Topics in Operator Algebras: Algebraic Conformal Field Theory

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0 Notations

$\mathbb{N} = \{0, 1, 2, \dots\}$. $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. $\overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \leqslant r\}$. $\mathbb{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$.

If X is a complex manifold, we let $\mathscr{O}(X)$ denote the set of holomorphic functions $f : X \to \mathbb{C}$.

Unless otherwise stated, an **unbounded operator** $T : \mathcal{H} \to \mathcal{K}$ (where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces) denotes a linear map from a dense linear subspace $\mathscr{D}(T) \subset \mathcal{H}$ to $\mathcal{H}$. $\mathscr{D}(T)$ is called the **domain** of T. We let T^* be the adjoint of T. In practice, we are also interested in T^* defined on a dense subspace of its domain $\mathscr{D}(T^*)$. We call its restriction a **formal adjoint** of T and denote it by $T^\dagger$.

Given a Hilbert space $\mathcal{H}$, its inner product is denoted by $(\xi, \eta) \in \mathcal{H}^2 \mapsto \langle \xi | \eta \rangle$. We assume that it is linear on the first variable and antilinear on the second one. (Namely, we are following mathematician's convention.)

Whenever we write $\langle \xi, \eta \rangle$, we understand that it is linear on both variables. E.g. $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual space.

If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, we let

$$\mathfrak{L}(\mathcal{H}, \mathcal{K}) = \{\text{Bounded linear maps } \mathcal{H} \to \mathcal{K}\} \qquad \mathfrak{L}(\mathcal{H}) = \mathfrak{L}(\mathcal{H}, \mathcal{H}) \tag{0.1}$$

If V, W are vector spaces, we let

$$\mathrm{Hom}(V, W) = \{\text{Linear maps } V \to W\} \qquad \mathrm{End}(V) = \mathrm{Hom}(V, V) \tag{0.2}$$

An unbounded operator $T : \mathcal{H} \to \mathcal{H}$ denotes a linear map $\mathscr{D}(T) \to \mathcal{H}$ where $\mathscr{D}(T)$ is a dense linear subspace of $\mathcal{H}$. We say that an unbounded operator T is **continuous** if it is continuous with respect to the norms on the domain and the codomain. Thus, "bounded" means continuous and $\mathscr{D}(T) = \mathcal{H}$.

If $z_\bullet = (z_1, \dots, z_k)$ are mutually commuting formal variables, for each $n_\bullet = (n_1, \dots, n_k) \in \mathbb{Z}^k$ we let

$$z_\bullet^{n_\bullet} = z_1^{n_1} \cdots z_k^{n_k}$$

For each vector space W, we let

$$W[[z_\bullet]] = \Big\{ \sum_{n_\bullet \in \mathbb{N}^k} w_{n_\bullet} z_\bullet^{n_\bullet} \Big\} \qquad W[[z_\bullet^{\pm 1}]] = \Big\{ \sum_{n_\bullet \in \mathbb{Z}^k} w_{n_\bullet} z_\bullet^{n_\bullet} \Big\}$$

$$W((z_\bullet)) = \Big\{ \sum_{n_\bullet \in \mathbb{Z}^k} w_{n_\bullet} z_\bullet^{n_\bullet} : w_{n_\bullet} = 0 \text{ when } n_1, \dots, n_k \ll 0 \Big\}$$

$$W[z_\bullet] = W((z_\bullet)) \cap W((z_\bullet^{-1})) = \text{polynomials of } z_\bullet \text{ with } W\text{-coefficients}$$

where $w_{n_\bullet} \in W$.

If X is a set, the n-fold **configuration space** $\mathrm{Conf}^n(X)$ is

$$\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X : x_i \neq x_j \text{ if } i \neq j\} \tag{0.3}$$

Definition 0.1. A map of complex vector spaces $T : V \to V'$ is called **antilinear** or **conjugate linear** if $T(a\xi + b\eta) = \overline{a}T\xi + \overline{b}T\eta$ for all $\xi, \eta \in V$ and $a, b \in \mathbb{C}$. If V and V' are (complex) inner product spaces, we say that T is **antiunitary** if it is am antiliear surjective and satisfies $\|T\xi\| = \|\xi\|$ for all $\xi \in V$, equivalently,

$$\langle T\xi|T\eta\rangle = \overline{\langle\xi|\eta\rangle} \equiv \langle\eta|\xi\rangle \tag{0.4}$$

for all $\xi, \eta \in V$.

For each $n \in \mathbb{Z}$, we let $\mathfrak{e}_n \in C^\infty(\mathbb{S}^1)$ be $\mathfrak{e}_n(z) = z^n$.

1 Introduction: PCT symmetry, Bisognano-Wichmann, Tomita-Takesaki

Algebraic quantum field theory (AQFT) is a mathematically rigorous approach to QFT using the language of functional analysis and operator algebras. The main subject of this course is 2d **algebraic conformal field theory (ACFT)**, namely, 2d CFT in the framework of AQFT.

1.1

Let $d \in \mathbb{Z}_+$. We first sketch the general picture of an $(1+d)$ dimensional Poincaré invariant QFT in the spirit of **Wightman axioms**. We consider Bosonic theory for simplicity.

We let $\mathbb{R}^{1,d}$ be the $(1+d)$-dimensional **Minkowski space**. So it is $\mathbb{R}^{1+d}$ but with metric tensor

$$ds^2 = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^d)^2 \tag{1.1}$$

Here x^0 denotes the time coordinate, and $x^1, \ldots, x^d$ denote the spatial coordinates. The (restricted) **Poincaré group** is

$$\mathrm{P}^+(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}^+(1,d)$$

Here, $\mathbb{R}^{1,d}$ acts by translation on $\mathbb{R}^{1,d}$. $\mathrm{SO}^+(1,d)$ is the (restricted) **Lorentz group**, the identity component of the (full) Lorentz group $\mathrm{O}(1,d)$ whose elements are invertible linear maps on $\mathbb{R}^{1,d}$ preserving the Minkowski metric.

Remark 1.1. Any $g \in \mathrm{O}(1,d)$ must have determinent ± 1. One can show that $\mathrm{SO}^+(1,d)$ is precisely the elements $g \in \mathrm{O}(1,d)$ such that $\det g = 1$, and that g does not change the direction of time (i.e., if $\mathbf{v} = (v_0, \ldots, v_d) \in \mathbb{R}^{1,d}$ satisfies $v_0 > 0$, then the first component of $g\mathbf{v}$ is > 0). See [Haag, Sec. I.2.1].

Definition 1.2. We say that $\mathbf{x} = (x_0, \ldots, x_d), \mathbf{y} = (y_0, \ldots, y_d) \in \mathbb{R}^{1,d}$ are **spacelike (separated)** if their Minkowski distance is negative, i.e.,

$$(x_0 - y_0)^2 < (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$$

1.2

A Poincaré invariant QFT consists of the following data:

(1) We have a Hilbert space $\mathcal{H}$.

(2) There is a (strongly continuous) projective unitary representation U of $\mathrm{P}^+(1,d)$ on $\mathcal{H}$. In particular, its restriction to the translation on the k-th component (where $k = 0,1,\dots,d$) gives a one parameter unitary group $x^k \in \mathbb{R} \mapsto \exp(\mathbf{i}x^k P_k)$ where P_k is a self-adjoint operator on $\mathcal{H}$.

(3) (Positive energy) The following are positive operators:

$$P_0 \geqslant 0 \qquad (P_0)^2 - (P_1)^2 - \cdots - (P_d)^2 \geqslant 0$$

The operator P_0 is called the **Hamiltonian** or the **energy operator**. $P_1,\dots,P_d$ are the momentum operators. $(P_0)^2 - (P_1)^2 - \cdots - (P_d)^2$ is the mass.

(4) We have a collection of **(quantum) fields** $\mathscr{Q}$, where each $\Phi \in \mathscr{Q}$ is an operator-valued function on $\mathbb{R}^{1,d}$. For each $\mathbf{x} \in \mathbb{R}^{1,d}$, $\Phi(x)$ is a "linear operator on $\mathcal{H}$".

(5) **(**Locality) If $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{R}^{1,d}$ are spacelike and $\Phi_1,\Phi_2 \in \mathscr{Q}$, then

$$[\Phi_1(x_1),\Phi_2(x_2)] = 0 \tag{1.2}$$

(6) (*-invariance) For each $\Phi \in \mathscr{Q}$, there exists $\Phi^\dagger \in \mathscr{Q}$ such that

$$\Phi(\mathbf{x})^\dagger = \Phi^\dagger(\mathbf{x}) \tag{1.3}$$

Moreover, $\Phi^{\dagger\dagger} = \Phi$.

(7) (Poincaré invariance) There is a distinguished unit vector[1] Ω, called the **vacuum vector**, such that

$$U(g)\Omega = \Omega \qquad \forall g \in \mathrm{P}^+(1,d)$$

Moreover, for each $g \in \mathrm{P}^+(1,d)$ and $\Phi \in \mathscr{Q}$, we have

$$U(g)\Phi(\mathbf{x})U(g)^{-1} = \Phi(g\mathbf{x}) \tag{1.4}$$

(8) (Cyclicity) Vectors of the form

$$\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega \tag{1.5}$$

(where $n \in \mathbb{N}$, $\mathbf{x}_1,\dots,\mathbf{x}_n \in \mathbb{R}^{1,d}$ are mutually spacelike, and $\Phi_1,\dots,\Phi_n \in \mathscr{Q}$) span a dense subspace of $\mathcal{H}$.

Remark 1.3. In some QFT, there is a factor (a function of $\mathbf{x}$) before $\Phi(g\mathbf{x})$ in the Poincaré invariance relation (1.4). Similarly, there is a factor before $\Phi^\dagger(\mathbf{x})$ in the $*$-invariance formula (1.3). We will encounter these more general covariance property later. In this section, we content ourselves with the simplest case that the factors are 1.

Remark 1.4. By the Poincaré invariance and the cyclicity, the action of $\mathrm{P}^+(1,d)$ is uniquely determined by $\mathscr{Q}$ by

$$U(g)\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(g\mathbf{x}_1)\cdots\Phi_n(g\mathbf{x}_n)\Omega \tag{1.6}$$

[1] A unit vector denotes a vector with length 1

1.3

Technically speaking, $\Phi(\mathbf{x})$ can not be viewed as a linear operator on $\mathcal{H}$. It cannot be defined even on a sufficiently large subspace of $\mathcal{H}$. One should think about **smeared fields**

$$\Phi(f) = \int_{\mathbb{R}^{1,d}} \Phi(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{1.7}$$

where $f \in C_c^\infty(\mathbb{R}^{1,d})$. (In contrast, we call $\Phi(\mathbf{x})$ a **pointed field**.) Then $\Phi(f)$ is usually a closable unbounded operator on $\mathcal{H}$ with dense domain $\mathscr{D}(\Phi(f))$. Moreover, $\mathscr{D}(\Phi(f))$ is preserved by any smeared operator $\Psi(g)$. Therefore, for any $f_1, \dots, f_n \in C_c^\infty(\mathbb{R}^{1,d})$ the following vector can be defined in $\mathcal{H}$:

$$\Phi_1(f_1) \cdots \Phi_n(f_n)\Omega \tag{1.8}$$

The precise meaning of cyclicity in Subsec. 1.2 means that vectors of the form (1.8) span a dense subspace of $\mathcal{H}$. Locality means that for $f_1, f_2 \in C_c^\infty(\mathbb{R}^{1,d})$ compactly supported in spacelike regions, on a reasonable dense subspace of $\mathcal{H}$ (e.g., the subspace spanned by (1.8)) we have

$$[\Phi_1(f_1), \Phi_2(f_2)] = 0 \tag{1.9}$$

The $*$-invariance means that

$$\langle \Phi(f)\xi | \eta \rangle = \langle \xi | \Phi^\dagger(f)\eta \rangle \tag{1.10}$$

for each ξ, η in the this good subspace.

1.4

In the remaining part of this section, if possible, we also understand $\Phi(\mathbf{x})$ as a smeared operator $\Phi(f)$ where $f \in C_c^\infty(\mathbb{R}^{1,d})$ satisfies $\int f = 1$ and is supported in a small region containing $\mathbf{x}$. Thus, $\Phi(\mathbf{x})$ can almost be viewed as a closable operator. Hence the expression (1.5) makes sense in $\mathcal{H}$.

We now explore the consequences of positive energy. As we will see, it implies that $\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega$, a function of $\mathbf{x}_\bullet$, can be analytically continued.

The fact that $P_0 \geqslant 0$ implies that when $t \leqslant 0$, e^{tP_0} is a bounded linear operator with operator norm $\leqslant 1$. Therefore, if τ belongs to

$$\mathfrak{I} = \{\mathrm{Im}\tau \geqslant 0\}$$

then $e^{\mathbf{i}\tau P_0} = e^{\mathbf{i}\mathrm{Re}\tau} \cdot e^{-\mathrm{Im}\tau}$ is bounded. Indeed, $\tau \in \mathfrak{I} \mapsto e^{\mathbf{i}\tau P_0}$ is continuous, and is holomorphic on $\mathrm{Int}\mathfrak{I}$.

Let $\mathbf{e}_0 = (1, 0, \ldots, 0)$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$ be distinct. By the Poincaré covariance, the relation

$$e^{\mathbf{i}\tau P_0}\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(\mathbf{x}_1 + \tau e_0)\cdots\Phi_n(\mathbf{x}_n + \tau e_0)\Omega \tag{1.11}$$

holds for all real τ. Moreover, the LHS is continuous on $\mathfrak{I}$ and holomorphic on $\mathrm{Int}\mathfrak{I}$. This suggests that the RHS of (1.11) can also be defined as an element of $\mathcal{H}$ when $\tau \in \mathfrak{I}$.

1.5

We shall further explore the question: for which $\mathbf{x}_i$ is in $\mathbb{C}^d$ can $\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega$ be reasonably defined as an element of $\mathcal{H}$?

Remark 1.5. We expect that the smeared fields should be defined on any P_0-**smooth vectors**, i.e., vectors in $\bigcap_{k\geqslant 0}\mathscr{D}(P_0^k)$. For each $r > 0$, since one can find $C_{k,r} \geqslant 0$ such that $\lambda^{2k} \leqslant C_{k,r}e^{2r\lambda}$ for all $\lambda \geqslant 0$, we conclude that

$$\mathrm{Rng}(e^{-rP_0}) \equiv \mathscr{D}(e^{rP_0}) \subset \bigcap_{k\geqslant 0}\mathscr{D}(P_0^k) \tag{1.12}$$

The above remark shows that $\Phi_1(\mathbf{x}_1)$, viewed as a smeared operator localized on a small neighborhood of $\mathbf{x}_1$, is definable on $e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\Omega = \Phi_2(\zeta_2\mathbf{e}_0 + \mathbf{x}_2)\Omega$ whenever $\mathrm{Im}\zeta_2 > 0$. Thus, heuristically, $(\zeta_1, \zeta_2) \mapsto e^{\mathbf{i}\zeta_1 P_0}\Phi_1(\mathbf{x}_1)e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\Omega$ should also be holomorphic on

$$\{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \mathrm{Im}\zeta_1, \mathrm{Im}\zeta_2 > 0\}$$

Repeating this procedure, we see that the holomorphicity holds for

$$e^{\mathbf{i}\zeta_1 P_0}\Phi_1(\mathbf{x}_1)e^{\mathbf{i}\zeta_2 P_0}\Phi_2(\mathbf{x}_2)\cdots e^{\mathbf{i}\zeta_n P_0}\Phi_n(\mathbf{x}_n)\Omega$$

when $\mathrm{Im}\zeta_i > 0$. By Poincaré covariance, the above expression equals

$$\Phi_1(\mathbf{x}_1 + \zeta_1\mathbf{e}_0)\Phi_2(\mathbf{x}_2 + (\zeta_1 + \zeta_2)\mathbf{e}_0)\cdots\Phi_n(\mathbf{x}_n + (\zeta_1 + \cdots + \zeta_n)\mathbf{e}_0)\Omega$$

Therefore,

$$(\zeta_1, \ldots, \zeta_n) \mapsto \Phi_1(\mathbf{x}_1 + \zeta_1\mathbf{e}_0)\cdots\Phi_n(\mathbf{x}_n + \zeta_n\mathbf{e}_0) \in \mathcal{H} \tag{1.13}$$

should be holomorphic on $\{\zeta_\bullet \in \mathbb{C}^n : 0 < \mathrm{Im}\zeta_1 < \cdots < \mathrm{Im}\zeta_n\}$.

By the locality axiom, the order of products of quantum fields can be exchanged. Thus, our expectation for a reasonable QFT includes the following condition:

Conclusion 1.6. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{1,d}$. Then (1.13) is holomorphic on

$$\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \mathrm{Im}\zeta_i > 0, \text{ and } \mathrm{Im}\zeta_i \neq \mathrm{Im}\zeta_j \text{ if } i \neq j\} \tag{1.14a}$$

Moreover, since (1.13) is also definable and continuous on

$$\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n : \mathbf{x}_1 + \zeta_1\mathbf{e}_1, \ldots, \mathbf{x}_n + \zeta_n\mathbf{e}_0 \text{ are mutually spacelike}\} \tag{1.14b}$$

we expect that the function (1.13) is continuous on the union of (1.14a) and (1.14b).

1.6

We have (informally) derived some consequences from the positivity of P_0. Note that since $P_0 \geqslant 0$, we have $U(g)P_0U(g)^{-1} \geqslant 0$ for each $g \in \mathrm{SO}^+(1,d)$. Since P_0 is the generator of the flow $t \in \mathbb{R} \mapsto t\mathbf{e}_0 \in \mathbb{R}^{1,d} \subset \mathrm{P}^+(1,d)$, $U(g)P_0U(g)^{-1}$ is the generator of the flow

$$t \in \mathbb{R} \mapsto g(t\mathbf{e}_0)g^{-1} = t \cdot g\mathbf{e}_0 \tag{1.15}$$

Therefore, if $g\mathbf{e}_0 = (a_0, \dots, a_n)$, then

$$U(g)P_0U(g)^{-1} = a_0P_0 + \cdots + a_nP_n \tag{1.16}$$

Hence the RHS must be positive. But what are all the possible $g\mathbf{e}_0$?

Remark 1.7. One can show that the orbit of $\mathbf{e}_0 = (1, 0, \dots, 0)$ under $\mathrm{SO}^+(1,d)$ is the upper hyperbola with diameter 1, i.e., the set of all $(a_0, \dots, a_n) \in \mathbb{R}^{1,d}$ satisfying

$$a_0 > 0 \qquad (a_0)^2 - (a_1)^2 - \cdots - (a_n)^2 = 1 \tag{1.17}$$

Thus $\sum_i a_iP_i \geqslant 0$ for all such $a_\bullet$. What are the consequences of this positivity?

1.7

To simplify the following discussions, we set $d = 2$ and

$$t = x^0 \qquad x = x^1$$

We further set

$$u = t - x \qquad v = t + x \tag{1.18}$$

so that

$$t = \frac{u+v}{2} \qquad x = \frac{-u+v}{2} \tag{1.19}$$

The Minkowski metric becomes

$$\boxed{(dt)^2 - (dx)^2 = du \cdot dv} \tag{1.20}$$

Then

$$(u,v) \text{ is spacelike to } (u',v') \qquad \Longleftrightarrow \qquad (u-u')(v-v') < 0 \tag{1.21}$$

For each $\Phi \in \mathscr{Q}$, we write

$$\widetilde{\Phi}(u,v) := \Phi(t,x) = \Phi\Big(\frac{u+v}{2}, \frac{-u+v}{2}\Big) \tag{1.22}$$

We let H_0 and H_1 be the self-adjoint operators such that

$$H_0 = P_0 - P_1 \qquad H_1 = P_0 + P_1$$

so that they are the generators of the flow $t \mapsto (t, -t)$ and $t \mapsto (t, t)$.

Remark 1.8. Since $\mathbb{R}^{1,d}$ is an abelian group, we know that P_i commutes with P_j. Hence H_0 commutes with H_1.

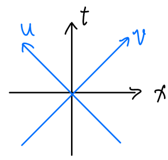

Figure 1.1. The coordinates u, v

1.8

The orbit of $\mathbf{e}_0$ under $\mathrm{SO}^+(1,1)$ is the unit upper hyperbola $(x^0)^2 - (x_1)^2 = 1, x^0 > 0$. Equivalently, it is $uv = 1, u > 0$. According to Subsec. 1.6, we conclude that $b_0 H_0 + b_1 H_1 \geqslant 0$ for each b_0, b_1 satisfying $b_0 b_1 = 1, b_0 > 0$ (equivalently, for each $b_0 > 0, b_1 > 0$). This implies

$$H_0 \geqslant 0 \qquad H_1 \geqslant 0 \tag{1.23}$$

Therefore, similar to the argument in Subsec. 1.5 (and specializing to the special case that $\mathbf{x}_1 = \cdots = \mathbf{x}_n = 0$), the holomorphicity of

$$(\zeta_\bullet, \gamma_\bullet) \mapsto e^{\mathbf{i}\zeta_1 H_0 + \mathbf{i}\gamma_1 H_1} \widetilde{\Phi}_1(0) e^{\mathbf{i}\zeta_2 H_0 + \mathbf{i}\gamma_2 H_1} \widetilde{\Phi}_2(0) \cdots e^{\mathbf{i}\zeta_n H_0 + \mathbf{i}\gamma_n H_1} \widetilde{\Phi}_n(0)\Omega$$

on the region $\mathrm{Im}\zeta_i > 0, \mathrm{Im}\gamma_i > 0$, together with locality, implies:

Conclusion 1.9. Let $\Phi_1, \ldots, \Phi_n \in \mathscr{Q}$. Then

$$(u_1, v_1, \ldots, u_n, v_n) \mapsto \widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}(u_n, v_n)\Omega \tag{1.24}$$

is holomorphic on

$$\{(u_\bullet, v_\bullet) \in \mathbb{C}^{2n} : \mathrm{Im}u_i > 0, \mathrm{Im}v_i > 0, \mathrm{Im}u_i \neq \mathrm{Im}u_j, \mathrm{Im}v_i \neq \mathrm{Im}v_j \text{ if } i \neq j\} \tag{1.25a}$$

and can be continuously extended to

$$\{(u_\bullet, v_\bullet) \in \mathbb{R}^{2n} : (u_i - u_j) \cdot (v_i - v_j) < 0 \text{ if } i \neq j\} \tag{1.25b}$$

Rigorously speaking, the above mentioned "continuity" of the extension should be understood in terms of distributions. Here, we ignore such subtlety and view pointed fields as smeared field in a small region.

1.9

We note that $\mathrm{diag}(-1, \pm 1)$ is not inside $\mathrm{SO}^+(1,1)$, since it reverses the time direction. Neither is $\mathrm{diag}(1, -1)$ in $\mathrm{SO}^+(1,1)$ because its determinant is negative. Consequently, the QFT is not necessarily symmetric under the following operations:

- **Time reversal** $t \mapsto -x$.
- **Parity transformation** $x \mapsto -x$.
- **PT transformation** $(t, x) \mapsto (-t, -x)$, the combination of time and parity inversions.

Mathematically, this means that the maps

$$\begin{aligned}
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega &\quad\mapsto\quad \Phi_1(-t_1, x_1) \cdots \Phi_n(-t_n, x_n)\Omega \\
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega &\quad\mapsto\quad \Phi_1(t_1, -x_1) \cdots \Phi_n(t_n, -x_n)\Omega \\
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega &\quad\mapsto\quad \Phi_1(-t_1, -x_1) \cdots \Phi_n(-t_n, -x_n)\Omega
\end{aligned}$$

(where $(t_1, x_1), \ldots, (t_n, x_n)$ are mutually spacelike) are not necessarily unitary. (Compare Rem. 1.4.) Simiarly, the QFT is not necessarily symmetric under **Charge conjugation** $\Phi \mapsto \Phi^\dagger$, which means that the map

$$\begin{aligned}
\Phi_1(t_1, x_1) \cdots \Phi_n(t_n, x_n)\Omega \quad\mapsto\quad & \Phi_n(t_n, x_n)^\dagger \cdots \Phi_1(t_1, x_1)^\dagger\Omega \\
& = \Phi_1^\dagger(t_1, x_1) \cdots \Phi_n^\dagger(t_n, x_n)\Omega
\end{aligned}$$

is not necessarily (anti)unitary. However, as we shall explain, the combination of PCT transformations is actually unitary, and hence is a symmetry of the QFT. This is called the PCT theorem.

1.10

To prove the PCT theorem, we shall first prove that the PT transformation, though not implemented by a unitary operator, is actually implemented by the analytic continuation of a one parameter unitary group.

Definition 1.10. The one parameter group $s \mapsto \Lambda(s) \in \mathrm{SO}^+(1,1)$ defined by

$$\Lambda(s)(u, v) = (e^{-s}u, e^s v) \tag{1.26}$$

is called the **Lorentz boost**. Equivalently,

$$\Lambda(s)\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}\begin{bmatrix} t \\ x \end{bmatrix} \tag{1.27}$$

Define the (open) **right wedge** $\mathcal{W}$ and **left wedge** $-\mathcal{W}$ by

$$\mathcal{W} = \{(u, v) \in \mathbb{R}^2 : v > 0, u < 0\} = \{(t, x) \in \mathbb{R}^{1,1} : -x < t < x\} \tag{1.28}$$

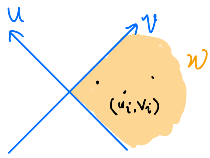

Figure 1.2.

Theorem 1.11 (PT theorem). *Let $(u_1, v_1), \dots, (u_n, v_n) \in \mathcal{W}$ be mutually spacelike (i.e. satisfying $(u_i - u_j)(v_i - v_j) < 0$ if $i \neq j$), cf. Fig. 1.2. Let $\Phi_1, \dots, \Phi_n \in \mathscr{Q}$. Let K be the self-adjoint generator of the Lorentz boost, i.e.,*

$$U(\Lambda(s)) = e^{\mathbf{i}sK}$$

Then $\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega$ belongs to the domain of $e^{-\pi K}$, and

$$e^{-\pi K}\Phi_1(\mathbf{x}_1) \cdots \Phi_n(\mathbf{x}_n)\Omega = \Phi_1(-\mathbf{x}_1) \cdots \Phi_n(-\mathbf{x}_n)\Omega \tag{1.29}$$

Equivalently, $\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega$ belongs to the domain of $e^{-\pi K}$, and

$$e^{-\pi K}\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega = \widetilde{\Phi}_1(-u_1, -v_1) \cdots \widetilde{\Phi}_n(-u_n, -v_n)\Omega \tag{1.30}$$

Note that the requirement that $(u_1, v_1), \dots, (u_n, v_n) \in \mathcal{W}$ are spacelike means, after relabeling the subscripts, that

$$0 < v_1 < \cdots < v_n \qquad 0 < -u_1 < \cdots < -u_n$$

Proof. This theorem relies on the following fact that we shall prove rigorously in the future:

- ⋆ Let $T \geqslant 0$ be a self-adjoint operator on $\mathcal{H}$ with $\mathrm{Ker}(T) = 0$. Let $r > 0$. Then $\xi \in \mathcal{H}$ belongs to $\mathscr{D}(T^r)$ iff the function $s \in \mathbb{R} \mapsto T^{\mathrm{i}s}\xi \in \mathcal{H}$ can be extended to a continuous function F on

$$\{z \in \mathbb{C} : -r \leqslant \mathrm{Im} z \leqslant 0\}$$

  and holomorphic on its interior. Moreover, for such ξ we have $F(-\mathbf{i}r) = T^r\xi$.

In fact, the function $F(z)$ is given by $z \mapsto T^z\xi$.

We shall apply this result to $T = e^{-K}$ and $r = \pi$. For that purpose, we must show that the $\mathcal{H}$-valued function of $s \in \mathbb{R}$ defined by

$$e^{\mathbf{i}\pi s}\widetilde{\Phi}_1(u_1, v_1) \cdots \widetilde{\Phi}_n(u_n, v_n)\Omega = \widetilde{\Phi}_1(e^{-s}u_1, e^s v_1) \cdots \widetilde{\Phi}_n(e^{-s}u_n, e^s v_n)\Omega$$

can be extended to a continuous function on

$$\{z \in \mathbb{C} : 0 \leqslant \mathrm{Im} z \leqslant \pi\}$$

and holomorphic on its interior.

In fact, we can construct this $\mathcal{H}$-valued function, which is

$$z \mapsto \widetilde{\Phi}_1(e^{-z}u_1, e^z v_1)\cdots\widetilde{\Phi}_n(e^{-z}u_n, e^z v_n)\Omega$$

noting that the conditions in Conc. 1.9 are fulfilled. In particular, the condition $0 < \mathrm{Im} < \pi$ is used to ensure that, since $u_i < 0, v_i > 0$, we have $\mathrm{Im}(e^{-z}u_i) > 0$ and $\mathrm{Im}(e^z v_i) > 0$ as required by (1.25a). The value of this function at $z = \mathbf{i}\pi$ equals the RHS of (1.30). Therefore the theorem is proved. □

1.11

Theorem 1.12 (PCT theorem). *We have an antiunitary map $\Theta : \mathcal{H} \to \mathcal{H}$, called the* ***PCT operator****, such that*

$$\Theta \cdot \Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega = \Phi_1(-\mathbf{x}_1)^\dagger\cdots\Phi_n(-\mathbf{x}_n)^\dagger\Omega \tag{1.31}$$

for any $\Phi_1, \dots, \Phi_n \in \mathscr{Q}$ and mutually spacelike $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Equivalently, Θ is defined by

$$\Theta \cdot \widetilde{\Phi}_1(u_1, v_1)\cdots\widetilde{\Phi}_n(u_n, v_n) = \widetilde{\Phi}_1(-u_1, -v_1)^\dagger\cdots\widetilde{\Phi}_n(-u_n, -v_n)^\dagger\Omega \tag{1.32}$$

Proof. The existence of an antilinear isometry Θ satisfying (1.32) is equivalent to showing that (cf. (0.4))

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_n(\mathbf{u}_n)\Omega|\widetilde{\Psi}_1(\mathbf{u}'_1)\cdots\widetilde{\Psi}_k(\mathbf{u}'_k)\Omega\rangle\\ =&\langle\widetilde{\Psi}_1(-\mathbf{u}'_1)^\dagger\cdots\widetilde{\Psi}_k(-\mathbf{u}'_k)^\dagger\Omega|\widetilde{\Phi}_1(-\mathbf{u}_1)^\dagger\cdots\widetilde{\Phi}_n(-\mathbf{u}_n)^\dagger\Omega\rangle\end{aligned} \tag{$\star$}$$

if $\mathbf{u}_1, \dots \mathbf{u}_n$ are spacelike, and $\mathbf{u}'_1, \dots \mathbf{u}'_k$ are spacelike. (We do not assume that, say, $\mathbf{u}_1$ and $\mathbf{u}'_1$ are spacelike.)

It suffices to prove this in the special case that $\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{u}'_1, \dots, \mathbf{u}'_k$ are mutually spacelike. Then the general case will follow that both sides of the above relation can be analytically continued to suitable regions as functions of $\mathbf{u}_1, \dots, \mathbf{u}_n$. For example, the fact that $H_0, H_1 \geqslant 0$ implies that

$$e^{\mathbf{i}\zeta H_0 + \mathbf{i}\gamma H_1}\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_n(\mathbf{u}_n)\Omega = \widetilde{\Phi}_1(\mathbf{u}_1 + (\zeta,\gamma))\cdots\widetilde{\Phi}_n(\mathbf{u}_n + (\zeta,\gamma))\Omega$$

is continuous on $\{(\zeta,\gamma) \in \mathbb{C}^2 : \mathrm{Im}\zeta \geqslant 0, \mathrm{Im}\gamma \geqslant 0\}$ and holomorphic on its interior.

Set $\Gamma_j = \Psi_j^\dagger$. Then ($\star$) is equivalent to

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\\ =&\langle\widetilde{\Phi}_1(-\mathbf{u}_1)\cdots\widetilde{\Phi}_1(-\mathbf{u}_n)\widetilde{\Gamma}_1(-\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(-\mathbf{u}'_k)\Omega|\Omega\rangle\end{aligned}$$

By the PT Thm. 1.11, this relation is equivalent to

$$\begin{aligned}&\langle\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\\ =&\langle e^{-\pi K}\widetilde{\Phi}_1(\mathbf{u}_1)\cdots\widetilde{\Phi}_1(\mathbf{u}_n)\widetilde{\Gamma}_1(\mathbf{u}'_1)\cdots\widetilde{\Gamma}_k(\mathbf{u}'_k)\Omega|\Omega\rangle\end{aligned}$$

But this of course holds since $e^{-\pi K}\Omega = \Omega$ by Poincaré invariance. □

1.12

Combining the PT Thm. 1.11 with the PCT Thm. 1.12, we conclude that $e^{-\pi K}$ is an injective positive operator, Θ is antinitary, and

$$\Theta e^{-\pi K} A\Omega = A^\dagger \Omega \tag{1.33a}$$

where A is a product of spacelike separated field in $\mathcal{W}$. The rigorous statement should be that

$$A = \Phi_1(f_1)\cdots\Phi_n(f_n)$$

where $\Phi_1, \dots, \Phi_n \in \mathscr{Q}$, and $f_i \in C_c^\infty(O_i)$ where $O_1, \dots, O_n \subset \mathcal{W}$ are open and mutually spacelike. If we let $\mathscr{A}(\mathcal{W})$ be the $*$-algebra generated by all such A, then by the Poincaré invariance, for each $g \in \mathrm{P}^+(1,d)$ we have

$$U(g)\mathscr{A}(\mathcal{W})U(g)^{-1} = \mathscr{A}(g\mathcal{W})$$

In particular, since for the Lorentz boost Λ we have $\Lambda(s)\mathcal{W} = \mathcal{W}$, we therefore have

$$e^{\mathbf{i}sK}\mathscr{A}(\mathcal{W})e^{-\mathbf{i}sK} = \mathscr{A}(\mathcal{W}) \tag{1.33b}$$

for all $s \in \mathbb{R}$. Since the PT transformation sends $\mathcal{W}$ to $-\mathcal{W}$, the definition of Θ clearly also implies

$$\Theta\mathscr{A}(\mathcal{W})\Theta^{-1} = \mathscr{A}(-\mathcal{W}) \tag{1.33c}$$

Note that since $\mathcal{W}$ is local to $-\mathcal{W}$, we have $[\mathscr{A}(\mathcal{W}), \mathscr{A}(-\mathcal{W})] = 0$. Therefore, $\Theta\mathscr{A}(\mathcal{W})\Theta$ is a subset of the (in some sense) commutant of $\mathscr{A}(\mathcal{W})$.

1.13

The set of formulas (1.33) is reminiscent of the Tomita-Takesaki theory, one of the deepest theories in the area of operator algebras. The setting is as follows.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Namely, $\mathcal{M}$ is a $*$-subalgebra of $\mathfrak{L}(\mathcal{H})$ closed under the "strong operator topology". (We will formally introduce von Neumann algebras in a later section.) Let $\Omega \in \mathcal{H}$ be a unit vector. Assume that Ω is **cyclic** (i.e. $\mathcal{M}\Omega$ is dense) and **separating** (i.e., if $x \in \mathcal{M}$ and $x\Omega = 0$ then $x = 0$) under $\mathcal{M}$. Then the **Tomita-Takesaki theorem** says that the linear map

$$S : \mathcal{M}\Omega \to \mathcal{M}\Omega \qquad x\Omega \mapsto x^*\Omega$$

is antilinear and closable. Denote its closure also by S, and consider its polar decomposition $S = J\Delta^{\frac{1}{2}}$ where Δ is a positive closed operator, and J is an antiunitary map. Then Δ is injective, we have $J^{-1} = J^* = J$, and

$$\Delta^{\mathbf{i}s}\mathcal{M}\Delta^{-\mathbf{i}s} = \mathcal{M} \qquad J\mathcal{M}J = \mathcal{M}'$$

where $\mathcal{M}'$ is the commutant $\{y \in \mathfrak{L}(\mathcal{H}) : xy = yx\ (\forall x \in \mathcal{M})\}$.We call Δ and J respectively the **modular operator** and the **modular conjugation**.

1.14

To relate the Tomita-Takesaki theory to QFT, one takes $\mathcal{M}$ to be $\mathfrak{A}(\mathcal{W})$, the von Neumann algebra generated by $\mathscr{A}(\mathcal{W})$. Note that the elements of $\mathscr{A}(\mathcal{W})$ are typically unbounded operators, whereas those of $\mathfrak{A}(\mathcal{W})$ are bounded. Thus, the meaning of "the von Neumann algebra generated by a set of closed/closable operators" should be clarified. This is an important notion, and we will study it in a later section.

To apply the setting of Tomita-Takesaki, one should first show that the vacuum vector is cyclic and separating under $\mathfrak{A}(\mathcal{W})$. This is not an easy task, although it is relatively easier to show that Ω is cyclic and separating under $\mathscr{A}(\mathcal{W})$. Moreover, we have

Theorem 1.13 (Bisognano-Wichmann). *Let Δ and J be the modular operator and the modular conjugation of $(\mathfrak{A}(\mathcal{W}),\Omega)$. Then $J=\Theta$ and $\Delta^{\frac{1}{2}}=e^{-\pi K}$.*

Since (1.33c) easily implies $\Theta\mathfrak{A}(\mathcal{W})\Theta^{-1}=\mathfrak{A}(-\mathcal{W})$, together with $J\mathcal{M}J^{-1}=\mathcal{M}'$ we obtain

$$\mathfrak{A}(\mathcal{W})'=\mathfrak{A}(-\mathcal{W}) \tag{1.34}$$

a version of **Haag duality**.

One of the main goals of this course is to give a rigorous and self-contained proof of the PCT theorem, the Bisognano-Wichmann theorem, and the Haag duality for 2d chiral conformal field theories.

1.15

For a general odd number $d>0$, the above results should be modified as follows. Let K be the generator of the **Lorentz boost**

$$\Lambda(s)=\left(\begin{array}{cc|ccc}\cosh s & \sinh s & & 0 & \\ \sinh s & \cosh s & & & \\ \hline & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1\end{array}\right)$$

Let $\Lambda(\mathbf{i}\pi)=\mathrm{diag}(-1,-1,1,\dots,1)$, which does not belong to $\mathrm{P}^+(1,d)$ since it reverses the time direction (although it has positive determinant). Then the PT Thm. 1.11 should be modified by replacing (1.29) with

$$e^{-\pi K}\Phi_1(\mathbf{x}_1)\cdots\Phi_n(\mathbf{x}_n)\Omega=\Phi_1(\Lambda(\mathbf{i}\pi)\mathbf{x}_1)\cdots\Phi_n(\Lambda(\mathbf{i}\pi)\mathbf{x}_n)\Omega \tag{1.35}$$

Let $\rho=\mathrm{diag}(1,1,-1,\dots,-1)$, which has determinant 1 (since d is odd) and hence belongs to $\mathrm{SO}^+(1,d)$. Then the PCT Thm. 1.12 still holds verbatim. Let

$$\mathcal{W}=\{(a_0,\dots,a_n)\in\mathbb{R}^{1,d}:-a_1<a_0<a_1\} \tag{1.36}$$

Then the **Bisognano-Wichmann theorem** says that $e^{-\pi K}$ is the modular operator of $(\mathfrak{A}(\mathcal{W}), \Omega)$, and $\Theta U(\rho)$ is the modular conjugation.

We refer the readers to [Haag, Sec. V.4.1] and the reference therein for a detailed study.

2 2d conformal field theory

2.1

We look at a 2d **unitary full conformal field theory** (unitary full CFT) $\mathcal{Q}$ on the **space-compactified Minkowski space**

$$\mathbb{R}_{\mathrm{c}}^{1,1} = \mathbb{R} \times \mathbb{S}^1 \qquad \text{with metric tensor } (dt)^2 - (dx)^2 = dudv$$

The space $\mathbb{R}_{\mathrm{c}}^{1,1}$ describes the propagation of the closed string $\{0\} \times \mathbb{S}^1$. Here, as in Subsec. 1.7, we write a general element of $\mathbb{R}_{\mathrm{c}}^{1,1}$ as $\mathbf{x} = (t, x)$, and write

$$u = t - x \quad v = t + x \qquad \text{so that} \qquad t = \frac{u+v}{2} \quad x = \frac{-u+v}{2}$$

The field operators are of the form $\Phi(\mathbf{x}) = \Phi(t, x)$. Recall that

$$\widetilde{\Phi}(u, v) := \Phi(t, x) = \Phi\Big(\frac{u+v}{2}, \frac{-u+v}{2}\Big)$$

Identifying $\mathbb{R}/2\pi\mathbb{Z} = \mathbb{S}^1$ via $\exp$, a field Φ can be viewed as an "operator valued function" on $\mathbb{R}^{1,1}$ satisfying

$$\Phi(t, x + 2\pi) = \Phi(t, x) \qquad \text{equivalently} \qquad \widetilde{\Phi}(u, v) = \widetilde{\Phi}(u - 2\pi, v + 2\pi) \tag{2.1}$$

The field operators are "acting on" a Hilbert space $\mathcal{H}$ with vacuum vector Ω.

Compared to the axioms for Poincaré invariant QFT in Subsec. 1.2, some changes should be made to describe a CFT. We still have the locality (1.2). Instead of considering $\mathrm{P}^+(1,1)$ we must consider the group of orientation-preserving, time-direction preserving, and conformal (i.e. angle-preserving) transforms on $\mathbb{R}_{\mathrm{c}}^{1,1}$. "Conformal" means that the diffeomorphism $g : \mathbb{R}_{\mathrm{c}}^{1,1} \to \mathbb{R}_{\mathrm{c}}^{1,1}$ satisfies

$$g^*(dudv) = \lambda(u, v)dudv$$

for a smooth function $\lambda : \mathbb{R}_{\mathrm{c}}^{1,1} \to \mathbb{R}_{>0}$. Our next goal is to classify such g.

2.2

Definition 2.1. We let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the group of orientation-preserving diffeomorphisms of $\mathbb{S}^1$. Equivalently, it is the group of smooth functions $f : \mathbb{S}^1 \to \mathbb{S}^1$ whose lift $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ satisfies for all $x \in \mathbb{R}$ that

$$\widetilde{f}(x + 2\pi) = \widetilde{f}(x) + 2\pi \qquad \widetilde{f}'(x) > 0 \tag{2.2}$$

Note that by the basics of covering spaces, any element of $\mathrm{Diff}^+(\mathbb{S}^1)$ can be lifted to $\widetilde{f}$ satisfying (2.2). Conversely, if $\widetilde{f}$ satisfies (2.2), then $\widetilde{f}$ gives rise to an injective smooth map $f : \mathbb{S}^1 \to \mathbb{S}^1$. (Note that $\widetilde{f}' > 0$ implies that $\widetilde{f}$ is strictly increasing.) Since $\widetilde{f}'(x) > 0$, the function f is injective, and the inverse function theorem shows that the compact set $f(\mathbb{S}^1)$ is open, and hence equals $\mathbb{S}^1$. Thus $f \in \mathrm{Diff}^+(\mathbb{S}^1)$.

Remark 2.2. Note that f uniquely determines $\widetilde{f}$ up to an $2\pi\mathbb{Z}$-addition, i.e., both $\widetilde{f}$ and $\widetilde{f} + 2n\pi$ (where $n \in \mathbb{Z}$) correspond to f. Therefore, if we let $\widetilde{\mathbf{Diff}^+}(\mathbb{S}^1)$ be the topological group formed by all $\widetilde{f}$ satisfying (2.2),[2] then we have an exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \longrightarrow \mathrm{Diff}^+(\mathbb{S}^1) \longrightarrow 1 \tag{2.3}$$

where $\mathbb{Z}$ is freely generated by $x \in \mathbb{R} \mapsto x + 2\pi$.

Note that the map $(\widetilde{f}, t) \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times [0, 1] \mapsto \widetilde{f}_t \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ defined by

$$\widetilde{f}_t(x) = (1 - t)\widetilde{f}(x) + tx$$

shows that $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ is contractible (to the identity element) and hence simply-connected. (Therefore $\mathrm{Diff}^+(\mathbb{S}^1)$ is connected.) We conclude that $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ is the universal cover of $\mathrm{Diff}^+(\mathbb{S}^1)$. $\square$

2.3

Theorem 2.3. *Under the coordinates (u, v), an orientation-preserving time-direction-preserving conformal transform g of $\mathbb{R}^{1,1}_{\mathrm{c}}$ is precisely of the form*

$$g(u, v) = (\alpha(u), \beta(v)) \tag{2.4}$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ belong to $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$.

Proof. Step 1. First, suppose that g is of the form (2.4). Then g gives a well-defined smooth map $\mathbb{R}^{1,1}_{\mathrm{c}} \to \mathbb{R}^{1,1}_{\mathrm{c}}$ because

$$g(u - 2\pi, v + 2\pi) = g(u, v) + (-2\pi, 2\pi) \tag{2.5}$$

One checks easily that g is a diffeomorphism (with inverse given by $(\alpha^{-1}(u), \beta^{-1}(v))$) preserving the orientation and the time direction. Since $g^*dudv = \alpha'(u)\beta'(v)dudv$, g is conformal.

[2]The topology is defined such that a net $\widetilde{f}_\alpha$ converges to $\widetilde{f}$ iff the n-th derivative $\widetilde{f}_\alpha^{(n)}$ converges uniformly to $\widetilde{f}^{(n)}$ for all $n \in \mathbb{N}$.

Step 2. Conversely, choose an orientation preserving conformal transform g. We lift g to a smooth conformal map $\mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ also denoted by $g = (\alpha, \beta)$. So $\alpha, \beta : \mathbb{R}^{1,1} \to \mathbb{R}$. Then, besides (2.5), g also satisfies:

$$\partial_u\alpha\partial_u\beta = 0 \qquad \partial_v\alpha\partial_v\beta = 0 \tag{a}$$
$$\partial_u\alpha\partial_v\beta + \partial_v\alpha\partial_u\beta > 0 \tag{b}$$
$$\partial_u\alpha\partial_v\beta - \partial_v\alpha\partial_u\beta > 0 \tag{c}$$

Here, (a) and (b) are due to the fact that

$$g^*(dudv) = (\partial_u\alpha du + \partial_v\alpha dv)(\partial_u\beta du + \partial_v\beta dv)$$

equals $\lambda(u,v)dudv$ for some smooth $\lambda : \mathbb{R}^{1,1} \to \mathbb{R}_{>0}$. (So λ is the LHS of (b).) Since g is orientation preserving, (c) follows from the computation

$$g^*(du \wedge dv) = (\partial_u\alpha\partial_v\beta - \partial_v\alpha\partial_u\beta)du \wedge dv$$

Step 3. By (a), at a given $p \in \mathbb{R}^{1,1}$, if $\partial_u\alpha \neq 0$, then $\partial_u\beta = 0$. Conversely, if at p we have $\partial_u\beta = 0$, then (b) shows that $\partial_u\alpha\partial_v\beta > 0$, and hence $\partial_u\alpha \neq 0$. Thus

$$\begin{aligned} \partial_u\alpha|_p \neq 0 \qquad &\Longleftrightarrow \qquad \partial_u\beta|_p = 0 \\ \partial_v\alpha|_p \neq 0 \qquad &\Longleftrightarrow \qquad \partial_v\beta|_p = 0 \end{aligned}$$

where the second equivalence follows from the same argument. Therefore, the set of p at which $\partial_v\alpha = 0$ is both open and closed, and hence must be either $\mathbb{R}^{1,1}$ or $\emptyset$. Similarly, either $\partial_u\beta = 0$ everywhere, or $\partial_u\beta \neq 0$ everywhere.

Let us prove that

$$\partial_v\alpha = 0 \qquad \partial_u\beta = 0$$

everywhere. Suppose the first is not true. Then by the previous paragraph, we have $\partial_v\alpha \neq 0$ and $\partial_v\beta = 0$ everywhere. Then (b) implies $\partial_v\alpha\partial_u\beta > 0$, and (c) implies $-\partial_v\alpha\partial_u\beta > 0$, impossible. So the first (and similarly the second) is true.

Step 4. Therefore, we can write $\alpha = \alpha(u)$ and $\beta = \beta(v)$, and we have $\alpha' \neq 0$ and $\beta' \neq 0$ everywhere. (b) implies that $\alpha'(u)\beta'(v) > 0$ for all u, v. Thus, either $\alpha' > 0$ and $\beta' > 0$ everywhere, or $\alpha' < 0$ and $\beta' < 0$ everywhere. The latter cannot happen, since g preserves the direction of time. Thus $\alpha' > 0$ and $\beta' > 0$ everywhere. Since g satisfies (2.5), we see that α satisfies (2.2). Similarly β satisfies (2.2). This finishes the proof. □

2.4

We let $\mathbf{Cf^+(\mathbb{R}_c^{1,1})}$ be the group of diffeomorphisms of $\mathbb{R}_c^{1,1}$ preserving the orientation and the time-direction. Then Thm. 2.3 says that any $g \in \mathrm{Cf}^+(\mathbb{R}_c^{1,1})$ can be represented by some $(\alpha, \beta) \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$.

However, (α, β) is not uniquely determined by g. Indeed, in Step 2 of the proof of Thm. 2.3 we have lifted g to a smooth map on $\mathbb{R}^{1,1}$. This lift is unique up to addition by $(-2\pi, 2\pi)\mathbb{Z}$ in the (u, v) coordinates (or $(0, 2\pi)\mathbb{Z}$ in the (t, x) coordinates). Thus, (α, β) are unique up to addition by $(-2\pi, 2\pi)\mathbb{Z}$. This non-uniqueness can be ignored once we pass to $(\breve{\alpha}, \breve{\beta})$, the projection of (α, β) into $\mathrm{Diff}^+(\mathbb{S}^1)^2$. Thus, we have a well-defined (continuous) surjective group homomorphism $\Gamma : \mathrm{Cf}^+(\mathbb{R}_c^{1,1}) \to \mathrm{Diff}^+(\mathbb{S}^1) \times \mathrm{Diff}^+(\mathbb{S}^1)$ sending g to $(\breve{\alpha}, \breve{\beta})$.

One checks easily that the kernel of this homomorphism is freely generated by $(2\pi, 0)$ (equivalently, by $(0, 2\pi)$) under the (u, v) coordinates, equivalently, by (π, π) under the (t, x) coordinates. Therefore, we have an exact sequence of groups

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cf}^+(\mathbb{R}_c^{1,1}) \xrightarrow{\Gamma} \mathrm{Diff}^+(\mathbb{S}^1)^2 \longrightarrow 1 \tag{2.6}$$

Since Γ is a covering map, we also have a covering map $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2 \twoheadrightarrow \mathrm{Cf}^+(\mathbb{R}_c^{1,1})$ such that the following diagram commutes

$$\begin{tikzcd} & \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2 \arrow[dl, two heads] \arrow[d, two heads] \\ \mathrm{Cf}^+(\mathbb{R}_c^{1,1}) \arrow[r, two heads, "\Gamma"'] & \mathrm{Diff}^+(\mathbb{S}^1)^2 \end{tikzcd} \tag{2.7}$$

2.5

Since we require that $\mathcal{Q}$ is a CFT with Hilbert space $\mathcal{H}$, we must have a **strongly continuous projective unitary representation** $\mathscr{U}$ of $\mathrm{Cf}^+(\mathbb{R}_c^{1,1})$. Namely,

$$\mathscr{U} : \mathrm{Cf}^+(\mathbb{R}_c^{1,1}) \to \mathrm{PU}(\mathcal{H})$$

is a continuous group homomorphism. Here, $\mathrm{PU}(\mathcal{H})$ is the quotient group (with quotient topology) $U(\mathcal{H})/\sim$ where $U(\mathcal{H})$ is the group of unitary operators of $\mathcal{H}$ (equipped with the strong operator topology), and $U_1 \simeq U_2$ iff $U_1 = \lambda U_2$ for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. We suppress the adjectives "strongly continuous" when no confusion arises.

By (2.7), $\mathscr{U}$ can be lifted to a projective unitary representation of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$ on $\mathcal{H}$. Since $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$ is simply connected, its projective unitary representations are (roughly) equivalent to the projective unitary representations of the Lie algebra of

$\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$, which is $\mathrm{Vec}(\mathbb{S}^1) \oplus \mathrm{Vec}(\mathbb{S}^1)$ where $\mathrm{Vec}(\mathbb{S}^1)$ is the Lie algebra of smooth real vector fields of $\mathbb{S}^1$.

The elements of $\mathrm{Vec}(\mathbb{S}^1)$ are of the form $f\partial_\theta$ where $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$ and ∂_θ is the unique vector field on $\mathbb{S}^1$ that is pulled back by $\exp(\mathbf{i}\cdot) : \mathbb{R} \to \mathbb{S}^1$ to $\partial_\theta \in \mathrm{Vec}(\mathbb{R}^1)$ where θ is the standard coordinate of $\mathbb{R}$ (sending x to x). The Lie bracket of $\mathrm{Vec}(\mathbb{S}^1)$ is the negative of the Lie derivative, i.e.

$$[f_1\partial_\theta, f_2\partial_\theta]_{\mathrm{Vec}(\mathbb{S}^1)} = (-f_1\partial_\theta f_2 + f_2\partial_\theta f_1)\partial_\theta$$

The negative choice is due to the fact that the group action of $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ on $h \in C^\infty(\mathbb{S}^1)$ is given by $h \circ g^{-1}$; however, the Lie derivative is defined by differentiating $g \mapsto h \circ g$.

2.6

The complexification $\mathrm{Vec}_{\mathbb{C}}(\mathbb{S}^1)$ of $\mathrm{Vec}(\mathbb{S}^1)$ is the Lie algebra of all $f\partial_\theta$ where $f \in C^\infty(\mathbb{S}^1) \equiv C^\infty(\mathbb{S}^1, \mathbb{C})$. Let $z = e^{\mathbf{i}\theta} \in C^\infty(\mathbb{S}^1)$, which is the inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{C}$. Then we can define $\partial_z \in \mathrm{Vec}_{\mathbb{C}}(\mathbb{S}^1)$ by

$$\partial_z = \frac{1}{\mathbf{i}z}\partial_\theta \qquad \text{so that} \qquad \partial_\theta = \mathbf{i}z\partial_z = \mathbf{i}e^{\mathbf{i}\theta}\partial_z \tag{2.8}$$

Then $\mathrm{Vec}_{\mathbb{C}}(\mathbb{S}^1)$ is a $*$-Lie algebra, i.e., a complex Lie algebra equipped with an involution $\dagger$. For $\mathrm{Vec}_{\mathbb{C}}(\mathbb{S}^1)$, the involution is defined by

$$(f\partial_\theta)^\dagger = -\overline{f}\partial_\theta$$

so that $\mathrm{Vec}(\mathbb{S}^1)$ is precisely the set of all $\mathfrak{x} \in \mathrm{Vec}_{\mathbb{C}}(\mathbb{S}^1)$ satisfying $\mathfrak{x}^\dagger = -\mathfrak{x}$. In particular, noting $\overline{z} = z^{-1}$ on $\mathbb{S}^1$, we have

$$(\partial_z)^\dagger = z^2\partial_z \tag{2.9}$$

$\mathrm{Vec}_{\mathbb{C}}(\mathbb{S}^1)$ contains a “sufficiently large" $*$-Lie subalgebra, the **Witt algebra** $\mathrm{Witt} = \mathrm{Span}_{\mathbb{C}}\{l_n : n \in \mathbb{Z}\}$, where

$$l_n = z^n\partial_z \tag{2.10}$$

One easily computes that $l_n^\dagger = l_{-n}$, and that

$$[l_m, l_n] = (m-n)l_{m+n}$$

Projective unitary representations of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ correspond to (honest) unitary representations of central extensions of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$, which (roughly) correspond to unitary representations of central extensions of Witt.

2.7

It can be shown that the central extensions of Witt are equivalent to the **Virasoro algebra Vir**. As a vector space, Vir has basis $\{C, L_n : n \in \mathbb{Z}\}$. These basis elements satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \qquad [L_n, C] = 0 \tag{2.11a}$$

$$L_n^\dagger = L_{-n} \qquad C^\dagger = C \tag{2.11b}$$

Thus, the projective unitary representation of $\widetilde{\mathrm{Diff}}^+(\mathbb{S}^1) \times \widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)$ on $\mathcal{H}$ can be described by a unitary representation of $\mathrm{Vir} \oplus \widehat{\mathrm{Vir}}$. Here, $\widehat{\mathrm{Vir}} = \{\widehat{C}, \widehat{L}_n : n \in \mathbb{Z}\}$ is isomorphic to Vir.

One can decompose a (unitary full) CFT $\mathcal{Q}$ into a "direct sum" of CFTs such that C and $\widehat{C}$ act as scalars $c, \widehat{c} \in \mathbb{R}$. (In fact, one can show that $c, \widehat{c} \geqslant 0$.) We call $(c, \widehat{c})$ the **central charge** of the CFT $\mathcal{Q}$. Since $L_0^\dagger = L_0$ and $\widehat{L}_0^\dagger = \widehat{L}_0$, one usually assume that $L_0, \widehat{L}_0$ act as self-adjoint operators on $\mathcal{H}$.

2.8

In a Poincaré invariant QFT, the vacuum vector is fixed by $\mathrm{P}^+(1,d)$. However, in our CFT $\mathcal{Q}$, the vacuum vector Ω is not fixed by $\mathrm{Cf}^+(\mathbb{R}^{1,1}_c)$. In terms of Vir, then $L_n\Omega$ is not necessarily zero for all n. This phenomenon is related to the fact that an arbitrary one-parameter subgroup $t \in \mathbb{R} \mapsto g_t \in \widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)$, when each g_t acts on $\mathbb{S}^1$ and hence can be viewed as a map $g_t : \mathbb{S}^1 \to \mathbb{P}^1$, does not have a sufficiently large domain for the analytic continuation $z \mapsto g_z$.

On the other hand, we do have

$$L_n\Omega = 0 \qquad \text{if } n = -1, 0, 1 \tag{2.12}$$

(and similarly $\widehat{L}_0\Omega = \widehat{L}_{\pm 1}\Omega = 0$). These $L_0, L_{\pm 1}$ span a Lie $*$-subalgebra

$$\mathfrak{sl}(2,\mathbb{C}) = \mathrm{Span}_{\mathbb{C}}\{L_0, L_{\pm 1}\}$$

with skew-symmetric part

$$\mathfrak{su}(2) := \{\mathfrak{x} \in \mathfrak{sl}(2,\mathbb{C}) : \mathfrak{x}^\dagger = -\mathfrak{x}\} = \mathrm{Span}_{\mathbb{R}}\Big\{\mathbf{i}l_0, \frac{l_1 - l_{-1}}{2}, \frac{\mathbf{i}(l_1 + l_{-1})}{2}\Big\} \tag{2.13}$$

As we will see in the future, the one-parameter group generated by $(l_1 - l_{-1})/2$ is related to the PCT symmetry of the CFT.

The Lie subgroup of $\widetilde{\mathrm{Diff}}^+(\mathbb{S}^1)$ with Lie algebra $\mathfrak{su}(2)$ is $\widetilde{\mathbf{PSU}(1,1)}$, , the universal cover of the **Möbius group** $\mathrm{PSU}(1,1)$ whose elements are linear fractional transforms

$$z \in \mathbb{P}^1 \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad \text{where } \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1$$

The condition $|\alpha|^2 - |\beta|^2 = 1$ is to ensure that the transform sends $\mathbb{S}^1$ to $\mathbb{S}^1$. The exact sequence (2.3) restricts to

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{PSU}}(1,1) \longrightarrow \mathrm{PSU}(1,1) \longrightarrow 1 \tag{2.14}$$

where $\mathbb{Z}$ is freely generated by "the anticlockwise rotation by 2π". Thus, the projective action $\mathcal{U}(g)$ of any g in $\widetilde{\mathrm{PSU}}(1,1) \times \widetilde{\mathrm{PSU}}(1,1) \subset \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1) \times \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ fixes Ω up to $\mathbb{S}^1$-multiplications. Now we choose $\mathcal{U}(g)$ to be the unique one such that $\mathcal{U}(g)\Omega = \Omega$. Then $\mathcal{U}$ gives an (honest) strongly-continuous unitary representation of $\widetilde{\mathrm{PSU}}(1,1) \times \widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}$ fixing Ω.

2.9

A field $\Phi \in \mathcal{Q}$ is called **chiral** (resp. **antichiral**) if $\widetilde{\Phi}$ depends only on u (resp. v) but not on v (resp. u). We let $\boldsymbol{\mathcal{V}}$ resp. $\widehat{\boldsymbol{\mathcal{V}}}$ be the set of chiral resp. anti chiral fields. They can be viewed as algebraic structures. (We will say more about such structures in the future.)

Let $\mathcal{H}_0$ (resp. $\widehat{\mathcal{H}}_0$) be the closure of the subspace spanned by $\varphi(f_1)\cdots\varphi(f_n)\Omega$ where each $f_i \in C_c^\infty(\mathbb{R}_c^{1,1})$ and $\varphi_i \in \mathcal{V}$ (resp. $\varphi_i \in \widehat{\mathcal{V}}$). Then $\mathcal{H}_0$ can be viewed as a (unitary) representation of $\mathcal{V}$, called the **vacuum representation**. Clearly $\Omega \in \mathcal{H}_0 \cap \widehat{\mathcal{H}}_0$.

A basic assumption of unitary full CFT is the existence of orthogonal decomposition

$$\mathcal{H} = \bigoplus_{i \in \mathfrak{I}} \mathcal{H}_i \otimes \widehat{\mathcal{H}}_i \qquad \supset \mathcal{H}_0 \otimes \widehat{\mathcal{H}}_0 \tag{2.15}$$

where each $\mathcal{H}_i$ (resp. $\widehat{\mathcal{H}}_i$) is an irreducible unitary representation of $\mathcal{V}$ (resp. $\widehat{\mathcal{V}}$). Here, $\bigoplus$ could be a finite, or infinite discrete, or even continuous (i.e. a direct integral). A large class of important CFTs are called **rational CFTs** , which means that the direct sum is finite. Here, $\mathcal{H}_0$ is identified with $\mathcal{H}_0 \otimes \Omega$ so that it can thus be viewed as a subspace of $\mathcal{H}$; similarly $\widehat{\mathcal{H}}_0 \simeq \Omega \otimes \widehat{\mathcal{H}}_0$. Therefore, with respect to the decomposition (2.15), the vacuum vector $\Omega \in \mathcal{H}$ can be written as $\Omega \otimes \Omega$.

2.10

From now on, we slightly change our notation a bit:

Convention 2.4. An element of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$ is not viewed as a function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, but rather a multivalued smooth function $f : \mathbb{S}^1 \to \mathbb{S}^1$ related to the original $\widetilde{f}$ by

$$f(e^{\mathbf{i}\theta}) = \widetilde{f}(\theta)$$

Following this convention, and similar to (2.8), we define

$$f'(e^{\mathbf{i}\theta}) \equiv \partial_z f(e^{\mathbf{i}\theta}) = \frac{\widetilde{f}'(\theta)}{\mathbf{i}e^{\mathbf{i}\theta}} \tag{2.16}$$

Similarly, for each $\Phi \in \mathcal{Q}$, we let

$$\mathring{\Phi}(e^{\mathbf{i}u}, e^{\mathbf{i}v}) \xlongequal{\text{def}} \widetilde{\Phi}(u,v) = \Phi\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) \tag{2.17}$$

viewing $\mathring{\Phi}$ as a multivalued function on $\mathbb{S}^1 \times \mathbb{S}^1$.

Although the projective unitary representation $\mathcal{U}$ of $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$ does not fix Ω up to $\mathbb{S}^1$-multiplications, similar to (1.4), there is a large class of $\Phi \in \mathcal{Q}$, called **primary fields**, satisfying the **conformal covariance property**: For each such Φ, there exist $\delta, \widehat{\delta} \in \mathbb{R}_{\geqslant 0}$ (called the **conformal weights** of Φ) such that for all $(g,h) \in \widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)^2$ and

$$\mathcal{U}(g,h)\mathring{\Phi}(e^{\mathbf{i}u}, e^{\mathbf{i}v})\mathcal{U}(g,h)^{-1} = g'(e^{\mathbf{i}u})^{\delta} h'(e^{\mathbf{i}v})^{\widehat{\delta}} \cdot \mathring{\Phi}(g(e^{\mathbf{i}u}), h(e^{\mathbf{i}v})) \tag{2.18}$$

in the sense of smeared operators.

2.11

In the special case that $\varphi \in \mathcal{V}$, then (2.1) says that $\mathring{\varphi}$ is a single-valued function on $\mathbb{S}^1$, and hence has a Fourier series expansion

$$\mathring{\varphi}(z) = \sum_{n \in \mathbb{Z}} \mathring{\varphi}_n z^{-n-1} \tag{2.19}$$

So $\mathring{\varphi}_n = \mathrm{Res}_{z=0}\mathring{\varphi}(z)z^n dz$. The derivative $\mathring{\varphi}'(z) = \partial_z \mathring{\varphi}(z)$ is understood in the usual way, i.e.,

$$\mathring{\varphi}'(z) = \sum_n (-n-1)\mathring{\varphi}_n z^{-n-2}$$

Now, writing $\mathcal{U}(g,1)$ as $\mathcal{U}(g)$, then for primary chiral φ, (2.18) becomes

$$\mathcal{U}(g)\mathring{\varphi}(z)\mathcal{U}(g)^{-1} = g'(z)^{\delta} \cdot \mathring{\varphi}(g(z)) \tag{2.20}$$

We simply call δ the **conformal weight** of the chiral field φ. If (2.20) only holds for $g \in \widetilde{\mathrm{PSU}}(1,1)$, we say that the chiral field φ is **quasi-primary**.

Remark 2.5. For each primary (resp. quasi-primary) chiral φ, and for each $m \in \mathbb{Z}$ (resp. $m = 0, \pm 1$), we have

$$[L_m, \mathring{\varphi}(z)] = z^{m+1}\mathring{\varphi}'(z) + \delta \cdot (m+1)z^m \mathring{\varphi}(z) \tag{2.21a}$$

Equivalently, for each $n \in \mathbb{Z}$ we have

$$[L_m, \mathring{\varphi}_n] = -(m+n+1)\mathring{\varphi}_{m+n} + \delta \cdot (m+1)\mathring{\varphi}_{m+n} \tag{2.21b}$$

Heuristic proof. Let $t \mapsto g_t$ be the one-parameter group generated by $\mathfrak{x} = \sum_m a_m l_m$ (a finite sum) satisfying $\mathfrak{x}^\dagger = -\mathfrak{x}$, i.e., $\overline{a_m} = -a_{-m}$. So $g_0(z) = z$ and $\partial_t g_t(z)\big|_{t=0} = \sum_m a_m z^{m+1}$. Set $X = \sum_m a_m L_m$. Then, informally, we have

$$\frac{d}{dt}\mathcal{U}(g_t)\mathring{\varphi}(z)\mathcal{U}(g_t)^{-1}\big|_{t=0} = [X, \mathring{\varphi}(z)]$$

Also

$$\frac{d}{dt}\mathring{\varphi}(g_t(z))\big|_{t=0} = \mathring{\varphi}'(z) \cdot \partial_t g_t(z)\big|_{t=0} = \sum_m a_m z^{m+1}\mathring{\varphi}'(z)$$

Since $\delta \cdot g_0'(z)^{\delta-1} = \delta \cdot \left(\frac{d}{dz}(z)\right)^{\delta-1} = \delta$, we have

$$\frac{d}{dt}g_t'(z)^\delta\big|_{t=0} = \delta \cdot g_0'(z)^{\delta-1} \cdot \partial_t g_t'(z)\big|_{t=0} = \delta\sum_m (a_m z^{m+1})' = \delta\sum_m (m+1)a_m z^m$$

Combining the above three results with (2.20), we get (2.21a). $\square$

3 Local fields and chiral algebras

In this section, we introduce a rigorous approach to the algebra $\mathcal{V}$ of chiral fields. We will give an axiomatic description of (the modes of) the chiral fields acting on $\mathbb{V}$, the dense subspace of $\mathcal{H}_0$ with finite L_0-spectra. (So $\mathcal{H}_0$ is the Hilbert space completion of $\mathbb{V}$.) Some of the proofs will be sketched or even omitted. But details can be found in [Gui-V] (especially Sec. 7 and 8).

3.1

Unless otherwise stated, we fix a complex inner product space $\mathbb{V}$ together with a diagonalizable operator $L_0 \in \mathrm{End}(\mathbb{V})$ such that the eigenvalues of L_0 belong to $\mathbb{N}$. Thus, we have orthogonal decomposition $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ where $\mathbb{V}(n) = \{v \in \mathbb{V} : L_0 v = nv\}$. If $v \in \mathbb{V}$, we say that v is **homogeneous** if $v \in \mathbb{V}(n)$ for some n; in that case we write

$$\mathrm{wt}(v) = n$$

The Hilbert space completion of $\mathbb{V}$ is denoted by $\mathcal{H}_{\mathbb{V}}$. We assume that each $\mathbb{V}(n)$ is finite-dimensional so that $\mathbb{V}(n)^{**} = \mathbb{V}(n)$. Define

$$\mathbb{V}^{\mathrm{ac}} = \prod_{n\in\mathbb{N}} \mathbb{V}(n)$$

the **algebraic completion** of $\mathbb{V}$. Then clearly

$$\mathbb{V} \subset \mathcal{H}_{\mathbb{V}} \subset \mathbb{V}^{\mathrm{ac}}$$

Note that L_0 acts on $\mathbb{V}^{\mathrm{ac}}$ in a canonical way by acting on each $\mathbb{V}(n)$ as $n \cdot \mathrm{id}$. Similarly, for each $q \in \mathbb{C}^\times$, q^{L_0} acts on $\mathbb{V}^{\mathrm{ac}}$.

For each $n \in \mathbb{N}$, we define the projection onto the n-th component

$$P_n : \mathbb{V}^{\mathrm{ac}} \to \mathbb{V}(n) \tag{3.1}$$

Then for any $\xi \in \mathbb{V}^{\mathrm{ac}}$, it is clear that

$$\xi \in \mathcal{H}_{\mathbb{V}} \qquad \Longleftrightarrow \qquad \sum_{n\in\mathbb{N}} \|P_n\xi\|^2 < +\infty \tag{3.2}$$

Note that L_0 and q^{L_0} commute with P_n. We also let

$$P_{\leqslant n} = \sum_{k\in\mathbb{N}, k\leqslant n} P_n \tag{3.3}$$

3.2

Definition 3.1. An **(homogeneous) field** on $\mathbb{V}$ is an element

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$$

(where each A_n is in $\mathrm{End}(\mathbb{V})$) satisfying

$$[L_0, A(z)] = \mathrm{wt}(A)\cdot A(z) + z\partial_z A(z) \tag{3.4a}$$

for some $\mathrm{wt}(A) \in \mathbb{N}$ (called the **(conformal) weight**) of $A(z)$; equivalently,

$$[L_0, A_n] = (\mathrm{wt}(A) - n - 1)A_n \tag{3.4b}$$

Remark 3.2. Note that by (3.4b), for each d, A_n restricts to

$$A_n : \mathbb{V}(d) \to \mathbb{V}(d + \mathrm{wt}(A) - n - 1) \tag{3.5}$$

Since no nonzero homogeneous vectors can have negative weights, we see that $A_n v = 0$ when $n \gg 0$, and that $\langle A_n \cdot |v\rangle = 0$ when $n \ll 0$. Thus

$$A(z)v \in \mathbb{V}((z)) \tag{3.6}$$

for each homogeneous $v \in \mathbb{V}$, and hence for all $v \in \mathbb{V}$. This is called the **lower truncation property**.

Note that A_n can be extended to $A_n^{\mathrm{tt}} : \mathbb{V}^{\mathrm{ac}} \to \mathbb{V}^{\mathrm{ac}}$. We abbreviate A_n^{tt} to A_n when no confusion arises.

Example 3.3. The field $\mathbf{1}(z) = \mathrm{id}_{\mathbb{V}}$ is called the **vacuum field**. By (3.4), we clearly have

$$\mathrm{wt}(\mathbf{1}) = 0$$

3.3

Let $A(z)$ be a homogeneous field. By (3.5), we have a well defined linear map $(A_n)^\dagger : \mathbb{V} \to \mathbb{V}$ being the formal adjoint of A_n, i.e.,

$$\langle A_n u|v\rangle = \langle u|(A_n)^\dagger v\rangle$$

This is because the restriction $A_n : \mathbb{V}(d) \to \mathbb{V}(d + \mathrm{wt}(A) - n - 1)$ has an adjoint due the the finite-dimensionality. Thus $(A_n)^\dagger$ restricts to

$$(A_n)^\dagger : \mathbb{V}(d) \to \mathbb{V}(d - \mathrm{wt}(A) + n + 1) \tag{3.7}$$

If z is a formal variable, we understand $\overline{z} \equiv z^\dagger$ as the formal conjugate of z. So $z, \overline{z}$ are mutually commuting formal variables.

Definition 3.4. Define the **quasi-primary contragredient $A^\theta(z)$** of $A(z)$ to be

$$A^\theta(z) = (-z^{-2})^{\mathrm{wt}(A)} A(\overline{z^{-1}})^\dagger = (-z^{-2})^{\mathrm{wt}(A)} \cdot \sum_{n\in\mathbb{Z}} (A_n)^\dagger z^{n+1} \tag{3.8}$$

One shows easily that

$$A_n^\theta = (-1)^{\mathrm{wt}(A)} \cdot (A_{-n-2+2\mathrm{wt}(A)})^\dagger \tag{3.9}$$

Comparing (3.9) with (3.7), we see that A_n^θ restricts to $\mathbb{V}(d) \to \mathbb{V}(d+\mathrm{wt}(A)-n-1)$. Hence A^θ is homogeneous with weight

$$\mathrm{wt}(A^\theta) = \mathrm{wt}(A) \tag{3.10}$$

One checks easily that $A^{\theta\theta} = A$.

The reason we need the extra term $(-z^{-2})^{\mathrm{wt}(A)}$ will be clear when studying PCT symmetry for chiral CFTs in the future. At present, we at least know that part of the reasons we need z^{-2} and its power $\mathrm{wt}(A)$ is because we want (3.10) to be true.

3.4

Remark 3.5. The field $A^\theta(z)$ can also be understood in the following way: For each $u, v \in \mathbb{V}$ we have

$$\langle A^\theta(z)u|v\rangle = (-z^{-2})^{\mathrm{wt}(A)} \langle u|A(\overline{z^{-1}})v\rangle \tag{3.11}$$

as elements of $\mathbb{C}[[z^{\pm 1}]]$. By (3.6), the LHS resp. RHS is in $\mathbb{C}((z))$ resp. $\mathbb{C}((z^{-1}))$, we conclude that (3.11) is in $\mathbb{C}[z^{\pm 1}]$. Similarly,

$$\langle A(z)u|v\rangle \in \mathbb{C}[z^{\pm 1}]$$

Thus, $z \in \mathbb{C}^\times \to \langle A(z)u|v\rangle \in \mathbb{C}$ is a holomorphic function with finite poles at $0, \infty$, and (3.11) holds in $\mathscr{O}(\mathbb{C}^\times)$. It follows that for each $m, n \in \mathbb{V}$,

$$z \in \mathbb{C}^\times \mapsto P_m A(z) P_n$$

is an $\mathrm{Hom}(\mathbb{V}(n), \mathbb{V}(m))$-valued holomorphic function.

Proposition 3.6. *Let $u, v \in \mathbb{V}$. Let A be a homogeneous field. Then for each $z, q \in \mathbb{C}^\times$ we have*

$$\langle q^{L_0} A(z) q^{-L_0} u|v\rangle = q^{\mathrm{wt}(A)} \cdot \langle A(qz)u|v\rangle \tag{3.12}$$

In short, we have $q^{L_0}A(z)q^{-L_0} = q^{\mathrm{wt}(A)}A(qz)$ as linear maps $\mathbb{V} \to \mathbb{V}^{\mathrm{ac}}$. Compare this with Eq. (2.20).

Proof. For each fixed $q \in \mathbb{C}^\times$, by expanding both sides of (3.12) as Laurent series of z, we see that (3.12) is equivalent to

$$\langle q^{L_0} A_n q^{-L_0} u|v\rangle = q^{\mathrm{wt}(A)-n-1} \langle A_n u|v\rangle \tag{3.13}$$

By linearity, it suffices to assume that u, v are homogenous. In that case, this relation follows immediately from (3.5). $\square$

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