CAPACITIES FROM THE CHIU-TAMARKIN COMPLEX

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To Claude Viterbo on the Occasion of his Sixtieth Birthday

ABSTRACT. In this paper, we construct a sequence $(c_k)_{k\in\mathbb{Z}_{\geq 1}}$ of symplectic capacities based on the Chiu-Tamarkin complex $C_{T,\ell}$, a $\mathbb{Z}/\ell\mathbb{Z}$-equivariant invariant coming from the microlocal theory of sheaves. We compute $(c_k)_{k\in\mathbb{Z}_{\geq 1}}$ for convex toric domains, which are the same as the Gutt-Hutchings capacities. On the other hand, our method works for the contact embedding problem. We define a sequence of "contact capacities" $([c]_k)_{k\in\mathbb{Z}_{\geq 1}}$ on the prequantized contact manifold $\mathbb{R}^{2d}\times S^1$, which could derive some embedding obstructions of prequantized convex toric domains.

0. INTRODUCTION

0.1. **Symplectic Embedding.** A symplectic manifold (X,ω) is a manifold with a non-degenerate closed 2-form ω. Classically it appears naturally as phase spaces in Hamiltonian Mechanics. An embedding $\varphi:(X,\omega)\hookrightarrow(X',\omega')$ is called symplectic if $\varphi^*\omega'=\omega$. A basic question in symplectic geometry is to decide when a symplectic embedding between two symplectic manifolds exists. The first result, the origin of the question, is the Gromov non-squeezing theorem:

Theorem 0.A ([Gro85]). Equip $\mathbb{R}^{2d}$ with the linear symplectic form. Let $B_{\pi r^2}=\{(x,p)\in\mathbb{R}^{2d}:\|x\|^2+\|p\|^2<r^2\}$, and $Z_{\pi R^2}=\{(x,p)\in\mathbb{R}^{2d}:x_1^2+p_1^2<R^2\}$.

If there is a symplectic embedding $\varphi:B_{\pi r^2}\to\mathbb{R}^{2d}$, such that $\varphi(B_{\pi r^2})\subset Z_{\pi R^2}$, then $r\leq R$.

A structure related to the embedding question is the so-called symplectic capacity.

Definition. Fix a class of symplectic manifolds $\mathcal{C}$, we say that a function $c:\mathcal{C}\to[0,\infty]$ is a symplectic capacity if the followings are true.

- (Monotonicity) If there is a symplectic embedding $\phi:(X,\omega)\hookrightarrow(X',\omega')$, then $c(X,\omega)\leq c(X',\omega')$.
- (Conformality) $c(X,r\omega)=rc(X,\omega)$, for $r\in\mathbb{R}_{>0}$.

Different capacities are defined. One example is the Gromov width, which is defined as

$$w(X,\omega)=\sup\{\pi r^2:B_{\pi r^2}\xhookrightarrow{symp}(X,\omega)\}.$$

However it is essential a re-formulation of the embedding question, and is hard to compute. There are other capacities defined by generating functions [Vit92], Hamiltonian dynamics [EH90], and J-holomorphic curves [Hut11,GH18,Kyl19b]. A great survey about symplectic capacities is [CHLS10].

By the definition of capacities, we can derive different numerical constrains of symplectic embeddings. When the dimension is 4, the ECH-capacity is a very effective tool. The

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ECH-capacity is defined by the embedded contact homology (ECH) [Hut11], which is a J-holomorphic curve type capacity. Taubes proved that the ECH-capacity is also a gauge theoretic invariant. McDuff shows that the ECH-capacity provides us the sharp obstruction of embeddings between ellipsoids of dimension 4 in [McD11]. In general, we may find in [Cri19] more results about symplectic embeddings in dimension 4 using the ECH-capacity.

When the dimension is greater than 4, we know fewer results. The Ekeland-Hofer capacity $(c_k^{\mathrm{EH}})_{k\in\mathbb{Z}_{\geq 1}}$ is a sequence of symplectic capacities defined for compact star-sharped domains in $\mathbb{R}^{2d}$ for all d, which is defined using Hamiltonian dynamics. The computation of c_k^{EH} is known for ellipsoids and poly-disks, say:

$$c_k^{\mathrm{EH}}(E(a_1,\dots,a_d)) = \min\left\{T : \sum_{i=1}^{d}\lfloor\frac{T}{a_i}\rfloor \geq k\right\},$$
$$c_k^{\mathrm{EH}}(D(a_1,\dots,a_d)) = ka_1,$$

where

$$E(a_1,\dots,a_d) = \left\{u\in\mathbb{C}^d : \sum_{i=1}^{d}\frac{\pi|u_i|^2}{a_i} < 1\right\},$$
$$D(a_1,\dots,a_d) = \left\{u\in\mathbb{C}^d : \pi|u_i|^2 < a_i, \forall i = 1,\dots,d\right\},$$
$$\text{with } 0 < a_1 \leq \cdots \leq a_d.$$

On the other hand, Gutt and Hutchings constructed a sequence of capacities $(c_k^{\mathrm{GH}})_{k\in\mathbb{Z}_{\geq 1}}$ in [GH18] based on the positive S^1-equivariant symplectic homology. They computed c_k^{GH} of convex toric domains and concave toric domains. Here a toric domain is an open set defined as follows:

$$X_\Omega = \left\{u\in\mathbb{C}^d : (\pi|u_1|^2,\dots,\pi|u_d|^2)\in\Omega\right\}$$

where $\Omega\subset\mathbb{R}^d_{\geq 0}$. We say X_Ω is convex if $\widehat{\Omega} = \{(x_1,\dots,x_d) : (|x_1|,\dots,|x_d|)\in\Omega\}$ is convex, and is concave if $\mathbb{R}^d_{\geq 0}\setminus\Omega$ is convex. For example, ellipsoids $E = X_{\Omega_E}$ and poly-disks $D = X_{\Omega_D}$ are convex toric domains, where

$$\Omega_E = \left\{\sum_{i=1}^{d}\frac{x_i}{a_i} < 1\right\}, \quad \Omega_D = \prod_{i=1}^{d}[0,a_i), \quad 0 < a_1 \leq \cdots \leq a_d.$$

Gutt and Hutchings computed c_k^{GH} of both convex and concave toric domains. For example, when X_Ω is convex, they showed that

$$c_k^{\mathrm{GH}}(X_\Omega) = \min\left\{\|v\|_\Omega^* : v\in\mathbb{N}^d, \sum_{i=1}^{d}v_i = k\right\} \tag{1}$$

where $\|v\|_\Omega^* = \max\{< v,w >: w\in\Omega\}$. So one can observe that for ellipsoids and poly-disks, $c_k^{\mathrm{GH}} = c_k^{\mathrm{EH}}$.

Unfortunately, even for ellipsoids, we know that the obstructions, given by Ekeland-Hofer capacities, and then the Gutt-Hutchings capacities, are not sharp. One new progress on higher dimensional embeddings is given by Siegel in [Kyl19b, Kyl19a]. Siegel studied further on the chain level structures of the S^1-equivariant symplectic homology(or the linearized contact homology), and then constructed some higher capacities using L_∞-structure on the corresponding chain complexes. He gave sharp obstructions for embeddings between some stabilized ellipsoids.

In this paper, we construct a sequence of capacities $(c_k)_{k\in\mathbb{Z}_{\geq 1}}$ on the class of admissible open sets with non-degenerate boundaries. Here, we say an open set $U \subset T^*\mathbb{R}^d$ is admissible if tools of sheaf theory work(see Definition 2.1). Examples of admissible open sets are bounded open sets and toric domains, most of our interests are included in.

Our main ingredient is the complex $C_{T,\ell}(U)$ defined by Tamarkin and Chiu in [Tam15, Chi17], where U is admissible, $T \geq 0$, and $\ell \in \mathbb{P} = \{\text{odd prime numbers}\}$. There is a structure of $\mathbb{K}[u]$-module on $H^*(C_{T,\ell}(U))$, and a distinguished class $\eta_{U,T,\ell} \in H^0(C_{T,\ell}(U))$. As a clumsy imitation of the definition of c_k^{GH} [GH18, Definition 4.1], we define (see Definition 2.11)

$$\mathrm{Spec}(U,k) \coloneqq \{T \geq 0 : \exists \ell_0 \in \mathbb{P}, \forall \ell \geq \ell_0, \exists \gamma_\ell \in H^*(C_{T,\ell}(U)), \quad \eta_{U,T,\ell} = u^k\gamma_\ell\},$$

and

$$c_k(U) \coloneqq \inf \mathrm{Spec}(U,k) \in [0,+\infty].$$

Then we show

Theorem 0.B (Theorem 2.12)**.** The functions $c_k : \mathcal{C}_{\text{admissible}} \to (0,\infty]$ satisfy the following:

(1) $c_k \leq c_{k+1}$ for all $k \in \mathbb{Z}_{\geq 1}$.

(2) If there is an inclusion of admissible open sets $U_1 \subset U_2$, then $c_k(U_1) \leq c_k(U_2)$.

(3) For a compactly supported Hamiltonian isotopy $\varphi_z : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, we have $c_k(U) = c_k(\varphi_z(U))$.

(4) If $U = \{H < 1\}$ is an admissible open set whose boundary $\partial U = \{H = 1\}$ is a regular level set and $c_k(U) < \infty$, then $c_k(U)$ is represented by the action of a closed characteristic in the boundary ∂U. Consequently, $c_k(rU) = r^2 c_k(U)$ for $r > 0$.

Moreover, based on the structural theorem, Theorem 3.5, of $H^*(C_{T,\ell}(X_\Omega))$, where X_Ω is a convex toric domain, we can compute $c_k(X_\Omega)$ as follows:

$$c_k(X_\Omega) = \min\left\{\|v\|_\Omega^* : v \in \mathbb{N}^d, \sum_{i=1}^d v_i = k\right\}. \tag{2}$$

Therefore, $c_k(X_\Omega) = c_k^{\mathrm{GH}}(X_\Omega)$ by (1) and (2).

Based on the computation on the convex toric domains, Gutt and Hutchings conjectured ([GH18, Conjecture 1.9]) that, for a bounded star shaped domain U and for all $k \in \mathbb{Z}_{\geq 1}$,

$$c_k^{\mathrm{EH}}(U) = c_k^{\mathrm{GH}}(U).$$

Comparing to our result, we hope the consistency could be extended to c_k as well.

Conjecture 0.C. For a bounded star shaped domain U and for all $k \in \mathbb{Z}_{\geq 1}$, there is

$$c_k^{\mathrm{EH}}(U) = c_k^{\mathrm{GH}}(U) = c_k(U).$$

Finally, let us describe some evidence about the conjecture. In fact, we will see later that $C_{T,\ell}(U)$ is defined from a projector P_U, which is unique no matter how it is defined. In this paper, We will exhibit the existence of P_U using the sheaf quantization of Guillermou-Kashiwara-Schapira, whose construction is dynamics natural [GKS12]. Therefore it may be related to the Ekeland-Hofer capacities. But on the other hand, Viterbo constructs a sheaf quantization of compact Lagrangian submanifold in cotangent bundles using the Lagrangian Floer theory [Vit19]. Hence, it is hopeful to construct a sheaf quantization using the Hamiltonian Floer theory, which is close to the theory of symplectic homology. Then one can expect that the capacities computed in this way are the Gutt-Hutchings capacities. So the uniqueness of P_U could show that all the capacities are the same.

0.2. **Contact Embedding.** Contact geometry is the odd-dimensional cousin of symplectic geometry. A (co-oriented) contact manifold (X, α) consists of a manifold X of dimension $2n+1$ and a 1-form α, such that $\alpha \wedge d\alpha^n \neq 0$. An embedding $\varphi : (X, \alpha) \hookrightarrow (X', \alpha')$ between two contact manifolds is called contact if $\varphi^*\alpha' = e^f \alpha$ for some function $f \in C^\infty(X)$. We also study the embedding question in contact geometry. The pioneering work of Eliashberg, Kim, and Polterovich [EKP06] promote our understanding about the contact embedding question a lot. Let us explain here.

A naive attempt is to study the non-squeezing problem in the 1-jet bundle $J^1(\mathbb{R}^d) = T^*\mathbb{R}^d \times \mathbb{R}$ equipped with the contact form $\alpha = dz + \mathbf{q}d\mathbf{p}$. But the re-scaling map $(\mathbf{q}, \mathbf{p}, z) \mapsto (r\mathbf{q}, r\mathbf{p}, rz)$, which is a contactomorphism, squeezes any open set into an arbitrary small neighborhood of the origin when r is big enough. This conformal naturality of 1-jet space illustrates us that we can study the prequantized space $T^*\mathbb{R}^d \times S^1$, where S^1 is a circle, equipped with a contact form $\alpha = d\theta + \frac{1}{2}(\mathbf{q}d\mathbf{p} - \mathbf{p}d\mathbf{q})$. But there is a global contactomorphism $F_N : T^*\mathbb{R}^d \times S^1 \to T^*\mathbb{R}^d \times S^1$ defined as follows: We use complex coordinates $T^*\mathbb{R}^d \cong \mathbb{C}^d$, and then $F_N(z, \theta) := (\nu(\theta)e^{2\pi N\theta}z, \theta)$, where $\nu(\theta) = (1 + N\pi|z|^2)^{-1/2}$. One can compute directly that F_N is still embedding any ball into arbitrary small neighborhood of $\{0\} \times S^1$ for N big enough. However we notice that F_N is not compactly supported.

So a better definition of contact squeezing is the following proposed in loc. cit.

Definition. [EKP06, p1636] Let (V, α) be a contact manifold. If $U_1, U_2 \subset V$ are two open subsets, we say that U_1 is squeezed into U_2 if there exists a compactly support contact isotopy $\varphi_s : \overline{U_1} \to V$, $s \in [0, 1]$, such that $\varphi_0 = \mathrm{Id}$, and $\varphi_1(\overline{U_1}) \subset U_2$.

An interesting phenomenon, which does not appear in the symplectic situation, is the scale of the ball will affect the validity of squeezing. Two results about both squeezing and non squeezing of contact balls $B_{\pi R^2} \times S^1$ are:

Theorem 0.D. (1) [EKP06, Theorem 1.3] Suppose $d \geq 2$. Then for all $0 < \pi r^2, \pi R^2 < 1$, one can squeeze the contact ball $B_{\pi R^2} \times S^1$ into $B_{\pi r^2} \times S^1$ whatever the relation between r and R is.

(2) [EKP06, Theorem 1.2] If there exists an integer $m \in [\pi r^2, \pi R^2]$, then $B_{\pi R^2} \times S^1$ cannot be squeezed into $B_{\pi r^2} \times S^1$.

About the large scale phenomenon, Eliashberg, Kim, and Polterovich give a very nice physical explanation using the quantization process. Then the only case left about the contact non-squeezing is: what will happen if there is an integer m such that $m < \pi r^2 < \pi R^2 < m + 1$? It is solved by Chiu using the microlocal theory of sheaves [Chi17], and by Fraser using technique of J-holomorphic curves [Fra16] in the spirit of [EKP06]. They proved the following:

Theorem 0.E ([Chi17, Fra16])**.** If $1 \leq \pi r^2 < \pi R^2$, then $B_{\pi R^2} \times S^1$ cannot be squeezed into $B_{\pi r^2} \times S^1$.

The second purpose of the paper is to explain how Chiu's work could be used for more embedding questions in the prequantization $T^*\mathbb{R}^d \times S^1$. In fact, the notion of admissible open sets still makes sense. But in view of the scale features above, we need to consider the so called big admissible open sets. Concretely, we say a contact admissible open set $U \subset T^*\mathbb{R}^d \times S^1$ is big if there is a ball $B_a \times S^1 \subset U$ where $a > 1$. Besides, invariance needs to be re-considered. It forces we restrict to $T = \ell$ in the contact case. Then the geometric information defining the symplectic capacities c_k could also be applied to define a version of "contact capacities", and our computation on convex toric domains still works. Specifically, for each $k \in \mathbb{Z}_{\geq 1}$, we define

$$[\mathrm{Spec}](U, k) := \{\ell_0 \in \mathbb{P} : \forall \ell \geq \ell_0, \exists \gamma_\ell \in H^*(C_\ell(U)), \quad \eta_{U,\ell} = u^k \gamma_\ell\},$$

and

$$[c]_k(U) := \min[\mathrm{Spec}](U, k) \in \mathbb{P}.$$

Then we have, Theorem 4.4, Theorem 4.7:

Theorem 0.F. The functions $[c]_k : \mathcal{C}_{\text{big admissible}} \to \mathbb{P}$ satisfy the following:

(1) $[c]_k \leq [c]_{k+1}$ for all $k \in \mathbb{Z}_{\geq 1}$.

(2) If there is an inclusion of admissible open sets $U_1 \subset U_2$, then $[c]_k(U_1) \leq [c]_k(U_2)$.

(3) For a compactly supported contact isotopy $\varphi_z : \mathbb{R}^{2d} \times S^1 \to \mathbb{R}^{2d} \times S^1$, we have $[c]_k(U) = [c]_k(\varphi_z(U))$.

(4) For a big contact convex toric domain $X_\Omega \times S^1 \subsetneq T^*\mathbb{R}^d \times S^1$, we have

$$[c]_k(X_\Omega \times S^1) = \inf \left\{ \ell \in \mathbb{P} : \exists z \in \Omega_\ell^\circ, \sum_{i=1}^d \lfloor -z_i \rfloor \geq k \right\}.$$

0.3. **Microlocal Theory of Sheaves and the Chiu-Tamarkin complex.** The main ingredient of our work is the microlocal theory of sheaves, introduced by Kashiwara and Schapira with motivation from algebraic analysis. We refer to [KS90]. The main idea we use in the paper is the notion of microsupport or singular support, which is defined as follows: For a ground field $\mathbb{K}$, let $D(X)$ be the derived category of complexes of sheaves of $\mathbb{K}$-vector spaces over X.

Definition. For $F \in D(X)$. The microsupport of F is

$$SS(F) = \overline{\left\{ (x,p) \in T^*X : \begin{array}{l} \text{There is a } C^1\text{-function } f \text{ near } x, \text{ such that,} \\ f(x) = 0,\ df(x) = p, \text{ and } \left(\mathrm{R}\Gamma_{\{f \geq 0\}} F\right)_x \neq 0. \end{array} \right\}}.$$

It is proved in [KS90] that $SS(F)$ is always a closed conic and coisotropic subset of T^*X. Moreover, when X is real analytic, $SS(F)$ is Lagrangian if and only if F is (weakly) constructible. This result inspires us that the sheaf theory plays its role in symplectic geometry and contact geometry. For instance, Tamarkin develops a new method to study displacibility of Lagrangians in [Tam13]. Guillermou gives sheafy proofs of Gromov-Eliashberg C^0-rigidity and of the result by Abouzaid and Kragh that closed exact Lagrangians in cotangent bundles are homotopically equivalent to the zero section. See [Gui12, Gui13, Gui16] and the survey [Gui19] about these topics. On the other hand, there are many works studying the category of sheaves from the point of view of the Fukaya category, see the work of Nadler and Zaslow on the compact Fukaya category [NZ09, Nad09]; and the work of Nadler [Nad16], and of Ganatra, Pardon, and Shende on the wrapped Fukaya category [GPS19].

Now, let us review some ideas of Tamarkin in [Tam13]. Tamarkin suggested to study the category of sheaves localized with respect to sheaves microsupported in non-positive direction, that is, the localization of $D(X \times \mathbb{R})$ with respect to the full thick subcategory $\{F : SS(F) \subset \{\tau \leq 0\}\}$. This localization is equivalent to the essential image of the functor $\star\mathbb{K}_{[0,\infty)} : D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$, where $\star : D(X \times \mathbb{R}) \times D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$ is the convolution. We denote these two equivalent categories by $\mathcal{D}(X)$ and call them the Tamarkin category of X. The category $\mathcal{D}(X)$ is triangulated.

Because the microsupport is conic, Tamarkin considers the following conification procedure: for a given closed set A in the cotangent bundle T^*X we set $\widehat{A} = \{(x, p, t, \tau) \in T^*(X \times \mathbb{R}) : (x, p/\tau) \in A, \tau > 0\}$. We are interested in the category of sheaves on $X \times \mathbb{R}$ microsupported in $\widehat{A}$, that is $F \in \mathcal{D}(X)$ such that $SS(F) \cap \{\tau > 0\} \subset \widehat{A}$, and we denote the category they form by $\mathcal{D}_A(X)$. Categorically, this category and its semi-orthogonal complement ${}^\perp\mathcal{D}_A(X)$ are completely determined by the projectors from $\mathcal{D}(X)$ onto them. Hopefully, we could understand the geometry of A from these projectors. One way to study these projectors is to represent them as convolution functors defined by kernels, for example $\star\mathbb{K}_{[0,\infty)}$ introduced by Tamarkin, or the *cut-off* functors of Kashiwara and Schapira [KS90]. An open set U whose projector $\mathcal{D}(X) \to \mathcal{D}_{U^c}(X)$ is represented by a convolution functor $P_U\star : \mathcal{D}(X) \to \mathcal{D}(X)$ is called admissible. We will see later that bounded open sets and toric domains are all admissible. One particularly interesting example is the open ball $U = B_{\pi R^2}$. Chiu constructed a kernel for $B_{\pi R^2}$ using the idea of generating functions in [Chi17], which is the main ingredient of his proof of contact non-squeezing.

Another ingredient of Chiu's proof is a complex $C_{T,\ell}(U) \in D_G(\mathbb{K})$ defined using P_U, where G is the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$, $\mathbb{K}$ is some field and $D_G(\mathbb{K})$ denotes the equivariant derived category (see Burnstein and Lunts book [BL94]). The complex $C_{T,\ell}(U)$ is an invariant of an admissible open set U for $\ell \in \mathbb{Z}_{\geq 1}$. We will see that $H^*(C_{T,\ell}(U))$ is a right graded module over the Yoneda algebra $A = \operatorname{Ext}^*_G(\mathbb{K}, \mathbb{K})$, which is non trivial if $\mathbb{K}$ is of characteristic ℓ. So the whole theory make sense when ℓ is a prime number. Let us assume ℓ to be an odd prime due to a technical reason. Under these conditions, we know $\operatorname{Ext}^*_G(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}[u, \theta]$, where $|u| = 2$, and $|\theta| = 1$, $\theta^2 = 0$.

In [Chi17] Chiu proved that $C_{T,\ell}(U)$ is a symplectic invariant of $U \subset T^*X$ when T is a non-negative real number, and is a contact invariant of $U \subset T^*X \times S^1$ when $T = \ell$. Besides he proved that $C_{T,\ell}(U)$ is functorial with respect to inclusions of admissible open sets. Then Chiu computed $C_{T,\ell}(B_{\pi R^2})$ when $\pi R^2 \geq 1$ by constructing a good kernel $P_{B_{\pi R^2}}$, and hence completed his proof of the contact non-squeezing theorem. Let us remark here that Chiu has used the $A = \operatorname{Ext}^*_G(\mathbb{K}, \mathbb{K})$ module structure essentially.

We will develop the idea of Chiu further. The third main result of this paper is as follows. For a toric domain $X_\Omega \subset \mathbb{R}^{2d}$, we construct a good kernel for X_Ω based on Chiu's construction and then study the Chiu-Tamarkin complex $C_{T,\ell}(X_\Omega)$ when X_Ω is a convex toric domain. We will show the following structural theorem.

Theorem 0.G (See more precise statements in Theorem 3.5)**.** For a convex toric domain $X_\Omega \subsetneq T^*\mathbb{R}^d$ we set $\Omega_T^\circ = \{z \in \mathbb{R}^d : T + \langle z, \zeta \rangle \geq 0, \forall \zeta \in \Omega\}$ and $\|\Omega_1^\circ\|_\infty = \max_{z \in \Omega_1^\circ} \|z\|_\infty$. Then, for $\ell \in \mathbb{P}$ such that $0 \leq T < \ell/\|\Omega_1^\circ\|_\infty$, we have:

- The minimal cohomology degree of $H^*(C_{T,\ell}(X_\Omega))$ is exactly $-2I(\Omega_T^\circ)$, i.e.,
$$H^*(C_{T,\ell}(X_\Omega)) \cong H^{\geq -2I(\Omega_T^\circ)}(C_{T,\ell}(X_\Omega))$$

and
$$H^{-2I(\Omega_T^\circ)}(C_{T,\ell}(X_\Omega)) \neq 0$$
where $I(\Omega_T^\circ) = \max_{z\in\Omega_T^\circ} I(z)$, $I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor$.

- $H^*(C_{T,\ell}(X_\Omega))$ is a finitely generated $\mathbb{K}[u]$ module, whose rank is 2, and the torsion part is located exactly in $[-2I(\Omega_T^\circ), -1]$.

0.4. **Organization and Conventions of the Paper.** We will review preliminary notions of sheaf theory in the first section. In the second section, we will the present main constructions, including kernels, the Chiu-Tamarkin complex, and the capacities; and demonstrate their basic properties. We will focus on toric domains, in the third section. We would like to exhibit all constructions and computations for toric domains therein. After then, we will state how our construction works for admissible open sets of the pre-quantized contact manifold $T^*\mathbb{R}^d \times S^1$ in the fourth section.

As a end of this section, let us introduce some notations.

Usually we use subscripts to represent elements in sets. For example, $x \in X$ is denoted by X_x. In some situations, we use underline $\underline{x} = (x_1, \cdots, x_n)$ to record elements in Cartesian products X^n. The product set itself is denoted by $X^n_{\underline{x}}$. For the Cartesian product X^n, we define $\delta_{X^n} : X \to X^n$ to be the diagonal map and its image is denoted by Δ_{X^n}.

Projection maps are always denoted by π, with a subscript that encode the fiber of the projection. For example, if there are two sets X_x and Y_y, two projections are
$$\pi_Y = \pi_y : X_x \times Y_y \to X_x, \quad \pi_X = \pi_x : X_x \times Y_y \to Y_y.$$

If we have a trivial vector bundle $X \times V_v$, its summation map is
$$\mathrm{id}_X \times s_V^n = \mathrm{id}_X \times s_v^n : X \times V^n \to X \times V,\ (x, v_1, \cdots, v_n) \mapsto (x, v_1 + \cdots + v_n).$$
In all cases, we will ignore the id_X and only use $s_V^n = s_v^n$ for simplicity.

When you see an integer k, it either a positive integer $k \in \mathbb{Z}_{\geq 1}$, or a periodic integer $k \in \mathbb{Z}/\ell\mathbb{Z}$. While for ℓ, it could be a positive integer $\ell \in \mathbb{Z}_{\geq 1}$; but in most of situations, we will assume ℓ to be an odd prime number, i.e., $\ell \in \mathbb{P} = \{\text{odd prime numbers}\}$.

For a manifold X, we always use $\mathbf{q} \in X$ to represent both points and local coordinates of X. Correspondingly, the canonical Darboux coordinate of T^*X will be denoted by $(\mathbf{q}, \mathbf{p})$.

One exception is $\mathbb{R}_t$, its cocoordinate is denoted bt $\tau \in T_t^*\mathbb{R}_t$. For a manifold X, the 1-jet space is $J^1(X) = T^*X \times \mathbb{R}_t$, which is a contact manifold equipped with the contact form $\alpha = dt + \mathbf{p}d\mathbf{q}$. The symplectization of $J^1(X)$ is identified with $T^*X \times T^*_{\tau>0}\mathbb{R}_t = T^*X \times \mathbb{R}_t \times \mathbb{R}_{\tau>0}$, equipped with the symplectic form $\omega = d\mathbf{p}\wedge d\mathbf{q} + d\tau \wedge dt$. The symplectic reduction of $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ with respect to the hypersurface $\{\tau = 1\}$ is denoted by ρ, which is identified with
$$\rho : T^*X \times T^*_{\tau>0}\mathbb{R}_t \to T^*X,\ (\mathbf{q}, \mathbf{p}, t, \tau) \mapsto (\mathbf{q}, \mathbf{p}/\tau). \tag{3}$$
We call it the Tamarkin's cone map. The map ρ factors through the symplectization map q tautologically:
$$\begin{tikzcd} T^*X \times T^*_{\tau>0}\mathbb{R}_t \arrow[r, "q"] \arrow[rr, bend right, "\rho"] & J^1(X) \arrow[r] & T^*X. \end{tikzcd}$$

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1. Reminder on Sheaves and Equivariant Sheaves

In this section, I would like to review notions and tools of sheaves we will use. Let $\mathbb{K}$ be the ground field. For a manifold X, let us denote $D(X)$ the derived category of complexes of sheaves of $\mathbb{K}$-vector spaces over X. Let us remark that we do not specify the boundedness of complexes we used in general. But in most of our applications, the complexes are locally bounded in the sense that their restrictions on relatively compact open sets are bounded.

1.1. Microsupport of Sheaves and Functorial Estimate.

We refer to [KS90] as main reference of the section. For a locally closed inclusion $i : Z \subset X$ and $F \in D(X)$ we set

$$F_Z = i_! i^{-1} F, \quad \mathrm{R}\Gamma_Z F = i_* i^! F.$$

Definition 1.1 ([KS90, Definition 5.1.2])**.** For $F \in D(X)$ the microsupport of F is

$$SS(F) = \overline{\left\{ (\mathbf{q}, \mathbf{p}) \in T^*X : \begin{array}{c} \text{There is a } C^1\text{-function } f \text{ near } \mathbf{q}, \text{ such that,} \\ f(\mathbf{q}) = 0,\ df(\mathbf{q}) = \mathbf{p}, \text{ and } \left(\mathrm{R}\Gamma_{\{f \geq 0\}} F\right)_{\mathbf{q}} \neq 0. \end{array} \right\}}.$$

By definition, $SS(F)$ is a closed subset of T^*X, conic with respect to the $\mathbb{R}_{>0}$-action on the fibers. There is a triangulated inequality for the microsupport: if there is a distinguished triangle $A \to B \to C \xrightarrow{+1}$, then $SS(A) \subset SS(B) \cup SS(C)$.

Let $f : X \to Y$ be a C^∞ map of manifolds. Then there is a diagram of cotangent map:

$$\begin{tikzcd} T^*X & X \times_Y T^*Y \arrow[l, "df^*"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi"] \\ & X \arrow[r, "f"] \arrow[from=ul, "\pi"'] & Y \end{tikzcd}$$

Definition 1.2. Let $f : X \to Y$ be a C^∞ map of manifolds, and $\Lambda \subset T^*Y$ be a conic subset. One says f is non-characteristic for Λ if $\forall (\mathbf{q}, \mathbf{p}) \in \Lambda$, and $df^*_{\mathbf{q}}(\mathbf{p}) = 0 \Rightarrow \mathbf{p} = 0$.

Then we list some functorial estimates we need.

Theorem 1.3 ([KS90, Theorem 5.4])**.** Let $f : X \to Y$ be a C^∞ map of manifolds, $F \in D(X), G \in D(Y)$. Let $\omega_{X/Y} = f^! \mathbb{K}_N$ be the dualizing complex.

(1) One has

$$\begin{aligned} &SS(F \boxtimes G) \subset SS(F) \times SS(G), \\ &SS(\mathrm{R}\mathcal{H}om(\pi_X^{-1} F, \pi_Y^{-1} G)) \subset SS(F)^a \times SS(G). \end{aligned}$$

(2) Assume f is proper on $\mathrm{supp}(F)$, then $SS(\mathrm{R}f_! F) \subset f_\pi (df^*)^{-1} (SS(F))$.

(3) Assume f is non-characteristic for $SS(G)$. Then the natural morphism $f^{-1} G \otimes \omega_{X/Y} \to f^! G$ is an isomorphism, and $SS(f^{-1} G) \cup SS(f^! G) \subset df^* f^{-1} (SS(G))$.

(4) Assume f is a submersion. Then $SS(F) \subset X \times_Y T^*Y$ if and only if $\forall j \in \mathbb{Z}$, the sheaves $\mathcal{H}^j(F)$ are locally constant on the fibres of f.

Corollary 1.4. Let $F_1, F_2 \in D(X)$.

(1) Assume $SS(F_1) \cap (-SS(F_2)) \subset 0_X$, then $SS(F_1 \otimes F_2) \subset SS(F_1) + SS(F_2)$.

(2) Assume $SS(F_1) \cap SS(F_2) \subset 0_X$, then $SS(\mathrm{R}\mathcal{H}om(F_2, F_1)) \subset (-SS(F_2)) + SS(F_1)$.

For the non-proper pushforward, we have

Theorem 1.5 ([Tam13, [Corollary 3.4]). Let V be a $\mathbb{R}$-vector space, $\pi_V : X \times V \to X$, and $\pi_V^\# : T^*X \times V \times V^* \to T^*V \times V^*$ be the corresponding projections, and $i : T^*X \to T^*V \times V^*$ be the inclusion. Then for $F \in D(X \times V)$, we have

$$SS(\pi_{V!}F), SS(\pi_{V*}F) \subset i^{-1}\overline{\pi_V^\#(SS(F))}.$$

1.2. **Convolution and Tamarkin Category.** Let X_1, X_2, X_3 be three manifolds, and V be a $\mathbb{R}$-vector space. Recall, $\pi_X : X \times Y \to Y$ is a projection whose fiber is X for arbitrary Y.

Definition 1.6. For $F \in D(X_1 \times X_2 \times V)$, $G \in D(X_2 \times X_3 \times V)$. The convolution is defined as

$$F \star G := \mathrm{R}s^2_{V!}(\pi^{-1}_{(X_3,v_2)}F \otimes \pi^{-1}_{(X_1,v_1)}G) \in D(X_1 \times X_3 \times V).$$

Similarly, for $F \in D(X_1 \times X_2)$, $G \in D(X_2 \times X_3)$, the composition is defined as

$$F \circ G := \mathrm{R}\pi_{X_2!}(\pi^{-1}_{X_3}F \otimes \pi^{-1}_{X_1}G) \in D(X_1 \times X_3).$$

A basic example here is for $0 \in V$, $F \in D(X \times V)$, then we have $F \star \mathbb{K}_0 \cong F$. So, the functor $\star\mathbb{K}_0$ plays the role of the identity functor. Besides, $\star$ and $\circ$ satisfy the following monoidal identities:

$$\begin{aligned} &(F_1 \star F_2) \star F_3 \cong F_1 \star (F_2 \star F_3), \quad (F_1 \circ F_2) \circ F_3 \cong F_1 \circ (F_2 \circ F_3), \\ &F_1 \star F_2 \cong F_2 \star F_1, \quad F_1 \circ F_2 \cong F_2 \circ F_1, \\ &(F_1 \star F_2) \circ F_3 \cong F_1 \star (F_2 \circ F_3). \end{aligned} \tag{4}$$

Here, the commutative identities is induced by the factor permute map $X_1 \times X_3 \cong X_3 \times X_1$.

In this paper, we are mainly concerned with the situation of $V = \mathbb{R}_t$.

Before going into further discussion, let us review the notion of semi-orthogonal decomposition of a triangulated category.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a thick full triangulated subcategory of $\mathcal{T}$. The left semi-orthogonal of $\mathcal{C}$ is defined by

$$^\perp\mathcal{C} := \{X \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(X, Y) = 0, \forall Y \in \mathcal{C}\}. \tag{5}$$

One can show that the following proposition holds, see [KS06, Chapter 4 and Exercise 10.15.].

Proposition 1.7. Using the above notations, we have the followings three equivalent properties:

(1) The inclusion $\mathcal{C} \to \mathcal{T}$ admits a left adjoint functor $L : \mathcal{T} \to \mathcal{C}$.

(2) There is an isomorphism $\mathcal{T}/\mathcal{C} \xrightarrow{\cong} {}^\perp\mathcal{C}$, where $\mathcal{T}/\mathcal{N}$ is the Verdier localization.

(3) There are two functors $P, Q : \mathcal{T} \to \mathcal{T}$, such that $\forall X \in \mathcal{T}$, we have the distingushed triangle:

$$P(X) \to X \to Q(X) \xrightarrow{+1},$$

such that $P(X) \in {}^{\perp}\mathcal{C}$, and $Q(X) \in \mathcal{C}$.

In this situation, we say one of these data gives a left semi-orthogonal decomposition of $\mathcal{T}$. One can verify, if one of the conditions here is satisfied, that $P^2 \cong P$, and $Q^2 \cong Q$. P, Q are called a pair of projectors associated to $\mathcal{C}$.

Now, let $\mathcal{T} = D(X \times \mathbb{R}_t)$, and $\mathcal{C} = \{F : SS(F) \subset \{\tau \leq 0\}\}$. The triangulated inequality of microsupport shows that $\mathcal{C}$ is a thick full triangulated subcategory of $\mathcal{T}$. Tamarkin constructs a pair of projectors associated to $\mathcal{C}$ given by convolution:

Theorem 1.8 ([Tam13])**.** The functors $F \mapsto F \star \mathbb{K}_{[0,\infty)}$, $F \mapsto F \star \mathbb{K}_{(0,\infty)}[1]$ on $D(X \times \mathbb{R}_t)$ and the excision triangle

$$\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0 \to \mathbb{K}_{(0,\infty)}[1] \xrightarrow{+1} .$$

give a left semi-orthogonal decomposition of $D(X \times \mathbb{R}_t)$ associated to $\mathcal{C}$. Namely, for $F \in D(X \times \mathbb{R}_t)$ we have the distinguished triangle

$$F \star \mathbb{K}_{[0,\infty)} \to F \to F \star \mathbb{K}_{(0,\infty)}[1] \xrightarrow{+1} \tag{6}$$

with $F \star \mathbb{K}_{[0,\infty)} \in {}^{\perp}\mathcal{C}$, $F \star \mathbb{K}_{(0,\infty)}[1] \in \mathcal{C}$.

One can also see [GS14, Proposition 3.24] for a proof and some generalizations of the proposition.

Definition 1.9. We define the Tamarkin category as

$$\mathcal{D}(X) = {}^{\perp}\{F : SS(F) \subset \{\tau \leq 0\}\} \cong D(X \times \mathbb{R}) / \{F : SS(F) \subset \{\tau \leq 0\}\} .$$

By proposition 1.7 and (6) , $F \in D(X \times \mathbb{R})$ is in $\mathcal{D}(X)$ if and only if $F \cong F \star \mathbb{K}_{[0,\infty)} \cong F \star \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$.

Consequently, the convolution functor $\star \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$ of the Tamarkin category $\mathcal{D}(X)$ coincides with the identity functor.

For $F \in \mathcal{D}(X)$, one can show $SS(F) \subset \{\tau \geq 0\}$ using microsupport estimates we mentioned last section, see [GS14, Proposition 3.17]. So it is helpful to define the Legendre microsupport and sectional microsupport of $F \in \mathcal{D}(X)$ as follows:

$$\begin{aligned} \mu s_L(F) &= q\left(SS(F) \cap \{\tau > 0\}\right) \subset J^1 X, \\ \mu s(F) &= \rho\left(SS(F) \cap \{\tau > 0\}\right) \subset T^* X, \end{aligned} \tag{7}$$

where ρ, p are defined in (3).

We recall that there exists a sheaf-theoretic Fourier transform called the Fourier-Sato transform. The Fourier-Sato transform defines a functor $D(V) \to D(V^*)$, where V is a real vector space and V^* is the dual of V. One can see [KS90, Section 3.7, Section 5.5] for more details. We want to mention that the Fourier-Sato gives an equivalence between $\mathbb{R}_{>0}$-equivariant sheaves. Tamarkin introduced a new version of the Fourier transform on the category $\mathcal{D}(V)$ which works also for non $\mathbb{R}_{>0}$-equivariant sheaves. We call it the Fourier-Sato-Tamarkin transform. For the relation between different versions of Fourier transforms, we refer [Gao17, DE03].

Definition 1.10. Let $FT = \{(z,\zeta,t) : t + \langle z,\zeta\rangle \geq 0\} \subset V \times V^* \times \mathbb{R}_t$. Then the Fourier-Sato-Tamarkin transform is defined as

$$FT : \mathcal{D}(V_z) \to \mathcal{D}(V^*_\zeta)$$
$$FT(F) = \widehat{F} := F \star \mathbb{K}_{FT}[\dim V],$$

One can see that FT is an equivalence in [Tam13, Theorem 3.5].

The important thing for us is the microsupport estimate under the Fourier-Sato-Tamarkin transform.

Theorem 1.11 ([Tam13, Theorem 3.6]). Let $\phi\colon T^*(V \times \mathbb{R}_t) \to T^*(V^* \times \mathbb{R}_t)$ be the isomorphism given by $\phi(z,\zeta,t,\tau) = (\zeta,-z,t,\tau)$, where we identify V^{**} with V naturally. Then we have the microsupport relation:

$$SS(\widehat{F}) \cap \{\tau > 0\} = \phi(SS(F)) \cap \{\tau > 0\},$$

and consequently

$$\mu s_L(\widehat{F}) = \phi(\mu s_L(F)). \tag{8}$$

They play a very important role in the construction of kernels associated to admissible open sets.

1.3. **Guillermou-Kashiwara-Schapira Sheaf Quantization.** As a sheaf pattern of Hamiltonian actions, we introduce the Guillermou-Kashiwara-Schapira sheaf quantization as a basic tool here.

Consider $\dot{T}^*Y$ as a symplectic manifold equipped with the Liouville symplectic form and with a $\mathbb{R}_{>0}$-action by dilation on cotangent fibers. If $\varphi : \dot{T}^*Y \times I \to \dot{T}^*Y$ is a $\mathbb{R}_{>0}$-equivariant symplectic isotopy, one can show that it must be Hamiltonian with a $\mathbb{R}_{>0}$-equivariant Hamiltonian function H.

Consider its total graph

$$\Lambda_\varphi := \left\{(z, -H_z \circ \varphi_z(\mathbf{q},\mathbf{p}), (\mathbf{q},\mathbf{p}), -\varphi_z(\mathbf{q},\mathbf{p})) : (\mathbf{q},\mathbf{p}) \in \dot{T}^*Y, z \in I\right\} \subset \dot{T}^*(I \times Y). \tag{9}$$

Then Guillermou, Kashiwara, and Schapira proved the following theorem:

Theorem 1.12 ([GKS12, Theorem 3.7]). Using the above notations there is a $K \in D(I \times Y \times Y)$ such that

(1) $SS(K) \subset \Lambda_{\widehat{\phi}} \cup 0_{I\times Y\times Y}$,

(2) $K_0 = \mathbb{K}_{\Delta_{Y^2}}$.

where $K_{z_0} = K|_{\{z=z_0\}}$.

If we set $K_z^{-1} = v^{-1}\mathrm{R}\mathcal{H}om(K_z, \omega_Y \overset{L}{\boxtimes} \mathbb{K}_Y)$, $v(m,n) = (n,m)$, $m,n \in Y$, $z \in I$, then

- $a)$ $\mathrm{supp}(K) \rightrightarrows I \times Y$ are both proper,
- $b)$ $K_z \circ K_z^{-1} \cong K_z^{-1} \circ K_z \cong \mathbb{K}_{\Delta_{Y^2}}$,
- $c)$ K is unique up to an unique isomorphism.

Consequently, $F \mapsto K_z \circ F$, $D(Y) \to D(Y)$ is a category equivalence for all $z \in I$, whose quasi inverse is $K_z^{-1} \circ F$.

Let us describe two situations where we will use the theorem.

I) Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy. Let $Y = X \times \mathbb{R}_t$. Then one can lift $\widehat{\varphi} : I \times \dot{T}^*Y \to \dot{T}^*Y$. Specifically, we have the following:

Proposition 1.13 ([GKS12, Proposition A.6])**.** Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy, whose Hamiltonian function is $H \in C^\infty(I \times T^*X)$.

There is a $\mathbb{R}_{>0}$-equivariant Hamiltonian isotopy $\widehat{\varphi} : I \times \dot{T}^*Y \to \dot{T}^*Y$ such that:

a) $\widehat{H} = \tau H(-, \rho(-))$ is a Hamiltonian function of $\widehat{\varphi}$.

b) The lifting $\widehat{\varphi}$ commutes with both the symplectization and the Tamarkin's cone map.

c) We can take

$$\begin{aligned} &\widehat{\varphi}(z, \mathbf{q}, t, \mathbf{p}, \tau) = (\tau.\varphi(z, \mathbf{q}, \mathbf{p}/\tau), t + u(z, \mathbf{q}, \mathbf{p}/\tau), \tau), && \tau \neq 0,\\ &\widehat{\varphi}(z, \mathbf{q}, t, \mathbf{p}, 0) = (\mathbf{q}, \mathbf{p}, t + v(z), 0), && \tau = 0, \end{aligned}$$

where $u \in C^\infty(I \times T^*X), v \in C^\infty(I)$. In fact, the proof shows $u(z, \mathbf{q}, \mathbf{p}) = S_H(z, \mathbf{q}, \mathbf{p}) = \int_0^z [\alpha(X_{H_x}) - H_x] \circ \phi_H^x(\mathbf{q}, \mathbf{p}) dx$ is the symplectic action function.

We call this $\widehat{\varphi}$ or $\widehat{\varphi}_z$ the conification of φ.

Remark 1.14. Noticed, it is easy to lift φ to $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ without the compactly supported assumption, but this is not enough to apply the Guillermou-Kashiwara-Schapira theorem. If we want to lift φ to $\dot{T}^*(X \times \mathbb{R}_t)$, we need the compactly supported condition.

Now, apply Theorem 1.12 to $\widehat{\varphi}$, there is a sheaf $K(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}_t^2)$. In our later application, we prefer to use only one t-variable. This is possible. Consider $m(t_1, t_2) = t_1 - t_2$, then [Gui19, Corollary 2.3.2] shows there is a unique $\mathcal{K}(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}_t)$ such that $K(\widehat{\varphi}) = m^{-1}\mathcal{K}(\widehat{\varphi})$, and $\mathcal{K}(\widehat{\varphi}) = \mathrm{R}m_! K(\widehat{\varphi})$. Then we can take $\mathcal{K} = \mathcal{K}(\widehat{\varphi})$ as the sheaf quantization of φ.

By the commutativity of the lifting with symplectization, we have the following estimates for the Legendrian microsupport and sectional microsupport of $\mathcal{K}(\widehat{\varphi})$:

$$\begin{aligned} \mu s_L(\mathcal{K}(\widehat{\varphi})) &\subset \{(z, -H(\mathbf{q}, \mathbf{p}), \mathbf{q}, \mathbf{p}, -\varphi_z(\mathbf{q}, \mathbf{p}), -S_H(z, \mathbf{q}, \mathbf{p})) : (z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\},\\ \mu s(\mathcal{K}(\widehat{\varphi})) &\subset \{(z, -H(\mathbf{q}, \mathbf{p}), \mathbf{q}, \mathbf{p}, -\varphi_z(\mathbf{q}, \mathbf{p})) : (z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\}. \end{aligned} \tag{10}$$

II) Let $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$ be a contact isotopy. One can lift φ to a $\mathbb{Z}$-equivariant contact isotopy of $J^1(X) = T^*X \times \mathbb{R}_t$, where $\mathbb{Z}$ acts by shifting t. Then it is easy to lift φ to the symplectization, $T^*X \times T^*_{\tau>0}\mathbb{R}_t$, of $J^1(X)$ to a $\mathbb{Z} \times \mathbb{R}_{>0}$ equivariant Hamiltonian isotopy $\widehat{\varphi} : I \times T^*X \times T^*_{\tau>0}\mathbb{R}_t \to T^*X \times T^*_{\tau>0}\mathbb{R}_t$. Similarly to the symplectic case, the compactly supported condition is necessary to extend $\widehat{\varphi}$ to whole $\dot{T}^*(X \times \mathbb{R}_t)$.

In this case, we still take the sheaf quantization $\mathcal{K} = \mathcal{K}(\widehat{\varphi})$ of $\widehat{\varphi}$ as sheaf quantization of φ. The only difference here is, the $\mathbb{Z}$-equivariant condition is inherited by the sheaf $\mathcal{K}(\widehat{\varphi})$. For more about this point, we refer [Chi17, Formula (26)] for readers.

1.4. **Equivariant Sheaves for Finite Group Actions.** Here, we review basic notions of equivariant sheaves. We refer to [BL94] for all details about the general theory of equivariant sheaves and equivariant derived categories. In this paper, we only use the theory of equivariant sheaves and equivariant category for a finite group (in fact, a cyclic group). So we assume G to be a finite group equipped with the discrete topology.

For a manifold X with a G action $\rho : G \times X \to X$, a G-equivariant sheaf is a pair (F, θ) where $F \in Sh(X)$ and $\theta : \rho^{-1}F \cong \pi_G^{-1}F$ is an isomorphism of sheaves satisfying the cocycle conditions:

$$d_0^{-1}\theta \circ d_2^{-1}\theta = d_1^{-1}\theta, \quad s_0^{-1}\theta = \mathrm{Id}_F.$$

Here

$$\begin{aligned} d_0(g,h,x) &= (h, g^{-1}x),\\ d_1(g,h,x) &= (gh, x),\\ d_2(g,h,x) &= (g,x),\\ s_0(x) &= (e,x). \end{aligned}$$

A sheaf morphism between two G-equivariant sheaves is equivariant if it commutes with the θ's. We let $Sh_G(X)$ be the category of G-equivariant sheaves. For example, when $X = \mathrm{pt}$, $Sh_G(X) \simeq \mathbb{K}[G] - Mod$, the category of all G-modules. The category of G-equivariant sheaves $Sh_G(X)$ is Abelian. Moreover, Grothendieck proved in the Tohoku paper that when G is finite, $Sh_G(X)$ admits enough injective objects. Therefore, the derived category $D(Sh_G(X))$ makes sense, which is treated as a naive version of equivariant derived category of sheaves.

For general topological groups, the naive version is not good as our expectation. A basic difference is the hom space $\mathrm{RHom}_{D(Sh_G(X))}(\mathbb{K}_X, \mathbb{K}_X)$ is not isomorphic to the equivariant cohomology of X. A more serious problem is how to define 6-operations with correct adjunction properties.

To resolve these problems, we must use the equivariant derived category $D_G(X)$ defined by Burnstein-Lunts, in where the expected isomorphism holds, and the correct 6-operations live. But for discrete groups, both the naive and advanced versions are equivalent, i.e., $D(Sh_G(X)) \simeq D_G(X)$. In particular, $D(\mathbb{K}[G] - Mod) \simeq D_G(\mathrm{pt})$. More luckily, there is a forgetful functor $D_G(X) \to D(X)$, which commutes with Grothendieck 6-operations. So for us, $D(Sh_G(X))$ is enough for our applications. In practice, we always write everything in the usual derived category and run the machine of equivariant derived category implicitly. As a rule of convenience, we only write a lower subscript G for all possible places to indicate that we are working on some version of equivariant categories without mentioning which one we are really working on.

2. Projectors, Chiu-Tamarkin complex, and Capacities.

2.1. **Projectors Associated to Open Sets in T^*X.**

In this subsection, we are going to study the categories related to sheaves microsupported in an open set $U \subset T^*X$. Next, we will construct kernels of the projectors onto these categories.

For a *closed* subset $Z \subset T^*X$ we define $\mathcal{D}_Z(X)$ as the full subcategory of $\mathcal{D}(X)$ consisting of the sheaves satisfying $\mu s(F) \subset Z$. For an *open* subset $U \subset T^*X$ we define $\mathcal{D}_U(X)$ to be the left semi-orthogonal complement of $\mathcal{D}_{T^*X\setminus U}(X)$ in $\mathcal{D}(X)$, i.e., $\mathcal{D}_U(X) = {}^\perp\mathcal{D}_{T^*X\setminus U}(X)$.

Now we have a diagram of inclusions

$$\mathcal{D}_{T^*X\setminus U}(X) \hookrightarrow \mathcal{D}(X) \hookleftarrow \mathcal{D}_U(X) \tag{11}$$

Following Tamarkin, we are looking for a convolution kernel to prove these inclusions admit suitable adjoint functors and give corresponding semi-orthogonal decomposition.

Definition 2.1. We say U is admissible if there is a distingushed triangle

$$P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U \xrightarrow{+1},$$

in $D(X^2 \times \mathbb{R}_t)$ such that the convolution functor $P_U\star$ is right adjoint to $\mathcal{D}_U(X) \hookrightarrow \mathcal{D}(X)$ and $Q_U\star$ is left adjoint to $\mathcal{D}_{T^*X\setminus U}(X) \hookrightarrow \mathcal{D}(X)$, i.e.,

$$\mathcal{D}_{T^*X\setminus U}(X) \xleftarrow{Q_U\star} \mathcal{D}(X) \xrightarrow{P_U\star} \mathcal{D}_U(X),$$

are two projectors.

Such a pair of sheaves (P_U, Q_U) together with the distinguished triangle give an orthogonal decomposition of $\mathcal{D}(X)$ by proposition 1.7. We also call the pair (P_U, Q_U) as (the kernels of) projectors associated with U.

In the following, we will present the functorial property, uniqueness of kernels, and existence of kernels for bounded open sets.

Proposition 2.2 (Functorial property, Theorem 4.7(2) of [Chi17])**.** Any inclusion $U_1 \subset U_2 \subset T^*X$ between admissible open subsets induces a morphism of distinguished triangle

$$\begin{tikzcd}
P_{U_1} \arrow[r] \arrow[d] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r] \arrow[d, equal] & Q_{U_1} \arrow[r, "+1"] \arrow[d] & {} \\
P_{U_2} \arrow[r] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r] & Q_{U_2} \arrow[r, "+1"] & {} .
\end{tikzcd}$$

These morphisms are natural with respect to inclusions of open sets. In particular, when $U_1 = U_2$ (but P_{U_1} and P_{U_2} are not the same initially), the morphism is an isomorphism of distingushed triangle(See [Zha20, Section 4.6] for example).

Next, let us study the existence of admissible open sets $U \subset T^*X$. In general, we can take a smooth Hamiltonian function H such that $U = \{H < 1\}$. Our tools to construct kernels are sheaf quantizations and the Fourier-Sato-Tamarkin transform.

For our later application for toric domains, let us state our idea in a more general form. Suppose there is a Hamiltonian $\mathbb{R}^m_z$-action on T^*X, i.e., a symplectic action $\varphi : \mathbb{R}^m_z \times T^*X \to T^*X$ with a moment map $\mu : T^*X \to (\mathbb{R}^m_z)^* = \mathbb{R}^m_\zeta$.

We assume there is a sheaf quantization $\mathcal{K} \in D(\mathbb{R}^m_z \times X^2 \times \mathbb{R}_t)$ associated with the Hamiltonian action in the sense:

$$\begin{aligned}
&\mathcal{K}|_{z=0} \cong \mathbb{K}_{\{0\}\times\Delta_{X^2}\times\{0\}} \\
&\mu s(\mathcal{K}) \subset \{(z, -\mu(\mathbf{q},\mathbf{p}), \mathbf{q}, \mathbf{p}, -\varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in \mathbb{R}^m_z \times T^*X\}.
\end{aligned} \tag{12}$$

Remark 2.3. One can see that when $m = 1$, it is exactly the single Hamiltonian situation, whose existence is discussed in subsection 1.3. In fact, as [GKS12, Remark 3.9] discussed, it exists for general m. A situation of particular interest is the standard torus action on $\mathbb{C}^d$, which will be discussed in subsection 3.1.

First of all, let us apply the Fourier-Sato-Tamarkin transform to the z-variable. That is $\widehat{\mathcal{K}} = \mathcal{K} \star \mathbb{K}_{FT}[m]$. So by (12) and (8), we have

$$\mu s(\widehat{\mathcal{K}}) \subset \{(\mu(\mathbf{q},\mathbf{p}), z, \mathbf{q}, \mathbf{p}, -\varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in \mathbb{R}^m_z \times T^*X\}. \tag{13}$$

Now, take $\Omega \subset \mathbb{R}^m_\zeta$ open. Intuitively, if we want to restrict the microsupport of some sheaves into $\{\mu \in \Omega\}$, we should cut-off the support of $\mathcal{K}$ into Ω in some way. The correct formulation of the idea is as follows. Consider the excision triangle:

$$\mathbb{K}_\Omega \to \mathbb{K}_{\mathbb{R}^m_\zeta} \to \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

After composition with $\widehat{\mathcal{K}}$, we obtain a distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

By the associativity of convolutions and compositions(formula(4)), we have $\widehat{\mathcal{K}} \circ F = (\mathcal{K} \star \mathbb{K}_{FT}[m]) \circ F \cong \mathcal{K} \star (\mathbb{K}_{FT}[m] \circ F)$. So because $\mathbb{K}_{FT}[m] \circ \mathbb{K}_{\mathbb{R}^m_\zeta} = \mathbb{K}_{\{z=0,\, t \geq 0\}}$, we have

$$(\widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta}) \cong \mathcal{K} \star \mathbb{K}_{\{z=0,\, t\geq 0\}} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)},$$

the last isomorphism comes from (12), i.e., $\mathcal{K}|_{z=0} \cong \mathbb{K}_{\Delta_{X^2} \times \{0\}}$. Therefore, we have the distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

Proposition 2.4. Suppose there is Hamiltonian $\mathbb{R}^m_z$-action φ on T^*X with a moment map $\mu : T^*X \to \mathbb{R}^m_\zeta$. We assume there is a sheaf quantization $\mathcal{K} \in D(\mathbb{R}^m_z \times X^2 \times \mathbb{R}_t)$ of the Hamiltonian action in the sense of (12). For an open subset $\Omega \subset \mathbb{R}^m_\zeta$ such that for any $\zeta \in \Omega$, $\mu^{-1}(\zeta)$ is compact, the open set $U = \mu^{-1}(\Omega) \subset T^*X$ is admissible.

More precisely, the pair of sheaves

$$P_U := \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega,\ Q_U := \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega}, \tag{14}$$

and the distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1}, \tag{15}$$

provide the kernels of projectors and the semi-orthogonal decomposition associated to U.

Proof. Our construction is a straight forward generalization of Chiu's result [Chi17, Theorem 3.11]. One only need to noticed that we consider a Hamiltonian $\mathbb{R}^m_z$-action more than a single Hamiltonian function, and we replace $(-\infty, R)$ in Chiu's paper by Ω. The properness condition of Ω is a technical condition which is automatically true in the situation of Chiu. One can check the proof of Chiu to confirm that our condition is enough to make sure the virtue of the semi-orthogonal decomposition without any other modification. □

Now let us state some corollaries.

Corollary 2.5. Bounded open sets U are admissible.

Proof. $T^*X \setminus U$ is a closed subset of T^*X. The smooth Urysohn's lemma(which is a corollary of the partition of unity) implies that there is a smooth function $H : T^*X \to [0,1]$, such that $U = \{H < 1\}$ and $T^*M \setminus U = \{H = 1\} = \{H \geq 1\}$.

Since U is bounded, the subsets $\{H = a\} \subset U$ with $a < 1$ are compact. Moreover dH has compact support. So we can take the GKS quantization $\mathcal{K}(\widehat{\varphi^H})$. Then the result follows from the proposition 2.4 by taking $\Omega = (-\infty, 1)$. □

The second corollary here is about the kernel of products of open sets.

Corollary 2.6. Suppose we have two bounded open sets $U_i \subset T^*X_i$, with two pairs of kernels (P_{U_i}, Q_{U_i}), $i = 1, 2$.

Then $U_1 \times U_2$ is admissible and $P_{U_1\times U_2} \cong \mathrm{R}s^2_{t!}(P_{U_1} \boxtimes P_{U_2})$.

Proof. Same as the last corollary, assume $U_i = \{H_i < 1\}$ for two Hamiltonian functions $H_i \in C^\infty(T^*X_i)$. So by the uniqueness of projectors, we can assume projectors associated to U_i are obtained from the GKS quantization $\mathcal{K}_i$, i.e.,

$$(P_{U_i}, Q_{U_i}) = (\widehat{\mathcal{K}}_i \circ \mathbb{K}_{(-\infty,1)}, \widehat{\mathcal{K}}_i \circ \mathbb{K}_{[1,\infty)}),\ i = 1, 2.$$

Now, consider the product Hamiltonian $\mathbb{R}^2_z$-action on $T^*(X_1 \times X_2)$ whose moment maps is $\mu = (H_1, H_2)$. Then $\mathcal{K} = \mathrm{R}s^2_{t!}(\mathcal{K}_1 \boxtimes \mathcal{K}_2)$ is a sheaf quantization of the Hamiltonian action in the sense of (12).

Observe that if we take $\Omega = \{\zeta = (\zeta_1, \zeta_2) : \zeta_1 < 1, \zeta_2 < 1\}$, then we have $U_1 \times U_2 = \mu^{-1}(\Omega)$. Consequently, the proposition 2.4 implies that $U_1 \times U_2$ is admissible by the following distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta\times\{t\geq 0\}} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^2_\zeta\setminus\Omega} \xrightarrow{+1} .$$

Subsequently, let us compute $\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega$.

Recall

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \mathbb{K}_{\{(z,\zeta,t):t+z\cdot\zeta\geq 0\}}[2] \circ \mathbb{K}_\Omega.$$

Noticed Ω is a open convex set. Therefore $\mathbb{K}_{\{(z,\zeta,t):t+z\cdot\zeta\geq 0\}}[2] \circ \mathbb{K}_\Omega$ is the constant sheaf $\mathbb{K}_{\Omega^\circ}$ supported on the Ω° polar cone of Ω, where

$$\Omega^\circ = \{(z, t) : t + z \cdot \zeta \geq 0, \forall \zeta \in \Omega\}.$$

In fact, $\mathbb{K}_{\{(z,\zeta,t):t+z\cdot\zeta\geq 0\}}[2] \circ \mathbb{K}_\Omega$ is the Fourier-Sato transform of the constant sheaf of the conification of Ω. The conification is a convex cone. The Fourier-Sato transform of a constant sheaf supported on a convex cone is the constant sheaf supported on the polar cone of the given cone. And a direct computation shows that the polar cone of the conification of Ω is exactly Ω°. Then our computation follows.

In particular, when $\Omega = \{\zeta_1 < 1, \zeta_2 < 1\}$, we have $\Omega^\circ = \{(z, t) : z = (z_1, z_2), , z_1 \leq 0, z_2 \leq 0, t \geq -(z_1 + z_2) \geq 0\}$. Moreover, $\mathbb{K}_{\Omega^\circ} \cong \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1\times\gamma_2})$, where $\gamma_i = \{(z_i, t) : t \geq -z_i \geq 0\}$.

Now we have

$$\begin{aligned}
\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega &\cong \mathcal{K} \star \mathbb{K}_{\{(z,\zeta,t):t+z\cdot\zeta\geq 0\}}[2] \circ \mathbb{K}_\Omega \\
&\cong \mathcal{K} \star \mathbb{K}_{\Omega^\circ} \\
&\cong \mathcal{K} \star \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1\times\gamma_2}) \\
&\cong \mathrm{R}s^2_{t!}(\mathcal{K}_1 \boxtimes \mathcal{K}_2) \star \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1\times\gamma_2}) \\
&\cong \mathrm{R}s^2_{t!}((\mathcal{K}_1 \star \mathbb{K}_{\gamma_1}) \boxtimes (\mathcal{K}_2 \star \mathbb{K}_{\gamma_2}))
\end{aligned}$$

Finally, noticed that $\mathbb{K}_{\{(z,t):t\geq -z\geq 0\}} \cong \mathbb{K}_{\{(z,\zeta,t):t+z\zeta\geq 0\}}[1] \circ \mathbb{K}_{(-\infty,1)}$, one can conclude that

$$P_{U_1\times U_2} \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathrm{R}s^2_{t!}((\mathcal{K}_1 \star \gamma_1) \boxtimes (\mathcal{K}_2 \star \gamma_2)) \cong \mathrm{R}s^2_{t!}(P_{U_1} \boxtimes P_{U_2}).$$

□

2.2. **Chiu-Tamarkin complex.** Let $G = \mathbb{Z}/\ell\mathbb{Z}$ be the finite cyclic group of order $\ell \in \mathbb{Z}_{\geq 1}$, X be a smooth orientable manifold of dimension d with a fixed orientation. In the following, we are going to study a G-equivariant invariant constructed from P_U.

Now take an admissible open set $U \subset T^*X$, and let P_U be the kernel associated to U. The manifold $(X^2 \times \mathbb{R}_t)^\ell$ admits a G-action induced by the cyclic permutation of ℓ factors.

There is an adjoint pair (α_T, β_T):

$$F \in D_G((X^2 \times \mathbb{R}_t)^\ell) \underset{\beta_T}{\overset{\alpha_T}{\rightleftarrows}} D_G(\mathrm{pt}) \ni F',$$

defined by:

$$\begin{aligned} \alpha_T(F) &= i_T^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}s^\ell_{t!}\tilde{\Delta}^{-1}(F) \\ \beta_T(F') &= \tilde{\Delta}_* s_t^{\ell!}\pi^!_{\underline{\mathbf{q}}} i_{T!}F', \end{aligned} \tag{16}$$

where $i_T : \{T\} \to \mathbb{R}$, and $\tilde{\Delta}(\underline{\mathbf{q}}, \underline{t}) = (\mathbf{q}_n, \mathbf{q}_1, \mathbf{q}_1, \cdots, \mathbf{q}_{n-1}, \mathbf{q}_{n-1}, \mathbf{q}_n, \underline{t})$ is a twisted diagonal.

Definition 2.7. Under these notations, we define the Chiu-Tamarkin complex as the derived Hom:

$$C_{T,\ell}(U) = \mathrm{RHom}_G\left(\alpha_T(P_U^{\boxtimes \ell}), \mathbb{K}[-d]\right) \cong \mathrm{RHom}_G\left(P_U^{\boxtimes \ell}, \beta_T\mathbb{K}[-d]\right).$$

We see $C_{T,\ell}(U)$ as an object of $D(\mathbb{K}[G]-Mod) \simeq D_G(\mathrm{pt})$. We set $A = \mathrm{Ext}^*_G(\mathbb{K}, \mathbb{K})$. Then $H^*(C_{T,\ell}(U))$ is a graded module over $A \cong \mathrm{Ext}^*_G(\mathbb{K}[-d], \mathbb{K}[-d])$ via the Yoneda product.

Remark 2.8. (1) For our convenience, the adjoint isomorphism here is denoted by :

$$N : \mathrm{RHom}_G\left(\alpha_T(P_U^{\boxtimes \ell}), \mathbb{K}[-d]\right) \xrightarrow{\cong} \mathrm{RHom}_G\left(P_U^{\boxtimes \ell}, \beta_T\mathbb{K}[-d]\right). \tag{17}$$

(2) $C_{T,\ell}(U)$ is mentioned by Tamarkin in [Tam15], and is defined explicitly by Chiu in [Chi17]. It looks slightly different from what Chiu concerned. But one can check directly that, because X is orientable, $\beta_T\mathbb{K}[-d]$ is exactly the constant sheaf supported on the twisted diagonal with a degree shift depending only on ℓ and $\dim X$. So the complex $C_{T,\ell}(U)$ is essentially the same as what Chiu defined.

Now, denote

$$F_{U,\ell} = \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}s^\ell_{t!}\tilde{\Delta}^{-1}\left(P_U^{\boxtimes \ell}\right). \tag{18}$$

Then we establish

$$C_{T,\ell}(U) = \mathrm{RHom}_G\left((F_{U,\ell})_T, \mathbb{K}[-d]\right). \tag{19}$$

Let us compute an example here. Recall $P_{T^*X} = \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$. So

$$\tilde{\Delta}^{-1}\left(P_{T^*X}^{\boxtimes \ell}\right) = \mathbb{K}_{\tilde{\Delta}^{-1}[(\Delta_{X^2} \times [0,\infty))^\ell]} = \mathbb{K}_{\Delta_{X^\ell} \times [0,\infty)^\ell}.$$

Then

$$F_{T^*X,\ell} = \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}s^\ell_{t!}(\mathbb{K}_{\Delta_{X^\ell} \times [0,\infty)^\ell}) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\mathbb{K}_{\Delta_{X^\ell} \times [0,\infty)}) = \mathrm{R}\Gamma_c(\Delta_{X^\ell}, \mathbb{K})_{[0,\infty)}$$

and G acts on $\mathrm{R}\Gamma_c(\Delta_{X^\ell}, \mathbb{K}) \cong \mathrm{R}\Gamma_c(X, \mathbb{K})$ trivially. It also shows us $C_{T,\ell}$ is non-trivial only when $T \geq 0$.

Since G acts trivially, we have

$$\begin{aligned} C_{T,\ell}(T^*X) &\cong \mathrm{RHom}_G(\mathrm{R}\Gamma_c(X, \mathbb{K}), \mathbb{K}[-d]) \\ &\cong \mathrm{RHom}_G(\mathbb{K}, \mathbb{K}) \otimes \mathrm{RHom}(\mathrm{R}\Gamma_c(X, \mathbb{K}), \mathbb{K}[-d]) \\ &\cong \mathrm{RHom}_G(\mathbb{K}, \mathbb{K}) \otimes \mathrm{R}\Gamma(X, \mathbb{K}), \end{aligned} \tag{20}$$

where the last isomorphism is Poincaré-Verdier duality (since X is orientable). Finally, for $T \geq 0$

$$H^*(C_{T,\ell}(T^*X)) \cong A \otimes H^*(T^*X, \mathbb{K}). \tag{21}$$

One of the most important theorem about the Chiu-Tamarkin complex is

Theorem 2.9 (Theorem 4.7 of [Chi17])**.** Let U, U_1, U_2 be admissible open sets and let $U_1 \overset{i}{\hookrightarrow} U_2$ be an inclusion. Then one has, for $T \geq 0$,

(1) There is a morphism $C_{T,\ell}(U_2) \xrightarrow{i^*} C_{T,\ell}(U_1)$, which is functorial with respect to inclusions of admissible open sets.

(2) Suppose there is a compactly supported Hamiltonian isotopy $\varphi : T^*X \times I \to T^*X$, then there is an isomorphism $\Phi_{z,T,\ell} : C_{T,\ell}(\varphi_z(U)) \cong C_{T,\ell}(U)$, for all $z \in I$. When $\varphi_z = \mathrm{id}$ for all $z \in I$, $\Phi_{z,T,\ell} = \mathrm{id}$.

For our application later, let us review the constructions here. The notations are the same as in Theorem 2.9.

(1) Recall, proposition 2.2 shows that we have a natural morphism $P_{U_1} \to P_{U_2}$. We deduce morphisms $P_{U_1}^{\boxtimes \ell} \to P_{U_2}^{\boxtimes \ell}$, then $\alpha_T(P_{U_1}^{\boxtimes \ell}) \to \alpha_T(P_{U_2}^{\boxtimes \ell})$ and then i^*. In fact, we have

$$F_{U_1,\ell} \xrightarrow{F_{i,\ell}} F_{U_2,\ell}. \tag{22}$$

(2) We take the GKS sheaf quantization $\mathcal{K} := \mathcal{K}(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}_t)$ of φ.

We denote by κ_z the "conjugation" functor on $D((X^2 \times \mathbb{R}_t)^l)$:

$$\kappa_z(F) := \mathcal{K}_z^{\boxtimes \ell} \star F \star \mathcal{K}_z^{-\boxtimes \ell}, \tag{23}$$

where $\mathcal{K}_z^{-\boxtimes \ell} := (\mathcal{K}_z^{-1})^{\boxtimes \ell}$. Chiu showed that

- $P_{\varphi_z(U)} \cong \mathcal{K}_z^{-1} \star P_U \star \mathcal{K}_z$, where $\mathcal{K}_z = \mathcal{K}|_z$.

Hence $\mathcal{K}_z^{\boxtimes \ell} \star P_{\varphi_z(U)}^{\boxtimes \ell} \star \mathcal{K}_z^{-\boxtimes \ell} \cong P_U^{\boxtimes \ell}$.

- There exists a unique isomorphism $g_z : \mathcal{K}_z^{\boxtimes \ell} \star \beta_T \mathbb{K}[-d] \star \mathcal{K}_z^{-\boxtimes \ell} \xrightarrow{\cong} \beta_T \mathbb{K}[-d]$ such that $g_0 = \mathrm{id}$.

Specifically, g_z comes from the following construction: Consider the whole family $[\beta_T \mathbb{K}[-d] \boxtimes \mathbb{K}_I]^{-1} \star_I \delta_z^{-1} \mathcal{K}^{\boxtimes \ell} \star_I [\beta_T \mathbb{K}[-d] \boxtimes \mathbb{K}_I]$, where $\delta_z : \mathbb{R}_z \to \mathbb{R}_z^\ell$ is the diagonal map, $\star_I$ is the convolution with respect to $\mathbb{R}_{\underline{t}}^\ell$ and relative to I. We refer [GKS12, Section 1.6] for the definition of this relative convolution. One can estimate the microsupport to verify that the microsupport of the family satisfied the condition of Theorem 1.12. So it defines a sheaf quantization of the Hamiltonian isotopy $\widehat{\varphi_z}^{\times \ell} : (\dot{T}^*(X \times \mathbb{R}))^\ell \circlearrowleft$. One the other hand, $\delta_z^{-1} \mathcal{K}^{\boxtimes \ell}$ is obviously a sheaf quantization of $\widehat{\varphi_z}^{\times \ell}$. So the uniqueness part of Theorem 1.12 shows there is a unique isomorphism

$$\tilde{g} : \delta_z^{-1} \mathcal{K}^{\boxtimes \ell} \xrightarrow{\cong} [\beta_T \mathbb{K}[-d] \boxtimes \mathbb{K}_I]^{-1} \star_I \delta_z^{-1} \mathcal{K}^{\boxtimes \ell} \star_I [\beta_T \mathbb{K}[-d] \boxtimes \mathbb{K}_I].$$

Here, please noticed to use the uniqueness, we need to extend the domain of $\widehat{\varphi_z}^{\times \ell}$, say $(\dot{T}^*(X \times \mathbb{R}))^\ell$, to $(\dot{T}^*((X \times \mathbb{R})^\ell)$. But because $\widehat{\varphi_z}^{\times \ell}$ is compactly supported, it is direct to extend it by the identity. One can take convolution to obtain

$$\begin{aligned} g \coloneqq & (\delta_z^{-1}\mathcal{K}^{\boxtimes\ell})^{-1} \star_I (\beta_T\mathbb{K}[-d]\boxtimes\mathbb{K}_I) \star_I \tilde{g} : \\ & (\delta_z^{-1}\mathcal{K}^{\boxtimes\ell})^{-1} \star_I (\beta_T\mathbb{K}[-d]\boxtimes\mathbb{K}_I) \star_I (\delta_z^{-1}\mathcal{K}^{\boxtimes\ell}) \xrightarrow{\cong} \beta_T\mathbb{K}[-d]\boxtimes\mathbb{K}_I, \end{aligned} \tag{24}$$

such that $g_z = g|_z$.

The $\Phi_{z,T,\ell}$ is defined as the composition $\Phi_{z,T,\ell} = N^{-1}\circ(g_{z*}\circ\kappa_z)\circ N$, i.e. $\Phi_{z,T,\ell}(f) = N^{-1}((g_z\circ\kappa_z)(N(f)))$, where N is given by the adjoint isomorphism (17), and $g_{z*} = g_z\circ -$. Sometimes, we use $\Phi_{z,T,\ell} = g_{z*}\circ\kappa_z$ too as an abuse of notation.

Taking into account the structure of $A = \operatorname{Ext}^*_G(\mathbb{K},\mathbb{K})$-modules, then we have

Proposition 2.10. Under the notations of Theorem 2.9, we have:

(1) $H^*(i^*)$ is a morphism of A-modules.

(2) $H^*(\Phi_{z,T,\ell})$ is an isomorphism of A-modules.

Proof. The first statement is direct from the definition of the Yoneda product. We prove the second statement. For $f \in H^*(C_{T,\ell}(\varphi_z(U)))$, $a \in \operatorname{Ext}^*_G(\mathbb{K},\mathbb{K})$, (ignoring degrees to simplify), we have

$$\Phi_{z,T,\ell}(a.f) = g_{z*}\,(\kappa_z(\beta_T(a)\circ f)) = g_{z*}(\kappa_z(\beta_T(a))\circ\kappa_z(f)) = g_z\circ\kappa_z(\beta_T(a))\circ\kappa_z(f),$$

and

$$a.\Phi_{z,T,\ell}(f) = \beta_T(a)\circ\Phi_{z,T,\ell}(f) = \beta_T(a)\circ g_{z*}(\kappa_z(f)) = \beta_T(a)\circ g_z\circ\kappa_z(f).$$

Then the problem is reduced to show $g_z\circ\kappa_z(\beta_T(a)) = \beta_T(a)\circ g_z$ for all $z\in I$.

We consider the whole families $g_z\circ\kappa_z(\beta_T(a))$, $\beta_T(a)\circ g_z$ and introduce the functor $\kappa = \delta_z^{-1}\mathcal{K}^{\boxtimes\ell}\star_I - \star_I \delta_z^{-1}\mathcal{K}^{-\boxtimes\ell}$. Then applying [GKS12, Proposition 3.12], we have the following isomorphism induced by the restriction $|_z$ for all $z\in I$:

$$\operatorname{Hom}_{D_G}(\kappa(\beta_T\mathbb{K}[-d]\boxtimes\mathbb{K}_I),\beta_T\mathbb{K}[-d]\boxtimes\mathbb{K}_I) \xrightarrow{\cong} \operatorname{Hom}_{D_G}(\kappa_z(\beta_T\mathbb{K}[-d]),\beta_T\mathbb{K}[-d]). \tag{25}$$

Now, one can verify directly that $g_0\circ\kappa_0(\beta_T(a)) = \beta_T(a)\circ g_0$ because $\mathcal{K}_0^{\boxtimes\ell} = \mathbb{K}_{\Delta_{X^{2\ell}}\times[0,\infty)^d}$. Then the isomorphism (25) shows that $g_z\circ\kappa_z(\beta_T(a)) = \beta_T(a)\circ g_z$ for all $z\in I$. ☐

2.3. **Capacities.** In this section, we assume that $\ell\in\mathbb{P}$ is an odd prime number and $\mathbb{K}$ is a finite field of characteristic ℓ. Under these conditions, the Yoneda algebra $A = \operatorname{Ext}^*_G(\mathbb{K},\mathbb{K})$ is isomorphic to $\mathbb{K}[u,\theta]$, where $|u| = 2, |\theta| = 1$, and $\theta^2 = 0$. Let V be a vector space of dimension d.

Now, we are going to define a sequence of capacities using the Chiu-Tamarkin complex for the class of admissible open sets in T^*V. For $U \xhookrightarrow{i_U} T^*V$ an admissible open subset, the morphism (22) induces a morphism of G-equivariant sheaves:

$$F_{U,\ell} \xrightarrow{F_{i_U,\ell}} F_{T^*V,\ell}.$$

Now, we first take the stalk at $T\geq 0$, and then use the orientation of V. There is a composition of morphisms:

$$\eta_{U,T,\ell} : (F_{U,\ell})_T \to (F_{T^*V,\ell})_T \cong \mathrm{R}\Gamma_c(V,\mathbb{K}) \cong \mathbb{K}[-d].$$

So $\eta_{U,T,\ell}\in \operatorname{Hom}_{D_G}((F_{U,\ell})_T,\mathbb{K}[-d]) = H^0(\mathrm{RHom}_G((F_{U,\ell})_T,\mathbb{K}[-d]))\cong H^0(C_{T,\ell}(U))$.

Definition 2.11. For an admissible open set U and $k \in \mathbb{Z}_{\geq 1}$ we define

$$\operatorname{Spec}(U,k) := \{T \geq 0 : \exists \ell_0 \in \mathbb{P}, \forall \ell \geq \ell_0, \exists \gamma_\ell \in H^*(C_{T,\ell}(U)), \quad \eta_{U,T,\ell} = u^k \gamma_\ell\}, \tag{26}$$

and

$$c_k(U) := \inf \operatorname{Spec}(U,k) \in [0,+\infty]. \tag{27}$$

In the following, we will prove step by step that $(c_k)_{k \in \mathbb{Z}_{\geq 1}}$ defines a sequence of non-trivial symplectic capacities.

Theorem 2.12. The functions $c_k : \mathcal{C}_{\text{admissible}} \to (0,\infty]$ satisfy the following:

(1) $c_k \leq c_{k+1}$ for all $k \in \mathbb{Z}_{\geq 1}$.

(2) If there is an inclusion of admissible open sets $U_1 \subset U_2$, then $c_k(U_1) \leq c_k(U_2)$.

(3) For a compactly supported Hamiltonian isotopy $\varphi_z : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, we have $c_k(U) = c_k(\varphi_z(U))$.

(4) If $U = \{H < 1\}$ is an admissible open set whose boundary $\partial U = \{H = 1\}$ is a regular level set and if $c_k(U) < \infty$, then $c_k(U)$ is represented by the action of a closed characteristic in the boundary ∂U. Consequently, $c_k(rU) = r^2 c_k(U)$ for $r > 0$.

The first thing is the invariance.

Lemma 2.13. Let $\varphi : T^*V \times I \to T^*V$ be a compactly supported Hamiltonian isotopy and U an admissible open set. Recall the A-module isomorphism, defined in Theorem 2.9,

$$H^*(\Phi_{z,T,\ell}) : H^*(C_{T,\ell}(\varphi_z(U))) \xrightarrow{\cong} H^*(C_{T,\ell}(U)).$$

Then $H^*(\Phi_{z,T,\ell})(\eta_{\varphi_z(U),T,\ell}) = \eta_{U,T,\ell}$ for all $z \in I$.

Proof. Recall $\eta_{\varphi_z(U),T,\ell}$ is the composition

$$\begin{tikzcd}
(F_{\varphi_z(U),\ell})_T \arrow[r] \arrow[d, equal] & (F_{T*V,\ell})_T \arrow[r, "\cong"] \arrow[d, equal] & \mathrm{R}\Gamma_c(V,\mathbb{K}) \arrow[r, "or"] & \mathbb{K}[-d], \\
\alpha_T(P^{\boxtimes \ell}_{\varphi_z(U)}) \arrow[r] & \alpha_T(P^{\boxtimes \ell}_{T^*V}) & &
\end{tikzcd}$$

where *or* is the isomorphism given by taking an orientation of $\Delta_{V^{2\ell}} \cong V$. Then

$$N^{-1}(\eta_{\varphi_z(U),T,\ell}) = \left[P^{\boxtimes \ell}_{\varphi_z(U)} \to \beta_T \alpha_T (P^{\boxtimes \ell}_{T^*V}) \xrightarrow{\beta_T(or)} \beta_T \mathbb{K}[-d] \right]$$

For the image $H^*(\Phi_{z,T,\ell})(N^{-1}(\eta_{\varphi_z(U),T,\ell}))$, we embedding it into a commutative diagram.

$$\begin{tikzcd}
P^\ell_U \arrow[r] \arrow[d, "\cong"] & \mathcal{K}_z^{-\boxtimes \ell} \star \beta_T \alpha_T (P^\ell_{T^*V}) \star \mathcal{K}_z^{\boxtimes \ell} \arrow[r, "\kappa_z(or)"] \arrow[d, "g_z"] & \mathcal{K}_z^{\boxtimes \ell} \star \beta_T \mathbb{K}[-d] \star \mathcal{K}_z^{-\boxtimes \ell} \arrow[d, "g_z"] \\
\mathcal{K}_z^{-\boxtimes \ell} \star P^\ell_{\varphi_z(U)} \star \mathcal{K}_z^{\boxtimes \ell} \arrow[r] & \beta_T \alpha_T (P^{\boxtimes \ell}_{T^*V}) \arrow[r, "or"] & \mathbb{K}[-d]
\end{tikzcd}$$

The left square is commutative by the natural isomorphism κ_z(See (23)), and commutativity of the right is essentially the same with proposition 2.10. On the other hand, the lower-left arrow is $N^{-1}(\eta_{U,T,\ell})$, and the upper-right arrow is $N^{-1}H^*(\Phi_{z,T,\ell})(\eta_{\varphi_z(U),T,\ell})$. Then the commutativivity of the diagram gives us $N^{-1}H^*(\Phi_{z,T,\ell})(\eta_{\varphi_z(U),T,\ell}) = N^{-1}(\eta_{U,T,\ell})$. That is $H^*(\Phi_{z,T,\ell})(\eta_{\varphi_z(U),T,\ell}) = \eta_{U,T,\ell}$. ∎

Proposition 2.14. For a compactly supported Hamiltonian isotopy $\varphi : T^*V \times I \to T^*V$ and an admissible open set U, we have $c_k(U) = c_k(\varphi_z(U))$ for all $z \in I$.

Proof. It is verified by the following equivalences.

$$\begin{aligned}
&T \in \mathrm{Spec}(\varphi_z(U), k)\\
\Leftrightarrow &\exists \ell_0, \text{ such that } \forall \ell \geq \ell_0, \exists \gamma_\ell \in H^*(C_{T,\ell}(\varphi_z(U))), \text{ and } \eta_{\varphi_z(U),T,\ell} = u^k\gamma_\ell\\
\Leftrightarrow &\exists \ell_0, \text{ such that } \forall \ell \geq \ell_0, \exists H^*(\Phi_{z,T,\ell})(\gamma_\ell) \in H^*(C_{T,\ell}(U)), \text{ and}\\
&\eta_{U,T,\ell} = H^*(\Phi_{z,T,\ell})(\eta_{\varphi_z(U),T,\ell}) = u^k H^*(\Phi_{z,T,\ell})(\gamma_\ell)\\
\Leftrightarrow &T \in \mathrm{Spec}(U, k)
\end{aligned}$$

The second equivalence comes from the proposition 2.10, and lemma 2.13. Then the result follows from (26). ☐

The following propositions follow directly from the definition and (22).

Proposition 2.15. For $k \in \mathbb{Z}_{\geq 1}$, U an admissible open set, we have $c_k(U) \leq c_{k+1}(U)$.

Proposition 2.16. For $k \in \mathbb{Z}_{\geq 1}$, U_1, U_2 are two admissible open sets. If there is an inclusion $i : U_1 \subset U_2$, then $c_k(U_1) \leq c_k(U_2)$.

We will see later, in corollary 3.8, that for a ball B_a, one can compute that $c_k(B_a) = ka$. Consequently, $c_k(U) > 0$ for all admissible subsets and $c_k(U) < \infty$ for bounded open sets(which are admissible by corollary 2.5).

Last, let us show the representing property.

Proposition 2.17. Suppose $U = \{H < 1\}$ is admissible such that $\partial U = \{H = 1\}$ is a regular level set of a Hamiltonian function H. If $c_k(U) < \infty$, then

$$c_k(U) \in \left\{ \left| \int_\gamma \mathbf{p}d\mathbf{q} \right| : \gamma \text{ is a closed orbit of } \varphi_z^H \right\}. \tag{28}$$

Consequently, for this U, we have $c_k(rU) = r^2 c_k(U)$ for $r \in \mathbb{R}_{>0}$.

Proof. Let $T = c_k(U)$. Suppose it is not given by the action of a closed characteristic.

The Liouville 1-form is non-degenerated here, so there is only finitely many closed characteristics with action less than $2T$. So there is a small $\varepsilon > 0$ such that there is no action happening in $[T - \varepsilon, T + \varepsilon]$.

However, one can estimate the microsupport of $F_{U,\ell}$ to see that

$$\mu s_L(F_{U,\ell}) \subset \left\{ t \in \mathbb{R} : t = \left| \int_\gamma \mathbf{p}d\mathbf{q} \right| \text{ for some closed orbit } \gamma \text{ of } \varphi_z^H \right\}.$$

Therefore $F_{U,\ell}$ is constant on $[T - \varepsilon, T + \varepsilon]$. Consequently, $(F_{U,\ell})_{T-\varepsilon} \cong (F_{U,\ell})_T$, and then $\eta_{U,T-\varepsilon,\ell} = \eta_{U,T,\ell}$ for all ℓ. Then we have that $c_k(U) \leq T - \varepsilon$, contradiction! ☐

Finally, let us remark about the computability of c_k. We define $c_k(U)$ using an arbitrary (but unique) kernel P_U. But in practice, we will choose a particular kernel. Usually, these kernels will admits some properties that are not so obvious from general existence results like proposition 2.4, and corollary 2.5.

We will see, in the section 3, that how to use a kernel associated to toric domains constructed from generating functions to compute capacities of convex toric domains.

2.4. **Geometry of $F_{U,\ell}$.** Here, we make a pause to discuss the geometry of the construction of the Chiu-Tamarkin complex. Following ideas of Chiu, we can resolve $F_{U,\ell} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}s^{\ell}_{t!}\tilde{\Delta}^{-1}\left(P_U^{\boxtimes \ell}\right)$ step in step.

Lemma 2.18. We assume $P_U \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega}$, where $\mathcal{K}$ is a sheaf quantization of some Hamiltonian $\mathbb{R}^m_z$ action on T^*X with moment map μ, $\Omega \subset \mathbb{R}^m_\zeta$ and $U = \mu^{-1}(\Omega)$. Then we have

$$\mathrm{R}s^{\ell}_{t!}\tilde{\Delta}^{-1}\left(P_U^{\boxtimes \ell}\right) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\mathrm{R}s^{\ell}_{z!}\left(\pi_{t_2}^{-1}\left(\mathrm{R}s^{\ell}_{t_1!}\tilde{\Delta}^{-1}\mathcal{K}^{\boxtimes \ell}\right) \otimes \pi^{-1}_{(\mathbf{q},t_1)}\mathrm{R}s^{\ell}_{t_2!}\widehat{\mathbb{K}_\Omega}^{\boxtimes \ell}\right).$$

Proof. It is a direct application of the proper base change and the projection formula. □

The lemma shows, as the construction itself, we can consider separately the Hamiltonian action and the cut-off by Ω. Let us study the Hamiltonian action first. In view of the lemma, it is convenient to define

$$\begin{aligned} L_\ell(\mathcal{K}) &:= \mathrm{R}s^{\ell}_{t!}(\tilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes \ell})), \\ \mathcal{L}_\ell(\mathcal{K}) &:= \mathrm{R}s^{\ell}_{z!}\, L_\ell(\mathcal{K}). \end{aligned} \tag{29}$$

The sheaf $L_\ell(\mathcal{K})$ encodes the information of discrete Hamiltonian loops. Precisely, we have

Proposition 2.19. With the notations (29) we have

(1) $\mu s(L_\ell(\mathcal{K}))$ is

$$\left\{(\underline{z},\underline{\zeta},\mathbf{q}_j,\mathbf{p}_j) : \begin{array}{l}\text{There exist } (\underline{z},\underline{\zeta},\mathbf{q}'_j,\mathbf{p}'_j,\mathbf{q}''_j,\mathbf{p}''_j) \text{ such that } (\mathbf{q}''_j,-\mathbf{p}''_j) = z_j.(\mathbf{q}'_j,\mathbf{p}'_j), \\ \mathbf{q}'_j = \mathbf{q}_{j+1},\ \mathbf{p}_j = \mathbf{p}'_j + \mathbf{p}''_{j+1}, \zeta = -\mu(\mathbf{q}'_j,\mathbf{p}'_j)\, j \in \mathbb{Z}/\ell\mathbb{Z}\end{array}\right\}$$

(2) $L_\ell(\mathcal{K}) \cong (s^{\ell}_z)^{-1}\mathrm{R}s^{\ell}_{z*}L_\ell(\mathcal{K}) \cong (s^{\ell}_z)^{-1}\mathcal{L}_\ell(\mathcal{K})$.

Proof. (1) It follows directly from the functorial estimate of microsupport. The equality follows that the righthand side is a connected Lagrangian submanifold.

(2) The first isomorphism follows the microsupport estimate. The second isomorphism follows from $s^{\ell}_z|_{supp(L_\ell(K))}$ is proper, which could be proven similarly to [GKS12, Proposition 3.2(i)].

□

We call both $L_\ell(\mathcal{K})$ and $\mathcal{L}_\ell(\mathcal{K})$ the sheaves of discrete Hamiltonian loops, and then the result of lemma 2.18 is written as

$$\begin{aligned} \mathrm{R}s^{\ell}_{t!}\tilde{\Delta}^{-1}\left(P_U^{\boxtimes \ell}\right) &\cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathcal{L}_\ell(\mathcal{K}) \otimes \pi^{-1}_{(\mathbf{q},t_1)}\mathrm{R}s^{\ell}_{(z,t_2)!}\widehat{\mathbb{K}_\Omega}^{\boxtimes \ell}\right) \\ &\cong \mathrm{R}\pi_{z!}\left(\mathcal{L}_\ell(\mathcal{K}) \star \mathrm{R}s^{\ell}_{(z,t_2)!}\widehat{\mathbb{K}_\Omega}^{\boxtimes \ell}\right). \end{aligned} \tag{30}$$

Next, let us study $\mathrm{R}s^{\ell}_{(z,t_2)!}\widehat{\mathbb{K}_\Omega}^{\boxtimes \ell}$. In general, the Fourier-Sato transformation is not easy to compute. But when Ω is an open convex set, the result is known. This condition is satisfied for two situations: (i) When we just consider one single Hamiltonian function H, and $\Omega = (-\infty, 1)$ such that $U = H^{-1}(\Omega)$. (ii) In the next section, we will assume Ω to be convex to study convex toric domains.

So, here, it is not so bad to assume convexity of Ω as our first glimpse. In this case $\widehat{\mathbb{K}_\Omega} \cong \mathbb{K}_{\Omega^\circ}$(see the same argument with corollary 2.6), where

$$\Omega^\circ = \{(z,t) : t + z\cdot\zeta \geq 0, \forall \zeta \in \Omega\}.$$

Moreover, Ω° is a closed convex cone containing the origin. So

$$\mathrm{R}s^{\ell}_{(z,t)!}\widehat{\mathbb{K}_\Omega}^{\boxtimes\ell} \cong \mathrm{R}s^{\ell}_{(z,t)!}\mathbb{K}_{\Omega^\circ}^{\boxtimes\ell} \cong \mathbb{K}_{\Omega^\circ}. \tag{31}$$

Consequently, (30) could be read as

$$\mathrm{R}s^{\ell}_{t!}\tilde{\Delta}^{-1}\left(P_U^{\boxtimes\ell}\right) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathcal{L}_\ell(\mathcal{K}) \otimes \pi^{-1}_{(\underline{\mathbf{q}},t_1)}\mathbb{K}_{\Omega^\circ}\right). \tag{32}$$

Therefore,

$$\begin{aligned} F_{U,\ell} &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}s^{\ell}_{t!}\tilde{\Delta}^{-1}\left(P_U^{\boxtimes\ell}\right) \\ &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathcal{L}_\ell(\mathcal{K}) \otimes \pi^{-1}_{(\underline{\mathbf{q}},t_1)}\mathbb{K}_{\Omega^\circ}\right) \\ &\cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(\mathcal{K}) \otimes \pi^{-1}_{t_1}\mathbb{K}_{\Omega^\circ}\right) \\ &\cong \mathrm{R}\pi_{z!}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(\mathcal{K}) \star \mathbb{K}_{\Omega^\circ}\right) \end{aligned} \tag{33}$$

From this formula, the study of $F_{U,\ell}$ is reduced to understand $\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(K)$, which could be thought as a cohomology sheaf of discrete loop space.

So far, we find two different ways to understand $F_{U,\ell}$. Initially, from the definition of $F_{U,\ell}$, we first cut off the energy of a Hamiltonian isotopy to obtain the kernels and then use the functor α_T to obtain cohomology of some discrete loop space. On the other hand, the results of this section shows, we can study discrete loops of a Hamiltonian isotopy first, and then cut off energy. The result of the section clarifies that these two ways are the same.

The second way is more direct than the first in many cases; we will see more about this point of view when doing computation for toric domains.

3. Toric domains

In this section, we would like to study toric domains.

For the Hamiltonian function $H(u) = \pi|u|^2$ on $\mathbb{C}_u$, whose Hamiltonian flow is the 2-dimensional rotation $\varphi_z(u) = e^{-2i\pi z}u$.

Consider the product action of single 2-dimensional rotations, i.e., the $\mathbb{R}^d_z$ action on $\mathbb{C}^d_u$, which is indeed a torus action, given by

$$z.(u_1,\cdots,u_n) = (e^{-2i\pi z_1}u_1,\cdots,e^{-2i\pi z_d}u_d).$$

The moment map of the standard Hamiltonian torus action is

$$\mu(u_1,\cdots,u_n) = (\pi|u_1|^2,\cdots,\pi|u_d|^2). \tag{34}$$

Definition 3.1. For $\Omega \subset \mathbb{R}^d_{\zeta\geq 0}$ open, we say $X_\Omega \coloneqq \mu^{-1}(\Omega)$ an (open) toric domain. We say X_Ω is a convex toric domain if $\widehat{\Omega} \coloneqq \{\zeta \in \mathbb{R}^d : (|\zeta_1|,\cdots,|\zeta_d|) \in \Omega\}$ is convex.

Remark 3.2. For any open subset $\Omega \subset \mathbb{R}^d_{\zeta\geq 0}$, one can take open set Ω^+ such that $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega^+ \subset \mathbb{R}^d$ and $\Omega^+ \cap \mathbb{R}^d_{\zeta\geq 0} = \Omega$. One can see $X_\Omega = \mu^{-1}(\Omega^+)$, and it is independent with choices of Ω^+. Moreover, when X_Ω is a convex toric domain, one can take Ω^+ to be convex(in the usual sense). In the following, we prefer to use Ω^+ instead of Ω, we will not distinguish them.

For example, we can take a sequences of real numbers $0 < a_1 \leq \cdots \leq a_d \leq +\infty$, let $\Omega_D = \{\zeta : \zeta_i < a_i, i = 1, \cdots, d\}$ and $\Omega_E = \{\zeta : \frac{\zeta_1}{a_1} + \cdots + \frac{\zeta_d}{a_d} < 1\}$. Then $X_{\Omega_D} = D(a_1, \cdots, a_d)$ is an open poly-disc and $X_{\Omega_E} = E(a_1, \cdots, a_d)$ is an open ellipsoid. They are all convex toric domains.

3.1. **Generating Function Model for Kernel of Toric Domains.** In [Chi17, Proposition 3.10], Chiu construct a sheaf quantization of Hamiltonian rotation in all dimensions, particularly for the 2-dimensional φ_z, say $\mathcal{S} \in D(\mathbb{R}_z \times \mathbb{R}_{q_1} \times \mathbb{R}_{q_2} \times \mathbb{R}_t)$. It possesses one more property than we stated for general sheaf quantizations(see (12)), say

$$\mathcal{S} \cong \mathrm{R}\pi_{(q_2,\cdots,q_M)!}\mathbb{K}_{\{(z,q_1,\cdots,q_{M+1},t):t+\sum_{j=1}^M S_H(z/M,q_j,q_{j+1})\geq 0\}}, \tag{35}$$

where M big enough such that $z/M \in (0, 1/4)$(we have the same property in negative direction), and S_H is the generating function of the Hamiltonian rotation, where

$$S_H(z, q_1, q_2) = \frac{q_1^2 + q_2^2}{2\tan(2\pi z)} - \frac{q_1 q_2}{\sin(2\pi z)}. \tag{36}$$

The formula (35) is essential when doing the computation of Chiu-Tamarkin complexes of convex toric domains.

Let

$$\mathcal{T} := \mathrm{R}s^d_{t!}(\mathcal{S}^{\boxtimes d}) \in D(\mathbb{R}^d_z \times \mathbb{R}^d_{\mathbf{q}_1} \times \mathbb{R}^d_{\mathbf{q}_2} \times \mathbb{R}_t). \tag{37}$$

So microsupport estimates show $\mathcal{T}$ is a sheaf quantization of the standard torus action in the sense of (12).

As a corollary of proposition 2.4, we have

Proposition 3.3. Toric domains X_Ω are admissible by the distinguished triangle:

$$\widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{V^2}\times\{t\geq 0\}} \to \widehat{\mathcal{T}} \circ \mathbb{K}_{\mathbb{R}^d_{\underline{\zeta}}\setminus\Omega} \xrightarrow{+1}, \tag{38}$$

and the pair of kernels

$$P_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_\Omega,\ Q_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_{\mathbb{R}^d_{\underline{\zeta}}\setminus\Omega} \tag{39}$$

This pair of kernels $(P_{X_\Omega}, Q_{X_\Omega})$ constructed from $\mathcal{T}$ is called the generating function model of the kernel associated to toric domains. As we studied in subsection 2.4, when X_Ω is a convex toric domain, we have

$$\widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \cong \mathcal{T} \star \mathbb{K}_{\Omega^\circ} \tag{40}$$

as the generating function model of the kernel of convex toric domains.

Example 3.4. Take a sequences of real numbers $0 < a_1 \leq \cdots \leq a_d$.

(1) Suppose $\Omega_D = \{\zeta : \zeta_i < a_i, i = 1, \cdots, d\}$, then $X_{\Omega_D} = D(a_1, \cdots, a_d)$ is the open poly-discs.

Let P_r be the kernel of open disc $\{\pi|u|^2 < r\}$ in $\mathbb{C}$, then corollary 2.6 applies and $P_D \cong \mathrm{R}s^d_{t!}(P_{a_1} \boxtimes \cdots \boxtimes P_{a_d})$

(2) Suppose $\Omega_E = \{\zeta : \frac{\zeta_1}{a_1} + \cdots + \frac{\zeta_d}{a_d} < 1\}$, then $X_{\Omega_E} = E(a_1, \cdots, a_d)$ is the open ellipsoid, and $\Omega_E^\circ = \{(z, t) : t \geq -a_1 z_1 = \cdots = -a_d z_d \geq 0\}$.

Let $e : \mathbb{R}_z \times \mathbb{R}_t \to \mathbb{R}_z^d \times \mathbb{R}_t, (z,t) \mapsto (a_1 z, \cdots, a_d z, t)$, then $\mathbb{K}_{\Omega_E^\circ} = \mathrm{R}e_! \mathbb{K}_{\{t \geq -z \geq 0\}}$. Therefore, the projection formula shows

$$P_E := P_{X_{\Omega_E}} \cong \mathcal{T} \star \mathrm{R}e_! \mathbb{K}_{\{t \geq -z \geq 0\}} \cong (e^{-1}\mathcal{T}) \star \mathbb{K}_{\{t \geq -z \geq 0\}} \cong \widehat{e^{-1}\mathcal{T}} \circ \mathbb{K}_{(-\infty,1)}$$

One can check directly that $e^{-1}\mathcal{T}$ is the sheaf quantization of the diagonal Hamiltonian rotation $\varphi_z(u) = (e^{\frac{-2i\pi z}{a_1}} u_1, \cdots, e^{\frac{-2i\pi z}{a_d}} u_d)$ in the sense of (12).

In particular, when $a_1 = \cdots = a_d = \pi R^2 > 0$, the construction is the same as the Chiu's for balls.

3.2. **Chiu-Tamarkin complex, and Capacities of Convex Toric Domains.** In this section, let us focus on convex toric domains. That is $X_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^d_{\geq 0}$ is an open set such that $\{\underline{\zeta} \in \mathbb{R}^d : (|\zeta_1|, \cdots, |\zeta_d|) \in \Omega\}$ is convex, in particular, Ω itself is convex too.

For such a Ω, we can a take open set Ω^+ such that $-\mathbb{R}^d_{\zeta \geq 0} \subset \Omega^+ \subset \mathbb{R}^d$ and $\Omega^+ \cap \mathbb{R}^d_{\zeta \geq 0} = \Omega$. We will see later in remark 3.14 that such a choice does not effect on the computation of $C_{T,\ell}(X_\Omega)$. So, we always choose one Ω^+ and we will not distinguish Ω and Ω^+.

One can verify that, under such conditions, the polar cone satisfies $\{O\} \times \mathbb{R}_{\geq 0} \subset \Omega^\circ \subset \mathbb{R}^d_{\leq 0} \times \mathbb{R}_{\geq 0}$. For $T \geq 0$, let $\Omega_T^\circ := \Omega^\circ \cap \{t = T\}$. Now, set $\|\Omega_T^\circ\|_\infty = \max_{z \in \Omega_T^\circ} \|z\|_\infty$, then $\|\Omega_T^\circ\|_\infty = T\|\Omega_1^\circ\|_\infty$.

So far, we are in a position to state our structural theorem of $H^*(C_{T,\ell}(X_\Omega))$, where X_Ω is a convex toric domain:

Theorem 3.5. For a convex toric domain $X_\Omega \subsetneq T^*V$, $\ell \in \mathbb{P}$, and $0 \leq T < \ell / \|\Omega_1^\circ\|_\infty$, we have:

- The minimal cohomology degree of $H^*(C_{T,\ell}(X_\Omega))$ is exactly $-2I(\Omega_T^\circ)$, i.e.,

$$H^*(C_{T,\ell}(X_\Omega)) \cong H^{\geq -2I(\Omega_T^\circ)}(C_{T,\ell}(X_\Omega))$$

and

$$H^{-2I(\Omega_T^\circ)}(C_{T,\ell}(X_\Omega)) \neq 0$$

where $I(\Omega_T^\circ) = \max_{z \in \Omega_T^\circ} I(z)$, and $I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor$.

- $H^*(C_{T,\ell}(X_\Omega))$ is a finitely generated $\mathbb{K}[u]$ module. The free part is isomorphic to $A = \mathbb{K}[u,\theta]$, so $H^*(C_{T,\ell}(X_\Omega))$ is of rank 2. The torsion part is located exactly in cohomology degree $[-2I(\Omega_T^\circ), -1]$. $H^*(C_{T,\ell}(X_\Omega))$ is torsion free when X_Ω is an open ellipsoid.

- For each $Z \in \Omega_T^\circ$, the inclusion $\overline{OZ} \subset \Omega_T^\circ$ induces a decomposition $\eta_{X_\Omega,T,\ell} = u^{I(Z)}\gamma_{Z,\ell}$ for a non-torsion element $\gamma_{Z,\ell} \in H^{-2I(Z)}(C_{T,\ell}(X_\Omega))$. In particular, $\eta_{X_\Omega,T,\ell}$ is non-zero.

Before proving it, let us use it to compute the capacities $c_k(X_\Omega)$.

Theorem 3.6. For a convex toric domain $X_\Omega \subsetneq T^*V$, we have

$$c_k(X_\Omega) = \inf \{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\}.$$

Proof. Let $S = \{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\}$, $L = \inf S$.

For $T \in S$, there is $Z \in \Omega_T^\circ$ such that $I(Z) = k$. Consider the closed inclusion of the segment $\overline{OZ} \subset \Omega_T^\circ$. Then Theorem 3.5 shows the closed inclusion induces a decomposition $\eta_{X_\Omega,T,\ell} = u^k \gamma_{Z,\ell}$, if we choose $\ell_0 > T\|\Omega_1^\circ\|_\infty$, and $\ell \geq \ell_0$. So $T \in \mathrm{Spec}(X_\Omega, k)$, and $L \geq c_k(X_\Omega)$.

Conversely, if $T \in \operatorname{Spec}(X_\Omega, k)$, there is an $\ell_0 \in \mathbb{P}$, such that for all $\ell \geq \ell_0$ there is a $\gamma_\ell \in H^*(C_{T,\ell}(X_\Omega))$, such that $\eta_{X_\Omega,T,\ell} = u^k\gamma_\ell$. Now, we can take ℓ big enough such that $T < \ell/\|\Omega_1^\circ\|_\infty$, then $\eta_{X_\Omega,T,\ell}$, and γ_ℓ are non-zero. Hence $0 = |\eta_{X_\Omega,T,\ell}| = 2k + |\gamma_\ell|$ shows $2k = -|\gamma_\ell|$. Therefore, the Theorem 3.5 shows $2k = -|\gamma_\ell| \leq 2I(\Omega_T^\circ)$. Hence $T \in S$, and $c_k(X_\Omega) \geq L$. □

Remark 3.7. Under the condition of Theorem 3.6, we have the following formula.

$$\min\left\{\|v\|_\Omega^* : v \in \mathbb{N}^d, \sum_{i=1}^d v_i = k\right\} = \inf\left\{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\right\}. \tag{41}$$

Then this formula together with (1) show that for convex toric domains, we have

$$c_k(X_\Omega) = c_k^{\mathrm{GH}}(X_\Omega).$$

Here, let us test the result by the example of ellipsoids.

Corollary 3.8. Let $X_\Omega = E(a_1, \cdots, a_d)$ is an ellipsoid. For $0 \leq T < \ell a_1$, set $T/\underline{a} = (T/a_1, \cdots, T/a_d)$. We have $H^*(C_{T,\ell}(E)) \cong u^{-I(T/\underline{a})}\mathbb{K}[u,\theta]$, $\eta_{E,T,\ell} \neq 0$, and $c_k(E) = \min\{T \geq 0 : \sum_{i=1}^d \lfloor T/a_i \rfloor \geq k\}$. In particular, $c_k(B_a) = ka$.

Proof. All statements are directly from the Theorem 3.5. □

The rest part of the section is going to prove the Theorem 3.5.

Let us define some notions here. Set $\gamma = (-\infty, 0]^d$, and $\gamma^a = -\gamma = [0,\infty)^d$. A subset $C \subset \mathbb{R}^d$ is called O star-sharped, if $\forall z \in C$, $\overline{Oz} \subset C$. So unions and intersections of O star-sharped sets are still O star-sharped. Examples of O star-sharped are convex set containing the original O, and $\gamma \setminus (\mathring{\gamma} + z)$ for $z \in \gamma$. More generally, for any O star-sharped set $C \subset \mathbb{R}^d$, let $C_\gamma = (C + \gamma^a) \cap \gamma$, then C_γ is still O star-sharped. Noticed, $(C_\gamma)_\gamma = C_\gamma$, so let us call an O star-sharped set C is γ-saturated if $C = C_\gamma$.

Remark 3.9. A related notion is the γ-topology, see [KS90, Section 3.5] and [KS18] for more about the definition and sheaf theory related to γ-topology. In fact, here we are considering the γ^a-topology on γ. But we will not use results related.

Now, review the the results subsection 3.1, and subsection 2.4, we have $P_{X_\Omega} \cong \mathcal{T} \star \mathbb{K}_{\Omega^\circ}$(The formula (40)) and $F_{X_\Omega,\ell} \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(\mathcal{T}) \otimes \pi_{t_1}^{-1}\mathbb{K}_{\Omega^\circ}\right)$(The formula (33)).

So let us study $\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(\mathcal{T})$ following ideas of Chiu. Recall, $\mathcal{T} \cong \mathrm{R}s^d_{t!}(\mathcal{S}^{\boxtimes d})$, where $\mathcal{S}$ is the sheaf quantization of Hamiltonian rotation in dimension 2. Use the Künneth formula, we have

$$\mathcal{L}_\ell(\mathcal{T}) \cong \mathrm{R}s^d_{t!}\mathcal{L}_\ell(\mathcal{S})^{\boxtimes d},$$
$$\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(\mathcal{T}) \cong \mathrm{R}s^d_{t!}\left(\mathrm{R}\pi_{\underline{q}!}\mathcal{L}_\ell(\mathcal{S})\right)^{\boxtimes d}.$$

Luckily, Chiu has computed $\mathrm{R}\pi_{\underline{q}!}\mathcal{L}_\ell(\mathcal{S})$, say:

Proposition 3.10. ([Chi17, Formula (38)])There is a sheaf $\mathcal{E}_\ell \in D^+_G(\mathbb{R}_z)$, and an isomorphism in $D^+_G(\mathbb{R}_z \times \mathbb{R}_t)$ such that

$$\mathrm{R}\pi_{\underline{q}!}\mathcal{L}_\ell(\mathcal{S}) \cong \mathcal{E}_\ell \boxtimes \mathbb{K}_{[0,\infty)}. \tag{42}$$

For any integer $M > 0$,

$$\mathcal{E}_\ell|_{(-M\ell/4,0)} \cong \mathrm{R}\pi_{\underline{q}!}\mathbb{K}_{\mathcal{W}} \tag{43}$$

with

$$\mathcal{W} = \{(z,\underline{q}) \in (-M\ell/4,0)\times\mathbb{R}^{M\ell} : \sum_{k\in\mathbb{Z}/M\ell\mathbb{Z}} S_H(z/M\ell, q_k, q_{k+1}) \geq 0\},$$

and

$$S_H(z,q_k,q_{k+1}) = \frac{q_k^2+q_{k+1}^2}{2\tan(2\pi z)} - \frac{q_kq_{k+1}}{\sin(2\pi z)}.$$

The G-action is induced by the linear action $(q_k)\mapsto(q_{k+M})$ of G on $\mathbb{R}^{M\ell}$, and G acts on (z,t) trivially.

For our convenience, let us make some remarks on $\mathcal{E}_\ell$ and $\mathcal{W}$. Noticed that $z/M\ell \in (-1/4,0)$, so $\sin(2\pi z/M\ell)<0$. One can rewrite $\mathcal{W}$ as follow:

$$\mathcal{W} = \left\{(z,\underline{q}) \in (-M\ell/4,0)\times\mathbb{R}^{M\ell} : \cos(2\pi z/M\ell)\sum_{k\in\mathbb{Z}/M\ell\mathbb{Z}} q_k^2 \leq \sum_{k\in\mathbb{Z}/M\ell\mathbb{Z}} q_kq_{k+1}\right\}.$$

Let us call

$$Q(z,\underline{q}) := \sum_{k\in\mathbb{Z}/M\ell\mathbb{Z}} \left(q_kq_{k+1} - \cos(2\pi z/M\ell)q_k^2\right) \tag{44}$$

$Q(0,\underline{q})$ is well defined. So we extend the definition of $\mathcal{W}$(use the same notation) to

$$\mathcal{W} = \{(z,\underline{q}) \in (-M\ell/4,0]\times\mathbb{R}^{M\ell} : Q(z,\underline{q})\geq 0\}. \tag{45}$$

For a given $z\in(-M\ell/4,0]$, let us denote $\mathcal{W}\cap\{z\}$ by $\mathcal{W}(z)$, and $Q_z = Q(z,-)$. Please notice that the map $z\mapsto\mathcal{W}(z)$ is a decreasing map with respect to the inclusion order.

By the fundamental inequality, we have $\sum_k q_k^2 \geq \sum_k q_kq_{k+1}$, and it takes equality when $q_1=\cdots=q_{M\ell}$. So

$$\mathcal{W}(0) = \{\underline{q}\in\mathbb{R}^{M\ell} : q_1=\cdots=q_{M\ell}\} = \Delta_{\mathbb{R}^{M\ell}}.$$

Finally, one can check $\mathcal{L}_\ell(\mathcal{S})|_{z=0} = \mathbb{K}_{\Delta_{\mathbb{R}^{M\ell}}}\boxtimes\mathbb{K}_{\{t\geq0\}}$ because $\mathcal{S}|_{z=0} = \mathbb{K}_{\Delta_{\mathbb{R}^2}}\boxtimes\mathbb{K}_{\{t\geq0\}}$.

So one can extend the isomorphism (43) to $z=0$, say $\mathcal{E}_\ell|_{(-M\ell/4,0]}\cong\mathrm{R}\pi_{\underline{q}!}\mathbb{K}_{\mathcal{W}}$, which is compatible with the definition of $\mathcal{E}_\ell$.

Consequently, if we take $\mathbf{q}=(q_1,\cdots,q_d)\in\mathbb{R}^d =: V_{\mathbf{q}}$, we have

$$\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathcal{L}_\ell(\mathcal{T}) \cong \mathcal{E}_\ell^{\boxtimes d}\boxtimes\mathbb{K}_{\{t\geq0\}}, \tag{46}$$

and

$$\mathcal{E}_\ell^{\boxtimes d}\Big|_{(-M\ell/4,0]^d} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\prod_{i=1}^d\mathcal{W}_i}, \tag{47}$$

here $\mathcal{W}_i$ means the i.th copy of one $\mathcal{W}$.

Before going into further computations, let us describe the $G=\mathbb{Z}/\ell\mathbb{Z}$-action. G acts on $\mathbb{R}^{M\ell}$ linearly by $(q_l)\mapsto(q_{l+M})$, and the quadratic form Q_z is invariant under this action for every z. So G acts on $\mathcal{W}$, and acts diagonally on $\prod_{i=1}^d\mathcal{W}_i$. The action of G on $(z,t)\in\mathbb{R}_z^d\times\mathbb{R}_t$ is trivial.

Now, let us compute $(F_{X_\Omega,\ell})_T$. Plese recall, for a locally closed inclusion $i: C\subset X$, $F_C = i_!i^{-1}F$ for $F\in D(X)$.

Lemma 3.11. For a compact O star-sharped set C, there is an integer $M > 0$ such that

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_C\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{U}(C), \mathbb{K}\right), \tag{48}$$

where

$$\mathcal{U}(C) = \bigcup_{z\in C}\prod_{i=1}^{d}\mathcal{W}_i(z_i), \tag{49}$$

$\mathcal{W}_i(z_i) = \mathcal{W}_i \cap \{z_i\}$. $\mathrm{R}\Gamma_c$ stands for the derived compactly supported global section. The section is taken as the usual cohomology and equips the G-action induced from the G-action on $\mathcal{W}$ we given above to be a G-module.

Proof. Compactness of C implies that there is $M > 0$ such that $C \subset (-M\ell/4, 0]^d$. Now we have $\mathcal{E}_\ell^{\boxtimes d}|_{(-M\ell/4,0]^d} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\prod_{i=1}^d \mathcal{W}_i}$, and then we obtain

$$\begin{aligned}\mathrm{R}\Gamma\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_C\right) &\cong \mathrm{R}\pi_{z!}\left[(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\prod_{i=1}^d \mathcal{W}_i})_C\right]\\ &\cong \mathrm{R}\pi_{z!}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{(\prod_{i=1}^d \mathcal{W}_i)\cap(V^{M\ell}\times C)}\\ &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}\pi_{z!}\mathbb{K}_{(\prod_{i=1}^d \mathcal{W}_i)\cap(V^{M\ell}\times C)}.\end{aligned}$$

Claim: When restrict to $(\prod_{i=1}^d \mathcal{W}_i)\cap(V^{M\ell}\times C)$, fiber of π_z are compact and contractible.

In fact, Chiu has proved that fiber of π_{z_i} over $\mathbb{R}^{M\ell}$ is a closed interval(See [Chi17, Lemma 4.10]. So fiber of π_z over $\prod_{i=1}^d \mathcal{W}_i$ is a closed cube containing the origin O. Hence, fiber of π_z over $(\prod_{i=1}^d \mathcal{W}_i)\cap(V^{M\ell}\times C)$ is a intersection of a closed cube containing the origin O with a compact O star-sharped set C, which is compact and contractible.

Consequently, the Vietoris-Begel theorem implies

$$\mathrm{R}\pi_{z!}\mathbb{K}_{(\prod_{i=1}^d \mathcal{W}_i)\cap(V^{M\ell}\times C)} \cong \mathbb{K}_{\pi_z((\prod_{i=1}^d \mathcal{W}_i)\cap(V^{M\ell}\times C))} = \mathbb{K}_{\mathcal{U}(C)}.$$

Therefore, $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_C\right) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\mathbb{K}_{\mathcal{U}(C)}) \cong \mathrm{R}\Gamma_c\left(\mathcal{U}(C), \mathbb{K}\right)$. ☐

Now (33) and (46) show

$$\begin{aligned}F_{X_\Omega,\ell} &\cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\mathcal{E}_\ell^{\boxtimes d}\boxtimes\mathbb{K}_{\{t_1\geq 0\}}\otimes\pi_{t_1}^{-1}\mathbb{K}_{\Omega^\circ}\right)\\ &\cong \mathrm{R}\pi_{z!}\left[(\mathcal{E}_\ell^{\boxtimes d}\boxtimes\mathbb{K}_{\{t\geq 0\}})_{\Omega^\circ}\right]\end{aligned} \tag{50}$$

As Ω_T° is compact and O star-sharped, lemma 3.11 and (50) show:

Corollary 3.12. For $T \geq 0$, we have

$$(F_{X_\Omega,\ell})_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{U}(\Omega_T^\circ), \mathbb{K}\right). \tag{51}$$

Another ingredient we need to take into account is the morphism $\eta_{X_\Omega,T,\ell} : (F_{X_\Omega,\ell})_T \to (F_{T^*V,\ell})_T \cong \mathbb{K}[-d]$. We can embed it into an excision triangle:

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ\setminus O}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \xrightarrow{\eta_{X_\Omega,T,\ell}} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_O\right) \xrightarrow{+1}.$$

A similar argument shows the cocone of $\eta_{X_\Omega,T,\ell}$ could be computed by cohomology of a sheaf. In fact, let $X = \{(z,\underline{\mathbf{q}}) \in \mathbb{R}_z^d \times V^{M\ell} : \mathbf{q} \in \mathcal{W}(z)\setminus\Delta_{V^{M\ell}}\}$, and $\mathcal{F}^d = \mathrm{R}\pi_{\mathbf{q}!}\mathbb{K}_X$, it defines $\mathcal{F}^d \in D_G(\mathbb{R}^d)$. Then we have:

Lemma 3.13. For a compact O star-sharped set C, there is

$$R\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{C\setminus O}\right)\cong R\Gamma_c\left(\mathbb{R}^d,\mathcal{F}_C^d\right). \tag{52}$$

Proof. Then a similar argument with lemma 3.11 shows that

$$R\Gamma_c\left(\mathbb{R}^d,\mathcal{F}_C^d\right)\cong R\Gamma_c\left(\bigcup_{z\in C}(\mathcal{W}(z)\setminus\Delta_{V^{M\ell}}),\mathbb{K}\right)=R\Gamma_c\left(\mathcal{U}(C)\setminus\Delta_{V^{M\ell}},\mathbb{K}\right)\cong R\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{C\setminus O}\right),$$

where the last isomorphism comes from lemma 3.11 and the excision sequence:

$$R\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{C\setminus O}\right)\to R\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{C}\right)\xrightarrow{\eta_{X_\Omega,T,\ell}}R\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{O}\right)\xrightarrow{+1}.$$

□

Remark 3.14. Because $z\mapsto\mathcal{W}(z)$ is a decreasing map, one can see that

$$\mathcal{U}(C)=\mathcal{U}(C_\gamma),\ \text{and}\ \mathcal{U}(C)\setminus\Delta_{V^{M\ell}}=\mathcal{U}(C_\gamma)\setminus\Delta_{V^{M\ell}}.$$

Then, later, we will mainly focus on γ-saturated O star-sharped sets.

Now, to understand topology of $\mathcal{U}(C)$(see (49)), let us spend some time on the quadratic form $Q_z=Q(z,-)$, for fixed $z\in(-\infty,0]$. The matrix of Q_z is a circulant matrix

$$A_z=\begin{pmatrix}-\cos(\frac{2\pi z}{M\ell}) & \frac{1}{2} & 0 & \cdots & \frac{1}{2}\\ \frac{1}{2} & -\cos(\frac{2\pi z}{M\ell}) & \frac{1}{2} & \cdots & 0\\ 0 & \frac{1}{2} & -\cos(\frac{2\pi z}{M\ell}) & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \frac{1}{2} & 0 & 0 & \cdots & -\cos(\frac{2\pi z}{M\ell})\end{pmatrix}$$

So one can diagonalize A_z unitarily using the discrete Fourier transform

$$(\omega^{ij})_{i,j=0,\cdots,M\ell-1},$$

where ω is a primitive $M\ell$th.-root of unity. In particular, eigenvalues of A_z are

$$\lambda_k(z)=\mathrm{Re}\left(\exp\left(\frac{2\pi k\sqrt{-1}}{M\ell}\right)\right)-\cos\left(\frac{2\pi z}{M\ell}\right)=\cos\left(\frac{2\pi k}{M\ell}\right)-\cos\left(\frac{2\pi z}{M\ell}\right), \tag{53}$$

where $k\in\mathbb{Z}/M\ell\mathbb{Z}$.

So A_z admits $\#\{k\in\mathbb{Z}/M\ell\mathbb{Z}:\lambda_k\geq 0\}=1+2\lfloor -z\rfloor$ non-negative eigenvalues if $M\ell$ is odd. Please noticed that, we assume ℓ is odd, so we only need to take M to be odd too.

Now, let $u_0=\langle\underline{q},(1,1,\cdots,1)\rangle\in\mathbb{R}$, and $u_k=\langle\underline{q},(1,\omega^k,\omega^{2k},\cdots)\rangle\in\mathbb{C}$, $k=1,2,\cdots M(\ell-1)$. Then the diagonal form of Q_z is

$$Q_z(u_0,\underline{u_k})=\lambda_0(z)u_0^2+\sum_{k=1}^{(M\ell-1)/2}\lambda_k(z)|u_k|^2,\ (u_0,\underline{u_k})\in\mathbb{R}\times\mathbb{C}^{\frac{M\ell-1}{2}}\cong\mathbb{R}^{M\ell}. \tag{54}$$

Therefore, $\mathcal{W}(z)=\{Q_z\geq 0\}$ is a quadratic cone of index $1+2\lfloor -z\rfloor$. In particular, $\mathcal{W}(0)=\Delta_{\mathbb{R}^{M\ell}}=\{(u_0,0,0):u_0\in\mathbb{R}\}$.

Next, we need to study the G-action on $\mathcal{W}(z)$. The action is easier to understand under the diagonal form, where the G-action is as follows: if we take $\mu=\omega^M$ a primitive ℓth. root of unity, then

$$\mu.(u_0,\underline{u_k})=(u_0,\underline{\mu^k u_k}). \tag{55}$$

Consequently, the fixed point sets $\mathcal{W}(z)^G$ is again a quadratic cone, whose index is $1 + 2\lfloor -z/\ell \rfloor$.

For a $z \in \mathbb{R}^d_{\geq 0}$, let $C_z = \prod_{i=1}^d (-z_i - 1, -z_i]$, and $Z_z = \prod_{i=1}^d (-z_i, 0]$. Let $\mathbb{1} = (1, \cdots, 1)$, and $e_i = (\delta_{ij})_{j=1}^d$, then $\mathbb{1}, e_i \in \mathbb{Z}^d_{\geq 0}$.

For a compact γ-saturated O star-sharped set C, let $J_C = \{v \in \mathbb{Z}^d_{\geq 0} : C \cap C_v \neq \varnothing\}$, and $\partial J_C = \{v \in J_C : C \cap (\overline{C_v} \setminus C_v) = \varnothing\}$. Compactness of C shows that both J_C and ∂J_C are finite sets.

Lemma 3.15. For a compact γ-saturated O star-sharped set C. Then $\partial J_C = \{v\}$ for some $v \in \mathbb{Z}^d_{\geq 0}$ if and only if $\overline{Z}_v \subset C \subset Z_{v+\mathbb{1}}$ for the same v.

Proof. When $\overline{Z}_v \subset C \subset Z_{v+\mathbb{1}}$, we can obtain $\partial J_C = \{v\}$ directly from

$$\overline{Z}_v \cap (\overline{C_w} \setminus C_w) \subset C \cap (\overline{C_w} \setminus C_w) \subset Z_{v+\mathbb{1}} \cap (\overline{C_w} \setminus C_w).$$

Conversely, when $\partial J_C = \{v\}$, then we have $v \in C$, so $C = C_\gamma$ implies $\overline{Z}_v = \{v\}_\gamma \subset C$. Now, suppose $C \nsubseteq Z_{v+\mathbb{1}}$, then there is a $z \in C$ such that $z_i = -v_i - 1$. Therefore, $v + e_i \in J_C$. Besides, $\partial J_C = \{v\}$, then $v + e_i \notin \partial J_C$. Hence, there is a j such that $v + e_i + e_j \in J_C$. The process could happen infinitely many times, when $\partial J_C = \{v\}$. But J_C is a finite set. Hence we get a contradiction! Then $C \subset Z_{v+\mathbb{1}}$. □

Lemma 3.16. Under the result of lemma 3.15, i.e., $\partial J_C = \{v\}$. We have $\mathrm{R}\Gamma(C, \mathcal{F}^d) \cong \mathrm{R}\Gamma(S^{2I(v)-1}, \mathbb{K})[-d-1]$, when $I(v) > 0$, and $\mathrm{R}\Gamma(C, \mathcal{F}^d) \cong 0$ when $I(v) = 0$.

Proof. By lemma 3.13, we have $\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_C) \cong \mathrm{R}\Gamma_c(\mathcal{U}(C) \setminus \Delta_{V^{M\ell}}, \mathbb{K})$. Let

$$\mathcal{V} := \mathcal{U}(C) \setminus \Delta_{V^{M\ell}} = \Big\{(u_i) = (u_{0,i}, \underline{u_{k,i}}) : u_{0,i} \neq 0, Q_{z_i}(u_i) \geq 0, z \in C\Big\},$$

here u_i is the coordinate diagonalize Q_{z_i} to (54).

We are going to study the proper homotopy type of $\mathcal{V}$.

When $I(v) > 0$. For $1 \leq m \leq (M\ell - 1)/2$, consider $h_m : \mathbb{R} \times \mathbb{C}^{\frac{M\ell-1}{2}} \times [0,1] \to \mathbb{R} \times \mathbb{C}^{\frac{M\ell-1}{2}}$,

$$h_m(u_0, u_+, u_-, t) = h_{m,t}(u_0, u_+, u_-) = (u_0, u_+, tu_-),$$

where $u_+ = (u_1, \cdots, u_m)$, $u_- = (u_{m+1}, u_{m+2}, \cdots)$.

Now, define $H_v : \left(\mathbb{R} \times \mathbb{C}^{\frac{M\ell-1}{2}}\right)^d \times [0,1] \to \left(\mathbb{R} \times \mathbb{C}^{\frac{M\ell-1}{2}}\right)^d$ by

$$H_{v,t} = h_{v_1,t} \times \cdots \times h_{v_d,t}.$$

Then we have following:

• $H_{v,t}(\mathcal{V}) \subset \mathcal{V}$. $u \in \mathcal{V}$ implies there is $z \in C$, such that for all $i = 1, \cdots, d$, we have $Q_{z_i}(u_i) \geq 0$. That is, in the diagonal form (54),

$$\sum_{k=0}^{v_i} \lambda_k(z_i)|u_{k,i}|^2 \geq \sum_{k \geq v_i+1} (-\lambda_k(z_i))|u_{k,i}|^2.$$

$\overline{Z}_v \subset C \subset Z_{v+\mathbb{1}}$ implies that $v_i \leq z_i < z_{i+1}$, then $\lambda_k(z_i) < 0$ for $k \geq v_i + 1$. So

$$\sum_{k=0}^{v_i} \lambda_k(z_i)|u_{k,i}|^2 \geq \sum_{k \geq v_i+1} (-\lambda_k(z_i))|u_{k,i}|^2 \geq t^2 \sum_{k \geq v_i+1} (-\lambda_k(z_i))|u_{k,i}|^2,$$

i.e., $Q_{z_i}(h_{v_i,t}(u_i)) \geq$. Hence $H_{v,t}(u) \in \mathcal{V}$.

- $H_v|_{\mathcal{V}}$ is proper. Take $u \in \mathcal{V}$, with $\sum_{k=0}^{v_i} |u_{k,i}|^2 + \sum_{k\geq v_i+1} |tu_{k,i}|^2 \leq R^2$, for all $i = 1, \cdots, d$. Obviously, $\sum_{k=0}^{v_i} |u_{k,i}|^2 \leq R^2$, for all $i = 1, \cdots, d$, and

$$\max_{\substack{k=1,\cdots,v_i \\ z\in C}} |\lambda_k(z_i)| R^2 \geq \sum_{k=0}^{v_i} \lambda_k(z_i)|u_{k,i}|^2 \geq \sum_{k\geq v_i+1} (-\lambda_k(z_i))|u_{k,i}|^2 \geq \min_{\substack{k\geq v_i+1 \\ z\in C}} |\lambda_k(z_i)| \sum_{k\geq v_i+1} |u_{k,i}|^2.$$

Because $\lambda_k(z_i) < 0$ for $k \geq v_i + 1$, and $z \in C$. We have $\min_{\substack{k\geq v_i+1 \\ z\in C}} |\lambda_k(z_i)| > 0$. Consequently,

$$\sum_{k\geq v_i+1} |u_{k,i}|^2 \leq \frac{\max\limits_{\substack{k=1,\cdots,v_i \\ z\in C}} |\lambda_k(z_i)|}{\min\limits_{\substack{k\geq v_i+1 \\ z\in C}} |\lambda_k(z_i)|} R^2.$$

That is $(H_v|_{\mathcal{V}})^{-1}$ (compact sets) $\subset$ compact sets.

- $$H_{v,0}(\mathcal{V}) = \{(\underline{u_0}, \underline{u_+}, 0) \in \mathbb{R}^d \times \mathbb{C}^{d\frac{M\ell-1}{2}} : \exists z \in C, Q_{z_i}(u_i) \geq 0, \forall i, (u_+, 0) \neq 0\}.$$

Now, because $v \in C$, then $Q_{v_i}(u_0, u_+, 0)$ is semi-positive, $\forall i$. Therefore $H_{v,0}(\mathcal{V}) = \mathbb{R}^d \times \left(\mathbb{C}^{I(v)} \setminus \{0\}\right) \times \{0\}$.

Therefore, $H_{v,0}(\mathcal{V}) \cong \mathbb{R}^d \times S^{2I(v)-1} \times \mathbb{R}_{>0}$ is a proper strong deformation retraction of $\mathcal{V}$.

When $v = 0$, the same idea shows $\mathcal{U}(C)$ is properly homotopic to $\Delta_{V^{M\ell}}$. Then $\mathrm{R\Gamma}(C, \mathcal{F}) \cong 0$ by excision. □

Lemma 3.17. For a compact γ-saturated O star-sharped set C. Let $I(C) = \max_{z\in C} I(z)$, if we assume further that $C \subset Z_{\ell\mathbb{1}}$. then

$$\mathrm{Ext}^*_G \left(\mathrm{R\Gamma}_c(\mathbb{R}^d, \mathcal{F}^d_C), \mathbb{K}[-d]\right),$$

is zero when $* \notin [-2I(C), -1]$, and it is finite dimensional as a graded $\mathbb{K}$-vector spaces.

Proof. Let us do an induction on $|J_C|$. Please notice that if $I(v) = I(C)$, then $v \in \partial J_C$. Because I is upper-semi continuous, the maximum $I(C)$ can be achieved by some v. Therefore, $|J_C| \geq |\partial J_C| \geq 1$.

If $|J_C| = 1$, that is $J_C = \{0\}$. Then the result follows from the lemma 3.16.

Now, suppose the result is true for C' such that $|J_{C'}| < |J_C|$.

For C given at the beginning, we need consider two possibility.

(1) If $|\partial J_C| = 1$, we apply the lemma 3.16. The result is trivial when $I(v) = 0$ for $v \in \partial J_C$. Then we assume $I(v) > 0$, and we have

$$\mathrm{R\Gamma}_c(\mathbb{R}^d, \mathcal{F}^d_C) \cong \mathrm{R\Gamma}_c(\mathcal{U}(C) \setminus \Delta_{V^{M\ell}}, \mathbb{K}) \cong \mathrm{R\Gamma}(S^{2I(v)-1}, \mathbb{K})[-d-1].$$

Now, because $C \subset Z_{\ell\mathbb{1}}$, all fixed points are $\Delta_{V^{M\ell}}$ by (55), which are removed from $\mathcal{U}(C)$. It means that G acts on $S^{2I(v)-1}$ freely because G does not have non-trivial subgroup. Accordingly,

$$\mathrm{Ext}^*_G \left(\mathrm{R\Gamma}_c(\mathbb{R}^d, \mathcal{F}^d_C), \mathbb{K}[-d]\right) \cong \mathrm{Ext}^*_G \left(\mathrm{R\Gamma}(S^{2I(v)-1}, \mathbb{K}), \mathbb{K}[1]\right) \cong \mathrm{Ext}^* \left(\mathrm{R\Gamma}(L^{2I(v)-1}, \mathbb{K}), \mathbb{K}\right)[1],$$

where $L^{2I(v)-1} = S^{2I(v)-1}/G$ is a Lens spaces of dimension $2I(v) - 1$.

Besides, $\mathrm{Ext}^{-q} \left(\mathrm{R\Gamma}(L^{2I(v)-1}, \mathbb{K}), \mathbb{K}\right)$ compute the (Borel-Moore) homology $H_q(L^{2I(v)-1})$. So, $H_q = \mathbb{K}$, when $q \in [0, 2I(v) - 1]$, and $H_q = 0$ for other q. Convert to cohomology degree, we have $\mathrm{Ext}^* \left(\mathrm{R\Gamma}(L^{2I(v)-1}, \mathbb{K}), \mathbb{K}\right)[1]$ is concentrated in $[-2I(v), -1]$.

The proof of this part is independent with our induction, so it could be applied to the second case.

(2) If $|\partial J_C| \geq 2$, take $v \in \partial J_C$ such that $I(v) = I(C)$. Then we can take $\epsilon > 0$ such that $C \cap (\gamma + (\epsilon \mathbb{1} - v)) \cap (\overline{Z}_{v+\mathbb{1}} \setminus Z_v) = \varnothing$. This is possible due to compactness of C. Define:

$$\begin{aligned} A &= [C \cap (\gamma + (\epsilon \mathbb{1} - v))]_\gamma, \\ B &= C \cap [\mathring{\gamma} \setminus (\gamma + (\epsilon \mathbb{1} - v))]. \end{aligned} \tag{56}$$

Then we have a closed covering $C = A \cup B$. Nevertheless, both A and B are compact γ-saturated O star-sharped sets. So does $A \cap B$.

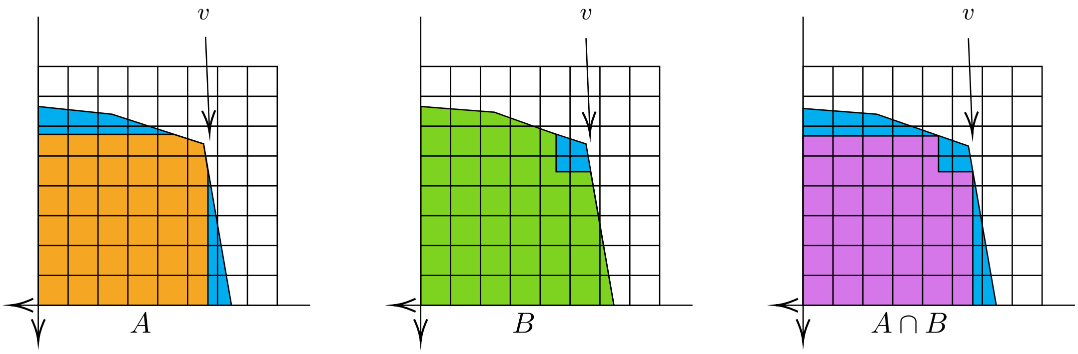

FIGURE 1. The picture illustrate the construction of A, B. C is the background blue set.

Then we have the Mayer-Vietoris triangle,

$$\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_C) \to \mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_A) \oplus \mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_B) \to \mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_{A\cap B}) \xrightarrow{+1}.$$

Next, we apply the $\mathrm{Ext}^*_G(-, \mathbb{K}[-d])$ to obtain a long exact sequence.

$$\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_{A\cap B}), \mathbb{K}[-d]\right) \to \begin{matrix} \mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_A), \mathbb{K}[-d]\right) \\ \bigoplus \\ \mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_B), \mathbb{K}[-d]\right) \end{matrix} \to \mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_C), \mathbb{K}[-d]\right) \xrightarrow{+1}. \tag{57}$$

By our construction (56), we have

- $|\partial J_A| = 1$, then we can apply the result of item 1. So that $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_A), \mathbb{K}[-d]\right)$ is concentrated in $[-2I(A), -1] \subset [-2I(C), -1]$.
- $|J_B| < |J_C|$, we can use the induction hypothesis, hence $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_B), \mathbb{K}[-d]\right)$ is concentrated in $[-2I(B), -1] \subset [-2I(C), -1]$.
- $|I(A \cap B)| < I(C)$, and $|J_{A\cap B}| < |J_C|$, because $A \cap B \subset A$, but $v \notin \partial J_{A\cap B}$. Then we can use the induction hypothesis, that $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_{A\cap B}), \mathbb{K}[-d]\right)$ is concentrated in $[-2I(A \cap B), -1] \subset [-2I(C) + 2, -1]$.

Therefore, one can apply the long exact sequence (57) to conclude that $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c(\mathbb{R}^d, \mathcal{F}^d_C), \mathbb{K}[-d]\right)$ is concentrated in $[-2I(C), -1]$. □

Now, we are in the position to prove the Theorem 3.5. Its proof is essentially a refinement of the computation of ball given by Chiu.

Proof of Theorem 3.5. The corollary 3.12 says that

$$(F_{X_\Omega,\ell})_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right).$$

Besides, we embedding $\eta_{X_\Omega,T,\ell}$ into an excision triangle that

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ\setminus O}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \xrightarrow{\eta_{X_\Omega,T,\ell}} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{O}\right) \xrightarrow{+1}. \tag{58}$$

When $0 \leq T < \ell/\|\Omega_1^\circ\|_\infty$, we have $\Omega_T^\circ \subset Z_{\ell\mathbb{1}}$ (A technical condition needed below). We will prove in lemma 3.17 that, $\mathrm{Ext}_G(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ\setminus O}\right), \mathbb{K}[-d])$ is a finite dimensional $\mathbb{K}$ graded vector space which is concentrated exactly in degrees $[-2I(\Omega_T^\circ), -1]$. One the other hand, $\mathrm{Ext}_G(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{O}\right), \mathbb{K}[-d]) \cong A$ is concentrated in $[0,\infty)$. Then the long exact sequence associate with (58) implies that $H^*(C_{T,\ell}(X_\Omega))$ is concentrated in $[-2I(\Omega_T^\circ), \infty)$.

Moreover, this estimate is sharp. In fact, if we take $Z \in \Omega_T^\circ$ such that $I(Z) = I(\Omega_T^\circ)$. Let us consider the morphism induced by the closed inclusion $j : \overline{OZ} \subset \Omega_T^\circ$, and we embed it into an excision triangle:

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ\setminus\overline{OZ}}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \xrightarrow{j} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right) \xrightarrow{+1}. \tag{59}$$

It means, the cocone of j is $\mathrm{R}\Gamma\left(\Omega_T^\circ \setminus \overline{OZ}, \mathcal{E}_\ell^{\boxtimes d}\right)$. But we can embed the cocone of j into another excision triangle, say:

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ\setminus\overline{OZ}}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ\setminus O}\right) \xrightarrow{j} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}\setminus O}\right) \xrightarrow{+1}. \tag{60}$$

Consequently, the lemma 3.17 and the long exact sequence associated with (60) show that $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}\setminus O}\right)$ is finite dimensional. After tensoring with $\otimes_{\mathbb{K}[u]}\mathbb{K}(u)$, we see that the closed inclusion induces an isomorphism $\mathrm{Ext}_G^*(j, \mathbb{K}[-d]) \otimes_{\mathbb{K}[u]} \mathbb{K}(u)$ between $\mathbb{K}(u)$-module:

$$H^*(C_{T,\ell})(X_\Omega) \otimes_{\mathbb{K}[u]} \mathbb{K}(u) \xrightarrow{\cong} \mathrm{Ext}_G^*\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right), \mathbb{K}[-d]\right) \otimes_{\mathbb{K}[u]} \mathbb{K}(u).$$

For $\mathrm{Ext}_G^*\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right), \mathbb{K}[-d]\right)$. $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{U}(\overline{OZ}), \mathbb{K}\right)$ by (48). But (49) shows, $\mathcal{U}(\overline{OZ}) = \prod_{i=1}^d \mathcal{W}_i(Z_i)$. So $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right) \cong \mathbb{K}[-d-2I(Z)]$, and

$$\mathrm{Ext}_G^*\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right), \mathbb{K}[-d]\right) \cong u^{-I(\Omega_T^\circ)}\mathbb{K}[u,\theta].$$

Then

$$\mathrm{Ext}_G^*\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right), \mathbb{K}[-d]\right) \otimes_{\mathbb{K}[u]} \mathbb{K}(u) \cong u^{-I(\Omega_T^\circ)}\mathbb{K}(u,\theta).$$

All our discussion show that $H^*(C_{T,\ell}(X_\Omega))$ is a finitely generated $\mathbb{K}[u]$-module, whose free part is $\mathbb{K}[u,\theta]$, and torsion part is concentrated exactly in degrees $[-2I(\Omega_T^\circ), -1]$.

When X_Ω is an ellipsoid, Ω_T° itself is a segment. So $H^*(C_{T,\ell})(X_\Omega)$ is torsion free.

The decomposition follows from the same idea for all $Z \in \Omega_T^\circ$. In fact, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \arrow[rr, "\eta_{X_\Omega,T,\ell}"] \arrow[dr, "\gamma_{Z,\ell}"] \arrow[d, equal, "(50)"] & & \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{O}\right) \arrow[d, equal, "(50)"] \\
(F_{X_\Omega})_T & \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\overline{OZ}}\right) \arrow[ur, "u^{I(Z)}"] & (F_{T^*V})_T
\end{tikzcd}$$

Now $\mathrm{Ext}_G^*(\gamma_{Z,\ell}, \mathbb{K}[-d]) \otimes_{\mathbb{K}[u]} \mathbb{K}(u)$ is an isomorphism of $\mathbb{K}(u)$-vector spaces in the same way. So $\gamma_{Z,\ell}$ is non-zero. □

Remark 3.18. Quillen study a more general localization question about equivariant cohomology in [Qui71]. In fact, he considered a localization with respect to an element in A, in which the generator θ is considered. The result of Quillen shows again that, module torsion, $H^*(C_{T,\ell}(X_\Omega))$ is isomorphic to A as A-module, which improves parts our result. But his result can not provide us the information about minimal cohomology degree, that is why we provide this proof, which is essentially independent with the result of Quillen.

We also hope to know what geometric information can be seen from θ. In this direction, let us mention the work of Granja, Karshon, Pabiniak, and Sandon [GKPS20], where they study some non-linear Maslov index involve θ.

At the end of the section, let us make a romantic description of the structure of

$$\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, ({\mathcal{E}_\ell}^{\boxtimes d})_C\right), \mathbb{K}[-d]\right),$$

where C is a γ-saturated O star-sharped set, and $C \subset Z_{\ell\mathbb{1}}$.

Due to our structural theorem, we know that $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, ({\mathcal{E}_\ell}^{\boxtimes d})_C\right), \mathbb{K}[-d]\right)$ could be given as follow: For each $v \in \partial J_C$, we have a copy of $A = \mathbb{K}[u, \theta]$, degree shifted by $-2I(v)$, we can think of it as a flower, and treat the non-negative degree part as the stem of the flower.

Next, we can imagine we have many flowers. Then we tie their stems, so we have a beautiful bouquet now. That is $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, ({\mathcal{E}_\ell}^{\boxtimes d})_C\right), \mathbb{K}[-d]\right)$. Moreover, the A-module structure seems like a water-drop is draping down from the top of blooms into the earth(maybe never reach).

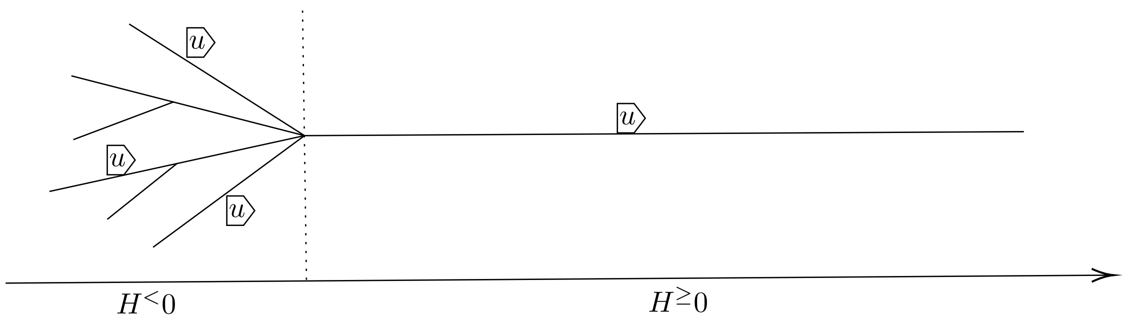

FIGURE 2. The shape of $\mathrm{Ext}^*_G\left(\mathrm{R}\Gamma_c\left(\mathbb{R}^d, ({\mathcal{E}_\ell}^{\boxtimes d})_C\right), \mathbb{K}[-d]\right)$

4. CONTACT GLEANINGS

In this section, let us review how to obtain embedding obstructions of admissible open sets in $T^*X \times S^1$.

For an open set $U \subset T^*X \times S^1$, we can lift it to $\tilde{U} \subset J^1(X)$ as a $\mathbb{Z}$-invariant open subset. In this way, we can discuss sheaves microsupported in $\mu_L(F) \subset J^1(X) \setminus \tilde{U}$. Then $\mathcal{D}_{J^1(X)\setminus\tilde{U}}(X)$ and its left semi-orthogonal complement are all well-formulated and the notion of admissibility is well-defined. In particular, our result in subsection 2.1 works very well. They show the uniqueness, functoriality, and the existence of kernels associated to admissible open sets in $T^*X \times S^1$.

Therefore, initially, the Chiu-Tamarkin complex can be defined, for admissible open sets $U \subset T^*V \times S^1$, without any modifications as:

$$C_{T,\ell}(U) = \mathrm{RHom}_G(P_U^{\boxtimes \ell}, \tilde{\Delta}_* s_t^{\ell !} \pi_{\underline{\mathbf{q}}}^! \mathbb{K}_{\{t=T\}}[-d]).$$

However, a problem here is the invariance. As Chiu discussed in [Chi17] $C_{T,\ell}(U)$ is a contact invariant only when $T = \ell \in \mathbb{Z}_{>0}$. That is

Theorem 4.1 ([Chi17, Theorem 4.7])**.** Let U, U_1, U_2 be contact admissible open sets and let $U_1 \overset{i}{\hookrightarrow} U_2$ be an inclusion. Then one has, for $\ell \in \mathbb{Z}_{>0}$,

(1) There is a morphism $C_{\ell,\ell}(U_2) \xrightarrow{i^*} C_{\ell,\ell}(U_1)$, which is functorial with respect to inclusions of admissible open sets.

(2) Suppose there is a compactly supported Hamiltonian isotopy $\varphi : T^*X \times I \to T^*X$, then there is an isomorphism $\Phi_{z,\ell,\ell} : C_{\ell,\ell}(\varphi_z(U)) \cong C_{\ell,\ell}(U)$, for all $z \in I$. When $\varphi_z = \mathrm{id}$ for all $z \in I$, $\Phi_{z,\ell,\ell} = \mathrm{id}$.

The proof based on the fact that for the contact isotopy φ, its lifting on $J^1(X)$ is $\mathbb{Z}$-equivariant. Then the sheaf quantization of φ is $\mathbb{Z}$-equivariant. Therefore, the isomorphism $\Phi_{z,T,\ell}$ can be defined only when $T = \ell$. Consequently, we can only use $T = \ell$ information of $F_{U,\ell}$ as contact invariants. Now, the same reason with the symplectic case, let us assume $\ell \in \mathbb{P}$, i.e., to be an odd prime. From now on, we only need one lower-subscript for contact admissible open sets U. That is $C_\ell(U) = C_{\ell,\ell}(U)$, and $\eta_{U,\ell,\ell} = \eta_{U,\ell}$.

For the definition of capacities, it is reasonable to require discrete spectrum now. That is

Definition 4.2. For an admissible open set $U \subset T^*V \times S^1$, $k \in \mathbb{Z}_{\geq 1}$. Define

$$[\mathrm{Spec}](U,k) \coloneqq \{\exists \ell_0 \in \mathbb{P} : \forall \ell \geq \ell_0, \exists \gamma_\ell \in H^*(C_\ell(U)), \quad \eta_{U,\ell} = u^k \gamma_\ell\},$$

and

$$[c]_k(U) \coloneqq \min[\mathrm{Spec}](U,k) \in \mathbb{P}.$$

Besides, we also need to take care of properties about $[c]_k$. Obviously, the invariance is still true. But to obtain non-trivial information, we must address some restrictions on size of domains. In fact, in the computation of $[c]_k(B_a)$, we need to take $T = \ell < a\ell$ to make sure $\eta_{U,\ell}$ is non-zero. Here, the constrain is read as $a > 1$. This fits into the framework of [EKP06] that a small contact ball can be squeezed into smaller contact balls. Therefore, we define

Definition 4.3. For a contact admissible open set $U \subset T^*V \times S^1$, we say it is big if there is a contact ball $B_a \times S^1 \subset U$ such that $a > 1$.

Then, we can make sure that the proofs of monotonicity of $[c]_k$ run well under the condition of big admissible open sets.

Unluckily, the proof of representing property is invalid now. In fact, it is even worse, because actions of closed characteristics are usually non-integer. In summary, we organize our discussions as a theorem here:

Theorem 4.4. The functions $[c]_k : \mathcal{C}_{\text{big admissible}} \to \mathbb{P}$ satisfy the following:

(1) $[c]_k \leq [c]_{k+1}$ for all $k \in \mathbb{Z}_{\geq 1}$. In fact, we have $[\mathrm{Spec}](U,k+1) \subset [\mathrm{Spec}](U,k)$.

(2) If there is an inclusion of admissible open sets $U_1 \subset U_2$, then $[c]_k(U_1) \leq [c]_k(U_2)$. In fact, we have $[\mathrm{Spec}](U_2, k) \subset [\mathrm{Spec}](U_1, k)$.

(3) For a compactly supported contact isotopy $\varphi_z : T^*V \times S^1 \to T^*V \times S^1$, then $[c]_k(U) = [c]_k(\phi_z(U))$, and $[\mathrm{Spec}](U, k) \subset [\mathrm{Spec}](\phi_z(U), k)$.

Remark 4.5. Maybe in the contact case, spectrum sets could provide us more interesting obstructions. So I state results of spectrum sets here.

Finally, let us discuss the problem of toric domains. In fact, here we do not need to change too much, because we have set everything up well now. We only need to change the statement of the structural theorem slightly.

Theorem 4.6. For a big contact convex toric domain $X_\Omega \times S^1 \subsetneq T^*V \times S^1$(that means $\|\Omega_1^\circ\|_\infty < 1$). Then we have:

- The minimal cohomology degree of $H^*(C_{T,\ell}(X_\Omega))$ is exactly $-2I(\Omega_T^\circ)$, i.e.,
$$H^*(C_{T,\ell}(X_\Omega)) \cong H^{\geq -2I(\Omega_T^\circ)}(C_{T,\ell}(X_\Omega))$$
and
$$H^{-2I(\Omega_T^\circ)}(C_{T,\ell}(X_\Omega)) \neq 0.$$
- $H^*(C_\ell(X_\Omega \times S^1))$ is a finitely generated $\mathbb{K}[u]$ module. The free part is isomorphic to $A = \mathbb{K}[u, \theta]$, $H^*(C_\ell(X_\Omega \times S^1))$ is of rank 2. The torsion part is located in cohomology degree $[-2I(\Omega_\ell^\circ), -1]$. $H^*(C_\ell(X_\Omega \times S^1))$ is torsion free when X_Ω is an open ellipsoid.
- For each $Z \in \Omega_\ell^\circ$, the inclusion $\overline{OZ} \subset \Omega_\ell^\circ$ induces a decomposition $\eta_{X_\Omega \times S^1, \ell} = u^{I(Z)}\gamma_{Z,\ell}$ for a non-torsion element $\gamma_{Z,\ell} \in H^{-2I(Z)}(C_\ell(X_\Omega \times S^1))$. In particular, $\eta_{X_\Omega \times S^1, \ell}$ is non-zero.

And,

Theorem 4.7. For a big contact convex toric domain $X_\Omega \times S^1 \subsetneq T^*V \times S^1$, we have
$$[c]_k(X_\Omega \times S^1) = \min \{\ell \geq 0 : \exists z \in \Omega_\ell^\circ, I(z) \geq k\}.$$

The result is much weaker than the symplectic case, but it is still interesting. For example, when we consider ellipsoids, we have
$$[c]_k(E \times S^1) = \min \left\{ \ell : \sum_{i=1}^{d} \left\lfloor \frac{\ell}{a_i} \right\rfloor \geq k \right\},$$
where $1 < a_1 \leq a_2 \leq \cdots \leq a_d$.

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