NON-LINEAR MICROLOCAL CUT-OFF FUNCTORS

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Abstract. To any conic closed set of a cotangent bundle, one can associate 4 functors on the category of sheaves, which are called non-linear microlocal cut-off functors. Here we explain their relation with the microlocal cut-off functor defined by Kashiwara and Schapira, and prove a microlocal cut-off lemma for non-linear microlocal cut-off functors, adapting inputs from symplectic geometry. We also prove two Künneth formulas and a functor classification result for categories of sheaves with microsupport conditions.

0. Introduction

In the study of microlocal theory of sheaves, a crucial tool is the microlocal cut-off lemma of Kashiwara and Schapira (see [KS90, Proposition 5.2.3, Lemma 6.1.5], [D'A96] and [Gui23, Chapter III]), which constructs certain functors that enable us to cut off the microsupport of sheaves functorially. On the other hand, in the study of symplectic topology, very similar functors were defined and studied for different purposes [Tam18, NS22, Kuo23].

Precisely, for a smooth manifold M, a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we will study the *non-linear microlocal cut-off functors* $L_U, L_Z, R_U, R_Z : \mathrm{Sh}(M) \to \mathrm{Sh}(M)$ together with fiber sequences of functors $L_U \to \mathrm{id} \to L_Z, , R_Z \to \mathrm{id} \to R_U$ defined using adjoint data of the split Verdier sequence $\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)$. See Section 3 for more details.

The main goal of this article is proving a non-linear microlocal cut-off lemma using the wrapping formula of L_Z introduced in [Kuo23].

Theorem A (Non-linear microlocal cut-off lemma, Theorem 3.5 below.)**.** *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have*

(1) (a) The morphisms $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms if and only if $SS(F) \subset Z$.
(b) The morphism $L_U(F) \to F$ is an isomorphism if and only if $F \in {}^\perp\mathrm{Sh}_Z(M)$, and the morphism $F \to R_U(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_Z(M)^\perp$.

(2) (a) The morphisms $L_U(F) \to F$ and $F \to R_U(F)$ are isomorphisms on U.
(b) If $0_M \subset Z$, the morphism $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms on $\mathrm{Int}(Z) \setminus 0_M$. Equivalently, we have $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$.

In fact, all of them except (2)-(b) follow directly from the definition. Only (2)-(b) of the theorem is a non-trivial fact.

To explain why the theorem generalizes the microlocal cut-off lemma in [KS90], we shall explain the relation between non-linear microlocal cut-off functors and microlocal cut-off functors in [KS90]. Here, we follow the notation of [Gui23, Section III.1] for microlocal cut-off functors.

We take $M = V$ to be a real vector space, $\lambda \subset V$, and $\gamma \subset V^*$ as pointed closed convex cones (we assume further that λ is proper). The comparison is summarized in Table 1, where the first column marks corresponding references, and the last column marks their position in this article. Functors in the same columns are isomorphic with corresponding U and Z.

Based on this comparison, we see that the non-linear microlocal cut-off lemma recovers [KS90, Proposition 5.2.3, Lemma 6.1.5], and [GS14, Proposition 3.17, 3.19, 3.20, 3.21]. We also mention that we say those functors are *non-linear* since we did not use the vector space structure on construction and proof comparing with Kashiwara-Schapira's.

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This article	Z	U	L_U	L_Z	R_U	R_Z	Definition 3.3
[Gui23, Section III.1]	$V \times \lambda^{\circ a}$		P'_λ	Q_λ	Q'_λ	P_λ	Proposition 5.1
[GS14, Section 3]		$V \times \mathrm{Int}(\gamma^\circ)$	L_γ		R_γ		Example 5.3

Table 1. Known microlocal cut-off functors.

We remark that our approach aims to explain the microlocal cut-off lemmas as globally as possible on T^*M using input from symplectic topology. Therefore, we did not consider the refined microlocal cut-off lemmas (see [KS90, Lemma 6.1.4, Proposition 6.1.6], [D'A96], and [Gui23, Section III.2, III.3]), which concern more on the local behavior of cut-off functors.

We also prove two Künneth formulas in Section 7 (with more precise statements), which remove the isotropic condition of [KL24, Theorem 1.1, 1.2] via a different approach.

Theorem B. *For two manifolds M, N, conic closed sets $Z \subset T^*M$ and $X \subset T^*N$, and we set $U = T^*M \setminus Z$ and $V = T^*N \setminus X$. We have*

$$\mathrm{Sh}_{X\times Z}(N \times M) \simeq \mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M), \quad \mathrm{Sh}(N; V) \otimes \mathrm{Sh}(M; U) \simeq \mathrm{Sh}(N \times M; V \times U).$$

Combine the dualizability of $\mathrm{Sh}_Z(M)$ and $\mathrm{Sh}(M; U)$ via [KSZ23, Remark 3.7], we have the following functor classification result:

$$\begin{aligned} &\mathrm{Sh}_{-X\times Z}(N \times M) \simeq \mathrm{Fun}^L(\mathrm{Sh}_X(N), \mathrm{Sh}_Z(M)) \simeq \mathrm{Fun}^R(\mathrm{Sh}_Z(M), \mathrm{Sh}_X(N))^{op}, \\ &\mathrm{Sh}(N \times M; -V \times U) \simeq \mathrm{Fun}^L(\mathrm{Sh}(N; V), \mathrm{Sh}(M; U)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(M; U), \mathrm{Sh}(N; V))^{op}. \end{aligned} \tag{0.1}$$

Category convention. In this article, a category means an ∞-category. We refer to [GR19, Chapter 1] for basics about higher algebra, and more details could be found in [Lur09, Lur17, Lur18].

In this article, we will fix a compactly generated rigid symmetric monoidal stable category $(\mathbf{k}, \otimes, 1)$. Here, rigidity means that the monoidal structure of $\mathbf{k}$ restricts to a symmetric monoidal structure on the category of compact objects $\mathbf{k}^c$ (in particular, the tensor unit 1 is compact) and all compact objects are dualizable. We denote $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the category of $\mathbf{k}$-linear presentable stable categories and $\mathbf{k}$-linear continuous functors. There exists a $\mathbf{k}$-relative Lurie tensor product that form a closed symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ such that the category of left adjoint funtors Fun^L serves as the internal hom, where we write $\otimes/\mathrm{Fun}^L$ directly instead of $\otimes_{\mathbf{k}}/\mathrm{Fun}^L_{\mathbf{k}}$. Rigidity ensures that plain adjoints of $\mathbf{k}$-linear functors are also $\mathbf{k}$-linear.

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1. Sheaves and integral kernel

For a a topological space X, we set $\mathrm{Sh}(X; \mathbf{k}) = \mathrm{Sh}(X)$ the category of $\mathbf{k}$-valued sheaves[Lur09, 7.3.3.1], which is also $\mathbf{k}$-linear. It is explained in [Vol21] that since $\mathbf{k}$ is a symmetric monoidal category, we have the $\mathbf{k}$-linear 6-functor formalism. We also refer to [Sch22] for the 6-functors formalism.

For a locally closed inclusion $i : Z \subset X$ and $F \in \mathrm{Sh}(X)$, we set $F_Z = i_! i^{-1} F$ and $\Gamma_Z F = i_* i^! F$. We denote the constant sheaf $1_X = a^* 1$, where a is the constant map $X \to \mathrm{pt}$, and denote $1_Z = (1_X)_Z \in \mathrm{Sh}(X)$ if there is no confusion.

The main reason for using ∞-category is the following classification of left adjoint functors, which is a combination of a sequence of Lurie's results. We refer to [KSZ23, Proposition 3.1] for a proof.

Proposition 1.1. *If H_1 is a locally compact Hausdorff space, then for all topological spaces H_2, we have the equivalence of categories*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2))$$

It is known that taking the right adjoint induces an equivalence of categories

$$\mathrm{Fun}^L(\mathrm{Sh}(H_1),\mathrm{Sh}(H_2)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2),\mathrm{Sh}(H_1))^{op},$$

where Fun^R stands for right adjoint functors.

However, for any convolution functor $\Phi_K : \mathrm{Sh}(H_1) \to \mathrm{Sh}(H_2)$ with $K \in \mathrm{Sh}(H_1 \times H_2)$, there exists an obvious right adjoint functor[1]

$$\Psi_K : \mathrm{Sh}(H_2) \to \mathrm{Sh}(H_1),\ F \mapsto p_{1*}\mathcal{H}om(K, p_2^! F).$$

Then we have the following equivalence of categories

Corollary 1.2. *Under the same condition of Proposition 1.1, we have*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2),\mathrm{Sh}(H_1))^{op},\ K \mapsto \Psi_K.$$

Remark 1.3. Proposition 1.1 and Corollary 1.2 are only true on the ∞-category level. In the classical theory of microlocal sheaves, many functors are built as triangulated convolution (nvolution) functors, which do not admit similar results. Especially, we do not know if the kernel of triangulated convolution (nvolution) functors is unique in general.

2. Microsupport of sheaves

From now on, we assume M is a smooth manifold. Regarding the microlocal theory of sheaves in the ∞-categorical setup, we remark that all arguments of [KS90] work well provided we have the non-characteristic deformation lemma [KS90, Proposition 2.7.2] for all sheaves. When $\mathbf{k}$ is compactly generated, the non-characteristic deformation lemma is proven for all hypersheaves ([RS18]), and then for all sheaves because of hypercompleteness for manifolds ([Lur09, 7.2.3.6, 7.2.1.12]).

To any object $F \in \mathrm{Sh}(M)$, one can associate a conic closed set $SS(F) \subset T^*M$ ([KS90, Definition 5.1.2]), which satisfies the following triangulated inequality: for a fiber sequence $A \to B \to C$, we have $SS(A) \subset SS(B) \cup SS(C)$.

We remark a further closure property of microsupport

Proposition 2.1 ([KS90, Exercise V.7])**.** *For a set of sheaves $F_\alpha \in \mathrm{Sh}(M)$ indexed by $\alpha \in A$, we have*

$$SS(\prod_\alpha F_\alpha) \cup SS(\bigoplus_\alpha F_\alpha) \subset \overline{\bigcup_\alpha SS(F_\alpha)}.$$

A proof can be find in [GV22b, Proposition 3.4.]. Importantly, the proof therein only use the non-characteristic deformation lemma and a geometric argument. In particular, the proof does not involve the microlocal cut-off lemma in [KS90, Section 5.2].

For $X_n \subset X$ with $n \in \mathbb{N}$, we set

$$\limsup_n X_n = \bigcap_{N\geq 1} \overline{\bigcup_{n \geq N} X_n} = \{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for infinitely many } n,\ x_n \to x\},$$
$$\liminf_n X_n = \{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for all } n,\ x_n \to x\}.$$

By definition, we have $\liminf_n X_n \subset \limsup_n X_n$. As a corollary of Proposition 2.1, we have

Corollary 2.2 ([GV22b, Proposition 6.26])**.** *For a functor $N(\mathbb{N}) \to \mathrm{Sh}(X)$, we have*

$$SS(\varinjlim_n F_n) \subset \liminf_n SS(F_n).$$

Remark 2.3. We can also denote $\mathbb{N}$ as a simplicial set which consists of all vertices together with the edges that join consecutive integers. Then the natural inclusion of simplicial sets $\mathbb{N} \to N(\mathbb{N})$ is cofinal. Therefore, we can compute the colimit using a mapping telescope construction as in triangulated categories, and the proof of Corollary 2.2 is the same as in the references.

For our later application, we present its proof here.

[1] We call it a nvolution functor since it is a dual of a convolution functor, while coconvolution should be the same with nvolution.

Proof. By Remark 2.3, and the fact that all strictly increasing sequences are cofinal in $\mathbb{N}$, we have cofiber sequences for all strictly increasing sequences $\{n_i\}_{i\in\mathbf{N}}$,

$$\bigoplus_i F_{n_i} \to \bigoplus_i F_{n_i} \to \varinjlim_n F_n.$$

We first take $n_i = i + N$ for all $N \geq 1$, and then we have $SS(\varinjlim_n F_n) \subset \limsup_n SS(F_n)$ by Proposition 2.1 and the triangulated inequality.

Now, take $x \notin \liminf_n SS(F_n)$. By the definition of $\liminf_n$, we can find a strictly increasing sequence n_i such that $x \notin \limsup_i SS(F_{n_i})$, then we have $x \notin SS(\varinjlim_i F_{n_i}) = SS(\varinjlim_n F_n)$. That is, $SS(\varinjlim_n F_n) \subset \liminf_n SS(F_n)$. $\square$

We need the following microsupport estimation of (co)nvolution functors:

Proposition 2.4 ([KL24, Lemma 4.4, Proposition 4.5])**.** *Let $K \in \mathrm{Sh}(N \times M)$ with $SS(K) \subset (-X) \times Z$ for conic closed sets $X \subset T^*N$, $Z \subset T^*M$, then we have, for $F \in \mathrm{Sh}(N)$,*

$$SS(\Phi_K(F)) \subset Z, \quad SS(\Psi_K(F)) \subset X.$$

3. Split Verdier sequence from microlocalization

Throughout this article, we set $Z \subset T^*M$ to be a conic closed set and $U = T^*M \setminus Z$ (which is a conic open set). We set $\mathrm{Sh}_Z(M)$ as the full subcategory of $\mathrm{Sh}(M)$ spaned by sheaves F with $SS(F) \subset Z$.

Proposition 3.1. *The category $\mathrm{Sh}_Z(M)$ is a stable subcategory of $\mathrm{Sh}(M)$ closed under small limits and colimits.*

In particular: 1) The inclusion $\iota : \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ admits both left and right adjoints. 2) $\mathrm{Sh}_Z(M)$ is $\mathbf{k}$-linear.

Proof. By the triangulated inequality of microsupport, we only need to show that $\mathrm{Sh}_Z(M)$ is closed under small products and coproducts, which follows from Proposition 2.1.

The inclusion ι admits both adjoints by the adjoint functor theorem [Lur09, 5.5.2.9]; $\mathrm{Sh}_Z(M)$ is presentable and $\mathbf{k}$-linear by [RS22] since ι is reflective and $\mathbf{k}$ is rigid. $\square$

Definition 3.2 ([KS90, Definition 6.1.1])**.** For the conic open set $U \subset T^*M$, we set

$$\mathrm{Sh}(M; U) := \mathrm{Sh}(M)/\mathrm{Sh}_Z(M).$$

We refer to [BGT13, Section 5], [NS18, Section 1.3] and [CDH+, Appendix A] for basic results of Verdier quotient of stable categories. Here, we say that $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$ is a *split Verdier sequence* if ι admits both left and right adjoints, see [CDH+, A.2.4, A.2.6]. We denote ${}^{\perp}\mathcal{C}$ (resp. $\mathcal{C}^{\perp}$) as the left (resp. right) semi-orthogonal complement of $\mathcal{C}$ in $\mathcal{D}$, and we can identify ${}^{\perp}\mathcal{C}$ (resp. $\mathcal{C}^{\perp}$) with $\mathcal{D}/\mathcal{C}$ via a (resp. co)continuous functor in the split case.

By Proposition 3.1, the inclusion $\iota : \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ defines a split Verdier sequence. Precisely, we have the diagram of functors:

$$\begin{tikzcd} \mathrm{Sh}_Z(M) \arrow[r, "\iota"] & \mathrm{Sh}(M) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j"] & \mathrm{Sh}(M; U). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{3.1}$$

Then we have adjunction pairs

$$L_U := j_! j \dashv j_* j =: R_U, \qquad L_Z := \iota\iota^* \dashv \iota\iota^! =: R_Z,$$

and the units/counits give us the following fiber sequences of functors on $\mathrm{Sh}(M)$

$$L_U \to \mathrm{id} \to L_Z, \quad R_Z \to \mathrm{id} \to R_U. \tag{3.2}$$

By Proposition 1.1 and Corollary 1.2, there exists a fiber sequence in $\mathrm{Sh}(M \times M)$

$$K_U \to 1_{\Delta_M} \to K_Z \tag{3.3}$$

that gives corresponding convolution/nvolution functors

$$L_U = \Phi_{K_U},\ L_Z = \Phi_{K_Z},\ R_U = \Psi_{K_U},\ R_Z = \Psi_{K_Z} \tag{3.4}$$

and corresponding natural transformations. Therefore, in parctise, to construct those functors and fiber sequences between them, we only need to write down the fiber sequence (3.3).

Definition 3.3. We call the functors L_U, L_Z, R_U, R_Z *non-linear microlocal cut-off functors.* Corresponding kernels K_U, K_Z are called microlocal cut-off kernels (or microlocal kernels for short).

For any object in $\mathrm{Sh}(M;U)$, the triangulated inequality implies that $SS_U([F]) \coloneqq SS(F) \cap U$ for any representative $F \in \mathrm{Sh}(M)$ is well-defined. In particular, we have $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for all $F \in \mathrm{Sh}(M)$. For conic closed sets $X \supset Z$, we consider the category $\frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$ as a full subcategory of $\mathrm{Sh}(M;U)$. Then the essential image of $\frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$ in $\mathrm{Sh}(M;U)$ is $\mathrm{Sh}_{X\cap U}(M;U)$, the full subcategory spanned by objects with $SS_U([F]) \subset X \cap U$. We will recall this construction in Section 6.

Example 3.4. Take an open set $W \subset M$, and set $U = T^*W \subset T^*M$ with $Z = T^*M \setminus T^*W$. We naturally identify $\mathrm{Sh}_Z(M)$ with sheaves supported in $M \setminus W$ since $\pi_{T^*M}(SS(F)) = \mathrm{supp} F$, and we can verify that the restriction functor $(\bullet)|_W : \mathrm{Sh}(M) \to \mathrm{Sh}(W)$ exhibits $\mathrm{Sh}(W)$ as the Verdier quotient $\mathrm{Sh}(M;T^*W)$. Then we have

$$L_U(F) = F_W,\ L_Z(F) = F_{M\setminus W},\ R_U(F) = \Gamma_W(F),\ R_Z(F) = \Gamma_{M\setminus W}(F),$$

and can take microlocal kernels as following:

$$K_U = 1_{\Delta_W} \to 1_{\Delta_M} \to K_Z = 1_{\Delta_{M\setminus W}}.$$

Therefore, the fiber sequences (3.2) are exactly the excision sequence for sheaves. This example also motivates the notation of functors in Equation (3.1).

In the end of this section, we state the main result of this article. Recall that we say a morphism $f : F \to G$ in $\mathrm{Sh}(M)$ is an isomorphism on an open set U if $SS(\mathrm{cofib}\,(f)) \cap U = \varnothing$, equivalently, f is an isomorphism in $\mathrm{Sh}(M;U)$ (cf. [KS90, Definition 6.1.1]).

Theorem 3.5. *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have*

(1) (a) The morphisms $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms if and only if $SS(F) \subset Z$.
(b) The morphism $L_U(F) \to F$ is an isomorphism if and only if $F \in {}^{\perp}\mathrm{Sh}_Z(M)$, and the morphism $F \to R_U(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_Z(M)^{\perp}$.

(2) (a) The morphisms $L_U(F) \to F$ and $F \to R_U(F)$ are isomorphisms on U.
(b) If $0_M \subset Z$, the morphism $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms on $\mathrm{Int}(Z) \setminus 0_M$. Equivalently, we have $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$.

Proof. (1)-(a) and (2)-(a) follows from definition of L_Z and L_U and fiber sequences (3.2). (1)-(b) follows from [CDH+, A.2.8]. For (2)-(b), we will prove in Proposition 4.4 that $SS(K_U) \subset (-\overline{U}) \times \overline{U} \cup 0_{M\times M}$ under the condition $0_M \subset Z$. Therefore, $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$ follows from Proposition 2.4. ☐

4. Wrapping formula of non-linaer microlocal cut-off functors

In this section, we present an explicit formula of microlocal kernels using Guillermou-Kashiwara-Schapira sheaf quantization [GKS12]. We recall the results of *loc.cit.* here.

Let $\dot{T}^*M$ be the complement of the zero section in T^*M, and, for subset $A \subset T^*M$, we set $\dot{A} = A \cap \dot{T}^*M$. In particular, we have the notion of $\dot{SS}(F)$ for $F \in \mathrm{Sh}(M)$.

Let $(I, 0)$ be a pointed interval. Consider a C^∞ conic symplectic isotopy

$$\phi : I \times \dot{T}^*M \to \dot{T}^*M$$

which is the identity at $0 \in I$. Such an isotopy is always the Hamiltonian flow for a unique conic function $H : I \times \dot{T}^*M \to \mathbb{R}$ and we set $\phi = \phi_H$ when emphasize the Hamiltonian functions. At fixed $z \in I$, we have the graph of ϕ_z:

$$\Lambda_{\phi_z} := \Big\{((q,-p),\phi_z(q,p)) : (q,p) \in \dot{T}^*M\Big\} \subset \dot{T}^*M \times \dot{T}^*M \subset \dot{T}^*(M \times M). \tag{4.1}$$

As for any of Hamiltonian isotopy, we may consider the Lagrangian graph, which by definition is a Lagrangian subset $\Lambda_\phi \subset T^*I \times \dot{T}^*(M \times M)$ with the property that $\Lambda_{\phi_{z_0}}$ is the symplectic reduction of Λ_ϕ along $\{z = z_0\}$. It is given by the formula:

$$\Lambda_\phi := \Big\{(z, -H(z,\phi_z(q,p)), (q,-p), \phi_z(q,p)) : z \in I, (q,p) \in \dot{T}^*Y\Big\} \tag{4.2}$$

Theorem 4.1 ([GKS12, Theorem 3.7, Prop. 4.8])**.** *For ϕ as above, there is a sheaf $K = K(\phi) \in \mathrm{Sh}(I \times Y^2)$ such that $\dot{SS}(K) \subset \Lambda_\phi$ and $K|_{\{0\}\times Y^2} \cong 1_{\Delta_Y}$. The pair $(K, K|_{\{0\}\times Y^2} \cong 1_{\Delta_Y})$ is unique up to unique isomorphism.*

Moreover, for isotopies $\phi_H, \phi_{H'}$ with $H' \leq H$, there's a map $K(\phi_{H'}) \to K(\phi_H)$. In particular, when $H \geq 0$, then there is a map $1_{I\times\Delta_Y} \to K(\phi_H)$.

Motivated by ideas of [Nad16, GPS18], it was shown in [Kuo23] that for any closed set $Z^\infty \subset S^*M$ and the conic closed set $Z = \mathbb{R}_{>0}Z^\infty \cup 0_M \subset T^*M$, the left and right adjoint inclusion $\iota : \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ can be computed 'by wrapping'. More precisely,

Theorem 4.2 ([Kuo23, Thm. 1.2])**.** *If ${H_n}$[2] is any increasing sequence of positive compactly supported Hamiltonians supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, then the adjoints of ι could be computed by*

$$\iota^* F = \varinjlim \Phi_{K(\phi_{H_n})|_1}(F), \quad \iota^! F = \varprojlim \Phi_{K(\phi_{-H_n})|_1}(F). \tag{4.3}$$

Moreover, the unit(resp. counit) of the adjoint is given by the map $1_{\Delta_M} \to \varinjlim K(\phi_{H_n})|_1$ (resp. $\varprojlim K(\phi_{-H_n})|_1 \to 1_{\Delta_M}$) which is induced by the continuation map $1_{\Delta_M} \to K(\phi_{H_n})|_1$ (resp. $K(\phi_{-H_n})|_1 \to 1_{\Delta_M}$) defined by positivity of H_n.

Remark 4.3. In fact, as explained in [KSZ23, Remark 6.5], the colimit in [Kuo23] is taken over an ∞-categorical 'wrapping category', but one can compute the colimit by a cofinal sequence as explained in [Kuo23, Lemma 3.31].

For $H \geq 0$, We set $K^\circ(\phi_{H_n}) = \mathrm{cofib}\,(1_{I\times\Delta_M} \to K(\phi_{H_n}))$.

Compose with the natural inclusion ι and combine with [Kuo23, Proposition 3.5], one can see that the fiber sequence (3.3) can be taken as

$$K_U = \varinjlim(K^\circ(\phi_{H_n})|_1) \to 1_{\Delta_M} \to K_Z = \varinjlim(K(\phi_{H_n})|_1). \tag{4.4}$$

To complete the proof of Theorem 3.5 (2)-(b), we prove the following microsupport estimation of $K^\circ(\phi_H)|_1$. We also notice that the requirement of $0_M \subset Z$ comes from the wrapping formula.

Proposition 4.4. *For a Hamiltonian function H supported in $U^\infty = U/\mathbb{R}_{>0}$, we have*

$$\dot{SS}(K^\circ(\phi_H)|_1) \subset (-U) \times U.$$

In particular, we have

$$\dot{SS}(K_U) \subset (-\overline{U}) \times \overline{U}.$$

Proof. By the triangulated inequality, we have that

$$\dot{SS}(K^\circ(\phi_H)|_1) \subset \Lambda_{\phi_{H,1}} \cup \Lambda_{\mathrm{id}}.$$

Since $\phi_{H,1}$ is compactly supported, there exists a maximal conic open set $W \subset T^*M$ at infinity such that $T^*M \setminus U \subset W$ such that $H|_W = 0$. Therefore, we have

$$\dot{SS}(K^\circ(\phi_H)|_1) \setminus (-W^c) \times W^c \subset \{(q,-p,q,p) : (q,p) \in W\}.$$

[2]Here, we do not distinguish a function on S^*M and its conic lifting on $\dot{T}^*M$.

Since $H|_W = 0$, the monotonicity morphism $1_{\Delta_M} \to K(\phi_H)|_1$ induces the identity map on microstalks at $(q, -p, q, p)$ for $(q, p) \in W$. Then we have that the microstalk of $K^\circ(\phi_H)|_1$ at the $(q, -p, q, p)$ is zero for $(q, p) \in W$. This implies that

$$\dot{SS}(K^\circ(\phi_H)|_1) \subset (-W^c) \times W^c \subset (-U) \times U.$$

The second statement follows from Corollary 2.2 and the wrapping formula (4.4). □

Using the wrapping formula, we can also prove the following microsupport estimation

Proposition 4.5. *If Z contains the zero section, then for any increasing sequence of positive compactly supported Hamiltonians H_n supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, we have*

$$SS(L_Z(F)) \subset \{x : \forall n \in \mathbb{N}, \exists x_n \in \dot{SS}(F), \text{ such that } \phi_{H_n,1}(x_n) \to x\} \cup 0_M,$$
$$SS(R_Z(F)) \subset \{x : \forall n \in \mathbb{N}, \exists x_n \in \dot{SS}(F), \text{ such that } \phi_{-H_n,1}(x_n) \to x\} \cup 0_M.$$

Proof. By the standard estimation of microsupport, we have

$$\dot{SS}(K(\phi)|_z \circ F) = \phi_z(\dot{SS}(F)). \tag{4.5}$$

For L_Z, the estimation follows immediately from Corollary 2.2 and Equation (4.5) using the wrapping formula Theorem 4.2.

For the proof of R_Z, we can not use Corollary 2.2 directly. However, we noticed that the proof of Corollary 2.2 only need the fact: for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, we have $\varinjlim_i F_{n_i} \simeq \varinjlim_n F_n$. It is not true for $\varprojlim$. Nevertheless, for H_n given, functions $\{H_{n_i}\}_{i\in\mathbb{N}}$ still go to infinity on $S^*M \setminus Z^\infty$ for any divergent subsequence $\{n_i\}_{i\in\mathbb{N}}$. Therefore, Theorem 4.2 implies, for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, that

$$R_Z(F) \simeq \varprojlim_i \Phi_{K(\phi_{-H_{n_i}})|_1}(F).$$

Then we can adapt the argument of Corollary 2.2 for the specific limit to show that

$$SS(R_Z(F)) \subset \liminf_n SS(\Phi_{K(\phi_{-H_n})|_1}(F)).$$

The required microsupport estimation follows from Equation (4.5). □

Remark 4.6. In this proposition, the only place we use the wrapping formula is in identifying $L_Z(F)$ with $\varinjlim \Phi_{K(\phi_{H_n})|_1}(F)$. On the other hand, it is possible to prove directly that $SS(\varinjlim \Phi_{K(\phi_{H_n})|_1}(F)) \subset Z$ using the estimation in the proposition, and then identify L_Z with $\varinjlim \Phi_{K(\phi_{H_n})|_1}$. Subsequenty, it offers us a new proof of the wrapping formula. Passing to adjoint, we get the wrapping formula of R_Z. We left the detail to interested readers.

Last, we write down the following limit formula.

Proposition 4.7. *For an increasing sequence of conic open sets U_n with $U = \cup_n U_n$. We set $Z_n = T^*M \setminus U_n$. Then there exists a colimit of fiber sequences in* $\mathrm{Sh}(M \times M)$*:*

$$K_U \to 1_{\Delta_M} \to K_Z \simeq \varinjlim [K_{U_n} \to 1_{\Delta_M} \to K_{Z_n}].$$

Proof. We refer to [Zha23, Appendix], where we prove a version of the proposition for triangulated Tamarkin categories. The construction therein only involves [Zha23, Proposition 1.12], which can be proven in the ∞-categorical setting directly. Moreover, we do not need the triangulated derivator argument here since the natural inclusions $\mathrm{Sh}_{Z_{n+1}}(M) \subset \mathrm{Sh}_{Z_n}(M)$ naturally induce a homotopy coherent diagram of kernels $\{K_{Z_n}\}_n$. Then we can run the proof therein since we only need Corollary 2.2. □

5. Kashiwara-Schapira microlocal cut-off functors

In this section, we study the microlocal cut-off functors defined by Kashiwara and Schapira. We will follow notation and formulation of [Gui23, Chapter III].

Let $M = V$ be a real vector space of dimension n, and we naturally identify $T^*V = V \times V^*$. A subset $\gamma \subset V$ is a cone if $\mathbb{R}_{>0}\gamma \subset \gamma$, we say γ is pointed if $0 \in \gamma$. We set $\gamma^a = -\gamma$. We say a cone γ is convex/closed if it is a convex/closed set, and is proper if $\gamma \cap \gamma^a = \{0\}$ (equivalently, γ contains no line). We define the dual cone $\gamma^\circ \subset V^*$ and $\widetilde{\gamma} \subset V \times V$ as

$$\gamma^\circ = \{l \in V^* : l(v) \geq 0, \forall v \in \gamma\}, \quad \widetilde{\gamma} = \{(x, y) \in V \times V : x - y \in \gamma\}.$$

For a pointed closed cone $\gamma \subset V$, we define 4 functors

$$\begin{aligned} P_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2*}(1_{\widetilde{\gamma}} \otimes q_1^* F), \\ Q_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2!}(\mathcal{H}om(1_{\widetilde{\gamma}^a}, q_1^! F)), \\ P'_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2!}(\mathcal{H}om(1_{\widetilde{\gamma}^a \setminus \Delta_V}[1], q_1^! F)), \\ Q'_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2*}(1_{\widetilde{\gamma} \setminus \Delta_V}[1] \otimes q_1^* F). \end{aligned} \tag{5.1}$$

For $\gamma = \{0\}$, we have $\widetilde{\{0\}} = \Delta_V$ and $P_{\{0\}}(F) \simeq Q_{\{0\}}(F) \simeq F$. Using the fiber sequence $1_{\widetilde{\gamma}} \to 1_{\Delta_V} \to 1_{\widetilde{\gamma} \setminus \Delta_V}[1]$ (and the same with γ^a), we obtain the fiber sequences of functors

$$P'_\gamma \to \mathrm{id} \to Q_\gamma, \quad P_\gamma \to \mathrm{id} \to Q'_\gamma. \tag{5.2}$$

Proposition 5.1. *Let γ be a pointed closed convex cone. For $Z = V \times \gamma^{\circ a}$ and $U = T^*M \setminus Z$, we have the isomorphisms of fiber sequences*

$$[P'_\gamma \to \mathrm{id} \to Q_\gamma] \simeq [L_U \to \mathrm{id} \to L_Z], \quad [P_\gamma \to \mathrm{id} \to Q'_\gamma] \simeq [R_Z \to \mathrm{id} \to R_U].$$

Remark 5.2. Originally, [KS90, Gui23] define triangulated functors $P_\gamma, Q_\gamma, P'_\gamma, Q'_\gamma$. Here, we use the same formula to define those functors on ∞-level, so they automatically descent to corresponding triangulated functors.

Proof. It is explained in [Gui23, Remark III.1.9] that, for the inclusion $\iota : \mathrm{Sh}_{V \times \gamma^{\circ a}}(V) \to \mathrm{Sh}(V)$, we have that $Q_\gamma = \iota\iota^*$ and $P_\gamma = \iota\iota^!$; and the natural transformations induced by $1_{\Delta_V} \to 1_{\widetilde{\gamma} \setminus \Delta_V}[1]$ and $1_{\widetilde{\gamma}^a} \to 1_{\Delta_V}$ are unit/counit of the corresponding adjunctions. ∎

By the virtue of Section 3, the existence of the cut-off functor appears as a purely categorical result. However, the advantage of Equation (5.1) is that the kernel is explicitly written (also no limit!). It is explained in [Gui23, III.1.5] one can write down kernels of P'_γ and Q_γ as Proposition 1.1 predicted, such that the fiber sequence $K_U \to 1_{\Delta_V} \to K_Z$ is given by the fiber sequence

$$K_U = \mathrm{D}'(1_{\widetilde{\gamma}^a \setminus \Delta_V})[n-1] \to 1_{\Delta_V} \to K_Z = \mathrm{D}'(1_{\widetilde{\gamma}^a})[n],$$

where $\mathrm{D}'(F) = \mathcal{H}om(F, 1_{V^2})$.

Example 5.3. We consider a variant. Let $M = V$ be a real vector spaces. Take a pointed closed convex proper cone $\gamma \subset V^*$, set $U = V \times \mathrm{Int}(\gamma^\circ)$ and $Z = V \times (V^* \setminus \mathrm{Int}(\gamma^\circ))$. The microlocal cut-off functors for U are studied by Tamarkin [Tam18] and Guillermou-Schapira [GS14] (where L_U, R_U are called L_γ, R_γ respectively.)

The fiber sequence of kernels is given by

$$K_U = 1_{\widetilde{\gamma}} \to 1_{\Delta_V} \to K_V = 1_{\widetilde{\gamma} \setminus \Delta_V}[1].$$

By Proposition 5.1, if we assume there exists a pointed closed convex cone λ such that $V^* \setminus \mathrm{Int}(\gamma^\circ) = \lambda^{\circ a}$, we have $L_U = P'_\lambda = L_\gamma, R_U = Q'_\lambda = R_\gamma$. Noticed that we can show by either computation of dual functor D' or abstract uniqueness of kernel that $K_U = 1_{\widetilde{\gamma}} \simeq \mathrm{D}'(1_{\widetilde{\lambda}^a \setminus \Delta_V})[n-1]$.

6. Results for pairs

In this section, we consider the following construction: Take two conic closed set $Z \subset X \subset T^*M$. Let $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, then $V \subset U$. Recall that $\mathrm{Sh}_{X\cap U}(M;U) = \frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$.

We consider the following 9-diagram of $\mathbf{k}$-linear categories:

$$\begin{tikzcd}
\mathrm{Sh}_Z(M) \arrow[r, hook] \arrow[d, equal] & \mathrm{Sh}_X(M) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Sh}_{X\cap U}(M;U) \arrow[d, hook] \\
\mathrm{Sh}_Z(M) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Sh}(M) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Sh}(M;U) \arrow[d, two heads] \\
0 \arrow[r, hook] & \mathrm{Sh}(M;V) \arrow[r, "\simeq"] & \mathrm{Sh}(M;U)\big/\mathrm{Sh}_{X\cap U}(M;U)
\end{tikzcd} \tag{6.1}$$

In this diagram, by the same argument as in Proposition 3.1, we have that all vertical and horizontal sequences are split Verdier sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathbf{k})$. We say a morphism $f : F \to G$ in $\mathrm{Sh}(M;U)$ is an isomorphism on an open set $V \subset U$ if f is an isomorphism in $\mathrm{Sh}(M;V)$. Because $\mathrm{Sh}_{X\cap U}(M;U)$ is a thick subcategory of $\mathrm{Sh}(M;U)$, we know that f is an isomorphism on V if and only if $SS_U(\mathrm{cofib}\,(f)) \cap V = \varnothing$.

Here, we write down adjoint functors in the last column precisely

$$\begin{tikzcd}
\mathrm{Sh}_{X\cap U}(M;V) \arrow[r, "\iota"] & \mathrm{Sh}(M;U) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j"] & \mathrm{Sh}(M;V). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd} \tag{6.2}$$

Then we have adjunction pairs

$$L_{(U,V)} \coloneqq j_!j \dashv j_*j =: R_{(U,V)}, \qquad L_{(Z,X)} \coloneqq \iota\iota^* \dashv \iota\iota^! =: R_{(Z,X)},$$

and the units/counits give us following fiber sequences of functors on $\mathrm{Sh}(M;U)$

$$L_{(U,V)} \to \mathrm{id} \to L_{(Z,X)}, \quad R_{(Z,X)} \to \mathrm{id} \to R_{(U,V)}. \tag{6.3}$$

Remark 6.1. By Equation (0.1), we can also represent functors in Equation (6.3) by integral kernels $K_{(Z,X)}, K_{(U,V)} \in \mathrm{Sh}(M \times M; (-U) \times U)$ and they are image of K_X, K_V under the quotient functor $\mathrm{Sh}(M \times M) \to \mathrm{Sh}(M \times M; (-U) \times U)$. The relative version of microsupport estimation Proposition 4.4 is also true.

Therefore, we can state the following version of cut-off lemma

Theorem 6.2. *For conic closed sets $Z \subset X \subset T^*M$, and $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, we have*

(1) (a) The morphisms $F \to L_{(Z,X)}(F)$ and $R_{(Z,X)}(F) \to F$ are isomorphisms if and only if $SS_U(F) \subset X \cap U$.
(b) The morphism $L_{(U,V)}(F) \to F$ is an isomorphism if and only if $F \in^{\perp} \mathrm{Sh}_{X\cap U}(M;U)$, and the morphism $F \to R_{(U,V)}(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_{X\cap U}(M;U)^{\perp}$.
(a) The morphisms $L_{(U,V)}(F) \to F$ and $F \to R_{(U,V)}(F)$ are isomorphisms on $U \cap V$.
(b) If $0_M \subset Z$, the morphism $F \to L_{(Z,X)}(F)$ and $R_{(Z,X)}(F) \to F$ are isomorphisms on $\mathrm{Int}(X) \cap U$. Equivalently, we have $SS_U(L_{(U,V)}(F)) \cup SS_U(R_{(U,V)}(F)) \subset \overline{V} \cap U$. Here $\overline{V}$ denotes the closure of V in T^*M.

Proof. Here, we notice that $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for $[F] \in \mathrm{Sh}(M;U)$. Then we apply the absolute version Theorem 3.5 to conclude. $\square$

Example 6.3. We consider the manifold $M \times \mathbb{R}_t$. We take a closed set $\mathcal{Z} \subset J^1M = T^*M \times \mathbb{R}$ and $\mathcal{U} = J^1M \setminus \mathcal{Z}$. We consider their conification in $T^*(M \times \mathbb{R}_t) = T^*M \times \mathbb{R}_t \times \mathbb{R}_\tau$. We set

$$\begin{aligned} &\Omega_{>0} = \{\tau > 0\}, \qquad \Omega_- = \{\tau \leq 0\}. \\ &U = \{(q, \tau p, t, \tau) : (q, p, t) \in \mathcal{U}, \tau > 0\}, \\ &Z_{>0} = \{(q, \tau p, t, \tau) : (q, p, t) \in \mathcal{Z}, \tau > 0\}, \\ &Z = Z_{>0} \cup \Omega_-. \end{aligned} \tag{6.4}$$

Then $\Omega_- \subset Z$ are two closed conic sets. In this case, the category

$$\mathrm{Sh}(M \times \mathbb{R}_t; U) \simeq \mathrm{Sh}(M \times \mathbb{R}_t; \Omega_{>0})/\mathrm{Sh}_{Z_{>0}}(M \times \mathbb{R}_t; \Omega_{>0})$$

is useful for contact topology of J^1M, since $\mathbb{R}_{>0}$ acts on $\Omega_{>0}$ freely, with the quotient $J^1M = T^*M \times \mathbb{R}_t$.

If we take $\mathcal{U} = U_0 \times \mathbb{R}$ for an open set $U_0 \subset T^*M$ and $\mathcal{Z} = Z_0 \times \mathbb{R}$ for the closed set $Z_0 = T^*M \setminus U_0$, the category $\mathrm{Sh}(M \times \mathbb{R}_t; U)$ is exactly the so-called Tamarkin categories $\mathcal{T}(U_0)$ (cf. [KSZ23]). Tamarkin categories are first defined in [Tam18], and then studied in [Vic13, Chi17, Ike19, AI20a, Zha20, AI22, GV22a, AGH+23, AIL23]. In this case, U admits an $\mathbb{R}$-action by translation along $\mathbb{R}_t$. Therefore, there exists an $\mathcal{T} := \mathcal{T}(\mathrm{pt})$ (which is an symmetric monoidal category [GS14, KSZ23]) action on $\mathcal{T}(U_0)$ that helps us understand the action filtration from symplectic geometry. The functors

$$L_{U_0}^{\mathcal{T}} := L_{(\Omega_{>0}, U)},\ L_{Z_0}^{\mathcal{T}} := L_{(Z, \Omega_-)}$$

are actually $\mathcal{T}$-linear. By the $\mathcal{T}$-linear left adjoint functors classification (see [KSZ23, Proposition 5.12], which is an enriched version of Remark 6.1), there exists a fiber sequences $K_{U_0}^{\mathcal{T}} \to 1_{\Delta_M}^{\mathcal{T}} \to K_{Z_0}^{\mathcal{T}}$ in $\mathcal{T}(T^*(M \times M)) = \mathrm{Sh}(M \times M; \mathcal{T})$ with

$$L_{U_0}^{\mathcal{T}} = \Phi_{K_{U_0}^{\mathcal{T}}}^{\mathcal{T}},\ L_{Z_0}^{\mathcal{T}} = \Phi_{K_{Z_0}^{\mathcal{T}}}^{\mathcal{T}},$$

where $\Phi^{\mathcal{T}}$ means the convolution functor defined using $\mathcal{T}$-linear 6-operators. These functors are studied in [Tam15, Chi17, Zha21, Zha23, KSZ23], where corresponding kernels are denoted by $K_{U_0}^{\mathcal{T}} = P_{U_0}, K_{Z_0}^{\mathcal{T}} = Q_{U_0}$. In particular, we are free to use Theorem 6.2 for Tamarkin categories.

In general, if the open set $\mathcal{U} \subset J^1M$ admits a $\mathbb{G}$-action for a subgroup $\mathbb{G} \subset \mathbb{R}$ via translation, Asano, Ike and Kuwagaki introduce an equivariant version of categories, see [AI20b, IK23]. If $\mathbb{G}$ is a discrete subgroup of $\mathbb{R}$, which means $\mathbb{G} \simeq \mathbb{Z}$, the equivariant version can be understood by the framework of this article by identifying $\mathbb{R}/\mathbb{G} = S^1$ and considering the category $\mathrm{Sh}(M \times S^1; \Omega_+)$ (which still makes sense.) However, if we treat $\mathbb{G} = \mathbb{R}_a$ as a discrete group acts on the topological group $\mathbb{R}_t$, the situation becomes more delicate and we will not discuss related categories and cut-off lemma in this article (while we still believe it is true.)

7. Künneth formula

Here, we present some computations of Lurie tensor products, which generalize the result of [KL24, Theorem 1.2], where an isotropic condition is needed. We take manifolds N, M, conic closed sets $Z \subset T^*M$ and $X \subset T^*N$, and we set $U = T^*M \setminus Z$ and $V = T^*N \setminus X$.

To start with, we recall that $\mathcal{C} \in \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ is dualizable if it is a dualizable object with respect to the symmetric monoidal structure defined by the Lurie tensor product, and its dual is denoted by $\mathcal{C}^\vee$. Efimov proves the following important fact we need to use: for dualizable category $\mathcal{C}$, the functor $\mathcal{C} \otimes -$ is preserve fully-faithfulness [Efi24, Theorem 2.2]. Then we recall that:

Lemma 7.1 ([KSZ23, Remark 3.7])**.** *The category* $\mathrm{Sh}_Z(M)$ *is dualizable with dual* $\mathrm{Sh}_{-Z}(M)$*. The category* $\mathrm{Sh}(M; U)$ *is dualizable with the dual* $\mathrm{Sh}(M; -U)$*.*

Remark 7.2. In the following propositions, the box tensor $\boxtimes$ of kernels should be understood after identifying $(N \times M)^2 \simeq N^2 \times M^2$ via $((n_1, m_1), (n_2, m_2)) \mapsto (n_1, n_2, m_1, m_2)$.

Theorem 7.3. *We have that* $K_X \boxtimes K_Z \simeq K_{X \times Z}$*, and* $\mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \simeq \mathrm{Sh}_{X \times Z}(N \times M)$*.*

Proof. We first assume that both X and Z containing the zero section.

For the first statement. We set $K := K_X \boxtimes K_Z$. It is clear that $\Phi_K \circ \Phi_K \simeq \Phi_K$, so we only need to prove that $F \in \mathrm{Sh}_{X\times Z}(N \times M)$ if and only if $\Phi_K(F) \simeq F$.

For $F \in \mathrm{Sh}(N \times M)$ with $\Phi_K(F) \simeq F$, we can write $F = \varinjlim A \boxtimes B$, and then we have $F \simeq \varinjlim \Phi_{K_X}(A) \boxtimes \Phi_{K_Z}(B)$. So $SS(F) = SS(\varinjlim \Phi_{K_X}(A) \boxtimes \Phi_{K_Z}(B)) \subset \overline{SS(\Phi_{K_X}(A)) \times SS(\Phi_{K_Z}(B))} \subset X \times Z$.

Conversely, we need to show that K fixes $F \in \mathrm{Sh}_{X\times Z}(N \times M)$. We use the wrapping formula Theorem 4.2, and adapt the argument of [AGH+23, Lemma 4]. For cofinal sequences of conic non-negative Hamiltonian functions $H_{\lambda_1(n_1)}$ supported in U and $H_{\lambda_2(n_2)}$ supported in V with index sequences $\lambda_i : \mathbb{N} \to \mathbb{N}$. We have that $K \simeq \varinjlim_{(n_1,n_2)} K(\phi_{H_{\lambda_1(n_1)}})|_1 \boxtimes K(\phi_{H_{\lambda_2(n_2)}})|_1$. Now, for the 1-parameter version of GKS quantization $K(\phi_{H_{\lambda_1(n_1)}})$ and $K(\phi_{H_{\lambda_2}(n_2)})$ as Theorem 4.1 given. By a parameter version of(4.5), it is directly to see that for $W = \Phi_{K(\phi_{H_{\lambda_1(n_1)}}) \boxtimes K(\phi_{H_{\lambda_2(n_2)}})}(F) \in \mathrm{Sh}(N \times \mathbb{R} \times M \times \mathbb{R})$, we have

$$SS(W) \subset \left\{ (q_1, z_1, p_1, \zeta_1, q_2, z_2, p_2, \zeta_2) : \begin{array}{l} \exists (q_1', p_1', q_2', p_2') \in SS(F), \\ (q_i, p_i) = \phi_{z_i}^{H_{\lambda_i(n_i)}}(q_i', p_i'), \zeta_i = -H_{\lambda_i(n_i)}(q_i', p_i'). \end{array} \right\}.$$

Now, since $SS(F) \subset X \times Z$, we have that $SS(W) \subset T^*(N \times M) \times 0_{\mathbb{R}^2}$. Then by [KS90, Proposition 5.4.5], we have $F = W|_{z_1=z_2=0} \simeq W|_{z_1=z_2=1} = \Phi_{K(\phi_{H_{\lambda_1(n_1)}})|_1 \boxtimes K(\phi_{H_{\lambda_2(n_2)}})|_1}(F)$. We can further verify that the isomorphism is commute with continuation map given by non-negativity of $H_{\lambda_i(n_i)}$ (see Theorem 4.2). Then we taking limit with respect to (n_1, n_2) to see that $\Phi_K(F) \simeq F$.

Consequently, we have $K_{X\times Z} \simeq K = K_X \boxtimes K_Z$ by uniqueness of kernel.

For the second statement, under the natural identification $\mathrm{Sh}(N) \otimes \mathrm{Sh}(M) = \mathrm{Sh}(N \times M)$ (see [Vol21, Proposition 2.30]), it remains to verify that the essential image of the functor

$$\mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \to \mathrm{Sh}(N) \otimes \mathrm{Sh}(M) = \mathrm{Sh}(N \times M), \quad F \otimes G \mapsto F \boxtimes G$$

is $\mathrm{Sh}_{X\times Z}(N \times M)$. In fact, any object F in $\mathrm{Sh}_{X\times Z}(N \times M)$ can be written as a colimit of some $F = \varinjlim F_\alpha \boxtimes G_\alpha \in \mathrm{Sh}(N \times M)$. We should show that the F_α can be chosen in $\mathrm{Sh}_X(N)$ and G_α can be chosen to be in $\mathrm{Sh}_Z(M)$. By definition of $K_{X\times Z}$, we have $F \simeq \Phi_{K_{X\times Z}}(F)$. As we already prove that $K_{X\times Z} \simeq K_X \boxtimes K_Z$, we have

$$F \simeq \Phi_{K_{X\times Z}}(\varinjlim F_\alpha \boxtimes G_\alpha) \simeq \varinjlim \Phi_{K_X \boxtimes K_Z}(F_\alpha \boxtimes G_\alpha) = \varinjlim \Phi_{K_X}(F_\alpha) \boxtimes \Phi_{K_Z}(G_\alpha).$$

Now, we remove the assumption that X, Z contain the zero section. In this case, we set $Z_0 = Z \cap 0_M$ and $\tilde{Z} = Z \cup 0_M$ (similarly for X_0 and $\tilde{X}$). Then we use the result for $\tilde{X}$ and $\tilde{Z}$, which contain zero sections, from above; and conclude by noticing that $SS(F) \subset Z$ if and only if $SS(F) \subset \tilde{Z}$ and $\mathrm{supp} F \subset Z_0$, and $F_n \otimes G_m \simeq (F \boxtimes G)_{(n,m)}$ for $(n, m) \in N \times M$. □

Corollary 7.4. *We have an equivalence of fiber sequences of categories*

$$\begin{aligned} &\mathrm{Sh}_{X\times Z}(N \times M) \to \mathrm{Sh}_{X\times T^*M}(N \times M) \to \mathrm{Sh}_{X\times U}(N \times M; T^*N \times U) \\ \simeq &\mathrm{Sh}_X(N) \otimes (\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)). \end{aligned}$$

Proof. The functor $- \mapsto \mathrm{Sh}_X(N) \otimes -$ preserves colimits and fully-faithfulness, so we have the equivalence of Verdier sequences

$$\begin{aligned} &\mathrm{Sh}_X(N) \otimes (\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)) \\ \simeq &\mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \to \mathrm{Sh}_X(N) \otimes \mathrm{Sh}(M) \to \mathrm{Sh}_X(N) \otimes \mathrm{Sh}(M; U). \end{aligned}$$ □

Theorem 7.5. *We have an equivalence*

$$\mathrm{Sh}(N; V) \otimes \mathrm{Sh}(M; U) \simeq \mathrm{Sh}(N \times M; V \times U).$$

In particular, we have that $K_V \boxtimes K_U \simeq K_{V\times U}$.

Proof. We fulfill the 9-diagram in Equation (6.1) by setting the manifold by $N \times M$, two conic closed sets are $T^*N \times Z \subset T^*N \times Z \cup X \times T^*M$. Therefore, we have the following equivalence

$$\mathrm{Sh}(N \times M; V \times U) \simeq \mathrm{Sh}(N \times M; T^*M \times U)/\mathrm{Sh}_{X\times U}(N \times M; T^*M \times U).$$

Now, we apply Corollary 7.4 twice to see that

$$[\mathrm{Sh}_{X\times U}(N\times M; T^*N\times U)\hookrightarrow \mathrm{Sh}(N\times M; T^*N\times U)] = [\mathrm{Sh}_X(N)\hookrightarrow \mathrm{Sh}(N)]\otimes \mathrm{Sh}(M;U).$$

Consequently, we have

$$\mathrm{Sh}(N\times M; T^*N\times U)/\mathrm{Sh}_{X\times U}(N\times M; T^*N\times U)\simeq \mathrm{Sh}(N;V)\otimes \mathrm{Sh}(M;U). \qquad \square$$

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