A REMARK ON CONTINUOUS K-THEORY AND FOURIER-SATO TRANSFORMATION

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Abstract. In this notes, we prove a generalization of Efimov's computation for the universal localizing invariant of categories of sheaves with certain microsupport constrain. The proof based certain categorical equivalences given by the Fourier-Sato transformation, which is different with the original one. As an application, we compute the universal localizing invariant of an almost quasi-coherent sheaf category of the Novikov toric scheme introduced by Vaintrob.

1. Introduction

In [Efi24], Efimov introduces an algebraic K-theory for a class of large categories, say dualizable stable categories, which extends the usual nonconnective algebraic K-theory defined for compactly generated categories. In general, the construction enable us to extend localizing invariants of small categories to dualizable stable categories. In particular, for the universal (finitary) localizing invariant $\mathcal{U}_{loc} : \mathrm{Cat}^{\mathrm{Ex}} \to \mathrm{Mot}_{\mathrm{loc}}$, there exists a canonical extension $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}} : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Mot}_{\mathrm{loc}}$, where $\mathrm{Mot}_{\mathrm{loc}}$ is the category of non-commutative motives.

However, as the construction involves computations about the so-called Calkin category that is not easy to describe, computation of the continuous-version of localizing invariants is even harder, and few computational results are known. One of a distinguished from them is the following:

Theorem 1.1 ([Efi24, Theorem 6.11])**.** *Let X be a locally compact Hausdorff space and $\underline{\mathcal{C}}$ is a presheaf on X with values in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$. Then the category $\mathrm{Sh}(X;\underline{\mathcal{C}})$ is dualizable and we have the following natural isomorphism in $\mathrm{Mot}_{\mathrm{loc}}$:*

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}(X;\underline{\mathcal{C}})) \simeq \Gamma_c(X, (\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}\underline{\mathcal{C}})^{\sharp}).$$

Another interesting example concern categories of sheaves with microsupport constrain. For a manifold M and any $F \in \mathrm{Sh}(M)$, Kashiwara and Schapira introduced a conic closed set $\mathrm{SS}(F) \subset T^*M$ in [KS90], which is called the microsupport of sheaves. For a conic closed set $Z \subset T^*M$, we denote $\mathrm{Sh}_Z(M;\mathcal{C})$ the full-subcategory of sheaves whose microsupport are bounded by Z. Then $\mathrm{Sh}_Z(M;\mathcal{C})$ is dualizable since it is a reflexive subcategory of $\mathrm{Sh}(M;\mathcal{C})$.

Theorem 1.2 ([Efi24, Proposition 4.21])**.** *Let $\mathcal{C}$ be a dualizable stable category. Then for the category $\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C})$, which is known to be dualizable stable, we have the following natural equivalence in $\mathrm{Mot}_{\mathrm{loc}}$:*

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C})) \simeq 0.$$

It was shared with us by Alexander I. Efimov, during the *Masterclass: Continuous K-theory* in University of Copenhagen on June 2024, that Theorem 1.2 is still true for a finite dimensional real vector spaces V with the microsupport constrain $Z = V \times \gamma$ for a non-zero proper closed convex cone γ (i.e. Equation (1.1) below). We thank for his generosity. One can prove the high dimensional version in the same way as the 1-dimensional version using a V-indexed semi-orthogonal decomposition.

Remark 1.3. There are many applications of Theorem 1.2 as explained in [Efi24]. For example, Theorem 4.28, Theorem 6.1 and the most important one the ω_1-presentability of $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ in Proposition C.4. in loc. cit.

In view of results of the present article we explained below, they are all corollary of Theorem 1.1. On other other hand, one may also deduce Theorem 1.1 from Theorem 1.2 as explained in [Efi22]. So, our result can also be treated as part of the proof of the equivalence of those two theorems.

New results. In this article, using the Fourier-Sato transformation, we directly identify certain categories of sheaves with microsupport constrain with certain categories without microsupport constrain. Those facts are well-known to experts, however, the interesting part is that we can use them to deduce a generalization of Theorem 1.2 directly from Theorem 1.1.

One of our main result is

Theorem 1.4 (Theorem 4.4 (2-b) below)**.** *Let $\mathcal{C}$ be a dualizable stable category. For a finite dimensional real vector space V and a conic closed set $X \subset V^\vee$, we have the equivalence*

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C})) \simeq \Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})).$$

In particular, if we pick $X = \gamma$ a non-zero proper convex closed cone. Then we deduce from a direct cohomology computation that

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times\gamma}(V;\mathcal{C})) \simeq \Gamma_c(\,\gamma\,;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})) = 0, \tag{1.1}$$

which is the straightforward generalization of Theorem 1.2.

Remark 1.5. Here, we explain logic dependence of results.

The originally proof of Theorem 1.2 (and Equation (1.1)) use the microlocal cut-off lemma of Kashiwara and Schapira to identify the corresponding categories to sheaves over the so-called $\gamma^\vee$-topology (which is non-Hausdorff!), and then it is concluded by a semi-orthogonal decomposition of corresponding presheaf categories. No other machine in microlocal sheaf theory are involved.

Our proof enjoy more machine from the microlocal sheaf theory, it is not surprise that we use the microlocal cut-off lemma somewhere. However, we will not construct any semi-orthogonal decomposition, which makes our proof different from the original one.

Remark 1.6. As a byproduct, we may also discuss the definition of microsupport with dualizable stable coefficient categories. This would have independent interests.

Next, we present an application. For a fan[1] Σ in $\mathbb{R}^n$, Vaintrob constructs a non-Noetherian $\mathbf{k}$-scheme ($\mathbf{k}$ is a discrete ring), the so-called Novikov toric scheme, X_Σ^{Nov}

[1]We do not ask the fan to be rational with respect to a fixed lattice in $\mathbb{R}^n$.

and a subscheme $\partial_{\Sigma}^{\mathrm{Nov}}$ defined by an idempotent ideal sheaf in [Vai17]. Then we can discuss the category of almost coherent sheaves on the almost content $(X_{\Sigma}^{\mathrm{Nov}}, \partial_{\Sigma}^{\mathrm{Nov}})$. If Σ is rational, $X_{\Sigma}^{\mathrm{Nov}}$ is strongly related to the infinite root stack $\sqrt[\infty]{X_{\Sigma}}$ of the usual toric variety. We refer to [KZ25] for more details.

We have the following result, which is first proven by Vaintrob and then by Kuwagaki and the author [KZ25] using different method.

Theorem 1.7. *For a fan Σ and $\mathrm{Mod}_{\mathbf{k}}$ the category of $\mathbf{k}$-modules, we have*

$$\mathrm{aQCoh}_{\mathbb{T}^{\mathrm{Nov}}}(X_{\Sigma}^{\mathrm{Nov}}, \partial_{\Sigma}) \simeq \mathrm{Sh}_{\mathbb{R}^n \times |\Sigma|}(\mathbb{R}^n; \mathrm{Mod}_{\mathbf{k}}).$$

Then as an application of Theorem 1.4, we have

Corollary 1.8. *For a fan Σ, we have*

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{aQCoh}_{\mathbb{T}^{\mathrm{Nov}}}(X_{\Sigma}^{\mathrm{Nov}}, \partial_{\Sigma})) \simeq \Gamma_c(|\Sigma|; \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Mod}_{\mathbf{k}})).$$

Category convention. In this article, categories always mean ∞-categories. We denote $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ the category of dualizable stable categories consists of presentable stable categories that are dualizable with respect to Lurie tensor product and strongly continuous functors between them. In particular, compactly generated stable categories are dualizable. We denote Sp the category of spectrum. And we set ω the countable cardinality.

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2. Sheaves and microsupport

For a bi-complete category $\mathcal{C}$ and a topological space X, we denote the category of $\mathcal{C}$-value sheaves to be $\mathrm{Sh}(X; \mathcal{C})$ [Lur09, 7.3.3.1], it is explained in [Vol21] that in this case we have $\mathrm{Sh}(X; \mathrm{Sp}) \otimes \mathcal{C} \simeq \mathrm{Sh}(X; \mathcal{C})$.

It is explain in loc. cit. that in the $\mathcal{C}$-valued sheaf setting we can define $f_*, f_!$ and f^* and $f^!$; when $\mathcal{C}$ is in addition symmetric monoidal, we can define $\otimes_{\mathcal{C}}$ and the $\mathcal{H}om_{\mathcal{C}}$ as right adjoint of $\otimes_{\mathcal{C}}$, and then we have a full 6-functor formalism. We also refer to[Sch22] for the 6-functors formalism.

Regarding on microlocal theory of sheaves in the ∞-category setup, we remark that, based on [RS18], all arguments of [KS90] work well when $\mathcal{C}$ is compactly generated. Therefore, we can develop the microlocal sheaf theory with compactly generated coefficient, for example $\mathcal{C} = \mathrm{Sp}$ very well.

However, as [Efi24, Remark 4.24] suggested, we may develop a microlocal sheaf theory with more general coefficient.

Here, we will present the Ω-lens definition. Let M be a smooth manifold, and we set $\dot{T}^*M = T^*M \setminus 0_M$.

Definition 2.1. [GV24, Definition 3.1] Let $\Omega \subset \dot{T}^*M$ be an open conic subset. We call Ω-*lens* a locally closed subset C of M with the following properties: $\overline{C}$ is compact and there exists an open neighborhood U of $\overline{C}$ and a function $g\colon U \times [0,1] \to \mathbb{R}$

(1) $dg_t(x) \in \Omega$ for all $(x,t) \in U \times [0,1]$, where $g_t = g|_{U \times \{t\}}$,
(2) $\{g_t < 0\} \subset \{g_{t'} < 0\}$ if $t \leq t'$,

(3) the hypersurfaces $\{g_t = 0\}$ coincide on $U \setminus \overline{C}$,
(4) $C = \{g_1 < 0\} \setminus \{g_0 < 0\}$.

Definition 2.2. For a category $\mathcal{C}$ admits small limits and for $F \in \mathrm{Sh}(M;\mathcal{C})$, we define $\dot{T}^*M \setminus \dot{\mathrm{SS}}_{\mathcal{C}}(F)$ as the maximal open set $\Omega \subset \dot{T}^*M$ such that, for any Ω-lens given by a smooth function g as Definition 2.1, we have the restriction morphism

$$\Gamma(\{g_1 < 0\}, F) \to \Gamma(\{g_0 < 0\}, F)$$

is an equivalence.

Then we define $\mathrm{SS}_{\mathcal{C}}(F) = \dot{\mathrm{SS}}_{\mathcal{C}}(F) \cup \mathrm{supp}(F)$. If it $\mathcal{C}$ is clear, we denote $\mathrm{SS}(F)$ for simplification.

When $\mathcal{C} = \mathrm{Sp}$, there exists a point-wise definition for microsupport, say $\mathrm{SS}_{KS}(F) \subset T^*M$, as explained in [KS90, RS18]. The argument therein shows that $\mathrm{SS}_{KS}(F) \cap 0_M = \mathrm{supp}(F)$. Then we have that $\mathrm{SS}_{KS}(F) \cap 0_M = \mathrm{supp}(F) = \mathrm{SS}_{\mathrm{Sp}}(F) \cap 0_M$. For the the non-zero part, we have the following.

Lemma 2.3. [GV24, Lemma 3.2, 3.3] *Let $F \in \mathrm{Sh}(M;\mathrm{Sp})$ and let $\Omega \subset T^*M \setminus 0_M$ be an open conic subset. Then following are equivalent: 1) $\dot{\mathrm{SS}}_{KS}(F) \cap \Omega = \emptyset$, 2) $\mathrm{Hom}(1_C, F) \simeq 0$ for any Ω-lens C, 3) $\Gamma_c(M; 1_C \otimes F) \simeq 0$ for any $(-\Omega)$-lens C.*

Corollary 2.4. *For $F \in \mathrm{Sh}(M;\mathrm{Sp})$, we have $\mathrm{SS}_{KS}(F)\mathrm{SS}_{\mathrm{Sp}}(F)$.*

For a conic closed subset $Z \subset T^*M$, we set $\mathrm{Sh}_Z(M;\mathrm{Sp})$ the full subcategory of $\mathrm{Sh}(M;\mathcal{C})$ spanned by F with $\mathrm{SS}_{\mathcal{C}}(F) \subset Z$.

By Lemma 2.3, we know that the inclusion $i_Z : \mathrm{Sh}_Z(M;\mathrm{Sp}) \to \mathrm{Sh}(M;\mathrm{Sp})$ admits both left and right adjoints. Then for any dualizable stable category $\mathcal{C}$, under the natural identification $\mathrm{Sh}(M;\mathrm{Sp}) \otimes \mathcal{C} \simeq \mathrm{Sh}(M;\mathcal{C})$, we have $\mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C} \to \mathrm{Sh}(M;\mathcal{C})$ is fully-faithful. Moreover, we have

Proposition 2.5 ([Efi24, Remark 4.24])**.** *For a dualizable stable category $\mathcal{C}$ and a conic closed subset $Z \subset T^*M$, the essential image of the functor $\mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C} \to \mathrm{Sh}(M;\mathcal{C})$ is identified with $\mathrm{Sh}_Z(M;\mathcal{C})$. Equivalently, we have*

$$\mathrm{Sh}_Z(M;\mathcal{C}) \simeq \mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C}.$$

Proof. Here, we know that $i_Z^l : \mathrm{Sh}(M;\mathrm{Sp}) \to \mathrm{Sh}_Z(M;\mathrm{Sp})$ is a left Bousfield localization in the sense its right adjoint i_Z is fully-faithful by definition. Moreover, it is localize at all morphisms $\mathbb{S}_{\{g_1<0\}} \to \mathbb{S}_{\{g_0<0\}}$ for a $\dot{T}^*M \setminus Z$-lense by definition.

Now, as $\mathcal{C}$ is dualizable, we may identify $i_Z^l \otimes \mathcal{C}$ with the following left Bousfield localization

$$\mathrm{Fun}^L(\mathcal{C}^\vee, \mathrm{Sh}(M;\mathrm{Sp})) \to \mathrm{Fun}^L(\mathcal{C}^\vee, \mathrm{Sh}_Z(M;\mathrm{Sp})),$$

where morphisms $c_{\{g_1<0\}} \to c_{\{g_0<0\}}$ for all $c \in \mathcal{C}$ are inverted via the identification $\mathcal{C} \simeq \mathcal{C} \otimes \mathrm{Sp}$. Subsequently, we have $\mathrm{Sh}_Z(M;\mathcal{C}) \simeq \mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{C}$. $\square$

Consequently, all microsupport estimation results in [KS90] are correct for $\mathcal{C}$-valued sheaves since they are correct for Sp-valued sheaves.

Now, we consider the following category introduced by Tamarkin. For a dualizable stable category $\mathcal{C}$, we set

$$\mathcal{T}_{\mathcal{C}} := \mathrm{Sh}(\mathbb{R};\mathcal{C})/\mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathcal{C}).$$

Use the bi-fiber sequence $\mathcal{T}_{\mathrm{Sp}} \to \mathrm{Sh}(\mathbb{R};\mathrm{Sp}) \to \mathrm{Sh}_{\mathbb{R}\times[0,\infty)}(\mathbb{R};\mathrm{Sp})$, we have that $\mathcal{T}_{\mathcal{C}} \simeq \mathcal{T}_{\mathrm{Sp}} \otimes \mathcal{C}$, and for a smooth manifold M we have

$$\mathrm{Sh}(M;\mathrm{Sp}) \otimes \mathcal{T}_{\mathcal{C}} \simeq \mathrm{Sh}(M;\mathcal{T}_{\mathcal{C}}) \simeq \mathrm{Sh}(M \times \mathbb{R};\mathcal{C})/\mathrm{Sh}_{T^*M\times\mathbb{R}\times[0,\infty)}(M \times \mathbb{R};\mathcal{C}).$$

Therefore, for $[F] \in \mathrm{Sh}(M;\mathcal{T}_{\mathcal{C}})$, we can discuss the positive part of microsupport, say

$$SS_+([F]) \coloneqq SS(F) \cap \{\tau > 0\}$$

is a well-defined closed conic subset of $T^*M \times \mathbb{R} \times (0,\infty)$.

The construction is introduced by Tamarkin for study non-conic subsets $Z \subset T^*M$. Precisely, for a subset $Z \subset T^*M$, we define its cone as

$$\widehat{Z} \coloneqq \{(q,p,t,\tau) \in T^*M \times \mathbb{R} \times (0,\infty) : (q, p/\tau) \in Z\}.$$

The conic set $\widehat{Z}$ is closed in $T^*M \times \mathbb{R} \times (0,\infty)$ if Z is closed in T^*M, and in this case we denote $\mathrm{Sh}_{\widehat{Z}}(M;\mathcal{T}_{\mathcal{C}})$ the full subcategory spanned by F with $SS_+([F]) \subset \widehat{Z}$.

Proposition 2.6. *For a conic closed set $Z \subset T^*M$, we have*

$$\mathrm{Sh}_{\widehat{Z}}(M;\mathcal{T}_{\mathcal{C}}) \simeq \mathrm{Sh}_Z(M;\mathrm{Sp}) \otimes \mathcal{T}_{\mathcal{C}} \simeq \mathrm{Sh}_Z(M;\mathcal{C}) \otimes \mathcal{T}_{\mathrm{Sp}}.$$

Proof. If Z is a already conic, then we have $\widehat{Z} = Z \times \mathbb{R} \times (0,\infty)$. Therefore, we have

$$\mathrm{Sh}_{\widehat{Z}}(M;\mathcal{T}_{\mathcal{C}}) = \mathrm{Sh}_{Z\times\mathbb{R}\times(0,\infty)}(M \times \mathbb{R};\mathcal{C}).$$

Then we can apply the Kunneth formula for category of sheaves with general microsupport condition, see [KSZ23] or [Zha25]. In loc. cit., we prove the Kunneth formula for compactly generated rigid symmetric monoidal $\mathcal{C}$. In particular, for $\mathcal{C} = \mathrm{Sp}$. As a result, we have

$$\mathrm{Sh}_{\widehat{Z}}(M;\mathcal{T}_{\mathrm{Sp}}) = \mathrm{Sh}_{Z\times\mathbb{R}\times(0,\infty)}(M\times\mathbb{R};\mathrm{Sp}) = \mathrm{Sh}_Z(M;\mathrm{Sp})\otimes\mathrm{Sh}_{\mathbb{R}\times(0,\infty)}(\mathbb{R};\mathrm{Sp}) = \mathrm{Sh}_Z(M;\mathrm{Sp})\otimes\mathcal{T}_{\mathrm{Sp}}.$$

Then for more general coefficient $\mathcal{C}$, the result is a formal consequence of Proposition 2.5. □

3. Fourier-Sato-Tamarkin transform

In this section, we only consider $\mathcal{C} = \mathrm{Sp}$. This is harmless to our discussion due to Proposition 2.5.

The Fourier-Sato transform was first introduced by Sato in [SKK73]. We refer to [KS90, Section 3.7] for more the relevant discussion. The Fourier-Sato transform gives an equivalence between $\mathbb{R}_{>0}$-equivariant sheaves on V and V^* for real vector space V. To adapt to various non-equivariant situation, one can consider some variants of the Fourier-Sato transform. We refer to [D'A13, Gao17] for more discussion on their definition and the comparison between them.

In [Tam18], Tamarkin introduces a variant of the Fourier-Sato transform $\mathrm{Sh}(V;\mathcal{T}) \to \mathrm{Sh}(V^\vee;\mathcal{T})$ that induces equivalence for not necessarily $\mathbb{R}_{>0}$-equivariant sheaves. We call it the Fourier-Sato-Tamarkin transform.

Let $Leg(V) = \{(z,\zeta,t,s) : t - s + \langle z,\zeta\rangle \geq 0\} \subset V \times V^\vee \times \mathbb{R}^2$. We consider

$$\begin{aligned}
&\mathbb{S}_{Leg(V)} \in \mathrm{Sh}(V \times V^\vee \times \mathbb{R}^2;\mathrm{Sp}),\\
&p_V : V \times V^\vee \times \mathbb{R}_t \times \mathbb{R}_s \to V \times \mathbb{R}_s,\\
&p_{V^\vee} : V \times V^\vee \times \mathbb{R}_t \times \mathbb{R}_s \to V^\vee \times \mathbb{R}_t.
\end{aligned}$$

Definition 3.1. The Fourier-Sato-Tamarkin transform is defined as the functor

$$\begin{aligned} &\mathrm{FST} : \mathrm{Sh}(V; \mathfrak{T}_{\mathrm{Sp}}) \to \mathrm{Sh}(V^\vee; \mathfrak{T}_{\mathrm{Sp}}), \\ &\mathrm{FST}(F) \coloneqq p_{V^\vee !}(p_V^* F \otimes_{\mathrm{Sp}} \mathbb{S}_{Leg(V)})[\dim V]. \end{aligned}$$

It is proven in [Tam18, Theorem 3.5] that the Fourier-Sato-Tamarkin transform FST is an equivalence of categories.

Theorem 3.2. *The Fourier-Sato-Tamarkin transform* FST *is an equivalence of categories: For a closed set $X \subset V^\vee$, we have*

$$\mathrm{Sh}_{\widehat{V \times X}}(V; \mathfrak{T}_{\mathrm{Sp}}) \simeq \mathrm{Sh}_{\widehat{X \times V}}(V^\vee; \mathfrak{T}_{\mathrm{Sp}}).$$

Then we deduce the following result.

Proposition 3.3. *Let V be a finite dimensional real vector space and $X \subset V^\vee$ be a closed set. We have*

$$\mathrm{Sh}_{\widehat{V \times X}}(V; \mathfrak{T}_{\mathrm{Sp}}) \simeq \mathrm{Sh}(X; \mathfrak{T}_{\mathrm{Sp}}).$$

Proof. By Theorem 3.2, we have $\mathrm{Sh}_{\widehat{V \times X}}(V; \mathfrak{T}_{\mathrm{Sp}}) \simeq \mathrm{Sh}_{\widehat{X \times V}}(V^\vee; \mathfrak{T}_{\mathrm{Sp}})$. We notice that $X \times V$ is a conic closed set in $V^\vee \times V$. Then $\mathrm{Sh}_{\widehat{X \times V}}(V^\vee; \mathfrak{T}_{\mathrm{Sp}}) \simeq \mathrm{Sh}_{X \times V}(V^\vee; \mathrm{Sp}) \otimes \mathfrak{T}_{\mathrm{Sp}}$ by Proposition 2.6.

Lastly, recall that $\mathrm{Sh}_{X \times V}(V^\vee; \mathrm{Sp})$ consists of sheaves $H \in \mathrm{Sh}(V^\vee; \mathrm{Sp})$ with the usual microsupport bound $SS(H) \subset X \times V$. Therefore, since $SS(H) \cap 0_{V^\vee} = \mathrm{supp}(H)$, we have that $SS(H) \subset X \times V$ if and only if H is supported in X. □

4. Localizing invariants

Let $\mathcal{E}$ be an accessible stable category (not necessarily cocomplete). Recall that a functor $F : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$ is a continues localizing invariant if the following conditions hold:

(i) $F(0) = 0$;
(ii) for any bi-fiber sequence of the form
$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$
in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ the sequence
$$F(\mathcal{A}) \to F(\mathcal{B}) \to F(\mathcal{C})$$
is a fiber sequence in $\mathcal{E}$.

Roughly speaking, the main result of [Efi24] is that a localizing invariant is determined by its value on compactly generated categories, and all localizing invariants come in this way.

Remark 4.1. In general, we should be careful on accessibility of localizing invariants. Here, we will only discuss finitary localizing invariants, i.e. those commute with filtered colimits.

Among all finitary localizing invariants, there is a universal one, which is originally studied by [BGT13] on compactly generated categories. Then extended to dualizable categories by [Efi24].

More precisely, there is a ω-accessible stable category $\mathrm{Mot}_{\mathrm{loc}}$ that is called the category of non-commutative motive and a universal finitary localizing invariant $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}} : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Mot}_{\mathrm{loc}}$ such that $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ is initial among all finitary localizing invariants:

$$\mathrm{Fun}^L(\mathrm{Mot}_{\mathrm{loc}}, \mathcal{E}) \simeq \mathrm{Fun}_{\mathrm{loc},\omega}(\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}, \mathcal{E}), \quad G \mapsto F = G \circ \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}.$$

Therefore, for many properties that are true for $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ are automatically true for all finitary localizing invariants.

The following well-known trick is clear by definition.

Lemma 4.2. *For a localizing invariant $F : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$ and a dualizable category $\mathcal{C}$, we have $F(\mathcal{C} \otimes -) : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$ is a localizing invariant.*

Now, we will discuss localizing invariants about $\mathrm{Sh}_{V \times X}(V; \mathcal{C}) \simeq \mathrm{Sh}_{V \times X}(V; \mathrm{Sp}) \otimes \mathcal{C}$. Then by Lemma 4.2, we may consider $\mathcal{C} = \mathrm{Sp}$ from the beginning.

To start with, we present a proof of the 1-dimension Theorem 1.2 based on Theorem 1.1. The idea is to reverse the process of the original proof presented in [Efi22].

Proposition 4.3. *We have $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R} \times [0,\infty)}(\mathbb{R}; \mathrm{Sp})) \simeq 0$ and $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{\mathcal{C}}) \simeq \Omega \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})$.*

Proof. One can check that we have the following Cartesian square of dualizable category. One may also see [KZ25] for a detailed proof of its generalization.

$$\begin{tikzcd}
\mathrm{Sh}(\mathbb{R}; \mathrm{Sp}) \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathbb{R} \times [0,\infty)}(\mathbb{R}; \mathrm{Sp}) \arrow[d] \\
\mathrm{Sh}_{\mathbb{R} \times (-\infty,0]}(\mathbb{R}; \mathrm{Sp}) \arrow[r] & \mathrm{Sh}_{\mathbb{R} \times \{0\}}(\mathbb{R}; \mathrm{Sp}).
\end{tikzcd}$$

The map $x \mapsto -x$ induces $\mathrm{Sh}_{\mathbb{R} \times [0,\infty)}(\mathbb{R}; \mathrm{Sp}) \simeq \mathrm{Sh}_{\mathbb{R} \times (-\infty,0]}(\mathbb{R}; \mathrm{Sp})$. Therefore, by [Efi24, Proposition 4.11], we have the fiber sequence in $\mathrm{Mot}_{\mathrm{loc}}$:

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R} \times [0,\infty)}(\mathbb{R}; \mathrm{Sp}))^{\oplus 2} \to \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R} \times \{0\}}(\mathbb{R}; \mathrm{Sp})) \to \Sigma \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}(\mathbb{R}; \mathrm{Sp})).$$

One can directly check that the second morphism is induced by the loop-suspension adjunction of $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sp})$, which is an equivalence. Then we have $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\mathbb{R} \times [0,\infty)}(\mathbb{R}; \mathrm{Sp})) \simeq 0$.

For the next, we consider the bi-fiber sequence

$$\mathcal{T}_{\mathrm{Sp}} \to \mathrm{Sh}_{\mathbb{R} \times [0,\infty)}(\mathbb{R}; \mathrm{Sp}) \to \mathrm{Sp}.$$

Then it follows directly from that $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ is a localizing invariant and the first equivalence. $\square$

Now, we can state our theorems.

Theorem 4.4. *Let V be a finite dimensional real vector space and a closed set $X \subset V^\vee$.*

(1) We have the equivalence:

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{\widehat{V \times X}}(V; \mathcal{T}_{\mathcal{C}})) \simeq \Gamma_c(X; \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{\mathcal{C}})) \simeq \Omega \Gamma_c(X; \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C}));$$

(2) If X is in addition conic, then we have

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V \times X}(V; \mathcal{C})) \simeq \Gamma_c(X; \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})).$$

In particular, all those results are true by replacing $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ with any finitary localizing invariants.

Proof. (1) This is a direct corollary of Theorem 3.2, Theorem 1.1 and Proposition 4.3.
(2) By Proposition 2.6, when X is conic, we have

$$\mathrm{Sh}_{\widehat{V\times X}}(V;\mathcal{T}_{\mathcal{C}}) \simeq \mathrm{Sh}_{V\times X}(V;\mathcal{C}) \otimes \mathcal{T}_{\mathrm{Sp}}.$$

By Lemma 4.2 and (1), we have for any localizing invariants F that $F(\mathcal{T}_{\mathrm{Sp}}) \simeq \Omega F(\mathrm{Sp})$. Here, we take $F = \mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C}) \otimes -)$ to get

$$\begin{aligned}\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C})) \simeq & \Sigma\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}_{V\times X}(V;\mathcal{C}) \otimes \mathcal{T}_{\mathrm{Sp}}) \\ \simeq & \Sigma\Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{T}_{\mathrm{Sp}} \otimes \mathcal{C})) \\ \simeq & \Sigma\Gamma_c(X;\Omega\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})) \\ \simeq & \Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})),\end{aligned}$$

where the second equivalence follows from (2-a) and the last equivalence follows from Proposition 4.3. Then we conclude the proof. $\square$

For a closed set $X \subset V^\vee$ and $U = V^\vee \setminus X$, we can consider the quotient category (see [KSZ23, Section 5] for more details)

$$\mathcal{T}(V \times U;\mathcal{C}) \coloneqq \mathrm{Sh}(V;\mathcal{T}_{\mathcal{C}})/\mathrm{Sh}_{\widehat{V\times X}}(V;\mathcal{T}_{\mathcal{C}}),$$

and, if X is conic and then U is also conic, we can consider

$$\mathrm{Sh}(V, V \times U;\mathcal{C}) \coloneqq \mathrm{Sh}(V;\mathcal{C})/\mathrm{Sh}_{V\times X}(V;\mathcal{C})$$

as in [KS90, Definition 6.1.1].

Then we have

Corollary 4.5. *Let V be a finite dimensional real vector space and a closed set $X \subset V^\vee$ with $U = V^\vee \setminus X$.*
(1) We have the equivalence:

$$\mathcal{T}(V \times U;\mathcal{C}) \simeq \Omega\Gamma_c(U;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C}));$$

(2) If X is in addition conic, then we have

$$\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathrm{Sh}(V, V \times U;\mathcal{C})) \simeq \Gamma_c(U;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})).$$

In particular, all those results are true by replacing $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ with any finitary localizing invariants.

Proof. We only explain (2), (1) follows in the same way.

By Lemma 2.3, we know the inclusion is strongly continuous $\mathrm{Sh}_{V\times X}(V;\mathcal{C}) \to \mathrm{Sh}(V;\mathcal{C})$. Then by definition we have a fiber sequence in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$

$$\mathrm{Sh}(V, V \times U;\mathcal{C}) \to \mathrm{Sh}(V;\mathcal{C}) \to \mathrm{Sh}_{V\times X}(V;\mathcal{C}).$$

Applying $\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}$ to the sequence, then the result follows from the fiber sequence

$$\Gamma_c(U;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})) \to \Gamma_c(V^\vee;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})) \to \Gamma_c(X;\mathcal{U}_{\mathrm{loc}}^{\mathrm{cont}}(\mathcal{C})).$$

$\square$

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