

$$d_\infty((x_1, \dots, x_N), (y_1, \dots, y_N)) := \max\{d(x_1, y_1), \dots, d(x_N, y_N)\}$$
$$d_p((x_1, \dots, x_N), (y_1, \dots, y_N)) := \sqrt[p]{d(x_1, y_1)^p + \cdots + d(x_N, y_N)}$$

for all $x_i, y_i \in X_i$. These metrics are equivalent. We equip $X_1 \times \cdots \times X_N$ with any metric equivalent to l^∞ and l^p.

Remark 1.4.4. Recall that if $f : X \to Y$ is a map of topological spaces, and $X = \bigcup_{i \in I} U_i$ is an open cover of X, then f is continuous iff $f|_{U_i} : U_i \to Y$ is continuous for any $i \in I$.

Definition 1.4.5. Let $f : X \to Y$ be a map where $(Y, \mathcal{T}_Y)$ is a topological space. The **pullback topology** on X is defined to be

$$f^*\mathcal{T}_Y := f^{-1}(\mathcal{T}_Y) = \{f^{-1}(V) : V \in \mathcal{T}_Y\}$$

Then, a net (x_α) in X converges under $f^*\mathcal{T}_Y$ to x iff

$$\lim_\alpha f(x_\alpha) = f(x)$$

1.4.2 Semicontinuous functions

Let X be a topological space.

Definition 1.4.6. We say that $f : X \to \overline{\mathbb{R}}$ is called **lower semicontinuous** if $f^{-1}(a, +\infty]$ is open for each $a \in \overline{\mathbb{R}}$. We say that f is **upper semicontinuous** if $f^{-1}[-\infty, a)$ is open for each $a \in \overline{\mathbb{R}}$.

Example 1.4.7. Let $A \subset X$. Then χ_A is lower semicontinuous iff A is open.

Proposition 1.4.8. *Let $(f_i)_{i \in I}$ be a family of lower semicontinuous functions $X \to \overline{\mathbb{R}}$. Let $f(x) = \sup_{i \in I} f_i(x)$. Then $f : X \to \overline{\mathbb{R}}$ is lower semicontinuous.*

Proof. Let $x \in f^{-1}(a, +\infty]$. Since $f(x) > a$, there exists i such that $f_i(x) > a$. Since f_i is lower semi-continuous, there exists $U \in \mathrm{Nbh}(x)$ such that $f_i|_U > a$ (e.g. $U = f_i^{-1}(a, +\infty]$), and hence $f|_U > a$. So $U \subset f^{-1}(a, +\infty]$. We have thus proved that any $x \in f^{-1}(a, +\infty]$ is an interior point of $f^{-1}(a, +\infty]$. $\square$

Proposition 1.4.9. *Let $f : X \to \overline{\mathbb{R}}$. Then the following are equivalent.*

(1) f is lower semicontinuous.

(2) For each $x \in X$ and each net (x_α) in X converging to x, we have $\liminf_\alpha f(x_\alpha) \geqslant f(x)$.

(3) For each $x \in X$ and each net (x_α) in X converging to x, we have $\limsup_\alpha f(x_\alpha) \geqslant f(x)$.

Proof. (1)⇒(2): Assume (1). Let $x \in X$, and let (x_α) be a net converging to x. If $\lambda \in \mathbb{R}$ satisfies $\lambda < f(x)$, then by (1), $f^{-1}(\lambda, +\infty]$ is a neighborhood of x. Therefore, x_α is eventually in $f^{-1}(\lambda, +\infty]$, and hence $f(x_\alpha)$ is eventually $> \lambda$. Therefore $\liminf_\alpha f(x_\alpha) \geqslant \lambda$. Since λ is arbitrary, we must have $\liminf_\alpha f(x_\alpha) \geqslant f(x)$. This proves (2).

(2)⇒(1): This is obvious.

¬(1)⇒¬(3): Assume that f is not lower semicontinuous. Then there exists $\lambda \in\in [-\infty, +\infty)$ such that $f^{-1}(\lambda, +\infty]$ is not open, and hence has non-interior point x. Therefore, $f(x) > \lambda$, and for any $U \in \mathrm{Nbh}_X(x)$ there exists $x_U \in U$ such that $f(x_U) \leqslant \lambda$. So $(x_U)_{U \in \mathrm{Nbh}_X(x)}$ is a net in X converging to x, and $\limsup_U f(x_U) \leqslant \lambda < f(x)$. Hence (3) is false. □

1.4.3 Product topology and pointwise convergence

Let $(X_\alpha)_{\alpha\in\mathscr{A}}$ be a family of topological spaces. Elements of the product space

$$S = \prod_{\alpha\in\mathscr{A}} X_\alpha$$

are denoted by $x = (x_\alpha)_{\alpha\in\mathscr{A}}$. Let

$$\pi_\alpha : S \to X_\alpha \qquad x \mapsto x(\alpha)$$

It is easy to check that

$$\begin{aligned}\mathcal{B} = \Big\{ \prod_{\alpha\in\mathscr{A}} U_\alpha &: \text{each } U_\alpha \text{ is open in } X_\alpha,\\ &U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \Big\}\\ =&\Big\{ \bigcap_{\alpha\in E} \pi_\alpha^{-1}(U_\alpha) : E \in \mathrm{fin}(2^{\mathscr{A}}),\ U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in E \Big\}\end{aligned}$$

is a base for a topology, namely, for each $W_1, W_2 \in \mathcal{B}$ and $x \in W_1 \cap W_2$, there exists $W_3 \in \mathcal{B}$ such that $W_3 \subset W_1 \cap W_2$. Therefore, $\mathcal{B}$ generates a topology.

Definition 1.4.10. The topology of S generated by $\mathcal{B}$ is called the **product topology** or **pointwise convergence topology** of S. Unless otherwise stated, the product of a family of topological spaces is equipped with the product topology.

Remark 1.4.11. If each X_α is Hausdorff, then S is clearly Hausdorff.

Theorem 1.4.12. *Let $(x_\mu)_{\mu\in I}$ be a net in S, and let $x \in S$. Then the following conditions are equivalent:*

(a) $\lim\limits_{\mu\in I} x_\mu = x$ *under the product topology.*

(b) $(x_\mu)_{\mu\in I}$ *converges pointwise to x, namely, for each $\alpha \in \mathscr{A}$ we have* $\lim\limits_{\mu\in I} x_\mu(\alpha) = x(\alpha)$ *in X_α.*

Proof. (a)⇒(b): Fix $\alpha \in \mathscr{A}$. For each open $U_\alpha \subset X_\alpha$, we have $\pi^{-1}(U_\alpha) \in \mathcal{B}$. Therefore,

$$\pi_\alpha : S \to X_\alpha \qquad \text{is continuous} \tag{1.17}$$

Thus, if $\lim_\mu x_\mu = x$, then $\lim_\mu \pi_\alpha(x_\mu) = \pi_\alpha(x)$. This proves (b).

(b)⇒(a): Assume (b). Choose any $W \in \mathcal{B}$ containing x. Then there exists $E \in \mathrm{fin}(2^{\mathscr{A}})$ such that $W = \bigcap_{\alpha\in E} \pi_\alpha^{-1}(U_\alpha)$, where each $U_\alpha \subset X_\alpha$ is open and containing x_α. For such $\alpha \in E$, since $\lim_\mu x_\mu(\alpha) = x(\alpha)$, we know that $(x_\mu(\alpha))$ is μ-eventually in U_α. Therefore, since E is finite, we conclude that (x_μ) is eventually in W. This proves (a). □

Corollary 1.4.13. *Let Z be a topological space. Suppose that for each $\alpha \in \mathscr{A}$, a map $f_\alpha : Z \to X_\alpha$ is chosen. Then*

$$\bigvee_{\alpha\in\mathscr{A}} f_\alpha : Z \to \prod_{\alpha\in\mathscr{A}} X_\alpha \qquad z \mapsto (f_\alpha(z))_{\alpha\in\mathscr{A}} \tag{1.18}$$

is continuous iff f_α is continuous for each $\alpha \in \mathscr{A}$.

Proof. If $F := \bigvee_{\alpha\in\mathscr{A}} f_\alpha$ is continuous, then since π_α is continuous, $f_\alpha = \pi_\alpha \circ f_\alpha$ is also continuous. Conversely, suppose that each f_α is continuous. Let (z_i) be a net in Z converging to $z \in Z$. For each α, since f_α is continuous, we see that $\lim_i f_\alpha(z_i) = f_\alpha(z)$. By Thm. 1.4.12, $F(z_i)$ converges to $F(z)$. This proves that F is continuous. □

Proposition 1.4.14. *Suppose that $\mathscr{A}$ is countable. If each X_α is second countable, then S is second countable. If each X_α is metrizable, then S is metrizable.*

Proof. If $\mathcal{U}_\alpha$ is a base of the topology of X_α, then

$$\mathcal{U} := \Big\{ \bigcap_{\alpha\in E} \pi_\alpha^{-1}(U_\alpha) : E \in \mathrm{fin}(2^{\mathscr{A}}), U_\alpha \in \mathcal{U}_\alpha \Big\}$$

is a base of the the product topology, which is countable if each $\mathcal{U}_\alpha$ is countable.

Now assume that each X_α is equipped with a metric d_α. Fix any $R \in \mathbb{R}_{>0}$, and let $\widetilde{d}_\alpha$ be metric on X_α inducing the same topology as d_α, and satisfies $d_\alpha \leqslant R$. For example,

$$\widetilde{d}_\alpha(x_\alpha, y_\alpha) = \min\{d_\alpha(x_\alpha, y_\alpha), R\} \qquad \text{for each } x_\alpha, y_\alpha \in X_\alpha \tag{1.19a}$$

Let $\nu : \mathscr{A} \to \mathbb{Z}_+$ be an injective map, and define a metric d on S by

$$d(x, y) = \sum_{\alpha \in \mathscr{A}} 2^{-\nu(\alpha)} \widetilde{d}_\alpha(x(\alpha), y(\alpha)) \qquad \text{for each } x, y \in S \tag{1.19b}$$

One shows easily that a net (x_μ) in S converging to $x \in S$ iff $\lim_\mu \widetilde{d}_\alpha(x_\mu(\alpha), x(\alpha)) = 0$ for all $\alpha \in \mathscr{A}$. Therefore, by Thm. 1.4.12, d induces the product topology. □

Theorem 1.4.15 (Tychonoff theorem). *Assume that X_α is compact for each $\alpha \in \mathscr{A}$. Then S is compact.*

⋆ *Proof.* Assume WLOG that $\mathscr{A}$ is non-empty, that each X_α is non-empty. Let $(x_\mu)_{\mu \in I}$ be a net in S. We want to show that $(x_\mu)_{\mu \in I}$ has a cluster point.

For each $\mathscr{E} \subset \mathscr{A}$, let $S_\mathscr{E} = \prod_{\alpha \in \mathscr{E}} X_\alpha$. For each $x \in S_\mathscr{E}$, we write $\mathrm{Dom}(x) = \mathscr{E}$. For each $\mathscr{E} \subset \mathscr{F} \subset \mathscr{A}$ and $y \in S_\mathscr{F}$, let $y|_\mathscr{E} = (y(\alpha))_{\alpha \in \mathscr{E}}$. Let

$$P = \bigcup_{\mathscr{E} \subset \mathscr{A}} \{x \in S_\mathscr{E} : x \text{ is a cluster point of } (x_\mu|_\mathscr{E})_{\mu \in I} \text{ in } S_\mathscr{E}\}$$

equipped with the partial order "$\subset$". In other words, if $x, y \in P$, then $x \leqslant y$ means that $\mathrm{Dom}(x) \subset \mathrm{Dom}(y)$ and $x = y|_\mathscr{E}$.

Since each X_α is compact, P is clearly non-empty. Let us show that every totally ordered non-empty subset $Q \subset P$ has an upper bound in P, so that Zorn's lemma can be applied. Let x be the union of all elements of Q. Thus $x \in S_\mathscr{E}$ where $\mathscr{E} = \bigcup_{y \in Q} \mathrm{Dom}(y)$, and we have $x|_{\mathrm{Dom}(y)} = y$ for each $y \in Q$.

To show that x is a cluster point of $(x_\mu|_\mathscr{E})_{\mu \in I}$ in $S_\mathscr{E}$, we pick any neighborhood of x in $S_\mathscr{E}$, which, after shrinking if necessary, is of the form $W = \prod_{\alpha \in \mathscr{E}} U_\alpha$ where each $U_\alpha \subset X_\alpha$ is open, and there exists $K \in \mathrm{fin}(2^\mathscr{E})$ such that $U_\alpha = X_\alpha$ whenever $\alpha \notin K$. Since $\mathscr{E} = \bigcup_{y \in Q} \mathrm{Dom}(y)$, there exists $y \in Q$ such that $K \subset \mathrm{Dom}(y)$. Namely, $(x_\mu|_{\mathrm{Dom}(y)})_{\mu \in I}$ has cluster point y, and $K \subset \mathrm{Dom}(y)$. Therefore $(x_\mu|_K)_{\mu \in I}$ has cluster point $y|_K$ (which equals $x|_K$ because $x|_{\mathrm{Dom}(y)} = y$), and hence is frequently in $\prod_{\alpha \in K} U_\alpha$. Thus $(x_\mu|_\mathscr{E})_{\mu \in I}$ is frequently in W. This finishes the proof that $x \in P$. Clearly x is an upper bound of Q.

Now we can apply Zorn's lemma, which claims that P has a maximal element $x \in P$. The proof of the Tychonoff theorem will be finished by showing that $\mathscr{E} := \mathrm{Dom}(x)$ equals $\mathscr{A}$. Suppose not. Choose $\beta \in \mathscr{A} \backslash \mathscr{E}$. Since $x \in P$, there is a subnet $(x_{\mu_\nu}|_\mathscr{E})_{\nu \in J}$ of $(x_\mu|_\mathscr{E})_{\mu \in I}$ converging pointwise to x. Since X_β is compact, $(x_{\mu_\nu}(\beta))_{\nu \in J}$ has a converging subnet $(x_{\mu_{\nu_\upsilon}}(\beta))_{\upsilon \in L}$. Define $\widetilde{x} \in S_{\mathscr{E} \cup \{\beta\}}$ to be x when restricted to $\mathscr{E}$, and $\widetilde{x}(\beta) := \lim_\upsilon x_{\mu_{\nu_\upsilon}}(\beta)$. Then $\widetilde{x} \in P$, and $\widetilde{x}$ is strictly larger than x, contradicting the maximality of x. □

Remark 1.4.16. If $\mathscr{A}$ is a countable set, and if each X_α is compact and second-countable, the **diagonal method** can be used in place of Zorn's lemma to prove that S (which is second countable by Prop. 1.4.14).

We consider the case that $\mathscr{A} = \mathbb{Z}_+$. (The case that $\mathscr{A}$ is finite is even simpler.) Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in S. We construct inductively a double sequence $(x_{m,n})_{m,n\in\mathbb{Z}_+}$ in S as follows. Since X_1 is sequentially compact, (x_n) has subsequence $(x_{1,n})_{n\in\mathbb{Z}_+}$ whose first component $(x_{1,n}(1))_{n\in\mathbb{Z}_+}$ converges to some $x(1) \in X_1$. Suppose that $(x_{m-1,n})_{n\in\mathbb{Z}_+}$ has been constructed (where $m-1 \geqslant 1$). Since X_m is sequentially compact, $(x_{m-1,n})_{n\in\mathbb{Z}_+}$ has a subsequence $(x_{m,n})_{n\in\mathbb{Z}_+}$ whose m-th component $(x_{m,n}(m))_{n\in\mathbb{Z}_+}$ to some $x(m) \in S$. In this way, the double sequence $(x_{m,n})$ in S and the element $x \in S$ are constructed. One checks easily that $(x_{n,n})_{n\in\mathbb{Z}_+}$ is a subsequence of (x_n) converging to x.

We have thus proved that S is sequentially compact. By Thm. 1.3.7, S is compact. □

1.4.4 Precompact sets

Let X be a Hausdorff space.

Definition 1.4.17. Let $A \subset X$. We say that A is **precompact** relative to X and write

$$A \Subset X$$

if $\mathrm{Cl}_X(A)$ is compact, equivalently, if A is contained in a compact subset of X.

Recall that a subset of a compact Hausdorff space is closed iff it is compact.

Proof of equivalence. "$\Rightarrow$": Obvious. "$\Leftarrow$": Let $B \subset X$ be compact and containing A. Then B is closed in X. So $\mathrm{Cl}_X(A) \subset B$. Since $\mathrm{Cl}_X(A)$ is closed in X and hence closed in B, it is compact. □

Remark 1.4.18. Let $W \subset X$. Then for each $A \subset W$, we have

$$A \Subset W \qquad \Longleftrightarrow \qquad A \Subset X \text{ and } \mathrm{Cl}_X(A) \subset W$$

When either side is true, we have $\mathrm{Cl}_W(A) = \mathrm{Cl}_X(A)$. Thus, both $\mathrm{Cl}_W(A)$ and $\mathrm{Cl}_X(A)$ can be denoted unambiguously by $\overline{A}$.

In practice, we often choose W to be an open subset of X.

Proof. "$\Leftarrow$": $\mathrm{Cl}_X(A)$ is a compact set inside W and contains A. So $A \Subset W$.

"$\Rightarrow$": We have a compact set B such that $A \subset B \subset W$. So $A \Subset X$. Since B is closed in any larger set, we have $\mathrm{Cl}_X(A) \subset B$ and hence $\mathrm{Cl}_X(A) \subset W$.

It is obvious that $\mathrm{Cl}_W(A) \subset \mathrm{Cl}_X(A)$. Assume $A \Subset W$. Then $\mathrm{Cl}_W(A)$ is compact. In the above paragraph, if we choose $B = \mathrm{Cl}_W(A)$. then we have $\mathrm{Cl}_X(A) \subset B = \mathrm{Cl}_X(A)$. This proves $\mathrm{Cl}_W(A) = \mathrm{Cl}_X(A)$. □

Remark 1.4.19. Let U be an open subset of X. Let $f \in C_c(U, \mathcal{V})$. Then by zero-extension, f can be viewed as an element of $C_c(X, \mathcal{V})$ supported in U. Briefly speaking, we have

$$C_c(U, \mathcal{V}) \subset C_c(X, \mathcal{V})$$

Moreover, for each $f \in C_c(U, \mathcal{V})$, we have

$$\mathrm{Supp}_U(f) = \mathrm{Supp}_X(f)$$

Proof. Let f take value 0 outside U. Let $K = \mathrm{Supp}_U(f)$, which is compact by assumption. Since $f|_U$ is continuous and $f|_{K^c} = 0$ are continuous, and since $X = U \cup K^c$ is an open cover on X, f is continuous. By the Rem. 1.4.18, we have $\mathrm{Supp}_U(f) = \mathrm{Supp}_X(f)$. Therefore $f \in C_c(X, \mathcal{V})$. $\square$

Under the setting of Rem. 1.4.19, it is clear that

$$C_c(U, \mathcal{V}) = \{f \in C_c(X, \mathcal{V}) : \mathrm{Supp}_X(f) \subset U\} \tag{1.20}$$

Proposition 1.4.20. *Assume that X is a metric space, and let $A \subset X$. Then A is precompact iff every sequence (x_n) in A has a subsequence converging to some $x \in X$.*

Proof. The direction "$\Rightarrow$" follows from the sequential compactness of $\overline{A}$. Conversely, assume that every sequence in A has a subsequence convering in X. Let us prove that $\overline{A}$ is sequentially compact. Let (x_n) be a sequence in $\overline{A}$. For each n, choose $y_n \in A$ such that $d(x_n, y_n) < 1/n$. By assumption, (y_n) has a subsequence (y_{n_k}) converging to some $x \in X$. One easily checks that (x_{n_k}) converges to x. $\square$

1.4.5 LCH spaces

Let X be LCH.

Proposition 1.4.21. *Any closed or open subset of X is LCH.*

Proof. See [Gui-A, Subsec. 8.6.2]. $\square$

Corollary 1.4.22. *Let $W \subset X$ be an open subset. Let $K \subset W$ be compact. Then there exists an open subset U of X such that $K \subset U \Subset W$.*

Proof. The case that K is a single point follows from the fact that W is LCH, cf. Prop. 1.4.21. The general case follows from the compactness of K. $\square$

Corollary 1.4.23. *Let K_1, K_2 be mutually disjoint compact subsets of X. Then there exist open subsets U_1, U_2 of X such that $K_1 \subset U_1$ and $K_2 \subset U_2$.*

Proof. This corollary in fact holds even without the assumption that X is locally compact, and its proof is a straightforward exercise in point-set topology. However, it also follows directly from the results established above. Indeed, by Prop. 1.4.21, $X\backslash K_2$ is LCH. Therefore, by Cor. 1.4.22, there exists an open set U_1 such that $K_1 \subset U_1 \Subset X\backslash K_2$. Let $U_2 = X\backslash \overline{U}_1$. □

Theorem 1.4.24 (Urysohn's lemma). *Let $K \subset X$ be compact. Then there exists a (continuous)* ***Urysohn function*** *f with respect to K and X, i.e., $f \in C_c(X, [0,1])$ and $f|_K = 1$.*

Proof. See [Gui-A, Sec. 15.4]. □

Remark 1.4.25. Urysohn's lemma can be used in the following way. Suppose that $K \subset U \subset X$ where K is compact and U is open in X. By Prop. 1.4.21, U is LCH. Therefore, by Thm. 1.4.24, there exists $f \in C_c(U, [0,1])$ such that $f|_K = 1$. By Rem. 1.4.19, f can be viewed as an element of $C_c(X, [0,1])$ satisfying $f|_K = 1$ and $\mathrm{Supp}(f) \subset U$.

Theorem 1.4.26. *Let K be a compact subset of X. Let $\mathfrak{U} = (U_1, \dots, U_n)$ be a finite collection of open subsets of X covering K (i.e. $K \subset U_1 \cup \cdots \cup U_n$). Then there exist $h_i \in C_c(U_i, \mathbb{R}_{\geqslant 0})$ (for all $1 \leqslant i \leqslant n$) satisfying the following conditions:*

(1) $0 \leqslant \displaystyle\sum_{i=1}^n h_i \leqslant 1$ *on* X.

(2) $\displaystyle\sum_{i=1}^n h_i\Big|_K = 1$.

Such $h_1, \dots, h_n$ are called a ***partition of unity of K subordinate to $\mathfrak{U}$****.*

In fact, $h_1, \dots, h_n$ should be viewed as a partition of the Urysohn function $h := h_1 + \cdots + h_n$.

Proof. See [Gui-A, Sec. 15.4]. Note that condition (1) is not stated in some textbooks on partitions of unity. However, even if (1) is not initially satisfied, one can enforce it by setting $g(x) = \max\{\sum_i h_i(x), 1\}$ and replacing each h_i with h_i/g. □

Theorem 1.4.27 (Tietze extension theorem). *Let K be a compact subset of X. Let $f \in C(K, \mathbb{F})$. Then there exists $\widetilde{f} \in C_c(X, \mathbb{F})$ such that $\widetilde{f}|_K = f$, and that $\|\widetilde{f}\|_{l^\infty(X)} = \|f\|_{l^\infty(K)}$.*

Proof. See [Gui-A, Sec. 15.4]. □

Definition 1.4.28. We let

$$C_0(X, \mathcal{V}) = \begin{cases} \{f \in C(X, \mathcal{V}) : \lim_{x\to\infty} \|f(x)\| = 0\} & \text{if } X \text{ is not compact} \\ C(X, \mathcal{V}) = C_c(X, \mathcal{V}) & \text{if } X \text{ is compact} \end{cases}$$

where $\widehat{X} = X \cup \{\infty\}$ is the one-point compactification of X. Equivalently, $C_0(X, \mathcal{V})$ is the set of all $f \in C(X, \mathcal{V})$ such that for any $\varepsilon > 0$ there exists a compact $K \subset X$ such that $\|f\|_{l^\infty(X\setminus K)} < \varepsilon$. See [Gui-A, Subsec. 15.8.1] for more discussions. For each $E \subset \mathcal{V}$, we let

$$C_0(X, E) = C_0(X, V) \cap E^X$$

Remark 1.4.29. $C_0(X, \mathcal{V})$ is the l^∞-closure of $C_c(X, \mathcal{V})$ in $C(X, \mathcal{V})$.

Proof. One easily shows that $C_0(X, \mathcal{V})$ is closed in $C(X, \mathcal{V})$. To show that $C_c(X, \mathcal{V})$ is dense in $C_0(X, \mathcal{V})$, we choose any $f \in C_0(X, \mathcal{V})$. Then for each $\varepsilon > 0$ there exists a compact $K \subset X$ such that $\|f\|_{l^\infty(K^c)} < \varepsilon$. By Urysohn's lemma, there exists $h \in C_c(X, [0, 1])$ such that $h|_K = 1$. Then $\|hf\|_{l^\infty(K^c)} < \varepsilon$, and hence $\|f - hf\|_{l^\infty(X)} < 2\varepsilon$. This finishes the proof, since $hf \in C_c(X, \mathcal{V})$. □

Remark 1.4.30. Suppose that X is second countable. Then X is Lindelöf. Therefore, X has a countable open cover $\mathfrak{U} = (U_n)_{n\in\mathbb{Z}_+}$ whose members U_n are precompact open subsets of X. In particular, X is σ-compact, since $X = \bigcup_{n\in\mathbb{Z}_+} \overline{U_n}$ where each $\overline{U_n}$ is compact.

1.4.6 Equicontinuity and the Arzelà-Ascoli theorem

Let X be a topological space. Let V be a normed vector space.

Definition 1.4.31. Let $(f_\alpha)_{\alpha\in\mathscr{A}}$ be a family of functions $X \to V$. We say that $(f_\alpha)_{\alpha\in\mathscr{A}}$ is **equicontinuous at $x \in X$** if for each $\varepsilon > 0$ there exists $U_x \in \mathrm{Nbh}_X(x)$ such that

$$\|f_\alpha(y) - f_\alpha(x)\| \leqslant \varepsilon \qquad \text{for all } \alpha \in \mathscr{A}, y \in U_x \tag{1.21}$$

This is equivalent to saying that

$$\lim_{y\to x} \sup_\alpha \|f_\alpha(y) - f_\alpha(x)\| = 0$$

We say that $(f_\alpha)_{\alpha\in\mathscr{A}}$ is an **equicontinuous family of functions** if it is equicontinuous at every point of X.

Theorem 1.4.32. *Let $(f_\alpha)_{\alpha\mathscr{A}}$ be an equicontinuous net of functions $X \to V$ converging pointwise to some $f : X \to V$. Then f is continuous. Moreover, if X is compact, then $(f_\alpha)_{\alpha\mathscr{A}}$ converges uniformly to f.*

Proof. For each $x \in X$ and $\varepsilon > 0$, choose $U_x \in \mathrm{Nbh}_X(x)$ such that (1.21) holds. Applying $\lim_\alpha$, we see that $\|f(y) - f(x)\| \leqslant \varepsilon$ for all $\alpha \in \mathscr{A}$ and $y \in U_x$. This proves that f is continuous at every $x \in X$.

Next, assume that X is compact. Then there exist $x_1, \ldots, x_n \in X$ such that $X = U_1 \cup \cdots \cup U_n$ if we set $U_i = U_{x_i}$. For each $y \in X$, choose i such that $y \in U_i$. Then

$$\begin{aligned}&\|f_\alpha(y) - f(y)\| \leqslant \|f_\alpha(y) - f_\alpha(x_i)\| + \|f_\alpha(x_i) - f(x_i)\| + \|f(x_i) - f(y)\|\\ \leqslant &2\varepsilon + \|f_\alpha(x_i) - f(x_i)\|\end{aligned}$$

and hence

$$\sup_{y\in X} \|f_\alpha(y) - f(y)\| \leqslant 2\varepsilon + \sum_{i=1}^n \|f_\alpha(x_i) - f(x_i)\|$$

Since f_α converges to f at $x_1, \ldots, x_n$, we conclude that

$$\limsup_\alpha \sup_{y\in X} \|f_\alpha(y) - f(y)\| \leqslant 2\varepsilon$$

for all $\varepsilon > 0$. Hence $\lim_\alpha \sup_{y\in X} \|f_\alpha(y) - f(y)\| = 0$. □

Theorem 1.4.33 (Arzelà-Ascoli theorem). *Assume that X is LCH. Let $\mathscr{F}$ be an equicontinuous set of functions $X \to \mathbb{F}$. If X is non-compact, we assume that*

$$\lim_{x\to\infty} \sup_{f\in\mathscr{F}} |f(x)| = 0 \tag{1.22}$$

which means that for each $\varepsilon > 0$, there exists a compact $K \subset X$ such that

$$|f(x)| \leqslant \varepsilon \qquad \text{for all } f \in \mathscr{F}, x \in X\backslash K$$

*Assume that $\mathscr{F}$ is **pointwise bounded**, i.e., $\sup_{f\in\mathscr{F}} |f(x)| < +\infty$ for each $x \in X$. Then $\mathscr{F}$ is precompact in $C_0(X, \mathbb{F})$ (under the l^∞-norm).*

Proof. It is easy to check that $\overline{\mathscr{F}}$, the closure of $\mathscr{F}$, satisfies the same properties as $\mathscr{F}$. Therefore, replacing $\mathscr{F}$ with its closure, we assume WLOG that $\mathscr{F}$ is closed.

We first consider the case that X is compact. Since $\mathscr{F}$ is pointwise bounded, it can be viewed as a subset of $\prod_{x\in X} D_x$ where D_x is a compact set in $\mathbb{F}$. Therefore, by the Tychonoff Thm. 1.4.15, every net (f_α) in $\mathscr{F}$ has a pointwise convergent subnet (f_β). Since (f_β) is equicontinuous and X is compact, by Thm. 1.4.32, (f_β) converges uniformly to some $f \in C(X, \mathbb{F})$. Since $\mathscr{F}$ is closed, we have $f \in \mathscr{F}$.5 This proves that $\mathscr{F}$ is compact.

Next, assume that X is not compact. Let $\widehat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Extend each $f\mathscr{F}$ to $\widehat{X} \to \mathbb{F}$ by setting $f(0) = 0$. Then $\mathscr{F}$ is a pointwise bounded and equicontiuous family of functions $\widehat{X} \to \mathbb{C}$. By the previous paragraph, every net in $\mathscr{F}$ has a uniformly convergent subnet. Thus, $\mathscr{F}$ is compact. □

1.5 $*$-algebras and the Stone-Weierstrass theorem

Recall that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In this section, we let $\mathbb{K}$ be any subfield of $\mathbb{C}$ closed under complex conjugation, such as $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Q} + \mathbf{i}\mathbb{Q}$.

Definition 1.5.1. A **$\mathbb{K}$-algebra** is defined to be a ring $\mathscr{A}$ (not necessarily having 1) that is also a $\mathbb{K}$-vector space, together with a bilinear map

$$\mathscr{A} \times \mathscr{A} \to \mathscr{A} \qquad (x, y) \mapsto xy$$

satisfying the associacitity rule

$$(xy)z = x(yz) \qquad \text{for all } x, y, z \in \mathscr{A}$$

A $\mathbb{K}$-algebra is called **unital** if $\mathscr{A}$, as a ring, has a multiplicative identity 1. In this case, we write $\lambda \cdot 1$ as λ if $\lambda \in \mathbb{K}$.

A $\mathbb{K}$-algebra is called **commutative** or **abelian** if $xy = yx$ for all $x, y \in \mathscr{A}$.

If $\mathscr{A}$ is a $\mathbb{K}$-algebra, then a **($\mathbb{K}$-)subalgebra** is a subset $\mathscr{B}$ which is invariant under the ring addition, ring multiplication, and scalar multiplication. (Namely, $\mathscr{B}$ is a subring and also a subspace of $\mathscr{A}$.) If $\mathscr{A}$ is unital, then a **unital ($\mathbb{K}$-)subalgebra** of $\mathscr{A}$ is a $\mathbb{K}$-subalgebra containing the identity of $\mathscr{A}$. $\square$

Remark 1.5.2. A unital $\mathbb{K}$-algebra $\mathscr{A}$ can equivalently be described as a ring with identity, together with a ring homomorphism $\mathbb{C} \to Z(\mathscr{A})$ where $Z(\mathscr{A})$ is the **center** of $\mathscr{A}$, i.e.

$$Z(\mathscr{A}) = \{x \in \mathscr{A} : xy = yx \text{ for every } y \in \mathscr{A}\}$$

We leave the verification of this equivalence to the reader.

Example 1.5.3. If V is a $\mathbb{F}$-vector space, then $\mathrm{End}(V)$, the set of $\mathbb{F}$ linear maps $V \to V$, is naturally an $\mathbb{F}$-algebra. If V is a normed vector space, then $\mathfrak{L}(V)$ is an $\mathbb{F}$-algebra.

Definition 1.5.4. A ***-$\mathbb{K}$-algebra** is defined to be a $\mathbb{K}$-algebra together with an **antilinear map** $* : \mathscr{A} \to \mathscr{A}$ sending x to x^* (where "antilinear" means that for every $a, b \in \mathbb{C}$ and $x, y \in \mathscr{A}$ we have $(ax + by)^* = \overline{a}x^* + \overline{b}y^*$) such that for every $x, y \in \mathscr{A}$, we have

$$(x^*)^* = x \qquad (xy)^* = y^* x^*$$

Note that $*$ must be bijective. We call $*$ an **involution**. A ***-$\mathbb{K}$-subalgebra** $\mathscr{B}$ is defined to be a subalgebra satisfying $x \in \mathscr{B}$ iff $x^* \in \mathscr{B}$. If $\mathscr{A}$ is a unital algebra with unit $\mathbf{1}$, we say that $\mathscr{A}$ is a **unital *-$\mathbb{K}$-algebra** if $\mathscr{A}$ is equipped with an involution $* : \mathscr{A} \to \mathscr{A}$ such that $\mathscr{A}$ is a *-algebra, and that

$$\mathbf{1}^* = \mathbf{1}$$

A unital *-subalgebra is a unital subalegbra and also a *-subalgebra.

Convention 1.5.5. We omit "$\mathbb{K}$-" when $\mathbb{K}$ is $\mathbb{C}$. For example, a **unital *-algebra** means a unital $*$-$\mathbb{C}$-algebra.

Example 1.5.6. The set of complex $n \times n$ matrices $\mathbb{C}^{n\times n}$ is naturally a unital $*$-algebra if for every $A \in \mathbb{C}^{n\times n}$ we define $A^* = \overline{A}^{\mathrm{t}}$, the complex conjugate of the transpose of A.

Example 1.5.7. Let X be a set. Then $\mathbb{K}^X$ is naturally a unital $\mathbb{K}$-algebra, and $l^\infty(X,\mathbb{K})$ is its unital $\mathbb{K}$-subalgebra. If X is a topological space, then $C(X,\mathbb{K})$ is a unital $\mathbb{K}$-subalgebra of $\mathbb{K}^X$. If X is compact, then $C(X,\mathbb{K})$ is a unital $\mathbb{K}$-subalgebra of $l^\infty(X,\mathbb{K})$.

Example 1.5.8. Let X be a set. Then $\mathbb{C}^X$ is a unital *-algebra if for every $f \in \mathbb{C}^X$ we define

$$f^* : X \to \mathbb{C} \qquad \boldsymbol{f^*(x)} = \overline{f(x)} \tag{1.23}$$

Then $\mathbb{K}^X$ and $l^\infty(X,\mathbb{K})$ are unital *-$\mathbb{K}$-subalgebras of $\mathbb{C}^X$.

Assume that X is a compact topological space. Then $C(X,\mathbb{F})$ is a unital *-$\mathbb{F}$-subalgebra of $l^\infty(X,\mathbb{F})$. If $f_1,\dots,f_n \in C(X,\mathbb{F})$, then $\mathbb{F}[f_1,\dots,f_n]$, the set of polynomials of $f_1,\dots,f_n$ with coefficients in $\mathbb{F}$, is a unital $\mathbb{F}$-subalgebra of $C(X,\mathbb{F})$. And $\mathbb{F}[f_1,f_1^*,\dots,f_n,f_n^*]$ is a unital *-$\mathbb{F}$-subalgebra of $C(X,\mathbb{F})$. $\square$

More generally, we have:

Example 1.5.9. Let $\mathscr{A}$ be an abelian unital $\mathbb{K}$-algebra. Let $\mathfrak{S} \subset \mathscr{A}$. Then

$$\mathbb{K}\langle\mathfrak{S}\rangle = \mathrm{Span}_{\mathbb{K}}\{x_1^{n_1}\cdots x_k^{n_k} : k \in \mathbb{Z}_+, x_i \in \mathfrak{S}, n_i \in \mathbb{N}\} \tag{1.24}$$

the set of (possibly non-commutative) polynomials of elements in $\mathfrak{S}$, is the smallest unital $\mathbb{K}$-subalgebra containing $\mathfrak{S}$, called the **unital $\mathbb{K}$-subalgebra generated by** $\mathfrak{S}$. (Here, we understand $x^0 = 1$ if $x \in \mathscr{A}$.) Thus, if $\mathscr{A}$ is an abelian unital *-algebra, then $\mathbb{C}\langle\mathfrak{S}\cup\mathfrak{S}^*\rangle$ (where $\mathfrak{S}^* = \{x^* : x \in \mathfrak{S}\}$) is the smallest unital *-algebra containing $\mathfrak{S}$, called the **unital *-$\mathbb{K}$-subalgebra generated by** $\mathfrak{S}$.

Definition 1.5.10. Let X be sets. Let $(f_\alpha)_{\alpha\in\mathfrak{A}}$ be a family of maps where $f_\alpha : X \to Y_\alpha$ and Y_α is a set. We say that $(f_\alpha)_{\alpha\in\mathfrak{A}}$ **separates the points of X** if for any distinct $x_1, x_2 \in X$ there exists $\alpha \in \mathfrak{A}$ such that $f_\alpha(x_1) \neq f_\alpha(x_2)$. Equivalently, the map

$$\bigvee_{\alpha\in\mathfrak{A}} f_\alpha : X \to \prod_{\alpha\in\mathfrak{A}} Y_\alpha \qquad x \mapsto (f_\alpha(x))_{\alpha\in\mathfrak{A}} \tag{1.25}$$

is injective.

Example 1.5.11. Let X be an LCH space. Then $C_c(X,[0,1])$ separates the points of X.

Proof. Choose any distinct points $x, y \in X$. By Urysohn's lemma (Rem. 1.4.25), there exists $f \in C_c(X, [0,1])$ such that $f(x) = 1$ and $\mathrm{Supp}(f) \subset X\backslash\{y\}$. So f separates x, y. □

Theorem 1.5.12 (Stone-Weierstrass theorem). *Let X be a compact Hausdorff space. Let $\mathfrak{S} \subset C(X, \mathbb{F})$. Suppose that $\mathfrak{S}$ separates the points of X. Then the $*$-$\mathbb{F}$-subalgebra $\mathbb{F}\langle \mathfrak{S} \cup \mathfrak{S}^* \rangle$ generated by $\mathfrak{S}$ is dense in $C(X, \mathbb{F})$ under the l^∞-norm.*

Note that if $\mathbb{F} = \mathbb{R}$, then $\mathfrak{S}^* = \mathfrak{S}$ by (1.23).

If $\mathbb{F} = \mathbb{C}$, then since $(\mathbb{Q} + \mathbf{i}\mathbb{Q})\langle \mathfrak{S} \cup \mathfrak{S}^* \rangle$ is l^∞-dense in $\mathbb{C}\langle \mathfrak{S} \cup \mathfrak{S}^* \rangle$, it is clear that $(\mathbb{Q} + \mathbf{i}\mathbb{Q})\langle \mathfrak{S} \cup \mathfrak{S}^* \rangle$ is l^∞-dense in $C(X)$. Similarly, if $\mathbb{F} = \mathbb{R}$, then $\mathbb{Q}\langle \mathfrak{S} \rangle$ is l^∞-dense in $C(X, \mathbb{R})$.

Proof. See [Gui-A, Ch. 15]. □

The following application of the Stone-Weierstrass theorem will be used in the study of weak-* topology, particularly in the proof of Thm. 2.6.6. Recall that $C(X, \mathbb{F})$ is equipped with the l^∞-norm.

Theorem 1.5.13. *Let X be a compact Hausdorff space. Then the following are equivalent:*

(a) X is metrizable.

(b) X is second countable.

(c) There is a sequence $(f_n)_{n \in \mathbb{Z}_+}$ in $C(X, \mathbb{F})$ separating the points of X.

(d) $C(X, \mathbb{F})$ is separable.

The Stone-Weierstrass theorem will be used in the direction (c)⇒(d). The equivalence of (a,b,c) does not rely on the Stone-Weierstrass theorem. Moreover, condition (a) will seldom be used in this course: while metrizability is a convenient notion for beginners in analysis, the second countability condition is more flexible and broadly applicable.

Proof. (a)⇒(b): Fix a metric on X. By the compactness, for each $n \in \mathbb{Z}_+$, X is covered by finitely many open balls with radius $1/n$. One checks easily that the collection of all these open balls for all $n \in \mathbb{Z}_+$ is a countable basis of the topology of X.

(b)⇒(c): Since X is second countable, we can choose an infinite countable base $(U_n)_{n \in \mathbb{Z}}$ of the topology. For each $m, n \in \mathbb{Z}_+$, if $U_n \Subset U_m$, we choose $f_{m,n} \in C_c(U_m, [0,1]) \subset C_c(X, [0,1])$ such that $f|_{\overline{U}_n} = 1$ (which exists by Urysohn's lemma); otherwise, we let $f_{m,n} = 0$.

Let us prove that $\{f_{m,n} : m, n \in \mathbb{Z}_+\}$ separates the points of X: Choose distinct $x, y \in X$. Since $X\backslash\{y\} \in \mathrm{Nbh}_X(x)$, there exists U_m containing x and is contained in $X\backslash\{y\}$. By Cor. 1.4.22, there exists n such that $\{x\} \subset U_n \Subset U_m$. Then $f_{m,n}(x) = 1$

and $f_{m,n}(y) = 0$.

(c)⇒(a,b): Since (f_n) separates points, the map

$$\Phi = \bigvee_n f_n : X \to \mathbb{F}^{\mathbb{Z}_+} \qquad x \mapsto (f_n(x))_{n \in \mathbb{Z}_+}$$

is injective. By Cor. 1.4.13, Φ is continuous. Since X is compact, the map Φ restricts to a homeomorphism $\Phi : X \to \Phi(X)$, where $\Phi(X)$ is equipped with the subspace topology of the product topology of $\mathbb{F}^{\mathbb{Z}_+}$. By Prop. 1.4.14, $\mathbb{F}^{\mathbb{Z}_+}$ is metrizable and second countable, so $\Phi(X)$, and hence X, is metrizable and second countable. This proves (a) and (b).

(c)⇒(d): Let $\mathbb{K} = \mathbb{F} \cap (\mathbb{Q} + \mathbf{i}\mathbb{Q})$. By Stone-Weierstrass, the countable set $\mathbb{K}[\{f_n : n \in \mathbb{Z}_+\}]$ is dense in $C(X, \mathbb{F})$. Thus $C(X, \mathbb{F})$ is separable.

(d)⇒(c): By Exp. 1.5.11, $C(X, \mathbb{F})$ separates the points of X. Therefore, any dense subset of $C(X, \mathbb{F})$ separates the points of X. Since $C(X, \mathbb{F})$ is separable, it has a countable dense subset separating the points of X. □

1.6 Review of measure theory: general facts

1.6.1 Some useful definitions and their basic properties

Definition 1.6.1. Let X be a set. Suppose that $\mathscr{C}$ is an $\mathbb{F}$-linear subspace of $\mathbb{F}^X$. A **positive linear functional** on $\mathscr{C}$ denotes an $\mathbb{F}$-linear map $\Lambda : \mathscr{C} \to \mathbb{F}$ such that $\Lambda(f) \geqslant 0$ for all $f \in \mathscr{C} \cap \mathbb{R}^X_{\geqslant 0}$.

Recall that if $(X, \mathfrak{M})$ is a measurable space, an **$\mathbb{F}$-valued simple function** on X is an $\mathbb{F}$-linear combination of characteristic functions over measurable sets; that is, an element of $\mathrm{Span}_{\mathbb{F}}\{\chi_E : E \in \mathfrak{M}\}$.

Definition 1.6.2. Let X be a set. Let $x \in X$. The **Dirac measure $\boldsymbol{\delta_x}$** of x is defined to be the measure $\delta_x : 2^X \to \overline{\mathbb{R}}_{\geqslant 0}$ satisfying $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ if $x \notin A$.

Definition 1.6.3. Let $(X, \mathcal{T}_X)$ be a topological space. Let $\mathfrak{M} \subset 2^X$ be a σ-algebra containing the Borel σ-algebra $\mathfrak{B}_X$. Let $\mu : \mathfrak{M} \to \overline{\mathbb{R}}_{\geqslant 0}$ be a measure. Assume that one of the following conditions holds:

(1) X is second countable.

(2) X is LCH, and $\mu|_{\mathfrak{B}_X}$ is a Radon measure.

The **support Supp($\boldsymbol{\mu}$)** is defined to be

$$\mathrm{Supp}(\mu) = \{x \in X : \mu(U) > 0 \text{ for each } U \in \mathrm{Nbh}_X(x)\}$$

Then $\mathrm{Supp}(\mu)$ is a closed subset of X, because we clearly have

$$X \backslash \mathrm{Supp}(\mu) = \bigcup_{U \in \mathcal{T}_X, \mu(U)=0} U$$

Moreover, we have $\mu(X \backslash \mathrm{Supp}(\mu)) = 0$. Thus, $\mathrm{Supp}(\mu)$ is the largest closed subset whose complement is μ-null.

Proof that $X \backslash \mathrm{Supp}(\mu)$ is null. It suffices to show that if a family of open subsets $(U_\alpha)_{\alpha \in \mathscr{A}}$ is null, then the union $U := \bigcup_\alpha U_\alpha$ is null.

Assume that condition (1) holds. Since any subset of a second countable space is second countable and hence Lindelöf, the set U is Lindelöf. So (U_α) has a countable subfamily covering U. Therefore, by the countable additivity, U is null.

Assume that condition (2) holds. Since Radon measures are inner regular on open sets (cf. Def. 1.7.3), $\mu(U)$ is the supremum of $\mu(K)$ where K runs through all compact subsets of U. Since K is compact, (U_α) has a finite subfamily covering K. Therefore K is null, and hence U is null. □

Lemma 1.6.4. *Let $\mu : \mathfrak{M} \to \overline{\mathbb{R}}_{\geqslant 0}$ be as in Def. 1.6.2, and assume that Condition (1) or (2) of Def. 1.6.2 holds. The following are equivalent:*

(a) $\mathrm{Supp}(\mu)$ *is a finite set.*

(b) μ *is a linear combination of Dirac measures (restricted to $\mathfrak{M}$).*

Proof. (b)⇒(a): This is obvious.

(a)⇒(b): Write $E = \mathrm{Supp}(\mu)$. Choose any measurable $f : X \to \overline{\mathbb{R}}_{\geqslant 0}$. Then, since $\mu|_{X \backslash E} = 0$, the integral of any measurable function $g : X \to \overline{\mathbb{R}}_{\geqslant 0}$ vanishing ourside E is zero. In particular, we can choose g to be the unique one such that $g + \sum_{x \in E} f(x)\chi_{\{x\}} = f$. Therefore

$$\int_X f d\mu = \int_E \sum_{x \in E} f(x)\chi_{\{x\}} d\mu = \sum_{x \in E} f(x) \cdot \mu(\{x\})$$

This shows that $\mu = \sum_{x \in E} \mu(\{x\})\delta_x$. □

1.6.2 Radon-Nikodym derivatives

Fix a measurable space $(X, \mathfrak{M})$.

Definition 1.6.5. Let $\mu, \nu : \mathfrak{M} \to [0, +\infty]$ are measures. We say that ν is **absolutely continuous** with respect to μ and write $\boldsymbol{\nu \ll \mu}$ if any μ-null set is ν-null. We say that $h \in \mathcal{L}(X, \overline{\mathbb{R}}_{\geqslant 0})$ is a **Radon-Nikodym derivative** of ν with respect to μ if

$$\int_X f d\nu = \int_X f h d\mu \quad \text{for all } f \in \mathcal{L}(X, \overline{\mathbb{R}}_{\geqslant 0})$$

By MCT, the above condition is equivalent to

$$\nu(E) = \int_E h d\mu \qquad \text{for all } E \in \mathfrak{M}$$

We write $\boldsymbol{d\nu = h d\mu}$.

Remark 1.6.6. If μ is σ-finite, and if h_1, h_2 are both Radon-Nikodym derivatives of ν with respect to μ, then $h_1(x) = h_2(x)$ for μ-a.e. $x \in X$.

Proof. It suffices to assume that $\mu(X) < +\infty$. For each $k \in \mathbb{N}$, let

$$A_k = \{x \in X : h_1(x) < h_2(x) \text{ and } h_1(x) \leqslant k\}$$

Then $\int_{A_k} h_1 d\mu \leqslant k\mu(X) < +\infty$, and

$$\int_{A_k} h_1 d\mu = \int_{A_k} d\nu = \int_{A_k} h_2 d\mu$$

Taking subtraction, we get $\int_{A_k} (h_2 - h_1) d\mu = 0$. Let $A = \bigcup_k A_k = \{x \in X : h_1(x) < h_2(x)\}$. By MCT, $\int_A (h_2 - h_1) d\mu = 0$. Since $h_2 - h_1 \geqslant 0$ on A, we conclude $h_2 - h_1 = 0$ μ-a.e. on A, and hence $\mu(A) = 0$. Similarly, $\mu(B) = 0$ where $B = \{x \in X : h_1(x) > h_2(x)\}$. □

Remark 1.6.7. If ν is σ-finite, and if $d\nu = h d\mu$, then $h(x) < +\infty$ for μ-a.e. $x \in X$.

Proof. Let $A = \{x \in A : h(x) = +\infty\}$. Since ν is σ-finite, we can write $A = \bigcup_{k \in \mathbb{N}} A_k$ where $A_k \in \mathfrak{M}$ and $\nu(A_k) < +\infty$. Since $\nu(A_k) = \int_{A_k} h d\mu = +\infty \mu(A_k)$, we have $\mu(A_k) = 0$, and hence $\mu(A) = 0$. □

Theorem 1.6.8 (Radon-Nikodym theorem). *Assume that $\mu, \nu : \mathfrak{M} \to [0, +\infty]$ are σ-finite measures. Then $\nu \ll \mu$ iff ν has a Radon-Nikodym derivative with respect to μ.*

Proof. "$\Leftarrow$" is obvious. Let us prove "$\Rightarrow$". It is easy to reduce to the case that $\mu(X), \nu(X) < +\infty$. Let $d\psi = d\mu + d\nu$. So $\mu, \nu \leqslant \psi$. Therefore, the linear functional

$$\Lambda : L^2(X, \psi) \to \mathbb{C} \qquad \xi \mapsto \int_X \xi d\mu$$

is bounded. Since $L^2(X, \psi)$ is a Hilbert space (Thm. 1.6.10), by the Riesz-Fréchet theorem, there exists $f \in L^2(X, \psi)$ such that $\int_X \xi d\nu = \int_X \xi f d\psi$ for all $\xi \in L^2(X, \psi)$. Since Λ sends positive functions to $\mathbb{R}_{\geqslant 0}$, after adding an ψ-a.e. function to ξ, we have $\psi \geqslant 0$ everywhere.

We have found $f \in \mathcal{L}(X, \mathbb{R}_{\geqslant 0})$ such that $d\mu = f d\psi$. Similarly, we have $g \in \mathcal{L}(X, \mathbb{R}_{\geqslant 0})$ such that $d\nu = g d\psi$. Since $\mu \leqslant \psi \ll \mu$, we have $f > 0$ outside a ψ-null set Δ. Let $h = g/f$ outside Δ, and $h = 0$ on Δ. Then $d\nu = h d\mu$. □

1.6.3 L^p-spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $1 \leqslant p, q \leqslant +\infty$ such that $p^{-1} + q^{-1} = 1$.

Theorem 1.6.9. *Let $1 \leqslant p < +\infty$. Then the set of integrable $\mathbb{F}$-valued simple functions is dense in $L^p(X, \mu, \mathbb{F})$. In other words,*

$$\{\chi_E : E \subset \mathfrak{M}, \mu(E) < +\infty\}$$

spans a dense subspace of $L^p(X, \mu, \mathbb{F})$.

Proof. See [Gui-A, Sec. 27.2]. □

Theorem 1.6.10 (Riesz-Fischer theorem, the modern form**).** *The normed vector space $L^p(X, \mu, \mathbb{F})$ is (Cauchy) complete. Moreover, any Cauchy sequence in $L^p(X, \mu, \mathbb{F})$ has a subseqence converging μ-a.e..*

Proof. See [Gui-A, Sec. 27.3]. □

Lemma 1.6.11. *Assume that (X, μ) is σ-finite. Let $\mathcal{S}_+$ be the set of simple functions $X \to \mathbb{R}_{\geqslant 0}$. Then for each $f \in \mathcal{L}(X, \overline{\mathbb{R}}_{\geqslant 0})$ we have*

$$\|f\|_{L^p(X,\mu)} = \sup\Big\{\int_X fgd\mu : g \in \mathcal{S}_+, \|g\|_{L^q(X,\mu)} \leqslant 1\Big\} \tag{1.26}$$

Consequently, for each $f \in \mathcal{L}(X, \mathbb{C})$ we have

$$\|f\|_{L^p(X,\mu)} = \sup\Big\{\int_X |fg| : g \in L^q(X, \mu), \|g\|_q \leqslant 1\Big\} \tag{1.27}$$

Proof. By Hölder's inequality, we have "$\geqslant$". To prove "$\leqslant$", we note that (1.27) follows immediately from (1.26) by writing $f = u|f|$ where $u \in \mathcal{L}(X, \mathbb{S}^1)$ and applying (1.26) to $|f|$. Thus, in the following, we assume $f \in \mathcal{L}(X, \overline{\mathbb{R}}_{\geqslant 0})$.

Case $1 < p < +\infty$: Choose an increasing sequence (f_n) (i.e. $f_1 \leqslant f_2 \leqslant \cdots$) in $\mathcal{S}_+$ converging pointwise to f such that each f_n vanishes outside a measurable μ-finite set. Let $g_n = (f_n)^{p-1}$. After removing the first several terms, we assume $\|g_n\|_{L^q} > 0$ for all n. Then

$$0 < \|g_n\|_q = \|f_n\|_p^{p/q} < +\infty$$

By MCT, we have $\lim_n \|g_n\|_p = \|f\|_p^{p/q}$ and $\lim_n \int_X fg_n = \|f\|_p^p$. Thus, if $\|f\|_p < +\infty$, then

$$\lim_n \|g_n\|_q^{-1} \int_X fg_n = \|f\|_p^{-p/q} \cdot \|f\|_p^p = \|f\|_p$$

This proves (1.26) when $\|f\|_p < +\infty$. If $\|f\|_p = +\infty$, then, by MCT, $\|f_n\|_p < +\infty$ can be sufficiently large. Applying (1.26) to f_n, we obtain $g \in \mathcal{S}_+$ such that $\|g\|_q \leqslant 1$

and $\int f_n g$ is sufficiently large, and hence $\int fg$ is sufficiently large. Thus (1.26) holds again.

Case $p = 1$: Let $g = 1$.

Case $p = +\infty$: Write $X = \bigcup_{n\in\mathbb{N}} \Omega_n$ where $\Omega_n \in \mathfrak{M}$ and $\mu(\Omega_n) < +\infty$. Choose any $0 \leqslant \lambda < \|f\|_\infty$. Then $A := \{|f| > \lambda\}$ satisfies $\mu(A) > 0$. Thus, there exists n such that $0 < \mu(A \cap \Omega_n) < +\infty$. Let $g = \chi_{A\cap\Omega_n}/\mu(A \cap \Omega_n)$. Then $g \in \mathcal{S}_+$, $\|g\|_1 = 1$, and $\int fg \geqslant \lambda$. This proves (1.26). $\square$

Theorem 1.6.12. *Assume that (X, μ) is σ-finite. Assume $1 < p \leqslant +\infty$. Then we have an isomorphism of normed vector spaces*

$$\Psi : L^p(X, \mu, \mathbb{F}) \to L^q(X, \mu, \mathbb{F})^* \qquad f \mapsto \Big(g \in L^q(X, \mu, \mathbb{F}) \mapsto \int_X fg d\mu\Big) \tag{1.28}$$

When $p < +\infty$, the assumption on σ-finiteness can be removed. See [Fol-R, Sec. 6.2]. When $p = 2$, this is simply due to the completeness of $L^2(X, \mu, \mathbb{F})$ and the Riesz-Fréchet theorem.

Proof. By Hölder's inequality and Lem. 1.6.11, Ψ is an isometry. Let us show that any $\Lambda \in L^q(X, \mu, \mathbb{F})^*$ belongs to the range of Ψ.

Step 1. By considering the real and imaginary parts, we can first assume that Λ is real, i.e., $\Lambda(f) \in \mathbb{R}$ for any $f \in L^q(X, \mu, \mathbb{R}_{\geqslant 0})$.

Let us define $\mathbb{R}_{\geqslant 0}$-linear maps $\Lambda^+, \Lambda^- : L^q(X, \mu, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ with operator norms $\leqslant \|\Lambda\|$, i.e.,

$$\|\Lambda^\pm(g)\| \leqslant \|\Lambda\| \cdot \|g\|_q \qquad \text{for all } g \in L^q(X, \mu, \mathbb{R}_{\geqslant 0}) \tag{1.29}$$

and let us check that

$$\Lambda(g) = \Lambda^+(g) - \Lambda^-(g) \qquad \text{for all } g \in L^q(X, \mu, \mathbb{R}_{\geqslant 0}) \tag{1.30}$$

Eq. (1.30) is called the **Jordan decomposition** of Λ.

Define the $\Lambda^\pm : L^q(X, \mu, \mathbb{R}_{\geqslant 0}) \to \overline{\mathbb{R}}$ by sending each $g \in L^q(X, \mu, \mathbb{R}_{\geqslant 0})$ to

$$\Lambda^\pm(g) = \sup\{\pm\Lambda(h) : h \in L^q(X, \mu, \mathbb{R}_{\geqslant 0}), h \leqslant g\} \tag{1.31}$$

Since $0 \leqslant g$, we clearly have $\Lambda^+(g) \geqslant 0$. Since Λ is bounded and $\|h\|_q \leqslant \|g\|_q$, we clearly have $\|\Lambda^+(g)\| \leqslant \|\Lambda\| \cdot \|g\|_q$. In particular, Λ^+ has range in $\mathbb{R}_{\geqslant 0}$. Since $\Lambda^\pm = (-\Lambda)^\mp$, a similar property holds for Λ^-. Thus, we have checked (1.29).

Clearly, for each $f, g \in L^1(X, \mu, \mathbb{R}_{\geqslant 0})$, we have $\Lambda^+(f + g) \geqslant \Lambda^+(f) + \Lambda^+(g)$. To prove the other direction, choose any $h \in L^q(X, \mu, \mathbb{R}_{\geqslant 0})$ such that $h \leqslant f + g$. Let $h_1 = fh/(f+g)$ and $h_2 = gh/(f+g)$, understood to be zero where the denominator vanishes. Then $h_1, h_2 \in L^1(X, \mu, \mathbb{R}_{\geqslant 0})$ and $h_1 \leqslant f$ and $h_2 \leqslant g$. This proves $\Lambda^+(f + g) \leqslant \Lambda^+(f) + \Lambda^+(g)$. Thus Λ^+ (and similarly Λ^-) is $\mathbb{R}_{\geqslant 0}$-linear.

From (1.31), one easily checks $\Lambda(g)+\Lambda^-(g)\leqslant\Lambda^+(g)$ for each $f\in L^q(X,\mu,\mathbb{R}_{\geqslant 0})$. Replacing Λ with $-\Lambda$, we get $-\Lambda(g)+\Lambda^+(g)\leqslant\Lambda^-(g)$. Thus (1.30) holds.

Step 2. Let us prove that Λ^+ is represented by some $f^+\in L^p(X,\mu,\mathbb{R}_{\geqslant 0})$, namely,

$$\Lambda^+(g)=\int_X f^+gd\mu\qquad\text{for all } g\in L^q(X,\mu,\mathbb{R}_{\geqslant 0})\tag{1.32}$$

Then, similarly, Λ^- is represented by some $f^-\in L^p(X,\mu,\mathbb{R}_{\geqslant 0})$. Thus Λ is represented by f^+-f^-, finishing the proof.

Write $X=\bigsqcup_n X_n$ where $\mu(X_n)<+\infty$. Suppose that we can find $f_n^+\in L^p(X_n,\mu)$ representing $\Lambda^+|_{L^q(X_n,\mu)}$, then we can define $f^+:X\to\mathbb{R}_{\geqslant 0}$ such that $f^+|_{X_n}=f_n$ for all n. Clearly f^+ represents Λ^+. In particular, by Lem. 1.6.11 and (1.29), $\|f^+\|_p\leqslant\|\Lambda\|<+\infty$. Thus $f\in L^p(X,\mu)$.

Therefore, according to the previous paragraph, we may assume at the beginning that $\mu(X)<+\infty$. Define

$$\nu:\mathfrak{M}\to[0,+\infty]\qquad E\mapsto\Lambda(\chi_E)$$

Then one checks easily that ν is a measure,[1] and that $\nu\ll\mu$. Therefore, by the Radon-Nikodym Thm. 1.6.8, there exists $f^+\in\mathcal{L}(X,\mathbb{R}_{\geqslant 0})$ such that $d\nu=f^+d\mu$. Thus

$$\Lambda^+(g)=\int_X gd\nu=\int_X f^+gd\mu\quad\text{for each simple function } g\in L^q(X,\mu,\mathbb{R}_{\geqslant 0})\tag{1.33}$$

Lem. 1.6.11 and (1.29) then imply $\|f^+\|_p\leqslant\|\Lambda\|<+\infty$, and hence $f\in L^p(X,\mu,\mathbb{R}_{\geqslant 0})$.

Finally, for $g\in L^q(X,\mu,\mathbb{R}_{\geqslant 0})$, find an increasing sequence of simple functions $g_n\in L^q(X,\mu,\mathbb{R}_{\geqslant 0})$ converging pointwise to g. By (1.29), $\Lambda^+(g-g_n)\leqslant\|\Lambda\|\cdot\|g-g_n\|_q$ where the RHS converges to zero by DCT. By MCT, $\int_X f^+g_nd\mu\to\int_X f^+gd\mu$. Thus, by (1.33), we conclude (1.32). □

1.7 Review of measure theory: Radon measures

1.7.1 Radon measures and the Riesz-Markov representation theorem

Let X be LCH. The reference for this subsection is [Gui-A, Ch. 25].

Definition 1.7.1. Let $\mathfrak{M}\subset 2^X$ be a σ algebra containing $\mathfrak{B}_X$, and let $\mu:\mathfrak{M}\to\overline{\mathbb{R}}_{\geqslant 0}$ be a measure. Let $E\in\mathfrak{M}$. We say that μ is **outer regular** on E if

$$\mu(E)=\inf\{\mu(U):U\supset E, U\text{ is open}\}$$

[1] To check the countable additivity, we let $E_1\subset E_2\subset\cdots$ be measurable and $E=\bigcup_n E_n$. Let $F_n=E\backslash E_n$. By (1.29), $\nu(F_n)\leqslant\|\Lambda\|\mu(F_n)^{\frac{1}{q}}\to 0$. Thus $\nu(E_n)\to\nu(E)$.

We say that μ is **inner regular** on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}$$

We say that μ is **regular** on E if μ is both outer and inner regular on E.

Lemma 1.7.2. *Let $\mu : \mathfrak{B}_X \to \overline{\mathbb{R}}_{\geqslant 0}$ be a Borel measure. Let $U \subset X$ be open. Then*

$$\sup\left\{\mu(K) : K \subset U, K \text{ is compact}\right\} = \sup\left\{\int_X f d\mu : f \in C_c(U, [0,1])\right\}$$

Therefore, μ is inner regular on U iff

$$\mu(U) = \sup\left\{\int_X f d\mu : f \in C_c(U, [0,1])\right\}$$

Proof. Let A, B denote the LHS and the RHS. If $f \in C_c(U, [0,1])$, then setting $K = \mathrm{Supp}(f)$, we have $\mu(K) = \int_X \chi_K d\mu \geqslant \int_X f d\mu$. This proves $A \geqslant B$.

Conversely, let $K \subset U$. By Urysohn's lemma, there exists $f \in C_c(U, [0,1])$ such that $f|_K = 1$. So $\mu(K) = \int_X \chi_K d\mu \leqslant \int_X f d\mu$. This proves $A \leqslant B$. □

Definition 1.7.3. A Borel measure $\mu : \mathfrak{B}_X \to \overline{\mathbb{R}}_{\geqslant 0}$ is called a **Radon measure** if the following conditions are satisfied:

(a) μ is outer regular on Borel sets.

(b) μ is inner regular on open sets. Equivalently, for each open $U \subset X$, we have

$$\mu(U) = \sup\left\{\int_X f d\mu : f \in C_c(U, [0,1])\right\} \tag{1.34}$$

(c) $\mu(K) < +\infty$ if K is a compact subset of X. Equivalently, for each $f \in C_c(X, \mathbb{R}_{\geqslant 0})$ we have

$$\int_X f d\mu < +\infty \tag{1.35}$$

Proof of equivalence. The equivalence in (b) is due to Lem. 1.7.2. The equivalence in (c) can be proved in a similar way to Lem. 1.7.2. □

Note that Radon measures are determined by their integrals against functions in $C_c(X, [0,1])$. Indeed, by (a), μ is determined by its values on open sets. (b) shows that those values on open sets are determined by the integrals $\int_X f d\mu$ where $f \in C_c(X, [0,1])$.

Remark 1.7.4. There exist canonical bijections among:

- $\mathbb{R}_{\geqslant 0}$-linear maps $C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$
- Positive linear functionals on $C_c(X, \mathbb{R})$.
- Positive linear functionals on $C_c(X) = C_c(X, \mathbb{C})$.

Proof. An $\mathbb{R}_{\geqslant 0}$-linear map $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ can be extended uniquely to a linear map $\Lambda : C_c(X, \mathbb{R}) \to \mathbb{R}$ due to the following Lem. 1.7.5. The latter can be extended to a linear functional on $C_c(X)$ by setting

$$\Lambda(f) = \Lambda(\mathrm{Re}f) + \mathbf{i}\Lambda(\mathrm{Im}f) \tag{1.36}$$

for all $C_c(X)$. □

Lemma 1.7.5. *Let K be an $\mathbb{R}_{\geqslant 0}$-linear subspace of an $\mathbb{R}$-vector space V. Let W be an $\mathbb{R}$-linear space. Let $\Gamma : K \to W$ be an $\mathbb{R}_{\geqslant 0}$-linear map. Suppose that $V = \mathrm{Span}_{\mathbb{R}} K$. Then Γ can be extended uniquely to an $\mathbb{R}$-linear map $\Lambda : V \to W$.*

Proof. The uniqueness is obvious. To prove the existence, note that any $v \in V$ can be written as

$$v = v^+ - v^-$$

where $v^+, v^- \in K$. (Proof: Since $V = \mathrm{Span}_{\mathbb{R}} K$, we have $v = a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n$ where each u_i, w_j are in K, and each a_i, b_j are in $\mathbb{R}_{\geqslant 0}$. One sets $v^+ = \sum_i a_iu_i$ and $v^- = \sum_j b_jw_j$.) We then define $\Lambda(v) = \Gamma(v^+) - \Gamma(v^-)$.

Let us show that this gives a well-defined map $\Lambda : V \to W$. Assume that $v = w^+ - w^-$ where $w^+, w^- \in K$. Then $\Gamma(v^+) - \Gamma(v^-) = \Gamma(w^+) - \Gamma(w^-)$ iff $\Gamma(v^+) + \Gamma(w^-) = \Gamma(v^-) + \Gamma(w^+)$, iff (by the additivity of Γ) $\Gamma(v^+ + w^-) = \Gamma(v^- + w^+)$. The last statement is true because $v^+ - v^- = w^+ - w^-$ implies $v^+ + w^- = v^- + w^+$.

It is easy to see that Λ is additive. If $c \geqslant 0$, then $cv = cv^+ - cv^-$ where $cv^+, cv^- \in K$. So $\Lambda(cv) = \Gamma(cv^+) - \Gamma(cv^-)$, which (by the $\mathbb{R}_{\geqslant 0}$-linearity of Γ) equals $c\Gamma(v^+) - c\Gamma(v^-) = c\Lambda(v)$. Since $-v = v^- - v^+$, we have $\Lambda(-v) = \Gamma(v^-) - \Gamma(v^+) = -\Lambda(v)$. Hence $\Lambda(-cv) = c\Lambda(-v) = -c\Lambda(v)$. This proves that Λ commutes with the $\mathbb{R}$-multiplication. □

Theorem 1.7.6 (Riesz-Markov representation theorem). *For every positive linear $\Lambda : C_c(X, \mathbb{F}) \to \mathbb{F}$ there exists a unique Radon measure $\mu : \mathfrak{B}_X \to \overline{\mathbb{R}}_{\geqslant 0}$ such that*

$$\Lambda(f) = \int_X f d\mu \tag{1.37}$$

for all $f \in C_c(X, \mathbb{F})$. Moreover, every Radon measure on X arises from some Λ in this way.

In addition, the operator norm $\|\Lambda\|$ equals $\mu(X)$. Therefore, Λ is bounded iff μ is a finite measure.

Proof. See [Gui-A, Sec. 25.3] for the first paragraph. The second paragraph asserts that

$$\sup_{f\in\overline{B}_{C_c(X)}(0,1)} |\Lambda(f)| = \mu(X)$$

The inequality "$\leqslant$" is obvious. The reverse inequality "$\geqslant$" follows from (1.34). $\square$

1.7.2 Basic properties of Radon measures

Theorem 1.7.7. *Let μ be a Radon measure (or its completion) on X. Then μ is regular on any measurable set E satisfying $\mu(E) < +\infty$.*

Proof. See [Gui-A, Sec. 25.4]. A sketch of the proof (different from that in [Gui-A]) is as follows.

Assume WLOG that E is Borel. Since Radon measures are outer regular on Borel sets, it remains to prove that μ is inner regular on E. Pick an open set $\mu(U)$ such that $\mu(U\backslash E)$ is small. Since μ is inner regular on U, there is a compact $K \subset U$ such that $\mu(U\backslash K)$ is small. However, K is not necessarily contained in E.

To fix this issue, we note that since μ is outer regular on $U\backslash E$, we can find an open set $V \subset U$ containing $U\backslash E$ whose measure is close to $\mu(U\backslash E)$. In particular, $\mu(V)$ is small. Then $K\backslash V$ is a compact subset of E whose measure is close to $\mu(E)$. $\square$

Theorem 1.7.8. *Assume that X is second countable. Let μ be a Borel measure on X. Then μ is Radon iff $\mu(K) < +\infty$ for any compact $K \subset X$.*

In particular, a finite Borel measure on $\mathbb{R}^n$ (where $n \in \mathbb{N}$) is Radon.

Proof. See [Gui-A, Sec. 25.5]. $\square$

1.7.3 Approximation and density

The main reference for this subsection is [Gui-A, Sec. 27.2].

Theorem 1.7.9 (Lusin's theorem). *Let X be LCH. Let μ be a Radon measure (or its completion) on X with σ-algebra $\mathfrak{M}$. Let $f : X \to \mathbb{F}$ be measurable. Let $A \in \mathfrak{M}$ such that $\mu(A) < +\infty$. Then for each $\varepsilon > 0$ there exists a compact $K \subset A$ such that $\mu(A\backslash K) < \varepsilon$ and that $f|_K : K \to \mathbb{F}$ is continuous.*

With the help of the Tietze extension Thm. 1.4.27, Lusin's theorem implies that for each $\varepsilon > 0$ there exist a compact $K \subset A$ and some $\widetilde{f} \in C_c(X,\mathbb{F})$ such that $\widetilde{f}|_K = f|_K$ and $\mu(A\backslash K) < \varepsilon$.

Proof. See [Gui-A, Sec. 25.4]. $\square$

Theorem 1.7.10. *Let $1 \leqslant p < +\infty$. Let μ be a Radon measure (or its completion) on an LCH space X. Then, under the L^p-norm, the space $C_c(X,\mathbb{F})$ is dense in $L^p(X,\mu,\mathbb{F})$. More precisely, the map $f \in C_c(X,\mathbb{F}) \mapsto f \in L^p(X,\mu,\mathbb{F})$ has dense range.*

Proof. See [Gui-A, Sec. 27.2]. □

Remark 1.7.11. One easily checks that

$$\begin{aligned}&\mathrm{Span}_{\mathbb{F}}\{\chi_I : I \subset \mathbb{R} \text{ is a bounded interval}\}\\ =&\mathrm{Span}_{\mathbb{F}}\{\chi_I : I \subset \mathbb{R} \text{ is a compact interval}\}\\ =&\mathrm{Span}_{\mathbb{F}}\{\chi_I : I \subset \mathbb{R} \text{ is a bounded open interval}\}\end{aligned}$$

An element in these sets is called an $\mathbb{F}$-valued **step function**. Moreover, one checks that

$$\begin{aligned}\{\text{right-continuous } \mathbb{F}\text{-valued step functions}\} &= \mathrm{Span}_{\mathbb{F}}\{\chi_{[a,b)} : a,b \in \mathbb{R}\}\\ \{\text{left-continuous } \mathbb{F}\text{-valued step functions}\} &= \mathrm{Span}_{\mathbb{F}}\{\chi_{(a,b]} : a,b \in \mathbb{R}\}\end{aligned}$$

Theorem 1.7.12. *Let $1 \leqslant p < +\infty$. Let μ be a Radon measure (or its completion) on $\mathbb{R}$. Then each of the following classes of functions form a dense subset of $L^p(\mathbb{R},\mu,\mathbb{F})$:*

(a) Right-continuous $\mathbb{F}$-valued step functions.

(b) Left-continuous $\mathbb{F}$-valued step functions.

(c) Elements of $\mathrm{Span}_{\mathbb{F}}\{\chi_{(-\infty,b]} : b \in \mathbb{R}\}$.

(d) Elements of $\mathrm{Span}_{\mathbb{F}}\{\chi_{(-\infty,b)} : b \in \mathbb{R}\}$.

Proof. With the help of Thm. 1.7.10, the density of (a) and (b) can be proved by approximating a function $f \in C_c(X,\mathbb{F})$ with left/right-continuous step functions. See [Gui-A, Sec. 27.2] for details.

Since (a)$\subset$(c) and (b)$\subset$(d), the density of (c) and (d) follows. □

Theorem 1.7.13. *Let $1 \leqslant p < +\infty$. Let μ be a Radon measure (or its completion) on a second countable LCH space X. Then $L^p(X,\mu,\mathbb{F})$ is separable.*

Proof. See [Gui-A, Sec. 27.2]. □

1.7.4 Complex Radon measures

Definition 1.7.14. If X is a set and $\mathfrak{M} \subset 2^X$ is a σ-algebra, a **complex measure** (resp. **signed measure**) is a function $\mathfrak{M} \to \mathbb{C}$ (resp. $\mathfrak{M} \to \mathbb{R}$) that can be written as a $\mathbb{C}$-linear (resp. $\mathbb{R}$-linear) combination of finite measures on $\mathfrak{M}$.

We now assume that X is LCH.

Definition 1.7.15. A complex (resp. signed) measure on $\mathfrak{B}_X$ is called **Radon** if it is a $\mathbb{C}$-linear (resp. $\mathbb{R}$-linear) combination of finite Radon measures.

Suppose that μ is a complex Radon measure on X. Then similar to the proof of Rem. 1.7.4, for each $f \in C_0(X)$, we can extend the $\mathbb{R}_{\geqslant 0}$-linear functional $f \mapsto \int_X f d\mu$, where μ are finite Radon measures, to $\mu \mapsto \int_X f d\mu$ for all complex Radon measures μ. This gives a $\mathbb{C}$-bilinear map

$$(f, \mu) \mapsto \int_X f d\mu \qquad \in \mathbb{C}$$

for $f \in C_0(X)$ and complex Radon measures μ.

Theorem 1.7.16 (Riesz-Markov representation theorem). *Let $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{F} = \mathbb{R}$.) Then the elements of $C_c(X, \mathbb{F})^*$ are precisely linear functionals*

$$\Lambda : C_c(X, \mathbb{F}) \to \mathbb{F} \qquad f \mapsto \int_X f d\mu$$

where μ is complex (resp. signed) Radon measure on X.

Proof. It suffices to assume that Λ is real, i.e., sending $C_c(X, \mathbb{R})$ into $\mathbb{R}$. Similar to the proof of Thm. 1.6.12, one writes $\Lambda = \Lambda^+ - \Lambda^-$ where $\Lambda^\pm$ are positive. Then apply Thm. 1.7.6 to $\Lambda^\pm$. See [Gui-A, Subsec. 25.10.2] for details. $\square$

Remark 1.7.17. Since $C_c(X, \mathbb{F})$ is l^∞-dense in $C_0(X, \mathbb{F})$, by Cor. 2.4.3, the dual spaces $C_c(X, \mathbb{F})^*$ and $C_0(X, \mathbb{F})^*$ are canonically identified. Therefore, Thm. 1.7.16 holds verbatim if $C_c(X, \mathbb{F})$ is replaced by $C_0(X, \mathbb{F})$.

1.8 Basic facts about increasing functions

1.8.1 Notation

If $I \subset \mathbb{R}$ is a proper interval, a function $\rho : I \to \mathbb{R}$ is called **increasing** if it is non-decreasing, i.e., $\rho(x) \leqslant \rho(y)$ whenever $x, y \in I$ and $x \leqslant y$. For each $t \in \mathbb{R}$, let

$$I_{\leqslant t} = I \cap (-\infty, t] \quad I_{<t} = I \cap (-\infty, t) \quad I_{\geqslant t} = I \cap [t, +\infty) \quad I_{>t} = I \cap (t, +\infty)$$

Suppose that $a = \inf I$ and $b = \sup I$. Let $\rho : I \to \mathbb{R}$ be increasing. If $x \in (a, b)$, then the left and right limits[2]

$$\rho(x^-) = \lim_{y \to x^-} \rho(y) \qquad \rho(x^+) = \lim_{y \to x^+} \rho(y) \tag{1.38}$$

exist, and

$$\rho(x^-) \leqslant \rho(x) \leqslant \rho(x^+)$$

[2]When taking the limit $\lim_{y \to x^\pm}$, we do not allow y to be equal to x.

If $a \in I$, then $\rho(a^+)$ exists, and $\rho(a) \leqslant \rho(a^+)$. If $b \in I$, then $\rho(b^-)$ exists, and $\rho(b^-) \leqslant \rho(b)$. Let

$$\Omega_\rho = \{x \in (a,b) : \rho|_{(a,b)} \text{ is continuous at } x\}$$

Then for each $x \in (a,b)$, we have

$$x \in \Omega_\rho \quad \Leftrightarrow \quad \rho(x^-) = \rho(x^+) \quad \Leftrightarrow \quad \rho(x^-) = \rho(x) = \rho(x^+) \tag{1.39}$$

1.8.2 Basic properties of increasing functions

Let $I \subset \mathbb{R}$ be a proper interval with $a = \inf I, b = \sup I$.

Proposition 1.8.1. *If $\rho : I \to \mathbb{R}$ is increasing, then $I \backslash \Omega_\rho$ is countable.*

Proof. Replacing ρ with $\arctan \circ \rho$, we may assume that ρ is bounded. Let $C = \mathrm{diam}(\rho(I)) = \sup_{x,y \in I} |\rho(x) - \rho(y)|$. Let $A = (a,b) \backslash \Omega_\rho$. Then for each $B \in \mathrm{fin}(2^A)$, we have

$$\sum_{x \in B} (\rho(x^+) - \rho(x^-)) \leqslant C$$

Applying $\lim_B$, we get $\sum_{x \in A} (\rho(x^+) - \rho(x^-)) \leqslant C < +\infty$. Therefore A is countable. $\square$

Definition 1.8.2. Let $\rho : I \to \mathbb{R}$. The **right-continuous normalization** of ρ is the function $\widetilde{\rho} : I \to \mathbb{R}$ defined by

$$\widetilde{\rho}(x) = \begin{cases} \rho(x^+) & \text{if } x < b \\ \rho(b) & \text{if } x = b \end{cases}$$

The function $\widetilde{\rho}$ is clearly increasing and right-continuous. Moreover, $\widetilde{\rho}$ clearly agrees with ρ on Ω_ρ. Therefore, $\widetilde{\rho}$ and ρ are almost equal, as defined by the following proposition.

Proposition 1.8.3. *Let $\rho_1, \rho_2 : I \to \mathbb{R}$ be increasing. Then the following are equivalent:*

(a) There exists a dense subset $E \subset I$ such that $\rho_1|_E = \rho_2|_E$.

(b) $\Omega_{\rho_1} = \Omega_{\rho_2}$, and $\rho_1|_{\Omega_{\rho_1}} = \rho_2|_{\Omega_{\rho_2}}$.

(c) The right-continuous normalizations of ρ_1 and ρ_2 agree on $I_{<b}$.

*If any of these statements are true, we say that ρ_1, ρ_2 are **almost equal**.*

Proof. (a)⇒(b): Assume (a). Choose any $x \in I$. If $x > a$ then

$$\rho_1(x^-) = \lim_{E \ni y \to x^-} \rho_1(y) = \lim_{E \ni y \to x^-} \rho_2(y) = \rho_2(x^-) \tag{1.40a}$$

Similarly, if $x < b$ then

$$\rho_1(x^+) = \rho_2(x^+) \tag{1.40b}$$

Thus (b) follows from (1.39).

(b)⇒(a): By Prop. 1.8.1, $E := (a, b) \cap \Omega_{\rho_1}$ is a dense subset of (a, b).

(b)⇔(c): Let $\widetilde{\rho}_i$ be the right continuous normalization of ρ_i. Then by (a)⇒(b), we have $\Omega_{\rho_i} = \Omega_{\widetilde{\rho}_i}$ and $\rho_i|_{\Omega_{\rho_i}} = \widetilde{\rho}_i|_{\Omega_{\widetilde{\rho}_i}}$. Therefore, (b) holds iff

$$\Omega_{\widetilde{\rho}_1} = \Omega_{\widetilde{\rho}_2} \qquad \text{and} \qquad \widetilde{\rho}_1|_{\Omega_{\widetilde{\rho}_1}} = \widetilde{\rho}_2|_{\Omega_{\widetilde{\rho}_2}} \tag{1.41}$$

Clearly (c) implies (1.41). Suppose that (1.41) is true. Then for each $x \in I_{<b}$ we have

$$\widetilde{\rho}_1(x) = \widetilde{\rho}_1(x^+) \xlongequal{(1.40b)} \widetilde{\rho}_2(x^+) = \widetilde{\rho}_2(x)$$

Thus (1.41) implies (c). Therefore (b) and (c) are equivalent. □

1.9 The Stieltjes integral

1.9.1 Definitions and basic properties

In this subsection, we fix a proper interval $I \subset \mathbb{R}$, and let $\rho : I \to \mathbb{R}_{\geqslant 0}$ be an increasing function.

Definition 1.9.1. Let J be any proper bounded interval. Let $a = \inf J, b = \sup J$. A **partition** of the interval J is defined to be an element of the form

$$\sigma = \{a_0, a_1, \ldots, a_n \in [a, b] : a_0 = a < a_1 < a_2 < \cdots < a_n = b, n \in \mathbb{Z}_+\} \tag{1.42}$$

The **mesh** of σ is defined to be

$$\max\{a_i - a_{i=1} : i = 1, \ldots, n\}$$

If $\sigma, \sigma' \in \mathrm{fin}(2^J)$ are partitions of J, we say that σ' is a **refinement** of σ (or that σ' is **finer than** σ), if $\sigma \subset \sigma'$. In this case, we also write

$$\sigma \prec \sigma'$$

We define $\mathcal{P}(J)$ to be

$$\mathcal{P}(J) = \{\text{partitions of } J\}$$

Remark 1.9.2. If $\sigma, \sigma' \in \mathcal{P}(J)$, then clearly $\sigma \cup \sigma' \in \mathcal{P}(J)$ and $\sigma, \sigma' \prec \sigma \cup \sigma$. Therefore, $\prec$ is a partial order on $\mathcal{P}(J)$. We call $\sigma \cup \sigma'$ the **common refinement** of σ and σ'.

Definition 1.9.3. A **tagged partition** of I is an ordered pair

$$(\sigma, \xi_\bullet) = \big(\{a_0 = a < a_1 < \cdots < a_n = b\}, (\xi_1, \ldots, \xi_n)\big) \tag{1.43}$$

where $\sigma \in \mathcal{P}(J)$ and

$$\xi_i \in (a_{j-1}, a_j]$$

for all $1 \leqslant j \leqslant n$. The set

$$\mathcal{Q}(J) = \{\text{tagged partitions of } J\}$$

equipped with the preorder $\prec$ defined by

$$(\sigma, \xi_\bullet) \prec (\sigma', \xi'_\bullet) \qquad \Longleftrightarrow \qquad \sigma \subset \sigma' \tag{1.44}$$

is a directed set.

Definition 1.9.4. Let V be a Banach space. Assume $[a, b] \subset I$ and $a < b$. Let $f \in C([a, b], V)$. For each $(\sigma, \xi_\bullet) \in \mathcal{Q}(I)$, define the **Stieltjes sum**

$$S_\rho(f, \sigma, \xi_\bullet) = \sum_{j \geqslant 1} f(\xi_j)\big(\rho(a_j) - \rho(a_{j-1})\big)$$

abbreviated to $S(f, \sigma, \xi_\bullet)$ when no confusion arises. The **Stieltjes integral** on $(a, b]$ is defined to be the limit of the net $(S_\rho(f, \sigma, \xi_\bullet))_{(\sigma, \xi_\bullet) \in \mathcal{Q}([a,b])}$:

$$\int_{(a,b]} f d\rho = \lim_{(\sigma, \xi_\bullet) \in \mathcal{Q}(I)} S_\rho(f, \sigma, \xi_\bullet) \tag{1.45}$$

The **Stieltjes integral** on $[a, b]$ is defined to be

$$\int_{[a,b]} f d\rho = f(a)\rho(a) + \int_{(a,b]} f d\rho \tag{1.46}$$

Note that when $f(a) \neq 0$, the integral $\int_{(a,b]} f d\rho$ depends not only on $\rho|_{(a,b]}$ but also on the value $\rho(a)$. On the other hand, it is clear that

$$\int_{(a,b]} f d\rho = \int_{(a,b]} f d\rho|_{[a,b]} \qquad \int_{[a,b]} f d\rho = \int_{[a,b]} f d\rho|_{[a,b]} \tag{1.47}$$

Proof of the convergence of (1.45). Since f is uniformly continuous, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(y)\| \leqslant \varepsilon$ for all $x, y \in [a, b]$ and $|x - y| \leqslant \delta$. Choose any tagged partition $(\sigma, \xi_\bullet)$ of $[a, b]$ with mesh $\leqslant \delta$. Then one easily sees that for any $(\sigma', \xi'_\bullet) \succ (\sigma, \xi)$ we have

$$\|S(f, \sigma', \xi'_\bullet) - S(f, \sigma, \xi_\bullet)\| \leqslant \varepsilon(\rho(b) - \rho(a))$$

Therefore, the net $(S(f, \sigma, \xi_\bullet))_{(\sigma, \xi_\bullet) \in \mathcal{Q}(I)}$ is Cauchy. So it must converge because V is complete. □

Remark 1.9.5. The above proof implies the following useful fact: Let $f \in [a,b]$. Let $\varepsilon, \delta > 0$ such that $\|f(x) - f(y)\| \leqslant \varepsilon$ for all $x, y \in [a,b]$ satisfying $|x-y| \leqslant \delta$. Then for each tagged partition $(\sigma, \xi_\bullet)$ of $[a,b]$ with mesh $\leqslant \delta$, we have

$$\Big\| \int_{(a,b]} f d\rho - S_\rho(f,\sigma,\xi_\bullet) \Big\| \leqslant \varepsilon(\rho(b) - \rho(a)) \tag{1.48}$$

and hence

$$\Big\| \int_{[a,b]} f d\rho - f(a)\rho(a) - S_\rho(f,\sigma,\xi_\bullet) \Big\| \leqslant \varepsilon(\rho(b) - \rho(a)) \tag{1.49}$$

Example 1.9.6. The integrals of the constant function 1 are

$$\int_{(a,b]} d\rho = \rho(b) - \rho(a) \qquad \int_{[a,b]} d\rho = \rho(b)$$

Example 1.9.7. Suppose that $\rho|_{(a,b]} = 1$. Then

$$\int_{(a,b]} f d\rho = f(a)(1-\rho(a)) \qquad \int_{[a,b]} f d\rho = f(a)$$

In particular, if $\rho|_{[a,b]} = 1$, then $\displaystyle\int_{(a,b]} f d\rho = 0$ and $\displaystyle\int_{[a,b]} f d\rho = f(a)$.

Remark 1.9.8. It is easy to see that

$$\Lambda : C([a,b], V) \to V \qquad f \mapsto \int_{[a,b]} f d\rho$$

is linear. Moreover, since $\|S(f,\sigma,\xi_\bullet)\| \leqslant (\rho(b) - \rho(a))\|f\|_{l^\infty}$ and hence $\|f(a)\rho(a) + S(f,\sigma,\xi_\bullet)\| \leqslant \rho(b)\|f\|_{l^\infty}$, the operator norm $\|\Lambda\|$ satisfies $\|\Lambda\| \leqslant \rho(b)$, that is

$$\Big\| \int_{[a,b]} f d\rho \Big\| \leqslant \rho(b)\|f\|_{l^\infty} \qquad \text{for all } f \in C([a,b], V)$$

In particular, Λ is bounded.

Remark 1.9.9. It is easy to check that $\rho \mapsto \int_{(a,b]} f d\rho$ and $\rho \mapsto \int_{[a,b]} f d\rho$ are $\mathbb{R}_{\geqslant 0}$-linear over increasing functions $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$. Moreover, if $c \in (a,b)$, one easily shows

$$\int_{(a,b]} f d\rho = \int_{(a,c]} f d\rho + \int_{(c,b]} f d\rho \tag{1.50}$$

by considering tagged partitions finer than $\{a, c, b\}$.

Exp. 1.9.7 suggests that the value of $\int_{[a,b]} f d\rho$ is independent of $\rho(a)$:

Lemma 1.9.10. *Suppose that $\rho_1, \rho_2 : [a,b] \to \mathbb{R}_{\geqslant 0}$ are increasing and satisfies $\rho_1|_{(a,b]} = \rho_2|_{(a,b]}$. Then for each $f \in C([a,b], V)$ we have $\displaystyle\int_{[a,b]} f d\rho_1 = \int_{[a,b]} f d\rho_2$.*

See Thm. 1.9.12 for a generalization of this lemma.

Proof. Assume WLOG that $\rho_1(a) \leqslant \rho_2(a)$. Let $\lambda = \rho_2(a) - \rho_1(a)$. Then $\rho_2 - \rho_1 = \lambda \cdot \chi_{\{a\}} = \lambda \cdot (1 - \chi_{(a,b]})$, and hence $\rho_1 + \lambda = \rho_2 + \lambda \cdot \chi_{(a,b]}$. By Rem. 1.9.9,

$$\int_{[a,b]} f d\rho_1 + \lambda \int_{[a,b]} f d1 = \int_{[a,b]} f d\rho_2 + \lambda \int_{[a,b]} f d\chi_{(a,b]}$$

By Exp. 1.9.9, we obtain $\int_{[a,b]} f d\rho_1 = \int_{[a,b]} f d\rho_2$. □

1.9.2 Dependence of the Stieltjes integral on ρ

Let $I \subset \mathbb{R}$ be a proper interval, and let $a = \inf I$ and $b = \sup I$. Note that I is not assumed to be bounded.

Definition 1.9.11. For each $f \in C_c(I, V)$ and each increasing $\rho : I \to \mathbb{R}$, we can still define the **Stieltjes integral**

$$\int_I f d\rho := \int_J f d\rho$$

where J is any compact sub-interval of I containing $\mathrm{Supp}_I(f)$. The value of the integral is clearly independent of the choice of such J. Moreover, this definition is compatible with the definitions of $\int_{[a,b]} f d\rho$ and $\int_{(a,b]} f d\rho$ in Def. 1.9.4.

Theorem 1.9.12. *Let $\rho_1, \rho_2 : I \to \mathbb{R}_{\geqslant 0}$ be increasing functions satisfying the following condition:*

- *ρ_1 and ρ_2 are almost equal, and $\rho_1(b) = \rho_2(b)$ if $b \in I$. (By Prop. 1.8.3, this is equivalent to that ρ_1, ρ_2 have the same right-continuous normalization.)*

Then for each $f \in C_c(I, V)$, we have

$$\int_I f d\rho_1 = \int_I f d\rho_2$$

Proof. By Lem. 1.9.10, we may assume that $\rho_1(a) = \rho_2(a)$ if $a \in I$.

Fix $f \in C_c(I, V)$. Choose $\alpha, \beta \in \mathbb{R}$ satisfying $\mathrm{Supp}_I(f) \subset [\alpha, \beta] \subset I$. Due to the assumption on ρ_1, ρ_2, we may slightly enlarge the compact interval $J := [\alpha, \beta]$ so that

$$\rho_1(\alpha) = \rho_2(\alpha) \qquad \rho_1(\beta) = \rho_2(\beta)$$

(When $a \in I$ resp. $b \in I$, one can even set $\alpha = a$ resp. $\beta = b$.) Then $\int_I f d\rho_i = \int_J f d\rho_i$.

Let $C = \max\{\rho_i(\beta) - \rho_i(\alpha) : i = 1, 2\}$. Choose any $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $x, y \in I$ and $|x - y| \leqslant \delta$. Choose a tagged partition

$$(\sigma, \xi_\bullet) = \big(\{a_0 = \alpha < a_1 < \cdots < a_n = \beta\}, (\xi_1, \ldots, \xi_n)\big)$$

of J with mesh $< \delta$. Moreover, due to the assumption on ρ_1, ρ_2, by a slight adjustment, we may assume that $\rho_1(a_j) = \rho_2(a_j)$ for each $0 \leqslant j \leqslant n$. This implies

$$S_{\rho_1}(f, \sigma, \xi_\bullet) = S_{\rho_2}(f, \sigma, \xi_\bullet)$$

Therefore, by Rem. 1.9.5, we obtain

$$\Big\| \int_J f d\rho_1 - \int_J f d\rho_2 \Big\| \leqslant 2\varepsilon \cdot C$$

This completes the proof by choosing arbitrary ε. □

Theorem 1.9.13. *Let $\rho_1, \rho_2 : I \to \mathbb{R}_{\geqslant 0}$ be bounded increasing functions satisfying*

$$\lim_{x \to a^+} \rho_1(x) = \lim_{x \to a^+} \rho_2(x) = 0 \qquad \text{if } a \notin I \tag{1.51}$$

Then the following are equivalent:

(1) *ρ_1 and ρ_2 are almost equal, and $\rho_1(b) = \rho_2(b)$ if $b \in I$. (By Prop. 1.8.3, this is equivalent to that ρ_1, ρ_2 have the same right-continuous normalization.)*

(2) *For each $f \in C_c(I, \mathbb{F})$ we have*

$$\int_I f d\rho_1 = \int_I f d\rho_2$$

Proof. By Thm. 1.9.12, we have "(1)⇒(2)". Assume (2). Let us prove (1). Let $\widetilde{\rho}_i$ be the right-normalization of ρ_i. By "(1)⇒(2)", we have $\int_I f d\rho_i = \int_I f d\widetilde{\rho}_i$. Therefore, to prove (1), it suffices to assume that ρ_1 and ρ_2 are right-continuous on I.

We shall prove (1) by choosing an arbitrary bounded increasing right-continuous $\rho : I \to \mathbb{R}_{\geqslant 0}$, and show that for each $x \in I$, the value $\rho(x)$ can be recovered from the integrals $\int_I f d\rho$ where $f \in C_c(I, \mathbb{R})$.

Case 1: Assume $a \notin I$ and $a < x < b$. For each real numbers v, y satisfying

$$a < v < x < y < b$$

choose $\varphi_{v,y} \in C_c(I, [0, 1])$ satisfying

$$\chi_{[v,x]} \leqslant \varphi_{v,y} \leqslant \chi_{(a,y]}$$

Choose $u \in (a,v)$ such that $\varphi_{v,y}$ vanishes outside $[u,y]$. Then by Rem. 1.9.9,

$$\int_I \varphi_{v,y}d\rho = \int_{[u,y]} \varphi_{v,y}d\rho = \varphi_{v,y}(u) + \int_{(u,v]} \varphi_{v,y}d\rho + \int_{(v,x]} \varphi_{v,y}d\rho + \int_{(x,y]} \varphi_{v,y}d\rho$$
$$= \int_{(u,v]} \varphi_{v,y}d\rho + \rho(x) - \rho(v) + \int_{(x,y]} \varphi_{v,y}d\rho$$

where Exp. 1.9.6 is used in the last equality. By Rem. 1.9.5, we have $\int_{(u,v]} \varphi_{v,y}d\rho \leqslant \rho(v) - \rho(u) \leqslant \rho(v)$ and $\int_{(x,y]} \varphi_{v,y}d\rho \leqslant (\rho(y) - \rho(x))$. Since ρ is right-continuous and satisfies (1.51), we have

$$\lim_{v \searrow a^+} \rho(v) = \lim_{y \searrow x^+} (\rho(y) - \rho(x)) = 0$$

Therefore, the above calculation of $\int_I \varphi_{v,y}d\rho$ shows

$$\lim_{\substack{v \searrow a^+ \\ y \searrow x^+}} \int_I \varphi_{v,y}d\rho = \lim_{v \searrow a^+} (\rho(x) - \rho(v)) = \rho(x)$$

Case 2: Assume $a \in I$ and $a \leqslant x < b$. For each $y \in (x,b)$, choose $\varphi_y \in C_c(I,[0,1])$ such that $\chi_{[a,x]} \leqslant \varphi_y \leqslant \chi_{[a,y]}$. Similar to the argument in Case 1, one shows

$$\int_I \varphi_y d\rho = \int_{[a,x]} \varphi_y d\rho + \int_{(x,y]} \varphi_y d\rho = \rho(x) + \int_{(x,y]} \varphi_y d\rho$$

where Exp. 1.9.6 is used. By Rem. 1.9.5, $\int_{(x,y]} \varphi_y d\rho \leqslant \rho(y) - \rho(x)$. Therefore, the right-continuity of ρ implies

$$\lim_{y \searrow x^+} \int_I \varphi_y d\rho = \rho(x)$$

Case 3: Assume $I = (a,b]$ and $x = b$. For each $v \in (a,x)$, choose $\varphi_v \in C_c(I,[0,1])$ such that $\chi_{[v,b]} \leqslant \varphi_v \leqslant \chi_I$. Similar to the argument above,

$$\lim_{v \searrow a^+} \int_I \varphi_v d\rho = \rho(b)$$

Case 4: Assume $I = [a,b]$ and $x = b$. Then $\displaystyle\int_I d\rho = \rho(b)$. $\square$

Remark 1.9.14. The assumption (1.51) imposes little restriction. Indeed, suppose $a \notin I$. Then for each $f \in C_c(I)$, since there exists $v \in \mathbb{R}_{>a}$ such that f vanishes on $(a,v]$, for any constant $\varkappa \in \mathbb{R}$ with $\rho + \varkappa \geqslant 0$, we clearly have

$$\int_I f d\rho = \int_I f d(\rho + \varkappa) \tag{1.52}$$

Therefore, when $a \notin I$, given any two increasing functions $\rho_1, \rho_2 : I \to \mathbb{R}_{\geqslant 0}$, we can freely add constants to ρ_1 and ρ_2 to ensure that (1.51) holds.

1.10 The Riesz representation theorem via the Stieltjes integral

In this section, we fix a proper interval $I \subset \mathbb{R}$, and let $a = \inf I$ and $b = \sup I$.

1.10.1 The positive case

Theorem 1.10.1 (Riesz representation theorem). *We have a bijection between:*

(a) A bounded increasing right-continuous function $\rho : I \to \mathbb{R}_{\geqslant 0}$ satisfying $\lim_{x\to a^+} \rho(a) = 0$ if $a \notin I$.

(b) A bounded positive linear functional $\Lambda : C_c(I, \mathbb{F}) \to \mathbb{F}$.

Λ is determined by ρ by

$$\Lambda : C(I, \mathbb{F}) \to \mathbb{F} \qquad f \mapsto \int_I f d\rho \tag{1.53}$$

ρ is determined by Λ by

$$\rho(x) = \mu(I_{\leqslant x}) \qquad \text{for all } x \in I \tag{1.54}$$

where μ is the finite Borel measure on I associated to Λ as in the Riesz-Markov representation Thm. 1.7.6.

Note that by Thm. 1.7.8, finite Borel measures on I and finite Radon measures on I coincide.

Proof. Step 1. Thm. 1.7.6 establishes the equivalence between a bounded positive linear Λ and a finite Borel measure μ. We let prove the equivalence between the radon measures μ and the functions ρ satisfying (a).

More precisely, given a Radon measure μ on I, let $\rho_\mu : I \to \mathbb{R}_{\geqslant 0}$ be defined by (1.54), that is, for each $x \in I$ we have

$$\rho_\mu(x) = \mu(I_{\leqslant x}) \tag{1.55}$$

Then ρ_μ is clearly bounded and increasing. By DCT, ρ_μ is right-continuous, and we have $\lim_{x\to a^-} \rho(x) = 0$ when $a \notin I$. Therefore, ρ_μ satisfies (a).

Conversely, given any ρ satisfying (a), let μ_ρ be the unique Radon measure corresponding to ρ via (1.53), i.e., for each $f \in C_c(I, \mathbb{F})$ we have

$$\int_I f d\mu_\rho = \int_I f d\rho \tag{1.56}$$

By Rem. 1.9.8, the linear functional $f \in C_c(I, \mathbb{F}) \mapsto \int_I f d\rho$ is bounded with operator norm $\leqslant \sup_{x\in I} \rho(x)$. Thus, μ_ρ is a finite measure.

We want to show that $\Phi : \rho \mapsto \mu_\rho$ and $\Psi : \mu \mapsto \rho_\mu$ are inverses of each other. By Thm. 1.9.13, the map Φ is injective. Therefore, it suffices to prove that $\Phi \circ \Psi = \mathrm{id}$, i.e., that $\mu_{\rho_\mu} = \mu$. This means proving

$$\int_I f d\mu = \int_I f d\rho_\mu \tag{1.57}$$

for each $f \in C_c(I, \mathbb{F})$.

Step 2. Let us fix $f \in C_c(I, \mathbb{F})$ and prove (1.57). Choose $\alpha, \beta \in \mathbb{R}$ such that $J := [\alpha, \beta]$ is a sub-interval of I containing $\mathrm{Supp}_I(f)$. Choose any $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $x, y \in I$ and $|x - y| \leqslant \delta$. Choose a tagged partition

$$(\sigma, \xi_\bullet) = \big(\{a_0 = \alpha < a_1 < \cdots < a_n = \beta\}, (\xi_1, \ldots, \xi_n)\big)$$

of J with mesh $\leqslant \delta$. By Rem. 1.9.5, we have

$$\Big|\int_J f d\rho_\mu - f(\alpha)\rho_\mu(\alpha) - S_{\rho_\mu}(f, \sigma, \xi_\bullet)\Big| \leqslant \varepsilon(\rho_\mu(\beta) - \rho_\mu(\alpha)) = \varepsilon \cdot \mu((\alpha, \beta]) \tag{1.58}$$

Also, we have $\|f - g\|_{l^\infty(I)} \leqslant \varepsilon$ where

$$g = f(\alpha)\chi_{\{a\}} + \sum_{i=1}^{n} f(\xi_i)\chi_{(a_{i-1}, a_i]}$$

By (1.54), we have

$$\mu(\{\alpha\}) = \rho_\mu(\alpha) - \mu(I_{<\alpha}) \qquad \mu((a_{i-1}, a_i]) = \rho_\mu(a_i) - \rho_\mu(a_{i-1})$$

Note that if $f(\alpha) \neq 0$, then by $\mathrm{Supp}_I(f) \subset [\alpha, \beta]$, we must have $\alpha = a \in I$ and hence $I_{<\alpha} = \varnothing$. Therefore, we must have

$$\int_I g d\mu = f(\alpha)\rho_\mu(\alpha) + S_{\rho_\mu}(f, \sigma, \xi_\bullet)$$

Combining this fact with $\|f - g\|_{l^\infty(I)} \leqslant \varepsilon$, we get

$$\Big|\int_I f d\mu - f(\alpha)\rho_\mu(\alpha) - S_{\rho_\mu}(f, \mu, \xi_\bullet)\Big| \leqslant \varepsilon \cdot \mu(J)$$

This inequality, together with (1.58), implies

$$\Big|\int_I f d\mu - \int_I f d\rho_\mu\Big| \leqslant 2\varepsilon \cdot \mu(J)$$

Since ε is arbitrary, we conclude (1.57). $\square$

1.10.2 The general case

Definition 1.10.2. A real-valued function $I \to \mathbb{F}$ is called of **bounded variation** (or simply **BV**) if it is an $\mathbb{F}$-linear combination of bounded increasing functions $I \to \mathbb{R}_{\geqslant 0}$. The space of BV functions from I to $\mathbb{F}$ is denoted by $BV(I, \mathbb{F})$.

Remark 1.10.3. By Rem. 1.9.8 and 1.9.9, we have an $\mathbb{R}_{\geqslant 0}$-bilinear functional

$$(f, \rho) \mapsto \int_I f d\rho \qquad \in \mathbb{R}_{\geqslant 0}$$

for $f \in C_c(I, \mathbb{R}_{\geqslant 0})$ and bounded increasing $\rho : I \to \mathbb{R}_{\geqslant 0}$. Similar to the proof of Rem. 1.7.4, it can be extended to a positive bililinear functional

$$C_c(I, \mathbb{C}) \times BV(I, \mathbb{C}) \to \mathbb{C} \qquad (f, \rho) \mapsto \int_I f d\rho$$

Theorem 1.10.4 (Riesz representation theorem). *The elements of the dual space $C_c(I, \mathbb{F})^*$ are precisely linear functionals of the form*

$$\Lambda : C(I, \mathbb{F}) \to \mathbb{F} \qquad f \mapsto \int_I f d\rho$$

where $\rho \in BV(I, \mathbb{F})$. Moreover, the BV function ρ can be chosen such that it is right-continuous on I, and that $\lim_{x \to a^+} \rho(x) = 0$ if $a \notin I$.

Proof. This is immediate from Thm. 1.7.16 and 1.10.1. $\square$

2 Normed vector spaces and their dual spaces

2.1 The origin of dual spaces in the calculus of variations

Linear functional analysis treats function spaces as linear spaces with appropriate geometric/topological structures and analytic properties. In the foundational theory of functional analysis, two analytic properties are especially important: (Cauchy) completeness and duality. In this course, our focus is primarily on normed vector spaces V. For such spaces, Cauchy completeness is interpreted in the same way as in any metric space. Duality, on the other hand, refers to the natural identification of V as the dual space V^* of another normed vector space U.

Many early results in functional analysis were related to duality, while the significance of completeness was not immediately recognized. In fact, the history of functional analysis experienced a paradigm shift from the study of (scalar-valued) functionals to linear maps between vector spaces. Specifically, attention moved from continuous bilinear maps of the form $U \times V \to \mathbb{F}$ to the analysis of continuous linear maps $V \to W$, where U, V, W are normed vector spaces. With this shift, completeness became increasingly central to modern analysis. See Sec. 2.5 for further illustrations.

The early part of this course will also focus more on dual spaces. If V is a normed $\mathbb{F}$-vector space, then the **dual space** $V^* = \mathfrak{L}(V, \mathbb{F})$ is defined to be the space of bounded (i.e. continuous) linear maps $V \to \mathbb{F}$. One of the major themes in early functional analysis was the characterization of dual spaces of various function spaces under appropriate norms. Among the most notable results are F. Riesz's characterization of $C([a, b], \mathbb{R})^*$ (cf. Thm. 1.10.4) in [Rie09, Rie11], and his proof that $L^q([a, b], m, \mathbb{R})^* \simeq L^p([a, b], m, \mathbb{R})$ (cf. Thm. 1.6.12) in [Rie10]. These results highlight a profound connection between dual spaces and measure/integration theory. Nevertheless, the study of dual spaces originally arose from a somewhat different field: the calculus of variations in the 19th century.

Consider a nonlinear functional $S : f \mapsto S(f) \in \mathbb{R}$, for example, of the form

$$S(f) = \int_a^b L(f(t), f'(t), \ldots, f^{(r)}(t))dt$$

where L is a "nice" real valued function with r-variables, and f is defined on $[a, b]$. If we perturb f slightly by a variation η, then the corresponding change in S can be approximated by

$$\delta S[f, \eta] := S(f + \eta) - S(f) \approx \int_a^b \beta_f(t) \cdot \eta(t)dt \tag{2.1}$$

where $\beta_f : [a, b] \to \mathbb{R}$ is a function depending on f. This function should be interpreted loosely. In some cases, it may involve delta functions or similar objects that are not functions in the classical sense, but rather distributions:

Example 2.1.1. Consider the case where L is smooth and $r = 1$, i.e.

$$S(f) = \int_a^b L(f(t), f'(t))dt$$

(For example, $L(x, y) = T(y) - V(x)$ where $T(y) = \frac{1}{2}my^2$ the kinetic energy for the mass $m \in \mathbb{R}_{>0}$, and $V(x)$ is the potential energy at x.) Then

$$\begin{aligned}\delta S[f, \eta] = &\int_a^b L(f + \eta, f' + \eta') \approx \int_a^b (\partial_x L(f, f')\eta + \partial_y L(f, \eta)\eta') \\ =&\partial_y L(f, f')\eta\Big|_a^b + \int_a^b (\partial_x L(f, f') - \partial_t\partial_y L(f, f'))\eta\end{aligned}$$

If we assume that the function f and its variation η always vanish at the endpoints a, b, then we obtain (2.1) with

$$\beta_f(t) = \partial_x L(f(t), f'(t)) - \partial_t\partial_y L(f(t), f'(t))$$

The equation $\beta_f = 0$ is called the **Euler-Lagrange equation**.

However, if no boundary conditions are imposed on the endpoints, then the term $\partial_y L(f, f')\eta\big|_a^b$ is not necessarily zero. As a result, we have

$$\beta_f = L(f(b), f'(b))\delta_b - L(f(a), f'(a))\delta_a + \partial_x L(f, f') - \partial_t\partial_y(f, f')$$

where, for each $c \in \mathbb{R}$, δ_c is the "**delta function**" at c, namely, the imaginary function $\mathbb{R} \to \mathbb{R}_{\geqslant 0}$ vanishing outside c and satisfying $\int_{\mathbb{R}} \delta_c = 1$. The situation becomes even more singular if we define S by

$$S(f) = \sum_{i=1}^n \lambda_i f(c_i) + \int_a^b L(f(t), f'(t))dt$$

where $\lambda_i \in \mathbb{R}$ and $a < c_i < b$, then

$$\beta_f = \sum_{i=1}^n \lambda_i\delta_{c_i} + L(f(b), f'(b))\delta_b - L(f(a), f'(a))\delta_a + \partial_x L(f, f') - \partial_t\partial_y(f, f')$$

This raises the question: what should the function β_f, alternatively the integral operator $\eta \mapsto \int_a^b \beta_f\eta$, actually look like in the general case? □

It is in this context that the problem of classifying bounded linear functionals on $C([a, b], \mathbb{R})$, originally posed by Hadamard in 1903, should be understood. Recall that if V, W are normed vector spaces, $\Omega \subset V$ is open, and $S : \Omega \to W$ is a map, one says that S is differentiable at $f \in \Omega$ if

$$S(f + \eta) - S(f) = \Lambda(\eta) + o(\eta)$$

where $\Lambda : V \to W$ is a bounded linear operator (called the **differential** of S at f), and $\lim_{\|\eta\|\to 0} o(\eta)/\|\eta\| = 0$. In the calculus of variations, one sets $W = \mathbb{F}$. Then $\Lambda \in V^*$. One can thus understand $\eta \mapsto \delta S[f, \eta]$ as a bounded linear functional on a function space V equipped with a suitable norm.

The problem of expressing $\delta S[f, \eta]$ as an integral involving η is therefore transformed to the problem of characterizing the dual space V^*. More precisely, the space V—and in particular its norm—is not fixed in advance. The situation is not that one starts with a given normed space and is then asked to characterize its dual. Rather, the task is to find an appropriate norm on a suitable function space V such that the bounded linear functionals on V, once studied and classified as integrals, are well-suited to capturing the variation of S.[1] The two perspectives on $\delta S[f, \eta]$—as a bounded linear functional on V, and as an integral involving η—together offer a deeper and more complete understanding of the variation of S.

More discussion of the relationship between dual spaces and the calculus of variations can be found in [Gray84].

2.2 Moment problems: a bridge between integral theory and dual spaces

The theory of dual spaces would not have reached its current depth and sophistication if it were developed solely within the framework of the calculus of variations. For instance, Riesz's classification of the duals of $C([a, b])$ and $L^p([a, b], m)$ would have been impossible without the Lebesgue and Stieltjes integrals. In fact, the very form of Riesz's theorems presents a striking connection between integration theory and dual spaces.

But why should such a connection exist in the first place? The way this relationship appears in Riesz's theorems calls for a deeper explanation. My short answer is this: it is the moment problems that form the bridge between integration theory and the theory of dual spaces. (Readers may jump ahead to Subsection 2.2.5 for the detailed final conclusion.)

To clarify my point, consider the first major example of a duality theorem: the identification $(L^2)^* \simeq L^2$ proved by Riesz and Fréchet in 1907:

Theorem 2.2.1 (Riesz-Fréchet theorem, the classical form**).** *We have a linear isomorphism*

$$\Lambda : L^2\big([-\pi, \pi], \frac{m}{2\pi}\big) \to L^2\big([-\pi, \pi], \frac{m}{2\pi}\big)^*$$

[1]The same function space V, when equipped with different norms, leads to different classifications of bounded linear functionals. For example, let $V = C([a, b])$. If the norm is l^∞, then by Thm. 1.10.4, the bounded linear functionals are the Stieltjes integrals with respect to BV functions. If the norm is L^2, then by Exp. 2.4.4, the bounded linear functionals are those of the form $f \mapsto \int fgdm$ where $g \in L^2([a, b], m)$.

$$\langle \Lambda(f), g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} fg\,dm$$

In fact, Riesz studied L^2 spaces several years before introducing the more general L^p spaces. His interest in L^2 spaces was clearly influenced by Hilbert's earlier work on the Hilbert space $l^2(\mathbb{Z})$ and its applications to the theory of integral equations. It was Hilbert's insights that served as the crucial bridge leading to the Riesz-Fréchet Thm. 2.2.1—the first major result linking Lebesgue integration with dual spaces.

As I will explain in the following, Hilbert's role in this development is best understood through the lens of moment problems.

2.2.1 Moment problems and dual spaces

Let me begin by introducing moment problems and explaining how they relate to dual spaces—particularly to the characterization of dual spaces in terms of integral representations.

Problem 2.2.2 (**Moment problem**, original version)**.** Let (ξ_n) be a sequence of scalar-valued functions defined on a space, e.g., an interval $I \subset \mathbb{R}$. Choose a sequence of scalars (c_n) satisfying certain conditions. Find a scalar valued function f on I such that for all n, we have

$$\int \xi_n f = c_n \qquad \text{resp.} \qquad \int \xi_n df = c_n \tag{2.2}$$

The numbers $c_1, c_2, \dots$ are called the **moments** of f resp. df.

There are two typical types of moment problems:

- **Trigonometric moment problem**: Here $I = \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, and $\xi_n(x) = e^{-\mathbf{i}nx}$ for $n \in \mathbb{Z}$. The problem then amounts to finding a function f with prescribed Fourier coefficients $c_1, c_2, \dots$.

- **Polynomial moment problem**: Here $I \subset \mathbb{R}$ is an interval, not necessarily bounded, and $\xi_n(x) = x^n$ for $n \in \mathbb{N}$. One is asked to find an increasing or BV function f such that df has moments $c_1, c_2, \dots$.

Many (but not all) moment problems can be reformulated in the language of bounded linear functionals and dual spaces as follows:

Problem 2.2.3 (**Moment problem**, dual space version)**.** Let (ξ_n) be a sequence in a normed vector space V, and let (c_n) be a sequence of scalars. Suppose that there exists $M \in \mathbb{R}_{\geqslant 0}$ such that

$$\Big|\sum_n a_n c_n\Big| \leqslant M \Big\|\sum_n a_n \xi_n\Big\| \tag{2.3}$$

for each sequence of scalars (a_n) with finitely many nonzero terms. Find $\varphi \in V^*$ such that

$$\langle \xi_n, \varphi \rangle = c_n \qquad \text{for all } n \tag{2.4}$$

Remark 2.2.4. Note that (2.3) is necessary for the existence of φ satisfying (2.4), because

$$\Big| \sum_n a_n c_n \Big| = \Big| \Big\langle \sum_n a_n \xi_n, \varphi \Big\rangle \Big| \leqslant \|\varphi\| \cdot \Big\| \sum_n a_n \xi_n \Big\|$$

where $\|\varphi\|$ is the operator norm. Hence (2.3) holds for any M satisfying $\|\varphi\| \leqslant M$.

Conversely, if we know that $V_0 = \mathrm{Span}\{\xi_n\}$ is dense in V, then (2.3) guarantees that the linear functional

$$\varphi : V_0 \to \mathbb{F} \qquad \sum_n a_n \xi_n \mapsto \sum_n a_n c_n$$

is well-defined and bounded, with operator norm $\|\varphi\| \leqslant M$. By boundedness, φ extends uniquely to a bounded linear functional on all of V, cf. Thm. 2.4.2. Therefore, Problem 2.2.3 can always be solved.

The case where V_0 is not dense in V is more subtle and will be treated in detail in a later chapter. □

Once Problem 2.2.3 is resolved—for example, when $\mathrm{Span}\{\xi_n\}$ is dense in V—Problem 2.2.2 can be solved by answering the following:

Problem 2.2.5 (Characterization of the dual space). Characterize the elements of V^* as precisely those linear functionals $\varphi : V \to \mathbb{F}$ of the form

$$\langle \xi, \varphi \rangle = \int \xi f \ \text{ resp. } \ \int \xi df$$

(for all $\xi \in V$), where f is a function satisfying suitable regularity or integrability conditions.

Conversely, Problem 2.2.5 can be reduced to the moment Problem 2.2.2 by choosing a densely-spanning (ξ_n) and taking $c_n = \langle \xi_n, \varphi \rangle$. The solution to Problem 2.2.2 then yields a function f such that $\langle \xi_n, f \rangle = \langle \xi_n, \varphi \rangle$. By the density of $\mathrm{Span}\{\xi_n\}$ in V, it follows that φ is represented by f.

Thus, we conclude that when (ξ_n) spans a dense subspace of V, the moment problem (Problem 2.2.2) and the characterization of dual spaces (Problem 2.2.5) are equivalent.

2.2.2 Moment problems and integral theory/function theory

In the remainder of this section, we focus on the case where the sequence of functions (ξ_n) is "sufficiently rich", for example, when it spans a dense subspace of V in Problem 2.2.3. Under this assumption, the function f (resp. df) in Problem 2.2.2 or the functional φ in Problem 2.2.3 is uniquely determined by the moments (c_n). Therefore, (c_n) can be understood as the **coordinates** of f (resp. df) and φ under the **coordinate system** (ξ_n).

We now explain how the moment problems connect to integral theory—in other words, to **function theory**. A central theme in function theory is the approximation of abstract or complicated functions by simpler, more elementary ones. This motivation often arises from practical mathematical problems, particularly those originating in physics, where one seeks to express the solution as a series of elementary functions, such as a power series or a Fourier series. The question of how such series should converge—uniformly, pointwise, or in some other sense—and what kinds of functions they can approximate was a central focus of function theory in the 18th and 19th centuries.

The first step in understanding and solving the approximation problem is to analyze the corresponding moment problem. A typical scenario unfolds as follows. In the setting of Problem 2.2.2, suppose there exists a sequence of elementary functions (f_n) such that

$$\int \xi_k f_n \quad \text{resp.} \quad \int \xi_k df_n \quad = c_k \qquad \text{when } |k| \leqslant |n| \tag{2.5}$$

This situation arises, for instance, in the study of continued fractions and polynomial moments, where $\xi_k(x) = x^k$. In the case of Fourier series, an even stronger condition holds:

$$\int \xi_k f_n \quad \text{resp.} \quad \int \xi_k df_n \quad = \begin{cases} c_k & \text{if } |k| \leqslant |n| \\ 0 & \text{if } |k| > |n| \end{cases} \tag{2.6}$$

where $\xi_k(x) = e^{-\mathbf{i}kx}$ and $f_n(x) = \sum_{|k|\leqslant n} c_k e^{\mathbf{i}kx}$. The approximation problem asks:

Problem 2.2.6. Does the sequence (f_n) converge to some function f? If so, in what sense does it converge?

To approach this problem, observe that if such a function f exists, and if the integral commutes with the convergence of sequence of functions, then

$$\int \xi_k f = \int \lim_{|n|\to\infty} \xi_k f_n = \lim_{|n|\to\infty} \int \xi_k f_n \overset{(2.5)}{=\!=} c_k \tag{2.7a}$$

resp.

$$\int \xi_k df = \int \lim_{|n|\to\infty} \xi_k df_n = \lim_{|n|\to\infty} \int \xi_k df_n \overset{(2.5)}{=\!=} c_k \tag{2.7b}$$

Therefore, the first step in solving Problem 2.2.6 is to find a function f solving the moment Problem 2.2.2. Once such an f is found, the next step is to prove that the sequence (f_n) converges to f, and investigate the mode of convergence.

Historically, the understanding of convergence, the properties of the limiting function f, and the integrals appearing in (2.7) was often insufficient to resolve the approximation problem at the outset. In many cases, addressing the approximation problem required the development of new theories of integration or the extension of the class of integrable functions. Both the Lebesgue and Stieltjes integrals emerged from such needs. For instance, the challenges posed by Fourier series played a central role in motivating the development of the Riemann and later the Lebesgue integral. See [Jah, Ch. 6, 9] and [Haw-L] for a detailed discussion of how Fourier series drove this evolution. The connection between continued fractions and the Stieltjes integral will be explored in Ch. 4.

Function theory	Moment Problems	Dual spaces
Lebesgue integral & Fourier series	Fourier coefficients	$L^2([a,b],m)^*$
Stieltjes integral & Continued fractions	Polynomial moments	$C([a,b])^*$

Table 2.1: The origin of moment problems in function theory

2.2.3 Convergence of functions, moments, and linear functionals

In the previous subsection, we noted that solving moment problems determines the function f that appears in Problem 2.2.6. But can the moment problem perspective also help us understand the convergence of f_n to f? Or conversely, can the convergence behavior of f_n toward f offer deeper insight into the structure of moment problems themselves? Thanks to Hilbert's foundational work on the Hilbert space $l^2(\mathbb{Z})$—especially his groundbreaking 1906 paper [Hil06]—the answer is yes.[2]

A key concept introduced by Hilbert in [Hil06] is **weak convergence**: If (ψ_n) is a sequence in $l^2(\mathbb{Z})$ with uniformly bounded norm, i.e.,

$$\sup_n \|\psi_n\|_2 < +\infty \tag{2.8}$$

we say that (ψ_n) converges weakly to $\psi \in l^2(\mathbb{Z})$ if it converges pointwise $\mathbb{Z}$, i.e.,

$$\lim_n \psi_n(k) = \psi(k) \qquad \text{for all } k \in \mathbb{Z} \tag{2.9}$$

[2]Indeed, Hilbert originally worked with the real Hilbert space $l^2(\mathbb{Z},\mathbb{R})$, rather than the complex one $l^2(\mathbb{Z}) = l^2(\mathbb{Z},\mathbb{C})$. For clarity and simplicity, however, we will work with $l^2(\mathbb{Z})$ in what follows.

Since $l^2(\mathbb{Z})$ is typically interpreted as the space of Fourier series of L^2-integrable functions, Hilbert's notion of weak convergence corresponds to the (pointwise) convergence of Fourier coefficients. That is,

$$\lim_n \widehat{f}_n(k) = \widehat{f}(k) \qquad \text{for all } k \in \mathbb{Z}$$

where f_n and f are L^2-integrable functions on $[-\pi, \pi]$.[3]

The notion of weak convergence—later extended to weak-* convergence—provided a fundamentally new insight into the study of moment problems and their connection to dual spaces and function theory/integral theory. Since Fourier coefficients are simply trigonometric moments, the weak convergence described by (2.9) can be understood as the (pointwise) **convergence of moments**, which means, in the setting of the moment Problem 2.2.2, that

$$\lim_n \int \xi_k f_n \quad \text{resp.} \quad \lim_n \int \xi_k df_n \quad = c_n \qquad \text{for all } k \tag{2.10}$$

The translation of (2.10) into the setting of the dual space version of the moment Problem 2.2.3 is straightforward: One considers a sequence (φ_k) in V^* such that $\lim_n \langle \xi_k, \varphi_n \rangle = \langle \xi_k, \varphi \rangle$ holds for all k. Since we have assumed at the beginning of Subsec. 2.2.2 that (ξ_n) spans a dense subspace of V, it follows from (2.8) that this convergence of moments is equivalent to the **weak-* convergence** of (φ_n) to φ. That is, we say that (φ_n) converges weak-* to φ if

$$\lim_n \langle \xi, \varphi_n \rangle = \langle \xi, \varphi \rangle \qquad \text{for all } \xi \in V \tag{2.11}$$

Thus, the second and third columns of Table 2.2 are equivalent. See Thm. 2.6.2 for the formal statement of this equivalence.

On the other hand, (2.10) generalizes the condition (2.5), which, as previously mentioned, arises naturally in the study of Fourier series and continued fractions. As such, its function-theoretic interpretation—highlighted by the following theorems—provides a general framework for understanding the convergence of the sequence (f_n) to f in Problem 2.2.6.

Theorem 2.2.7. *Let $1 < p \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$. Let (f_n) be a uniformly L^p-norm bounded sequence in $L^p([a,b], m)$. Suppose that (f_n) converges pointwise to f. Then we have $f \in L^p([a,b], m)$. Moreover, (f_n) converges weak-* to f, which means that $\lim_n \int f_n g dm = \int f g dm$ for all $g \in L^q([a,b], m)$.*

Proof. See Thm. 2.7.2. □

[3]Hilbert himself did not initially connect $l^2(\mathbb{Z})$ with the Lebesgue integral. The precise relationship between $l^2(\mathbb{Z})$ and $L^2([-\pi, \pi], \frac{m}{2\pi})$ was later clarified by Riesz and Fischer in 1907.

Theorem 2.2.8. *Let $1 < p \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$. Let (f_n) be a uniformly L^p-norm bounded sequence in $L^p([a,b],m)$. Then (f_n) converges weak-* to some element $f \in L^p([a,b],m)$ iff the limit*

$$F(x) := \lim_n \int_a^x f_n dm \tag{2.12}$$

exists for every $x \in [a,b]$. When (f_n) converges weak- to $f \in L^p([a,b],m)$, for each $x \in [a,b]$ we have*

$$F(x) = \int_a^x f dm \tag{2.13}$$

Proof. If (f_n) converges weak-* to f, then $\lim_n \int f_n \chi_{[a,x]} = \int f_n \chi_{[a,x]}$, which implies that $F(x)$ exists and equals $\int_a^x f dm$.

The other direction is more difficult. Indeed, it is almost equivalent to the duality $L^p([a,b],m) \simeq L^q([a,b],m)^*$. See Thm. 2.7.1. □

Theorem 2.2.9. *Let (ρ_n) be a uniformly l^∞-bounded sequence of increasing functions $[a,b] \to \mathbb{R}_{\geqslant 0}$. The following are true.*

1. *Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ be bounded and increasing. Then $(d\rho_n)$ converges weak-* to $d\rho$ iff (ρ_n) converges pointwise to ρ at b and at any point where $\rho|_{(a,b)}$ is continuous.*
2. *$(d\rho_n)$ converges weak-* to $d\rho$ for some bounded increasing $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ iff (ρ_n) converges pointwise at b and on a dense subset of I.*

By saying that $(d\rho_n)$ converges weak-* to $d\rho$, we mean $\lim_n \int g d\rho_n = \int g d\rho$ for all $g \in C([a,b],m)$.

Proof. See Thm. 2.9.6 and Cor. 2.9.7. □

The above theorems establish an intimate connection between the (pointwise) convergence of moments and the pointwise convergence of the antiderivatives of a sequence of functions.[4] Our understanding of convergence from various perspectives can thus be summarized in Table 2.2.

Function theory	Moment Problems	Dual spaces
Pointwise convergence of (antiderivatives of) a sequence of functions	Pointwise convergence of moments	Weak-* convergence

Table 2.2: Equivalence of convergence notions

[4]We are viewing ρ_n and ρ as the antiderivatives of $d\rho_n$ and $d\rho$.

2.2.4 Equivalence of the first and second columns of Table 2.2

Thm. 2.2.8 and 2.2.9, which establish the equivalence of the first and second columns of Table 2.2, are not easy to prove. In fact, proving Thm. 2.2.8 typically requires the duality $L^p([a,b]) \simeq L^q([a,b])^*$, or at least techniques closely related to those used in establishing this duality.

Therefore, the solvability of the moment problems (Problems 2.2.2 and 2.2.3)—in other words, the solvability of Problem 2.2.5 concerning the characterization of dual spaces—is closely related to the equivalence between the first and second columns of Table 2.2. This close connection rests on the following principle:

Principle 2.2.10. Usually, if V is a normed vector space consisting of functions, any element φ of V^* can be weak-* approximated by elementary functions with uniformly bounded norms. More precisely, there exists a sequence (or a net) of elementary functions (f_n) such that the operator norms of the linear functionals $\xi \in V \mapsto \int \xi f_n$ are uniformly bounded, and

$$\lim_n \int \xi f_n = \langle \xi, \varphi \rangle \qquad \text{for all } \xi \in V$$

Remark 2.2.11. Here is how, with the help of Principle 2.2.10, the characterization of V^* can be derived from the equivalence of the first and second columns of Table 2.2:

By this principle, for each $\varphi \in V^*$, we can select a sequence (f_n) approximating weak-* to φ. Since the second column of Table 2.2 implies the first column, the sequence (f_n) converges to some function f in the sense described in the first column of Table 2.2. Then, by the equivalence of the three modes of convergence in that table, it follows that (f_n) converges weak-* to f. Consequently, φ is represented by integration against f, thereby solving the problem of characterizing the dual space V^*. □

The idea outlined in Rem. 2.2.11 is roughly the approach Riesz employed in 1907 to solve the following trigonometric moment problem.

Theorem 2.2.12 (Riesz-Fischer theorem, Riesz's original version)**.** [5] *For each $(c_k)_{k\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$, there is an (automatically unique) $f \in L^2([-\pi,\pi], \frac{m}{2\pi})$ whose Fourier series is equal to (c_k).*

[5] The modern interpretation of the Riesz-Fischer theorem as stating that $L^2(X,\mu)$ (or more generally $L^p(X,\mu)$) is Cauchy-complete for any measure space (X,μ) has led to a significant misunderstanding. In fact, while Fischer formulated the theorem for $L^2([-\pi,\pi], \frac{m}{2\pi})$ in terms of Cauchy sequences, Riesz understood it quite differently—through the lens of moment problems.

Therefore, once Riesz realized that solving moment problems is equivalent to the characterization of dual spaces, he immediately obtained the Riesz-Fréchet Thm. 2.2.1. As we have emphasized at the beginning of Sec. 2.1, completeness and duality are fundamentally distinct properties, each serving distinct purposes and arising from different considerations. The fact that they coincide in the case of inner product spaces is purely a coincidence.

Riesz's idea of the proof. [6] Choose $(c_k)_{k\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$. One aims to solve the moment problem that there exists $f \in L^2$ such that $\frac{1}{2\pi}\int fe_{-k} = c_k$ for all $k \in \mathbb{Z}$, where $e_k(x) = e^{\mathbf{i}kx}$. For each $n \in \mathbb{N}$, let

$$f_n = \sum_{-n\leqslant k\leqslant n} c_k e_k$$

Then (f_n) converges weak-* to the bounded linear functional $\varphi \in (L^2)^*$ satisfying $\langle e_{-k}, \varphi\rangle = c_k$ for all k. (This is an instance of Principle 2.2.10.) [7]

On the other hand, the property $\sum_k |c_k|^2 < +\infty$ implies that the antiderivatives of (f_n) converge pointwise to some function F in the sense of (2.12). This establishes the convergence described in the first column of Table 2.2.

Then, applying the fundamental theorem of calculus for the Lebesgue integral, Riesz deduced the convergence in the second column of Table 2.2 for the derivative function $f := F'$ (which exists a.e. and is L^2) and for another densely spanning set of functions—the set $\{\chi_{[a,x]} : x \in [a,b]\}$.[8] Namely, he obtained

$$\langle\chi_{[a,x]}, f\rangle = \lim_n \langle\chi_{[a,x]}, f_n\rangle \qquad \text{for all } x \in [a,b]$$

Therefore, since the second column of Table 2.2 is equivalent to the third, (f_n) converges weak-* to f. Thus φ is represented by f, which implies that f solves the desired moment problem—since φ does. $\square$

Note that the fundamental theorem of calculus for the Lebesgue integral is crucial to the above proof. Likewise, the Radon-Nikodym Thm. 1.6.8—a modern form of the fundamental theorem of calculus—also plays a central role in the proof of Theorem 1.6.12, which establishes the duality 1.6.12 on $L^p(X,\mu) \simeq L^q(X,\mu)^*$. This reinforces the point that the characterization of dual spaces is deeply connected to the equivalence between the first and second columns of Table 2.2.

See [Gui-A, Sec. 27.3] for further discussion on the relationship between the classical and modern proofs of the duality $L^p \simeq (L^q)^*$, the connection between this duality and the completeness of L^p-spaces, and the role of derivatives—both in the classical sense and in the form of Radon-Nikodym derivatives—in this context.

2.2.5 Conclusion

We now summarize the discussion so far by addressing the question posed at the beginning of this section: Why are dual spaces related to integral theory?

[6] See [Haw-L, Ch. 6].

[7] Riesz's original proof does not use the language of linear functionals.

[8] The fact that the fundamental theorem of calculus for the Lebesgue integral—one of the deepest results in measure theory—is used here highlights how non-trivial the equivalence between the first and second columns of Table 2.2 really is.

More specifically, from the mathematical-historical perspective, why is it possible to characterize the dual spaces of $L^p(X, \mu)$ and $C(X)$?

Function theory	Moment Problems	Dual spaces
	Solving moment problems	Characterizing V^*
Related by $\Updownarrow$ Principle 2.2.10		
Pointwise convergence of (antiderivatives of) a sequence of functions	Pointwise convergence of moments	Weak-* convergence

Table 2.3: The cells in each row are equivalent

The answer, in my view, is captured in Table 2.3: The power of the Lebesgue and Stieltjes integrals lies in their ability to establish the equivalence between the two gray cells in that table. Once this equivalence is established, with the help of Principle 2.2.10, the characterization of dual spaces in terms of integrals becomes straightforward.

But why are these two integrals powerful enough to establish the equivalence between the two gray cells in Table 2.3?—Because both the Lebesgue and Stieltjes integrals arise from the study of moment problems, which in turn are rooted in the corresponding approximation problems, as illustrated in Table 2.1. The emphasis of these integral theories on the commutativity of limits and integration anticipates the equivalence of the two gray cells.

In light of the equivalences in Table 2.3, the Lebesgue integral, as the completion of the Riemann integral, can be interpreted as the weak-* completion of trigonometric functions and continuous functions. Similarly, the Stieltjes integral, as the completion of finite sums, can be viewed as the weak-* completion of discrete spectra—a perspective that will be one of the main themes of Ch. 4 and 5. See Table 2.4.

Completion of Integrals	Extension of classes of functions	**Weak-* completion**
Riemann integral $\cap$ Lebesgue integral	Continuous functions $\cap$ Measurable functions	of continuous functions
Finite sum $\cap$ Stieltjes integral	Discrete spectra $\cap$ Continuous spectra	of discrete spectra

Table 2.4

Side note. A common viewpoint—motivated by the completeness of L^1-spaces—regards the Lebesgue integral and the Lebesgue measurable/integrable functions as the Cauchy completion of Riemann integrals and continuous functions. In my view, this perspective is not only historically inaccurate, but also mathematically misleading.

Historically, the first L^p-space considered is $L^2([a,b],m)$, due to its close relation with $l^2(\mathbb{Z})$, the space of trigonometric moments of L^2-integrable functions. The space $l^2(\mathbb{Z})$ was introduced by Hilbert in [Hil06], where weak convergence (equivalently, pointwise convergence of moments) plays a central role in his proof of the Hilbert-Schmidt theorem. In [Rie10], Riesz studied the space $L^p([a,b],m)$ for $1 < p < +\infty$, and in particular proved the duality $L^p([a,b],m) \simeq L^q([a,b],m)^*$. The completeness of $L^p([a,b],m)$ follows as a corollary. However, $L^1([a,b],m)$ was not considered, likely due to its lack of a satisfactory duality theory. This clearly shows that duality was originally viewed as more fundamental than Cauchy completeness.

Mathematically, to perform a Cauchy completion, one needs a norm, which in this context is defined via an integral. Yet, while integrals are linear functionals, norms only satisfy the subadditivity. As a result, norms and Cauchy completions do not provide the right conceptual framework for understanding the nature of the Lebesgue integral from a functional-analytic perspective.

The more appropriate viewpoint is to regard the Lebesgue integral as arising from weak-* completion, not Cauchy completion.

2.3 Bounded multilinear maps

2.3.1 Seminorms, norms, and normed vector spaces

Definition 2.3.1. If V is an $\mathbb{F}$-vector space, a function $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ is called a **seminorm** if

$$\|av\| = |a|\cdot\|v\| \qquad \|u+v\| \leqslant \|u\| + \|v\| \qquad \text{for any } u,v\in V \text{ and } a\in\mathbb{F} \tag{2.14}$$

A seminorm is called a **norm** if any $v \in V$ satisfying $\|v\| = 0$ is the zero vector 0. A vector space V, equipped with a norm, is called a **normed vector space**.

If V is a normed vector space, then a **normed vector subspace** of V denotes a linear subspace $U \subset V$ equipped with the norm inherited from V, i.e., the restriction of V's norm to U.

We say that V is **separable** if it is so under the **norm topology**, namely, the topology induced by the metric $d(u,v) = \|u-v\|$. □

Remark 2.3.2. In Def. 2.3.1, the condition $\|av\| = |a|\cdot\|v\|$ can be weakened to

$$\|av\| \leqslant |a|\cdot\|v\| \qquad \text{for any } v\in V \text{ and } a\in\mathbb{F} \tag{2.15}$$

Therefore, (2.14) can be weakened to

$$\|au+bv\| \leqslant |a|\cdot\|u\| + |b|\cdot\|v\| \qquad \text{for any } u,v\in V \text{ and } a,b\in\mathbb{F} \tag{2.16}$$

Proof. Suppose that (2.15) is true. Then we clearly have $\|av\| = |a| \cdot \|v\|$ when $a = 0$. Suppose that $a \neq 0$. Then $\|v\| = \|a^{-1}av\| \leqslant |a|^{-1}\|av\|$, and hence $\|av\| \geqslant |a| \cdot \|v\|$. Therefore $\|av\| = |a| \cdot \|v\|$. □

Remark 2.3.3. The norm function $\| \cdot \| : V \to \mathbb{R}_{\geqslant 0}$ is continuous. This is because

$$\|u\| - \|v\| \leqslant \|u - v\| \tag{2.17}$$

Therefore, if (v_α) is a net in V converging (in norm) to v, then

$$\|v\| = \lim_\alpha \|v_\alpha\|$$

Proposition 2.3.4. *Let $\| \cdot \|_V$ be a seminorm on an $\mathbb{F}$-vector space V. Let $V_0 = \{v \in V : \|v\|_V = 0\}$. Then V_0 is a linear subspace on V, and there is a (clearly unique) norm $\| \cdot \|_{V/V_0}$ on the quotient space V/V_0 such that*

$$\|v + V_0\|_{V/V_0} = \|v\|_V \qquad \textit{for all } v \in V \tag{2.18}$$

In the future, unless otherwise stated, we will always equip V/V_0 with this norm $\| \cdot \|_{V/V_0}$.

Proof. We abbreviate $\| \cdot \|_V$ to $\| \cdot \|$. If $u, v \in V_0$ and $a, b \in \mathbb{F}$, then

$$\|au + bv\| \leqslant |a|\|u\| + |b|\|v\| = 0$$

This shows that V_0 is a linear subspace of V. On the other hand, if $u, v \in V$ satisfy $u + V_0 = v + V_0$, then $u - v \in V_0$, and hence

$$\|v\| = \|u + v - u\| \leqslant \|u\| + \|v - u\| = \|u\|$$

Similarly, $\|u\| \leqslant \|v\|$. Therefore $\|u\| = \|v\|$. This implies that we have a well-defined function $\| \cdot \|_{V/V_0} : V/V_0 \to \mathbb{R}_{\geqslant 0}$ satisfying (2.18).

If $u, v \in V$ and $a, b \in \mathbb{F}$, then

$$\|a(u + V_0) + b(v + V_0)\|_{V/V_0} = \|au + bv + V_0\|_{V/V_0} = \|au + bv\| \leqslant |a|\|u\| + |b|\|v\|$$

□

2.3.2 Bounded multilinear maps

In the rest of this section, $V_1, V_2, \ldots$ and U, V, W all denote normed $\mathbb{F}$-vector spaces.

Definition 2.3.5. Let $N \in \mathbb{Z}_+$. A map $T : V_1 \times \cdots \times V_N \to W$ is called a **multilinear map** if for each $1 \leqslant i \leqslant N$ and each fixed $v_j \in V_j$ (for all $j \neq i$), the map

$$v_i \in V_i \mapsto T(v_1, \dots, v_N) \in W$$

is $\mathbb{F}$-linear. We let

$$\mathrm{Lin}(V_1 \times \cdots \times V_N, W) = \{\text{multilinear maps } V_1 \times \cdots \times V_N \to W\}$$

For each $T \in \mathrm{Lin}(V_1 \times \cdots \times V_N, W)$, we define the **operator norm**

$$\|T\| := \|T\|_{l^\infty(\overline{B}_{V_1}(0,1)\times\cdots\times\overline{B}_{V_N}(0,1),W)} = \sup_{v_1\in\overline{B}_{V_1}(0,1),\dots,v_N\in\overline{B}_{V_N}(0,1)} \|T(v_1, \dots, v_N)\|$$

We say that T is **bounded** if $\|T\| < +\infty$.

Definition 2.3.6. We let

$$\mathfrak{L}(V_1 \times \cdots \times V_N, W) := \{\text{bounded multilinear maps } V_1 \times \cdots \times V_N \to W\} \qquad (2.19)$$

viewed as an $\mathbb{F}$-linear subspace of $W^{V_1\times\cdots\times V_N}$. We let

$$\mathfrak{L}(V) := \mathfrak{L}(V, V) \qquad V^* := \mathfrak{L}(V, \mathbb{F})$$

Elements of $\mathfrak{L}(V)$ are called **bounded linear operators on** V. An element $T \in \mathfrak{L}(V)$ is called **invertible** if there exists $T^{-1} \in \mathfrak{L}(V)$ such that

$$TT^{-1} = T^{-1}T = \mathrm{id}_V$$

The space V^* is called the **dual space** of V.

Remark 2.3.7. In this course, the most frequently encountered cases of (2.19) are $\mathfrak{L}(V)$, V^*, and $\mathfrak{L}(U \times V, \mathbb{F})$. In Ch. 5, we also consider spaces such as $\mathfrak{L}(U \times V \times V_*, \mathbb{F})$, where V_* is a normed vector space with dual space V. In such cases, Prop. 2.5.1 gives isomorphisms

$$\mathfrak{L}(U \times V \times V_*, \mathbb{F}) \simeq \mathfrak{L}(U, \mathfrak{L}(V \times V_*, \mathbb{F})) \simeq \mathfrak{L}(U, \mathfrak{L}(V))$$

Remark 2.3.8. $\|T\|$ is the smallest element in $\overline{\mathbb{R}}_{\geqslant 0}$ satisfying

$$\|T(v_1, \dots, v_N)\| \leqslant \|T\| \cdot \|v_1\| \cdots \|v_N\| \qquad (2.20)$$

Proof. If one of $v_1, \dots, v_N$ is zero, then $T(v_1, \dots, v_N) = 0$ by the multilinearity, and hence (2.20) holds. So we assume that $v_1, \dots, v_N$ are all non-zero. So their norms are all nonzero. Since $v_i/\|v_i\| \in B_{V_i}(0, 1)$, we have

$$\Big\|T\Big(\frac{v_1}{\|v_1\|}, \cdots, \frac{v_N}{\|v_N\|}\Big)\Big\| \leqslant \|T\|$$

which implies (2.20) by the multilinearity.

We have proved that $\|T\|$ satisfies (2.20). Now, suppose that $C \in \overline{\mathbb{R}}_{\geqslant 0}$ and

$$\|T(v_1, \dots, v_N)\| \leqslant C \cdot \|v_1\| \cdots \|v_N\|$$

for all $v_i \in V_i$. Taking $v_i \in \overline{B}_V(0, 1)$, we see that $\|T\| \leqslant C$. $\square$

Recall Def. 1.4.3.

Proposition 2.3.9. *Let $T : V_1 \times \cdots \times V_N \to W$ be multilinear. The following are equivalent.*

(a) T is continuous.

(b) T is continuous at $0 \times \cdots \times 0$.

(c) T is bounded.

(d) T is Lipschitz continuous on $\overline{B}_{V_1}(0,R) \times \cdots \times \overline{B}_{V_N}(0,R)$ for every $R \in \mathbb{R}_{>0}$.

(e) T is Lipschitz continuous on $\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1)$.

Moreover, if T is bounded, and if $V_1 \times \cdots \times V_N$ is equipped with the l^∞-product metric, then the Lipschitz constant in (d) can be chosen to be $NR^{N-1}\|T\|$.

What matters about the Lipschitz constant above is not its exact formula, but the implication it carries: namely, that any family (T_α) in $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ satisfying $\sup_\alpha \|T_\alpha\| < +\infty$, when restricted to a bounded subset of $V_1 \times \cdots \times V_N$, admits a uniform Lipschitz constant.

Proof. Clearly (a)⇒(b).

(b)⇒(c): Assume (b). Then $0 \times \cdots \times 0$ is an interior point of $T^{-1}(B_W(0,1))$, and hence contains $B_{V_1}(0,2\delta_1) \times \cdots \times B_{V_N}(0,2\delta_N)$ for some $\delta_1,\dots,\delta_N > 0$. So T sends $\overline{B}_{V_1}(0,\delta_1) \times \cdots \times \overline{B}_{V_N}(0,\delta_N)$ (which equals $\delta_1\overline{B}_{V_1}(0,1) \times \cdots \times \delta_N\overline{B}_{V_N}(0,1)$) into $B_W(0,1)$. By multilinearity, T sends $\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1)$ into $B_W(0,\delta_1^{-1}\cdots\delta_N^{-1})$. This proves (c).

(c)⇒(d): Assume (c). Choose $v_i \in \overline{B}_{V_i}(0,R_i)$. Then, for each $\xi_i \in \overline{B}_{V_i}(0,R_i)$,

$$\begin{aligned}
&\|T(\xi_1,\dots,\xi_N) - T(v_1,\dots,v_N)\| \\
\leqslant&\|T(\xi_1 - v_1,\xi_2,\xi_3,\dots,\xi_N)\| + \|T(v_1,\xi_2 - v_2,\xi_3,\dots,\xi_N)\| \\
&+ \|T(v_1,v_2,\xi_3 - v_3,\dots,\xi_N)\| + \cdots + \|T(v_1,v_2,v_3,\dots,\xi_N - v_N)\| \\
\leqslant&NR^{N-1}\|T\| \cdot \max\{\|\xi_1 - v_1\|,\dots,\|\xi_N - v_N\|\}
\end{aligned}$$

where (2.20) is used in the last inequality. Thus T has Lipschitz constant $NR^{N-1}\|T\|$.

(e)⇔(d): This is clear by scaling the vectors.

(d)⇒(f): This is clear from Rem. 1.4.4. □

Corollary 2.3.10. *Let $T \in \mathfrak{L}(V,W)$. Then $\mathrm{Ker}(T)$ is a closed linear subspace of V.*

Proof. By Prop. 2.3.9, T is continuous. Since the preimage of any closed set under a continuous map is closed, $\mathrm{Ker}(T) = T^{-1}(0)$ is closed. □

Example 2.3.11. A linear map $T : V \to W$ is called a **linear isometry** if it is an isometry of metric spaces, i.e., $\|Tv_1 - Tv_2\| = \|v_1 - v_2\|$ for all $v_1, v_2 \in V$. This is clearly equivalent to

$$\|Tv\| = \|v\| \qquad \text{for all } v \in V$$

A linear isometry is clearly bounded with operator norm $\|T\| = 1$ (unless when $V = \{0\}$). Moreover, a linear isometry is clearly injective. A linear isometry $T : V \to W$ which is also surjective (and hence bijective) is called an **isomorphism of normed vector spaces**. In that case, we say that the normed vector spaces V, W are **isomorphic**.

Remark 2.3.12. Suppose that $\Phi : V \to W$ is a linear map of vector spaces, and W is a normed vector space. Then V has a seminorm defined by

$$\|v\|_V := \|\Phi(v)\|_V$$

Equip $V/\mathrm{Ker}\Phi$ with the norm defined by Prop. 2.3.4. Then Φ descends to a linear map $\widetilde{\Phi} : V/\mathrm{Ker}\Phi \to W$, which is clearly a linear isometry.

Example 2.3.13. Let $1 \leqslant p \leqslant +\infty$, let X be an LCH space, let μ be a Radon measure (or its completion) on X. Let $\Phi : C_c(X, \mathbb{F}) \to L^p(X, \mu, \mathbb{F})$ be the obvious map. Then Φ descends to a linear isometry of normed vector spaces

$$C_c(X, \mathbb{F})/\{f \in C_c(X, \mathbb{F}) : f = 0 \ \mu\text{-a.e.}\} \longrightarrow L^p(X, \mu, \mathbb{F}) \tag{2.21}$$

Now assume $p < +\infty$. Then by Thm. 1.7.10, the map (2.21) has dense range. This is often expressed by saying that $C_c(X, \mathbb{F})/\{f \in C_c(X, \mathbb{F}) : f = 0 \ \mu\text{-a.e.}\}$ is dense in $L^p(X, \mu, \mathbb{F})$, or simply that $C_c(X, \mathbb{F})$ is dense in $L^p(X, \mu, \mathbb{F})$.

2.4 Fundamental properties of bounded multilinear maps

Let $V_1, V_2, \dots, U, V, W$ be normed vector spaces. In this section, we establish several fundamental properties of bounded multilinear maps that will be used frequently throughout the course. We first note the elementary fact:

Remark 2.4.1. Let U be a linear subspace of V. Let $R \in \mathbb{R}_{>0}$. Then U is dense in V iff $\overline{B}_U(0, R)$ is dense in $\overline{B}_V(0, R)$.

Proof. The direction "$\Leftarrow$" is obvious. Let us prove "$\Rightarrow$". Let $\xi \in \overline{B}_V(0, R)$, choose a sequence (ξ_n) in U converging to ξ. Assume WLOG that $\xi \neq 0$ and $R \in \mathbb{R}_{>0}$; otherwise, the approximation is obvious. Since the norm function is continuous, $\|\xi_n\| \to \|\xi\|$. In particular, $\|\xi_n\|$ is eventually nonzero. Thus $\frac{\|\xi\|}{\|\xi_n\|}\xi_n \to \xi$. $\square$

Recall that two sequences $(x_n), (y_n)$ in a metric space X is called **Cauchy equivalent** if $\lim_n d(x_n, y_n) = 0$.

Theorem 2.4.2. *Suppose that W is complete. For each i, let U_i be a dense linear subspace of V_i. Then we have an isomorphism of normed vector spaces*

$$\begin{gathered}\mathfrak{L}(V_1 \times \cdots \times V_N, W) \xrightarrow{\simeq} \mathfrak{L}(U_1 \times \cdots \times U_N, W) \\ T \mapsto T\big|_{U_1 \times \cdots \times U_N}\end{gathered} \tag{2.22}$$

Proof. Denote the map (2.22) by Φ which is clearly linear. By Rem. 2.4.1, $\overline{B}_{U_1}(0,1) \times \cdots \times \overline{B}_{U_N}(0,1)$ is dense in $\overline{B}_{U_1}(0,1) \times \cdots \times \overline{B}_{U_N}(0,1)$. This shows that Ψ is a linear isometry, i.e., T and $T\big|_{U_1 \times \cdots \times U_N}$ have the same operator norm.

We now show that Φ is surjective. Here, the completeness of W is need. Let $T \in \mathfrak{L}(U_1 \times \cdots \times U_N, W)$. We want to extend T to a bounded multilinear map $V_1 \times \cdots \times V_N \to W$. We only need to extend T on the first component, i.e., extend T to a bounded multilinear $V_1 \times U_2 \times U_3 \times \cdots \times U_N \to W$. Then, a similar argument applies to the second component extend T to a bounded multilinear $V_1 \times V_2 \times U_3 \times \cdots \times U_N \to W$. By repeating this procedure, we obtain bounded multilinear $V_1 \times \cdots \times V_N \to W$ extending T.

Let $\xi \in V_1, u_2 \in U_2, \ldots, u_N \in U_N$. Let (ξ_n) be a sequence in U_1 converging to ξ. In particular, (x_n) is a Cauchy sequence. By Rem. 2.3.8, $T(\xi_n, v_2, \ldots, v_N)$ is a Cauchy sequence in W. Therefore, by the completeness of W, $T(\xi_n, v_2, \ldots, v_N)$ converges to some element, which we denote by $T(\xi, v_2, \ldots, v_N)$.

Let us show that the definition of $T(\xi, v_2, \ldots, v_N)$ is independent of the choice of sequence converging to ξ. Suppose that (ξ_n') is another sequence converging to ξ. Then (ξ_n) and (ξ_n') are Cauchy equivalent. By Rem. 2.3.8, $T(\xi_n, v_2, \ldots, v_N)$ and $T(\xi_n', v_2, \ldots, v_N)$ are Cauchy equivalent. So they converge to the same element.

Thus, we have defined a map $T : V_1 \times U_2 \times \cdots \times U_N \to W$. We leave it to the reader to check that T is bounded multi-linear map. □

Corollary 2.4.3. *Let U be a dense linear subspace of V. Then we have an isomorphism of normed vector spaces*

$$V^* \xrightarrow{\simeq} U^* \qquad \varphi \mapsto \varphi|_U \tag{2.23}$$

Proof. This follows immediate from Thm. 2.4.2. □

Example 2.4.4. Let $1 \leqslant q < +\infty$ and $p^{-1} + q^{-1} = 1$. Let X be an LCH space. Let μ be a Radon measure (or its completion) on X. By Exp. 2.3.13, the L^q-seminorm on $C_c(X, \mathbb{F})$ descends to the L^q-norm on $V = C_c(X, \mathbb{F})/\{f \in C_c(X, \mathbb{F}) : f = 0\ \mu\text{-a.e.}\}$, and V is dense in $L^p(X, \mu)$. Therefore, by Thm. 1.6.12 and Cor. 2.4.3, the map (1.28) gives an isomorphism of normed vector spaces $V^* \simeq L^p(X, \mu)$.

The following Prop. 2.4.5 and Thm. 1.9.13 will imply Thm. 2.6.2, which establishes the equivalence of the second and third columns of Table 2.2.

Proposition 2.4.5. *For each i, let E_i be a densely spanning subset of V_i. Let (T_α) be a net in $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ with* ***uniformly bounded operator norms****, i.e.,* $\sup_\alpha \|T_\alpha\| < +\infty$*. Choose $T \in \mathfrak{L}(V_1 \times \cdots \times V_N, W)$, and assume that (T_α) converges pointwise on $E_1 \times \cdots \times E_N$ to T. Then (T_α) converges pointwise on $V_1 \times \cdots \times V_N$ to T.*

Proof. Let $U_i = \mathrm{Span}(E_i)$, which is dense in V_i. Then (T_α) converges pointwise on $U_1 \times \cdots \times U_N$ to T.

Choose any $\xi_i \in V_i$. Choose $R \in \mathbb{R}_{>0}$ such that $\|\xi_i\| \leqslant R$ for each i. Since $\sup_\alpha \|T_\alpha\| < +\infty$, by Prop. 2.3.9, $\{T_\alpha, T : \alpha \in I\}$ has a uniform Lipschitz constant $C \in \mathbb{R}_{\geqslant 0}$ (with respect to the l^∞-product metric) when restricted to $\overline{B}_{V_1}(0, R) \times \cdots \times \overline{B}_{V_N}(0, R)$. By Rem. 2.4.1, for each $\varepsilon > 0$, there exists $v_i \in \overline{B}_{U_i}(0, R)$ such that $\|\xi_i - v_i\| \leqslant \varepsilon$. Then

$$
\begin{aligned}
&\limsup_\alpha \|T(\xi_1, \dots, \xi_N) - T_\alpha(\xi_1, \dots, \xi_N)\| \\
\leqslant &\limsup_\alpha \|T(v_1, \dots, v_N) - T_\alpha(v_1, \dots, v_N)\| + 2C\varepsilon = 2C\varepsilon
\end{aligned}
$$

Since ε is arbitrary, we conclude that $T_\alpha(\xi_1, \dots, \xi_N) \to T(\xi_1, \dots, \xi_N)$. □

Theorem 2.4.6. *Suppose that W is complete. For each i, let E_i be a densely spanning subset of V_i. Let (T_α) be a net in $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ satisfying $\sup_\alpha \|T_\alpha\| < +\infty$. Suppose that (T_α) converges pointwise on $E_1 \times \cdots \times E_N$. Then (T_α) converges pointwise on $V_1 \times \cdots \times V_N$ to some $T \in \mathfrak{L}(V_1 \times \cdots \times V_N, W)$, and*

$$
\|T\| \leqslant \liminf_\alpha \|T_\alpha\| \tag{2.24}
$$

Inequality (2.24) is sometimes referred to as **Fatou's lemma**.

Proof. Let $U_i = \mathrm{Span}(E_i)$, which is dense in V_i. Let $T : U_1 \times \cdots \times U_N \to W$ be the pointwise limit of $(T_\alpha)_{\alpha \in I}$ restricted to $U_1 \times \cdots \times U_N$, which is clearly linear. Moreover, for each $v_i \in \overline{B}_{U_i}(0, 1)$ we have

$$
\|T(v_1, \dots, v_N)\| = \liminf_\alpha \|T_\alpha(v_1, \dots, v_N)\| \leqslant \liminf_\alpha \|T_\alpha\|
$$

Taking sup over all $v_i \in \overline{B}_{U_i}(0, 1)$, we see that $\|T\| \leqslant \sup_\alpha \|T_\alpha\| < +\infty$. In particular, $T \in \mathfrak{L}(U_1 \times \cdots \times U_N, W)$. By Thm. 2.4.2, T can be extended to a bounded multilinear map $T : V_1 \times \cdots \times V_N \to W$ with $\|T\|$ unchanged. By Prop. 2.4.5, this extended T is the pointwise limit of (T_α) on the whole domain $V_1 \times \cdots \times V_N$. □

Remark 2.4.7. Recall that if X is a set, then $l^\infty(X, W)$, equipped with the l^∞-norm, is a normed vector space.

By the definition of operator norms, we have a linear isometry of normed vector spaces

$$
\begin{gathered}
\mathfrak{L}(V_1 \times \cdots \times V_N, W) \to l^\infty(\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1), W) \\
T \mapsto T|_{\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1)}
\end{gathered}
\tag{2.25}
$$

Therefore, by identifying $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ with its image under (2.25), we view $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ as a normed vector subspace of $l^\infty(\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}, W)$.

Consequently, if (T_α) is a net in $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$, and if $T \in \mathfrak{L}(V_1 \times \cdots \times V_N, W)$, then $\lim_\alpha \|T - T_\alpha\| = 0$ is equivalent to that (T_α) converges uniformly to T on $\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1)$. □

Theorem 2.4.8. *$\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ is a closed linear subspace of $l^\infty(\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1), W)$.*

Proof. Let $T \in l^\infty(\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1), W)$ be the limit of a sequence (T_n) in $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$. Then (T_n) converges uniformly on $\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1)$ to T. By scaling the vectors, we see that (T_n) converges uniformly on $\overline{B}_{V_1}(0,R) \times \cdots \times \overline{B}_{V_N}(0,R)$ for any $R > 0$. Let $T : V_1 \times \cdots \times V_N \to W$ be the pointwise limit of (T_n), which automatically extends the original T defined on $\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}(0,1)$.

Since each T_n is multilinear, clearly T is multilinear. Thus $T \in \mathfrak{L}(V_1 \times \cdots \times V_N, W)$. This proves that $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ is a closed. □

Corollary 2.4.9. *Suppose that W is complete. Then $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ is complete.*

Proof. Since W is complete, by the following Prop. 2.4.10, $l^\infty(\overline{B}_{V_1}(0,1) \times \cdots \times \overline{B}_{V_N}, W)$ is complete. Since any closed subset of a complete space is complete, by Thm. 2.4.8, $\mathfrak{L}(V_1 \times \cdots \times V_N, W)$ is complete. □

Proposition 2.4.10. *Suppose that W is complete. Then for each $1 \leqslant p \leqslant +\infty$, the normed vector space $l^p(X, W)$ is complete.*

Proof. Let (f_n) be a Cauchy sequence in $l^p(X, W)$. Then for each $x \in X$, $(f_n(x))$ is a Cauchy sequence in W, and hence converges to some $f(x) \in W$. This defines $f : X \to W$.

Case $p = +\infty$: For each $\varepsilon > 0$, choose $N \in \mathbb{Z}_+$ such that for all $m, n \geqslant N$ we have $\|f_n - f_m\|_{l^\infty} \leqslant \varepsilon$, i.e., $\|f_n(x) - f_m(x)\| \leqslant \varepsilon$ for every $x \in X$. Applying $\lim_{m\to\infty}$, we get $\|f_n(x) - f(x)\| \leqslant \varepsilon$ for all $x \in X$ and $n \geqslant N$. Thus, for all $n \geqslant N$ we have $\|f_n - f\|_{l^\infty} \leqslant \varepsilon$; in particular, we have $f \in l^\infty(X, W)$. Thus $\|f_n - f\|_{l^\infty} \to 0$.

Case $p < +\infty$: For each $\varepsilon > 0$, choose $N \in \mathbb{Z}_+$ such that for all $m, n \geqslant N$ we have $\|f_n - f_m\|_{l^p(X)} \leqslant \varepsilon$, equivalently, $\|f_n - f_m\|_{l^p(A)} \leqslant \varepsilon$ for each $A \in \mathrm{fin}(2^X)$. Applying $\lim_{m\to\infty}$, we get $\|f_n - f\|_{l^p(A)} \leqslant \varepsilon$ for all $n \geqslant N$ and $A \in \mathrm{fin}(2^X)$. Thus $\|f_n - f\|_{l^p(X)} \leqslant \varepsilon$ for all $n \geqslant N$; in particular, we have $f \in l^p(X, W)$. This proves $\|f_n - f\|_p \to 0$. □

Corollary 2.4.11. *The dual space V^*, equipped with the operator norm, is complete.*

Proof. This follows immediately from Cor. 2.4.9. □

2.5 The roles of completeness and duality

Let $V_1, \ldots, V_N$ and V, W be normed vector spaces.

2.5.1 The role of Cauchy completeness

In functional analysis, Cauchy completeness plays two primary roles:

1. Completeness as a domain property, where it is often used in conjunction with the Baire category theorem.

2. Completeness as a codomain property, which ensures that linear operators can be restricted from the whole space to a dense subspace without loss. Thm. 2.4.2 and 2.4.6 are typical examples illustrating this usage.

Among these two, completeness as a codomain is the more widely encountered in practice. This suggests that the recognition and widespread appreciation of **Cauchy completeness** in function spaces developed alongside the study of linear operators—that is, linear maps from V to W—rather than with linear, bilinear, or multilinear functionals, such as $V \times W \to \mathbb{F}$. In the early days of functional analysis, particularly in Hilbert's foundational work [Hil06], the dominant perspective was centered not on linear operators, but on bilinear forms and linear functionals. Within this (bi)linear framework, completeness is not required—indeed, in Thm 2.4.2, 2.4.6, and Corollary 2.4.9, when $W = \mathbb{F}$, none of the remaining vector spaces involved (namely $V_1, \dots, V_N$) are assumed to be complete.

Historically, the focus on bilinear forms gradually gave way to the linear operator viewpoint. As this shift took place, Cauchy completeness came to occupy a central role in functional analysis. The fact that the bilinear form or multilinear functional viewpoint can be reformulated in terms of linear operators is a consequence of the following elementary observation:

Proposition 2.5.1. *Let $U_1, \dots, U_M$ be normed vector spaces. Then we have an isomorphism of normed vector spaces*

$$\begin{gathered}\mathfrak{L}(U_1 \times \cdots \times U_M \times V_1 \times \cdots \times V_N, W) \xrightarrow{\simeq} \mathfrak{L}(U_1 \times \cdots \times U_M, \mathfrak{L}(V_1 \times \cdots \times V_N, W)) \\ T \mapsto \Big((u_1, \dots, u_M) \mapsto T(u_1, \dots, u_M, -, \dots, -)\Big)\end{gathered} \tag{2.26}$$

where $T(u_1, \dots, u_M, -, \dots, -)$ denotes the multilinear map $V_1 \times \cdots \times V_N \to W$ sending $(v_1, \dots, v_N)$ to $T(u_1, \dots, u_M, v_1, \dots, v_N)$.

Proof. It is easy to verify that the second line of (2.26) defines a linear isomorphism

$$\begin{aligned}\Psi :&\mathrm{Lin}(U_1 \times \cdots \times U_M \times V_1 \times \cdots \times V_N, W) \\ &\xrightarrow{\simeq} \mathrm{Lin}(U_1 \times \cdots \times U_M, \mathrm{Lin}(V_1 \times \cdots \times V_N, W))\end{aligned}$$

To explain the idea of comparing the operator norms, we assume for simplicity that $M = N = 1$, and write $U_1 = U$ and $V_1 = V$.

Choose any $T \in \mathrm{Lin}(U \times V, W)$. Then $\Psi(T) : U \to \mathrm{Lin}(V, W)$ sends each $u \in V$ to the linear map

$$\Psi(T)(u) : v \in \mathrm{Lin}(V, W) \mapsto T(u, v)$$

Thus, for each $u \in U$ and $v \in V$, we have

$$\|T(u, v)\| = \|\Psi(T)(u)(v)\| \leqslant \|\Psi(T)(u)\| \cdot \|v\| \leqslant \|\Psi(T)\| \cdot \|u\| \cdot \|v\|$$

This proves $\|T\| \leqslant \|\Psi(T)\|$. Conversely, for each $u \in U$,

$$\begin{aligned}\|\Psi(T)(u)\| &= \sup_{v \in \overline{B}_V(0,1)} \|\Psi(T)(u)(v)\| = \sup_{v \in \overline{B}_V(0,1)} \|T(u, v)\| \\ &\leqslant \sup_{v \in \overline{B}_V(0,1)} \|T\| \cdot \|u\| \cdot \|v\| = \|T\| \cdot \|u\|\end{aligned}$$

This proves $\|\Psi(T)\| \leqslant \|T\|$.

We have proved that $\|\Psi(T)\| = \|T\|$. In particular, if T is bounded, then $\Psi(T)(u)$ is bounded for each $u \in U$, and $\Psi(T)$ is bounded. Conversely, if $\Psi(T)(u)$ is bounded for each u, and if $\Psi(T)$ is bounded, then T is bounded. This proves that Ψ restricts to the linear isomorphism (2.26), which is an isometry because $\|\Psi(T)\| = \|T\|$. □

2.5.2 The role of duality

The following two corollaries follow immediate from Prop. 2.5.1.

Corollary 2.5.2. *We have an isomorphism of normed vector spaces*

$$\mathfrak{L}(U \times V, \mathbb{F}) \xrightarrow{\simeq} \mathfrak{L}(U, V^*) \qquad T \mapsto \big(u \mapsto T(u, -)\big) \tag{2.27}$$

Corollary 2.5.3. *Suppose that V is the dual space of another normed vector space V_*. Then we have an isomorphism of normed vector spaces*

$$\mathfrak{L}(V \times V_*, \mathbb{F}) \xrightarrow{\simeq} \mathfrak{L}(V) \qquad T \mapsto \big(v \mapsto T(v, -)\big) \tag{2.28}$$

In Sec. 2.1 and 2.2, we explored the motivation for introducing dual spaces from the perspectives of the calculus of variations and moment problems. Cor. 2.5.3 now offers yet another compelling reason for the study of duality: when a space V possesses a **dual structure**—specifically, when V is the dual of some normed space V_*—it allows us to approach problems from both the bilinear form and linear operator perspectives.

What are the respective advantages of these two viewpoints? To address this, I would like to revisit the arguments presented in [Gui-A], particularly in the Introduction and in Ch. 21 and 25 of [Gui-A]:

1. The bilinear form framework allows us to draw upon the full strength of measure theory. In fact, measure theory can be understood as a method of **monotone convergence extension**—a procedure for extending linear functionals in such a way that the monotone convergence theorem (or its variants) holds. This type of extension aligns naturally with the structure of bilinear forms.

2. The space $\mathfrak{L}(V)$ of bounded linear operators on V is not just a vector space but also an algebra, with multiplication given by composition. This algebraic structure enables the use of **symbolic calculus**, a technique developed in the mid-19th century in the study of linear algebras, and it connects directly to the representation-theoretic perspectives that flourished in the 20th century.

As discussed in [Gui-A, Sec. 25.8, 25.9], and as we will also explore in Ch. 5, Riesz's spectral theorem provides a striking example of how these two advantages can be fruitfully combined.

2.6 Dual spaces and the weak-* topology

Let $V_1, V_2, \dots, U, V, W$ be normed $\mathbb{F}$-vector spaces.

Definition 2.6.1. By viewing V^* as a subset of $\mathbb{C}^V$, the subspace topology on V^* inherited from the product topology of $\mathbb{C}^V$ is called the **weak-* topology** on V^*. By Thm. 1.4.12, this is the unique topology such that for any net (φ_α) in V^* and any $\varphi \in V$, the net (φ_α) **converges weak-*** to φ—that is, converges to φ in the weak-* topology—iff

$$\lim_\alpha \langle \varphi_\alpha, v\rangle = \langle \varphi, v\rangle \qquad \text{for any } v \in V \tag{2.29}$$

Since $\mathbb{C}^V$ is Hausdorff, the weak-* topology is also Hausdorff.

Weak-* topology is mainly considered for closed balls of V^*, rather than the whole dual space V^*, because for such subsets, pointwise convergence of moments is equivalent to weak-* convergence—that is, the second and third columns of Table 2.2 are equivalent. This equivalence is formally stated in the following theorem.

Theorem 2.6.2. *Suppose that E is a densely spanning subset of V. Let (φ_α) be a net in V^* satisfying $\sup_\alpha \|\varphi_\alpha\| < +\infty$. Then (φ_α) converges weak-* in V^* iff the limit $\lim_\alpha \langle \varphi_\alpha, v\rangle$ exists for any $v \in E$.*

Moreover, if $\varphi \in V^*$ satisfies that

$$\lim_\alpha \langle \varphi_\alpha, v\rangle = \langle \varphi, v\rangle \qquad \textit{for any } v \in E$$

then (φ_α) converges weak- to φ.*

Proof. This is clear from Prop. 2.4.5 and Thm. 2.4.6. □

Remark 2.6.3. Let U be a dense linear subspace of V. (For example, take $V = C_0(X, \mathbb{F})$ and $U = C_c(X, \mathbb{F})$.) Recall the canonical isomorphism $V^* \simeq U^*$ given in Cor. 2.4.3. Then by Prop. 2.6.2, for each $R \in \mathbb{R}_{\geqslant 0}$, the weak-* topology on $\overline{B}_{V^*}(0, R)$ agrees with the weak-* topology on $\overline{B}_{U^*}(0, R)$. However, the weak-* topology on V^* is in general not equal to the weak-* topology on U^*.

In Prop. 2.6.2, one might further ask whether a net (φ_α) in $\overline{B}_{V^*}(0, R)$ that converges weak-* has its limit also in $\overline{B}_{V^*}(0, R)$. The answer is yes:

Proposition 2.6.4 (Fatou's lemma for weak-* convergence). *Let (φ_α) be a net in V^* converging weak-* to some $\varphi \in V^*$. Then*

$$\|\varphi\| \leqslant \liminf_\alpha \|\varphi_\alpha\| \tag{2.30}$$

In other words, the norm function $\|\cdot\| : V^* \to \mathbb{R}_{\geqslant 0}$ is lower semicontinuous with respect to the weak- topology on V^*.*

In contrast, if (φ_α) converges in the operator norm to φ, then $\|\varphi\| = \lim_\alpha \|\varphi_\alpha\|$. Cf. Rem. 2.3.3.

Proof. For each $v \in \overline{B}_V(0, 1)$, we have

$$|\langle \varphi, v\rangle| = \lim_\alpha |\langle \varphi_\alpha, v\rangle| = \liminf_\alpha |\langle \varphi_\alpha, v\rangle| \leqslant \liminf_\alpha \|\varphi_\alpha\| \cdot \|v\| = \|\varphi_\alpha\|$$

Applying $\sup_{v \in \overline{B}_V(0,1)}$ to the LHS above yields (2.30). (See also Thm. 2.4.6.) □

Theorem 2.6.5 (Banach-Alaoglu theorem). $\overline{B}_{V^*}(0, 1)$ *is* ***weak-* compact****—that is, it is compact in the weak-* topology.*

Thus, $\overline{B}_{V^*}(0, 1)$ is a compact Hausdorff space.

First proof. Let (φ_α) be a net $\overline{B}_{V^*}(0, 1)$. Since $|\langle \varphi_\alpha, v\rangle| \leqslant \|v\|$ for each $v \in V$, we can view (φ_α) as a net in

$$S = \prod_{v \in V} \overline{B}_{\mathbb{F}}(0, \|v\|)$$

By Tychonoff's Thm. 1.4.15, S is compact. Therefore, (φ_α) has a subnet (φ_{α_μ}) converging pointwise on V to some function $\varphi : V \to \mathbb{F}$. The function φ is clearly linear and satisfies $\|\varphi\| \leqslant \sup_\mu \|\varphi_{\alpha_\mu}\| \leqslant 1$, cf. Thm. 2.4.6. Thus (φ_{α_μ}) converges weak-* to $\varphi \in \overline{B}_{V^*}(0, 1)$. This finishes the proof that $\overline{B}_{V^*}(0, 1)$ is compact. □

The above proof relies on Tychonoff's theorem, which in turn relies on Zorn's lemma. When V is separable, one can prove the Banach-Alaoglu theorem without using Zorn's lemma:

Second proof assuming that V ***is separable.*** Let E be a countable dense subset of V. Then

$$\Phi : \overline{B}_{V^*}(0,1) \to \mathbb{F}^E \qquad \varphi \mapsto \varphi|_E$$

is injective. Moreover, if (φ_α) is a net in $\overline{B}_{V^*}(0,1)$ and $\varphi \in \overline{B}_{V^*}(0,1)$, then Prop. 2.4.5 indicates that (φ_α) converges weak-* to φ iff (φ_α) converges pointwise on E to φ. Therefore, Φ restricts to a homeomorphism from $\overline{B}_{V^*(0,1)}$ to its image. Thus, since $\mathbb{F}^E$ is second countable (cf. Prop. 1.4.14), so is any subset—in particular, $\overline{B}_{V^*}(0,1)$.

Therefore, by Thm. 1.3.7, showing that $\overline{B}_{V^*}(0,1)$ is compact is equivalent to showing that it is sequentially compact. Let (φ_n) be a sequence in $\overline{B}_{V^*}(0,1)$. By the diagonal method (cf. Rem. 1.4.16), (φ_n) has a subsequence (φ_{n_k}) converging pointwise on E. Thm. 2.4.6 now implies that (φ_{n_k}) converges weak-* to some $\varphi \in \overline{B}_{V^*}(0,1)$. □

The above proof shows that if V is separable, then $\overline{B}_{V^*}(0,1)$ is second-countable and therefore sequentially compact under the weak-* topology. The converse is also true:

Theorem 2.6.6. *The following statements are equivalent.*

(a) The normed vector space V is separable.

(b) When equipped with the weak- topology, the compact Hausdorff space $\overline{B}_{V^*}(0,1)$ is second countable.*

Proof. (a)⇒(b) has been proved above. Here, we give a more direct argument of the equivalence (a)⇒(b). By the following Lem. 2.6.7, V can be viewed as a subset of $C(X,\mathbb{F})$ where $X = \overline{B}_{V^*}(0,1)$ is compact by Banach-Alaoglu. Clearly V separates the points of X. Therefore, if V is separable, then X is second countable by (c)⇒(b) of Thm. 1.5.13. Conversely, if X is second countable, then $C(X,\mathbb{F})$ is separable the (c)⇒(d) of Thm. 1.5.13. Therefore, the subset V of $C(X,\mathbb{F})$ is also separable. □

Lemma 2.6.7. *For each $\varphi \in V$, the function*

$$\overline{B}_{V^*}(0,1) \to \mathbb{F} \qquad \varphi \mapsto \langle \varphi, v\rangle$$

is continuous with respect to the weak- topology.*

Proof. This is clear by (2.29). □

2.7 Weak-* convergence in L^p-spaces

Let $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space.[9] Let $I \subset \mathbb{R}$ be a closed proper interval. Let $1 < p \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$.

We identify $L^p(X, \mu, \mathbb{F})$ with the dual space $L^q(X, \mu, \mathbb{F})^*$ via the isomorphism described in Thm. 1.6.12. This defines the **weak-* topology on $\boldsymbol{L^p(X, \mu, \mathbb{F})}$** . In particular, a net (f_α) in $L^p(X, \mu, \mathbb{F})$ converges weak-* to $f \in L^p(X, \mu, \mathbb{F})$ iff

$$\lim_\alpha \int_X f_\alpha g d\mu = \int_X f g d\mu \qquad \text{for all } g \in L^q(X, \mu, \mathbb{F})$$

2.7.1 Pointwise convergence and weak-* convergence

Let us prove Thm. 2.2.8 in a slightly more general setting. Note that a finite Borel measure μ on an interval $I \subset \mathbb{R}$ can be extended by zero to a finite Borel measure on $\mathbb{R}$, which is Radon by Thm. 1.7.8. Therefore, to generalize Thm. 2.2.8, it suffices to consider finite Borel (equivalently, finite Radon) measures on $\mathbb{R}$.

Theorem 2.7.1. *Let μ be a finite Borel measure on $\mathbb{R}$. Let (f_α) be a net in $L^p(\mathbb{R}, \mu, \mathbb{F})$ satisfying $\sup_\alpha \|f_\alpha\|_{L^p} < +\infty$. Then (f_α) converges weak-* to some element $f \in L^p(\mathbb{R}, \mu, \mathbb{F})$ iff the following limit exists for every $x \in \mathbb{R}$:*

$$F(x) := \lim_\alpha \int_{(-\infty, x]} f_\alpha d\mu \tag{2.31}$$

When (f_α) converges weak- to $f \in L^p(\mathbb{R}, \mu, \mathbb{F})$, for each $x \in \mathbb{R}$ we have*

$$F(x) = \int_{(-\infty, x]} f d\mu \tag{2.32}$$

Note that since μ is finite, the constant function 1 belongs to L^q. Therefore, by Hölder's inequality, any function in $L^p(\mathbb{R}, \mu, \mathbb{F})$ is integrable.

Proof. First, assume that (f_α) converges weak-* to f in $L^p(\mathbb{R}, \mu, \mathbb{F})$. Then for each $x \in \mathbb{R}$, we have $\lim_\alpha \int f_\alpha \chi_{(-\infty, x]} d\mu = \int f \chi_{(-\infty, x]} d\mu$. This proves that (2.31) exists and (2.32) holds.

Next, we assume that (2.31) exists for every x. In the following, we give two proofs of the weak-* convergence of (f_α).

First proof. Let $\varphi_\alpha \in L^q(\mathbb{R}, \mu, \mathbb{F})^*$ be the linear functional associated to f_α, i.e., $\langle \varphi_\alpha, g \rangle = \int f_\alpha g d\mu$ for each $g \in L^q$. By assumption, φ_α converges when evaluated with any member of

$$\mathcal{E} = \mathrm{Span}_{\mathbb{F}}\{\chi_{(-\infty, x]} : x \in \mathbb{R}\}$$

[9] The condition on σ-finiteness can be removed at least when $p = 2$. See the paragraph after Thm. 1.6.12.

By Thm. 1.7.12, $\mathcal{E}$ is dense in L^q. Therefore, since

$$\sup_\alpha \|\varphi_\alpha\| = \sup_\alpha \|f_\alpha\|_p < +\infty$$

by Thm. 2.6.2, (φ_α) converges weak-* to some $\varphi \in (L^q)^*$. By Thm. 1.6.12, φ is represented by some $f \in L^p(\mathbb{R}, \mu, \mathbb{F})$. Thus (f_α) converges weak-* to f.

Second proof. In this proof, we use the fact that any bounded closed ball of $L^p(\mathbb{R}, \mu, \mathbb{F})$ is weak-* compact, which is due to Thm. 1.6.12 and the Banach-Alaoglu theorem.

Since $\sup_\alpha \|f_\alpha\|_p < +\infty$, the net (f_α) has a subnet (f_{α_ν}) converging weak-* to some $f \in L^p$. By the first paragraph, for each $x \in \mathbb{R}$ we have

$$\lim_\nu \int_{(-\infty,x]} f_{\alpha_\nu} d\mu = \int_{(-\infty,x]} f d\mu$$

Since (2.31) converges, we conclude

$$\lim_\alpha \int_{(-\infty,x]} f_\alpha d\mu = \int_{(-\infty,x]} f d\mu$$

That is, if we let $\varphi_\alpha \in (L^q)^*$ represent f_α and let $\varphi \in (L^q)^*$ represent f, then (φ_α) converges to φ when evaluated on $\mathcal{E}$. By Thm. 1.7.12, $\mathcal{E}$ is dense in L^q. Therefore, by Thm. 2.6.2, (φ_α) converges weak-* to φ. That is, (f_α) converges weak-* to f. □

We now present another connection between pointwise convergence and weak-* convergence.

Theorem 2.7.2. *Let (f_n) be a sequence in $L^p(X, \mu, \mathbb{F})$ satisfying $\sup_n \|f_n\|_p < +\infty$. Suppose that (f_n) converges pointwise to f. Then $f \in L^p(X, \mu, \mathbb{F})$, and (f_n) converges weak-* to f.*

Proof. By Fatou's lemma, we have $f \in L^p$, since

$$\int |f|^p \leqslant \liminf_n \int |f_n|^p < +\infty$$

Thm. 2.7.1 suggests that when $X = \mathbb{R}$ and μ is a finite Borel measure, to prove that (f_n) converges weak-* to f, it suffices to verify that $\lim_n \int_{(-\infty,x]} f_n = \int_{(-\infty,x]} f$ for each $x \in \mathbb{R}$. Motivated by this, we claim that in the general case, it suffices to prove

$$\lim_n \int_E f_n d\mu = \int_E f d\mu \tag{2.33}$$

for each $E \in \mathfrak{M}$ satisfying $\mu(E) < +\infty$. (Note that any L^p function is integrable in E by Hölder's inequality.) Indeed, suppose (2.33) is true. Then, by the density of integrable simple functions in L^p (Thm. 1.6.9), and by Thm. 2.6.2, the sequence (f_n) converges weak-* to f.

Let us prove (2.33). For each $\lambda \geqslant 0$, let $\alpha_\lambda : \mathbb{R}_{\geqslant 0} \to [0,1]$ be the (continuous) piecewise linear increasing function such that $\alpha_\lambda|_{[0,\lambda]} = 0$ and $\alpha_\lambda|_{[\lambda+1,+\infty)} = 1$. Let $\beta_\lambda = 1 - \alpha_\lambda$. Since $0 \leqslant \alpha_\lambda \leqslant \chi_{[\lambda,+\infty)}$, we have

$$\lambda^{p-1}\alpha_\lambda \leqslant \lambda^{p-1}\chi_{[\lambda,+\infty)} \leqslant x^{p-1}$$

where x denotes the identity function $\mathrm{id} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. Hence $\lambda^{p-1}x\alpha_\lambda \leqslant x^p$, which implies $\lambda^{p-1}|f_n|(\alpha_\lambda \circ |f_n|) \leqslant |f_n|^p$. Let $C := \sup_n \int_E |f_n|^p$. Then

$$\lambda^{p-1} \sup_n \int_E |f_n| \cdot (\alpha_\lambda \circ |f_n|) \leqslant \sup_n \int_E |f_n|^p = C < +\infty$$

Therefore, (f_n) is **uniformly integrable** on E, which means that for each $\varepsilon > 0$ we have

$$\sup_n \int_E |f_n| \cdot (\alpha_\lambda \circ |f_n|) \leqslant \varepsilon \qquad \text{for sufficiently large } \lambda$$

Since $|f| \cdot (\alpha_\lambda \circ |f|)$ decreases to 0 as $\lambda \to +\infty$, and since $\int_E |f| < +\infty$ (due to Hölder's inequality), by DCT or MCT,

$$\int_E |f| \cdot (\alpha_\lambda \circ |f|) \leqslant \varepsilon \qquad \text{for sufficiently large } \lambda$$

On the other hand, since $0 \leqslant x\beta_\lambda \leqslant \lambda + 1$, and since $\lim_n \beta_\lambda \circ |f_n|$ converges pointwise to $\beta_\lambda \circ |f|$ (due to the continuity of β_λ), by DCT we have

$$\lim_n \int_E f_n \cdot (\beta_\lambda \circ |f_n|) = \int_E f \cdot (\beta_\lambda \circ |f|)$$

Therefore, since $\alpha_\lambda + \beta_\lambda = 1$,

$$\begin{aligned}\limsup_n \Big|\int_E f_n - \int_E f\Big| \leqslant{}& \limsup_n \Big|\int_E f_n \cdot (\beta_\lambda \circ |f_n|) - \int_E f \cdot (\beta_\lambda \circ |f|)\Big| \\ &+ \limsup_n \int_E |f_n| \cdot (\alpha_\lambda \circ |f_n|) + \int_E |f| \cdot (\alpha_\lambda \circ |f|)\end{aligned}$$

where the RHS is $\leqslant 2\varepsilon$ for sufficiently large λ. □

2.7.2 Weak-* approximation by elementary functions

Let X be an LCH space, and let μ be a Radon measure (or its completion) on X with σ-algebra $\mathfrak{M}$. We assume that μ is σ-finite. This condition holds, for example, when X is σ-compact (in particular, when X is second countable; cf. Rem. 1.4.30.)

In this subsection, we examine Principle 2.2.10 in the context of L^p-spaces. We begin with the following observation:

Remark 2.7.3. Let V be a normed vector space, and let U be a linear subspace of V^*. Let $R \in \mathbb{R}_{>0}$. By Rem. 2.4.1, U is norm-dense in V^* iff $\overline{B}_U(0,R)$ is norm-dense in $\overline{B}_{V^*}(0,R)$.

It is clear from linearity that if $\overline{B}_U(0,R)$ is weak-* dense in $\overline{B}_{V^*}(0,R)$, then U is weak-* dense in V^*. However, the weak-* density of U in V^* does not imply the weak-* density of $\overline{B}_U(0,R)$ in $\overline{B}_{V^*}(0,R)$. Therefore, when studying weak-* approximation in V^*, we aim—when possible—to approximate any $\varphi \in V^*$ by a net (φ_α) in U such that $\|\varphi_\alpha\| \leqslant \|\varphi\|$. This ensures not only convergence but also control of norms. □

Theorem 2.7.4. *The closed unit ball of $C_c(X,\mathbb{F})$ is weak-* dense in the closed unit ball of $L^p(X,\mu,\mathbb{F})$. More precisely, the obvious map $C_c(X,\mathbb{F}) \to L^p(X,\mu,\mathbb{F})$ sends $\overline{B}_{C_c(X,\mathbb{F})}(0,1)$ to a weak-* dense subset of $\overline{B}_{L^p(X,\mu,\mathbb{F})}(0,1)$.*

Proof. By Thm. 1.7.10, if $p < +\infty$, then $\overline{B}_{C_c(X,\mathbb{F})}(0,1)$ is norm-dense in $\overline{B}_{L^p(X,\mu,\mathbb{F})}(0,1)$, and hence also weak-* dense.

Now, we assume $p = +\infty$. let $\mathscr{I}$ be the directed set

$$\begin{aligned}\mathscr{I} &= \{(\mathcal{G},\varepsilon) : \mathcal{G} \in \mathrm{fin}(2^{C_c(X,\mathbb{F})}), \varepsilon \in \mathbb{R}_{\geqslant 0}\}\\ (\mathcal{G}_1,\varepsilon_1) \leqslant (\mathcal{G}_2,\varepsilon_2) &\qquad \text{means} \qquad \mathcal{G}_1 \subset \mathcal{G}_2, \varepsilon_1 \geqslant \varepsilon_2\end{aligned}$$

Fix any $f \in \overline{B}_{L^\infty(X,\mu,\mathbb{F})}(0,1)$. By adding a μ-a.e. zero function to f, we assume that $\|f\|_{l^\infty(X)} = \|f\|_{L^\infty(X,\mu,\mathbb{F})} \leqslant 1$. We claim that for any $(\mathcal{G},\varepsilon) \in \mathscr{I}$, there exists $f_{\mathcal{G},\varepsilon} \in \overline{B}_{C_c(X,\mathbb{F})}(0,1)$ such that

$$\Big|\int_X (f - f_{\mathcal{G},\varepsilon}) g d\mu\Big| \leqslant \varepsilon \qquad \text{for all } g \in \mathcal{G}$$

If this is true, then $(f_{\mathcal{G},\varepsilon})_{(\mathcal{G},\varepsilon)\in\mathscr{I}}$ converges to f when integrated against any element of $C_c(X,\mathbb{F})$. Since $C_c(X,\mathbb{F})$ is dense in $L^1(X,\mu,\mathbb{F})$ (Thm. 1.7.10), it follows from Thm. 2.6.2 that $(f_{\mathcal{G},\varepsilon})_{(\mathcal{G},\varepsilon)\in\mathscr{I}}$ converges weak-* to f, finishing the proof.

Let us prove the claim. We write $\mathcal{G} = \{g_1,\dots,g_n\}$. Let $A_i = \mathrm{Supp}(g_i)$ and $A = A_1 \cup \cdots \cup A_n$. Since A is compact, we have $\mu(A) < +\infty$. Let $M = \|g_1\|_\infty + \cdots + \|g_n\|_\infty$. By Lusin's Thm. 1.7.9 and the Tietze extension Thm. 1.4.27, there exist a compact set $K \subset A$ and a function $f_{\mathcal{G},\varepsilon} \in C_c(X,\mathbb{F})$ satisfying

$$f_{\mathcal{G},\varepsilon}|_K = f|_K \qquad \|f_{\mathcal{G},\varepsilon}\|_{l^\infty} = \|f\|_{l^\infty} \qquad \mu(A\backslash K) \leqslant \varepsilon/2M$$

Recall that $\|f\|_{l^\infty} \leqslant 1$. Thus, for each $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}&\Big|\int_X (f - f_{\mathcal{G},\varepsilon}) g_i\Big| = \Big|\int_{A\backslash K} (f - f_{\mathcal{G},\varepsilon}) g_i\Big| \leqslant M\int_{A\backslash K}(|f| + |f_{\mathcal{G},\varepsilon}|)\\ \leqslant& 2M\cdot\mu(A\backslash K) \leqslant \varepsilon\end{aligned}$$

□

Corollary 2.7.5. *Let μ be a finite Borel measure on $\mathbb{S}^1$. Let $U = \mathrm{Span}_{\mathbb{C}}\{e_n : n \in \mathbb{Z}\}$ where $e_n : z \in \mathbb{S}^1 \mapsto z^n \in \mathbb{C}$. Then for each $f \in L^p(\mathbb{S}^1, \mu)$, there exists a sequence (f_n) in U converging weak-* to f and satisfying $\sup_n \|f\|_{L^p} \leqslant \|f\|_{L^p}$.*

Proof. By Thm. 1.7.13, the normed vector space $V = L^q(\mathbb{S}^1, \mu)$ is separable. Therefore, by Thm. 2.6.6, the weak-* topology of $\overline{B}_{L^q(\mathbb{S}^1,\mu)}(0,1)$ is second countable (and hence first countable). Therefore, by Prop. 1.2.17 and Rem. 1.2.22, to prove the corollary, it suffices to show that $\overline{B}_U(0,1)$ is weak-* dense in $\overline{B}_{L^q(\mathbb{S}^1,\mu)}(0,1)$.

By Thm. 2.7.4, $\overline{B}_{C(\mathbb{S}^1)}(0,1)$ is weak-* dense in $\overline{B}_{L^q(\mathbb{S}^1,\mu)}(0,1)$. By the Stone-Weierstrass Thm. 1.5.12, U is l^∞-dense (and hence L^p-dense) in $C(\mathbb{S}^1)$. Thus, $\overline{B}_U(0,1)$ is L^p-norm-dense (and hence weak-* dense) in $\overline{B}_{C(\mathbb{S}^1)}(0,1)$. This finishes the proof. □

2.8 Weak-* convergence in l^p-spaces

Let X be a set, and let $1 \leqslant p \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$. In this section, we prove the equivalence of the first two columns of Table 2.2 for $V = L^q(X, \mathbb{F})$, cf. Thm. 2.8.5. The most important case is when X is countable and $p = q = 2$. For example, $l^2(\mathbb{Z}^n)$ corresponds to the space of Fourier coefficients of L^2-functions on $\mathbb{T}^n := (\mathbb{S}^1)^n$.

2.8.1 The linear isometry $l^p(X, \mathbb{F}) \to l^q(X, \mathbb{F})^*$

Proposition 2.8.1. *Assume that $1 \leqslant p < +\infty$. Then $C_c(X, \mathbb{F})$ is dense in $l^p(X, \mathbb{F})$, where*

$$C_c(X, \mathbb{F}) := \{f \in \mathbb{F}^X : \mathrm{Supp}(f) \textit{ is a finite set}\} \tag{2.34}$$

The notation of $C_c(X, \mathbb{F})$ in (2.34) is compatible with our usual notation for LCH spaces if X is equipped with the discrete topology $\mathcal{T}_X = 2^X$.

Proof. Choose $f \in l^p(X, \mathbb{F})$. Then, since

$$\lim_{A \in \mathrm{fin}(2^X)} \sum_A |f|^p = \sum_X |f|^p$$

we have

$$\lim_{A \in \mathrm{fin}(2^X)} \|f - f\chi_A\|_{l^p}^p = \lim_{A \in \mathrm{fin}(2^X)} \sum_{X \setminus A} |f| = \sum_X |f|^p - \lim_{A \in \mathrm{fin}(2^X)} \sum_A |f|^p = 0$$

Thus, $(f\chi_A)_{A \in \mathrm{fin}(2^X)}$ is a net in $C_c(X, \mathbb{F})$ converging to f. □

Remark 2.8.2. We have a linear map

$$\begin{aligned}\Psi : l^p(X,\mathbb{F}) &\rightarrow l^q(X,\mathbb{F})^* \\ f &\mapsto \Big(g \in l^q(X,\mathbb{F}) \mapsto \sum_{x\in X} f(x)g(x)\Big)\end{aligned} \tag{2.35}$$

Indeed, by Hölder's inequality, for each $A \in \mathrm{fin}(2^X)$,

$$\Big|\sum_A fg\Big| \leqslant \sum_A |fg| \leqslant \|f\|_{l^p(A)} \cdot \|g\|_{l^q(X)} \leqslant \|f\|_{l^p(X)} \cdot \|g\|_{l^q(X)}$$

Applying $\lim_A$, we see that $\sum_X fg$ is absolutely convergence (i.e. $\sum_X |fg| < +\infty$), and

$$\Big|\sum_X fg\Big| \leqslant \sum_X |fg| \leqslant \|f\|_{l^p(X)} \cdot \|g\|_{l^q(X)}$$

This justifies the claim that Ψ has range in $l^q(X,\mathbb{F})^*$ (rather than just in $\mathrm{Lin}(l^q(X,\mathbb{F}),\mathbb{F})$), and that $\|\Psi\| \leqslant 1$.

Proposition 2.8.3. *The map Ψ in (2.35) is a linear isometry.*

Proof. We already know $\|\Psi\| \leqslant 1$, and we want to show $\|\Psi\| = 1$.

Case $p < +\infty$: By Prop. 2.8.1 and Thm. 2.4.2, we have $\|\Psi\| = \|\Psi|_{C_c(X,\mathbb{F})}\|$. Therefore, it suffices to show that $\|\Psi(f)\| = \|f\|$ for each $f \in C_c(X,\mathbb{F})$. We assume WLOG that $f \neq 0$. Then

$$\langle \Psi(f), g\rangle = \|f\|_{l^p} \cdot \|g\|_{l^q}$$

if we write $f = u|f|$ (where $u : X \rightarrow \mathbb{S}^1$) and let $g = \overline{u} \cdot |f|^{p-1}$. Since $\|\Psi(f)\| \cdot \|g\|_{l^q} \geqslant |\langle \Psi(f), g\rangle|$ and $\|g\|_{l^q} > 0$, we conclude that $\|\Psi(f)\| \geqslant \|f\|_{l^p}$, and hence $\|\Psi(f)\| = \|f\|_{l^p}$.

Case $p = +\infty$: For each $0 \leqslant \lambda < 1$, let $x \in X$ such that $|f(x)| \geqslant \lambda\|f\|_{l^\infty}$. Take $g = \chi_{\{x\}}$. Then

$$\langle \Psi(f), g\rangle = \lambda\|f\|_{l^p} \cdot \|g\|_{l^q}$$

and hence $\|\Psi(f)\| \geqslant \lambda\|f\|_{l^p}$. Since λ is arbitrary, we conclude $\|\Psi(f)\| = \|f\|_{l^p}$. $\square$

2.8.2 Weak-* convergence in $l^p(X,\mathbb{F})$

Definition 2.8.4. Assume that $1 < p \leqslant +\infty$. The **weak-* topology on $l^p(X,\mathbb{F})$** is defined to be the pullback topology via the (injective) map $\Phi : l^p(X,\mathbb{F}) \rightarrow l^q(X,\mathbb{F})^*$ of the weak-* topology of $l^q(X,\mathbb{F})^*$. In other words, a net (f_α) in $l^p(X,\mathbb{F})$ converges weak-* to $f \in l^p(X,\mathbb{F})$ iff for each $g \in l^q(X,\mathbb{F})$ we have

$$\lim_\alpha \sum_X f_\alpha g = \sum_X fg \tag{2.36}$$

Theorem 2.8.5. *Assume $1 < p \leqslant +\infty$. Let (f_α) be a net in $L^p(X,\mathbb{F})$ satisfying $\sup_\alpha \|f_\alpha\|_{l^p} < +\infty$. Then (f_α) converges weak-* to some $f \in l^p(X,\mathbb{F})$ iff $\lim_\alpha f_\alpha(x)$ converges for each $x \in X$.*

Moreover, if (f_α) converges weak- to f, then $f(x) = \lim_\alpha f_\alpha(x)$ for each $x \in X$.*

Consequently, if $p > 1$ and (f_α) is a uniformly l^p-bounded net in $L^p(X,\mathbb{F})$ converging pointwise to $f : X \to \mathbb{F}$, then $f \in l^p(X,\mathbb{F})$. (Indeed, by Thm. 2.8.5, (f_α) converges weak-* to some $\widetilde{f} \in l^p(X,\mathbb{F})$, and $\widetilde{f}$ is the pointwise limit of (f_α). Therefore $f = \widetilde{f}$ belongs to $l^p(X,\mathbb{F})$.)

However, as we will see below, this conclusion must in fact be established first in order to complete the proof of Thm. 2.8.5

Proof. First, assume that (f_α) converges weak-* to $f \in l^p(X,\mathbb{F})$. Applying (2.36) to $g = \chi_{\{x\}}$ (for each $x \in X$), we see that (f_α) converges pointwise to f.

Conversely, assume that (f_α) converges pointwise on X. Let $f \in \mathbb{F}^X$ be the pointwise limit of (f_α). Recall that $C = \sup_\alpha \|f_\alpha\|_{l^p}$ is finite. We claim that $f \in l^p(X,\mathbb{F})$. Indeed, if $p = +\infty$, then for each $x \in X$, we have

$$|f(x)| = \lim_\alpha |f_\alpha(x)| \leqslant \sup_\alpha \|f_\alpha\|_{l^\infty} < +\infty$$

If $p < +\infty$, then for each $A \in \mathrm{fin}(2^X)$,

$$\sum_A |f|^p = \lim_\alpha \sum_A |f_\alpha|^p \leqslant \sup_\alpha \|f_\alpha\|_{l^p}^p \leqslant C^p$$

Applying $\lim_A$, we see that $\sum_X |f|^p \leqslant C^p$, and hence $f \in l^p(X,\mathbb{F})$.

Let Ψ be as in (2.35). By Prop. 2.8.1, $C_c(X,\mathbb{F})$ is dense in $L^q(X,\mathbb{F})$. Therefore, to show that (f_α) converges weak-* to f, by Thm. 2.6.2 and the observation that

$$\sup_\alpha \|\Psi(f_\alpha)\| = \sup_\alpha \|f_\alpha\|_{l^p} < +\infty$$

it suffices to show that $\langle \Psi(f_\alpha), g\rangle$ converges to $\langle \Psi(f), g\rangle$ (that is, $\sum f_\alpha g$ converges to $\sum fg$) for each $g \in C_c(X,\mathbb{F})$. But this follows from the fact that (f_α) converges pointwise to f. $\square$

As an application of Thm. 2.8.5, we prove a variant of Prop. 2.8.1.

Proposition 2.8.6. *Let $1 < p \leqslant +\infty$. Then $\overline{B}_{C_c(X,\mathbb{F})}(0,1)$ is weak-* dense in $\overline{B}_{l^\infty(X,\mathbb{F})}$.*

Proof. Let $f \in \overline{B}_{l^\infty(X,\mathbb{F})}$. Then $(f\chi_A)_{A\in\mathrm{fin}(2^X)}$ is a net in $\overline{B}_{C_c(X,\mathbb{F})}(0,1)$ converging pointwise to f. By Thm. 2.8.5, this net converges weak-* to f. $\square$

2.8.3 The isomorphism $l^p(X,\mathbb{F}) \simeq l^q(X,\mathbb{F})^*$

Now that the equivalence of the first two columns of Table 2.2 for $V = L^q(X,\mathbb{F})$ has been established in Thm. 2.8.5 for $p > 1$, we can prove the isomorphism $l^p(X,\mathbb{F}) \simeq l^q(X,\mathbb{F})^*$ by following the strategy outlined in Rem. 2.2.11.

Of course, at least when X is countable, this isomorphism is a special case of the duality $L^p(X,\mu,\mathbb{F}) \simeq L^q(X,\mu,\mathbb{F})^*$ from Thm. 1.6.12, by taking $\mu : 2^X \to [0,+\infty]$ to be the counting measure. However, there are good reasons to study the proof of $l^q(X,\mathbb{F})^* \simeq l^p(X,\mathbb{F})$ independently.

First, the proof of Thm. 1.6.12 is significantly more involved than the direct proof in the l^p setting. Whenever a result admits a simpler proof in a special case, it is worthwhile to examine that proof directly. Second, Thm. 1.6.12 depends crucially on the Radon–Nikodym Thm. 1.6.8, which in turn can be derived from the Riesz-Fréchet Theorem. The latter can be proved with the help of the isomorphism $l^2(X,\mathbb{F}) \simeq l^2(X,\mathbb{F})^*$. Third, since the proof below follows the idea in Rem. 2.2.11, it also serves as another concrete illustration of Table 2.3.

Theorem 2.8.7. *Assume that $1 < p \leqslant +\infty$. Then the map $\Psi : l^p(X,\mathbb{F}) \to l^q(X,\mathbb{F})^*$ is an isomorphism of normed vector spaces.*

Proof. By Prop. 2.8.3, it remains to show that Ψ is surjective. Choose $\varphi \in l^q(X,\mathbb{F})^*$. We want to find $f \in l^p(X,\mathbb{F})$ such that $\Psi(f) = \varphi$.

Step 1. In this step, we verify Principle 2.2.10, which says in the current setting that φ can be weak-* approximated by a uniformly bounded net in $C_c(X,\mathbb{F})$.

For each $A \in \mathrm{fin}(2^X)$, let $\Psi_A : l^p(A,\mathbb{F}) \to l^q(A,\mathbb{F})^*$ be defined as in (2.35), which is a linear isometry by Prop. 2.8.3. Moreover, since $l^p(A,\mathbb{F})$ and $l^q(A,\mathbb{F})^*$ both have dimension $|A|$, Ψ_A is an isomorphism. Therefore, there exists $f_A \in C_c(X,\mathbb{F})$, supported in A, such that

$$\Psi_A(f_A) = \varphi|_{l^q(A,\mathbb{F})}$$

This relation clearly shows that

$$\lim_{A\in\mathrm{fin}(2^X)} \langle \Psi(f_A), g\rangle = \langle \varphi, g\rangle$$

for each g of the form $\chi_{\{x\}}$ where $x \in X$, and hence for each $g \in C_c(X,\mathbb{F})$. Moreover, the net $(\Psi(f_A))_{A\in\mathrm{fin}(2^X)}$ is uniformly bounded, since

$$\|\Psi(f_A)\|_{l^p} = \|\varphi|_{l^q(A,\mathbb{F})}\| \leqslant \|\varphi\|$$

Therefore, since $C_c(X,\mathbb{F})$ is dense in $l^q(X,\mathbb{F})$ (cf. Prop. 2.8.1), by Thm. 2.6.2, the net $(\Psi(f_A))_{A\in\mathrm{fin}(2^X)}$ converges weak-* to φ. In other words, $(f_A)_{A\in\mathrm{fin}(2^X)}$ is a uniformly l^p-bounded net in $C_c(X,\mathbb{F})$ converging weak-* to φ.

Step 2. For each $x \in X$, the limit

$$\lim_{A \in \mathrm{fin}(2^X)} f_A(x) = \lim_{A \in \mathrm{fin}(2^X)} \sum_X f_A \chi_{\{x\}}$$

converges by the weak-* convergence of $(f_A)_{A \in \mathrm{fin}(2^X)}$. Therefore, since $(f_A)_{A \in \mathrm{fin}(2^X)}$ is a uniformly bounded, by Thm. 2.8.5, the net $(f_A)_{A \in \mathrm{fin}(2^X)}$ converge weak-* to some $f \in l^p(X, \mathbb{F})$. Thus $\varphi = \Psi(f)$. □

2.9 Weak-* convergence of distribution functions

In this section, we fix a proper interval $I \subset \mathbb{R}$, and let $a = \inf I, b = \sup I$. We use freely the notation in Subsec. 1.8.1. In particular, for each function ρ on I, we let

$$\Omega_\rho = \{x \in (a, b) : \rho|_{(a,b)} \text{ is continuous at } x\}$$

A family of functions (ρ_α) from I to $\mathbb{R}$ is called **uniformly bounded** if $\sup_\alpha \|\rho_\alpha\|_{l^\infty(I,\mathbb{R})} < +\infty$.

The goal of this section is to prove Thm. 2.2.9, which characterizes the relationship between pointwise convergence and weak-* convergence for increasing functions. To this end, we begin with several preparatory results concerning the pointwise convergence of such functions.

2.9.1 Almost convergence of increasing functions

Lemma 2.9.1. *Let (ρ_α) be a uniformly bounded net of increasing functions $I \to \mathbb{R}_{\geqslant 0}$. Suppose that (ρ_α) converges pointwise on a dense subset $E \subset I$. Then there exists a bounded increasing function $\rho : I \to \mathbb{R}_{\geqslant 0}$ such that (ρ_α) converges pointwise on E to ρ.*

Proof. Let $\rho : E \to \mathbb{R}_{\geqslant 0}$ be the pointwise limit of (ρ_α), which is clearly bounded and increasing. Extend ρ to a function $\rho : I_{<b} \cup (E \cap \{b\}) \to \mathbb{R}_{\geqslant 0}$ by setting

$$\rho(x) = \lim_{E \ni y \to x^+} \rho(y)$$

if $x \in I \backslash E$. Extend ρ further to $\rho : I \to \mathbb{R}_{\geqslant 0}$ by setting $\rho(b) = \lim_{x \to b^-} \rho(x)$ if $b \in I \backslash E$. Then ρ is bounded and increasing, and (ρ_α) converges pointwise to ρ on E. □

Proposition 2.9.2. *Let (ρ_α) be a uniformly bounded net of increasing functions $I \to \mathbb{R}_{\geqslant 0}$. Let $\rho : I \to \mathbb{R}_{\geqslant 0}$ be increasing. Then the following are equivalent:*

(a) There exists a dense subset $E \subset I$ such that (ρ_α) converges pointwise on E to ρ.

(b) The net (ρ_α) converges pointwise on Ω_ρ to ρ.

*If either of these two statements are true, we say that (ρ_α) **almost converges** to ρ.*

Proof. Since Ω_ρ is dense (Prop. 1.8.1), clearly (b) implies (a).

Now assume (a). Choose any $x \in \Omega_\rho$. We will show that every convergent subnet $(\rho_{\alpha_\nu}(x))$ of $(\rho_\alpha(x))$ converges to $\rho(x)$. This will immediately imply (b).

By Lem. 2.9.1, there exists an increasing function $\widetilde{\rho} : I \to \mathbb{R}_{\geqslant 0}$ such that (ρ_{α_ν}) converges on $E \cup \{x\}$ to $\widetilde{\rho}$. Since (ρ_{α_ν}) converges pointwise on E to ρ, the functions ρ and $\widetilde{\rho}$ agree on E. Namely, ρ and $\widetilde{\rho}$ are almost equal. Therefore, by Prop. 1.8.3, ρ and $\widetilde{\rho}$ agree on Ω_ρ, and in particular at x. This proves $\lim_\nu \rho_{\alpha_\nu}(x) = \rho(x)$. □

The following theorem can be viewed as a concrete manifestation of the Banach-Alaoglu Thm. 2.6.5 in the setting of $C_c(I)^*$. It will be used in the proof of Thm. 2.9.6.

Theorem 2.9.3 (Helly selection theorem). *Let (ρ_α) be a uniformly bounded net (resp. sequence) of increasing functions $I \to \mathbb{R}_{\geqslant 0}$. Then (ρ_α) admits a pointwise convergent subnet (resp. subsequence).*

Proof. The existence of a pointwise convergent subnet follows directly from the Tychonoff Thm. 1.4.15. Therefore, let us assume that (ρ_α) is a sequence (ρ_n). Let $E = I \cap \mathbb{Q}$. Then, by the diagonal method (cf. Rem. 1.4.16), (ρ_n) has a subsequence (ρ_{n_k}) converging pointwise on E. By Lem. 2.9.1, there exists a bounded increasing $\rho : I \to \mathbb{R}_{\geqslant 0}$ such that (ρ_{n_k}) converges pointwise on E to ρ. Therefore, by Prop. 2.9.2, (ρ_{n_k}) converges pointwise on Ω_ρ to ρ. Since $I \backslash \Omega_\rho$ is countable, by the diagonal method again, (ρ_{n_k}) has a subsequence converging pointwise on $I \backslash \Omega_\rho$, and hence on I. □

2.9.2 Almost convergence and weak-* convergence

Definition 2.9.4. Let (ρ_α) be a net in $BV(I, \mathbb{F})$. Let $\rho \in BV(I, \mathbb{F})$. Let Λ_α and Λ be the elements of $C_c(I, \mathbb{F})^*$ corresponding to ρ_α and ρ, respectively, via the Riesz representation Thm. 1.10.4. We say that the net $(d\rho_\alpha)$ **converges weak-*** to $d\rho$ if (Λ_α) converges weak-* to Λ. Namely, for each $f \in C_c(I, \mathbb{F})$, we have

$$\lim_\alpha \int_I f d\rho_\alpha = \int_I f d\rho \tag{2.37}$$

Remark 2.9.5. Suppose that (ρ_α) is a uniformly bounded net of increasing functions $I \to \mathbb{R}_{\geqslant 0}$. Let $\rho : I \to \mathbb{R}_{\geqslant 0}$ be increasing. Since $C_c(I, \mathbb{F})$ is dense in $C_0(I, \mathbb{F})$, by Thm. 2.6.2, the net $(d\rho_\alpha)$ converges weak-* to $d\rho$ iff (2.37) holds for any $f \in C_0(I, \mathbb{F})$.

The following Thm. 2.9.6 is parallel to Thm. 2.7.1. However, unlike Thm. 2.7.1 whose proof relies on the isomorphism $L^p \simeq (L^q)^*$, Thm. 2.9.6 does not rely on the Riesz representation theorem.

Theorem 2.9.6. *Let $(\rho_\alpha)_{\alpha \in \mathscr{A}}$ be a uniformly bounded net of bounded increasing functions $I \to \mathbb{R}_{\geqslant 0}$. Let $\rho : I \to \mathbb{R}_{\geqslant 0}$ be bounded and increasing. Then the following are equivalent:*

(a) *There exists a bounded family* $(\varkappa_\alpha)_{\alpha\in\mathscr{A}}$ *in* $\mathbb{R}$ *(assumed to be zero if* $a \in I$*) satisfying the following conditions:*

- $(\rho_\alpha + \varkappa_\alpha)$ *almost converges to* ρ*.*
- $\lim_\alpha(\rho_\alpha(b) + \varkappa_\alpha) = \rho(b)$ *if* $b \in I$*.*

(b) *The net* $(d\rho_\alpha)$ *converges weak-* to* $d\rho$*.*

The boundedness of $(\varkappa_\alpha)_{\alpha\in\mathscr{A}}$ means that $\sup_\alpha |\varkappa_\alpha| < +\infty$.

Proof. (a)⇒(b): Assume (a). We verify (2.37) for each $f \in C_c(I, \mathbb{F})$, which established (b). Recall from Rem. 1.9.14 that if $a \notin I$, adding constants to ρ_α and ρ does not affect the values of $\int_I f d\rho_\alpha$ and $\int_I f d\rho$.

Since (ρ_α) is uniformly bounded $(\varkappa_\alpha)$ is bounded, there exists $c \geqslant 0$ such that $\rho_\alpha + \varkappa_\alpha + c \geqslant 0$ for all α. Therefore, replacing ρ_α with $\rho_\alpha + \varkappa_\alpha + c$ and ρ with $\rho + c$, we assume that there exists a dense subset $E \subset I$ such that (ρ_α) converges pointwise on E to ρ, and that $b \in E$ if $b \in I$.

Choose any $f \in C_c(I, \mathbb{F})$. Choose $u, v \in \mathbb{R}$ satisfying $\mathrm{Supp}_I(f) \subset [u, v] \subset I$, and let $J = [u, v]$. By increasing v if possible, we may assume that $v \in E$. (When $b \in I$, one simply choose $v = b$.)

In the case where $a \in I$, by Lem. 1.9.10, the values of $\int_J f d\rho_\alpha$ and $\int_J f d\rho$ remain unchanged if we change the values of $\rho_\alpha(a)$ and $\rho(a)$ to 0. Therefore, we may assume that $\rho_\alpha(a) = \rho(a) = 0$ (so that a can be included to E), and we may also choose $u = a$. In the case where $a \notin I$, by the density of E, we can slightly decrease u so that $u \in E$. To summarize, whether a or b belongs to I or not, we can assume

$$u, v \in E$$

Since f is uniformly continuous, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ for each $x, y \in I$ satisfying $|x - y| \leqslant \delta$. Choose a tagged partition

$$(\sigma, \xi_\bullet) = \big(\{a_0 = u < a_1 < \cdots < a_n = v\}, (\xi_1, \dots, \xi_n)\big)$$

of J with mesh $< \delta$. Since E is dense, by a slight adjustment, we may assume that $a_0, a_1, \dots, a_n \in E$. This implies

$$\lim_\alpha f(u)\rho_\alpha(u) = f(u)\rho(u) \qquad \lim_\alpha S_{\rho_\alpha}(f, \sigma, \xi_\bullet) = S_\rho(f, \sigma, \xi_\bullet)$$

Therefore, if we let $C = \sup\{\rho_\alpha(v) - \rho_\alpha(u), \rho(v) - \rho(u) : \alpha \in \mathscr{A}\}$, then Rem. 1.9.5 implies

$$\limsup_\alpha \Big|\int_J f d\rho_\alpha - \int_J f d\rho\Big| \leqslant 2\varepsilon \cdot C$$

This finishes the proof of (2.37).

(b)⇒(a): Assume (b). We first consider the case where $a \notin I$. Fix $t \in \Omega_\rho$, and let

$$\varkappa_\alpha = \rho(t) - \rho_\alpha(t)$$

Then $(\varkappa_\alpha)$ is bounded. Therefore, $(\rho_\alpha + \varkappa_\alpha)$ is uniformly bounded, and hence there exists $c \geqslant 0$ such that $\rho_\alpha + \varkappa_\alpha + c \geqslant 0$ for all α. Replacing ρ_α with $\rho_\alpha + c$ and ρ with $\rho + c$, we assume that $\rho_\alpha + \varkappa_\alpha \geqslant 0$ for all α. (Of course, we still have $\rho \geqslant 0$.)

Choose any $x \in \Omega_\rho$. To show that $(\rho_\alpha(x) + \varkappa_\alpha)_\alpha$ converges to $\rho(x)$, it suffices to show that every convergent subnet $(\rho_\beta(x) + \varkappa_\beta)_\beta$ converges to $\rho(x)$.

By the Helly selection Thm. 2.9.3, the net of functions $(\rho_\beta + \varkappa_\beta)_\beta$ has a pointwise convergent subnet $(\rho_\gamma + \varkappa_\gamma)_\gamma$. Let $\widetilde{\rho} : I \to \overline{\mathbb{R}}_{\geqslant 0}$ be the pointwise limit of this subnet, which is clear bounded and increasing. By (a)⇒(b), the net $(d(\rho_\gamma + \varkappa_\gamma))_\gamma$ converges weak-* to $d\widetilde{\rho}$. By assumption, it also converges weak-* to $d\rho$. Therefore, we have $\int_I f d\widetilde{\rho} = \int_I f d\rho$ for each $f \in C_c(I)$.

By Thm. 1.9.13 (and noting Rem. 1.9.14), we have

$$\widetilde{\rho} - \lim_{y \to a^+} \widetilde{\rho}(y) = \rho - \lim_{y \to a^+} \rho(y) \qquad \text{on } \Omega_\rho$$

In other words, there exists a constant $c \in \mathbb{R}$ such that

$$\widetilde{\rho} + c = \rho \qquad \text{on } \Omega_\rho \tag{2.38}$$

Since $\rho_\alpha(t) + \varkappa_\alpha = \rho(t)$ is constant over α, and since its subnet $(\rho_\gamma(t) + \varkappa_\gamma)_\gamma$ converges to $\widetilde{\rho}(t)$, we conclude $\widetilde{\rho}(t) = \rho(t)$. Therefore, since $t \in \Omega_\rho$, by (2.38), we have $c = 0$. Since $x \in \Omega_\rho$, by (2.38), we obtain $\widetilde{\rho}(x) = \rho(x)$. This proves that $(\rho_\gamma(x) + \varkappa_\gamma)_\gamma$ converges to $\rho(x)$, and hence $(\rho_\beta(x) + \varkappa_\beta)_\beta$ converges to $\rho(x)$.

Now consider the case where $a \in I$. We set $\varkappa_\alpha = 0$. Similar to the above argument, we choose any $x \in \Omega_\rho$, choose a subnet ρ_β converging at x, and further choose a subnet ρ_γ converging pointwise on I to $\widetilde{\rho} : I \to \overline{\mathbb{R}}_{\geqslant 0}$. By (a)⇒(b), we have $\int_I f d\widetilde{\rho} = \int_I f d\rho$ for each $f \in C_c(I)$. Consequently, Thm. 1.9.13 implies that $\widetilde{\rho} = \rho$ on Ω_ρ. Since $x \in \Omega_\rho$, we obtain again $\lim_\beta \rho_\beta(x) = \lim_\gamma \rho_\gamma(x) = \widetilde{\rho}(x) = \rho(x)$. Therefore $(\rho_\alpha(x))_\alpha$ converges to $\rho(x)$ for each $x \in \Omega_\rho$. □

Corollary 2.9.7. *Let $(\rho_\alpha)_{\alpha \in \mathscr{A}}$ be a uniformly bounded net of increasing functions $I \to \mathbb{R}_{\geqslant 0}$. Then the following are equivalent:*

(1) There exists a bounded family $(\varkappa_\alpha)_{\alpha \in \mathscr{A}}$ in $\mathbb{R}$ (assumed to be zero if $a \in I$) such that $(\rho_\alpha + \varkappa_\alpha)$ converges pointwise on a dense subset $E \subset I$, and also at b if $b \in I$.

(2) There exists a bounded increasing $\rho : I \to \mathbb{R}_{\geqslant 0}$ such that $(d\rho_\alpha)_{\alpha \in \mathscr{A}}$ converges weak- to $d\rho$.*

Proof. "(2)⇒(1)" follows immediately from Thm. 2.9.6. Conversely, assume (1). By Lem. 2.9.1, there exists a bounded increasing $\rho : I \to \mathbb{R}_{\geqslant 0}$ such that $(\rho_\alpha + \varkappa_\alpha)$ converges pointwise on $E \cup \{I \cap \{b\}\}$ to ρ. Then Thm. 2.9.6 implies (2). □

2.10 Weak-* approximation of Radon measures by Dirac measures

Fix an LCH space X. Recall that we have assumed throughout the notes that $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let

$$\begin{aligned} \mathcal{RM}(X, \overline{\mathbb{R}}_{\geqslant 0}) &= \{\text{Radon measures on } X\} \\ \mathcal{RM}(X, \mathbb{R}_{\geqslant 0}) &= \{\text{finite Radon measures on } X\} \\ \mathcal{RM}(X, \mathbb{R}) &= \{\text{signed Radon measures on } X\} \\ \mathcal{RM}(X, \mathbb{C}) &= \{\text{complex Radon measures on } X\} \end{aligned} \tag{2.39}$$

which are vectors spaces over $\overline{\mathbb{R}}_{\geqslant 0}, \mathbb{R}_{\geqslant 0}, \mathbb{R}, \mathbb{C}$ respectively. Note the inclusion relation

$$\mathcal{RM}(X, \mathbb{R}_{\geqslant 0}) \subset \mathcal{RM}(X, \overline{\mathbb{R}}_{\geqslant 0}) \qquad \mathcal{RM}(X, \mathbb{R}_{\geqslant 0}) \subset \mathcal{RM}(X, \mathbb{R}) \subset \mathcal{RM}(X, \mathbb{C})$$

Recall that for each $x \in X$, the Dirac measure at x is denoted by δ_x.

The goal of this section is to prove Principle 2.2.10 for $V = C_c(X, \mathbb{F})$. In this context, elementary functions are understood as linear combinations of Dirac measures. When X is an interval $I \subset \mathbb{R}$, these elementary functions correspond to bounded increasing functions $I \to \mathbb{R}_{\geqslant 0}$ whose ranges are finite sets.

2.10.1 Definitions and basic properties

Definition 2.10.1. Recall the $\mathbb{F}$-linear isomorphism

$$\mathcal{RM}(X, \mathbb{F}) \simeq C_c(X, \mathbb{F})^*$$

defined by the Riesz-Markov representation Thm. 1.7.16. The pullback of the operator norm on $C_c(X, \mathbb{F})^*$ to $\mu \in \mathcal{RM}(X, \mathbb{F})$ is called the **total variation** of μ, and is denoted by $\|\mu\|$. In other words,

$$\|\mu\| = \sup \Big\{ \Big| \int f d\mu \Big| : f \in C_c(X, \mathbb{F}), |f| \leqslant 1 \Big\}$$

A family of complex Radon measures $(\mu_\alpha)_{\alpha \in \mathscr{A}}$ is called **uniformly bounded** if

$$\sup_{\alpha \in \mathscr{A}} \|\mu_\alpha\| < +\infty$$

The weak-* topology on $C_c(X, \mathbb{F})^*$ defines the **weak-* topology on $\mathcal{RM}(X, \mathbb{F})$**. Thus, if (μ_α) is a uniformly bounded net in $\mathcal{RM}(X, \mathbb{F})$, and if $\mu \in \mathcal{RM}(X, \mathbb{F})$, then (μ_α) converges weak-* to μ [10] iff for each $f \in C_c(X, \mathbb{F})$ we have

$$\lim_\alpha \int_X f d\mu_\alpha = \int_X f d\mu \tag{2.40}$$

By Thm. 2.6.2, (μ_α) converges weak-* to μ iff (2.40) holds for all $f \in C_0(X, \mathbb{F})$.

[10] We also say that $(d\mu_\alpha)$ converges weak-* to $d\mu$.

Example 2.10.2. By Thm. 1.7.6, if $\mu \in \mathcal{RM}(X, \mathbb{R}_{\geqslant 0})$, then

$$\|\mu\| = \mu(X)$$

Example 2.10.3. Let $E \subset X$ be a finite set, and let $c : E \to \mathbb{F}$ be a function. Then

$$\Big\| \sum_{x \in E} c(x)\delta_x \Big\| = \sum_{x \in E} |c(x)| \tag{2.41}$$

Proof. Let $\mu = \sum_{x \in E} c(x)\delta_x$. By Exp. 2.10.2, we have $\|\delta_x\| = 1$. Since norms satisfy the sub-additivity, we have

$$\|\mu\| \leqslant \sum_{x \in E} |c(x)| \cdot \|\delta_x\| = \sum_{x \in E} |c(x)|$$

By Urysohn's lemma, there exists $f \in C_c(X, \mathbb{F})$ such that $\|f\|_{l^\infty} \leqslant 1$, and that for each $x \in E$, we have $|f(x)| = 1$ and $f(x)c(x) = |c(x)|$. Then $\int_X f d\mu = \sum_{x \in E} |c(x)|$. This proves $\|\mu\| \geqslant \sum_{x \in E} |c(x)|$. $\square$

Lemma 2.10.4. *Let $\mu \in \mathcal{RM}(X, \mathbb{F})$. Let $A_1, \ldots, A_k$ be mutually disjoint Borel subsets of X. Then*

$$\|\mu\| \geqslant \sum_{j=1}^{k} |\mu(A_j)|$$

Proof. Since μ is a linear combination of finite Radon measures, there exists $\widehat{\mu} \in \mathcal{RM}(X, \mathbb{R}_{\geqslant 0})$ such that $|\mu(A)| \leqslant \widehat{\mu}(A)$ for each Borel $A \subset X$. Since Radon measures are regular on Borel sets with finite measures (Thm. 1.7.7), for each $\varepsilon > 0$ there exists compact $K_j \subset A_j$ such that $\widehat{\mu}(A_j \backslash K_j) \leqslant \varepsilon$.

By Cor. 1.4.23, there exist mutually disjoint open subsets $U_1, \ldots, U_n \subset X$ such that $U_j \supset K_j$. Since $\widehat{\mu}$ is regular on K_j, we may assume that $\widehat{\mu}(U_j \backslash K_j) < \varepsilon$. By Urysohn's lemma, there exists $f_j \in C_c(U_j, \mathbb{F})$ such that $|f_j| \leqslant 1$, that $f_j|_{K_j}$ equals a constant $c_j \in \mathbb{F}$, and that $c_j \mu(K_j) = |\mu(K_j)|$. Let $f = f_1 + \cdots f_k$, which is an element of $C_c(X, \mathbb{F})$ satisfying $|f| \leqslant 1$. Then

$$\int_{\bigcup_j K_j} f d\mu = \sum_j |\mu(K_j)| \qquad \Big| \int_{X \backslash \bigcup_j K_j} f d\mu \Big| \leqslant k\varepsilon$$

Since $|\mu(A_j) - \mu(K_j)| = |\mu(A_j \backslash K_j)| \leqslant \widehat{\mu}(A_j \backslash K_j) \leqslant \varepsilon$, we obtain $|\mu(K_j)| \geqslant |\mu(A_j)| - \varepsilon$, and hence

$$\|\mu\| \geqslant \Big| \int_X f d\mu \Big| \geqslant \Big| \int_{\bigcup_j K_j} f d\mu \Big| - \Big| \int_{X \backslash \bigcup_j K_j} f d\mu \Big| \geqslant \sum_j |\mu(A_j)| - 2k\varepsilon$$

Since ε is arbitrary, we obtain the desired inequality. $\square$

2.10.2 Approximation of Radon measures by Dirac measures

In this section, we let $\mathbb{K} \in \{\mathbb{R}_{\geqslant 0}, \mathbb{R}, \mathbb{C}\}$.

Theorem 2.10.5. *Define*

$$\mathcal{D}(X, \mathbb{K}) = \mathrm{Span}_{\mathbb{K}}\{\delta_x : x \in X\}$$

Then the closed unit ball of $\mathcal{D}(X, \mathbb{K})$ is weak- dense in the closed unit ball of $\mathcal{RM}(X, \mathbb{K})$. In other words, $\overline{B}_{\mathcal{D}(X,\mathbb{K})}(0, 1)$ is weak-* dense in $\overline{B}_{\mathcal{RM}(X,\mathbb{K})}(0, 1)$.*

Proof. Fix $\mu \in \mathcal{RM}(X, \mathbb{K})$ satisfying $\|\mu\| \leqslant 1$. Similar to the proof of Thm. 2.7.4, we let $\mathscr{I}$ be the directed set

$$\mathscr{I} = \{(\mathcal{G}, \varepsilon) : \mathcal{G} \in \mathrm{fin}(2^{C_c(X,\mathbb{K})}), \varepsilon \in \mathbb{R}_{\geqslant 0}\}$$
$$(\mathcal{G}_1, \varepsilon_1) \leqslant (\mathcal{G}_2, \varepsilon_2) \qquad \text{means} \qquad \mathcal{G}_1 \subset \mathcal{G}_2, \varepsilon_1 \geqslant \varepsilon_2$$

We claim that for any $(\mathcal{G}, \varepsilon) \in \mathscr{I}$, there exists $\mu_{\mathcal{G},\varepsilon} \in \overline{B}_{\mathcal{D}(X,\mathbb{K})}(0, 1)$ such that

$$\Big|\int_X f d\mu - \int_X f d\mu_{\mathcal{G},\varepsilon}\Big| \leqslant \varepsilon \qquad \text{for all } f \in \mathcal{G}$$

If this is true, then $(\mu_{\mathcal{G},\varepsilon})_{(\mathcal{G},\varepsilon)\in\mathscr{I}}$ is a net in $\overline{B}_{\mathcal{D}(X,\mathbb{K})}(0, 1)$ converging weak-* to μ. This will finish the proof.

Let us prove the claim. Since μ is a linear combination of finite Radon measures, there exists $\widehat{\mu} \in \mathcal{RM}(X, \mathbb{R}_{\geqslant 0})$ such that

$$\Big|\int_X g d\mu\Big| \leqslant \int_X |g| d\widehat{\mu}$$

for each bounded Borel function $g : X \to \mathbb{C}$.

Let $K \subset X$ be compact and containing $\mathrm{Supp}(f)$ for all $f \in \mathcal{G}$. By the compactness of K, there exist open sets $U_1, \dots, U_k \subset X$ whose union contains K, such that $\mathrm{diam}(f(U_j)) \leqslant \varepsilon/\widehat{\mu}(K)$ for each j and $f \in \mathcal{G}$. Choose a Borel set $A_j \subset U_j$ such that $K = A_1 \sqcup \cdots \sqcup A_k$.[11] Choose any $x_j \in A_j$, and let

$$\mu_{\mathcal{G},\varepsilon} = \sum_{j=1}^{k} \mu_j(A_j)\delta_{x_j} \tag{2.42}$$

Then, for each $f \in \mathcal{G}$,

$$\Big|\int_X f d(\mu - \mu_{\mathcal{G},\varepsilon})\Big| \leqslant \sum_{j=1}^{k} \Big|\int_{A_i} f d(\mu - \mu_{\mathcal{G},\varepsilon})\Big| = \sum_{j=1}^{k} \Big|\int_{A_i} f d\mu - \mu_j(A_j) f(x_i)\Big|$$

[11]For example, take $A_1 = K \cap U_1$ and $A_j = K \cap U_j \backslash (U_1 \cup \cdots \cup U_{j-1})$ if $j > 1$.

$$= \sum_{j=1}^{k} \Big| \int_{A_i} (f - f(x_j)) d\mu \Big| \leqslant \sum_{j=1}^{k} \int_{A_j} |f - f(x_j)| d\widehat{\mu} \leqslant \frac{\varepsilon}{\widehat{\mu}(K)} \sum_{j=1}^{k} \widehat{\mu}(A_j) = \varepsilon$$

This proves the desired inequality. Moreover, by Exp. 2.10.3 and Lem. 2.10.4,

$$\|\mu_{\mathcal{G},\varepsilon}\| = \sum_{j=1}^{k} |\mu_j(A_j)| \leqslant \|\mu\| \leqslant 1$$

This proves that $\mu_{\mathcal{G},\varepsilon} \in \overline{B}_{\mathcal{D}(X,\mathbb{K})}(0,1)$. □

The proof of Thm. 2.10.5 immediately implies:

Theorem 2.10.6. *For each $\mu \in \mathcal{RM}(X,\mathbb{C})$, we have*

$$\|\mu\| = \sup \Big\{ \sum_{j=1}^{k} |\mu(A_j)| : k \in \mathbb{Z}_+, \text{ and } A_1, \dots, A_k \in \mathfrak{B}_X \text{ are mutually disjoint} \Big\} \tag{2.43}$$

Proof. Lem. 2.10.4 implies "$\geqslant$". Let us prove "$\leqslant$". Let $(\mu_{\mathcal{G},\varepsilon})_{(\mathcal{G},\varepsilon)\in\mathfrak{I}}$ be the net in $\mathcal{D}(X,\mathbb{C})$ converging weak-* to μ and satisfying $\|\mu_{\mathcal{G},\varepsilon}\| \leqslant \|\mu\|$. Each $\mu_{\mathcal{G},\varepsilon}$ is of the form (2.42), by Lem. 2.10.3, the RHS of (2.43) is $\geqslant \|\mu_{\mathcal{G},\varepsilon}\|$. By Fatou's lemma for weak-* convergence (Prop. 2.6.4), the RHS of (2.43) is $\geqslant \|\mu\|$. □

3 Basics of inner product spaces

3.1 Sesquilinear forms

Let V be $\mathbb{C}$-vector spaces.

3.1.1 Sesquilinear forms

Definition 3.1.1. A map of $\mathbb{C}$-vector spaces $T : V \to W$ is called **antilinear** or **conjugate linear** if for every $a, b \in \mathbb{F}$ and $u, v \in V$ we have

$$T(au + bv) = \overline{a}u + \overline{b}v$$

where $\overline{a}, \overline{b}$ are the complex conjugates of a, b.

Definition 3.1.2. A function $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ (sending $u \times v \in V^2$ to $\langle u|v\rangle$) is called a **sesquilinear form** if it is antilinear on the first variable, and linear on the second one.[1] Namely, for each $a, b \in \mathbb{C}$ and $u, v, w \in V$ we have

$$\langle au + bv|w\rangle = \overline{a}\langle u|w\rangle + \overline{b}\langle v|w\rangle \qquad \langle w|au + bv\rangle = a\langle w|u\rangle + b\langle w|v\rangle$$

More generally, if V, W are complex vector spaces, a map $V \times W \to \mathbb{C}$ is also called **sesquilinear** if it is antilinear on the V-component and linear on the W-component. The function

$$V \to \mathbb{C} \qquad v \mapsto \langle v|v\rangle$$

is called the **quadratic form** associated to the sesquilinear form $\langle\cdot|\cdot\rangle$.

Notice the difference between the notations $\langle u|v\rangle$ and $\langle u, v\rangle$: the latter always means a bilinear form, i.e., a function which is linear on both variables.

Remark 3.1.3. For each sesquilinear form $\langle\cdot|\cdot\rangle$ on V, we have the **polarization identity**

$$\begin{aligned}\langle u|v\rangle &= \frac{1}{4}\sum_{t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}} \langle u + e^{\mathbf{i}t}v|u + e^{\mathbf{i}t}v\rangle e^{\mathbf{i}t} \\ &= \frac{1}{4}\Big(\langle u + v|u + v\rangle - \langle u - v|u - v\rangle + \mathbf{i}\langle u + \mathbf{i}v|u + \mathbf{i}v\rangle - \mathbf{i}\langle u - \mathbf{i}v|u - \mathbf{i}v\rangle\Big)\end{aligned} \tag{3.1}$$

Therefore, sesquilinear forms are determined by their associated quadratic forms.

Definition 3.1.4. Let $\omega(\cdot|\cdot) : V \times W \to \mathbb{C}$ be a sesquilinear form. The **adjoint sesequilinear form ω^*** is defined to be

$$\omega^* : W \times V \to \mathbb{C} \qquad \omega^*(w|v) = \overline{\omega(v|w)}$$

[1] This is different from [Gui-A], where the second variable is assumed to be antilinear

Definition 3.1.5. A sesquilinear form $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ is called a **Hermitian form** if is equal to it adjoint, namely,

$$\langle v|u\rangle = \overline{\langle u|v\rangle} \qquad \text{for each } u, v \in V$$

Proposition 3.1.6. *Let $\langle\cdot|\cdot\rangle$ be a sesquilinear form on V. The following are equivalent:*

(1) $\langle\cdot|\cdot\rangle$ is a Hermitian form.

(2) The quadratic form associated to $\langle\cdot|\cdot\rangle$ is real-valued, that is, for each $v \in V$ we have $\langle v|v\rangle \in \mathbb{R}$.

Proof. Let $\omega = \langle\cdot|\cdot\rangle$. By the polarization identity, we have $\omega^* = \omega$ iff $\omega^*(v|v) = \omega(v|v)$ (i.e. $\overline{\omega(v|v)} = \omega(v|v)$) for each $v \in V$. □

3.1.2 Positive sesquilinear forms

Definition 3.1.7. A sesquilinear form $\langle\cdot|\cdot\rangle$ on V is called **positive semi-definite** (or simply **positive**) and written as $\langle\cdot|\cdot\rangle \geqslant 0$, if $\langle v|v\rangle \geqslant 0$ for all $v \in V$. If a positive sesquilinear form $\langle\cdot|\cdot\rangle$ on V is fixed, we define

$$\|v\| = \sqrt{\langle v|v\rangle} \qquad \text{for all } v \in V \tag{3.2}$$

Then it is clear that $\|\lambda v\| = |\lambda| \cdot \|v\|$ for each $v \in V$ and $\lambda \in \mathbb{C}$. A vector $v \in V$ satisfying $\|v\| = 1$ is called a **unit vector**.

By Prop. 3.1.6, a positive sesquilinear form is Hermitian. More generally, we have the following definition:

Definition 3.1.8. Let ω_1, ω_2 be Hermitian forms on V. We write

$$\boldsymbol{\omega_1 \leqslant \omega_2}$$

(equivalently, $\omega_2 \geqslant \omega_1$) if the (real-valued) quadratic forms associated to ω_1 and ω_2 satisfy the corresponding inequality, that is,

$$\omega_1(\xi|\xi) \leqslant \omega_2(\xi|\xi) \qquad \text{for each } \xi \in V$$

Thus, "$\leqslant$" defines a partial order on the set of sesquilinear forms on V. Moreover, the meaning of $0 \leqslant \omega$ agrees that in Def. 3.1.7

Theorem 3.1.9 (Cauchy-Schwarz inequality). *Let $\langle\cdot|\cdot\rangle$ be a positive sesquilinear form on V. Then for each $u, v \in V$ we have*

$$|\langle u|v\rangle| \leqslant \|u\| \cdot \|v\|$$

Proof. By linear algebra, if $f : \mathbb{R}^2 \to \mathbb{R}$ is a quadratic form

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2$$

where $a, b, c \in \mathbb{R}$, then $f \geqslant 0$ iff $a \geqslant 0, b \geqslant 0$ and

$$ac - b^2 \equiv \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \geqslant 0$$

In fact, we only need the fact that if $f \geqslant 0$ then $ac - b^2 \geqslant 0$. To see this, note that if f is not always 0, then one of a, c must be nonzero; otherwise, $f(x,y) = 2bxy$ cannot be always $\geqslant 0$. Thus, assume WLOG that $a \neq 0$. Then $f(x,1) = ax^2 + 2bx + c = a(x + b/a)^2 + c - b^2/a$, which implies $a > 0$ and $c - b^2/a \geqslant 0$, and hence $ac - b^2 \geqslant 0$.

Now, we let $f : \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ be the quadratic form defined by pulling back the form $\xi \in V \mapsto \langle \xi | \xi \rangle$ via the map $(x,y) \in \mathbb{R}^2 \mapsto xu + yv \in V$, that is,

$$f(x,y) = \langle xu + yv | xu + yv \rangle = \|u\|^2 \cdot x^2 + 2\mathrm{Re}\langle u|v \rangle \cdot xy + \|v\|^2 \cdot y^2$$

Then, the above paragraph shows that $\|u\|^2 \cdot \|v\|^2 - (\mathrm{Re}\langle u|v\rangle)^2 \geqslant 0$, equivalently,

$$|\mathrm{Re}\langle u|v\rangle| \leqslant \|u\| \cdot \|v\|$$

Choose $\lambda \in \mathbb{S}^1$ such that $\lambda \langle u|v \rangle \in \mathbb{R}$. Since the above inequality holds when v is replaced by λv, we get

$$|\langle u|v\rangle| = |\mathrm{Re}\langle u|\lambda v\rangle| \leqslant \|u\| \cdot \|\lambda v\| = \|u\| \cdot \|v\|$$

$\square$

Corollary 3.1.10. *Let $\langle \cdot | \cdot \rangle$ be a positive sesquilinear form on V. Then we have*

$$\{v \in V : \|v\| = 0\} = \{v \in V : \langle v|\xi\rangle = 0 \text{ for all } \xi \in V\}$$

where the RHS is clearly a linear subspace of V. We call this space the ***null space*** *of $\langle \cdot | \cdot \rangle$.*

Proof. Let $v \in V$. If $\langle v|V\rangle = 0$, then $\|v\|^2 = \langle v|v\rangle = 0$. Conversely, if $\|v\| = 0$, then by the Cauchy-Schwarz inequality, for each $\xi \in V$ we have $|\langle u|\xi\rangle| \leqslant \|u\| \cdot \|\xi\| = 0$. $\square$

Corollary 3.1.11. *Let $\langle \cdot | \cdot \rangle$ be a positive sesquilinear form on V. Then $v \in V \mapsto \|v\| \in \mathbb{R}_{\geqslant 0}$ is a seminorm on V.*

Proof. It remains to check the subadditivity: for each $u, v \in V$, the Cauchy-Schwarz inequality imlies

$$\begin{aligned} \|u+v\|^2 &= \langle u+v|u+v\rangle = \|u\|^2 + 2\mathrm{Re}\langle u|v\rangle + \|u\|^2 \\ &\leqslant \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2 \end{aligned}$$

$\square$

3.2 Inner product spaces and bounded sesquilinear forms

3.2.1 Inner product spaces

Definition 3.2.1. Let $\langle\cdot|\cdot\rangle$ be a positive sesquilinear form on a $\mathbb{C}$-vector space V. We call $\langle\cdot|\cdot\rangle$ an **inner product** if it is **non-degenerate**, i.e., the null space is 0. We call the pair $(V,\langle\cdot|\cdot\rangle)$ (or simply call V) an **inner product space** or a **pre-Hilbert space** .

Exercise 3.2.2. Let $\langle\cdot|\cdot\rangle$ be a positive sesquilinear form on V with null space $\mathscr{N}$. Prove that there is a (necessarily unique) inner product $\langle\cdot|\cdot\rangle_{V/\mathscr{N}}$ on the quotient space $V/\mathscr{N}$ such that for any $u,v\in V$, the cosets $u+\mathscr{N}$ and $v+\mathscr{N}$ satisfy

$$\langle u+\mathscr{N}|v+\mathscr{N}\rangle_{V/\mathscr{N}}=\langle u|v\rangle$$

Example 3.2.3. Let X be a set. Then $l^2(X)=l^2(X,\mathbb{C})$ is an inner product space, where

$$\langle f|g\rangle=\sum_{x\in X}\overline{f(x)}g(x)\qquad\text{for any } f,g\in l^2(X)$$

Example 3.2.4. Let (X,μ) be a measure space. Then $L^2(X,\mu)$ is an inner product space, where

$$\langle f|g\rangle=\int_X\overline{f}gd\mu\qquad\text{for any } f,g\in L^2(X,\mu)$$

Remark 3.2.5. By Rem. 3.1.11, an inner product space V is equipped with the norm defined by $\|v\|=\sqrt{\langle v|v\rangle}$. In particular, V is a metric space with metric $d(u,v)=\|u-v\|$. The topology on V induced by this metric is called the **norm topology** of V.

Remark 3.2.6. Let V,W be inner product spaces. If $T:V\to V$ is a linear map, then T is an isometry of metric spaces iff T is an isometry of normed vector spaces, i.e.,

$$\langle Tv|Tv\rangle=\langle v|v\rangle\qquad\text{for all } v\in V$$

By the polarization identity, this is equivalent to

$$\langle Tu|Tv\rangle=\langle u|v\rangle\qquad\text{for all } u,v\in V$$

A surjective linear isometry $T:V\to W$ is called a **unitary map**. If $T:V\to W$ is unitary, we say that V,W are **isomorphic inner product spaces** (or that V,W are **unitarily equivalent**).

Similarly, if $T:V\to V$ is antilinear map between inner product spaces, then T is an isometry of metric spaces iff

$$\langle Tv|Tv\rangle=\langle v|v\rangle\qquad\text{for all } v\in V$$

By the polarization identity, this is equivalent to

$$\langle Tu|Tv\rangle = \langle v|u\rangle \qquad \text{for all } u, v \in V$$

A surjective antilinear isometry $T : V \to W$ is called an **antiunitary map**. If $T : V \to W$ is antiunitary, we say that V and W are **antiunitarily equivalent**. □

3.2.2 Bounded sesquilinear forms

Let V, W be inner product spaces.

Definition 3.2.7. The **(complex) conjugate** of V is the inner product space $V^{\complement}$ defined as follows. The elements of $V^{\complement}$ correspond bijectively to those of V by the map

$$\complement : V \to V^{\complement} \qquad v \mapsto v^{\complement} \equiv \overline{v}$$

where $v^{\complement} \equiv \overline{v}$ is an abstract element, called the **conjugate** of v. Moreover, the structure of an inner product space on $V^{\complement}$ is defined in such a way that $\complement$ is antiunitary. In other words, for each $u, v \in V$ and $a, b \in \mathbb{C}$, we have

$$\overline{a} \cdot \overline{u} + \overline{b} \cdot \overline{v} := \overline{au + bv}$$
$$\langle \overline{u}|\overline{v}\rangle_{V^{\complement}} := \overline{\langle u|v\rangle_V} = \langle v|u\rangle_V$$

The conjugate of $V^{\complement}$ is defined to be V, that is,

$$(V^{\complement})^{\complement} = V$$

Moreover, the conjugate map $\complement : V^{\complement} \to V$ is defined by

$$\complement : V^{\complement} \to V \qquad \overline{v} \mapsto v$$

Thus $\overline{\overline{v}} = v$ for each $v \in V$. □

Remark 3.2.8. An antilinear map $T : V \to W$ is equivalent to the linear map

$$V \to W^{\complement} \qquad v \mapsto \overline{Tv} \tag{3.3a}$$

and is also equivalent to the linear map

$$V^{\complement} \to W \qquad \overline{v} \mapsto Tv \tag{3.3b}$$

It is clear that T is an antilinear isometry (resp. antiunitary) iff (3.3a) is a linear isometry (resp. unitary) iff (3.3b) is a linear isometry (resp. unitary).

Remark 3.2.9. A sesquilinear form $\omega : V \times W \to \mathbb{C}$ is equivalent to a bilinear form

$$\widetilde{\omega} : V^{\complement} \times W \to \mathbb{C} \qquad (\overline{v}, w) \mapsto \langle v|w\rangle$$

Unless otherwise stated, we always view ω and $\widetilde{\omega}$ as the same.

Definition 3.2.10. Let $\omega : V \times W \to \mathbb{C}$ be a sesquilinear form. The **norm** $\|\omega\|$ is defined to be the norm of the associated bilinear form $V^{\complement} \times W \to \mathbb{C}$. Therefore,

$$\|\omega\| = \sup_{v\in\overline{B}_V(0,1), w\in\overline{B}_W(0,1)} |\omega(u|v)|$$

Recalling the notation (2.19), we let

$$\mathfrak{Ses}(V|W) := \mathfrak{L}(V^{\complement} \times W, \mathbb{C})$$

which is the space of bounded sesquilinear forms $V \times W \to \mathbb{C}$. We write

$$\mathfrak{Ses}(V) := \mathfrak{Ses}(V|V)$$

The elements of $\mathfrak{Ses}(V|W)$ (resp. $\mathfrak{Ses}(V)$) are called **bounded sesquilinear forms** on $V \times W$ (resp. on V).

Example 3.2.11. The inner product

$$\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C} \qquad (u, v) \mapsto \langle u|v\rangle$$

has norm 1, and hence belongs to $\mathfrak{Ses}(V)$. Therefore, by Prop. 2.3.9, this map is continuous.

The following useful property says that a sesquilinear form is bounded iff the associated quadratic form is bounded.

Proposition 3.2.12. *Let ω be sesquilinear form on V. Let $M \in \mathbb{R}_{\geqslant 0}$. Assume that*

$$|\omega(\xi|\xi)| \leqslant M\|\xi\|^2$$

for each $\xi \in V$. Then $\|\omega\| \leqslant 4M$.

Proof. Choose any $\xi, \eta \in \overline{B}_V(0,1)$. For each $\lambda \in \mathbb{S}^1$, we have

$$|\omega(\xi + \lambda\eta|\xi + \lambda\eta)| \leqslant M\|\xi + \lambda\eta\|^2 \leqslant M(\|\xi\| + \|\eta\|)^2 \leqslant 4M$$

Therefore, by the polarization identity (3.1),

$$|\omega(\xi|\eta)| = \frac{1}{4}\Big\| \sum_{t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}} \omega(\xi + e^{\mathbf{i}t}\eta|\xi + e^{\mathbf{i}t}\eta)e^{\mathbf{i}t}\Big\| \leqslant 4M$$

□

3.3 Orthogonality

Let V be an inner product spaces.

3.3.1 Orthogonal and orthonormal vectors

Definition 3.3.1. A set $\mathfrak{S}$ of vectors of V are called **orthogonal** if $\langle u|v\rangle = 0$ for any distinct $u, v \in V$. An orthogonal set $\mathfrak{S}$ is called **orthonormal** if $\|v\| = 1$ for all $v \in V$.

Remark 3.3.2. We will also talk about an **orthogonal** resp. **orthonormal family of vectors** $(e_i)_{i\in I}$. This means that $\langle e_i|e_j\rangle = 0$ for any distinct $i, j \in I$ (resp. $\langle e_i|e_j\rangle = \delta_{i,j}$ for any $i, j \in I$).

In particular, two vectors $u, v \in V$ are called orthogonal and written as

$$u \perp v$$

when $\langle u|v\rangle = 0$. A fundamental fact about orthogonal vectors is

Proposition 3.3.3 (Pythagorean identity). *Suppose that $u, v \in V$ are orthogonal. Then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 \tag{3.4}$$

In particular,

$$\|v\| \leqslant \|u + v\| \tag{3.5}$$

Proof. $\|u + v\|^2 = \langle u + v|u + v\rangle = \langle u|u\rangle + \langle v|v\rangle + 2\mathrm{Re}\langle u|v\rangle = \langle u|u\rangle + \langle v|v\rangle$. □

Note that by applying (3.4) repeatedly, we see that if $v_1, \ldots, v_n \in V$ are orthogonal, then

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2 \tag{3.6}$$

Remark 3.3.4. Suppose that $\mathfrak{S}$ is an orthonormal set of vectors of V. Then $\mathfrak{S}$ is clearly linearly independent. (If $e_1, \ldots, e_n \in \mathfrak{S}$ and $\sum_i a_i e_i = 0$, then $a_j = \sum_i \langle e_j|a_i e_i\rangle = \langle e_j|0\rangle = 0$.) Thus, by linear algebra, if $\mathfrak{S} = \{e_1, \ldots, e_n\}$ is finite, then one can find uniquely $a_1, \ldots, a_n \in \mathbb{C}$ and $u \in V$ such that $v = a_1 e_1 + \cdots + a_n e_n + u$ and that u is orthogonal to $e_1, \ldots, e_n$. The expressions of $a_1, \ldots, a_n, u$ can be expressed explicitly:

Proposition 3.3.5 (Gram-Schmidt). *Let $e_1, \ldots, e_n$ be orthonormal vectors in V. Let $v \in V$. Then*

$$v - \sum_{i=1}^{n} e_i \cdot \langle e_i|v\rangle \tag{3.7}$$

is orthogonal to $e_1, \ldots, e_n$.

Proof. This is a direct calculation and is left to the readers. □

Remark 3.3.6. "Gram-Schmidt" usually refers to the following process. Let $v_1, \ldots, v_n$ be a set of linearly independent vectors of V. Then there is an algorithm of finding an orthonormal basis of $U = \mathrm{Span}\{v_1, \ldots, v_n\}$: Let $e_1 = v_1/\|v_1\|$. Suppose that a set of orthonormal vectors $e_1, \ldots, e_k$ in U have been found. Then e_{k+1} is defined by $\widetilde{v}_{k+1}/\|\widetilde{v}_{k+1}\|$ where $\widetilde{v}_{k+1} = v_{k+1} - \sum_{i=1}^k e_i \cdot \langle e_i|v_{k+1}\rangle$.

Combining Pythagorean with Gram-Schmidt, we have:

Corollary 3.3.7 (Bessel's inequality). *Let $(e_i)_{i\in I}$ be a family of orthonormal vectors of V. Then for each $v \in V$ we have*

$$\sum_{i\in I} |\langle e_i|v\rangle|^2 \leqslant \|v\|^2 \tag{3.8}$$

In particular, the set $\{i \in I : \langle e_i|v\rangle \neq 0\}$ is countable.

Proof. The LHS of (3.8) is $\lim_{J\in\mathrm{fin}(2^I)} \sum_{j\in J} |\langle e_j|v\rangle|^2$. Thus, it suffices to show that for each $J \in \mathrm{fin}(2^I)$ we have $\sum_{j\in J} |\langle e_j|v\rangle|^2 \leqslant \|v\|^2$. Let

$$u_1 = \sum_{j\in J} e_j \cdot \langle e_j|v\rangle \qquad u_2 = v - u_1$$

(Namely, $v = u_1 + u_2$ is the orthogonal decomposition of v with respect to $\mathrm{Span}\{e_j : j \in J\}$.) By Gram-Schmidt, we have $\langle u_1|u_2\rangle = 0$. By Pythagorean, we have $\|u_1\|^2 \leqslant \|v\|^2$. But Pythagorean (3.6) also implies

$$\|u_1\|^2 = \sum_{j\in J} |\langle e_j|v\rangle|^2$$

The last statement about countability follows from Prop. 1.2.43. □

3.3.2 Orthogonal decomposition

Definition 3.3.8. Let U be a linear subspace of V. Let $v \in V$. An **orthogonal decomposition** of v with respect to U is an expression of the form

$$v = u + w \qquad \text{where } u \in U \text{ and } w \perp U$$

Orthogonal decompositions of v are unique if exist. We call u the **orthogonal projection** of v onto U.

Proof of uniqueness. Suppose that $v = u' + w'$ is another orthogonal decomposition. Then $u - u'$ equals $w' - w$. Let $\xi = u - u'$. Then $\xi \in U$ and $\xi \perp U$. So $\langle \xi|\xi\rangle = 0$, and hence $\xi = 0$. So $u = u'$ and $w = w'$. □

Definition 3.3.9. Let U be a linear subspace of V. We say that V **has a projection onto** U if every vector has an orthogonal decomposition with respect to U. In that case, we define the map

$$P : V \to V$$

determined by the fact that each $v \in V$ has orthogonal decomposition $v = Pv + (v - Pv)$ where $Pv \in U$ and $v - Pv \perp U$. Clearly P is linear. By the Pythagorean identity, we have $\|Pv\| \leqslant \|v\|$, and hence

$$\|P\| \leqslant 1$$

Thus $P \in \mathfrak{L}(V)$. We say that P is the **projection (operator) associated to** U.

Example 3.3.10. Let $e_1, \dots, e_n$ be orthonormal vectors of V. Let $U = \mathrm{Span}\{e_1, \dots, e_n\}$. Choose any $v \in V$. Then by Gram-Schmidt,

$$v = u + w \qquad \text{where } u = \sum_{i=1}^{n} e_i \cdot \langle e_i | v \rangle \text{ and } w = v - u \tag{3.9}$$

is the orthogonal decomposition of v with respect to U. Therefore, the projection operator associated to U is

$$V \to V \qquad v \mapsto \sum_{i=1}^{n} e_i \cdot \langle e_i | v \rangle$$

Proposition 3.3.11. *Let U be a linear subspace of V. Suppose that $v \in V$ has orthogonal decomposition $v = u + w$ with respect to U. Then*

$$\|v - u\| = \inf_{\xi \in U} \|v - \xi\| \tag{3.10}$$

Proof. Clearly "$\geqslant$" holds. Choose any $\xi \in U$. Then $v - \xi = v - u + u - \xi = w + (u - \xi)$. Since $u - \xi \in U$, we have $w \perp u - \xi$. Thus, by Pythagorean, we have $\|w\| \leqslant \|v - \xi\|$. □

3.3.3 Direct sums and orthogonal decomposition

Next, we give a more explicit description of orthogonal decomposition in terms of direct sum.

Definition 3.3.12. Let $V_1, \dots, V_n$ be inner product spaces. Their **direct sum** $\boldsymbol{V_1 \oplus \cdots \oplus V_n}$ is an inner product space defined as follows. As a set, $V_1 \oplus \cdots \oplus V_n$ equals $V_1 \times \cdots \times V_n$. So it consists of elements of the form $(v_1, \dots, v_n)$ where $v_i \in V_i$. We write $(v_1, \dots, v_n)$ as $v_1 \oplus \cdots \oplus v_n$. The linear structure is defined by

$$(v_1 \oplus \cdots \oplus v_n) + (v_1' \oplus \cdots \oplus v_n') = (v_1 + v_1') \oplus \cdots \oplus (v_n + v_n')$$

$$a(v_1 \oplus \cdots \oplus v_n) = av_1 \oplus \cdots \oplus av_n$$

where $v_i, v_i' \in V_i$ and $a \in \mathbb{C}$. The inner product is defined by

$$\langle v_1 \oplus \cdots \oplus v_n | v_1' \oplus \cdots \oplus v_n' \rangle = \langle v_1 | v_1' \rangle + \cdots + \langle v_n | v_n' \rangle$$

We view V_i as an inner product subspace of $V_1 \oplus \cdots \oplus V_n$ by identifying $v_i \in V_i$ with $0 \oplus \cdots \oplus v_i \oplus \cdots \oplus 0 \in V_1 \oplus \cdots \oplus V_n$. Then, it is clear that $V_i \perp V_j$ if $i \neq j$.

Remark 3.3.13. Suppose that $U_1, \dots, U_n$ are mutually orthogonal linear subspaces of V. Then we clearly have a linear isometry

$$U_1 \oplus \cdots \oplus U_n \longrightarrow V \qquad u_1 \oplus \cdots \oplus u_n \mapsto u_1 + \cdots + u_n \tag{3.11}$$

Therefore, if V is spanned by $U_1, \dots, U_n$, then (3.11) is surjective, and hence is an isomorphism of normed vector spaces. In that case, we say that (3.11) is the **canonical isomorphism** from $U_1 \oplus \cdots \oplus U_n$ to V. With abuse of notation, we also say that V "is" the direct sum $U_1 \oplus \cdots \oplus U_n$.

Example 3.3.14. Let U_1, U_2 be inner product spaces and $V = U_1 \oplus U_2$. Then V has a projection onto U_1. The projection operator associated to U_1 is defined by sending each $u_1 \oplus u_2$ to u_1.

We now show that any projection is unitarily equivalent to the one given in Exp. 3.3.14.

Definition 3.3.15. If U is a linear subspace of V, we define the **orthogonal complement** of U (in V) to be

$$U^\perp = \{\xi \in V : \langle \xi | u \rangle = 0 \text{ for all } u \in U\}$$

Remark 3.3.16. Let U be a linear subspace of V. Then $U^\perp$ is closed in V, since it is the kernel of the bounded linear map $\xi \in V \mapsto \langle \xi | u \rangle$. Moreover, by the continuity of $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$, a vector of V is orthogonal to U iff it is orthogonal to $\overline{U} = \mathrm{Cl}_V(U)$, that is,

$$U^\perp = \overline{U}^\perp$$

Example 3.3.17. If U_1, U_2 are inner product spaces, then U_1 and U_2 are the orthogonal complements of each other in $U_1 \oplus U_2$.

Proposition 3.3.18. *Let U be a linear subspace of V. Suppose that V has a projection onto U, and let P the projection operator onto U. Then V is canonically isomorphic to $U \oplus U^\perp$. Moreover, identifying $U \oplus U^\perp$ with V (by identifying $u \oplus v$ with $u + v$ if $u \in U, v \in U^\perp$), then*

$$P : U \oplus U^\perp \to U \oplus U^\perp \qquad u \oplus v \mapsto u = u \oplus 0$$

Consequently, $1 - P$ is the projection of V onto $U^\perp$, and we have

$$\mathrm{Rng}(P) = \mathrm{Ker}(1 - P) = U \qquad \mathrm{Ker}(P) = \mathrm{Rng}(1 - P) = U^\perp \tag{3.12}$$

It follows from $V = U \oplus U^\perp$ that U is the orthogonal complement of $U^\perp$, i.e., $U = U^{\perp\perp}$.

Proof. The surjectivity of the linear isometry

$$U \oplus U^\perp \to V \qquad u \oplus v \mapsto u + v$$

follows from the fact that V has a projection onto U. Clearly P sends $u + v$ to u. The rest of this proposition is obvious. $\square$

Corollary 3.3.19. *Suppose that U is a finite-dimensional linear subspace of V. Then U is closed in V.*

Proof. By Exp. 3.3.10, there is a projection operator P of V onto U. By Prop. 3.3.18, U is the orthogonal complement of $U^\perp$, and hence is closed. $\square$

Corollary 3.3.20. *Let U be a linear subspace of V, and suppose that V has a projection onto U. Let P be the projection operator associated to U. Then $P^2 = P$, and ω_P is positive.*

Proof. By Prop. 3.3.18, we assume that $V = U \oplus U^\perp$, and P sends each $\xi \oplus \eta \in U \oplus U^\perp$ to $\xi = \xi \oplus 0$. Then it is easy to verify that $P^2 = P$. Moreover, $\omega_P(\xi \oplus \eta) = \|\xi\|^2 \geqslant 0$. $\square$

3.3.4 Orthonormal basis

Definition 3.3.21. A set $\mathfrak{S}$ (or a family $(e_i)_{i\in I}$) of orthonormal vectors of V is called an **orthonormal basis** of V if it spans a dense subspace of V.

Example 3.3.22. If X is a set, by Prop. 2.8.1, $l^2(X)$ has an orthonormal basis $(\chi_{\{x\}})_{x\in X}$.

Example 3.3.23. If V is separable, then V has a countable orthonormal basis.

Proof. Let $\{v_1, v_2, \dots\}$ be a dense subset of V where $v_1 \neq 0$. Then by Gram-Schmidt (Rem. 3.3.6), we can find $e_1, e_2, \dots \in V$ such that the set $\{e_1, e_2, \dots\}$ is orthnormal (after removing the duplicated terms), and that $\mathrm{Span}\{v_1, \dots, v_n\} = \mathrm{Span}\{e_1, \dots, e_n\}$ for each n. Then $\{e_1, e_2, \dots\}$ clearly spans a dense subspace of V. $\square$

We remark that there are non-separable and non-complete inner product spaces that do not have orthonormal bases. See [Gud74].

Theorem 3.3.24. *Suppose that $(e_i)_{i\in I}$ is an orthonormal basis of V. Then for each $v \in V$, the RHS of the following converges (under the norm of V) to the LHS:*

$$v = \sum_{i\in I} e_i \cdot \langle e_i | v \rangle \tag{3.13}$$

Proof. Note that for $J \in \operatorname{fin}(2^I)$, the expression

$$\Big\| v - \sum_{j\in J} e_j \cdot \langle e_j|v\rangle \Big\|^2 = \|v\|^2 - \sum_{j\in J} |\langle e_j|v\rangle|^2$$

decreases when J increases. Thus, it suffices to prove that the $\inf_{J\in\operatorname{fin}(2^I)}$ of this expression is 0.

By assumption, we can find $J \in \operatorname{fin}(2^I)$ and $(\lambda_j)_{j\in J}$ in $\mathbb{C}$ such that $\|v - \sum_{j\in J} \lambda_j e_j\|$ is small enough. On the other hand, applying Prop. 3.3.11 to the orthogonal projection $v = u + w$ where $w = \sum_{j\in J} e_j \cdot \langle e_j|v\rangle$ (cf. Exp. 3.3.10), we have

$$\Big\| v - \sum_{j\in J} e_j \cdot \langle e_j|v\rangle \Big\| \leqslant \Big\| v - \sum_{j\in J} \lambda_j e_j \Big\| \tag{3.14}$$

Thus, the infimum of the LHS over $J \in \operatorname{fin}(2^I)$ is zero. □

Corollary 3.3.25 (Parseval's identity). *Suppose that $(e_i)_{i\in I}$ is an orthonormal basis of V. Then for each $u, v \in V$ we have*

$$\langle u|v\rangle = \sum_{i\in I} \langle u|e_i\rangle \cdot \langle e_i|v\rangle \tag{3.15}$$

In particular,

$$\|v\|^2 = \sum_{i\in I} |\langle e_i|v\rangle|^2 \tag{3.16}$$

Proof. By Thm. 3.3.24, $v = \lim_{J\in\operatorname{fin}(2^I)} v_J$ where $v_J = \sum_{j\in J} e_j \cdot \langle v|e_j\rangle$. By the continuity of $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ (Exp. 3.2.11), we have

$$\langle u|v\rangle = \lim_{J\in\operatorname{fin}(2^I)} \langle u|v_J\rangle = \lim_{J\in\operatorname{fin}(2^I)} \sum_{j\in J} \langle u|e_j\rangle \cdot \langle e_j|v\rangle = \sum_{i\in I} \langle u|e_i\rangle \cdot \langle e_i|v\rangle$$

□

Corollary 3.3.26. *Suppose that $(e_x)_{x\in X}$ is an orthonormal basis of V. Then there is a linear isometry*

$$\Phi : V \to l^2(X) \qquad v \mapsto (\langle e_x|v\rangle)_{x\in X} \tag{3.17}$$

whose range is dense in $l^2(X)$.

Proof. Parseval's identity shows that $(\langle e_x|v\rangle)_{x\in X}$ has finite l^2-norm $\|v\|$. So the map Φ defined by (3.17) is clearly a linear isometry. The density of the range of Φ follows from the fact that $l^2(X)$ contains all $\chi_{\{x\}} = \Phi(e_x)$, and that $\operatorname{Span}\{\chi_{\{x\}} : x \in X\}$ is dense in $l^2(X)$ (cf. Prop. 2.8.1). □

3.4 Hilbert spaces

Theorem 3.4.1. *Let $\mathcal{H}$ be an inner product space. Then the following three conditions are equivalent:*

(a) $\mathcal{H}$ is (Cauchy) complete.

(b) For each orthonormal family $(e_i)_{i\in I}$ in $\mathcal{H}$, and for each family $(a_i)_{i\in I}$ in $\mathbb{C}$ satisfying $\sum_{i\in I}|a_i|^2<+\infty$, the unordered sum $\sum_{i\in I}a_ie_i$ converges (under the norm of $\mathcal{H}$).

(c) $\mathcal{H}$ is unitarily equivalent to $l^2(X)$ for some set X.

*If $\mathcal{H}$ satisfies any of these conditions, we say that $\mathcal{H}$ is a **Hilbert space**.*

Proof. (c)⇒(a): By Thm. 2.8.7, $l^2(X)$ is the dual space of $l^2(X)$. Since any dual space is complete (Cor. 2.4.11), $l^2(X)$ is complete.

(a)⇒(b): Since $\sum_i|a_i|^2<+\infty$, for each $\varepsilon>0$ there exists $J\in\mathrm{fin}(2^I)$ such that for all finite $K\subset I\backslash J$ we have $\sum_{k\in K}|a_k|^2<\varepsilon$, and hence, by the Pythagorean identity,

$$\Big\Vert\sum_{k\in K}a_ke_k\Big\Vert^2=\sum_{k\in K}\Vert a_ke_k\Vert^2<\varepsilon$$

Thus $(\sum_{j\in J}a_je_j)_{J\in\mathrm{fin}(2^I)}$ is a Cauchy net. By the completeness of $\mathcal{H}$, we see that $\sum_{i\in I}a_ie_i$ converges.

(b)⇒(c): Assume (b). We first show that $\mathcal{H}$ has an orthonormal basis. By Zorn's lemma, we can find a maximal (with respect to the partial order $\subset$) set of orthonormal vectors, written as a family $(e_i)_{i\in I}$. The maximality implies that every nonzero vector $\xi\in\mathcal{H}$ is not orthogonal to some e_i. (Otherwise, $\{e_i:i\in I\}$ can be extended to $\{e_i:i\in I\}\cup\{\xi/\Vert\xi\Vert\}$.)

Let us prove that $(e_i)_{i\in I}$ is an orthonormal basis. Suppose not. Then $U=\mathrm{Span}\{e_i:i\in I\}$ is not dense in $\mathcal{H}$. Let $\xi\in\mathcal{H}\backslash\overline{U}$. By Bessel's inequality, we have

$$\sum_{i\in I}|\langle e_i|\xi\rangle|^2<+\infty$$

Therefore, by (b),

$$\sum_{i\in I}e_i\cdot\langle e_i|\xi\rangle \tag{3.18}$$

converges to some vector $\eta\in\mathcal{H}$. By the continuity of $\langle\cdot|\cdot\rangle$ (Exp. 3.2.11), we see that $\langle e_i|\eta\rangle=\langle e_i|\xi\rangle$ for all i, and hence

$$\langle e_i|\xi-\eta\rangle=0\qquad\text{for all } i\in I \tag{3.19}$$

Since $\eta \in \overline{U}$ and $\xi \notin \overline{U}$, we conclude that $\xi - \eta$ is a nonzero vector orthogonal to all e_i. This contradicts the maximality of $(e_i)_{i\in I}$.

Now we have an orthonormal basis $(e_i)_{i\in I}$. By Cor. 3.3.26, we have a linear isometry

$$\Phi : \mathcal{H} \to l^2(I) \qquad \xi \mapsto \big(\langle e_i|\xi\rangle\big)_{i\in I}$$

with dense range. If $(a_i)_{i\in I}$ belongs to $l^2(I)$, by (b), the unordered sum $\sum_{i\in I} a_i e_i$ converges to some $\xi \in \mathcal{H}$. Clearly $\Phi(\xi) = (a_i)_{i\in I}$. This proves that Φ is surjective, and hence is a unitary map. So $\mathcal{H} \simeq l^2(I)$. $\square$

In the proof of Thm. 3.4.1, we use Zorn's lemma to show that every Hilbert space $\mathcal{H}$ admits an orthonormal basis. The same argument yields a stronger result:

Proposition 3.4.2. *Let $\mathcal{H}$ be a Hilbert space. Then any orthonormal family of vectors in $\mathcal{H}$ can be extended to an orthonormal basis.*

When $\mathcal{H}$ is separable, this proposition can be proved without invoking Zorn's lemma, by applying mathematical induction together with the Gram-Schmidt process (Rem. 3.3.6). We leave the details of the proof of Prop. 3.4.2 to the reader.

Example 3.4.3. By Thm. 3.4.1, if X is a set, then $l^2(X)$ is a Hilbert space.

Example 3.4.4. Let (X, μ) be a measure space. By the Riesz-Fischer Thm. 1.6.10, the inner product space $L^2(X, \mu)$ is a Hilbert space.

Example 3.4.5. If V is a closed linear subspace of $\mathcal{H}$ whose inner product is inherited from that of $\mathcal{H}$, then V is a Hilbert space. This is either due to Thm. 3.4.1-(b), or due to the fact that a closed subset of a complete metric space is complete.

Corollary 3.4.6. *Every Hilbert space $\mathcal{H}$ has an orthonormal basis. Moreover, $\mathcal{H}$ is separable iff the orthonormal basis can be chosen to be countable.*

Proof. That $\mathcal{H}$ has an orthonormal basis follows from the proof of Thm. 3.4.1 or from the fact that $l^2(X)$ has an orthonormal basis $(\chi_{\{x\}})_{x\in X}$. If X is countable, then $l^2(X)$ has dense subset $\mathrm{Span}_{\mathbb{Q}+\mathbf{i}\mathbb{Q}}\{\chi_{\{x\}} : x \in X\}$ and hence is separable. Conversely, we have proved in Exp. 3.3.23 that every separable inner product space has a countable orthonormal basis. $\square$

Theorem 3.4.7. *Let $(e_x)_{x\in X}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Then we have a unitary map*

$$\mathcal{H} \xrightarrow{\simeq} l^2(X) \qquad \xi \mapsto \big(\langle e_x|\xi\rangle\big)_{x\in X} \tag{3.20}$$

Proof. This is clear from the proof of Thm. 3.4.1. $\square$

Theorem 3.4.8. *Let V be a closed linear subspace of $\mathcal{H}$. Then $\mathcal{H}$ has a projection onto V. Consequently, by Prop. 3.3.18, $V \oplus V^\perp$ is canonically isomorphic to $\mathcal{H}$.*

Proof. By Exp. 3.4.5, V is a Hilbert space, and hence admits an orthonormal basis $(e_i)_{i\in I}$. For each $\xi \in \mathcal{H}$, since Bessel's inequality implies $\sum_i |\langle e_i|\xi\rangle|^2 < \|\xi\|^2 < +\infty$, by Thm. 3.4.1-(b), the following sum converges:

$$P\xi = \sum_{i\in I} e_i \cdot \langle e_i|\xi\rangle$$

and is clearly in V. Similar to the argument around (3.19), $\xi - P\xi$ is orthogonal to every e_i. Hence $V_0 := \mathrm{Span}\{e_i : i \in I\}$ is orthogonal to $\xi - P\xi$, i.e., $\xi - P\xi \in V_0^\perp$. Since V is the closure of V_0, by Rem. 3.3.16, we have $\xi - P\xi \in V^\perp$. Therefore, $\xi = P\xi + (\xi - P\xi)$ is the orthogonal decomposition of ξ with respect to V. □

Corollary 3.4.9. *Let V be a linear subspace of $\mathcal{H}$. Then $(V^\perp)^\perp = \mathrm{Cl}_{\mathcal{H}}(V)$.*

Note that since $V^\perp$ is closed, Cor. 3.4.9 implies $V^{\perp\perp\perp} = V^\perp$.

Proof. By Rem. 3.3.16, we have $V^\perp = \overline{V}^\perp$. By Thm. 3.4.8, $\mathcal{H}$ has a projection onto $\overline{V}$. Therefore, by Prop. 3.3.18, we have $\mathcal{H} = \overline{V} \oplus \overline{V}^\perp = \overline{V} \oplus V^\perp$. Therefore, $\overline{V}$ is the orthogonal complement of $V^\perp$. □

Corollary 3.4.10. *Let V be a linear subspace of $\mathcal{H}$. Then V is dense in $\mathcal{H}$ iff $V^\perp = \{0\}$.*

Proof. If V is dense, then $V^\perp = \overline{V}^\perp = \mathcal{H}^\perp = 0$. Conversely, if $V^\perp = \{0\}$, then $V^{\perp\perp} = 0^\perp = \mathcal{H}$. By Cor. 3.4.9, we have $\overline{V} = V^{\perp\perp} = \mathcal{H}$. Hence V is dense. □

3.5 Bounded linear maps and bounded sesquilinear forms

In this section, we let U, V, W be inner product spaces.

In Subsec. 2.5.2, we discussed the close relationship between bounded linear maps and bounded bilinear forms in the general setting of normed vector spaces. This connection allows us to combine the strengths of both perspectives. One key advantage of the perspective of linear operators is that the space $\mathfrak{L}(V)$ is particularly well-suited for symbolic calculus.

In this section, we explore this relationship in the context of inner product spaces and Hilbert spaces. We will see that the passage from $\mathfrak{L}(V)$ to bounded sesquilinear forms fundamentally relies on the Riesz-Fréchet theorem, a pivotal result that enables this correspondence.

3.5.1 The Riesz-Fréchet representation theorem

Definition 3.5.1. If $T \in \mathrm{Lin}(V, W)$, we let ω_T be the sesquilinear form

$$\omega_T : W \times V \to \mathbb{C} \qquad (w, v) \mapsto \langle w|Tv\rangle$$

Proposition 3.5.2. *For each $T \in \mathrm{Lin}(V,W)$, we have*

$$\|T\| = \|\omega_T\|$$

Consequently, T is bounded iff ω_T is so, and the map $T \in \mathrm{Lin}(V,W) \mapsto \omega_T$ is injective.

Proof. For each $v \in V, w \in W$, we have

$$|\omega_T(w|v)| = |\langle w|Tv\rangle| \leqslant \|Tv\| \cdot \|w\| \leqslant \|T\| \cdot \|v\| \cdot \|w\|$$

Applying $\sup$ over all v, w in the closed unit balls, we get $\|\omega_T\| \leqslant \|T\|$. Moreover,

$$\|Tv\|^2 = \omega_T(Tv|v) \leqslant \|\omega_T\| \cdot \|Tv\| \cdot \|v\|$$

and hence $\|Tv\| \leqslant \|\omega_T\| \cdot \|v\|$. Applying $\sup$ over all v in the closed unit ball, we get $\|T\| \leqslant \|\omega_T\|$. $\square$

By Prop. 3.5.2, the map $T \in \mathrm{Lin}(V,W)$ restricts to a linear isometry of normed vector spaces

$$\mathfrak{L}(V,W) \to \mathfrak{Ses}(W|V) \qquad T \mapsto \omega_T \tag{3.21}$$

On the other hand, Cor. 2.5.2 implies

$$\mathfrak{Ses}(W|V) = \mathfrak{L}(W^{\complement} \times V, \mathbb{C}) \simeq \mathfrak{L}(V, (W^{\complement})^*)$$

and hence a linear isometry

$$\mathfrak{L}(V,W) \to \mathfrak{L}(V, (W^{\complement})^*) \tag{3.22}$$

Exercise 3.5.3. Show that the map (3.22) sends each $T \in \mathfrak{L}(V,W)$ to $\Phi \circ T$, where $\Phi : W \to (W^{\complement})^*$ is defined below.

Theorem 3.5.4 (Riesz-Fréchet representation theorem). *The following map is a linear isometry:*

$$\Phi : W \to (W^{\complement})^* \qquad \xi \mapsto \langle\overline{\xi}|-\rangle \tag{3.23a}$$

where $\langle\overline{\xi}|-\rangle$ denotes the bounded linear functional

$$\langle\overline{\xi}|-\rangle : W^{\complement} \to \mathbb{C} \qquad \overline{w} \mapsto \langle\overline{\xi}|\overline{w}\rangle_{W^{\complement}} = \langle w|\xi\rangle_W \tag{3.23b}$$

Moreover, W is a Hilbert space iff Φ is surjective (and hence an isomorphism of normed vector spaces).

In other words, Φ is determined by the fact that for each $w, \xi \in W$,

$$\langle\overline{w}, \Phi\xi\rangle = \langle w|\xi\rangle \tag{3.24}$$

Proof. First, note that for each $\xi \in W$,

$$\|\xi\| = \sup_{w \in \overline{B}_W(0,1)} |\langle w|\xi\rangle| \tag{3.25}$$

Indeed, the Cauchy-Schwarz inequality implies "$\geqslant$ 0". The equality can be achieved by choosing $w = \xi/\|\xi\|$ if $\xi \neq 0$. Therefore,

$$\|\Phi(\xi)\| = \sup_{\overline{w} \in \overline{B}_{W^{\complement}}(0,1)} |\langle \overline{w}, \Phi(\xi)\rangle| = \sup_{w \in \overline{B}_W(0,1)} |\langle w|\xi\rangle| = \|\xi\|$$

This proves that Φ is a linear isometry.

If Φ is surjective, then the normed vector space W is isomorphic to the dual space $(W^{\complement})^*$ where the latter is complete by Cor. 2.4.11. Therefore, W is a Hilbert space.

Conversely, assume that W is a Hilbert space. Then we can assume that $W = l^2(X)$ for some set X. The surjectivity of Φ then follows from the surjectivity of the map

$$l^2(X) \to l^2(X)^* \qquad \xi \mapsto \langle -, \xi\rangle$$

due to Thm. 2.8.7. $\square$

Definition 3.5.5. The map Φ in Thm. 3.5.4 is called the **Riesz isometry** of W. If W is a Hilbert space, then Φ is called the **Riesz isomorphism** of W. An equivalent description of Φ is as follows: In view of the isomorphism

$$\mathfrak{Ses}(W) = \mathfrak{L}(W^{\complement} \times W, \mathbb{C}) \simeq \mathfrak{L}(W, (W^{\complement})^*)$$

due to Cor. 2.5.2, the Riesz isometry Φ is the element of $\mathfrak{L}(W, (W^{\complement})^*)$ corresponding the the inner product $\langle\cdot|\cdot\rangle_W$ as an elemet of $\mathfrak{Ses}(W)$.

3.5.2 Equivalence between bounded linear maps and bounded sesquilinear forms

With the help of the Riesz-Fréchet theorem, we can establish the equivalence between bounded linear maps and bounded sesquilinear forms.

Theorem 3.5.6. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Then we have an isomorphism of normed vector spaces*

$$\mathfrak{L}(\mathcal{H}, \mathcal{K}) \xrightarrow{\simeq} \mathfrak{Ses}(\mathcal{K}|\mathcal{H}) \qquad T \mapsto \omega_T \tag{3.26}$$

In particular, when $\mathcal{H} = \mathcal{K}$, the above isomorphism becomes

$$\mathfrak{L}(\mathcal{H}) \xrightarrow{\simeq} \mathfrak{Ses}(\mathcal{H}) \qquad T \mapsto \omega_T \tag{3.27}$$

Proof. By Cor. 2.5.2, we have

$$\mathfrak{L}(\mathcal{H}, (\mathcal{K}^{\complement})^*) \simeq \mathfrak{L}(\mathcal{K}^{\complement} \times \mathcal{H}, \mathbb{C}) = \mathfrak{Ses}(\mathcal{K}|\mathcal{H})$$

where each $S \in \mathfrak{L}(\mathcal{H}, (\mathcal{K}^{\complement})^*)$ corresponds to the bounded bilinear form

$$\mathcal{K}^{\complement} \times \mathcal{H} \to \mathbb{C} \qquad (\overline{\eta}, \xi) \mapsto \langle \overline{\eta}, S\xi \rangle$$

equivalently, the bounded sesquilinear form

$$\mathcal{K} \times \mathcal{H} \to \mathbb{C} \qquad (\eta, \xi) \mapsto \langle \overline{\eta}, S\xi \rangle$$

Now, suppose that $S = \Phi \circ T$ where $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$, and $\Phi : \mathcal{K} \xrightarrow{\simeq} (\mathcal{K}^{\complement})^*$ is the Riesz-isomorphism of $\mathcal{K}$ defined in Thm. 3.5.4. Then $\langle \overline{\eta} | \Phi\mu \rangle = \langle \eta | \mu \rangle$ for each $\mu, \eta \in \mathcal{K}$, and hence

$$\langle \overline{\eta}, S\xi \rangle = \langle \overline{\eta}, \Phi \circ T\xi \rangle = \langle \eta | T\xi \rangle = \omega_T(\eta|\xi)$$

Therefore, the isomorphism

$$\mathfrak{L}(\mathcal{H}, \mathcal{K}) \xrightarrow[\simeq]{T \mapsto \Phi \circ T} \mathfrak{L}(\mathcal{H}, (\mathcal{K}^{\complement})^*) \simeq \mathfrak{Ses}(\mathcal{K}|\mathcal{H})$$

sends T to ω_T. □

3.5.3 Adjoint operators, self-adjoint operators and positive operators

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. With the help of Thm. 3.5.6, we can define adjoint operators:

Definition 3.5.7. Recall that for each $\omega \in \mathfrak{Ses}(\mathcal{K}|\mathcal{H})$, the **adjoint sesquilinear form** $\omega^* \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$ is defined by $\omega^*(\xi|\eta) = \overline{\omega(\eta|\xi)}$ for each $\xi \in \mathcal{H}, \eta \in \mathcal{K}$. It is clear that

$$\|\omega^*\| = \|\omega\|$$

Now, for each $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$, define the **adjoint operator** $T^* \in \mathfrak{L}(\mathcal{K}, \mathcal{H})$ such that

$$\omega_{T^*} = (\omega_T)^*$$

More explicitly, T^* is determined by the fact that for each $\xi \in \mathcal{H}, \eta \in \mathcal{K}$,

$$\langle \eta | T\xi \rangle = \langle T^*\eta | \xi \rangle$$

Then, we clearly also have $\|T\| = \|T^*\|$.

Exercise 3.5.8. Show that

$$*:\mathfrak{L}(\mathcal{H},\mathcal{K})\rightarrow\mathfrak{L}(\mathcal{K},\mathcal{H})\qquad T\mapsto T^*$$

is a bijective antilinear map, and that $(T^*)^*=T$. Prove that if $\mathcal{M}$ is a Hilbert space and $T\in\mathfrak{L}(\mathcal{H},\mathcal{K})$, $S\in\mathfrak{L}(\mathcal{K},\mathcal{M})$, then

$$(ST)^*=T^*S^*$$

Definition 3.5.9. A bounded linear operator $T\in\mathfrak{L}(\mathcal{H})$ is called **self-adjoint** if $T=T^*$, equivalently, if ω_T is Hermitian.

Definition 3.5.10. Let $A,B\in\mathfrak{L}(\mathcal{H})$ be self-adjoint. We write

$$\boldsymbol{A\leqslant B}$$

if $\omega_A\leqslant\omega_B$ in the sense of Def. 3.1.8, that is, $\langle\xi|A\xi\rangle\leqslant\langle\xi|B\xi\rangle$ for all $\xi\in\mathcal{H}$. We say that $A\in\mathfrak{L}(\mathcal{H})$ is **positive** if $A\geqslant 0$, equivalently, if ω_A is positive.

Example 3.5.11. Let $A\in\mathfrak{L}(\mathcal{H},\mathcal{K})$. Then $A^*A\in\mathfrak{L}(\mathcal{H})$ is positive, because

$$\langle\xi|A^*A\xi\rangle=\|A\xi\|^2\geqslant 0$$

Example 3.5.12. Let $A\in\mathfrak{L}(\mathcal{H})$, and let $a\geqslant 0$ such that $\|A\|\leqslant a$. Then $-a\leqslant A\leqslant a$.

Proof. Since $|\langle\eta|A\xi\rangle|\leqslant a\|\eta\|\cdot\|\xi\|$, we obtain $-a\|\xi\|^2\leqslant\langle\xi|A\xi\rangle\leqslant a\|\xi\|^2$, and hence $-a\leqslant A\leqslant a$. □

3.5.4 Composition of bounded linear operators and bounded sesquilinear forms

Let $\mathcal{H},\mathcal{K},\mathcal{M}$ be Hilbert spaces.

One of the major advantages of working with bounded linear operators rather than bounded sesquilinear forms is the ease with which one can handle problems involving operator composition. This does not mean, however, that a notion of composition cannot be defined on the side of sesquilinear forms. In fact, the following lemma illustrates how such a composition can be defined.

Lemma 3.5.13. *Let $T\in\mathfrak{L}(\mathcal{K},\mathcal{H})$ and $S\in\mathfrak{L}(\mathcal{M},\mathcal{K})$. Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{K}$. Then for each $\xi\in\mathcal{M}$, we have*

$$T\circ S\xi=\sum_{i\in I}Te_i\cdot\langle e_i|S\xi\rangle \tag{3.28}$$

where the unordered sum on the RHS converges in norm to the LHS.

Proof. By Thm. 3.3.24, we have $S\xi=\sum_i e_i\cdot\langle e_i|S\xi\rangle$. Therefore, by the linearity and the continuity of T, we get (3.28). □

Definition 3.5.14. Let $\omega \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$ and $\sigma \in \mathfrak{Ses}(\mathcal{K}|\mathcal{M})$. Then the **composition** $\boldsymbol{\omega \circ \sigma}$ is the element of $\mathfrak{Ses}(\mathcal{M}|\mathcal{H})$ defined by [2]

$$(\omega \circ \sigma)(\psi|\xi) = \sum_{i \in I} \omega(\psi|e_i) \cdot \sigma(e_i|\xi) \qquad \text{for all } \psi \in \mathcal{H}, \xi \in \mathcal{M}$$

where $(e_i)_{i \in I}$ is a basis of $\mathcal{K}$. This definition is independent of the choice of basis (and applies even to bounded sesquilinear forms on inner product spaces). Moreover, by Lem. 3.5.13, for each $T \in \mathfrak{L}(\mathcal{K}, \mathcal{H})$ and $S \in \mathfrak{L}(\mathcal{M}, \mathcal{K})$, we have

$$\omega_{T \circ S} = \omega_T \circ \omega_S$$

However, many properties about composition that are straightforward from the perspective of bounded linear operators become far less transparent when viewed in terms of sesquilinear forms. For instance, consider the following basic result:

Proposition 3.5.15. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be normed vector spaces. Let $T \in \mathfrak{L}(\mathcal{V}, \mathcal{W})$ and $S \in \mathfrak{L}(\mathcal{U}, \mathcal{V})$. Then*

$$\|TS\| \leqslant \|T\| \cdot \|S\|$$

Proof. Apply $\sup$ over all $\xi \in \overline{B}_{\mathcal{U}}(0,1)$ to

$$\|TS\xi\| \leqslant \|T\| \cdot \|S\xi\| \leqslant \|T\| \cdot \|S\| \cdot \|\xi\|$$

□

Corollary 3.5.16. *Let $T \in \mathfrak{L}(\mathcal{H})$. Let $\Omega = \{z \in \mathbb{C} : |z| > \|T\|\}$. Then for each $z \in \Omega$, the operator $z - T$ is invertible (cf. Def. 2.3.6). Moreover, for each $\xi, \eta \in \mathcal{H}$, the function*

$$z \in \Omega \mapsto \langle \eta | (z-T)^{-1}\xi \rangle = \omega_{(z-T)^{-1}}(\eta|\xi)$$

is holomorphic.

The expression $(z-T)^{-1}$ is called the **resolvent** of T.

Proof. By Prop. 3.5.15, we have $\|T^k\| \leqslant \|T\|^k$. Therefore, if $z \in \Omega$, then

$$\sum_{k=0}^{\infty} \|z^{-k-1}T^k\| \leqslant \sum_{k=0}^{\infty} |z|^{-k-1}\|T\|^k < +\infty$$

Therefore, if we define

$$S_n(z) = \sum_{k=0}^{n} z^{-k-1}T^k \tag{3.29}$$

[2]This definition clearly also applies when $\mathcal{H}$ and $\mathcal{K}$ are merely inner product spaces admitting orthonormal bases, in particular, when $\mathcal{H}, \mathcal{K}$ are separable inner product spaces (cf. Exp. 3.3.23).

Then $(S_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in the normed vector space $\mathfrak{L}(\mathcal{H})$. By Cor. 2.4.9, $\mathfrak{L}(\mathcal{H}) \simeq \mathfrak{Ses}(\mathcal{H})$ is complete. Therefore, $(S_n(z))$ converges under the operator norm to some $S(z) \in \mathfrak{L}(\mathcal{H})$. Since

$$(z-T)S_n(z) = S_n(z)\cdot(z-T) = 1 - z^{-n-1}T^{n+1}$$

and since $\|z^{-n-1}T^{n+1}\| \leqslant |z|^{-n-1}\|T\|^{n+1} \to 0$, we have $(z-T)S(z) = S(z)(z-T) = 1$. This proves that $z-T$ is invertible.

For each $\xi, \eta \in \mathcal{H}$, and for each compact $K \subset \Omega$, we have

$$\sup_{z\in K}\sum_{k=0}^{\infty}|z^{-k-1}\langle\eta|T^k\xi\rangle| \leqslant \sup_{z\in K}\sum_{k=0}^{\infty}|z|^{-k-1}\|T\|^k\cdot\|\eta\|\cdot\|\xi\| < +\infty$$

Therefore, the series of functions

$$z\in\Omega \mapsto \sum_{k=0}^{\infty}\langle\eta|z^{-k-1}T^k\xi\rangle$$

converges absolutely and uniformly on compact subsets of Ω. Since the limit of this series of functions is $z \in \Omega \mapsto \omega_{S_n(z)}(\eta|\xi)$, the latter is holomorphic. $\square$

Before we explore further examples, let us examine another foundational perspective that played a central role in the early development of functional analysis: the viewpoint of bounded matrices.

3.6 Bounded matrices

Let $\mathcal{H}, \mathcal{K}, \mathcal{M}$ be Hilbert spaces.

As mentioned in Subsec. 2.5.1, early developments in functional analysis focused primarily on bounded sesquilinear forms rather than bounded linear operators. Closely tied to this approach was the study of infinite matrices, which provided a concrete representation of these abstract objects.

The notion of boundedness was first defined in this matrix context. Hilbert introduced this concept in [Hil06], where he also introduced the space $l^2(\mathbb{Z})$. As established in Prop. 2.3.9, boundedness in the context of linear maps or sesquilinear forms is equivalent to (Lipschitz) continuity. However, as we will see below, in the setting of infinite matrices, boundedness takes on a stronger meaning—it implies equicontinuity. More precisely, it ensures that a family of linear maps or sesquilinear forms shares a uniform Lipschitz constant. See Step 2 of the proof of Thm. 3.6.3.

This distinction highlights a deeper philosophical insight under the perspective of infinite matrices: a bounded linear operator or sesquilinear form is regarded as the limit of a sequence (or net) of finite-rank operators or forms. This philosophy is central to our treatment of spectral theory in Ch. 5, and it resonates not only with the historical approaches of Hilbert and F. Riesz, but also with the viewpoint that the Stieltjes integral arises as the weak-* completion of finite sums (see Table 2.4, Sec. 4.1, and Sec. 4.2). For this reason, it is worthwhile to study bounded matrices and their relationship to bounded linear operators and sesquilinear forms.

Definition 3.6.1. Let X, Y be sets. The elements of $\mathbb{C}^{I\times J}$, which are of the form

$$A = (A(x,y))_{x\in X, y\in Y} \qquad \text{where } A_{x,y}\in\mathbb{C}$$

are called ***I* × *J* (complex) matrices**. For each $A\in\mathbb{C}^{X\times Y}$, the **norm $\|A\|$** is defined to be

$$\|A\| = \sup_{I\in\mathrm{fin}(2^X), J\in\mathrm{fin}(2^Y)} \|A_{I,J}\| \tag{3.30a}$$

where each $\|A_{I,J}\|$ is defined by

$$\|A_{I,J}\| = \sup_{\substack{\xi\in\overline{B}_{l^2(I)}(0,1)\\ \eta\in\overline{B}_{l^2(J)}(0,1)}} \Big|\sum_{x\in I, y\in J} \overline{\xi(x)}A(x,y)\eta(y)\Big| \tag{3.30b}$$

We say that A is **bounded** if $\|A\| < +\infty$.

Definition 3.6.2. Suppose that $(e_x)_{x\in X}$ and $(e_y)_{y\in Y}$ are orthonormal basis of $\mathcal{H}$ and $\mathcal{K}$, respectively. The **matrix representation** of each $\omega\in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$ is the element $[\omega]\in\mathbb{C}^{X\times Y}$ defined by

$$[\omega](x,y) = \omega(e_x|e_y) \qquad \text{for each } x\in X, y\in Y$$

If $\omega = \omega_T$ where $T\in\mathfrak{L}(\mathcal{K},\mathcal{H})$, we also say that $[\omega]$ is the **matrix representation** of T and write it as $[T]$. In other words, we say that $[T]\in\mathbb{C}^{X\times Y}$ is the matrix representation of T if

$$[T](x,y) = \langle e_x|Te_y\rangle \qquad \text{for each } x\in X, y\in Y$$

Theorem 3.6.3. *Suppose that $(e_x)_{x\in X}$ and $(e_y)_{y\in Y}$ are orthonormal basis of $\mathcal{H}$ and $\mathcal{K}$, respectively. Then the linear map*

$$\mathfrak{Ses}(\mathcal{H}|\mathcal{K}) \to \mathbb{C}^{X\times Y} \qquad \omega\mapsto[\omega] \tag{3.31}$$

is injective, and its range is the set of all bounded matrices. Moreover, for each $\omega\in\mathfrak{Ses}(\mathcal{H}|\mathcal{K})$, we have

$$\|\omega\| = \|[\omega]\|$$

Proof. Step 1. We assume WLOG that $\mathcal{H} = l^2(X), \mathcal{K} = l^2(Y)$ and $e_x = \{\chi_{\{x\}}\}_{x\in X}$ and $e_y = \{\chi_{\{y\}}\}_{y\in Y}$. Recall that $C_c(X)$ is dense in $l^2(X)$ and $C_c(Y)$ is dense in $l^2(Y)$.

Let $\omega \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$. From (3.30), it is clear that $\|[\omega]\|$ is the norm of the restriction of ω to $C_c(X) \times C_c(Y)$. Therefore, by the continuity of ω, we have $\|[\omega]\| = \|\omega\| < +\infty$. In particular, we have proved that the matrices in the range of (3.31) are bounded. Moreover, if $[\omega] = 0$, then $\|\omega\| = \|\omega\| = 0$, and hence $\omega = 0$. This proves that (3.31) is injective.

Step 2. Choose any bounded $A \in \mathbb{C}^{I\times J}$. We want to find $\omega \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$ such that $[\omega] = A$.

For each $I \in \mathrm{fin}(2^X)$ and $J \in \mathrm{fin}(2^Y)$, let $A_{I,J} \in \mathbb{C}^{X\times Y}$ be the **truncation of A by I, J**. In other words, for each $x \in X, y \in Y$,

$$A_{I,J}(x,y) = \begin{cases} A(x,y) & \text{if } x \in I, y \in J \\ 0 & \text{otherwise} \end{cases}$$

Then there exists $\omega_{I,J} \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$ whose matrix representation is $A_{I,J}$, namely,

$$\omega_{I,J}(\xi|\eta) = \sum_{x\in I, y\in J} \overline{\xi(x)} A(x,y)\eta(y)$$

We clearly have $\|\omega_{I,J}\| = \|A_{I,J}\|$.

Consider the net $(\omega_{I,J})_{I\times J\in\mathrm{fin}(2^X)\times\mathrm{fin}(2^Y)}$ of sesquilinear forms. The assumption $\|A\| < +\infty$ implies that $\sup_{I,J}\|\omega_{I,J}\| < +\infty$. Moreover, since

$$\lim_{I\in\mathrm{fin}(2^X), J\in\mathrm{fin}(2^Y)} \omega_{I,J}(\chi_{\{x\}}|\chi_{\{y\}}) = A(x,y) \qquad \text{for all } x \in X, y \in Y \tag{3.32}$$

the net $(\omega_{I,J})$ converges pointwise on $C_c(X) \times C_c(Y)$. Therefore, by Thm. 2.4.6, the net $(\omega_{I,J})$ converges pointwise on $\mathcal{H} \times \mathcal{K}$ to some $\omega \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K})$. By (3.32), we have $[\omega] = A$. □

Definition 3.6.4. Let X, Y, Z be sets. Let $A \in \mathbb{C}^{X\times Y}$ and $B \in \mathbb{C}^{Y\times Z}$ be bounded matrices. Then the **matrix multiplication** $AB \in \mathbb{C}^{X\times Z}$ is defined to be

$$(AB)(x,z) = \sum_{y\in Y} A(x,y)B(y,z)$$

where the RHS is convergent for each $x \in X, z \in Z$. This definition is clearly compatible with Def. 3.5.14, that is, if $\mathcal{H}, \mathcal{K}, \mathcal{M}$ have orthonormal basis $(e_x)_{x\in X}, (e_y)_{y\in Y}, (e_z)_{z\in Z}$ respectively, and if $\omega \in \mathfrak{Ses}(\mathcal{H}|\mathcal{K}), \sigma \in \mathfrak{Ses}(\mathcal{K}|\mathcal{M})$, then the corresponding matrix representations satisfy

$$[\omega \circ \sigma] = [\omega] \cdot [\sigma]$$

We now return to the topic discussed at the end of Sec. 3.5.4: the subtlety of defining and understanding composition on the side of bounded sesquilinear forms—a subtlety that also arises in the context of bounded matrices. For simplicity, we restrict attention to a fixed Hilbert space $\mathcal{H}$ with orthonormal basis $(e_x)_{x\in X}$.

For $R, S, T \in \mathfrak{L}(\mathcal{H})$, associativity of composition,

$$(RS)T = R(ST)$$

is almost tautological. However, when working with bounded sesquilinear forms or bounded matrices, associativity is far less transparent. To see this, consider $\sigma, \omega, \tau \in \mathfrak{Ses}(\mathcal{H})$. Then the associativity $(\sigma\omega)\tau = \sigma(\omega\tau)$ amounts to the commutativity of the two unordered sums: for all $\xi, \eta \in \mathcal{H}$,

$$\sum_{y\in X}\sum_{x\in X}\sigma(\xi|e_x)\omega(e_x|e_y)\tau(e_y|\eta) = \sum_{x\in X}\sum_{y\in X}\sigma(\xi|e_x)\omega(e_x|e_y)\tau(e_y|\eta) \tag{3.33}$$

Similarly, if $A, B, C \in \mathbb{C}^{X\times X}$ are bounded matrices, associativity of matrix multiplications means that for each $i, j \in X$,

$$\sum_{y\in X}\sum_{x\in X}A(i,x)B(x,y)C(y,j) = \sum_{x\in X}\sum_{y\in X}A(i,x)B(x,y)C(y,j) \tag{3.34}$$

At first glance, the commutativity of $\sum_{x\in X}$ and $\sum_{y\in X}$ is not at all obvious.

The issue of commutativity of unordered sums—which appears in the frameworks of sesquilinear forms and matrices—disappears in the perspective of linear maps. Where is this Fubini-type property hidden in the linear map viewpoint? And how can one understand the commutativity of such unordered sums in a more general context? We will answer this question in the next section.

3.7 SOT and WOT

Let U, V, W be inner product spaces.

3.7.1 Convergence of vectors

Definition 3.7.1. The **weak topology** on V is defined to be the pullback of the weak-* topology on $(V^{\complement})^*$ by the Riesz isometry $V \to (V^{\complement})^*$. Therefore, a net (ξ_α) in V converges weakly to $\xi \in V$ iff

$$\lim_\alpha \langle\eta|\xi_\alpha\rangle = \langle\eta|\xi\rangle \tag{3.35}$$

holds for each $\eta \in V$

It is clear that norm convergence implies weak convergence.

Remark 3.7.2. Let (ξ_α) be a uniformly bounded net in V, and let $\xi\in V$. Let U be a dense linear subspace of V. Applying Thm. 2.6.2 to the images of (ξ_α) and ξ in $(V^{\complement})^*$, we see that (ξ_α) converges weakly to ξ iff (3.35) holds for each $\eta\in U$.

Proposition 3.7.3 (Fatou's lemma for weak convergence). *Let (ξ_α) be a net in V converging weakly to $\xi\in V$. Then*

$$\|\xi\|\leqslant\liminf_\alpha\|\xi_\alpha\| \tag{3.36}$$

Moreover, (ξ_α) converges in norm to ξ iff $\lim_\alpha\|\xi_\alpha\|$ converges to $\|\xi\|$.

Proof. The inequality (3.36) follows from Prop. 2.6.4 and the fact that the Riesz isometry $V\to(V^{\complement})^*$ is an isometry. But it can also be proved directly:

$$\|\xi\|^2=\lim_\alpha|\langle\xi|\xi_\alpha\rangle|=\liminf_\alpha|\langle\xi|\xi_\alpha\rangle|\leqslant\liminf_\alpha\|\xi\|\cdot\|\xi_\alpha\|$$

If (ξ_α) converges in norm to ξ, then by the continuity of $\|\cdot\|:V\to\mathbb{R}_{\geqslant 0}$, $\lim_\alpha\|\xi_\alpha\|$ converges to $\|\xi\|$. Conversely, suppose that $\lim_\alpha\|\xi_\alpha\|=\|\xi\|$. Then, since $\langle\xi|\xi_\alpha\rangle\to\langle\xi|\xi\rangle$, we have

$$\langle\xi-\xi_\alpha|\xi-\xi_\alpha\rangle=\|\xi\|^2-2\Re\langle\xi|\xi_\alpha\rangle+\|\xi_\alpha\|^2\to\|\xi\|^2-2\Re\|\xi\|^2+\|\xi\|^2=0$$

This shows that $\|\xi-\xi_\alpha\|\to 0$. □

3.7.2 Convergence of operators

Recall that the **norm topology** on $\mathfrak{L}(V,W)$ is the topology determined by the operator norm on $\mathfrak{L}(V,W)$.

Definition 3.7.4. The **strong operator topology (SOT)** on $\mathfrak{L}(V,W)$ is defined to be the pullback of the product topology on W^V by the inclusion map

$$\mathfrak{L}(V,W)\hookrightarrow W^V$$

The **weak topology** on $\mathfrak{Ses}(W|V)$ is defined to be the pullback of the product topology on $\mathbb{C}^{W\times V}$ by the inclusion map

$$\mathfrak{Ses}(W|V)\hookrightarrow\mathbb{C}^{W\times V}$$

The **weak operator topology (WOT)** on $\mathfrak{L}(V,W)$ is defined to be the pullback of the weak topology on $\mathfrak{Ses}(W|V)$ by the linear isometry

$$\mathfrak{L}(V,W)\to\mathfrak{Ses}(W|V)\qquad T\mapsto\omega_T$$

Remark 3.7.5. Let (T_α) be a net in $\mathfrak{L}(V,W)$ and $T\in\mathfrak{L}(V,W)$. Then (T_α) converges in SOT to T iff

$$\lim_\alpha T_\alpha\xi = T\xi \tag{3.37}$$

holds for each $\xi\in V$. (T_α) converges in WOT to T iff

$$\lim_\alpha\langle\eta|T_\alpha\xi\rangle = \langle\eta|T\xi\rangle \tag{3.38}$$

holds for each $\xi\in V,\eta\in W$. It is clear that

$$\text{convergence in norm} \quad\Rightarrow\quad \text{convergence in SOT} \quad\Rightarrow\quad \text{convergence in WOT}$$

Remark 3.7.6. Let (T_α) be a net in $\mathfrak{L}(V,W)$ and $T\in\mathfrak{L}(V,W)$. Then it is clear that (T_α) converges in WOT to T iff

$$\lim_\alpha T\xi_\alpha \quad \text{converges weakly to} \quad \xi$$

for each $\xi\in V$. By Prop. 3.7.3, (T_α) converges in SOT to T iff the following two conditions hold:

(1) (T_α) converges in WOT to T.

(2) $\lim_\alpha\lVert T_\alpha\xi\rVert = \lVert T\xi\rVert$ for each $\xi\in V$.

Example 3.7.7. Suppose that V has a basis basis $(e_x)_{x\in X}$. For each $I\in\mathrm{fin}(2^X)$, let P_I be the projection of V onto $V_I=\mathrm{Span}\{e_x:x\in I\}$, that is,

$$P_I\xi=\sum_{x\in I}e_x\cdot\langle e_x|\xi\rangle \qquad \text{for all } \xi\in V$$

Then by Thm. 3.3.24, $\lim_{I\in\mathrm{fin}(2^X)}P_I$ converges strongly to $\mathbf{1}$.

However, if X is infinite, then $\lim_{I\in\mathrm{fin}(2^X)}P_I$ does not converge in norm to $\mathbf{1}$. Indeed, for each I, choose $x\in X\backslash I$. Then $(\mathbf{1}-P_I)e_x=e_x$, and hence

$$\lVert\mathbf{1}-P_I\rVert\geqslant 1$$

Indeed, by Prop. 3.3.18, $\mathbf{1}-P_I$ is the projection operator associated to $U_I^\perp$, and hence has operator norm 1. □

3.7.3 SOT and composition of operators

Let us examine the commutativity of unordered sums from Sec. 3.6 through the lens of SOT.

Proposition 3.7.8. *Let $(T_\alpha)_{\alpha\in\mathscr{A}}$ be a net in $\mathfrak{L}(V,W)$ converging strongly to $T \in \mathfrak{L}(V,W)$. Let $(\xi_\beta)_{\beta\in\mathscr{B}}$ be a net in V converging strongly to $\xi \in V$. Then*

$$\lim_\alpha \lim_\beta T_\alpha \xi_\beta = \lim_\beta \lim_\alpha T_\alpha \xi_\beta = T\xi \tag{3.39}$$

Proof. We compute that

$$\lim_\alpha \lim_\beta T_\alpha \xi_\beta = \lim_\alpha T_\alpha \xi = T\xi$$

and

$$\lim_\beta \lim_\alpha T_\alpha \xi_\beta = \lim_\beta T\xi_\beta = T\xi$$

where the last equality is due to the continuity of T. □

Corollary 3.7.9. *Let $(T_\alpha)_{\alpha\in\mathscr{A}}$ be a net in $\mathfrak{L}(V,W)$ converging strongly to $T \in \mathfrak{L}(V,W)$. Let $(S_\beta)_{\beta\in\mathscr{B}}$ be a net in $\mathfrak{L}(U,V)$ converging strongly to $S \in \mathfrak{L}(U,V)$. Then for each $u \in U$ we have*

$$\lim_\alpha \lim_\beta T_\alpha S_\beta u = \lim_\beta \lim_\alpha T_\alpha S_\beta u = TSu \tag{3.40}$$

Proof. Apply Prop. 3.7.8 to $\xi_\beta = S_\beta u$. □

Cor. 3.7.9 can be easily generalized to products of more than two nets of operators. We leave the details to the reader.

Example 3.7.10. Let $\mathcal{H}$ be a Hilbert space with basis $(e_x)_{x\in X}$. Let $(P_I)_{I\in\mathrm{fin}(2^X)}$ be the net of projections where P_I projects $\mathcal{H}$ onto $V_I = \mathrm{Span}\{e_x : x \in I\}$. By Exp. 3.7.7, $\lim_I P_I$ converges strongly to 1. Choose any $R,S,T \in \mathfrak{L}(\mathcal{H})$. Then $\lim_I P_I S$ converges strongly to S, and $\lim_I P_I T$ converges strongly to T. Therefore, by Cor. 3.7.9, for each $\eta \in \mathcal{H}$ we have

$$\lim_{J\in\mathrm{fin}(2^X)} \lim_{I\in\mathrm{fin}(2^X)} RP_ISP_JT\eta = \lim_{I\in\mathrm{fin}(2^X)} \lim_{J\in\mathrm{fin}(2^X)} RP_ISP_JT\eta = RST\eta$$

Therefore, for each $\xi,\eta \in \mathcal{H}$ we have

$$\lim_{J\in\mathrm{fin}(2^X)} \lim_{I\in\mathrm{fin}(2^X)} \langle\xi|RP_ISP_JT\eta\rangle = \lim_{I\in\mathrm{fin}(2^X)} \lim_{J\in\mathrm{fin}(2^X)} \langle\xi|RP_ISP_JT\eta\rangle = \langle\xi|RST\eta\rangle \tag{3.41}$$

The commutativity of the two iterated limits in (3.41) is equivalent to the commutativity of the two limits in (3.33).

Next, we introduce an elementary yet fundamental property of SOT. This property was first highlighted by Riesz in [Rie11] as a key tool in his proof of the spectral theorem for bounded self-adjoint operators. In Ch. 5, we will also rely on it in our study of spectral theory. This property roughly says that in Cor. 3.7.9, if the nets of operators are uniformly bounded, and if $\mathscr{A} = \mathscr{B}$, then $\lim_\alpha T_\alpha S_\alpha$ converges strongly to TS.

Theorem 3.7.11. *Let $(T_\alpha)_{\alpha\in\mathscr{A}}$ be a net in $\mathfrak{L}(V,W)$ satisfying $\sup_\alpha \|T_\alpha\| < +\infty$ and converging in SOT to $T \in \mathfrak{L}(V,W)$. Let $(\xi_\beta)_{\beta\in\mathscr{B}}$ be a net in V converging to $\xi \in V$. Then the double net $(T_\alpha\xi_\beta)_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}}$ converges to $T\xi$, that is,*

$$\lim_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} T_\alpha\xi_\beta = T\xi$$

In particular, if $\mathscr{A} = \mathscr{B}$, since $(T_\alpha\xi_\alpha)_{\alpha\in\mathscr{A}}$ is a subnet of $(T_\alpha\xi_\beta)_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{A}}$ (defined by $\alpha \in \mathscr{A} \mapsto (\alpha,\alpha) \in \mathscr{A}$), we conclude that

$$\lim_\alpha T_\alpha\xi_\alpha = T\xi$$

Proof. Let $C = \sup_\alpha \|T_\alpha\|$, which is finite. We compute that

$$\|T\xi - T_\alpha\xi_\beta\| \leqslant \|T\xi - T_\alpha\xi\| + \|T_\alpha\xi - T_\alpha\xi_\beta\| \leqslant \|T\xi - T_\alpha\xi\| + C\|\xi - \xi_\beta\|$$

where the RHS converges to 0 under $\lim_{\alpha,\beta}$. $\square$

Corollary 3.7.12. *Let $V_0, V_1, \dots, V_k$ be inner product spaces. For each $1 \leqslant i \leqslant k$, let $(T^i_{\alpha_i})_{\alpha_i\in\mathscr{A}_i}$ be a net in $\mathfrak{L}(V_{i-1}, V_i)$ converging in SOT to $T^i \in \mathfrak{L}(V_{i-1}, V_i)$. Assume that*

$$\sup_{\alpha_i\in\mathscr{A}_i} \|T^i_{\alpha_i}\| < +\infty \qquad \text{for all } 2 \leqslant i \leqslant k$$

Then for each $\xi \in V_0$, we have

$$\lim_{(\alpha_1,\dots,\alpha_k)\in\mathscr{A}_1\times\cdots\times\mathscr{A}_k} T^k_{\alpha_k}\cdots T^1_{\alpha_1}\xi = T^k\cdots T^1\xi$$

Similar to Thm. 3.7.11, if we also assume $\mathscr{A}_1 = \cdots = \mathscr{A}_k = \mathscr{A}$, then

$$\lim_{\alpha\in\mathscr{A}} T^k_\alpha\cdots T^1_\alpha\xi = T^k\cdots T^1\xi$$

Proof. This follows immediately from Thm. 3.7.11. $\square$

Corollary 3.7.13. *Let $(T_\alpha)_{\alpha\in\mathscr{A}}$ be a net in $\mathfrak{L}(V)$ satisfying $\sup_\alpha \|T_\alpha\| < +\infty$ and converging in SOT to $T \in \mathfrak{L}(V)$. Then for each polynomial $f \in \mathbb{C}[z]$ and $\xi \in V$,*

$$\lim_\alpha f(T_\alpha)\xi = f(T)\xi$$

Proof. This is clear by Cor. 3.7.12. $\square$

Example 3.7.14. In Exp. 3.7.10, since $\sup_{I\in\mathrm{fin}(2^X)} \|E_I\| = 1 < +\infty$, by Cor. 3.7.12 we have

$$\lim_{I\in\mathrm{fin}(2^X)} RP_ISP_IT\eta = RST\eta \qquad \text{for each } \eta \in \mathcal{H}$$

Example 3.7.15. Let $\mathcal{H}$ be a Hilbert space with basis $(e_x)_{x\in X}$. For each $I \in \mathrm{fin}(2^X)$, let P_I be the projection onto $U_I = \mathrm{Span}\{e_x : x \in I\}$. By Thm. 3.7.11, $\lim_I P_ITP_I$ converges strongly to T, and

$$\sup_I \|P_ITP_I\| \leqslant \sup_I \|P_I \cdot \|T\| \cdot \|P_I\| \leqslant \|T\| < +\infty$$

Therefore, by Cor. 3.7.13, for each $T \in \mathfrak{L}(H)$ and $f \in \mathbb{C}[z]$,

$$\lim_{I\in\mathrm{fin}(2^X)} f(P_ITP_I) \qquad \text{converges strongly to } f(T)$$

3.8 Problems

Let V be an inner product space.

Problem 3.8.1. Let ω be a Hermitian form on V. Prove the following sharpened polarization identity: For each $\xi,\eta\in V$ we have

$$\omega(\xi|\eta)=\frac{1}{4}\big(\omega(\xi+\eta|\xi+\eta)-\omega(\xi-\eta|\xi-\eta)\big) \tag{3.42}$$

Conclude that if $M\in\mathbb{R}_{\geqslant 0}$ and

$$|\omega(\xi|\xi)|\leqslant M\|\xi\|^2$$

for each $\xi\in V$, then $\|\omega\|\leqslant M$. This sharpens Prop. 3.2.12 in the Hermitian case.

Hint. The strategy differs slightly from that of Prop. 3.2.12. In particular, apply (3.42) to the inner product on V at an appropriate step. □

4 The polynomial moment problem: a prehistory of spectral theory

4.1 Divergent series and the birth of the Stieltjes integral

4.1.1 Divergent series and the Padé approximation

In 1894, Stieltjes introduced the Stieltjes integral in [Sti94] (see [Sti-C, Vol. II] for an English translation) for the purpose of studying continued fractions. One key motivation for investigating continued fractions was to better understand the behavior of the series

$$\frac{c_0}{z}+\frac{c_1}{z^2}+\frac{c_2}{z^3}+\cdots \qquad \text{where each } c_n \geqslant 0 \tag{4.1}$$

in the divergent case—that is, when $\sum_n c_n r^n = +\infty$ for each $r>0$. The core idea is that even when the power series in (4.1) diverges, it may still define a meaningful function outside a closed interval $I\subset\mathbb{R}$, provided we adopt a different notion of convergence.

Specifically, under suitable conditions on the sequence $(c_n)_{n\in\mathbb{N}}$, one can construct a sequence of rational functions $(f_n(z))_{n\in\mathbb{N}}$,

$$f_n(z)=\frac{q_n(z)}{p_{n+1}(z)} \qquad \text{where } p_{n+1},q_n\in\mathbb{C}[z], \deg p_{n+1}=n+1, \deg q_n=n \tag{4.2}$$

Moreover, when $|z|$ is sufficiently large, $f_n(z)$ has Laurent expansion

$$f_n(z)=\sum_{m\in\mathbb{N}} c_{n,m}z^{-m-1} \qquad \text{with } c_{n,m}=c_n \text{ when } m\leqslant 2n+1 \tag{4.3a}$$

In other words,

$$f_n(z)=\frac{c_0}{z}+\frac{c_1}{z^2}+\cdots+\frac{c_{2n+1}}{z^{2n+2}}+\frac{?}{z^{2n+3}}+\cdots \tag{4.3b}$$

The sequence (f_n) (or a subsequence thereof) converges locally uniformly on $\mathbb{C}\backslash I$ to a holomorphic function f. This approximation is referred to as a **Padé approximation**. In such cases, we say that the holomorphic function $f(z)$ represents the series (4.1) and write

$$f(z)\sim\frac{c_0}{z}+\frac{c_1}{z^2}+\frac{c_2}{z^3}+\cdots$$

4.1.2 Stieltjes integral as the weak-* completion of finite sum

With the help of Padé approximation, we can understand how the Stieltjes integral naturally arises as the weak-* completion of finite sums. As we will see in the following sections, under suitable assumptions on the sequence (c_n), each rational function $f_n(z)$ has only simple poles. Consequently, $f_n(z)$ admits the representation

$$f_n(z) = \sum_i \frac{a_{n,i}}{z - \lambda_{n,i}} \tag{4.4}$$

where the sum is finite. Moreover, we have $a_{n,i} \geqslant 0$ and $\lambda_{n_i} \in I$.

The general Stieltjes integral appears when one tries to understand the behavior of the finite sum on the RHS of (4.4) under the limit $n \to +\infty$. To understand this behavior, we define an increasing function $\rho_n : I \to \mathbb{R}_{\geqslant 0}$ by setting

$$\rho_n(x) = \sum_{\lambda_{n,i} \leqslant x} a_{n,i}$$

That is, ρ_n is the right-continuous increasing function associated to the measure $\sum_i a_{n,i}\delta_{\lambda_{n,i}}$. Then (4.4) can be rewritten as

$$f_n(z) = \int_I \frac{d\rho_n(x)}{z - x}$$

The sequence (ρ_n) is indeed uniformly bounded. Therefore, by passing to a subsequence if necessary, we may assume that (ρ_n) almost converges to some increasing right-continuous function ρ. Therefore, by Thm. 2.9.6, (ρ_n) converges weak-* to ρ. It follows that (f_n) converges locally uniformly on $\mathbb{C}\backslash I$ to

$$f(z) = \int_I \frac{d\rho(x)}{z - x} \tag{4.5}$$

This gives a holomorphic function f on $\mathbb{C}\backslash I$ representing the series 2.9.3.

Definition 4.1.1. For each increasing function $\rho : I \to \overline{\mathbb{R}}_{\geqslant 0}$, the function $f(z)$ defined by (4.5) is called the **Stieltjes transform** of ρ. If μ is a finite Borel measure on I, we also call the function

$$\mathbb{C}\backslash I \to \mathbb{C} \qquad z \mapsto \int_I \frac{d\mu(x)}{z - x}$$

the **Stieltjes transform** of μ.

We have thus seen that, via Padé approximation, the Stieltjes integral with respect to a general increasing function emerges as the (weak-*) limit of finite sums. This marks the first historical appearance of approximating a continuous spectrum by discrete spectra.

In the next section, we will see that the approximation of ρ by the sequence (ρ_n) can be interpreted as the finite-rank approximation of a (not necessarily bounded) Hermitian operator. Since this type of approximation plays a central role in the development of spectral theory by Hilbert and Riesz, it is crucial to understand its origin in the study of divergent series and its connection with Padé approximation.

4.1.3 The polynomial moment problem

We now investigate the following question: What assumption should we impose on $(c_n)_{n\in\mathbb{N}}$ so that the above strategy can be carried out? Note that when $|z| > |\lambda_{n,i}|$, we have

$$(z-\lambda_{n,i})^{-1} = \sum_{m\in\mathbb{N}} z^{-m-1}(\lambda_{n,i})^m$$

Therefore, (4.4) becomes $f_n(z) = \sum_{m\in\mathbb{N}}\sum_i z^{-m-1}\cdot a_{n,i}(\lambda_{n,i})^m$, and hence

$$f_n(z) = \sum_{m\in\mathbb{N}} z^{-m-1}\cdot\int_I x^m d\rho_n$$

Comparing this with (4.3), we obtain

$$c_{n,m} = \int_I x^m d\rho_n$$

On the one hand, (4.3) shows that $\lim_n c_{n,m} = c_n$. On the other hand, the weak-* convergence of $(d\rho_n)$ to $d\rho$ actually implies that $\lim_n \int_I x^m d\rho_n(\lambda)$ converges to $\int_I x^m d\rho(\lambda)$. Thus, we obtain

$$c_m = \int_I x^m d\rho \qquad \text{for all } m\in\mathbb{N} \tag{4.6}$$

Therefore, a necessary condition for the above strategy to work is the existence of an increasing function ρ satisfying (4.6). As we shall see in the next section, this condition is also sufficient. In this way, the problem of representing a divergent series and approximating it using rational functions is closely related to the polynomial moment problem.

4.2 Padé approximation via the finite-rank approximation of Hermitian operators

In this section, we let $I\subset\mathbb{R}$ be a proper closed interval. For simplicity, we assume that I is one of the following intervals

$$\mathbb{R} \qquad [0,+\infty) \qquad [0,1]$$

Let $\mathfrak{Rr}(I)$ be the set of functions $\rho: I\to\mathbb{R}_{\geqslant 0}$ satisfying condition (a) of the Riesz representation Thm. 1.10.1. That is,

$$\begin{aligned}\mathfrak{Rr}(I) = \{&\text{Increasing right continuous function } \rho: I\to\mathbb{R}_{\geqslant 0}\\ &\text{satisfying } \lim_{x\to-\infty}\rho(x) = 0 \text{ if } I=\mathbb{R}\}\end{aligned} \tag{4.7}$$

Fix a family $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

4.2.1 The goal

Problem 4.2.1 (Polynomial moment problem). Does there exists $\rho \in \mathcal{R}r\,(I)$ such that

$$c_n = \int_I x^n d\rho \qquad \text{for each } n \in \mathbb{N} \tag{4.8}$$

where the RHS is integrable, i.e., $\int_I |x|^n d\rho < +\infty$? Depending on whther I is $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, or $[0,1]$, this problem is referred to as the **Hamburger moment problem**, the **Stieltjes moment problem**, or the **Hausdorff moment problem**, respectively.

The goal of this section is to give a complete solution of this problem; see Thm. 4.2.9. Moreover, when the moment problem is solvable for the given sequence (c_n), we will find $\rho \in \mathcal{R}r\,(I)$ whose Stieltjes transform

$$f(z) = \int_I \frac{d\rho(x)}{z-x}$$

represents the series $\sum_{n=0}^{\infty} c_n z^{-n-1}$ in the sense of Padé approximation. That is, there is a sequence of rational functions $(f_n(z))_{n\in\mathbb{N}}$ satisfying (4.3), and a subsequence $(f_{n_k}(z))$ converging locally uniformly on $\mathbb{C}\backslash I$ to $f(z)$. [1] Thus, the problem of representing the (possibly divergent) series $\sum_{n=0}^{\infty} c_n z^{-n-1}$ for such (c_n) is solved. See Thm. 4.2.21.

The classical construction of the sequence (f_n) relies crucially on the idea of orthogonal polynomials. The approach presented in this section reformulates that classical method using the language of inner product spaces—a modern reinterpretation shaped by the development of spectral theory in Hilbert spaces. Of course, this reformulation is a retrospective abstraction that emerged only after the development of spectral theory in Hilbert spaces. In this section, we adopt this modern perspective, while aiming to present it in a way that remains mindful of its historical origins. In the following sections, we will explain how this approach connects to continued fractions and the classical formulation using orthogonal polynomials.

4.2.2 Preliminary

Let us clarify the meaning of (4.8), since we have so far defined $\int_I f d\rho$ only for $f \in C_c(I)$.

Definition 4.2.2. Let f be a Borel function from I to $\mathbb{C}$ or $\overline{\mathbb{R}}_{\geqslant 0}$. Let $\rho \in \mathcal{R}r\,(I)$. Let μ_ρ be the finite Borel measure on I associated to ρ as in the Riesz representation

[1]When the solution to the moment problem is unique, the original sequence (f_n) indeed converges locally uniformly to f. The uniqueness aspect of moment problems, however, will not be addressed in this course.

Thm. 1.10.1. We define the **Stieltjes integral** $\int_I f d\rho$ to be

$$\int_I f d\rho := \int_I f \mu_\rho$$

provided that the integral on the RHS exists.

Remark 4.2.3. When $f : I \to \mathbb{R}_{\geqslant 0}$ is continuous, the computation of the integral $\int_I f d\rho$ can be reduced to those of compactedly supported continuous functions. Indeed, for each $\lambda \geqslant 0$, let $\beta_\lambda : \mathbb{R} \to [0,1]$ be the (continuous) piecewise linear functions such that

$$\beta_\lambda|_{[-\lambda,\lambda]} = 1 \qquad \beta_\lambda|_{(-\infty,-\lambda-1]\cup[\lambda+1,+\infty)} = 0$$

Then $f\beta_\lambda \in C_c(I)$, and $\int_I f\beta_\lambda d\rho$ is increasing as λ increases. Therefore, by MCT, we have

$$\int_I f d\rho = \lim_{\lambda\to+\infty} \int_I f\beta_\lambda d\rho$$

4.2.3 The Hankel matrix

Definition 4.2.4. Let $H \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$. Associate to H the unique sesquilinear form

$$\langle\cdot|\cdot\rangle : \mathbb{C}[x] \times \mathbb{C}[x] \to \mathbb{C} \qquad (x^m, x^n) \mapsto H(m,n)$$

We say that H is **Hermitian** (resp. **positive, positive definite**) if $\langle\cdot|\cdot\rangle$ is Hermitian (resp. positive (semi-definite), positive definite).

For each $n \in \mathbb{N}$, the ***n*-th truncation H_n** of H is the $(n+1) \times (n+1)$ matrix defined by the first $(n+1)$ rows and columns of H.

Remark 4.2.5. By linear algebra, H is positive definite iff $\det H_n > 0$ for each n; H is positive (semi-definite) iff the determinant of each principal submatrix is $\geqslant 0$.

Definition 4.2.6. The **Hankel matrix *H*** of $(c_n)_{n\in\mathbb{N}}$ is defined by

$$H(m,n) = c_{m+n}$$

That is,

$$H = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \cdots \\ c_1 & c_2 & c_3 & c_4 & \cdots \\ c_2 & c_3 & c_4 & c_5 & \cdots \\ c_3 & c_4 & c_5 & c_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Equivalently, the associated sesquilinear form is determined by

$$\langle f|g\rangle = \langle 1|\overline{f}g\rangle \qquad \text{for each } f,g \in \mathbb{C}[x] \tag{4.9a}$$
$$\langle 1|x^n\rangle = c_n \qquad \text{for each } n \in \mathbb{N} \tag{4.9b}$$

Since each c_n is real, H is Hermitian.

We also let $H' \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be the defined by $H'(m,n) = c_{m+n+1}$, that is,

$$H' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots \\ c_2 & c_3 & c_4 & c_5 & \cdots \\ c_3 & c_4 & c_5 & c_6 & \cdots \\ c_4 & c_5 & c_6 & c_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In other words, its associated sesquilinear form is

$$\mathbb{C}[x] \times \mathbb{C}[x] \to \mathbb{C} \qquad (f,g) \mapsto \langle f|xg\rangle \tag{4.10}$$

Since $c_n \in \mathbb{R}$, H' is also Hermitian. It follows that

$$\mathbb{C}[x] \times \mathbb{C}[x] \to \mathbb{C} \qquad (f,g) \mapsto \langle f|(1-x)g\rangle \tag{4.11}$$

is the (Hermitian) sesquilinear form associated to $H - H'$. □

Remark 4.2.7. Note that by (4.9a), for each $f,g,h \in \mathbb{C}[x]$ we have

$$\langle f|hg\rangle = \langle \overline{h}f|g\rangle \tag{4.12}$$

since $\langle f|hg\rangle = \langle 1|\overline{f}hg\rangle = \langle 1|\overline{\overline{h}f}g\rangle = \langle \overline{h}f|g\rangle$.

In the rest of this section, we always let H denote the Hankel matrix of $(c_n)_{n\in\mathbb{N}}$, and equip $\mathbb{C}[x]$ with the sesquilinear form associated to H.

Proposition 4.2.8. *Suppose that Pb. 4.2.1 has a solution ρ. Then H must be positive, and the following are equivalent:*

(1) H is positive definite.

(2) The range of the solution ρ is not a finite set (equivalently, the associated measure μ_ρ is not supported in a finite set, cf. Lem. 1.6.4).

Moreover, if $I = [0,+\infty)$, then H' is positve; if $I = [0,1]$, then both H' and $H - H'$ are positive.

Proof. Let ρ be a solution of Pb. 4.2.1. Then for each $f, g \in \mathbb{C}[x]$, we have

$$\langle f|g\rangle = \int_I \overline{f} g d\rho \tag{4.13}$$

since this is clearly true when $f = x^m, g = x^n$. Thus $\langle f|f\rangle = \int_I |f|^2 d\rho \geqslant 0$.

If H is not non-degenerated, then there exists $0 \neq f \in \mathbb{C}[x]$ such that $\langle f|f\rangle = 0$, and hence $\int_I |f|^2 d\rho = 0$. From this, one easily checks that $\mathrm{Supp}(\rho)$ is a subset of the finite set $f^{-1}(0)$. Conversely, if μ_ρ is supported in a finite set $E \subset I$, we choose a non-zero $f \in \mathbb{C}[x]$ such that $f|_E = 0$. Then $\langle f|f\rangle = \int_I |f|^2 d\mu_\rho = 0$. This shows that H is not non-degenerate. This proves the equivalence of (1) and (2).

If $I = [0, +\infty)$, then for each $f \in \mathbb{C}[x]$, since $x|f|^2 \geqslant 0$ on I, we have

$$\langle f|xf\rangle = \int_{[0,+\infty)} x|f|^2 d\rho \geqslant 0$$

Therefore, H' is positive. Similarly, if $I = [0, 1]$, then for each $f \in \mathbb{C}[x]$, both $x|f|^2$ and $(1 - x)|f|^2$ are $\geqslant 0$ on I. Therefore

$$\langle f|xf\rangle = \int_{[0,1]} x|f|^2 d\rho \geqslant 0 \qquad \langle f|(1-x)f\rangle = \int_{[0,1]} (1-x)|f|^2 d\rho \geqslant 0$$

This proves that both H' and $H - H'$ are positive. □

Therefore, to solve the polynomial moment problem, we should at least assume that H is positive. Indeed, Prop. 4.2.8 implies a half of the following theorem.

Theorem 4.2.9. *Let H be the Hankel matrix for $(c_n)_{n\in\mathbb{N}}$. The following are true*

1. *The Hamburger moment problem (i.e. $I = \mathbb{R}$) has a solution iff H is positive.*
2. *The Stieltjes moment problem (i.e. $I = [0, +\infty)$) has a solution iff H, H' are positive.*
3. *The Hausdorff moment problem (i.e. $I = [0, 1]$) has a solution iff $H, H', H - H'$ are positive.*

Proof. The direction "$\Rightarrow$" follows from Prop. 4.2.8. The direction "$\Rightarrow$" will follow immediately from Thm. 4.2.20 once the latter has been established. □

4.2.4 An abstract formulation of the Hankel matrix and its positivity

Assume that H is positive. In this subsection, we express the positivity of $H, H', H - H'$ in terms of the positivity of certain Hermitian operator T and the operator $1 - T$ on an inner product space V.

Theorem 4.2.10. *Then there exists an inner product space V, a vector $\Omega \in V$, and a linear operator $T \in \mathrm{Lin}(V)$ satisfying the following properties:*

(1) For each $n \in \mathbb{N}$ we have

$$\langle \Omega | T^n \Omega \rangle = c_n \tag{4.14}$$

(2) ω_T is Hermitian, i.e., $\langle \eta | T\xi \rangle = \langle T\eta | \xi \rangle$ for each $\xi, \eta \in V$.

*(3) (**Algebraic cyclicity**) V is spanned by $T^n\Omega$ for all $n \in \mathbb{N}$.*

Note that (4.14) implies

$$\|\Omega\|^2 = c_0 \tag{4.15}$$

Proof. Recall that H determines a positive sesquilinear form $\langle \cdot | \cdot \rangle$ on $\mathbb{C}[x]$. By Cor. 3.1.10, the null space

$$\mathscr{N} = \{ f \in \mathbb{C}[x] : \|f\|^2 = 0 \} \tag{4.16}$$

is a linear subspace of $\mathbb{C}[x]$, and $\langle \cdot | \cdot \rangle$ descends to an inner product of

$$V = \mathbb{C}[x]/\mathscr{N}$$

Let

$$T : \mathbb{C}[x] \to \mathbb{C}[x] \qquad f \mapsto xf$$

If $f \in \mathscr{N}$, by Cauchy-Schwarz (cf. Thm. 3.1.9) and Rem. 4.2.7,

$$0 \leqslant \langle xf | xf \rangle = \langle x^2 f | f \rangle \leqslant \|x^2 f\| \cdot \|f\| = 0$$

and hence $xf \in \mathscr{N}$. Therefore, the above map T descends to a linear map

$$T : V \to V \qquad f + \mathscr{N} \mapsto xf + \mathscr{N} \tag{4.17}$$

Let $\Omega \in V$ be

$$\Omega = 1 + \mathscr{N}$$

Then $\langle \Omega | T^n \Omega \rangle = \langle 1 | x^n \rangle = c_n$. By Rem. 4.2.7, ω_T is Hermitian. The algebraic cyclicity is obvious. $\square$

Remark 4.2.11. From the above proof, it is clear that for each $f, g \in \mathbb{C}[x]$, we have

$$\langle f(T)\Omega | g(T)\Omega \rangle = \langle f | g \rangle$$

Therefore, $\omega_T \geqslant 0$ holds iff $\langle f(T)\Omega | Tf(T)\Omega \rangle \geqslant 0$ for all $f \in \mathbb{C}[x]$, iff $\langle f | xf \rangle \geqslant 0$ for all $f \in \mathbb{C}[x]$, iff H' is positive. Similarly, $\omega_{1-T} \geqslant 0$ iff $H - H'$ is positive.

Remark 4.2.12. The triple (V, Ω, T) satisfying the requirements in Thm. 4.2.10 are unique up to unitary operators. Namely, if $(\widetilde{V}, \widetilde{\Omega}, \widetilde{T})$ is another triple satisfying the requirements in Thm. 4.2.10, then there is a (necessarily unique) unitary operator $\Phi : V \to \widetilde{V}$ satisfying $\Phi\Omega = \widetilde{\Omega}$ and $\Phi T = \widetilde{T}\Phi$.

Proof. For each $f \in \mathbb{C}[x]$, since ω_T is Hermitian, we have $\langle \eta | f(T)\xi \rangle = \langle \overline{f}(T)\eta | \xi \rangle$ for each $\xi, \eta \in V$. A similar property holds for $\widetilde{T}$. Therefore,

$$\big\langle f(T)\Omega \big| f(T)\Omega \big\rangle = \big\langle \Omega \big| |f|^2(T)\Omega \big\rangle = \big\langle \widetilde{\Omega} \big| |f|^2(\widetilde{T})\widetilde{\Omega} \big\rangle = \big\langle f(\widetilde{T})\widetilde{\Omega} \big| f(\widetilde{T})\widetilde{\Omega} \big\rangle$$

In particular, $f(T)\Omega = 0$ iff $f(\widetilde{T})\widetilde{\Omega} = 0$. Therefore, by the algebraic cyclicity, we have a unitary map

$$\Phi : V \to \widetilde{V} \qquad f(T)\Omega \mapsto f(\widetilde{T})\widetilde{\Omega} \tag{4.18}$$

This map clearly satisfies the desired property.

Conversely, if Φ satisfies the requirements in the remark, then Φ must send $f(T)\Omega$ to $f(\widetilde{T})\widetilde{\Omega}$. Therefore, such Φ must be unique. $\square$

4.2.5 Solving the moment problem: the degenerate case

Assume $H \geqslant 0$.

In this subsection, we explain how the moment problem can be solved when $\dim V < +\infty$. This happens when H is degenerate. In that case, if we let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be the polynomial with lowest degree satisfying $\langle f|f \rangle = 0$, then $f(T)\Omega = 0$. Then V has basis $\Omega, T\Omega, \cdots, T^{n-1}\Omega$, and f is a minimal polynomial of T on V. Conversely, it is also clear that if $\dim V < +\infty$, then H is degenerate.

Assume $\dim V < +\infty$. Then T is a Hermitian operator on the finite-dimensional inner product space V, and hence can be diagonalized. More precisely, there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Te_j = \lambda_j e_j$ for all $1 \leqslant j \leqslant n$, where $\lambda_j \in \mathbb{R}$. Write

$$\Omega = a_1 e_1 + \cdots + a_n e_n$$

where $a_j \in \mathbb{C}$. Then, for each $f \in \mathbb{C}[x]$, we have

$$f(T)\Omega = \sum_{j=1}^{n} a_j f(\lambda_j) e_j$$

and hence

$$\langle \Omega | f(T)\Omega \rangle = \sum_{j=1}^{n} f(\lambda_j)|a_j|^2 = \int_{\mathbb{R}} f d\rho$$

where $\rho : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is an increasing function corresponding to the measure $\mu = \sum_{j=1}^n |a_j|^2 \delta_{\lambda_j}$. In particular, by Thm. 4.2.10-(1), we conclude

$$\int_{\mathbb{R}} x^n d\rho = \langle \Omega | T^n \Omega \rangle = c_n$$

Thus, the Hamburger moment problem is solved.

Next, assume $I = [0, +\infty)$ and $H' \geqslant 0$. By Rem. 4.2.11, T is positive, and hence each λ_j belongs to $[0, +\infty)$. Therefore, the measure μ constructed in the above paragraph is supported in $[0, +\infty)$, and hence $\int_{[0,+\infty)} x^n d\rho = c_n$. This solves the Stieltjes moment problem.

Finally, assume $I = [0, 1]$ and $H', H - H'$ are positive. By Rem. 4.2.11, T and $1 - T$ are positive. Therefore, each λ_j belongs to $[0, 1]$, and hence $\int_{[0,1]} x^n d\rho = c_n$. This solves the Hausdorff moment problem.

To summarize, we have proved Thm. 4.2.9 when $\dim V < +\infty$, equivalently, when H is positive but not positive definite.

4.2.6 Finite-rank approximation of T

In this subsection, we assume the following condition.

Condition 4.2.13. *Assume that $H \geqslant 0$. Moreover:*

- *If $I = [0, +\infty)$, assume that $H' \geqslant 0$ (equivalently, $\omega_T \geqslant 0$, cf. Rem. 4.2.11).*
- *If $I = [0, 1]$, assume that $H', H - H'$ are both positive (equivalently, ω_T and ω_{1-T} are positive, cf. Rem. 4.2.11).*

In this subsection, we construct a uniformly bounded sequence (ρ_n) in (I) whose Stieltjes transforms—or those of a subsequence—will provide the Padé approximation of a function representing the series $\sum_{n \in \mathbb{N}} c_n z^{-n-1}$.

Definition 4.2.14. Let T be as in Thm. 4.2.10. For each $n \in \mathbb{N}$, let

$$E_n \text{ be the projection of } V \text{ onto } V_n = \mathrm{Span}\{\Omega, T\Omega, \dots, T^n\Omega\}$$

Then we clearly have

$$\Omega \in V_0 \qquad TV_n \subset V_{n+1} \tag{4.19}$$

Remark 4.2.15. Since ω_T and ω_{E_n} are Hermitian (Cor. 3.3.20), for each n and $\xi \in V$ we have

$$\langle \xi | E_n T E_n \xi \rangle = \langle T E_n \xi | T E_n \xi \rangle = \omega_T(E_n \xi | E_n \xi) \geqslant 0$$

It follows that ω_{PTP} is Hermitian. Similarly, if $H' \geqslant 0$ (equivalently, $\omega_T \geqslant 0$), then

$$\langle \xi | E_n T E_n \xi \rangle = \langle E_n \xi | T E_n \xi \rangle \geqslant 0$$

and hence $\omega_{E_n T E_n} \geqslant 0$. If both H' and $H - H'$ are positive, then

$$\langle \xi | E_n T E_n \xi \rangle = \langle E_n \xi | T E_n \xi \rangle \geqslant 0 \qquad \langle \xi | E_n (1-T) E_n \xi \rangle = \langle E_n \xi | (1-T) E_n \xi \rangle \geqslant 0$$

and hence $\omega_{E_n T E_n}, \omega_{E_n(1-T)E_n}$ are positive.

Definition 4.2.16. For each $n \in \mathbb{N}$, define a family $(c_{n,m})_{m \in \mathbb{N}}$ in $\mathbb{R}$ by

$$c_{n,m} = \langle \Omega | (E_n T E_n)^m \Omega \rangle \tag{4.20}$$

Remark 4.2.17. We have

$$c_{n,m} = c_m \qquad \text{if } m \leqslant 2n+1 \tag{4.21}$$

Consequently,

$$\lim_{n \to \infty} c_{n,m} = c_m \tag{4.22}$$

Proof. By (4.19), we have

$$(E_n T E_n)^k \Omega = T^k \Omega \in V_k \qquad \text{if } 0 \leqslant k \leqslant n \tag{4.23a}$$
$$(E_n T E_n)^{n+1} \Omega = E_n T^{n+1} \Omega \tag{4.23b}$$

where the first line is proved by induction on k. Thus, for each $m \leqslant 2n+1$, writing $m = a + b$ where $a, b \in \mathbb{N}$ and $a \leqslant n$ and $b \leqslant n+1$, we have

$$\begin{aligned}\langle \Omega | (E_n T E_n)^m \Omega \rangle &= \langle (E_n T E_n)^a \Omega | (E_n T E_n)^b \Omega \rangle = \langle T^a \Omega | E_n T^b \Omega \rangle = \langle E_n T^a \Omega | T^b \Omega \rangle \\ =& \langle T^a \Omega | T^b \Omega \rangle = \langle \Omega | T^{a+b} \Omega \rangle = c_m\end{aligned}$$

□

Proposition 4.2.18. *For each $n \in \mathbb{N}$, there exists $\rho_n \in \mathcal{R}\mathrm{r}(I)$ such that $\mathrm{Rng}(\rho_n)$ is a finite subset of $[0, c_0]$, and that for all $m \in \mathbb{N}$ we have*

$$c_{n,m} = \int_I x^m d\rho_n \tag{4.24}$$

Proof. We view $E_n T E_n$ as a linear operator T_n on V_n, i.e.,

$$T_n := E_n T E_n \big|_{V_n} \tag{4.25}$$

By Rem. 4.2.15, T_n is Hermitian. Therefore, by linear algebra, V_n has an orthonormal basis $e_{n,0}, e_{n,1}, \ldots$ such that

$$T_n e_{n,i} = \lambda_{n,i} e_{n,i} \qquad \text{for all } i \tag{4.26}$$

where $\lambda_{n,i} \in \mathbb{R}$. Moreover, by Rem. 4.2.15, if $I = \mathbb{R}_{\geqslant 0}$ then $T_n \geqslant 0$, and hence $\lambda_{n,i} \geqslant 0$; if $I = [0,1]$ then $0 \leqslant T_n \leqslant \mathrm{id}_{V_n}$, and hence $0 \leqslant \lambda_{n,i} \leqslant 1$. It follows that

$$\lambda_{n,i} \in I$$

in all cases. Write

$$\Omega = \sum_i a_{n,i} e_{n,i} \tag{4.27}$$

where $a_n \in \mathbb{C}$. (So $a_{n,i} = \langle e_{n,i} | \Omega \rangle$.) Thus

$$c_{n,m} = \langle \Omega | (T_n)^m \Omega \rangle = \sum_i (\lambda_{n,i})^m \cdot |a_{n,i}|^2 = \int_I x^m d\rho_n$$

where ρ_n is the unique element in $\mathscr{B}\!\mathit{r}\,(I)$ whose associated finite Borel measure is $\sum_{i=0}^n |a_{n,i}|^2 \delta_{\lambda_{n,i}}$, that is,

$$\rho_n(x) = \sum_{\substack{\text{all } i \text{ satisfying} \\ \lambda_{n,i} \leqslant x}} |a_{n,i}|^2 \tag{4.28}$$

In particular, by Parseval's identity, we have

$$0 \leqslant \rho_n(x) \leqslant \sum_i |a_{n,i}|^2 = \|\Omega\|^2 \xlongequal{(4.15)} c_0$$

□

4.2.7 From discrete spectra to continuous spectra

We continue to assume Condition 4.2.13. In this subsection, we solve the moment problem (i.e., complete the proof of Thm. 4.2.9) by proving Thm. 4.2.20.

Definition 4.2.19. Let $(\rho_n)_{n \in \mathbb{N}}$ be the uniformly bounded sequence in $\mathscr{B}\!\mathit{r}\,(I)$ described by Prop. 4.2.18. By Helly's selection Thm. 2.9.3, (ρ_n) has a subsequence $(\rho_{n_k})_{k \in \mathbb{N}}$ converging pointwise to some increasing function $\widetilde{\rho} : I \to \mathbb{R}_{\geqslant 0}$. Define $\rho \in \mathscr{B}\!\mathit{r}\,(I)$ as follows.

- If I is $\mathbb{R}_{\geqslant 0}$ or $[0,1]$, we let ρ be the right-continuous normalization of $\widetilde{\rho}$.
- If I is $\mathbb{R}$, we let ρ be the right-continuous normalization of $\widetilde{\rho} - \lim_{x \to -\infty} \widetilde{\rho}(x)$.

Then by Thm. 1.9.13 and Rem. 1.9.14, $d\rho$ and $d\widetilde{\rho}$ represent the same element of $C_c(I, \mathbb{F})^*$. By Thm. 2.9.6, the sequence $(d\rho_{n_k})$ converges weak-* to $d\rho$.

Theorem 4.2.20. *Assume Condition 4.2.13, and let $\rho \in \mathfrak{Kr}(I)$ be as in Def. 4.2.19. Then for each $m \in \mathbb{N}$, we have*

$$c_m = \int_I x^m d\rho$$

where the RHS is integrable.

Proof. The easiest case is where $I = [0,1]$. In that case, $x^m \in C_c(I)$. Therefore, by the weak-* convergence of $(d\rho_{n_k})$ to $d\rho$,

$$\int_I x^m d\rho = \lim_k \int_I x^m d\rho_{n_k} \xlongequal{(4.24)} \lim_k c_{n_k,m} \xlongequal{(4.22)} c_m$$

Next, we consider the case where $I = \mathbb{R}$. Let β_λ be as in Rem. 4.2.3, and let $\alpha_\lambda = 1 - \beta_\lambda$. Let

$$0 \leqslant \alpha_\lambda \leqslant \chi_{J_\lambda} \qquad \text{where } J_\lambda = (-\infty, -\lambda] \cup [\lambda, +\infty)$$

We first consider the case where m is even (so that $x^m \geqslant 0$). Then $\int_I x^m d\rho$ exists as an element of $\overline{\mathbb{R}}_{\geqslant 0}$. Moreover, we have

$$\lambda^2 \alpha_\lambda \cdot x^m \leqslant \lambda^2 \chi_{J_\lambda} \cdot x^m \leqslant x^{m+2} \tag{4.29}$$

Therefore,

$$\int_{\mathbb{R}} \alpha_\lambda \cdot x^m d\rho_{n_k} \leqslant \lambda^{-2} \int_{\mathbb{R}} x^{m+2} d\rho_{n_k}$$

By (4.21), the RHS above is equal to c_{m+2} when $1 + 2n_k \geqslant m$. Therefore,

$$\lim_{\lambda \to +\infty} \sup_k \int_{\mathbb{R}} \alpha_\lambda \cdot x^m d\rho_{n_k} = 0 \tag{4.30}$$

Since $\beta_\lambda \cdot x^m$ is compactly supported, by the weak-* convergence of $(d\rho_{n_k})$ to $d\rho$,

$$\int_{\mathbb{R}} \beta_\lambda \cdot x^m d\rho = \lim_k \int_{\mathbb{R}} \beta_\lambda \cdot x^m d\rho_{n_k} \tag{4.31}$$

By (4.30) and (4.31) and the fact that $\alpha_\lambda + \beta_\lambda = 1$, for each $\varepsilon > 0$, we have

$$\left| \int_{\mathbb{R}} \beta_\lambda \cdot x^m d\rho - \lim_k \int_{\mathbb{R}} x^m d\rho_{n_k} \right| \leqslant \varepsilon \qquad \text{for sufficiently large } \lambda$$

Therefore, by (4.21) and (4.22), we have

$$\left| \int_{\mathbb{R}} \beta_\lambda \cdot x^m d\rho - c_m \right| \leqslant \varepsilon \qquad \text{for sufficiently large } \lambda \tag{4.32}$$

Since $\beta_\lambda \cdot x^m$ is increasing and converging pointwise to x^m as $\lambda \nearrow +\infty$, by MCT, we conclude that $\int_{\mathbb{R}} x^m d\rho = c_m$. In particular, x^m is $d\rho$-integrable.

Next, assume that m is odd. Since $|x|^m \leqslant 1 + x^{m+1}$ and since both $1 = x^0$ and x^{m+1} have been proved to be $d\rho$-integrable, we conclude that x^m is $d\rho$-integrable.

Similar to (4.29), we have

$$\lambda\alpha_\lambda \cdot x^m \leqslant \lambda\chi_{J_\lambda} \cdot x^m \leqslant x^{m+1}$$

Therefore, a similar argument as above shows that for each $\varepsilon > 0$, (4.32) holds for the odd number m. By DCT, we conclude that $\int_{\mathbb{R}} x^m d\rho = c_m$.

The proof for the case that $I = \mathbb{R}_{\geqslant 0}$ is similar and is left to the reader. This case is even simpler, since $x^m \geqslant 0$ for all m. □

4.2.8 Padé approximation and the representation of divergence series

We continue to assume Condition 4.2.13. Let $(\rho_n)_{n\in\mathbb{N}}$ be described by Prop. 4.2.18. Let (ρ_{n_k}) be a subsequence as in Def. 4.2.19. That is, there exists $\rho \in \mathcal{R}r(I)$ such that $(d\rho_{n_k})$ converges weak-* to $d\rho$.

Theorem 4.2.21. *Let $f_n, f : \mathbb{C}\backslash I \to \mathbb{C}$ be the Stieltjes transforms of ρ_n and ρ, respectively. That is,*

$$f_n(z) = \int_I \frac{d\rho_n(x)}{z-x} \qquad f(x) = \int_I \frac{d\rho(x)}{z-x}$$

Then each $f_n(z)$ has the Laurent expansion

$$f_n(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_{2n+1}}{z^{2n+2}} + \frac{?}{z^{2n+3}} + \cdots \tag{4.33}$$

when $|z|$ is sufficiently large. Moreover, (f_{n_k}) converges locally uniformly on $\mathbb{C}\backslash I$ to f.

Therefore, the subsequence (f_{n_k}) provides a Padé approximation of f, and hence

$$f \sim \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots$$

Proof. Recall from Prop. 4.2.18 that $c_{m,n} = \int_I x^m d\rho_n$. Therefore, by the argument in Subsec. 4.1.3, $f_n(z)$ has the Laurent expansion $\sum_n c_{m,n} z^{-n-1}$. Combined with Rem. 4.2.17, this establishes (4.33).

For each $z \in \mathbb{C}\backslash I$, let $f_z \in C_0(I)$ be defined by $f_z(x) = (z-x)^{-1}$. For each $z \in \mathbb{C}\backslash I$, one easily checks that $\lim_{\zeta\to z} f_\zeta$ converges uniformly on I to f_z. Thus, we have a continuous map

$$\Phi : \mathbb{C}\backslash I \to C_0(I) \qquad z \mapsto f_z$$

where $C_0(I)$ is equipped with the l^∞-norm. Therefore, for each compact $K \subset \mathbb{C}\backslash I$, the family $\Phi(K)$ is compact in $C_0(I)$. Since $(d\rho_{n_k})$ converges weak-* to $d\rho$ as linear functionals on $C_0(I)$ (cf. Rem. 2.9.5), it follows from Thm. 4.2.22 below that $(d\rho_{n_k})$ converges uniformly to $d\rho$ when evaluated on functions in $\Phi(K)$. This proves that (f_{n_k}) converges locally uniformly to f. $\square$

Theorem 4.2.22. *Let $\mathcal{V}, \mathcal{W}$ be normed vector spaces. Let (T_α) be a net in $\mathfrak{L}(\mathcal{V}, \mathcal{W})$ satisfying $\sup_\alpha \|T_\alpha\| < +\infty$ and converging pointwise to some $T \in \mathfrak{L}(\mathcal{V}, \mathcal{W})$. Let K be a precompact subset of V. Then (T_α) converges uniformly on K to T. That is,*

$$\lim_\alpha \sup_{\xi \in K} \|T\xi - T_\alpha \xi\| = 0$$

Proof. Replacing K with $\overline{K}$, we assume that K is compact. Since $C := \sup_\alpha \|T_\alpha\| < +\infty$ is a uniform Lipschitz of (T_α), the family (T_α) is equicontinuous. Therefore, by Thm. 1.4.32, (T_α) converges uniformly on K to T. $\square$

4.3 Padé approximation via orthogonal polynomials

4.3.1 The setting

Fix $I \in \{\mathbb{R}, \mathbb{R}_{\geqslant 0}, [0,1]\}$, and choose a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying Condition 4.2.13. Moreover, we assume that the Hankel matrix H of (c_n) is positive-definite. Therefore, the triple (V, Ω, T) in Thm. 4.2.10 can be described as follows: V is the vector space $\mathbb{C}[x]$ together with the inner product determined by H, the cyclic vector Ω is chosen to be the constant 1, and T is the multiplication by x. We assume for simplicity that

$$c_0 = 1$$

Therefore, $\|\Omega\| = 1$.

Recall that E_n is the projection operator of V onto $V_n = \mathrm{Span}\{1, x, \dots, x^n\} = \mathrm{Span}\{\Omega, T\Omega, \dots, T^n\Omega\}$. By Rem. 4.2.15,

$$T_n := E_n T E_n\big|_{V_n}$$

is a self-adjoint operator on V_n. Note that

$$\dim V_n = n + 1$$

Recall (4.7) for the meaning of $\mathscr{R}r(I)$. Let $(\rho_n)_{n\in\mathbb{N}}$ be the sequence in $\mathscr{R}r(I)$ constructed in the proof of Prop. 4.2.18. That is,

$$\rho_n(x) = \sum_{\substack{\text{all } i \text{ satisfying} \\ \lambda_{n,i} \leqslant x}} |\langle e_{n,i} | \Omega \rangle|^2 \tag{4.34a}$$

where $e_{n,0}, \ldots, e_{n,n}$ form an orthonormal basis of V_n such that

$$T_n e_{n,i} = \lambda_{n,i} e_{n,i} \qquad \text{for all } i \tag{4.34b}$$

and $\lambda_{n,i} \in I$. As in Thm. 4.2.21, we let $f_n(z)$ be the Stieltjes transform of ρ_n, i.e.,

$$f_n(z) = \int_I \frac{d\rho_n(x)}{z - x}$$

Since ρ_n has finite range, $f_n(z)$ is a rational function.

4.3.2 Expressing $f_n(z)$ as a quotient of determinants

As shown in Thm. 4.2.21, a subsequence of (f_n) forms a Padé approximation to a holomorphic function f representing the series $\sum_{n\in\mathbb{N}} c_n z^{-n-1}$. The goal of this section is to provide an elementary description of f_n in terms of orthogonal polynomials. This description not only offers insight into the historical development of Padé approximation but is also essential for connecting Padé approximation to continued fractions, as we will see in the next section.

Proposition 4.3.1. *For sufficiently large $|z|$, we have*

$$f_n(z) = \big\langle \Omega \big| (z - T_n)^{-1} \Omega \big\rangle$$

Proof. By (4.34), we have $(z - T_n)^{-1} e_{n,i} = (z - \lambda_{n,i})^{-1} e_{n,i}$. Hence, the relation $\Omega = \sum_{i=0}^n e_{n,i} \cdot \langle e_{n,i} | \Omega \rangle$ implies

$$\big\langle \Omega \big| (z - T_n)^{-1} \Omega \big\rangle = \sum_i |\langle e_{n,i} | \Omega \rangle|^2 \cdot (z - \lambda_{n,i})^{-1} = \int_I \frac{d\rho_n(x)}{z - x}$$

□

Therefore, if we extend the unit vector Ω to an orthonormal basis of V_n, then by Cramer's rule, $f_n(z)$ can be expressed as a quotient

$$f_n(z) = \frac{\widetilde{q}_n(z)}{\widetilde{p}_{n+1}(z)} \tag{4.35}$$

where $\widetilde{p}_{n+1}(z) = \det(z - T_n)$, and $\widetilde{q}_n(z)$ is a minor of $z - T_n$ of order n. In particular, we have

$$\deg \widetilde{p}_{n+1} = n + 1 \qquad \deg \widetilde{q}_n = n$$

Remarkably, the sequence $(\widetilde{p}_n)_{n\in\mathbb{N}}$ turns out to be orthogonal polynomials, as we will explain below.

4.3.3 Orthogonal polynomials

Definition 4.3.2. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}[x]$. We say that (p_n) are **orthogonal polynomials** (resp. **orthonormal polynomials**) with respect to $(c_n)_{n\in\mathbb{N}}$ if the following conditions are satisfied:

(1) $\deg p_n = n$.

(2) (p_n) is orthogonal (resp. orthonormal) in the inner product space $\mathbb{C}[x]$ defined by the Hankel matrix H of c_n. Equivalently, $(p_n(T)\Omega)_{n\in\mathbb{N}}$ is an orthonormal (resp. orthogonal) basis of V.

Unless otherwise stated, we also assume that

$$\text{the leading coefficient of } p_n \text{ is} > 0 \tag{4.36}$$

We say that p_n is **monic** if the leading coefficient of p_n is 1.

Remark 4.3.3. The orthonormal polynomials (p_n) are uniquely determined by (c_n), and can be constructed by the Gram-Schmidt process. Since $c_0 = 1$, it is clear that

$$p_0 = 1$$

Moreover, since each c_n is real, the Gram-Schmidt process indicates that all the coefficients of p_n are real numbers.

Similarly, the monic orthogonal polynomials $(\widetilde{p}_n)$ are uniquely determined by (c_n). □

Remark 4.3.4. Moreover, if $\rho \in \mathfrak{Rr}(I)$ solves the polynomial moment Problem 4.2.1 for (c_n), then by (4.13), condition (2) of Def. 4.3.2 is equivalent to

$$\int_I \overline{p_m} p_n d\rho = \delta_{m,n} \qquad \text{for each } m, n \in \mathbb{N} \tag{4.37}$$

(But note that $\overline{p_m} = p_m$.) In that case, we also say that (ρ_n) are **orthonormal polynomials** with respect to ρ.

Theorem 4.3.5. *Let $\widetilde{p}_{n+1}(z) = \det(z - T_n)$ and $\widetilde{p}_0(z) = 1$. Then $(\widetilde{p}_n(x))_{n\in\mathbb{N}}$ are the (unique) monic orthogonal polynomials with respect to (c_n).*

Proof. We want to show that $\widetilde{p}_{n+1}$ is orthogonal to V_n, equivalently, that $\widetilde{p}_{n+1}(T)\Omega \perp V_n$.

Applying the Cayley-Hamilton theorem to T_n, we have $\widetilde{p}_{n+1}(T_n) = 0$, equivalently, $\widetilde{p}_{n+1}(E_nTE_n) = 0$. Therefore, if $\widetilde{p}_{n+1}(x) = \sum_{k=0}^{n+1} \gamma_k x^k$ where $\gamma_i \in \mathbb{R}$, then

$$\sum_{k=0}^{n+1} \gamma_k (E_nTE_n)^k \Omega = 0$$

Together with (4.23), this implies

$$\sum_{k=0}^{n+1}\gamma_k E_n T^k\Omega = 0$$

and hence $E_n\widetilde{p}_{n+1}(T)\Omega = 0$. This proves $\widetilde{p}_{n+1}(T)\Omega \perp V_n$. $\square$

Corollary 4.3.6. *We have*

$$\det(z - T_n) = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n+1} \\ c_1 & c_2 & \cdots & c_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n+1} \\ 1 & z & \cdots & z^{n+1} \end{vmatrix} \tag{4.38}$$

Proof. Denote the RHS by $\widetilde{p}_{n+1}(z)$. Then $\widetilde{p}_{n+1}(z)$ is a monic polynomial of degree $n+1$. Moreover, if we let $D_{i,j}$ be the (i,j)-th minor of the determinant on the RHS of (4.38), then for each $0 \leqslant k \leqslant n$, we have

$$\langle x^k|\widetilde{p}_{n+1}(x)\rangle = \sum_{i=0}^{n+1} c_{k+i}D_{n+2,i+1} = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n+1} \\ c_1 & c_2 & \cdots & c_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n+1} \\ c_k & c_{1+k} & \cdots & c_{n+k} \end{vmatrix} = 0$$

This proves that $\widetilde{p}_{n+1} \perp V_n$. $\square$

4.3.4 The Jacobi matrix

We now study the numerator $\widetilde{q}_n(z)$ in (4.35). As discussed in Subsec. 4.3.2, $\widetilde{q}_n(z)$ is a minor of $z - T_n$ of order n under the orthonormal basis $p_0, \dots, p_n$ of V_n. To compute this minor, let us find the matrix representation of T under this basis.

Definition 4.3.7. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ with $a_n > 0$ for each n. Define $J, J^+ \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ by

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ 0 & 0 & 0 & a_3 & b_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \qquad J^+ = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_3 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_4 & a_4 & \cdots \\ 0 & 0 & 0 & a_4 & b_5 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

The matrix J is called the **Jacobi matrix** for (a_n) and (b_n). We also call (a_n) and (b_n), the **off-diagonal sequence** and the **diagonal sequence** of J, respectively.

By this definition, J^+ is the Jacobi matrix for

$$(a_n)^+ = (a_1, a_2, \dots) \qquad (b_n)^+ = (b_1, b_2, \dots) \tag{4.39}$$

Theorem 4.3.8. *There is a bijection between:*

(1) A sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ where $c_n = 1$, and the associated Hankel matrix H is positive-definite.[2]

(2) A pair of sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ where $a_n > 0$ for each n.

The bijection is described as follows.

- *Given (c_n) satisfying (1), let $(p_n)_{n\in\mathbb{N}}$ be the orthonormal polynomials with respect to (c_n). Then the Jacobi matrix J for (a_n) and (b_n) is the matrix representation of $T : V \to V$ under (p_n).*

- *Given (a_n) and (b_n) satisfying (2), then c_n is the $(0,0)$-entry (i.e., the top-left entry) of J^n.*

The Jacobi matrix J for $(a_n), (b_n)$ is called the **Jacobi matrix associated to the Hankel matrix of (c_n)**.

Remark 4.3.9. The fact that J is the matrix representation of T under (p_n) can be made explicit as follows: The sequence $(p_n)_{n\in\mathbb{N}}$ satisfies the **three-term recurrence relation**

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1}p_{n-1}(x) \qquad \text{for all } n \in \mathbb{N} \tag{4.40a}$$

where we set $p_{-1}(x) = 0$, and let a_{-1} be any number. Rewriting this relation as

$$a_n p_{n+1}(x) = (x - b_n)p_n(x) - a_{n-1}p_{n-1}(x)$$

and noting that

$$p_{-1}(x) = 0 \qquad p_0(x) = 1 \tag{4.40b}$$

we see that (p_n) is uniquely determined by (a_n) and (b_n) through (4.40).

Proof of Thm. 4.3.8. Step 1. Given (c_n), let J be the matrix representation of T with respect to the orthonormal polynomials (p_n). Since $\deg p_k = k$, we have

$$xp_n(x) = a_n p_{n+1}(x) + b_n x^n + ?x^{n-1} + \cdots + ?x + ? \tag{4.41}$$

[2] In other words, (c_n) satisfies the assumptions in Subsec. 4.3.1 for the case $I = \mathbb{R}$.

where $a_{n+1} \in \mathbb{R}_{>0}$ and $b_n \in \mathbb{R}$. Therefore, J is of the form

$$J = \begin{pmatrix} b_0 & ? & ? & ? & ? & \cdots \\ a_0 & b_1 & ? & ? & ? & \cdots \\ 0 & a_1 & b_2 & ? & ? & \cdots \\ 0 & 0 & a_2 & b_3 & ? & \cdots \\ 0 & 0 & 0 & a_3 & b_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Since ω_T is Hermitian, J must be of the form given in Def. 4.3.7. This establishes the map

$$(c_n) \mapsto (a_n), (b_n) \tag{4.42}$$

equivalently, the map $H \mapsto J$. Moreover, we compute

$$c_n = \langle 1|x^n\rangle = \langle \Omega|T^n\Omega\rangle = \langle e_0|J^n e_0\rangle$$

This shows that c_n is the $(0,0)$-entry of J^n. Consequently, the map (4.42) is injective.

Step 2. It remains to prove that the map (4.42) is surjective. Let J be the Jacobi matrix of (a_n) and (b_n) satisfying (2). Let $C_c(\mathbb{N})$ be the set of functions $\mathbb{N} \to \mathbb{C}$ with finite supports. The inner product on $C_c(\mathbb{N})$ is chosen to be the one inherited from that of $l^2(\mathbb{N})$. Then J can be viewed as a linear operator on $C_c(\mathbb{N})$. Moreover, it is clear that

$$J^n\chi_{\{0\}} = a_0 \cdots a_{n-1}\chi_{\{n\}} + ?\chi_{\{n-1\}} + \cdots + ?\chi_{\{0\}}$$

where $a_0 \cdots a_{n-1}$ is understood as 1 if $n = 0$. Therefore, $(J^n\chi_{\{0\}})_{n\in\mathbb{N}}$ is a basis of $C_c(\mathbb{N})$, and there exists $p_n \in \mathbb{R}[x]$ satisfying

$$p_n(x) = (a_0 \cdots a_{n-1})^{-1}x^n + ?x^{n-1} + \cdots + ?x + ? \tag{4.43a}$$

$$\chi_{\{n\}} = p_n(J)\chi_{\{0\}} \tag{4.43b}$$

We define

$$c_n = \langle \chi_{\{0\}}|J^n\chi_{\{0\}}\rangle \tag{4.44}$$

Then the sesquilinear form $\langle\cdot|\cdot\rangle$ on $V = \mathbb{C}[x]$ determined by (c_n) satisfies

$$\langle g|h\rangle = \langle 1|\overline{g}h\rangle \xlongequal{(4.44)} \langle \chi_{\{0\}}|\overline{g}(J)h(J)\chi_{\{0\}}\rangle = \langle g(J)\chi_{\{0\}}|h(J)\chi_{\{0\}}\rangle \tag{4.45}$$

for each $g, h \in \mathbb{C}[x]$. (Note that in the last equality of (4.45), we have used the fact that $\langle \eta|J\xi\rangle = \langle J\eta|\xi\rangle$ for each $\xi, \eta \in C_c(\mathbb{N})$.) Therefore, $\langle g|g\rangle \geqslant 0$. If $g \neq 0$, by the fact that $(J^n\chi_{\{0\}})_{n\in\mathbb{N}}$ is a basis, we have $g(J)\chi_{\{0\}} \neq 0$, and hence

$$\langle g|g\rangle = \langle g(J)\chi_{\{0\}}|g(J)\chi_{\{0\}}\rangle > 0$$

Therefore, $\langle\cdot|\cdot\rangle$ is positive-definite. Thus, (c_n) satisfies (1).

Let us show that the map (4.42) sends (c_n) to $(a_n),(b_n)$. By (4.45), we have a unitary map

$$\Phi : V = \mathbb{C}[x] \to C_c(\mathbb{N}) \qquad g = g(T)\Omega \mapsto g(J)\chi_{\{0\}} \tag{4.46}$$

By (4.43b), Φ sends $\chi_{\{n\}}$ to p_n. Therefore, since $(\chi_{\{n\}})$ is an orthonormal basis of $C_c(\mathbb{N})$, the sequence $(p_n)_{n\in\mathbb{N}}$ is an orthonormal basis of V. This, together with (4.43a), shows that (p_n) is the orthogonal polynomials with respect to (c_n).

From (4.43), we have $J\Phi = \Phi T$. Therefore, J is the matrix representation of T under (p_n). This finishes the proof that (4.42) sends (c_n) to $(a_n),(b_n)$. $\square$

Corollary 4.3.10. *Let $(p_n)_{n\in\mathbb{N}}$ be the orthonormal polynomials with respect to (c_n), and let (a_n) be the off-diagonal sequence of the Jacobi matrix associated to (c_n). Then the leading coefficient of p_n is $(a_0\cdots a_{n-1})^{-1}$, understood to be 1 if $n=0$.*

Proof. This is clear from the proof of Thm. 4.3.8, especially from (4.43). $\square$

Corollary 4.3.11. *Let (a_n) and (b_n) be sequences in $\mathbb{R}$ with $a_n > 0$. Let J be the associated Jacobi matrix. Let $(p_n)_{n\in\mathbb{N}}$ be the sequence of polynomials satisfying (4.40). Then (p_n) can be described by*

$$a_0\cdots a_n\cdot p_{n+1}(z) = \det(z - J_{n+1}) \qquad p_0(z) = 1 \tag{4.47}$$

where $J_{n+1}\in\mathbb{R}^{(n+1)\times(n+1)}$ is the matrix formed by taking the first $n+1$ rows and columns of J.

Proof. By Thm. 4.3.8, J is the Jacobi matrix associated to the positive-definite Hankel matrix of a sequence (c_n) satisfying $c_0 = 1$, and (p_n) are the orthonormal polynomial with respect to (c_n). Therefore, J_{n+1} is the matrix representation of T_n with respect to the orthonormal basis $p_0,\dots,p_n$ of V_n. It follows from Thm. 4.3.5 that the monic orthonormal polynomials $(\widetilde{p}_n)$ satisfy $\widetilde{p}_{n+1}(z) = \det(z - J_{n+1})$. By Cor. 4.3.10, we have $\widetilde{p}_n = a_0\cdots a_n p_{n+1}(z)$. This finishes the proof. $\square$

4.3.5 The main theorem

Recall the sequence of rational functions $(f_n(z))_{n\in\mathbb{N}}$ as in Subsec. 4.3.1.

Theorem 4.3.12. *Let J be the Jacobi matrix associated to the Hankel matrix of (c_n), and let (b_n) and (a_n) be the diagonal and off-diagonal sequences of J, respectively. Choose sequences of polynomials $(p_n(x))_{n\in\mathbb{N}}$ and $(q_n(x))_{n\in\mathbb{N}}$ determined by*

$$xp_n(x) = a_np_{n+1}(x) + b_np_n(x) + a_{n-1}p_{n-1}(x) \qquad p_{-1}(x) = 0 \quad p_0(x) = 1 \tag{4.48}$$

$$xq_n(x) = a_{n+1}q_{n+1}(x) + b_{n+1}q_n(x) + a_nq_{n-1}(x) \qquad q_{-1}(x) = 0 \quad q_0(x) = \frac{1}{a_0} \tag{4.49}$$

for each $n\in\mathbb{N}$. Then

$$f_n(z) = \frac{q_n(z)}{p_{n+1}(z)} \tag{4.50}$$

Proof. Let J_{n+1} (resp. J_n^+) be the $(n+1)\times(n+1)$ (resp. $n\times n$) matrix formed by taking the first $n+1$ (resp. first n) rows and columns of J (resp. J^+). By the description of J in Thm. 4.3.8, J_n is the matrix representation of T_n with respect to the orthonormal basis $p_0,\dots,p_n$. Therefore, since $p_0 = 1 = \Omega$, by Prop. 4.3.1 and Cramer's rule (or the inverse matrix formula), we have

$$f_n(z) = \frac{\det(z - J_n^+)}{\det(z - J_{n+1})} \tag{4.51}$$

Applying Cor. 4.3.11 to the Jacobi matrix J (resp. J^+) and the sequence (p_n) (resp. $(a_0 q_n)$), we see that $\det(z - J_{n+1}) = a_0\cdots a_n p_{n+1}$ (resp. $\det(z - J_n^+) = a_1\cdots a_n\cdot a_0 q_n(z)$). This establishes (4.50). □

4.4 Padé approximation via continued fractions

We continued to work in the setting of Subsec. 4.3.1 and freely use the notations recalled there. In particular, we consider the sequence of rational functions $(f_n(z))_{n\in\mathbb{N}}$ described by $f_n(z) = \langle\Omega|(z-T_n)^{-1}\Omega\rangle$ (cf. Prop. 4.3.1), which admits a subsequence that Padé-approximates a holomorphic function f representing the (possibly) divergent series $\sum_{n\in\mathbb{N}} c_n z^{-n-1}$.

In this section, we use Thm. 4.3.12 to express (f_n) as finite approximations of a continued fraction. In [Sti94], Stieltjes's reasoning proceeds in the opposite direction: he begins with a continued fraction, derives the three-term recurrence relation satisfied by its finite approximants, and then uses orthogonality to obtain the Padé approximation.

Theorem 4.4.1. *Let J be the Jacobi matrix associated to the Hankel matrix of (c_n), and let (b_n) and (a_n) be the diagonal and off-diagonal sequences of J, respectively. Then $f_n(z)$ is the n-th approximation of the continued fraction*

$$\cfrac{1}{z - b_0 - \cfrac{a_0^2}{z - b_1 - \cfrac{a_1^2}{z - b_2 - \cfrac{a_2^2}{z - b_3 - \ddots}}}}$$

That is, for each $n\in\mathbb{N}$, we have

$$f_n(z) = \cfrac{1}{z - b_0 - \cfrac{a_0^2}{\ddots - \cfrac{a_{n-2}^2}{z - b_{n-1} - \cfrac{a_{n-1}^2}{z - b_n}}}} \tag{4.52}$$

Proof. By Thm. 4.3.12, we have $f_n = q_n/p_{n+1}$ where (p_n) and (q_n) satisfy

$$a_n p_{n+1}(z) = (z-b_n)p_n(z) - a_{n-1}p_{n-1}(z) \qquad p_{-1}(z)=0 \quad p_0(z)=1 \tag{4.53a}$$

$$a_n q_n(z) = (z-b_n)q_{n-1}(z) - a_{n-1}q_{n-2}(z) \qquad q_{-1}(z)=0 \quad q_0(z)=\frac{1}{a_0} \tag{4.53b}$$

In particular, we have $p_1(z)=(b_0-z)/a_0$, and hence $f_0(z)=1/(z-b_0)$. This proves (4.52) for $n=0$. Note that (4.53b) originally holds only when $n\geqslant 0$. However, by setting $a_{-1}=1$ and $q_{-2}(z)=-1$, Eq. (4.53) also holds when $n=0$.

We denote the RHS of (4.52) by Υ_n. We view $(p_n),(q_n),(\Upsilon_n)$ as sequences of rational functions of $z,a_0,b_0,a_1,b_1,\dots$. Assume that (4.52) holds for $n-1$ where $n\in\mathbb{Z}_+$, i.e.,

$$\frac{q_{n-1}}{p_n}=\Upsilon_{n-1}$$

Note that Υ_n is obtained from Υ_{n-1} by replacing b_{n-1} with $b_{n-1}+\frac{a_{n-1}^2}{z-b_n}$. To prove that (4.52) holds for n, it suffices to show that $a_n p_{n+1}/(z-b_n)$ (resp. $a_n q_n/(z-b_n)$) is also obtained from p_n (resp. q_{n-1}) by replacing b_{n-1} with $b_{n-1}+\frac{a_{n-1}^2}{z-b_n}$.

Note that

$$a_{n-1}p_n = (z-b_{n-1})p_{n-1} - a_{n-2}p_{n-2} \tag{4.54}$$

and p_{n-1} does not involve b_{n-1}. Replacing the b_{n-1} on the RHS of (4.54) with $b_{n-1}+\frac{a_{n-1}^2}{z-b_n}$, what we want to prove is

$$a_n p_{n+1}/(z-b_n) = a_{n-1}^{-1}\Big(z-b_{n-1}-\frac{a_{n-1}^2}{z-b_n}\Big)p_{n-1} - a_{n-1}^{-1}a_{n-2}p_{n-2}$$

equivalently,

$$a_{n-1}a_n p_{n+1} = \Big((z-b_{n-1})(z-b_n)-a_{n-1}^2\Big)p_{n-1} - a_{n-2}(z-b_n)p_{n-2}$$

But this follows from (4.53a) and (4.54), since

$$\begin{aligned}
&a_{n-1}a_n p_{n+1} = (z-b_n)\cdot a_{n-1}p_n - a_{n-1}^2 p_{n-1}\\
=&(z-b_n)\cdot\big((z-b_{n-1})p_{n-1}-a_{n-2}p_{n-2}\big) - a_{n-1}^2 p_{n-1}\\
=&\Big((z-b_{n-1})(z-b_n)-a_{n-1}^2\Big)p_{n-1} - a_{n-2}(z-b_n)p_{n-2}
\end{aligned}$$

This proves the desired property for $a_n p_{n+1}/(z-b_n)$. A similar argument proves the desired property for $a_n q_n/(z-b_n)$. □

4.5 Application: an alternative proof of the Riesz representation theorem

In Sec. 2.2.1, we noted that solving a moment problem and characterizing a dual space are often equivalent tasks. For the Hamburger and Stieltjes moment problems, the interval I is non-compact, so there is no suitable norm on $C(I)$ that would allow us to reformulate the moment problem as a dual space problem. In contrast, the Hausdorff moment problem can be translated into such a characterization because I is compact in this case.

Since we have already solved all types of polynomial moment problems in Thm. 4.2.9, it is natural to expect that this theorem also yields an alternative proof of the main part of the Riesz representation Thm. 1.10.1 for a compact interval I, namely, the classification of positive linear functionals on $C(I)$. This is exactly what we will do in this section.

For that purpose, Thm. 2.4.2 must be adapted so that the moment-problem interpretation of the classification of *bounded* linear functionals (discussed in Sec. 2.2.1 and relying crucially on Thm. 2.4.2) admits an analogue for *positive* linear functionals. This analogue is stated as Thm. 4.5.2. To prove it we first establish a preliminary result which abstracts the elementary fact that any positive integral operator is bounded whenever the constant function 1 is integrable.

Proposition 4.5.1. *Let X be a set. Let $\mathscr{A}$ be a unital *-$\mathbb{F}$-subalgebra of $l^\infty(X,\mathbb{F})$. Let $\Lambda:\mathscr{A}\to\mathbb{C}$ be a linear map which is* ***positive*** *in the sense that $\Lambda(f)\geqslant 0$ for each $f\geqslant 0$. Then Λ is bounded with operator norm $\leqslant\Lambda(1)$.*

Recall from Exp. 4.5.2 that the involution $*$ on $l^\infty(X,\mathbb{F})$ is defined by $f^*=\overline{f}$.

Proof. In the case that $\mathbb{F}=\mathbb{R}$, since $-\|f\|_{l^\infty}\leqslant f\leqslant\|f\|_{l^\infty}$, we have and since $\Lambda(a)=a\Lambda(1)$ for each scalar a, we obtain $-\|f\|_{l^\infty}\cdot\Lambda(1)\leqslant\Lambda(f)\leqslant\|f\|_{l^\infty}\cdot\Lambda(1)$, and hence $|\Lambda(f)|\leqslant\|f\|_{l^\infty}\cdot\Lambda(1)$.

In the case that $\mathbb{F}=\mathbb{C}$, by the positivity of Λ, the sesequilinear form

$$\mathscr{A}\times\mathscr{A}\to\mathbb{C}\qquad(f,g)\mapsto\Lambda(f^*g)$$

is positive. Therefore, by Cauchy Schwarz,

$$|\Lambda(f)|^2\leqslant\Lambda(f^*f)\Lambda(1)\leqslant\|f\|_{l^\infty}^2\cdot\Lambda(1)^2$$

where the last inequality is due to $f^*f\leqslant\|f\|_{l^\infty}^2$. □

Theorem 4.5.2. *Let X be a set. Let $\mathscr{A}$ be a unital *-$\mathbb{F}$-subalgebra of $l^\infty(X,\mathbb{F})$ with l^∞-closure $\overline{\mathscr{A}}$. Suppose that for each $f\in\overline{\mathscr{A}}$ satisfying $f\geqslant 0$, there exists $g\in\overline{\mathscr{A}}$ such that $f=\overline{g}g$ (i.e. $f=g^*g$). Then we have an $\overline{\mathbb{R}}_{\geqslant 0}$-linear isomorphism*

$$\begin{gathered}\{\textit{positive linear functionals on }\overline{\mathscr{A}}\}\xrightarrow{\simeq}\{\textit{positive linear functionals on }\mathscr{A}\}\\ \Lambda\mapsto\Lambda|_{\mathscr{A}}\end{gathered}\tag{4.55}$$

Note that $\overline{\mathscr{A}}$ is also a unital *-$\mathbb{F}$-subalgebra of $l^\infty(X, \mathbb{F})$.

Proof. By Prop. 4.5.1, positive linear functionals on $\overline{\mathscr{A}}$ are bounded. Therefore, by Thm. 2.4.2, they are determined by their restrictions to $\mathscr{A}$. Hence the map (4.55) is injective.

To prove that (4.55) is surjective, we pick any positive linear functional $\Lambda : \mathscr{A} \to \mathbb{F}$, which is bounded (by Prop. 4.5.1) and hence can be extended to a bounded linear functional $\Lambda : \overline{\mathscr{A}} \to \mathbb{F}$ (by Thm. 2.4.2). Suppose that $f \in \overline{\mathscr{A}}$ satisfies $f \geqslant 0$. By assumption, $f = \overline{g}g$ for some $g\overline{\mathscr{A}}$. Therefore, there is a sequence (g_n) in $\mathscr{A}$ converging uniformly to g. So $(\overline{g_n}g_n)$ converges uniformly to f. Since Λ is continuous, and since $\Lambda(\overline{g_n}g_n) \geqslant 0$, we conclude that $\Lambda(f) \geqslant 0$. This proves that $\Lambda : \overline{\mathscr{A}} \to \mathbb{F}$ is a positive linear functional. □

Theorem 4.5.3 (Riesz representation theorem). *Let I be a compact interval. Then positive linear functionals on $C(I)$ are precisely linear functionals of the form*

$$C(I) \to \mathbb{C} \qquad f \mapsto \int_I f d\rho$$

where $\rho : I \to \mathbb{R}_{\geqslant 0}$ is increasing.

Recall from Thm. 1.9.13 that replacing ρ with its right-continuous normalization does not change the values of the Stieltjes integrals. Therefore, in the above theorem, one may assume that ρ is right-continuous.

Proof. Clearly, any increasing function $\rho : I \to \mathbb{R}_{\geqslant 0}$ defines a positive linear functional. Conversely, let $\Lambda : C(I) \to \mathbb{C}$ be a positive linear functional. We assume WLOG that $I = [0, 1]$. Define a sequence $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ by

$$c_n = \Lambda(x^n)$$

Then, in view of Def. 4.2.6, the Hankel matrix H of (c_n) determines the sesquilinear form $\langle \cdot | \cdot \rangle$ on $\mathbb{C}[x]$ satisfying $\langle 1 | f \rangle = \Lambda(f)$ for all $f \in \mathbb{C}[x]$, and hence

$$\langle f | g \rangle = \langle 1 | f^* g \rangle = \Lambda(f^* g)$$

for all $f, g \in \mathbb{C}[x]$. Therefore, the positivity of Λ implies that $\langle \cdot | \cdot \rangle$ is positive. Moreover, for each $f \in \mathbb{C}[x]$ we have

$$\langle xf | f \rangle = \Lambda(xf^* f) \geqslant 0 \qquad \langle (1-x)f | f \rangle = \Lambda((1-x)f^* f) \geqslant 0$$

because xf^*f and $(1-x)f^*f$ belong to $C([0,1], \mathbb{R}_{\geqslant 0})$. Therefore, (c_n) satisfies the assumption of the Hausdorff moment problem. Hence, by Thm. 4.2.9, there exists an increasing $\rho : [0,1] \to \mathbb{R}_{\geqslant 0}$ such that $c_n = \int_I x^n d\rho$ for all n, and hence

$$\Lambda(f) = \int_I f d\rho$$

for all $f \in \mathbb{C}[x]$. In other words, the positive linear operators Λ and $\int_I d\rho$ agree on $\mathbb{C}[x]$. By Thm. 4.5.2, they agree on the closure of $\mathbb{C}[x]$, which is $C(I)$ by Stone-Weierstrass. □

Remark 4.5.4. The above proof of the Riesz representation theorem aligns perfectly with Table 2.3. The reason is that the core of the proof of Thm. 4.5.3 is the solution of the polynomial moment problem, namely, the proof of Thm. 4.2.9. Our proof of Thm. 4.2.9 in Sec. 4.2 (especially Subsec. 4.2.7) is an excellent illustration of Table 2.3: Principle 2.2.10 is verified by approximating the linear functional

$$\Lambda : \mathbb{C}[x] \to \mathbb{C} \qquad x^m \mapsto c_m$$

with a sequence (ρ_n) of increasing functions (or a subsequence thereof), where each ρ_n (defined in (4.28)) has finite range. The equivalence between pointwise convergence of functions and convergence of moments—that is, the equivalence of the two shaded areas in Table 2.3—is captured by Thm. 2.9.6, which is invoked in Def. 4.2.19 in the process of solving the polynomial moment problem.

It is worth noting that our use of (ρ_n) to approximate Λ is an instance of approximating the infinite by the finite. This is not only because each ρ_n has finite range, but also because the definition of ρ_n arises from the diagonalization of the finite-rank operator T_n given in (4.25). In other words, approximating Λ by ρ_n is, at its core, an approximation of T (described in Thm. 4.2.10) by T_n.

Riesz, in contrast, proved the Riesz representation theorem using the method of linear extension rather than finite approximation. As we will discuss in Ch. 5, Riesz's treatment of the Riesz representation theorem and the spectral theorem of bounded self-adjoint operators marked a paradigm shift in functional analysis: the transition from finite approximation to linear extension. This paradigm shift will be one of the key themes of this course. □

5 The spectral theorem for bounded self-adjoint operators

5.1 Hilbert's spectral theorem

5.1.1 Introduction

After Stieltjes' pioneering work on continued fractions [Sti94], the Stieltjes integral once again came into prominence with Hilbert's spectral theorem for bounded symmetric bilinear forms [Hil06]. It is fair to say that without Hilbert's discovery of the spectral theorem—and its subsequent refinement by later mathematicians, most notably F. Riesz—the Stieltjes integral might never have become the central and influential concept it is today. The reason modern readers are often unfamiliar with the Stieltjes integral is simply that its theory has been fully absorbed into modern measure theory. One should not forget that Stieltjes integrals are equivalent to integrals over intervals with respect to finite Borel measures.

The formulation of the spectral theorem for bounded self-adjoint operators on Hilbert spaces has undergone significant evolution throughout history. In this chapter, we will encounter four versions:

- Hilbert's original version.
- Riesz's version.
- The Borel functional calculus version.
- The multiplication operator version.

The relationships among these formulations are not immediate, nor are they obviously equivalent. In fact, to meaningfully compare them, we must take a practical perspective, that is, consider the problems that these versions can solve or help illuminate. Among them, the Borel functional calculus version and the multiplication operator version are the more modern and practically useful formulations. However, to fully appreciate their significance, we must not overlook their historical development, particularly Hilbert's and Riesz's versions, as well as the background of the polynomial moment problem discussed in the previous chapter.

In [Hil06], Hilbert introduced the Hilbert space $l^2(\mathbb{Z})$ and proved the spectral theorem for bounded Hermitian forms on it [Hil06, Satz 31]. We first present a formulation of this theorem, slightly adapted to modern terminology, and then provide some comments.

5.1.2 The spectral theorem of Hilbert

Theorem 5.1.1 (Hilbert's spectral theorem). *Let V be a separable inner product space, and let $\omega \in \mathfrak{Ses}(V)$ be Hermitian. Choose $r \geqslant 0$ such that $\|\omega\| \leqslant r$. Then for each $\xi \in V$, there is an increasing function $\rho_\xi : [-r, r] \to \mathbb{R}_{\geqslant 0}$ such that for $z \in \mathbb{C}$ satisfying $|z| > r$, the resolvent form $(z - \omega)^{-1} \in \mathfrak{Ses}(V)$ satisfies*

$$(z-\omega)^{-1}(\xi|\xi) = \int_{[-r,r]} \frac{d\rho_\xi(\lambda)}{z-\lambda} \tag{5.1}$$

The resolvent form $(z - \omega)^{-1}$ will be defined in the proof, and is related to the resolvent operator introduced in Cor. 3.5.16. The precise relationship will be discussed after the proof.

Proof. Let $u_1, u_2, \ldots$ be an orthonormal basis of V. Similar to Def. 4.2.14, we let

$$V_n = \mathrm{Span}\{u_1, \ldots, u_n\} \qquad V_\infty = \bigcup_{n \in \mathbb{Z}_+} = \mathrm{Span}\{u_1, u_2, \ldots\}$$

We shall first establish (5.1) for ξ in dense subset of V, and then extend it to all $\xi \in V$.

Step 1. Similar to (4.25), we let $\omega_n \in \mathfrak{Ses}(V_n)$ be the restriction of ω to $V_n \times V_n$. Then ω_n is Hermitian, and satisfies $\|\omega_n\| \leqslant r$. Hence $\omega_n = \omega_{T_n}$ for some self-adjoint $T_n \in \mathrm{Lin}(V_n)$, that is,

$$\omega_n(\xi|\eta) = \langle \xi | T_n \eta \rangle \qquad \text{for each } \xi, \eta \in V_n$$

and $\|T_n\| \leqslant r$.

By the spectral theorem for Hermitian operators on finite-dimensional inner product spaces, T_n is diagonal. Therefore, there exist $\lambda_{n,1}, \ldots, \lambda_{n,n} \in \mathbb{R}$ and an orthonormal basis $e_{n,1}, \ldots, e_{n,n}$ of V_n such that

$$T_n e_{n,i} = \lambda_{n,i} e_{n,i} \qquad \text{for each } 1 \leqslant i \leqslant n$$

Since $\|T\| \leqslant r$, we have $|\lambda_{n,i}| = \|Te_{n,i}\| \leqslant \|T\| \leqslant r$, and hence

$$\lambda_{n,1}, \ldots, \lambda_{n,n} \in [-r, r]$$

Step 2. Let $\mathbb{K} = \mathbb{Q} + \mathbf{i}\mathbb{Q}$. Let U be a dense $\mathbb{K}$-linear subspace of V_∞ with countable cardinality, e.g.

$$U = \mathrm{Span}_{\mathbb{K}}\{u_1, u_2, \ldots\}$$

In this step, we define $(z - \omega)^{-1}$ as a bounded $\mathbb{K}$-sesquilinear form on U admitting an integral representation as in (5.1). Roughly speaking, $(z - \omega)^{-1}$ on U will be defined to be the limit of a convergent subsequence of the sesquilinear forms associated to $(z-T_n)^{-1}$. The details are as follows.

For each $\xi \in V_\infty$, we have $V_n \ni \xi$ for sufficiently large n. Moreover, for such n, we clearly have $(z-T_n)^{-1}\xi = \sum_{i=1}^n (z-\lambda_{n,i})^{-1} e_{n,i} \cdot \langle e_{n,i}|\xi\rangle$, and hence

$$\langle \eta|(z-T_n)^{-1}\xi\rangle = \sum_{i=1}^n (z-\lambda_{n,i})^{-1}\langle \xi|e_{n,i}\rangle \cdot \langle e_{n,i}|\eta\rangle \qquad \text{for each } \eta \in V \tag{5.2}$$

Similar to (4.28), for each $\xi \in V_\infty$, we let $\rho_{\xi,n} : [-r,r] \to \mathbb{R}_{\geqslant 0}$ be

$$\rho_{\xi,n}(x) = \sum_{\substack{\text{all } i \text{ satisfying} \\ \lambda_{n,i} \leqslant x}} |\langle e_{n,i}|\xi\rangle|^2 \qquad \text{for all } V_n \ni \xi \tag{5.3}$$

We set $\rho_{\xi,n} = 0$ if $V_n \not\ni \xi$. Then $(\rho_{\xi,n})_{n\in\mathbb{Z}_+}$ is a uniformly bounded sequence, since Bessel's inequality implies $0 \leqslant \rho_{\xi,n}(x) \leqslant \|\xi\|^2$. It follows from (5.2) that

$$\langle \xi|(z-T_n)^{-1}\xi\rangle = \int_{[-r,r]} \frac{d\rho_{\xi,n}(\lambda)}{z-\lambda} \qquad \text{for all } V_n \ni \xi$$

By Helly's selection Thm. 2.9.3 and the diagonal method (Rem. 1.4.16), there exist strictly positive integers $n_1 < n_2 < \dots$ such that [1]

$$\lim_k \rho_{\xi,n_k} \text{ converges pointwise to some function } \rho_\xi \qquad \text{for each } \xi \in U$$

with $0 \leqslant \rho_\xi \leqslant \|\xi\|^2$. By Thm. 2.9.6, for each $\xi \in U$, the sequence $(d\rho_{\xi,n_k})$ converges weak-* to $d\rho_\xi$ as positive linear functionals on $C([-r,r])$. Therefore

$$\lim_k \langle \xi|(z-T_{n_k})^{-1}\xi\rangle = \int_{[-r,r]} \frac{d\rho_\xi(\lambda)}{z-\lambda} \qquad \text{for all } \xi \in U \tag{5.4}$$

By the polarization identity (cf. Rem. 3.1.3), $\lim_k \langle \xi|(z-T_{n_k})^{-1}\eta\rangle$ converges for each $\xi,\eta \in U$. We can thus define a $\mathbb{K}$-sesquilinear form

$$(z-\omega)^{-1} : U \times U \to \mathbb{C} \qquad (z-\omega)^{-1}(\xi|\eta) = \lim_k \langle \xi|(z-T_{n_k})^{-1}\eta\rangle \tag{5.5}$$

By (5.4), the relation (5.1) is satisfied for all $\xi \in U$. Therefore, for each $\xi \in U$, since $0 \leqslant \rho_\xi \leqslant \|\xi\|$, we have

$$\big|(z-\omega)^{-1}(\xi|\xi)\big| \leqslant \|\xi\|^2 / \inf_{\lambda\in[-r,r]} |z-\lambda| \tag{5.6}$$

[1] When Hilbert wrote [Hil06], the Helly selection theorem had not yet been discovered. His argument proceeded as follows. He first applied the Arzelà-Ascoli theorem to obtain a uniformly convergent subsequence of the antiderivatives of $(\rho_{\xi,n})$. The limit of this subsequence is a convex function, and its derivative (which exists outside a countable set) is then taken as the definition of ρ_ξ.

By the proof of Prop. 3.2.12, we conclude that $(z-\omega)^{-1}$ is a bounded $\mathbb{K}$-sesquilinear form.

Step 3. From the proof of Thm. 2.4.2, we know that $(z-\omega)^{-1}$ can be uniquely extended to a bounded ($\mathbb{C}$-)sesquilinear form on V. We know that (5.1) holds for all $\xi \in U$. Let us establish the integral representation (5.1) for any $\xi \in V$.

Suppose that $\xi \in V\backslash U$. Let (ξ_n) be a sequence in U converging to ξ. From the above proof, we know that $0 \leqslant \rho_{\xi_n} \leqslant \|\xi_n\|$. In particular, the sequence (ρ_{ξ_n}) is bounded. By the Helly selection theorem, (ρ_{ξ_n}) has a subsequence $(\rho_{\xi_{n_k}})$ converging pointwise to some increasing $\rho : [-r,r] \to \mathbb{R}_{\geqslant 0}$. By Thm. 2.9.6, $(d\rho_{\xi_{n_k}})$ converges to $d\rho_\xi$ when integrated against the function $\lambda \in [-r,r] \mapsto 1/(z-\lambda)$. This establishes (5.1) for all $\xi \in V$. □

5.1.3 Q&A

We give some comments on Hilbert's spectral theorem (Thm. 5.1.1) in the form of Q&A.

Question 5.1.2. Assume that the inner product space V in Thm. 5.1.1 is a Hilbert space, so that there is a canonical isomorphism $\mathfrak{L}(V) \simeq \mathfrak{Ses}(V)$ (cf. Thm. 3.5.6). Write $\omega = \omega_T$ where $T \in \mathfrak{L}(V)$ is self-adjoint. Is $(z-\omega)^{-1}$ equal to the bounded sesquilinear form associated to the resolvent operator $(z-T)^{-1}$?

Answer. Yes, but this is not immediate. Let us first summarize how $(z-\omega)^{-1}$ is constructed in the proof of Thm. 5.1.1, now using the language of bounded operators.

Choose $r \in \mathbb{R}$ such that $r \geqslant \|\omega| = \|T\|$.As in Def. 4.2.14, we let E_n be the projection of V onto $V_n = \{u_1, \dots, u_n\}$. Then E_nTE_n is a self-adjoint operator on V with operator norm $\leqslant r$. For $|z| > r$, the limit

$$\lim_n (z - E_nTE_n)^{-1} \tag{5.7}$$

converges in WOT. However, this fact was not known at the time of Hilbert's work [Hil06].

As seen in the proof of Thm. 5.1.1, Hilbert's idea was instead to show that for each $\xi \in V$, a subsequence of (5.7) converges when evaluated in $\langle \xi | - \xi \rangle$. One then selects a subsequence that converges simultaneously on a sufficiently large countable set of vectors, and finally uses the uniform boundedness of the operator norms of $(z-E_nTE_n)^{-1}$to conclude convergence on all pairs of vectors in V. (Compare this with Prop. 2.4.5.)

In summary, $(z-\omega)^{-1}$ is defined to be the bounded sesquilinear form obtained as the limit along a suitable subsequence:

$$(z-\omega)^{-1}(\xi|\eta) = \lim_k \langle \xi | (z - E_{n_k}TE_{n_k})^{-1}\eta \rangle \tag{5.8}$$

where the subsequence is independent of the choice of z, ξ, η.

Thus, the question reduces to whether $(z-\omega)^{-1}$ coincides with the bounded sesquilinear form associated to $(z-T)^{-1}$. The answer is yes, once one shows that

$$\lim_n (z - E_n T E_n)^{-1} = (z-T)^{-1} \tag{5.9}$$

in SOT. This will be proved in Pb. 5.2. □

Question 5.1.3. The proofs of Hilbert's spectral theorem (Thm. 5.1.1) and of the polynomial moment problems (Thm. 4.2.9) share many similarities. For instance, both make use of finite-rank approximations of Hermitian operators of the form $E_n T E_n \to T$; in both cases, the increasing function ρ_n for $E_n T E_n$ is constructed by diagonalizing $E_n T E_n$; and in both, the increasing function ρ for T is obtained by taking a pointwise convergent subsequence of (ρ_n).

In view of these similarities, what exactly are the novelties in Hilbert's proof of his spectral theorem?

Answer. As noted in Subsec. 4.2.1, Stieltjes' treatment of the polynomial moment problem did not rely on the diagonalization theory of linear algebra. He obtained Padé approximations not via finite-rank approximations of Hermitian operators, but through continued fractions and detailed analyses of determinants and polynomials. It was Hilbert's work that established the connection between the polynomial moment problem, inner product spaces, and spectral theory, thereby allowing us to approach polynomial moment problems from the perspective of spectral theory, as developed in Ch. 4.

I should also mention that although the resolvent form $(z-\omega)^{-1}$ considered by Hilbert bears a striking resemblance to the Stieltjes transforms arising in the polynomial moment problem with divergent series as background (see Thm. 4.2.21)—indeed, this similarity justifies viewing the resolvent of an operator T as its Stieltjes transform—the notion of resolvent in functional analysis actually first appeared in Fredholm's study of integral equations [Fre03].

In [Fre03], Fredholm sought to analyze the solutions of integral equations of the form

$$f(x) + \int_0^1 K(x,y)f(y) = g(y)$$

where K, g are given continuous functions and f is the unkown solution. Fredholm considered the resolvent of the integral operator $S : C([0,1]) \to C([0,1])$ defined by $(Sf)(x) = \int_0^1 K(x,y)f(y)$. As in Question 5.1.2, this resolvent was not defined directly as the inverse operator of $z-S$, but rather as the limit of $(z-S_n)^{-1}$, where (S_n) is a sequence of finite-rank matrices (with increasing ranks) obtained by partioning the interval $[0,1]$. Moreover, the inverse $(z-S_n)^{-1}$ was expressed in terms of determinants, thanks to Cramer's rule/the inverse matrix formula. Fredholm studied the invertibility of $z - S$ by analyzing the zeros of the holomorphic function $\Delta(z) := \lim_n \det(z - S_n)$.

Thus, another major novelty of Hilbert's proof of the spectral theorem was the way it connected Fredholm's notion of the resolvent, developed in the study of integral equations, with the Stieltjes transforms arising from the polynomial moment problem and divergent series. □

Question 5.1.4. I noticed that the more modern versions of the spectral theorem we will encounter later (such as Riesz's version, the Borel functional calculus version, and the multiplication operator version) are more powerful and widely applicable than Hilbert's spectral theorem. In fact, Hilbert's version seems more like a special case of these modern results. So what is the significance of studying the proof of Hilbert's spectral theorem?

Answer. The proofs of modern spectral theorems share a common trait: they rely heavily on sophisticated algebraic machinery, often employing the Riesz representation theorem (for spaces of continuous functions) as a "black box" at critical junctures. Studying these proofs alone can leave learners puzzled:

- How did mathematicians first realize that the Riesz representation theorem could be used to prove spectral theorems?
- Why is it applied in this specific way?

While Hilbert's spectral theorem is less general than its modern counterparts, its proof avoids this complex abstraction. Instead, the connection with the Riesz representation theorem occupies most of the argument: As mentioned in Question 5.1.3, the proofs of Hilbert's spectral theorem (Thm. 5.1.1) and of the Hausdorff moment problem (cf. Thm. 4.2.9) run in close parallel. And as discussed in Subsec. 2.2.1 and Sec. 4.5, the latter problem is almost equivalent to classifying positive linear functionals on $C(I)$ for a compact interval I.

Thus, the real significance of Hilbert's spectral theorem is that it makes transparent why the Riesz representation theorem enters spectral theory in the first place. Hilbert's proof, the earliest version of the spectral theorem, can be viewed as a linear-algebraic reinterpretation of all the key steps in the proof of the Hausdorff moment problem—and hence as an almost equivalent reformulation of the Riesz representation theorem for $C(I)$. The additional layers found in modern spectral theorems—those not directly tied to the moment problem/Riesz representation paradigm—were introduced later, as part of the refinement and expansion of the theory. □

5.2 Towards Riesz's spectral theorem: projections

5.2.1 Hilbert's spectral theorem holds for inner product spaces

It is often said that one of the main differences between mathematicians and physicists in their approaches to the mathematics of quantum mechanics is that mathematicians stress the completeness of Hilbert spaces, emphasizing that they are more than just inner product spaces, whereas physicists find the notion of completeness largely irrelevant. Mathematicians commonly justify this emphasis by pointing out that the spectral theorem for self-adjoint operators requires completeness. This is certainly true for the spectral theorems developed after Hilbert. However, as we saw in Thm. 5.1.1, Hilbert's own spectral theorem already holds for general inner product spaces.—If this observation causes a degree of unease for the reader, then I have achieved my aim.

The fact that Hilbert's spectral theorem holds for all inner product spaces, while later versions hold only for Hilbert spaces, shows that Hilbert's version is less powerful and can therefore be established under weaker assumptions. Even so, completeness still plays a role in Hilbert's theorem—though not the Cauchy completeness of the inner product space. Rather, it is the weak-* completeness of Stieltjes integrals against increasing functions, already noted in Table 2.4. Since Hilbert's spectral theorem is the ancestor of all later versions, we may conclude that the truly central analytic condition underlying all spectral theorems is not the Cauchy completeness of Hilbert space (or, equivalently, the Riesz isomorphism $\mathcal{H} \simeq (\mathcal{H}^{\complement})^*$, cf. Thm. 3.5.4), but instead the weak-* completeness of increasing functions/finite Borel measures.

5.2.2 Why Riesz's spectral theorem requires Hilbert spaces

Riesz's spectral theorem, proved in [Rie13, Ch. V], is a significant improvement over Hilbert's. Beginning with this section, we prepare for the introduction of Riesz's version.

An important drawback of Hilbert's spectral theorem is that it is unclear how the increasing function ρ_ξ in Thm. 5.1.1 relies on ξ. The first highlight of Riesz's spectral theorem is that, under the assumption that $\mathcal{H}$ is a Hilbert space and $T \in \mathfrak{L}(\mathcal{H})$ is self-adjoint with $\|T\| \leqslant r$, he realizes ρ_ξ through an increasing net of projections $(E(\lambda))_{\lambda\in[-r,r]}$ associated to T (called the **spectral projections** of T), more precisely:

$$\langle \xi|E(\lambda)\xi\rangle = \rho_\xi(\lambda) \qquad \text{for all } \lambda \in [-r,r] \text{ and } \xi \in \mathcal{H}$$

As we will learn in this section, for each Hilbert space $\mathcal{H}$ there is an order-preserving bijection between projection operators and closed linear subspaces of $\mathcal{H}$, related by $P \mapsto \mathrm{Rng}(P)$. Therefore, Riesz's replacement of scalar-valued increasing functions ρ_ξ with projection-valued increasing functions E brings the geometry of Hilbert spaces into the formulation of the spectral theorem. In fact, when $\dim \mathcal{H} < +\infty$, the subspace associated to $E(\lambda)$ is spanned by eigenvectors of T with eigenvalues $\leqslant \lambda$.

Interestingly, even for non-complete inner product spaces, the correspondence $P \mapsto \mathrm{Rng}(P)$ is still injective (cf. the proof of Thm. 5.2.3), which is sufficient for the spectral theorem. The essential reason Riesz's spectral theorem applies only to Hilbert spaces and not to general inner product spaces is that he constructs $E(\lambda)$ by first constructing its associated sesquilinear form $\omega_{E(\lambda)}$, as we will see in Sec. 5.4. However, if an inner product space V is not complete, there are even no natural injective maps from the set of **projection forms** (i.e., bounded Hermitian forms ω satisfying $\omega \circ \omega = \omega$, cf. Def. 3.5.14 for the definition of $\omega \circ \omega$) to the set of linear subspaces of V.

To turn a projection form into a projection operator, one needs the isomorphism $\mathfrak{Ses}(\mathcal{H}) \simeq \mathfrak{L}(\mathcal{H})$ in Thm. 3.5.6, which holds only for Hilbert spaces $\mathcal{H}$ due to the Riesz-Fréchet Thm. 3.5.4. This is one major reason why completeness is essential in Riesz's spectral theorem—though not in the form of Cauchy completeness, but rather through the duality $\mathcal{H} \simeq (\mathcal{H}^{\complement})^*$.

In Sec. 5.4, we will see another fundamental way in which the duality $\mathcal{H} \simeq (\mathcal{H}^{\complement})^*$ enters Riesz's proof of the spectral theorem, once again through the isomorphism $\mathfrak{Ses}(\mathcal{H}) \simeq \mathfrak{L}(\mathcal{H})$.

5.2.3 Projections

Definition 5.2.1. A **projection operator** (or simply a **projection**) on an inner product V is an element $P \in \mathfrak{L}(V)$ such that ω_P is Hermitian (i.e. $\langle \xi | P\eta \rangle = \langle P\xi | \eta \rangle$ for all $\xi, \eta \in V$) and $P^2 = \mathrm{id}_V$. It is easy to check that

$$P^\perp := 1 - P$$

is also a projection.

Recall that if V is a Hilbert space, ω_P being Hermitian is equivalent to $P^* = P$.

Example 5.2.2. Let U be a linear subspace of an inner product space V, and suppose that V has a projection onto U. Then by Cor. 3.3.20, the projection operator associated to U is a projection.

In what follows, we will mainly discuss projections on Hilbert spaces, although many of the results extend naturally to general inner product spaces.

Theorem 5.2.3. *Let $\mathcal{H}$ be a Hilbert space. We have a bijection*

$$\{\textit{projections on } \mathcal{H}\} \xrightarrow{\simeq} \{\textit{closed linear subspaces of } \mathcal{H}\} \qquad P \mapsto \mathrm{Rng}(P) \tag{5.10}$$

Moreover, P is the projection operator associated to $\mathrm{Rng}(P)$ in the sense of Def. 3.3.9. That is, for each $\xi \in \mathcal{H}$, we have $P\xi \in \mathrm{Rng}(P)$ and $\xi - P\xi \in \mathrm{Rng}(P)^\perp$.

Proof. If P is a projection on $\mathcal{H}$, then clearly $1-P$ is also a projection (i.e. $1-P$ is self-adjoint and $(1-P)^2 = 1-P$). Moreover, we have

$$\mathrm{Rng}(P) = \mathrm{Ker}(1-P) \tag{5.11}$$

Indeed, for each $\xi \in \mathcal{H}$ we have $(1-P)P\xi = P\xi - P^2\xi = 0$ and hence $\mathrm{Rng}(P) \subset \mathrm{Ker}(1-P)$; if $(1-P)\xi = 0$, then $\xi = P\xi$, and hence $\mathrm{Ker}(1-P) \subset \mathrm{Rng}(P)$. This proves $\mathrm{Rng}(P) = \mathrm{Ker}(1-P)$. We have thus proved that $\mathrm{Rng}(P)$ is a closed linear subspace of $\mathcal{H}$, since the kernel of any bounded linear operator is a close linear subspace (Cor. 2.3.10).

If $\xi \in \mathcal{H}$, then clearly $P\xi \in \mathrm{Rng}(P)$. For each $\eta \in \mathcal{H}$, we have $\langle P\eta|\xi - P\xi\rangle = \langle \eta|P\xi - P^2\xi\rangle = 0$. Thus $\xi - P\xi \in \mathrm{Rng}(P)^\perp$. This proves that P is the (unique) projection associated to $\mathrm{Rng}(P)$. In particular, P is determined by $\mathrm{Rng}(P)$, and hence the map (5.10) is injective. The surjectivity follows from Thm. 3.4.8. □

Recall Def. 3.5.10 for the meaning of $A \leqslant B$ where A, B are bounded self-adjoint operators on a Hilbert space. The following property says that the bijection (5.10) is an isomorphism of partially ordered sets.

Theorem 5.2.4. *Let P, Q be projections on a Hilbert space $\mathcal{H}$. The following are equivalent.*

(1) $\mathrm{Rng}(P) \subset \mathrm{Rng}(Q)$.

(2) $QP = P$.

(2') $PQ = P$.

(3) $P \leqslant Q$, *namely,* $\langle\xi|P\xi\rangle \leqslant \langle\xi|Q\xi\rangle$ *for all* $\xi \in \mathcal{H}$.

(3') $\|P\xi\| \leqslant \|Q\xi\|$ *for all* $\xi \in \mathcal{H}$.

Proof. (1)⇒(2): Assume (1). Choose any $\xi \in \mathcal{H}$. Since $P\xi \in \mathrm{Rng}(Q)$, and since (by (5.11) or (3.12)) $\mathrm{Rng}(Q) = \mathrm{Ker}(Q-1)$, we have $(Q-1)P\xi = 0$. This proves (2).

(2)⇔(2'): This follows from $(QP)^* = PQ$.

(3)⇔(3'): This follows from $\|P\xi\|^2 = \langle P\xi|P\xi\rangle = \langle\xi|P^*P\xi\rangle = \langle\xi|P\xi\rangle$ and, similarly, $\|Q\xi\|^2 = \langle\xi|Q\xi\rangle$.

(2')⇒(3'): Since $\xi = P\xi + (1-P)\xi$ where $P\xi \perp (1-P)\xi$, by the Pythagorean identity, we have $\|P\xi\| \leqslant \|\xi\|$. Replacing ξ with $Q\xi$, we get $\|PQ\xi\| \leqslant \|Q\xi\|$. Therefore, (3') follows from (2').

(3')⇒(1'): Assume (3'). Then $\|Q\xi\| = 0$ implies $\|P\xi\| = 0$, i.e., $\mathrm{Ker}(Q) \subset \mathrm{Ker}(P)$. Therefore, $\mathrm{Ker}(P)^\perp \subset \mathrm{Ker}(Q)^\perp$. By (3.12), we have $\mathrm{Ker}(P)^\perp = \mathrm{Rng}(P)$ and $\mathrm{Ker}(Q)^\perp = \mathrm{Rng}(Q)$. This proves (1'). □

Corollary 5.2.5. *Let P, Q be projections on a Hilbert space $\mathcal{H}$. The following are equivalent.*

(1) $\mathrm{Rng}(P) \perp \mathrm{Rng}(Q)$.

(2) $QP = 0$.

(3) $PQ = 0$.

(4) $P + Q \leqslant 1$.

Proof. Note that (1) is equivalent to $\mathrm{Rng}(P) \subset \mathrm{Rng}(Q)^\perp$. By (3.12), we have $\mathrm{Rng}(Q)^\perp = \mathrm{Rng}(1-Q)$. Thus (1) is equivalent to $\mathrm{Rng}(P) \subset \mathrm{Rng}(1-Q)$, and hence (by Thm. 5.2.4) is equivalent to each of the following three conditions: $P(1-Q) = 0$, $(1-Q)P = 0$, and $P \leqslant 1 - Q$. □

Remark 5.2.6. Let P, Q be projections on a Hilbert space such that $P \leqslant Q$. Then $Q - P$ is the projection operator associated to $\mathrm{Rng}(Q) \cap \mathrm{Rng}(P)^\perp$, that is,

$$(Q-P)(\mathcal{H}) = Q(\mathcal{H}) \cap P(\mathcal{H})^\perp$$

Proof. Let $\mathcal{K} = Q(\mathcal{H})$. Since $P(\mathcal{H}) \subset \mathcal{K}$, $P|_\mathcal{K}$ and $Q|_\mathcal{K}$ are bounded linear operators on $\mathcal{K}$, with $Q|_\mathcal{K}$ being the identity operator of $\mathcal{K}$ and $P|_\mathcal{K}$ being the projection of $\mathcal{K}$ onto $P(\mathcal{H})$. By (3.12), the range of $\mathrm{id}_\mathcal{K} - P|_\mathcal{K} = Q|_\mathcal{K} - P|_\mathcal{K}$ (which equals $(Q-P)(\mathcal{H})$ because $(Q-P)Q = Q^2 - PQ = Q - P$) equals the orthogonal complement of $P(\mathcal{H})$ in $\mathcal{K}$. □

The convergence of an increasing net of projections also has a geometric meaning:

Theorem 5.2.7. *Let $(E_\alpha)_{\alpha\in I}$ be a net of projections on $\mathcal{H}$. Assume that (E_α) is increasing, i.e., $E_\alpha \leqslant E_\beta$ whenever $\alpha \leqslant \beta$. Let E be the projection operator such that*

$$\mathrm{Rng}(E) = \mathrm{Cl}_\mathcal{H}\Big(\bigcup_{\alpha\in I} \mathrm{Rng}(E_\alpha)\Big) \tag{5.12}$$

Then $E_\alpha \leqslant E$ for each α, and $\lim_\alpha E_\alpha$ converges in SOT to E.

Consequently, if an increasing net of projections (E_α) converges in WOT to some $F \in \mathfrak{L}(\mathcal{H})$, then clearly $F = E$. It follows that F is a projection, that (E_α) in SOT to E, and that $\mathrm{Rng}(F) = \mathrm{Cl}_\mathcal{H}\big(\bigcup_{\alpha\in I}\mathrm{Rng}(E_\alpha)\big)$.

Proof. Since $\mathrm{Rng}(E_\alpha) \subset \mathrm{Rng}(E)$, we have $E_\alpha \leqslant E$. Let $\xi \in \mathcal{H}$. By the definition of E, for each $\varepsilon > 0$ there exists $\alpha \in I$ such that

$$\|E\xi - E_\alpha\eta\| \leqslant \varepsilon$$

for some $\eta \in \mathcal{H}$. Since $E\xi - E_\alpha\eta = (E - E_\alpha)\xi + E_\alpha(\xi - \eta)$ with $(E-E_\alpha)\xi$ orthogonal to $E_\alpha(\xi - \eta)$ (cf. Rem. 5.2.6), the Pythagorean identity shows that $(E - E_\alpha)\xi\| \leqslant \|E\xi - E_\alpha\eta\|$ and hence

$$\|(E - E_\alpha)\xi\| \leqslant \varepsilon$$

Therefore, for each $\beta \geqslant \alpha$, we have $E - E_\beta \leqslant E - E_\alpha$ due to Rem. 5.2.6, and hence $\|(E - E_\beta)\xi\| \leqslant \varepsilon$. This proves $\lim_\alpha E_\alpha\xi = E\xi$. □

Corollary 5.2.8. *Let $(E_\alpha)_{\alpha\in I}$ be a decreasing net of projections on $\mathcal{H}$. Let E be the projection operator onto*

$$\mathrm{Rng}(E)=\bigcap_{\alpha\in I}\mathrm{Rng}(E_\alpha)$$

Then $E_\alpha \geqslant E$ for each α, and $\lim_\alpha E_\alpha$ converges in SOT to E.

Proof. Apply Thm. 5.2.7 to the increasing net $(E_\alpha^\perp)$ and use the following Exe. 5.2.9. □

Exercise 5.2.9. Let $(\mathcal{K}_\alpha)_{\alpha\in\mathscr{A}}$ be a family of closed linear subspaces of $\mathcal{H}$. Prove that

$$\mathrm{Cl}_{\mathcal{H}}\Big(\bigcup_\alpha \mathcal{K}_\alpha\Big)^\perp=\bigcap_\alpha \mathcal{K}_\alpha^\perp \qquad \Big(\bigcap_\alpha \mathcal{K}_\alpha\Big)^\perp=\mathrm{Cl}_{\mathcal{H}}\Big(\bigcup_\alpha \mathcal{K}_\alpha^\perp\Big) \tag{5.13}$$

(Hint: use Cor. 3.4.9.)

5.3 Towards Riesz's spectral theorem: monotone convergence extension

5.3.1 Paradigm shift: from finite approximation to linear extension

Another of Riesz's innovations on Hilbert's spectral theorem is his entirely different approach to the polynomial moment problem/the Riesz representation theorem. One year after his proof of the spectral theorem in [Rie13], Riesz gave a new proof of the Riesz representation theorem in [Rie14]; this proof draws on key steps from his treatment of the spectral theorem in [Rie13] and simplifies the method he originally used in [Rie11].

We will not discuss [Rie11], since it offers little insight for this course. Instead we compare the method used in [Rie13, Rie14]—which I call the monotone convergence extension—with the Stieltjes-Hilbert method for treating the moment problem. (Recall again that the Riesz representation theorem for $C(I)$, with I a compact interval, is roughly equivalent to the Hausdorff moment problem, cf. the answer to Question 5.1.4.) The transition from the Stieltjes-Hilbert method to Riesz's method marked a paradigm shift in the early development of functional analysis: the move from finite approximations to linear extensions. We explain this in more detail below.

As discussed in Sec. 2.2 and summarized in Table 2.3, the traditional approach to moment problems and to characterizing dual spaces proceeds in two steps: (1) establish the link between pointwise convergence of functions and convergence of moments; (2) show that any bounded (or positive) linear functional can be approximated in the weak-* topology by elementary functions. The treatment of polynomial moment problems in Ch. 4 exemplifies this strategy: the connection in step (1) is captured by Thm. 2.9.6 together with the Helly selection theorem (Thm. 2.9.3), while step (2) is achieved via Padé approximation, implemented as finite-rank approximations of Hermitian operators. This approach clearly belongs to the paradigm of finite approximation.

This finite-approximation paradigm gave way to the linear-extension paradigm, with F. Riesz as its prime mover. In this course we will exhibit two main patterns of the linear-extension paradigm:

- The **monotone convergence extension**, the main subject of this section (and treated in greater detail in [Gui-A, Ch. 24-25]), which is closely tied to integration theory.
- The **bounded linear extension**, which is intimately connected with convexity in normed spaces and with various forms of the Hahn-Banach theorem.

The monotone convergence extension may be regarded as a gift from integration theory to functional analysis. Whereas Lebesgue developed integration by first defining measurable sets, the approach of monotone convergence extension—originally introduced by Young [You10, You13] as an alternative to Lebesgue's approach that appealed to more conservative contemporaries [2]—builds the integral by enlarging the class of integrable functions in such a way that the integral satisfies the monotone convergence theorem.

5.3.2 Monotone convergence extension as a theorem

Fix $\mathbb{K} \in \{\mathbb{R}_{\geqslant 0}, \mathbb{R}, \mathbb{C}\}$. Let X be a topological space. Recall from Sec. 1.1 that

$$\mathscr{Bor}^{\mathrm{b}}(X, \mathbb{K}) = \{\text{bounded Borel functions } X \to \mathbb{K}\} \tag{5.14}$$

As usual, $\mathscr{Bor}^{\mathrm{b}}(X)$ denotes $\mathscr{Bor}^{\mathrm{b}}(X, \mathbb{C})$.

Definition 5.3.1. A positive linear functional $\Lambda : \mathscr{Bor}^{\mathrm{b}}(X, \mathbb{K}) \to \mathbb{K}$ is called **normal** [3] if it satisfies the monotone convergence theorem, that is, if (f_n) is an increasing sequence in $\mathscr{Bor}^{\mathrm{b}}(X, \mathbb{K})$ converging pointwise to $f \in \mathscr{Bor}^{\mathrm{b}}(X, \mathbb{K})$, then

$$\lim_n \Lambda(f_n) = \Lambda(f)$$

Proposition 5.3.2. *We have a bijective map*

$$\begin{aligned} \{\textit{finite Borel measures on } X\} &\xrightarrow{\simeq} \{\textit{normal positive linear functionals}\} \\ \mu &\mapsto \Lambda_\mu \end{aligned} \tag{5.15}$$

where $\Lambda_\mu(f) = \int_X f d\mu$ for each $f \in \mathscr{Bor}^{\mathrm{b}}(X, \mathbb{K})$.

[2] See [Pes, Sec. 6.6].

[3] This terminology is borrowed from the theory of von Neumann algebras.

Proof. Given each finite Borel measure μ, Λ_μ satisfies the MCT. Therefore the map (5.15) is well-defined. Since Λ_μ is determined by its values on $\mathscr{Bor}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$, and since each $f \in \mathscr{Bor}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ is the pointwise limit of an increasing sequence of simple functions, Λ_μ must be determined by the values $\Lambda_\mu(\chi_E) = \mu(E)$ for all any Borel set $E \subset X$. Therefore, the map (5.15) is injective.

To prove that (5.15) is surjective, we pick an arbitrary normal positive linear functional $\Lambda : X \to \mathbb{K}$. Then Λ being normal implies that $\mu : E \in \mathfrak{B}_X \mapsto \Lambda(\chi_E) \in \mathbb{R}_{\geqslant 0}$ is a (Borel) measure on X. So Λ and Λ_μ agree on simple functions. Since both Λ and Λ_μ and satisfy MCT, by the argument in the first paragraph, we conclude $\Lambda = \Lambda_\mu$. □

Prop. 5.3.2, which gives us a linear functional interpretation of measure theory, allows us to formulate the Riesz-Markov representation theorem (Thm. 1.7.6) for second-countable compact Hausdorff spaces in the form of monotone convergence extension.

Theorem 5.3.3 (Riesz-Markov representation theorem). *Let X be a second-countable compact Hausdorff space. Then we have an $\mathbb{R}_{\geqslant 0}$-linear isomorphism*

$$\begin{gathered}\{\textit{normal positive linear functionals } \mathscr{Bor}^{\mathrm{b}}(X,\mathbb{F}) \to \mathbb{F}\} \\ \downarrow \simeq \\ \{\textit{positive linear functionals } C(X,\mathbb{F}) \to \mathbb{F}\} \\ \Lambda \mapsto \Lambda|_{C(X,\mathbb{F})}\end{gathered} \tag{5.16}$$

Note that for second-countable compact Hausdorff spaces, finite Borel measures and finite Radon measures are synonymous (cf. Thm. 1.7.8).

Proof. This follows immediately from Thm. 1.7.6 and Prop. 5.3.2. □

Corollary 5.3.4 (Abstract Hausdorff moment theorem). *Let X be a second-countable compact Hausdorff space. Let $\mathscr{A}$ be a unital *-$\mathbb{F}$-subalgebra of $C(X,\mathbb{F})$ separating points of X. Then we have an $\mathbb{R}_{\geqslant 0}$-linear isomorphism*

$$\begin{gathered}\{\textit{normal positive linear functionals } \mathscr{Bor}^{\mathrm{b}}(F,\mathbb{K}) \to \mathbb{F}\} \\ \downarrow \simeq \\ \{\textit{positive linear functionals } \mathscr{A} \to \mathbb{F}\} \\ \Lambda \mapsto \Lambda|_{\mathscr{A}}\end{gathered} \tag{5.17}$$

Proof. By Stone-Weierstrass, the l^∞-closure of $\mathscr{A}$ is $C(X,\mathbb{F})$. Therefore, the corollary follows from Thm. 5.3.3 and 4.5.2. □

Remark 5.3.5. Suppose that X is a compact Hausdorff space, not necessarily second-countable. A positive linear functional $\Lambda : \mathscr{Bor}^{\mathrm{b}}(X,\mathbb{F}) \to \mathbb{F}$ is called **Radon** if there exists a (necessarily finite) Radon measure μ on X such that

$$\Lambda(f) = \int_X f d\mu \qquad \text{for all } f \in \mathscr{Bor}^{\mathrm{b}}(X,\mathbb{F})$$

It is clear that Thm. 5.3.3 and Cor. 5.3.4 can be generalized to this situation, with normal positive linear functionals replaced by Radon positive linear functionals.

5.3.3 Monotone convergence extension as a method

The connection between Thm. 5.3.3 and the monotone convergence extension is straightforward: the theorem asserts that every positive linear functional on $C(X,\mathbb{F})$ extends uniquely to a positive linear functional on $\mathscr{Bor}^{\mathrm{b}}(X,\mathbb{F})$ satisfying the monotone convergence theorem. However, the monotone convergence extension is not only the statement of a theorem, but also provides the mechanism for constructing the proof.

In what follows, we outline how this method of monotone convergence extension is applied to prove Thm. 5.3.3. For simplicity, we restrict attention to an $\mathbb{R}_{\geqslant 0}$-linear functional $\Lambda : C(X,\mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$, and explain how such a linear functional is extended.

The first main step is to extend Λ to an $\mathbb{R}_{\geqslant 0}$-linear functional on

$$\mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0}) = \{\text{bounded lower semicontinuous functions } X \to \mathbb{R}_{\geqslant 0}\}$$

by the following procedure of **monotone convergence extension**:

(1) For each $f \in \mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$, set

$$\Lambda(f) = \sup\{\Lambda(h) : h \leqslant f, h \in C(X,\mathbb{R}_{\geqslant 0})\}$$

(2) Prove the following version of the MCT: If (f_n) is a uniformly bounded increasing sequence in $\mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ converging pointwise to $f : X \to \mathbb{R}_{\geqslant 0}$. Then $\Lambda(f) = \lim_n \Lambda(f_n)$. (Note that $f \in \mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ by Prop. 1.4.8.)

(3) Show that any $f \in \mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ is the pointwise limit of an increasing sequence of functions in $C(X,\mathbb{R}_{\geqslant 0})$. Together with Step 2, this implies that the extended Λ is still $\mathbb{R}_{\geqslant 0}$-linear.

Readers familiar with measure theory will recognize that this is the same method used to define the integral on a measure space: one extends the integral from nonnegative simple functions to nonnegative measurable functions such that the MCT is satisfied. However, in [Rie13, Rie14], Riesz employed an equivalent but seemingly different procedure:

(a) Show that any $f \in \mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ is the pointwise limit of an increasing sequence of functions (f_n) in $C(X,\mathbb{R}_{\geqslant 0})$.

(b) Define $\Lambda(f)$ to be $\lim_n \Lambda(f_n)$ where (f_n) is any increasing sequence in $C(X,\mathbb{R}_{\geqslant 0})$ converging pointwise to $f \in \mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$.

(c) Show that $\Lambda(f)$ is well-defined, i.e., independent of the choice of (f_n) approximating f. (The linearity of Λ is obvious.)

Step (c) plays the role of Step (2) mentioned above, since the arguments required in both cases are essentially the same.

The above approach (adapted to nets so that it applies to general locally compact Hausdorff spaces) is used in [Gui-A, Ch. 25] to prove the Riesz-Markov representation Thm. 1.7.6. To complete the proof, **the second main step** is of course to extend Λ from $\mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ to $\mathscr{Bor}^{\mathrm{b}}(X,\mathbb{F})$. In [Gui-A], this extension is carried out via a more measurable-set–based approach rather than the monotone convergence extension. Nevertheless, it is possible to proceed using monotone convergence extension as follows.

First, extend Λ from $\mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$ to

$$\mathscr{C}_0=\{f+h: f\in\mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0}),h\geqslant 0,h=0\text{ almost everywhere}\}$$

by setting $\Lambda(f+h)=0$ (with "almost everywhere" interpreted appropriately [4]). Then apply steps (a)–(c) above to extend Λ from $\mathscr{C}_0$ to the class $\mathscr{C}_1$ of functions that are pointwise limits of increasing sequences in $\mathscr{C}_0$. Finally, using Lem. 1.7.5 and Eq. (1.36), extend Λ to $\mathscr{C}_2:=\mathrm{Span}_{\mathbb{F}}(\mathscr{C}_1)$. One then verifies that $\mathscr{C}_2$ coincides with the space of bounded measurable functions.

The above approach was in fact used by Riesz and Sz.-Nagy to construct the Lebesgue integral on a compact interval $I\subset\mathbb{R}$, though with nonnegative step functions in place of $\mathrm{LSC}^{\mathrm{b}}(X,\mathbb{R}_{\geqslant 0})$. See [RN, Sec. 16-22] or [Apo, Ch. 10]. However, to prove the Riesz representation theorem for $C(I,\mathbb{F})$—that is, to represent a positive linear functional

$$\Lambda: C(I,\mathbb{F})\to\mathbb{F}$$

by a Stieltjes-integral against an increasing function $\rho: I\to\mathbb{R}_{\geqslant 0}$—it suffices to extend Λ to $\mathrm{LSC}^{\mathrm{b}}(I,\mathbb{F})$ using the monotone convergence extension described in the first main step (namely, (a)-(c), or equivalently, (1)-(3)). By Lem. 1.7.5 and Eq. (1.36), Λ is further extended to $\mathrm{Span}_{\mathbb{F}}\mathrm{LSC}^{\mathrm{b}}(I,\mathbb{F})$, which then allows us to define a desired increasing function $\rho: I=[a,b]\to\mathbb{R}_{\geqslant 0}$ by

$$\rho(x)=\Lambda(\chi_{[a,x]})$$

where $\chi_{I_{\leqslant x}}$ is an upper semicontinuous function (and hence lies to $\mathrm{Span}_{\mathbb{F}}\mathrm{LSC}^{\mathrm{b}}(I,\mathbb{F})$). The MCT shows that this function ρ is right-continuous. This is precisely the approach taken by Riesz in [Rie13] to prove the Riesz representation theorem in the form needed for his spectral theorem.

For a more detailed discussion of the monotone convergence extension method, see [Gui-A, Ch. 25].

[4] One defines "almost everywhere" by defining a set $E\subset X$ to be null if it is contained in some open set $U\subset X$ with $\Lambda(\chi_U)=0$.

5.4 Riesz's spectral theorem: two paradigm shifts

5.4.1 Three paradigm shifts, and why Riesz's spectral theorem is related to the first two

The theme of this course is the three major paradigm shifts in functional analysis:

1. From finite approximations to linear extensions.

2. From (muti)linear forms to linear operators.

3. From duality to Cauchy completeness.

Riesz's proof of the spectral theorem in [Rie13] was a major milestone for the first two shifts. We have already discussed the first shift in detail in Sec. 5.3; we now turn to the second.

As discussed before (for instance, in Sec. 2.1 and 2.5), functional analysis moved its focus from scalar-valued functions (especially linear functionals and bilinear or sesquilinear forms) to vector-valued functions (linear operators acting on a normed or inner-product space V). This is the second paradigm shift mentioned above.

We have seen that Hilbert's spectral theorem (Thm. 5.1.1) is stated in the language of bilinear/sesquilinear forms. As noted in Sec. 5.2, one reason Riesz framed his spectral theorem in the language of linear operators is that projection operators correspond more naturally to linear subspaces of an inner-product space than do projection forms. Another reason—mentioned in Subsec. 2.5.2—is that symbolic calculus is easier to manipulate in the operator framework, i.e. one may replace the real or complex variable x in a function $f(x)$ by an operator or a sesquilinear form. We will explore this in more detail in this section and in Sec. 5.5. For now, we answer some questions that readers might naturally ask.

Question 5.4.1. What's the role played by symbolic calculus in Riesz's spectral theorem?

Answer. Let $T \in \mathfrak{L}(\mathcal{H})$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Starting with the polynomial functional calculus, i.e. the linear map

$$\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H}) \qquad x^n \mapsto T^n$$

Riesz applied the Riesz representation theorem to extend π_T to a homomorphism

$$\pi_T : \mathrm{SpanLSC}^{\mathrm{b}}([-r,r]) \to \mathfrak{L}(\mathcal{H})$$

where $r \in \mathbb{R}_{\geqslant 0}$ satisfies $r \geqslant \|T\|$. This extended linear map is not only linear but also multiplicative (i.e. $\pi_T(fg) = \pi_T(f)\pi_T(g)$) and preserves the involutions (i.e.

$\pi_T(\overline{f}) = \pi_T(f)^*$). Therefore, since for each $\lambda \in [-a, a]$ we have $\overline{\chi_{[a,\lambda]}} = \chi_{[a,\lambda]}$ and $\chi_{[a,\lambda]}^2 = \chi_{[a,\lambda]}$, the operator

$$E(\lambda) := \pi_T(\chi_{[a,\lambda]})$$

is a projection. This yields the construction of the spectral projections mentioned in Subsec. 5.2.2. □

Note that the above answer also explains how the Riesz representation theorem is used in the proof of Riesz's spectral theorem.

Question 5.4.2. Why does symbolic/functional calculus require the linear-operator perspective rather than the bilinear/sesquilinear form perspective?

Answer. Of course, to perform symbolic/functional calculus one must first define multiplication of operators, sesquilinear forms, or matrices. As noted in Subsec. 5.2.2, multiplication of bounded sesquilinear forms or bounded matrices can be defined as in Def. 3.5.14; this historical approach was indeed the one originally adopted by mathematicians.

The principal complexity with the sesquilinear-form and matrix perspectives—which does not arise in finite-dimensional linear algebra—is not their definition but the associativity of multiplication: one must address the Fubini-type issues for infinite sums appearing in (3.33) and (3.34). For bounded sesquilinear forms and bounded matrices, defining products in terms of orthonormal bases introduces many inconveniences that are absent in the finite-dimensional setting, whereas the operator framework avoids this subtlety. □

5.4.2 Riesz's spectral theorem, and why the third paradigm shift is missing

In the rest of this section, we fix a Hilbert space $\mathcal{H}$.

Definition 5.4.3. Let I be an interval. An **increasing net of projections** indexed by I is defined to be a function

$$E : I \to \{\text{projections on } \mathcal{H}\}$$

such that $E(\lambda) \leqslant E(\mu)$ (cf. Thm. 5.2.4) for all $\lambda, \mu \in I$ satisfying $\lambda \leqslant \mu$. We say that E is **right-continuous**, if for each $\lambda \in [a, b)$ we have

$$\lim_{\lambda \to \lambda_0} E(\lambda) = E(\lambda_0) \tag{5.18}$$

By Cor. 5.2.8, the SOT and WOT of the above limit are equivalent; moreover, this limit is equivalent to

$$\mathrm{Rng}(E(\lambda_0)) = \bigcap_{\lambda > \lambda_0} \mathrm{Rng}(E(\lambda))$$

The following theorem was proved in [Rie13, Ch. V, Sec. 94].

Theorem 5.4.4 (Riesz's spectral theorem). *Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint. Let $r \in \mathbb{R}_{\geqslant 0}$ such that $\|T\| \leqslant r$. Then there exists a right-continuous increasing net of projections $E : I \to \mathfrak{L}(\mathcal{H})$ (called the* ***spectral projections****) such that for any $f \in C([-r, r])$, the continuous functional calculus $f(T)$ satisfies*

$$f(T) = \int_{[-r,r]} f(\lambda) dE(\lambda) \tag{5.19}$$

Right-continuity is not essential; it is imposed to ensure uniqueness of the net E satisfying (5.19). We will not need this uniqueness in the course.

Neither the continuous functional calculus nor the integral on the RHS of (5.19) has been defined yet. We will do this in the following subsections.

5.5 Borel functional calculus

5.6 Problems

In this section, we fix Hilbert spaces $\mathcal{H}, \mathcal{K}$.

Definition 5.6.1. The **strong-* operator topology (SOT*)** on $\mathfrak{L}(\mathcal{H}, \mathcal{K})$ is defined as the pullback topology along the map

$$\mathfrak{L}(\mathcal{H}, \mathcal{K}) \to \mathfrak{L}(\mathcal{H}, \mathcal{K}) \times \mathfrak{L}(\mathcal{K}, \mathcal{H}) \qquad T \mapsto (T, T^*)$$

where the RHS carries the product topology of SOT on $\mathfrak{L}(\mathcal{H}, \mathcal{K})$ and on $\mathfrak{L}(\mathcal{K}, \mathcal{H})$, respectively. Equivalently, a net (T_α) in $\mathfrak{L}(\mathcal{H}, \mathcal{K})$ converges to $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$ in SOT* iff $(T_\alpha) \to T$ in SOT and simultaneously $(T_\alpha^*) \to T$ in SOT.

Problem 5.1. Let (T_α) be a net of bounded normal operators on $\mathcal{H}$. Let $T \in \mathfrak{L}(\mathcal{H})$ be normal. Assume that (T_α) converges in SOT to T. Prove that (T_α) converges in SOT* to T.

Hint. Recall Rem. 3.7.6. □

Problem 5.2. Let (T_α) be a net of bounded normal operators on $\mathcal{H}$ satisfying $\sup_\alpha \|T_\alpha\| \leqslant R$ for some $R \in \mathbb{R}_{\geqslant 0}$. Let $f \in C(\overline{B}_{\mathbb{C}}(0, R))$. Assume that (T_α) converges in SOT to a normal operator $T \in \mathfrak{L}(\mathcal{H})$. Prove that

$$\lim_\alpha f(T_\alpha) = f(T) \qquad \text{in SOT} \tag{5.20}$$

6 Unbounded operators

7 From completely continuous forms to compact operators

8 Convexity and bounded linear extensions

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