A REMARK ON CONTINUOUS K-THEORY AND FOURIER-SATO TRANSFORMATION

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Abstract. In this note, we prove a generalization of Efimov's computation for the universal localizing invariant of categories of sheaves with certain microsupport constraint. The proof base on certain categorical equivalences given by the Fourier-Sato transformation, which is different from the original proof. As an application, we compute the universal localizing invariant of the category of almost quasi-coherent sheaf on the Novikov toric scheme introduced by Vaintrob.

1. Introduction

In [Efi24], Efimov introduces an algebraic K-theory for a class of large categories, say dualizable stable categories, which extends the usual nonconnective algebraic K-theory defined for compactly generated categories. In general, the construction enables us to extend localizing invariants of small categories to dualizable stable categories. In particular, for the universal (finitary) localizing invariant $\mathcal{U}_{loc} : \mathrm{Cat}^{\mathrm{Ex}} \to \mathrm{Mot}_{\mathrm{loc}}$, there exists a canonical extension $\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \to \mathrm{Mot}_{\mathrm{loc}}$, where $\mathrm{Mot}_{\mathrm{loc}}$ is the category of non-commutative motives.

However, as the construction involves computations about the so-called Calkin category that is not easy to describe, computation of the continuous version of localizing invariants is even harder, and few computational results are known. One distinguished result among them is the following:

Theorem 1.1 ([Efi24, Theorem 6.11])**.** *Let X be a locally compact Hausdorff space and $\underline{\mathcal{C}}$ be a presheaf on X with values in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$. Then the category $\mathrm{Sh}(X;\underline{\mathcal{C}})$ is dualizable stable and we have the following natural isomorphism in $\mathrm{Mot}_{\mathrm{loc}}$:*

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathrm{Sh}(X;\underline{\mathcal{C}})) \simeq \Gamma_c(X, (\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}\underline{\mathcal{C}})^\sharp).$$

Another interesting example concerns categories of sheaves with microsupport constraints. For a manifold M and any $F \in \mathrm{Sh}(M)$, Kashiwara and Schapira introduced a conic closed set $\mathrm{SS}(F) \subset T^*M$ in [KS90], which is called the microsupport of sheaves. For a conic closed set $Z \subset T^*M$, we denote $\mathrm{Sh}_Z(M;\mathcal{C})$ the full subcategory of sheaves whose microsupport is bounded by Z. Then $\mathrm{Sh}_Z(M;\mathcal{C})$ is dualizable stable (when $\mathcal{C}$ is) since it is a reflexive subcategory of $\mathrm{Sh}(M;\mathcal{C})$.

Theorem 1.2 ([Efi24, Proposition 4.21])**.** *Let $\mathcal{C}$ be a dualizable stable category. Then for the category $\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C})$, which is known to be dualizable stable, we have the following natural equivalence in $\mathrm{Mot}_{\mathrm{loc}}$:*

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C})) \simeq 0.$$

It was shared with us by Alexander I. Efimov, during the *Masterclass: Continuous K-theory* in University of Copenhagen on June 2024, that Theorem 1.2 is still true for a finite dimensional real vector space V with the microsupport constraint $Z = V \times \gamma$ for a non-zero proper closed convex cone γ (i.e. Equation (1.1) below). We are grateful for his generosity. One can prove the high dimensional version in the same way as the 1-dimensional version using a V-indexed semi-orthogonal decomposition.

New results. In this article, using the Fourier-Sato transformation, we directly identify certain categories of sheaves with microsupport constraint with certain categories without microsupport constraint. Those facts are well-known to experts; however, the interesting part is that we can use them to deduce a generalization of Theorem 1.2 directly from Theorem 1.1.

To achieve our target, we give a definition of microsupport for more general coefficient categories. And in particular, we show that if $\mathcal{C}$ is presentable stable, then the definition inherits the most of nice properties deduced in [KS90]. In particular, we may develop a $\mathrm{Mot}_{\mathrm{loc}}$-valued microlocal sheaf theory without any difficulty. Those constructions may be of independent interests.

Our main result is

Theorem 1.3 (Theorem 4.4 (2-b) below). *Let $\mathcal{C}$ be a dualizable stable category. For a finite dimensional real vector space V and a conic closed set $X \subset V^\vee$, we have the equivalence*

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C})) \simeq \Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})).$$

In particular, if we pick $X = \gamma$, a non-zero proper convex closed cone. Then we deduce from a direct cohomology computation that

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times \gamma}(V;\mathcal{C})) \simeq \Gamma_c(\,\gamma\,;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})) = 0, \tag{1.1}$$

which is the straightforward generalization of Theorem 1.2.

Remark 1.4. The proof of Theorem 1.2 therein is better in the sense that it could be generalized to all accessible localizing invariants, in particular, this also works for Equation (1.1). However, so far we only know Theorem 1.3 works for finitary localizing invariants due to the state of Theorem 1.1.

Remark 1.5. Here, we explain logic dependence of results.

The original proof of Theorem 1.2 (and Equation (1.1)) uses the microlocal cut-off lemma of Kashiwara and Schapira to identify the corresponding categories to sheaves over the so-called $\gamma^\vee$-topology (which is non-Hausdorff!), and then it is concluded by a semi-orthogonal decomposition of corresponding presheaf categories. No other machinery in microlocal sheaf theory is involved.

Our proof uses more machinery from the microlocal sheaf theory, it is unsurprising that the microlocal cut-off lemma appears implicitly in our approach. However, we will not construct any semi-orthogonal decomposition, which makes our proof different from the original one.

Next, we present an application. For a fan[1] Σ in $\mathbb{R}^n$, Vaintrob constructs a non-Noetherian $\mathbf{k}$-scheme (where $\mathbf{k}$ is a discrete ring), the so-called Novikov toric scheme,

[1] We do not ask the fan to be rational with respect to a fixed lattice in $\mathbb{R}^n$.

X_Σ^{Nov} and a subscheme $\partial_\Sigma^{\mathrm{Nov}}$ defined by an idempotent ideal sheaf in [Vai17]. Then we can discuss the category of almost coherent sheaves on the almost content $(X_\Sigma^{\mathrm{Nov}}, \partial_\Sigma^{\mathrm{Nov}})$. If Σ is rational, X_Σ^{Nov} is strongly related to the infinite root stack $\sqrt[\infty]{X_\Sigma}$ of the usual toric variety. We refer to [KZ25] for more details.

We have the following result, which is first proven by Vaintrob and then by Kuwagaki and the author [KZ25] using a different method.

Theorem 1.6. *For a fan Σ and $\mathrm{Mod}_{\mathbf{k}}$ the category of $\mathbf{k}$-modules, we have*

$$\mathrm{aQCoh}_{\mathbb{T}^{\mathrm{Nov}}}(X_\Sigma^{\mathrm{Nov}}, \partial_\Sigma) \simeq \mathrm{Sh}_{\mathbb{R}^n \times |\Sigma|}(\mathbb{R}^n; \mathrm{Mod}_{\mathbf{k}}).$$

Then as an application of Theorem 1.3, we have

Corollary 1.7. *For a fan Σ, we have*

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{aQCoh}_{\mathbb{T}^{\mathrm{Nov}}}(X_\Sigma^{\mathrm{Nov}}, \partial_\Sigma)) \simeq \Gamma_c(|\Sigma|; \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Mod}_{\mathbf{k}})).$$

Remark 1.8. Instead of Theorem 1.3, one can also deduce the Corollary from (1.1) and [Efi24, Proposition 4.11] based on the fiber product decomposition of $\mathrm{aQCoh}_{\mathbb{T}^{\mathrm{Nov}}}(X_\Sigma^{\mathrm{Nov}}, \partial_\Sigma)$ explained in [KZ25].

Category convention. In this article, we always mean ∞-categories when referring to categories. We denote by $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ the category of dualizable stable categories consisting of presentable stable categories that are dualizable with respect to the Lurie tensor product and strongly continuous functors between them. In particular, compactly generated stable categories are dualizable. We denote Sp the category of spectra. We denote the countable cardinality by ω.

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2. Sheaves and microsupport

For a bi-complete category $\mathcal{C}$ and a topological space X, we denote the category of $\mathcal{C}$-valued sheaves by $\mathrm{Sh}(X; \mathcal{C})$ [Lur09, 7.3.3.1]. It is explained in [Vol21] that in this case we have an equivalence $\mathrm{Sh}(X; \mathrm{Sp}) \otimes \mathcal{C} \simeq \mathrm{Sh}(X; \mathcal{C})$.

It is further explained in loc. cit. that in the $\mathcal{C}$-valued sheaf setting, we can define the functors $f_*^{\mathcal{C}}, f_!^{\mathcal{C}}$, and $f_{\mathcal{C}}^*, f_{\mathcal{C}}^!$. When $\mathcal{C}$ is, in addition, symmetric monoidal, we can define the monoidal product $\otimes_{\mathcal{C}}$ and the internal hom $\mathcal{H}om_{\mathcal{C}}$ as the right adjoint of $\otimes_{\mathcal{C}}$, yielding the full six-functor formalism. We also refer to [Sch22] for further details on the six-functor formalism.

Regarding the microlocal theory of sheaves in the ∞-categorical setting, we remark that, based on [RS18], all arguments in [KS90] extend to the case where $\mathcal{C}$ is compactly generated. Therefore, the microlocal sheaf theory with compactly generated coefficients, for example $\mathcal{C} = \mathrm{Sp}$, can be developed without difficulty.

However, as suggested in [Efi24, Remark 4.24], it is possible to develop a microlocal sheaf theory with more general coefficients.

Here, we present the Ω-lens definition. Let M be a smooth manifold, and set $\dot{T}^*M = T^*M \setminus 0_M$.

Definition 2.1. [GV24, Definition 3.1] Let $\Omega \subset \dot{T}^*M$ be an open conic subset. We call a locally closed subset C of M an *Ω-lens* if the following conditions are satisfied: $\overline{C}$ is compact, and there exists an open neighborhood U of $\overline{C}$ and a function $g\colon U \times [0,1] \to \mathbb{R}$ such that

(1) $dg_t(x) \in \Omega$ for all $(x,t) \in U \times [0,1]$, where $g_t = g|_{U\times\{t\}}$;
(2) $\{g_t < 0\} \subset \{g_{t'} < 0\}$ if $t \leq t'$;
(3) the hypersurfaces $\{g_t = 0\}$ coincide on $U \setminus \overline{C}$;
(4) $C = \{g_1 < 0\} \setminus \{g_0 < 0\}$.

Definition 2.2. Let $\mathcal{C}$ be a category that admits small limits, and let $F \in \mathrm{Sh}(M;\mathcal{C})$. We define $\dot{T}^*M \setminus \dot{\mathrm{SS}}_{\mathcal{C}}(F)$ as the maximal open subset $\Omega \subset \dot{T}^*M$ such that, for any Ω-lens C defined by a smooth function g as in Definition 2.1, the restriction morphism

$$\Gamma(\{g_1 < 0\}, F) \to \Gamma(\{g_0 < 0\}, F)$$

is an equivalence.

We then define the microsupport of F by $\mathrm{SS}_{\mathcal{C}}(F) = \dot{\mathrm{SS}}_{\mathcal{C}}(F) \cup \mathrm{supp}(F)$. When the category $\mathcal{C}$ is clear from context, we simply write $\mathrm{SS}(F)$.

When $\mathcal{C} = \mathrm{Sp}$, there exists a pointwise definition of microsupport $\mathrm{SS}_{KS}(F) \subset T^*M$, as explained in [KS90, RS18]. The argument therein shows that $\mathrm{SS}_{KS}(F) \cap 0_M = \mathrm{supp}(F)$, and hence $\mathrm{SS}_{KS}(F) \cap 0_M = \mathrm{SS}_{\mathrm{Sp}}(F) \cap 0_M$. For the non-zero part, we have the following:

Lemma 2.3. [GV24, Lemma 3.2] *Let $F \in \mathrm{Sh}(M;\mathrm{Sp})$ and let $\Omega \subset T^*M \setminus 0_M$ be an open conic subset. Then $\dot{\mathrm{SS}}_{KS}(F) \cap \Omega = \emptyset$ if and only if $\mathrm{Hom}(1_C, F) \simeq 0$ for any Ω-lens C.*

Corollary 2.4. *For $F \in \mathrm{Sh}(M;\mathrm{Sp})$, we have $\mathrm{SS}_{KS}(F) = \mathrm{SS}_{\mathrm{Sp}}(F)$.*

On the other hand, based on [GV24, Lemma 3.3], we incorporate the covariant Verdier duality with microsupport:

Proposition 2.5. [KZ25, Theorem B.8][2] *For a stable category $\mathcal{C}$ that admits both small limits and colimits, we have $\dot{\mathrm{SS}}_{\mathcal{C}}(F) \cap \Omega = \emptyset$ if and only if $\Gamma_c(\{g_0 < 0\}, F) \to \Gamma_c(\{g_1 < 0\}, F)$ is an equivalence for any $-\Omega$-lens defined by g.*

For a conic closed subset $Z \subset T^*M$, we set $\mathrm{Sh}_Z(M;\mathcal{C})$ as the full subcategory of $\mathrm{Sh}(M;\mathcal{C})$ spanned by F with $\mathrm{SS}_{\mathcal{C}}(F) \subset Z$. The Definition 2.2 shows that $\mathrm{Sh}_Z(M;\mathcal{C})$ is closed under limits, and Proposition 2.5 shows that $\mathrm{Sh}_Z(M;\mathcal{C})$ is closed under colimits. Therefore, we know that the inclusion $i_Z^{\mathcal{C}} : \mathrm{Sh}_Z(M;\mathcal{C}) \to \mathrm{Sh}(M;\mathcal{C})$ admits both left and right adjoints.

In particular, for any presentable stable category $\mathcal{C}$, under the natural identification $\mathrm{Sh}(M;\mathrm{Sp}) \otimes \mathcal{C} \simeq \mathrm{Sh}(M;\mathcal{C})$, we have $i_Z^{\mathrm{Sp}} \otimes \mathcal{C} : \mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C} \to \mathrm{Sh}(M;\mathcal{C})$ is fully-faithful. Moreover, we have

Proposition 2.6 ([Efi24, Remark 4.24])**.** *For a presentable stable category $\mathcal{C}$ and a conic closed subset $Z \subset T^*M$, the essential image of the functor $\mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C} \to \mathrm{Sh}(M;\mathcal{C})$ is identified with $\mathrm{Sh}_Z(M;\mathcal{C})$. Equivalently, we have $i_Z^{\mathcal{C}} \simeq i_Z^{\mathrm{Sp}} \otimes \mathcal{C}$ and*

$$\mathrm{Sh}_Z(M;\mathcal{C}) \simeq \mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C}.$$

[2] In the arXiv version of the article, we require a symmetric monoidal structure for $\mathcal{C}$, which is unnecessary.

Proof. By Proposition 2.5, we have that $(i_Z^{\mathrm{Sp}})^l$ can be characterized as a presentable Bousfield localization that is local with respect to morphisms $\mathbb{S}_{\{g_0<0\}} \to \{g_1 < 0\}$ such that g_t defines a $-\dot{T}^*M \setminus Z$-lense. Let us denote by W_{-Z} the set of morphisms(since ω-lenses form a set).

We identify $\mathrm{Sh}(M;\mathcal{C}) = \mathrm{Fun}^L(\mathrm{Sh}(M;\mathrm{Sp}),\mathcal{C})$ and $\mathrm{Sh}_Z(M;\mathrm{Sp})\otimes\mathcal{C} = \mathrm{Fun}^L(\mathrm{Sh}_{-Z}(M;\mathrm{Sp}),\mathcal{C})$ (i.e. corresponding cosheaf categories) by dualizability, we have that $\mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C}$ consists exactly colimit preserving functors $\mathrm{Sh}(M;\mathrm{Sp}) \to \mathcal{C}$ that are local with respect to W_{-Z}. However, those W_{-Z}-local colimit preserving functors form exactly $\mathrm{Sh}_Z(M;\mathcal{C})$ in $\mathrm{Sh}(M;\mathcal{C}) = \mathrm{Fun}^L(\mathrm{Sh}(M;\mathrm{Sp}),\mathcal{C})$ by the definition of $\mathrm{Sh}_Z(M;\mathcal{C})$. The result then follows from the above discussion and [Lur09, Proposition 5.5.4.2]. □

Consequently, all microsupport estimation results in [KS90] are correct for $\mathcal{C}$-valued sheaves since they are correct for Sp-valued sheaves.

Remark 2.7. When discussing monoidal structures on $\mathcal{C}$ and related $\mathcal{C}$-linear dualizability, we need more constraints. For example, one can assume $\mathcal{C}$ is (dualizable stable) locally rigid. See, for example, [Ram24]. In this note, we only discuss the monoidal structure on Sp, which is known to be rigid.

Now, we consider the following category introduced by Tamarkin. For a dualizable stable category $\mathcal{C}$, we set

$$\mathcal{T}_{\mathcal{C}} \coloneqq \mathrm{Sh}(\mathbb{R};\mathcal{C})/\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C}).$$

Using the bi-fiber sequence $\mathcal{T}_{\mathrm{Sp}} \to \mathrm{Sh}(\mathbb{R};\mathrm{Sp}) \to \mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp})$, we have that $\mathcal{T}_{\mathcal{C}} \simeq \mathcal{T}_{\mathrm{Sp}} \otimes \mathcal{C}$, and for a smooth manifold M we have

$$\mathrm{Sh}(M;\mathrm{Sp}) \otimes \mathcal{T}_{\mathcal{C}} \simeq \mathrm{Sh}(M;\mathcal{T}_{\mathcal{C}}) \simeq \mathrm{Sh}(M \times \mathbb{R};\mathcal{C})/\mathrm{Sh}_{T^*M\times\mathbb{R}\times[0,\infty)}(M \times \mathbb{R};\mathcal{C}). \tag{2.1}$$

To avoid certain confusion, we denote the three equivalent categories by

$$\mathcal{T}(T^*M;\mathcal{C}).$$

We refer to [KSZ23] for both the motivation of this notation and the proof of the equivalence of above definitions (for the case $\mathcal{C} = \mathrm{Sp}$, and the general case follows from Proposition 2.6.

Therefore, for $[F] \in \mathcal{T}(T^*M;\mathcal{C})$, we can discuss the positive part of microsupport, say

$$SS_+([F]) \coloneqq SS(F) \cap \{\tau > 0\}$$

is a well-defined closed conic subset of $T^*M \times \mathbb{R} \times (0,\infty)$.

The construction is introduced by Tamarkin to study non-conic subsets $Z \subset T^*M$. Precisely, for a subset $Z \subset T^*M$, we define its cone as

$$\widehat{Z} \coloneqq \{(q,p,t,\tau) \in T^*M \times \mathbb{R} \times (0,\infty) : (q, p/\tau) \in Z\}.$$

The conic set $\widehat{Z}$ is closed in $T^*M \times \mathbb{R} \times (0,\infty)$ if Z is closed in T^*M, and in this case we denote $\mathcal{T}_Z(T^*M;\mathcal{C})$ the full subcategory spanned by those $[F]$ such that $SS_+([F]) \subset \widehat{Z}$.

Proposition 2.8. *For a conic closed set $Z \subset T^*M$, we have*

$$\mathcal{T}_Z(T^*M;\mathcal{C}) \simeq \mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{T}_{\mathcal{C}} \simeq \mathrm{Sh}_Z(M;\mathcal{C}) \otimes \mathcal{T}_{\mathrm{Sp}}.$$

Proof. If Z is already conic, then we have $\widehat{Z} = Z \times \mathbb{R} \times (0, \infty)$. Therefore, we have

$$\mathcal{T}_Z(T^*M; \mathcal{C}) = \mathrm{Sh}_{Z\times\mathbb{R}\times(0,\infty)}(M \times \mathbb{R}; \mathcal{C}).$$

Then we can apply the Künneth formula for the category of sheaves with general microsupport condition, see [KSZ23] or [Zha25]. In loc. cit., we prove the Kunneth formula for compactly generated rigid symmetric monoidal $\mathcal{C}$. In particular, for $\mathcal{C} = \mathrm{Sp}$. As a result, we have

$$\mathcal{T}_Z(T^*M; \mathcal{C}) = \mathrm{Sh}_{Z\times\mathbb{R}\times(0,\infty)}(M\times\mathbb{R}; \mathrm{Sp}) = \mathrm{Sh}_Z(M; \mathrm{Sp})\otimes\mathrm{Sh}_{\mathbb{R}\times(0,\infty)}(\mathbb{R}; \mathrm{Sp}) = \mathrm{Sh}_Z(M; \mathrm{Sp})\otimes\mathcal{T}_{\mathrm{Sp}}.$$

Then for more general coefficients $\mathcal{C}$, the result is a formal consequence of Proposition 2.6. $\square$

3. Fourier-Sato-Tamarkin transform

In this section, we only consider $\mathcal{C} = \mathrm{Sp}$. This restriction does not affect generality in our discussion due to Proposition 2.6.

The Fourier-Sato transform was first introduced by Sato in [SKK73]. We refer to [KS90, Section 3.7] for more the relevant discussion. The Fourier-Sato transform gives an equivalence between $\mathbb{R}_{>0}$-equivariant sheaves on V and $V^\vee$ for real vector space V. To adapt to various non-equivariant situations, one can consider some variants of the Fourier-Sato transform. We refer to [D'A13, Gao17] for more relevant discussion on their definition and the comparison between them.

In [Tam18], Tamarkin introduces a variant of the Fourier-Sato transform $\mathrm{Sh}(V; \mathcal{T}) \to \mathrm{Sh}(V^\vee; \mathcal{T})$ that induces an equivalence for sheaves that are not necessarily $\mathbb{R}_{>0}$-equivariant. We call it the Fourier-Sato-Tamarkin transform.

We naturally identify both T^*V and $T^*V^\vee$ with $V \times V^\vee$. Let $Leg(V) = \{(z, \zeta, t, s) : t - s + \langle z, \zeta\rangle \geq 0\} \subset V \times V^\vee \times \mathbb{R}^2$. We consider

$$\begin{aligned}
&\mathbb{S}_{Leg(V)} \in \mathrm{Sh}(V \times V^\vee \times \mathbb{R}^2; \mathrm{Sp}),\\
&p_V : V \times V^\vee \times \mathbb{R}_t \times \mathbb{R}_s \to V \times \mathbb{R}_s,\\
&p_{V^\vee} : V \times V^\vee \times \mathbb{R}_t \times \mathbb{R}_s \to V^\vee \times \mathbb{R}_t.
\end{aligned}$$

Definition 3.1. The Fourier-Sato-Tamarkin transform is defined as the functor

$$\begin{aligned}
&\mathrm{FST} : \mathcal{T}(T^*M; \mathrm{Sp}) \to \mathcal{T}(T^*V^\vee; \mathrm{Sp}),\\
&\mathrm{FST}(F) := p_{V^\vee!}(p_V^* F \otimes_{\mathrm{Sp}} \mathbb{S}_{Leg(V)})[\dim V].
\end{aligned}$$

It is proved in [Tam18, Theorem 3.5] that the Fourier-Sato-Tamarkin transform FST is an equivalence of categories.

Theorem 3.2. *The Fourier-Sato-Tamarkin transform* FST *is an equivalence of categories: For a closed set $X \subset V^\vee$, we have*

$$\mathcal{T}_{V\times X}(T^*V; \mathrm{Sp}) \simeq \mathcal{T}_{X\times V}(T^*V^\vee; \mathrm{Sp}).$$

This yields the following result

Proposition 3.3. *Let V be a finite dimensional real vector space and $X \subset V^\vee$ be a closed set. We have*

$$\mathcal{T}_{V\times X}(T^*V; \mathrm{Sp}) \simeq \mathrm{Sh}(X; \mathcal{T}_{\mathrm{Sp}}).$$

Proof. By Theorem 3.2, we have $\mathcal{T}_{V\times X}(T^*V;\mathrm{Sp}) \simeq \mathcal{T}_{X\times V}(T^*V^\vee;\mathrm{Sp})$. We notice that V is a conic closed set in $V = (V^\vee)^\vee$. Then $\mathcal{T}_{X\times V}(T^*V^\vee;\mathrm{Sp}) \simeq \mathrm{Sh}_{X\times V}(V^\vee;\mathrm{Sp}) \otimes \mathcal{T}_{\mathrm{Sp}}$ by Proposition 2.8.

Lastly, recall that $\mathrm{Sh}_{X\times V}(V^\vee;\mathrm{Sp})$ consists of sheaves $H \in \mathrm{Sh}(V^\vee;\mathrm{Sp})$ with the usual microsupport bound $\mathrm{SS}(H) \subset X \times V$. We notice that $\mathrm{SS}(H) \cap 0_{V^\vee} = \mathrm{supp}(H)$. Henceforth, we have $\mathrm{SS}(H) \subset X \times V$ if and only if H is supported in X. In particular, it means that $\mathrm{Sh}_{X\times V}(V^\vee;\mathrm{Sp}) \simeq \mathrm{Sh}(X;\mathrm{Sp})$. □

4. Localizing invariants

Let $\mathcal{E}$ be an accessible stable category (not necessarily cocomplete). Recall that a functor $F : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$ is a continuous localizing invariant if the following conditions hold:

(i) $F(0) = 0$;

(ii) for any bi-fiber sequence of the form
$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$
in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ the sequence
$$F(\mathcal{A}) \to F(\mathcal{B}) \to F(\mathcal{C})$$
is a fiber sequence in $\mathcal{E}$.

Roughly speaking, the main result of [Efi24] is that a localizing invariant is determined by its value on compactly generated categories, and all localizing invariants come in this way.

Remark 4.1. In general, we should be careful about accessibility of localizing invariants. Here, we will only discuss finitary localizing invariants, i.e. those commute with filtered colimits.

Among all finitary localizing invariants, there exists a universal one, which was originally studied by [BGT13] on compactly generated categories, and was later extended to dualizable stable categories by [Efi24].

More precisely, there exists an ω-accessible stable category $\mathrm{Mot}_{\mathrm{loc}}$ that is called the category of non-commutative motives and a universal finitary localizing invariant $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}} : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Mot}_{\mathrm{loc}}$ which is initial among all finitary localizing invariants:
$$\mathrm{Fun}^L(\mathrm{Mot}_{\mathrm{loc}}, \mathcal{E}) \simeq \mathrm{Fun}_{\mathrm{loc},\omega}(\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}, \mathcal{E}), \quad G \mapsto F = G \circ \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}.$$

Therefore, many properties of $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ are automatically shared by all finitary localizing invariants.

The following standard observation follows directly from the definition.

Lemma 4.2. *For a localizing invariant $F : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$ and a dualizable stable category $\mathcal{C}$, we have $F(\mathcal{C} \otimes -) : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$ is a localizing invariant.*

Now, we will discuss localizing invariants of $\mathrm{Sh}_{V\times X}(V;\mathcal{C}) \simeq \mathrm{Sh}_{V\times X}(V;\mathrm{Sp}) \otimes \mathcal{C}$. Then by Lemma 4.2, we may assume $\mathcal{C} = \mathrm{Sp}$ in the following proofs.

To start with, we present a proof of the 1-dimension Theorem 1.2 based on Theorem 1.1. The idea is to reverse the process of the original proof presented in [Efi22].

Proposition 4.3. *We have* $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C}))\simeq 0$ *and* $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{\mathcal{C}})\simeq\Omega\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})$*.*

Proof. One can check that we have the following Cartesian square of dualizable stable categories. See also [KZ25, Theorem 6.16] for a detailed proof of its generalization.

$$\begin{tikzcd}
\mathrm{Sh}(\mathbb{R};\mathrm{Sp}) \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp}) \arrow[d] \\
\mathrm{Sh}_{\mathbb{R}\times(-\infty,0]}(\mathbb{R};\mathrm{Sp}) \arrow[r] & \mathrm{Sh}_{\mathbb{R}\times\{0\}}(\mathbb{R};\mathrm{Sp}).
\end{tikzcd}$$

The map $x\mapsto -x$ identifies $\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp})\simeq\mathrm{Sh}_{\mathbb{R}\times(-\infty,0]}(\mathbb{R};\mathrm{Sp})$. Therefore, by [Efi24, Proposition 4.11], we have the fiber sequence in $\mathrm{Mot}_{\mathrm{loc}}$:

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp}))^{\oplus 2}\to\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R}\times\{0\}}(\mathbb{R};\mathrm{Sp}))\to\Sigma\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}(\mathbb{R};\mathrm{Sp})).$$

One can directly check that the second morphism is induced by the loop-suspension adjunction of $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sp})$, which yields an equivalence. Then we have $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp}))\simeq 0$.

For the next, we consider the bi-fiber sequence

$$\mathcal{T}_{\mathrm{Sp}}\to\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp})\to\mathrm{Sp}.$$

Then the second statement follows directly from the fact that $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ is a localizing invariant the first equivalence. □

Now, we can state our theorems.

Theorem 4.4. *Let V be a finite dimensional real vector space and and let $X\subset V^\vee$ be a closed set.*

(1) We have the equivalence:

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{V\times X}(T^*V;\mathrm{Sp}))\simeq\Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{\mathcal{C}}))\simeq\Omega\Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C}));$$

(2) If X is in addition conic, then we have

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C}))\simeq\Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})).$$

In particular, the same statements hold for any finitary localizing invariant in place of $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$.

Proof. (1) This is a direct corollary of Theorem 3.2, Theorem 1.1 and Proposition 4.3.
(2) By Proposition 2.8, when X is conic, we have

$$\mathcal{T}_{V\times X}(T^*V;\mathrm{Sp})\simeq\mathrm{Sh}_{V\times X}(V;\mathcal{C})\otimes\mathcal{T}_{\mathrm{Sp}}.$$

By Lemma 4.2 and (1), we have for any localizing invariants F that $F(\mathcal{T}_{\mathrm{Sp}})\simeq\Omega F(\mathrm{Sp})$. Taking $F=\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C})\otimes -)$, we obtain

$$\begin{aligned}
\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C}))&\simeq\Sigma\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C})\otimes\mathcal{T}_{\mathrm{Sp}})\\
&\simeq\Sigma\Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{\mathrm{Sp}}\otimes\mathcal{C}))\\
&\simeq\Sigma\Gamma_c(X;\Omega\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C}))\\
&\simeq\Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})),
\end{aligned}$$

where the second equivalence follows from (2-a) and the last equivalence follows from Proposition 4.3. Then we conclude the proof. □

Let $X \subset V^\vee$ be closed, and set $U = V^\vee \setminus X$, we can consider the quotient category (see [KSZ23, Section 5] for more details)

$$\mathcal{T}(V \times U; \mathcal{C}) \coloneqq \mathcal{T}(T^*V; \mathrm{Sp})/\mathcal{T}_{V\times X}(T^*V; \mathrm{Sp}),$$

and, if X is conic and then U is also conic, we can consider

$$\mathrm{Sh}(V, V \times U; \mathcal{C}) \coloneqq \mathrm{Sh}(V; \mathcal{C})/\mathrm{Sh}_{V\times X}(V; \mathcal{C})$$

as in [KS90, Definition 6.1.1].

We then obtain the following consequence:

Corollary 4.5. *Let V be a finite dimensional real vector space and a closed set $X \subset V^\vee$ with $U = V^\vee \setminus X$.*
(1) We have the equivalence:

$$\mathcal{T}(V \times U; \mathcal{C}) \simeq \Omega\Gamma_c(U; \mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathcal{C}));$$

(2) If X is in addition conic, then we have

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathrm{Sh}(V, V \times U; \mathcal{C})) \simeq \Gamma_c(U; \mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathcal{C})).$$

In particular, the same statements hold for any finitary localizing invariant in place of $\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}$.

Proof. We prove (2); the argument for (1) is analogous.

By Proposition 2.5, we know the quotient functor $\mathrm{Sh}(V; \mathcal{C}) \to \mathrm{Sh}_{V\times X}(V; \mathcal{C})$ is strongly continuous. This yields a fiber sequence in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$

$$\mathrm{Sh}(V, V \times U; \mathcal{C}) \to \mathrm{Sh}(V; \mathcal{C}) \to \mathrm{Sh}_{V\times X}(V; \mathcal{C}).$$

Applying $\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}$ to the sequence, then the result follows from the fiber sequence

$$\Gamma_c(U; \mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathcal{C})) \to \Gamma_c(V^\vee; \mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathcal{C})) \to \Gamma_c(X; \mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathcal{C})).$$ □

4.1. **Further questions.** At the end of this article, we mention that for a general cotangent bundle T^*M and an open set D, it is known that the Tamarkin category

$$\mathcal{T}(D; \mathcal{C}) = \mathcal{T}(T^*M; \mathcal{C})/\mathcal{T}_{T^*M\setminus D}(T^*M; \mathcal{C})$$

has important symplectic geometric information of the open symplectic manifold D, and it is conjectured that the filtered Fukaya category $\mathcal{F}^{fil}(D)$ should be a full subcategory of $\mathcal{T}(D; \mathcal{C})$.

For example, the Hochschild homology was studied in [Zha23, KSZ23]. It is proven that corresponding Hochschild homology is deeply related to the symplectic cohomology of U (when D has good contact boundary).

One may be naturally interested in the computation of

$$\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}(\mathcal{T}(D; \mathcal{C}))$$

or any specific finitary localizing invariants instead of $\mathcal{U}^{\mathrm{cont}}_{\mathrm{loc}}$. Here, our result is the first attempt for the question for $D = V \times U$ for a real vector space V and an open set $U \subset V^\vee$.

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