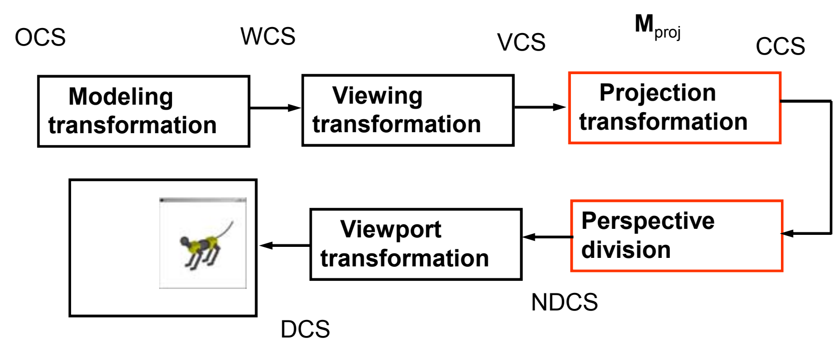
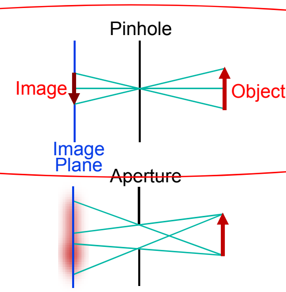
Graphics Pipeline



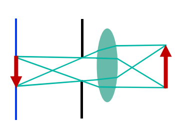
Reminder: Cameras (and the Eye)

Ideal pinhole

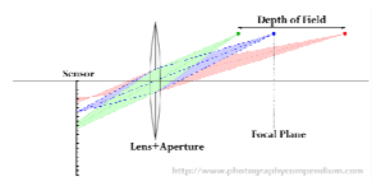


Aperture

Aperture + lens



Depth of field



Projection Transformations

Mapping: $T : R^n \rightarrow R^m$

Projection: n > m

We are interested in

$R^3 \rightarrow R^2$ or

$R^4 \rightarrow R^3$ in homogenous coordinates

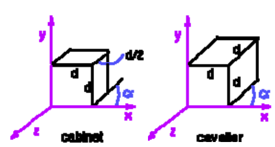
Planar Projections:

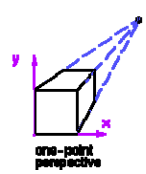
Projections onto a plane

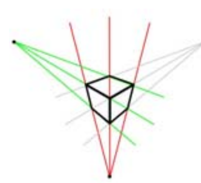
Taxonomy and Examples

Planar Projections

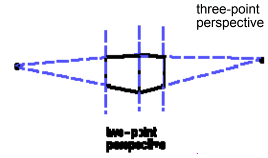
- Parallel
  - Orthographic
    - Top
    - Front
    - Side
    - Axonometric:
      - Isometric
      - Dimetric
      - Trimetric
  - Oblique
    - Cabinet
    - Cavalier
- Perspective
  - 1 Point
  - 2 Point
  - 3 Point

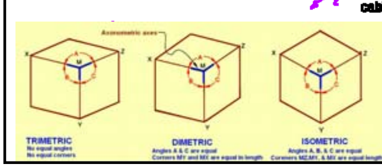






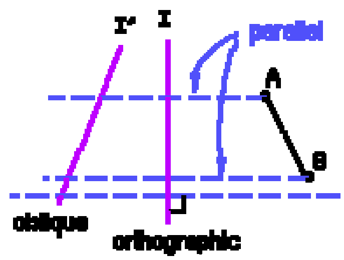
three-point perspective



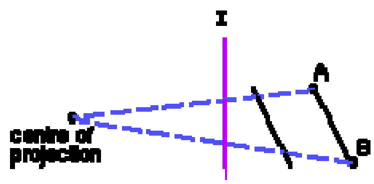


Basic Projections

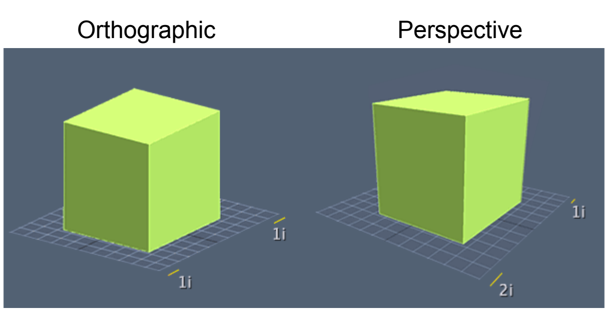
Parallel



Perspective



Orthographic vs Perspective



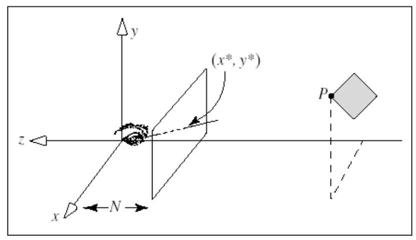
Camera Coordinate System

Camera at (0,0,0)

Looking at $-z$

Image plane *aka* ***near plane***

at $z = -N$

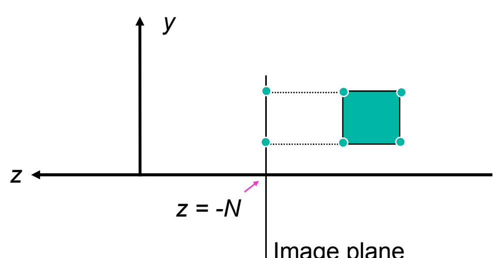


Basic Orthographic Projection

$P'_x = P_x$

$P'_y = P_y$

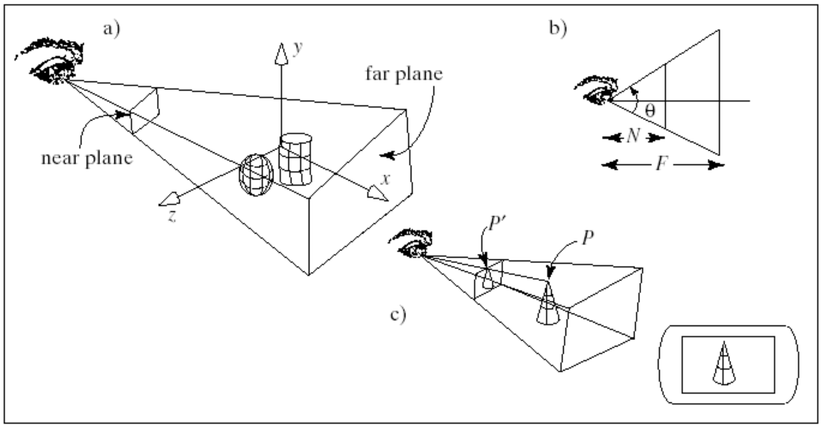
$P'_z = -N$



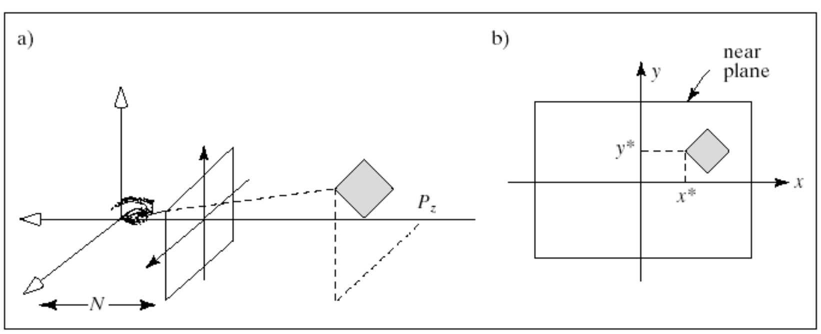
Matrix Form *($P' = \mathbf{M}P$):*

$$\begin{bmatrix} P'_x \\ P'_y \\ P'_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$$

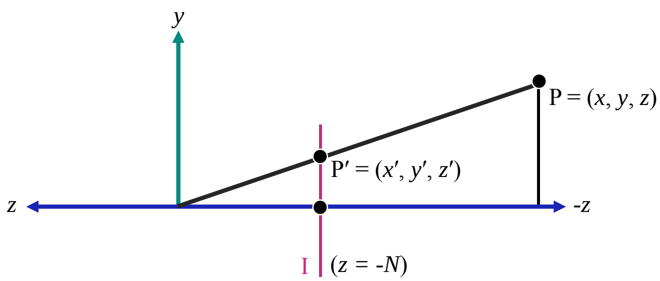
Perspective Projection



Perspective Projection of a Point



Basic Perspective Projection



Similar triangles

$$y'/N = y/-z \quad \Rightarrow \quad P_y' = P_y N / -P_z$$

Similarly $\quad P_x' = P_x N / -P_z$

$$P_z' = -N$$

This is a non-linear transformation!

In Homogeneous Matrix Form

Reminder:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \rightarrow \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \xrightarrow[\times w]{} \begin{bmatrix} wP_x \\ wP_y \\ wP_z \\ w \end{bmatrix} \xrightarrow[\div w]{\text{homogenize}} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

(a line in 4D space)

Perspective projection:

$$\begin{bmatrix} P_x' \\ P_y' \\ P_z' \\ 1 \end{bmatrix} = \begin{bmatrix} P_x N/(-P_z) \\ P_y N/(-P_z) \\ -N \\ 1 \end{bmatrix} \xrightarrow[-P_z/N]{\times} \begin{bmatrix} P_x \\ P_y \\ P_z \\ -P_z/N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/N & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$$

Therefore: $\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/N & 0 \end{bmatrix}}_{\text{Matrix } \mathbf{M}} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \xrightarrow[\div\, -P_z/N]{\text{and then: homogenize}} \begin{bmatrix} P_x' \\ P_y' \\ P_z' \\ 1 \end{bmatrix}$

$\div -P_z/N$

Homogenization step:
"Perspective Division"
(divide by $w = -P_z/N$)

Observations

- Projection undefined for $P_z = 0$
- If P is behind the eye,
  P_z changes sign
- Near plane N just scales the picture
- A line projects to a line
- Perspective foreshortening

$$P'_x = -N\frac{P_x}{P_z}$$
$$P'_y = -N\frac{P_y}{P_z}$$
$$P'_z = -N$$

Perspective Projection of a Line

$$L(t) = \mathbf{P} + \mathbf{v}t = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} t$$

Perspective Division & drop fourth coordinate $\longrightarrow$

Is it still a line?

$$\text{Original:}\quad L(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}P_x+v_xt\\P_y+v_yt\\P_z+v_zt\end{bmatrix}$$

$$\text{Projected:}\quad L'(t)=\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}-Nx/z\\-Ny/z\\-N\end{bmatrix}=\begin{bmatrix}-N(P_x+v_xt)/(P_z+v_zt)\\-N(P_y+v_yt)/(P_z+v_zt)\\-N\end{bmatrix}$$

$$x'=-N(P_x+v_xt)/(P_z+v_zt)\Rightarrow x'(P_z+v_zt)=-N(P_x+v_xt)\Rightarrow$$

$$x'P_z+x'v_zt=-NP_x-Nv_xt\Rightarrow\begin{cases}x'P_z+NP_x=-(x'v_z+Nv_x)t\\ \text{ and similarly for y:}\\ y'P_z+NP_y=-(y'v_z+Nv_y)t\end{cases}$$

Is it still a line? (cont'd)

$$\left.\begin{matrix}x'P_z+NP_x=-(x'v_z+Nv_x)t\\y'P_z+NP_y=-(y'v_z+Nv_y)t\end{matrix}\right|\Rightarrow\left.\begin{matrix}x'P_z+NP_x=-(x'v_z+Nv_x)t\\(y'v_z+Nv_y)t=-(y'P_z+NP_y)\end{matrix}\right|\Rightarrow$$

Multiply the two equations and divide through by *t*

$$(x'P_z+NP_x)(y'v_z+Nv_y)=(x'v_z+Nv_x)(y'P_z+NP_y)\Rightarrow$$

$$\cancel{x'P_zy'v_z}+x'P_zNv_y+NP_xy'v_z+N^2P_xv_y=\cancel{x'v_zy'P_z}+x'v_zNP_y+Nv_xy'P_z+N^2P_yv_x\Rightarrow$$

$$(P_zNc_y-v_zNP_y)x'+(NP_xv_z+Nv_xP_z)y'+N^2(P_xv_y+P_yv_x)=0\Rightarrow$$

$\Rightarrow$ $\boxed{ax'+by'+c=0}$ which is the equation of a line in the x'-y' plane

But is There a Difference?

$$\text{Original:}\quad L(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}P_x+v_xt\\P_y+v_yt\\P_z+v_zt\end{bmatrix}$$

$$\text{Projected:}\quad L'(t)=\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}-Nx/z\\-Ny/z\\-N\end{bmatrix}=\begin{bmatrix}-N(P_x+v_xt)/(P_z+v_zt)\\-N(P_y+v_yt)/(P_z+v_zt)\\-N\end{bmatrix}$$

But is There a Difference?

The "speed along the lines" if $v_z \neq 0$

$$\text{Original:}\quad L(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}P_x+v_xt\\P_y+v_yt\\P_z+v_zt\end{bmatrix}\Rightarrow \boxed{\frac{\partial L(t)}{\partial t}=\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}=\mathbf{v}}$$

$$\text{Projected:}\quad L'(t)=\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}-Nx/z\\-Ny/z\\-N\end{bmatrix}=\begin{bmatrix}-N(P_x+v_xt)/(P_z+v_zt)\\-N(P_y+v_yt)/(P_z+v_zt)\\-N\end{bmatrix}\Rightarrow$$

$$\frac{\partial x'}{\partial t}=-N\frac{\partial}{\partial t}((P_x+v_xt)/(P_z+v_zt))=-N\frac{v_x(P_z+v_zt)-(P_x+v_xt)v_z}{(P_z+v_zt)^2}=-N\frac{v_xP_z-P_xv_z}{(P_z+v_zt)^2}\Rightarrow$$

$$\boxed{\frac{\partial L'(t)}{\partial t}=\frac{-N}{(P_z+v_zt)^2}\begin{bmatrix}v_xP_z-P_xv_z\\v_yP_z-P_yv_z\\0\end{bmatrix}}$$

As time t tends to infinity, the speed along the projected line L' tends to zero

Effect of Perspective Projection on Lines

Line equations

$$\text{Original:}\quad L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z + v_z t \end{bmatrix}$$

$$\text{Projected:}\quad L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(P_x + v_x t)/(P_z + v_z t) \\ -N(P_y + v_y t)/(P_z + v_z t) \\ -N \end{bmatrix}$$

If lines in space are parallel to the view plane then:

$$v_z = 0 \rightarrow L'(t) = -\frac{N}{P_z} \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z \end{bmatrix}$$

slope of line: $\frac{v_y}{v_x}$ so, parallel lines parallel to the view plane remain parallel

Effect of Perspective Projection on Lines

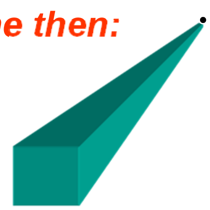
Line equations (again)

$$\text{Original:}\quad L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z + v_z t \end{bmatrix}$$

$$\text{Projected:}\quad L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(P_x + v_x t)/(P_z + v_z t) \\ -N(P_y + v_y t)/(P_z + v_z t) \\ -N \end{bmatrix}$$

If lines are not parallel to the view plane then:

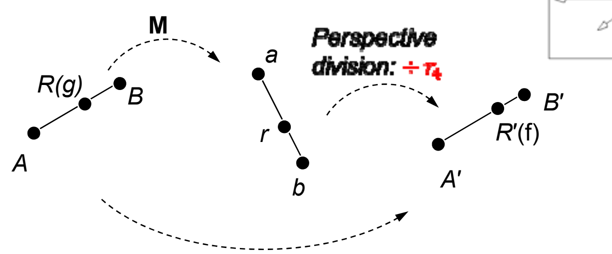
$$v_z \neq 0 \rightarrow \lim_{t \to \infty} L'(t) = \begin{bmatrix} -Nv_x/v_z \\ -Nv_y/v_z \\ -N \end{bmatrix}$$

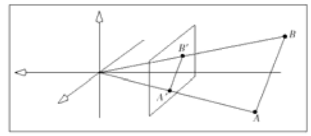


Lines converge to a **vanishing point**!

Foreshortening: In-Between Points on Perspective-Projected Lines

How do points on lines transform?





View coordinate system: $R(g) = (1 - g)A + gB$

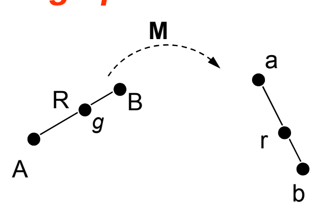
Projected homogeneous 4D: $r = \mathbf{M}R$

Projected homogeneous 3D: $R'(f) = (1 - f)A' + fB'$

g and f are <u>not</u> the same
What is the relationship between g and f?

First Step

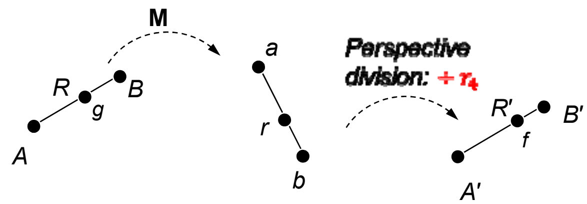
Viewing space to homogeneous space (4D)



$$R = (1-g)A + gB$$
$$r = \mathrm{M}R = \mathrm{M}[(1-g)A + gB] = (1-g)\mathrm{M}A + g\mathrm{M}B \Rightarrow$$
$$r = (1-g)a + gb$$
$$a = \mathrm{M}A = [a_1, a_2, a_3, a_4]^T$$
$$b = \mathrm{M}B = [b_1, b_2, b_3, b_4]^T$$

Second Step

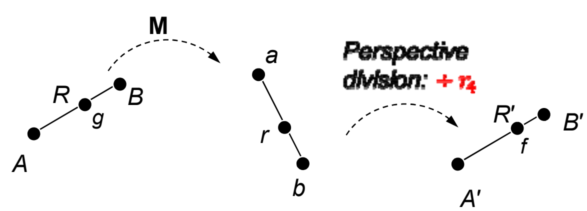
Perspective division



$$\left\{ \begin{array}{l} r = (1-g)a + gb \\ a = [a_1, a_2, a_3, a_4]^T \\ b = [b_1, b_2, b_3, b_4]^T \end{array} \right\} \Rightarrow \boxed{R'_1 = \frac{r_1}{r_4} = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4}}$$

And similarly for R'_2 *and* R'_3

Putting it Together



$$R'_1 = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4} = \frac{\text{lerp}(a_1, b_1, g)}{\text{lerp}(a_4, b_4, g)}$$

lerp: linear Interpolation
(done by hardware acceleration)

Furthermore:

$$R' = (1-f)A' + fB' \Rightarrow R'_1 = (1-f)A'_1 + fB'_1$$

$$R'_1 = (1-f)\frac{a_1}{a_4} + f\frac{b_1}{b_4} = \text{lerp}\left(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f\right)$$

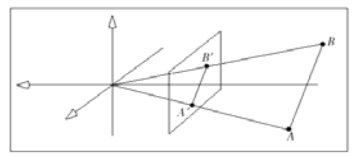
Relation Between the Fractions

$$\left.\begin{array}{l} R'_1(f) = \dfrac{\text{lerp}(a_1, b_1, g)}{\text{lerp}(a_4, b_4, g)} \\[2ex] R'_1(f) = \text{lerp}\left(\dfrac{a_1}{a_4}, \dfrac{b_1}{b_4}, f\right) \end{array}\right\} \Rightarrow g = \frac{f}{\text{lerp}(\frac{b_4}{a_4}, 1, f)}$$

substituting this in $R(g) = (1-g)A + gB$ yields

$$R_1 = \frac{\text{lerp}(\frac{A_1}{a_4}, \frac{B_1}{b_4}, f)}{\text{lerp}(\frac{1}{a_4}, \frac{1}{b_4}, f)}$$

similarly for R_2 & R_3



THIS MEANS: For a given *f* in **image space** and *A, B* in **viewing space,** we can find the corresponding *R (or g)* in **viewing space** using the above formula

This works if "*A*", "*B*" are **positions**, **texture coordinates**, **color**, **normals**, etc.

So, it is generally VERY useful during rasterization (to be covered later)

Summary

Perspective projection is <u>non-linear</u>

Lines project to lines

Parallel lines either project to parallel lines or they intersect at the vanishing point

Foreshortening of projected lines and the "Inbetweeness" relationship