NON-LINEAR MICROLOCAL CUT-OFF FUNCTORS

Abstract. We define the non-linear microlocal cut-off functors. Then we will explain its relation with microlocal cut-off functor defined by Kashiwara and Schapira, and prove a microlocal cut-off lemma for non-linear microlocal cut-off functors adapting inputs from symplectic geometry.

0. Introduction

In the study of microlocal theory of sheaves, a crucial tool is the microlocal cut-off lemma of Kashiwara and Schapira (see [KS90, Proposition 5.2.3, Lemma 6.1.5], [Gui23, Chapter III] and [D'A96]), which constructs certain functors that enable us to cut-off the microsupport of sheaves functorially. On the other hand, in the study of symplectic topology, very similar functors was defined and studied for different purposes [Tam18, NS22, Kuo23].

Precisely, for a smooth manifold M, a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we will define the *non-linear microlocal cut-off functors* $L_U, L_Z, R_U, R_Z : \mathrm{Sh}(M) \to \mathrm{Sh}(M)$ together with natural transformations $L_U \to \mathrm{id} \to L_Z$, $R_Z \to \mathrm{id} \to R_U$ using adjoint data of the split Verdier sequence $\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)$. See Section 4 for more details.

The main goal of this article is first explaining precise relations of those functors with known microlocal cut-off functors of Kashiwara-Schapira, and then prove a non-linear microlocal cut-off lemma using the wrapping formula of L_Z, R_Z introduced in [Kuo23]. Hence fore, most of results in this article are known or obvious to experts, but we did not find an affable reference.

Theorem (Non-linear microlocal cut-off lemma, Theorem 6.1 below.)**.** *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have*

(1) (a) $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms if and only if $SS(F) \subset Z$.
(b) $L_U(F) \to F$ is an isomorphism if and only if $F \in {}^{\perp}\mathrm{Sh}_Z(M)$, and $F \to R_U(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_Z(M)^{\perp}$.

(2) (a) The morphisms $L_U(F) \to F$ and $F \to R_U(F)$ are isomorphisms on U.
(b) If $0_M \subset Z$, the morphism $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms on $\mathrm{Int}(Z) \setminus 0_M$. Equivalently, we have $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$.

In fact, all of them except (2)-(b) follow directly from definition. Only (2)-(b) of the theorem is a non-trivial fact.

To explain why the theorem generalizes microlocal cut-off lemma in [KS90], we share explain the relation between non-linear microlocal cut-off functors with microlocal cut-off functors in [KS90]. Here, we follow notation of [Gui23, Section III.1] for microlocal cut-off functors.

We take $M = V$ to be a real vector space, $\lambda \subset V$ and $\gamma \subset V^*$ are pointed closed convex cones (we assume further that λ is proper). The comparison is summarized in the following table, where the first column marks corresponding references, the last column marks their position in this article. Functors in the same columns are isomorphic with corresponding U and Z.

This article	Z	U	L_U	L_Z	R_U	R_Z	Definition 4.4
[Gui23, Section III.1]	$V \times \lambda^{\circ a}$		P'_λ	Q_λ	Q'_λ	P_λ	Proposition 5.1
[GS14, Section 3]		$V \times \mathrm{Int}(\gamma^\circ)$	L_γ		R_γ		Example 5.3

Table 1. Known microlocal cut-off functors.

Based on this comparison, we see that the non-linear microlocal cut-off lemma recover [KS90, Proposition 5.2.3, Lemma 6.1.5], and [GS14, Proposition 3.17, 3.19, 3.20, 3.21.].

Date: May 29, 2024.

We also prove two Künneth formula in Section 8 (with more precise statements), which remove the isotropic condition of [KL24, Theorem 1.2] via a different approach.

Theorem A. *For two manifolds M, N, conic closed sets $Z \subset T^*M$ and $X \subset T^*N$ containing the zero sections, and we set $U = T^*M \setminus Z$ and $V = T^*N \setminus X$. Then we have*

$$\mathrm{Sh}_{X\times Z}(N \times M) \simeq \mathrm{Sh}_{XZ}(N) \otimes \mathrm{Sh}_Z(M), \quad \mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N; V) \simeq \mathrm{Sh}(M \times N; U \times V).$$

Combine the dualizibility of $\mathrm{Sh}_Z(M)$ and $\mathrm{Sh}(M; U)$ via [KSZ23, Remark 3.7], we have the following functor classification result:

$$\begin{aligned} &\mathrm{Sh}_{-Z\times X}(M \times N) \simeq \mathrm{Fun}^L(\mathrm{Sh}_Z(M), \mathrm{Sh}_X(N)) \simeq \mathrm{Fun}^R(\mathrm{Sh}_X(N), \mathrm{Sh}_Z(M))^{op}, \\ &\mathrm{Sh}(M \times N; -U \times V) \simeq \mathrm{Fun}^L(\mathrm{Sh}(M; U), \mathrm{Sh}(N; V)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(N; V), \mathrm{Sh}(M; U))^{op}. \end{aligned} \tag{0.1}$$

In the last section, we will discuss a relative case of the cut-off functors, and apply our results to Tamarkin categories.

Category convention. In this article, a category means an ∞-category and we will emphasize 1-category we need.

We refer to [GR19, Chapter 1] for basics about higher algebra, and more details could be find in [Lur09, Lur17, Lur18].

Let $\mathrm{Cat}^{\mathrm{Ex}}$ be the category of stable categories with exact functors between them. We denote Fun^{ex} the stable category of exact functors between stable categories.

We denote $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the category of presentable stable category and continuous functors. Lurie defines a closed symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, which is called the Lurie tensor product $\otimes$, and the internal hom is given by continuous functors Fun^L. For a symmetric monoidal presentable stable category $\mathcal{C}$ (i.e. $\mathcal{C}$ is a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$), we denote the category of $\mathcal{C}$-module in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathcal{C})$. Objects in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathcal{C})$ are called $\mathcal{C}$-linear categories, and morphisms in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathcal{C})$ are $\mathcal{C}$-module morphisms, i.e. $\mathcal{C}$-linear continuous functors (which automatically admit lax $\mathcal{C}$-linear right adjoints) such that its right adjoints are strict $\mathcal{C}$-linear. Lurie tensor product induces a relative tensor $\otimes_{\mathcal{C}}$ on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathcal{C})$, which is also closed with the internal hom $\mathrm{Fun}^L_{\mathcal{C}}$ of $\mathcal{C}$-module morphisms.

In this article, we will fix a compactly generated rigid symmetric monoidal stable category $(\mathbf{k}, \otimes, 1)$. In this case, rigidity means that the monoidal structure of $\mathbf{k}$ restrict to a symmetric monoidal structure on the category of compact objects $\mathbf{k}^c$ (in particular, the tensor unit 1 is compact) and all compact objects are dualizable. For the $\mathbf{k}$-relative Lurie tensor product, we will write $\otimes/\mathrm{Fun}^L$ directly instead of $\otimes_{\mathbf{k}}/\mathrm{Fun}^L_{\mathbf{k}}$. For example, for an E_∞ ring-spectrum $\mathbb{K}$, we can take $\mathbf{k} = \mathrm{Mod}_{\mathbb{K}}$, which is the category of $\mathbb{K}$-module. In particular, for the spherical spectrum $\mathbb{S}$, we have $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathrm{Mod}_{\mathbb{S}})$. When R is a discrete commutative ring, we abuse the notation $\mathrm{Mod}_R = \mathrm{Mod}_{\mathrm{H}R}$ for the the Eilenberg–MacLane spectrum $\mathrm{H}R$ (c.f. [Lur17, 7.1.2.13]).

Acknowledgements. This work is inspired by a discussion with Stéphane Guillermou and Vivek Shende's interests on microlocal cut-off functors. The author thanks them for their helpful discussion and encouragement. We also thank for Xin Jin, Christopher Kuo, Wenyuan Li and Matthias Scharitzer for helpful discussion. This work was supported by the Novo Nordisk Foundation grant NNF20OC0066298.

1. Sheaves and kernel calculus

For a $\mathbf{k}$-linear category $\mathcal{C}$ and a topological space X, we denote the category of $\mathcal{C}$-value sheaves to be $\mathrm{Sh}(X; \mathcal{C})$ [Lur09, 7.3.3.1], which is also $\mathbf{k}$-linear. It is explain by [Vol21] that, when $\mathcal{C}$ is a $\mathbf{k}$-linear symmetric monoidal category (i.e. a commutative algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathbf{k})$), we have the $\mathcal{C}$-linear 6-functor formalism, and we also refer to[Sch22] for the 6-functors formalism. In particular, we denote $\mathrm{Sh}(X; \mathbf{k})$ as $\mathrm{Sh}(X)$.

For a locally closed inclusion $i : Z \subset X$ and $F \in \mathrm{Sh}(X)$ we set

$$F_Z = i_! i^{-1} F, \quad \Gamma_Z F = i_* i^! F.$$

We denote the constant sheaf $1_X = a^*1$ where a is the constant map $X \to \mathrm{pt}$ and denote $1_Z = (1_X)_Z \in \mathrm{Sh}(X)$ if there is no confusion.

Remark 1.1. It is explained in [Sch22, Proposition 7.1] that, if X is paracompact and has a finite covering dimension, we have that $\mathrm{Sh}(X; \mathrm{Mod}_R) \simeq D(\mathrm{Sh}^\heartsuit(R_X))$ where $\mathrm{Sh}^\heartsuit(R_X)$ denote the abelian category of sheaves of R-modules (c.f. [KS90]) and D stands for the ∞-derived category. By similar arguments, we can verify that the ∞-categorical 6-functors, passing to the homotopy category, coincide with unbounded derived functors defined by Spaltenstein [Spa88].

The main reason for us to use ∞-category is the following classification of left adjoint functors, which is a combiation of a sequence of Lurie's results. We refer to [KSZ23, Proposition 3.1] for a proof.

Proposition 1.2. *If H_1 is a locally compact Hausdorff space, then for all topological spaces H_2, we have the equivalence of categories*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2))$$

that sends $K \in \mathrm{Sh}(H_1 \times H_2)$ to the convolution functor $\Phi_K = [F \mapsto p_{2!}(K \otimes p_1^* F)]$, where $p_i : H_1 \times H_2 \to H_i$ are projections. We call K the kernel of Φ_K

For locally compact Hausdorff spaces H_1, H_2, H_3, and two convolution functors with kernel $F \in Sh(H_1 \times H_2)$, $G \in Sh(H_2 \times H_3)$, we have $\Phi_G \circ \Phi_F \simeq \Phi_{F \circ G}$ for

$$F \circ G \coloneqq p_{13!}(p_{12}^* F \otimes p_{23}^* G) \in Sh(H_1 \times H_3),$$

where $p_{ij} : H_1 \times H_2 \times H_3 \to H_i \times H_j$ are projections.

It is known that, taking right adjoint induces an equivalence of categories

$$\mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2), \mathrm{Sh}(H_1))^{op},$$

where Fun^R stands for right adjoint functors.

However, for any convolution functor $\Phi_K : \mathrm{Sh}(H_1) \to \mathrm{Sh}(H_2)$ with $K \in \mathrm{Sh}(H_1 \times H_2)$, there exists an obvious right adjoint functor[1]

$$\Psi_K : \mathrm{Sh}(H_2) \to \mathrm{Sh}(H_1),\ F \mapsto p_{1*}\mathcal{H}om(K, p_2^! F),$$

such that $\Psi_F \circ \Psi_G \simeq \Psi_{F \circ G}$ for $F \in Sh(H_1 \times H_2)$, $G \in Sh(H_2 \times H_3)$.

Then we have the following equivalence of categories

Corollary 1.3. *Under the same condition of Proposition 1.2, we have*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2), \mathrm{Sh}(H_1))^{op},\ K \mapsto \Psi_K.$$

Remark 1.4. Proposition 1.2 and Corollary 1.3 are only true on ∞-category level. In classical theory of microlocal sheaves, many functors are built as triangulated convolution(nvolution) functors, which do not admit similar results, especially, we do not know if the kernel is unique in general.

2. Microsupport of sheaves

From now, we assume M is a smooth manifold. Regarding on microlocal theory of sheaves in the ∞-category setup, we remark that all arguments of [KS90] work well upon we have the non-characteristic deformation lemma [KS90, Proposition 2.7.2] for all sheaves. When $\mathbf{k}$ is compactly generated, the non-characteristic deformation lemma is proven for all hypersheaves ([RS18]), then for all sheaves because of hypercompleness for manifolds ([Lur09, 7.2.3.6, 7.2.1.12]).

Definition 2.1 ([KS90, Definition 5.1.2])**.** For $F \in \mathrm{Sh}(M)$ the microsupport of F is

$$SS(F) = \overline{\left\{ (\mathbf{q}, \mathbf{p}) \in T^*M : \begin{array}{l} \text{There is a } C^1\text{-function } f \text{ near } \mathbf{q} \text{ such that} \\ f(\mathbf{q}) = 0,\ df(\mathbf{q}) = \mathbf{p} \text{ and } \left(\Gamma_{\{f \geq 0\}} F\right)_{\mathbf{q}} \neq 0. \end{array} \right\}}.$$

[1] We call it a nvolution functor since it is a dual of a convolution functor, while coconvolution should be the same with nvolution.

By definition, $SS(F)$ is a conic closed subset of T^*M. There is a triangulated inequality for the microsupport: for a distinguished triangle $A \to B \to C \xrightarrow{+1}$, we have $SS(A) \subset SS(B) \cup SS(C)$.

We remark a further closure property of microsupport

Proposition 2.2 ([KS90, Exercise V.7]). *For a small collection of sheaves $F_\alpha \in \mathrm{Sh}(M)$ indexed by $\alpha \in A$, we have*

$$SS(\prod_\alpha F_\alpha) \cup SS(\bigoplus_\alpha F_\alpha) \subset \overline{\bigcup_\alpha SS(F_\alpha)}.$$

A proof can be find in [GV22b, Proposition 3.4.]. Importantly, the proof therein only use the non-characteristic deformation lemma and a geometric argument. In particular, it does not involve the microlocal cut-off lemma in [KS90, Section 5.2].

For $X_n \subset X$ with $n \in \mathbb{N}$, we set

$$\begin{aligned}\limsup_n X_n &= \bigcap_{N\geq 1} \overline{\bigcup_{n\geq N} X_n}\\ &= \{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for infinitely many } n,\, x_n \to x\},\\ \liminf_n X_n &= \{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for all } n,\, x_n \to x\}.\end{aligned}$$

By definition, we have $\liminf_n X_n \subset \limsup_n X_n$. As a corollary of Proposition 2.2, we have

Corollary 2.3 ([GV22b, Proposition 6.26]). *For an functor $N(\mathbb{N}) \to \mathrm{Sh}(X)$, we have*

$$SS(\varinjlim_n F_n) \subset \liminf_n SS(F_n).$$

Remark 2.4. We can also denote $\mathbb{N}$ as a simplicial set which consists of all vertices together with the edges which join consecutive integers, then the natural inclusion of simplicial sets $\mathbb{N} \to N(\mathbb{N})$ is cofinal. Therefore, we can compute the colimit using mapping telescope as in triangulated categories, and the proof of Corollary 2.3 is the same as references.

For our later application, we present its proof here.

Proof. By Remark 2.4, and the fact that all for all strictly increasing sequences are cofinal in $\mathbb{N}$, we have cofiber sequences, for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$,

$$\bigoplus_i F_{n_i} \to \bigoplus_i F_{n_i} \to \varinjlim_n F_n.$$

We first take $n_i = i + N$ for all $N \geq 1$, and then we have $SS(\varinjlim_n F_n) \subset \limsup_n SS(F_n)$ by Proposition 2.2 and the triangulated inequality.

Now, take $x \notin \liminf_n SS(F_n)$. By definition of $\liminf_n$, we can find a strictly increasing sequence n_i such that $x \notin \limsup_i SS(F_{n_i})$, then we have $x \notin SS(\varinjlim_i F_{n_i}) = SS(\varinjlim_n F_n)$. That is $SS(\varinjlim_n F_n) \subset \liminf_n SS(F_n)$. $\square$

We need the following microsupport estimation of (co)nvolution functors:

Proposition 2.5 ([KL24, Lemma 4.4, Proposition 4.5]). *Let $K \in \mathrm{Sh}(M \times M)$ with $SS(K) \subset (-A) \times B$ for conic closed sets $A, B \subset T^*M$, then we have, for $F \in \mathrm{Sh}(M)$,*

$$SS(\Phi_K(F)) \subset B, \quad SS(\Psi_K(F)) \subset A.$$

Remark 2.6. In [KL24, Proposition 4.5.], only the first statement is given. The second statement can be proven from [KL24, Lemma 4.4] based on the same consideration.

3. Verdier sequences

Now, we collect some facts about Verdier quotient for stable ∞-categories, whose triangulated version are [KS06, Exercise 10.15]. All results here are collected from [BGT13, Section 5], [NS18, Section 1.3] and [CDH+, Appendix A].

Definition 3.1. Let $\iota : \mathcal{C} \to \mathcal{D}$ be a fully faithful stable subcategory of $\mathcal{D}$.

(1) We say $\mathcal{C}$ is thick if $\mathcal{C}$ is stable under retraction.
(2) Let $W = \{\text{Morphisms in } \mathcal{D} \text{ whose cofiber is in } \mathcal{C}\}$. We define the Verdier quotient $j : \mathcal{D} \to \mathcal{D}/\mathcal{C} := \mathcal{C}[W^{-1}]$ as the Dwyer-Kan localization.

Proposition 3.2. *Let $\iota : \mathcal{C} \to \mathcal{D}$ be a thick stable subcategory of $\mathcal{D}$, then we have*

(1) $\mathcal{D}/\mathcal{C}$ is a cofiber of ι in $\mathrm{Cat}^{\mathrm{Ex}}$. For all $\mathcal{E} \in \mathrm{Cat}^{\mathrm{Ex}}$, we have $\mathrm{Fun}^{ex}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \xrightarrow{j^*} \mathrm{Fun}^{ex}(\mathcal{D}, \mathcal{E})$ is fully faithful whose essential image consisting of those functors which send all objects of $\mathcal{C}$ to 0. In particular, j is essential surjective.
(2) The fiber of j is $\mathcal{C}$.
(3) For $x, y \in \mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(j(x), j(y)) \simeq \varinjlim_{f:\mathcal{C}_{/y}} \mathrm{Hom}_{\mathcal{D}}(x, \mathrm{cofib}\,(f)) \simeq \varinjlim_{g:_{x\backslash}\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(\mathrm{fib}\,(g), y),$$

where the colimit is filtered. I.e., we can compute Hom *of the Verdier quotient using "roof".*
(4) If further $\iota : \mathcal{C} \to \mathcal{D}$ is in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathbf{k})$, then j is in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathbf{k})$.

In this case, we call the sequence $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$ a *Verdier sequence*. We have the equivalence of triangulated categories $h(\mathcal{D}/\mathcal{C}) \simeq h\mathcal{D}/h\mathcal{C}$, where $h\mathcal{D}/h\mathcal{C}$ is the triangulated Verdier quotient.

Proposition 3.3. *For a Verdier sequence $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$, we have that ι admits left(right) adjoint if and only if j admits left(right) adjoints, the left(right) adjoint (if it exists) of j is fully-faithful, and the essential image of the left(right) adjoint is identified with the left(right) orthogonal complement ${}^{\perp}\mathcal{C}$ ($\mathcal{C}^{\perp}$):*

$$^{\perp}\mathcal{C} := \{x \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}(x, y) \text{ is contractible for all } y \in \mathcal{C}\},$$
$$\mathcal{C}^{\perp} := \{x \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}(y, x) \text{ is contractible for all } y \in \mathcal{C}\}.$$

Moreover, the left(right) adjoint of j induces an equivalence $\mathcal{D}/\mathcal{C} \xrightarrow{\simeq} {}^{\perp}\mathcal{C}(\mathcal{C}^{\perp})$.

In this case, we say the Verdier sequence is *left(right) split*, and say it is split if both left and right split. It is also proven in [CDH+, Proposition A.2.11] that $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$ is a split Verdier sequence is equivalent to say that $(\mathcal{C}, \mathcal{D}/\mathcal{C})$ is a recollement of $\mathcal{D}$ in the sense of [Lur17, Definition A.8.1].

4. Split Verdier sequence from microlocalization

For a conic closed set $Z \subset T^*M$, we set $\mathrm{Sh}_Z(M)$ as the full subcategory of $\mathrm{Sh}(M)$ spaned by sheaves F with $SS(F) \subset Z$.

Proposition 4.1. *The category $\mathrm{Sh}_Z(M)$ is a stable subcategory of $\mathrm{Sh}(M)$ closed under small limits and colimits.*

In particular: 1) The inclusion $\iota : \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ admits both left and right adjoints. 2) $\mathrm{Sh}_Z(M)$ is $\mathbf{k}$-linear.

Proof. By the triangulated inequality of microsupport, we only need to know $\mathrm{Sh}_Z(M)$ is closed under small products and coproducts, which follows from Proposition 2.2.

The inclusion ι admits both adjoints by the adjoint functor theorem [Lur09, 5.5.2.9], and $\mathrm{Sh}_Z(M)$ is presentable and $\mathbf{k}$-linear by [RS22] since ι is reflextive and $\mathbf{k}$ is rigid. □

We set $U = T^*M \setminus Z$. Recall the following definition originally given in [KS90, Definition 6.1.1]

Definition 4.2. For the conic open set $U \subset T^*M$, we set

$$\mathrm{Sh}(M;U) \coloneqq \mathrm{Sh}(M)/\mathrm{Sh}_Z(M).$$

Remark 4.3. In [Gui12, NS22], out of necessity to study microsheaves, the category $\mathrm{Sh}(M;U)$ is also denoted by $\mu sh^{pre}(U)$.

By Proposition 4.1 and Proposition 3.3, we have that

$$\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M;U)$$

is a split Verdier sequence. Precisely, we write down the diagram of functors:

$$\begin{tikzcd} \mathrm{Sh}_Z(M) \arrow[r, "\iota"] & \mathrm{Sh}(M) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j"] & \mathrm{Sh}(M;U). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{4.1}$$

Then we have adjunction pairs

$$L_U \coloneqq j_! j \dashv j_* j \eqqcolon R_U, \qquad L_Z \coloneqq \iota\iota^* \dashv \iota\iota^! \eqqcolon R_Z,$$

and the unit/counits give us following fiber sequences of functors on $\mathrm{Sh}(M)$

$$\begin{aligned} L_U &\to \mathrm{id} \to L_Z, \\ R_Z &\to \mathrm{id} \to R_U. \end{aligned} \tag{4.2}$$

By Proposition 1.2 and Corollary 1.3, there exists a fiber sequence in $\mathrm{Sh}(M \times M)$

$$K_U \to 1_{\Delta_M} \to K_Z \tag{4.3}$$

that gives corresponding convolution/nvolution functors

$$L_U = \Phi_{K_U},\ L_Z = \Phi_{K_Z},\ R_U = \Psi_{K_U},\ R_Z = \Psi_{K_Z} \tag{4.4}$$

and corresponding natural transformations. Therefore, in parctise, to construct those functors and fiber sequences between them, we only need to write down the fiber sequence (4.3).

Definition 4.4. We say the functors L_U, L_Z, R_U, R_Z *non-linear microlocal cut-off functors.* Corresponding kernels K_U, K_Z are called microlocal cut-off kernels (or microlocal kernel for short).

Remark 4.5. Using counits/units in (4.2) ι is fully-faithful. We can check L_U, L_Z, R_U, R_Z are (co)projectors in the sense of [KS06, Section 4.1].

For any object in $\mathrm{Sh}(M;U)$, triangulated inequality implies that $SS_U([F]) \coloneqq SS(F) \cap U$ for any representative $F \in \mathrm{Sh}(M)$ is well defined. In particular, we have $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for all $F \in \mathrm{Sh}(M)$. For conic closed sets $X \supset Z$, we consider the category $\frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$ as a full subcategory of $\mathrm{Sh}(M;U)$. Then the essential image of $\frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$ in $\mathrm{Sh}(M;U)$ is $\mathrm{Sh}_{X\cap U}(M;U)$, the full subcategory spanned by objects with $SS_U(F) \subset X \cap U$.

Example 4.6. Take an open set $W \subset M$, and set $U = T^*W \subset T^*M$ with $Z = T^*M \backslash T^*W$. We naturally identifies $\mathrm{Sh}_Z(M)$ with sheaves supported in $M \setminus W$ since $\pi_{T^*M}(SS(F)) = \mathrm{supp} F$, and we can verify that the restriction functor $(\bullet)|_W : \mathrm{Sh}(M) \to \mathrm{Sh}(W)$ exhibits $\mathrm{Sh}(W)$ as the Verdier quotient $\mathrm{Sh}(M;T^*W)$. Then we have

$$L_U(F) = F_W,\ L_Z(F) = F_{M\backslash W},\ R_U(F) = \Gamma_W(F),\ R_Z(F) = \Gamma_{M\backslash W}(F),$$

and can take microlocal kernels as following:

$$K_U = 1_{\Delta_W} \to 1_{\Delta_M} \to K_Z = 1_{\Delta_{M\backslash W}}.$$

Therefore, the fiber sequences (4.2) are exactly the excision sequence for sheaves. This example also motivates the notation of functors in Equation (4.1).

5. Kashiwara-Schapira microlocal cut-off functors

In this section, we study the microlocal cut-off functors defined by Kashiwara and Schapira. We will follow notation and formulation of [Gui23, Chapter III].

We assume $M = V$ is a real vector space of dimension n and we naturally identify $T^*V = V \times V^*$. A subset $\gamma \subset V$ is a cone if $\mathbb{R}_{>0}\gamma \subset \gamma$, we say γ is pointed if $0 \in \gamma$. We set $\gamma^a = -\gamma$. We say a cone γ is convex/closed if it is a convex/closed set, and is proper if $\gamma \cap \gamma^a = \{0\}$ (equivalently, γ contains no line). We define the dual cone γ° as

$$\gamma^\circ = \{l \in V^* : l(v) \geq 0, \forall v \in \gamma\}.$$

We also set

$$\widetilde{\gamma} = \{(x, y) \in V^2 : x - y \in \gamma\}.$$

For a pointed closed cone $\gamma \subset V$, we define 4 functors

$$\begin{aligned} P_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2*}(1_{\widetilde{\gamma}} \otimes q_1^* F), \\ Q_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2!}(\mathcal{H}om(1_{\widetilde{\gamma}^a}, q_1^! F)), \\ P'_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2!}(\mathcal{H}om(1_{\widetilde{\gamma}^a \setminus \Delta_V}[1], q_1^! F)), \\ Q'_\gamma &: \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2*}(1_{\widetilde{\gamma} \setminus \Delta_V}[1] \otimes q_1^* F). \end{aligned} \tag{5.1}$$

For $\gamma = \{0\}$, we have $\widetilde{\{0\}} = \Delta_V$ and $P_{\{0\}}(F) \simeq Q_{\{0\}}(F) \simeq F$. Using the fiber sequence $1_{\widetilde{\gamma}} \to 1_{\Delta_V} \to 1_{\widetilde{\gamma} \setminus \Delta_V}[1]$ (and the same with γ^a), we obtain the fiber sequence of functors

$$\begin{aligned} P'_\gamma \to \mathrm{id} \to Q_\gamma, \\ P_\gamma \to \mathrm{id} \to Q'_\gamma. \end{aligned} \tag{5.2}$$

Proposition 5.1. *Let γ be a pointed closed convex cone. For $Z = V \times \gamma^{\circ a}$ and $U = T^*M \setminus Z$, we have the isomorphism of functors*

$$L_U = P'_\gamma, L_Z = Q_\gamma, R_Z = P_\gamma, R_U = Q'_\gamma,$$

and corresponding natutal transformations.

Remark 5.2. In the originally definition of [KS90, Gui23], functors $P_\gamma, Q_\gamma, P'_\gamma, Q'_\gamma$ are defined as triangulated functors. Here, we use the same formula to define those functors on ∞-level, so those functors automatically descent to the same triangulated functors in *loc.cit* by Remark 1.1.

Proof. It is explained in [Gui23, Remark III.1.9] that, for the inclusion $\iota : \mathrm{Sh}_{V \times \gamma^{\circ a}}(V) \to \mathrm{Sh}(V)$, we have that $Q_\gamma = \iota\iota^*$ and $P_\gamma = \iota\iota^!$; and the natural transform induced by inclusions $1_{\Delta_V} \to 1_{\widetilde{\gamma} \setminus \Delta_V}[1]$ and $1_{\widetilde{\gamma}^a} \to 1_{\Delta_V}$ are unit/counit of corresponding adjunctions. Then the result follows from the fiber sequence of functors (4.2), (5.2). ☐

By the virture of Section 4, exitence of cut-off functor appears as a purely categorical result. However, the main advantage of Equation (5.1) is that the kernel is explicitly written. One may observe that the kernels in Equation (5.1) are not written as Proposition 1.2 in a standard way. This is not a problem, it is explained in [Gui23, III.1.5] how to write down the kernel of P'_γ and Q_γ in the standard way, such that the fiber sequence $K_U \to 1_{\Delta_V} \to K_Z$ is given by the fiber sequence

$$K_U = \mathrm{D}'(1_{\widetilde{\gamma}^a \setminus \Delta_V})[n-1] \to 1_{\Delta_V} \to K_Z = \mathrm{D}'(1_{\widetilde{\gamma}^a})[n],$$

where $\mathrm{D}'(F) = \mathcal{H}om(F, 1_{V^2})$.

Example 5.3. We consider a variant situation. Let $M = V$ be a real vector spaces. Take a pointed closed convex proper cone $\gamma \subset V^*$, set $U = V \times \mathrm{Int}(\gamma^\circ)$ and $Z = V \times V^* \setminus \mathrm{Int}(\gamma^\circ)$. The microlocal cut-off functor for this U is studid by Tamarkin [Tam18] and Guillermou-Schapira [GS14] (where $L_U(R_U)$ is called $L_\gamma(R_\gamma)$ respectively.) In particular, when $V = \mathbb{R}$ and $\gamma = [0, \infty)$, corresponding functors are called Tamarkin projectors.

The fiber sequence of kernels could be given by

$$K_U = 1_{\widetilde{\gamma}} \to 1_{\Delta_V} \to K_V = 1_{\widetilde{\gamma} \setminus \Delta_V}[1].$$

By Proposition 5.1, if we assume there exists a pointed closed convex cone λ such that $V^* \setminus \mathrm{Int}(\gamma^\circ) = \lambda^{\circ a}$, we have

$$L_U = P'_\lambda = L_\gamma, \qquad R_U = Q'_\lambda = R_\gamma.$$

Noticed that we can show by either computation of dual functor D' or abstract uniqueness of kernel that $K_U = 1_{\widetilde{\gamma}} \simeq \mathrm{D}'(1_{\widetilde{\lambda}^a \setminus \Delta_V})[n-1]$.

6. Non-linaer microlocal cut-off lemma

Recall that we say a morphism $f : F \to G$ in $\mathrm{Sh}(M)$ is an isomorphism on an open set U if f is an isomorphism in $\mathrm{Sh}(M; U)$ (cf. [KS90, Definition 6.1.1]). Because $\mathrm{Sh}_Z(M)$ is a thick subcategory of $\mathrm{Sh}(M)$, we know that f is an isomorphism on U if and only if $SS(\mathrm{cofib}\,(f)) \cap U = \varnothing$.

Theorem 6.1. *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have*

(1) (a) $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms if and only if $SS(F) \subset Z$.
(b) $L_U(F) \to F$ is an isomorphism if and only if $F \in^\perp \mathrm{Sh}_Z(M)$, and $F \to R_U(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_Z(M)^\perp$.
(a) The morphisms $L_U(F) \to F$ and $F \to R_U(F)$ are isomorphisms on U.
(b) If $0_M \subset Z$, the morphism $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms on $\mathrm{Int}(Z) \setminus 0_M$. Equivalently, we have $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$.

Proof. (1)-(a) and (2)-(a) follows from definition of L_Z and L_U and fiber sequences (4.2). (1)-(b) follows from [CDH+, A.2.8]. For (2)-(b), we will prove in Proposition 7.5 that $SS(K_U) \subset (-\overline{U}) \times \overline{U} \cup 0_{M\times M}$ under the condition $0_M \subset Z$. Therefore, $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$ follows from Proposition 2.5. □

Remark 6.2. (1) By virtue of Corollary 8.4, this result generalize eassily to the relative case. (2) Based on results of Section 5, (1)-(a), (2)-(a) is [KS90, Proposition 5.2.3-(i), Lemma 6.1.5-(i)], (1)-(b) is explained in [Gui23, Section III.5], and (2)-(b) is essentially [KS90, Proposition 5.2.3-(ii), Lemma 6.1.5-(ii)]. Also, by Example 5.3, the theorem also recover [GS14, Proposition 3.17, 3.19, 3.20, 3.21.].

7. Wrapping formula of non-linaer microlocal cut-off functors

The goal of those section is present a explicit formula of microlocal kernels using Guillermou-Kashiwara-Schapira sheaf quantization [GKS12]. We recall the results of *loc.cit.* here.

Let $\dot{T}^*M$ be the complement of the zero section in T^*M, and, for subset $A \subset T^*M$, we set $\dot{A} = A \cap \dot{T}^*M$. In particular, we have the notion of $\dot{SS}(F)$ for $F \in \mathrm{Sh}(M)$.

Let $(I, 0)$ be a pointed interval. Consider a C^∞ conic symplectic isotopy

$$\phi : I \times \dot{T}^*M \to \dot{T}^*M$$

which is the identity at $0 \in I$. Such an isotopy is always the Hamiltonian flow for a unique conic function $H : I \times \dot{T}^*M \to \mathbb{R}$ and we set $\phi = \phi_H$ when emphasize the Hamiltonian functions. At fixed $z \in I$, we have the graph of ϕ_z:

$$\Lambda_{\phi_z} := \Big\{((q, -p), \phi_z(q, p)) : (q, p) \in \dot{T}^*M\Big\} \subset \dot{T}^*M \times \dot{T}^*M \subset \dot{T}^*(M \times M). \tag{7.1}$$

As for any of Hamiltonian isotopy, we may consider the Lagrangian graph, which by definition is a Lagrangian subset $\Lambda_\phi \subset T^*I \times \dot{T}^*(M \times M)$ with the property that $\Lambda_{\phi_{z_0}}$ is the symplectic reduction of Λ_ϕ along $\{z = z_0\}$. It is given by the formula:

$$\Lambda_\phi := \Big\{(z, -H(z, \phi_z(q, p)), (q, -p), \phi_z(q, p)) : z \in I, (q, p) \in \dot{T}^*Y\Big\} \tag{7.2}$$

Theorem 7.1 ([GKS12, Theorem 3.7, Prop. 4.8])**.** *For ϕ as above, there is a sheaf $K = K(\phi) \in \mathrm{Sh}(I \times Y^2)$ such that $\dot{SS}(K) = \Lambda_\phi$ and $K|_{\{0\}\times Y^2} \cong 1_{\Delta_Y}$. The pair $(K, K|_{\{0\}\times Y^2} \cong 1_{\Delta_Y})$ is unique up to unique isomorphism.*

Moreover, for isotopies $\phi_H, \phi_{H'}$ with $H' \leq H$, there's a map $K(\phi_{H'}) \to K(\phi_H)$. In particular, when $H \geq 0$, then there is a map $1_{I\times\Delta_Y} \to K(\phi_H)$.

Similar to Proposition 2.5, one can see that

$$\dot{SS}(\Phi_{K(\phi)|_z}(F)) = \phi_z(\dot{SS}(F)). \tag{7.3}$$

Remark 7.2. In [GKS12], only $\dot{SS}(K) \subset \Lambda_\phi$ is proven. However, as [Gui23, Remark II.1.2] explained, the equality follows from coisotropicness of microsupport.

Motivated by ideas of [Nad16, GPS18], it was shown in [Kuo23] that for any closed set $Z^\infty \subset S^*M$ and the conic closed set $Z = \mathbb{R}_{>0}Z^\infty \cup 0_M \subset T^*M$, the adjoints of the inclusion $\iota : \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ can be computed 'by wrapping'. More precisely,

Theorem 7.3 ([Kuo23, Thm. 1.2])**.** *If H_n is any increasing sequence of positive compactly supported Hamiltonians supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, then the adjoints of ι could be computed by*

$$\iota^* F = \varinjlim \Phi_{K(\phi_{H_n})|_1}(F), \quad \iota^! F = \varprojlim \Phi_{K(\phi_{-H_n})|_1}(F). \tag{7.4}$$

Moreover, the unit(counit) of the adjoint is given by the map $1_{\Delta_M} \to \varinjlim K(\phi_{H_n})|_1$ ($\varprojlim K(\phi_{-H_n})|_1 \to 1_{\Delta_M}$) which is induced by the continuation map $1_{\Delta_M} \to K(\phi_{H_n})|_1$ ($K(\phi_{-H_n})|_1 \to 1_{\Delta_M}$) defined by positivity of H_n.

Remark 7.4. In fact, as we explained in [KSZ23, Remark 6.5], the colimit in [Kuo23] is taken over an ∞-categorical 'wrapping category', but one can compute the colimit by a cofinal sequence as explained in [Kuo23, Lemma 3.31].

For $H \geq 0$, We set

$$K^\circ(\phi_{H_n}) = \mathrm{cofib}\,(1_{I\times\Delta_M} \to K(\phi_{H_n})). \tag{7.5}$$

Compose with the natural inclusion ι and combine with [Kuo23, Proposition 3.5], one can see that the fiber sequence (4.3) can be taken as

$$K_U = \varinjlim(K^\circ(\phi_{H_n})|_1) \to 1_{\Delta_M} \to K_Z = \varinjlim(K(\phi_{H_n})|_1). \tag{7.6}$$

To complete the proof of Theorem 6.1 (2)-(b), we start from a microsupport estimation of $K^\circ(\phi_H)|_1$. We also notice that the requirement of $0_M \subset Z$ comes from the wrapping formula.

Proposition 7.5. *For a Hamiltonian function H supported in $U^\infty = U/\mathbb{R}_{>0}$, we have*

$$\dot{SS}(K^\circ(\phi_H)|_1) \subset (-U) \times U.$$

In particular, we have

$$\dot{SS}(K_U) \subset (-\overline{U}) \times \overline{U}.$$

Proof. By the triangulated inequality, we have that

$$\dot{SS}(K^\circ(\phi_H)|_1) \subset \Lambda_{\phi_{H,1}} \cup \Lambda_{\mathrm{id}}.$$

Since $\phi_{H,1}$ is compactly supported, there exists a maximal conic open set $W \subset T^*M$ at infinity such that $T^*M \setminus U \subset W$ such that $H|_W = 0$. Therefore, we have

$$\dot{SS}(K^\circ(\phi_H)|_1) \setminus (-W^c) \times W^c \subset \{(q, -p, q, p) : (q, p) \in W\}.$$

The construction of the monotonicity morphism $1_{\Delta_M} \to K(\phi_H)|_1$ shows that the monotonicity morphism induces the identity map on microstalks at $(q, -p, q, p)$ for $(q, p) \in W$. Then we have that the microstalk of $K^\circ(\phi_H)|_1$ at the $(q, -p, q, p)$ is zero for $(q, p) \in W$. This implies that

$$\dot{SS}(K^\circ(\phi_H)|_1) \subset (-W^c) \times W^c \subset (-U) \times U.$$

The second statement follows from Corollary 2.3 and the wrapping formula Equation (7.6). ☐

***Proof** of Theorem 6.1 (2)-(b).* The estimation $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$ follows from Proposition 2.5, and Proposition 7.5. ☐

Using wrapping formula, we can also prove the following microsupport estimation

Proposition 7.6. *If Z containing the zero section, then for any increasing sequence of positive compactly supported Hamiltonians H_n supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, we have*

$$SS(L_Z(F)) \subset \{x : \forall n \in \mathbb{N}, \exists x_n \in \dot{SS}(F), \text{ such that } \phi_{H_n,1}(x_n) \to x\} \cup 0_M,$$
$$SS(R_Z(F)) \subset \{x : \forall n \in \mathbb{N}, \exists x_n \in \dot{SS}(F), \text{ such that } \phi_{-H_n,1}(x_n) \to x\} \cup 0_M.$$

Proof. For L_Z, the estimation follows immediately from Corollary 2.3 and Equation (7.3) using the wrapping formula Theorem 7.3.

For the proof of R_Z, we can not use Corollary 2.3 directly. However, we noticed that the proof of Corollary 2.3 only need the fact: for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, we have $\varinjlim_i F_{n_i} \simeq \varinjlim_n F_n$, which is in general not true for $\varprojlim$. Nevertheless, if H_n is a cofinal sequence, then for any divergent subsequence $\{n_i\}_{i\in\mathbb{N}}$, functions $\{H_{n_i}\}_{i\in\mathbb{N}}$ is still a cofinal sequence supported on $S^*M \setminus Z^\infty$. Therefore, Theorem 7.3 implies, for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, that

$$R_Z(F) \simeq \varprojlim_i \Phi_{K(\phi_{-H_{n_i}})|_1}(F).$$

Then we can adapt the argument of Corollary 2.3 for the specific limit to show that

$$SS(R_Z(F)) \subset \liminf_n SS(\Phi_{K(\phi_{-H_n})|_1}(F)).$$

The require microsupport estimation follows from Equation (7.3). □

Remark 7.7. In this proposition, the only place we use the wrapping formula is identifying $L_Z(F)$ with $\varinjlim \Phi_{K(\phi_{H_n})|_1}(F)$. On the other hand, we can prove directly that $SS(\varinjlim \Phi_{K(\phi_{H_n})|_1}(F)) \subset Z$, and then identify L_Z with $\varinjlim \Phi_{K(\phi_{H_n})|_1}$. Passing to adjoint, we get the wrapping formula of R_Z. We left the detail to interested readers.

8. Künneth formula

Here, we present two computations of Lurie tensor products, which generalize the result of [KL24, Theorem 1.2], where a isotropic condition is needed.

To start with, we recall that $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is dualizable if it is a dualizable object with respect to the symmetric monoidal structure defined by the Lurie tensor product, and its dual is denoted by $\mathcal{C}^\vee$. Efimov prove the following important fact we need to use: for dualizable category $\mathcal{C}$, the functor $- \otimes \mathcal{C}$ is flat [Efi24, Theorem 2.2]. Then we recall that:

Lemma 8.1 ([KSZ23, Remark 3.7])**.** *The category $\mathrm{Sh}_Z(M)$ is dualizable with dual $\mathrm{Sh}_{-Z}(M)$. The category $\mathrm{Sh}(M;U)$ is dualizable with the dual $\mathrm{Sh}(M;-U)$.*

For a manifold N, conic closed sets $Z \subset T^*M$ and $X \subset T^*N$ containing the zero sections, and we set $U = T^*M \setminus Z$ and $V = T^*N \setminus X$. We consider the split Verdier sequence

$$\mathrm{Sh}_{X\times Z}(N \times M) \to \mathrm{Sh}(N \times M) \to \mathrm{Sh}_{X\times U}(N \times M; T^*N \times U).$$

Remark 8.2. In the following propositions, the box tensor $\boxtimes$ of kernels should be understood after identifying $(N \times M)^2 \simeq N^2 \times M^2$ via $((n_1, m_1), (n_2, m_2)) \mapsto (n_1, n_2, m_1, m_2)$.

Proposition 8.3. *We have that $K_X \boxtimes K_Z \simeq K_{X\times Z}$.*

Proof. We set $K := K_X \boxtimes K_Z$. It is clear that $\Phi_K \circ \Phi_K \simeq \Phi_K$, so we only need to prove that $F \in \mathrm{Sh}_{X\times Z}(N \times M)$ if and only if $\Phi_K(F) \simeq F$.

For $F \in \mathrm{Sh}(N \times M)$ with $\Phi_K(F) \simeq F$, we can write $F = \varinjlim A \boxtimes B$, and then we have $F \simeq \varinjlim \Phi_{K_X}(A) \boxtimes \Phi_{K_Z}(B)$. So $SS(F) = SS(\varinjlim \Phi_{K_X}(A) \boxtimes \Phi_{K_Z}(B)) \subset \overline{SS(\Phi_{K_X}(A)) \times SS(\Phi_{K_Z}(B))} \subset X \times Z$.

Conversely, we need to show that K fixes $F \in \mathrm{Sh}_{X\times Z}(N \times M)$. We use the wrapping formula Theorem 7.3, and adapt the argument of [AGH+23, Lemma 4]. For cofinal sequences of conic Hamiltonian functions H_α supported in U and G_β supported in V. We have that

$K \simeq \varinjlim_{(\alpha,\beta)} K(\phi_{H_\alpha})|_1 \boxtimes K(\phi_{G_\beta})|_1$. Now, for the 1-parameter version of GKS quantization $K(\phi_{H_\alpha})$ and $K(\phi_{G_\beta})$ as Theorem 7.1 given. By a parameter version of(7.3), it is directly to see that for $W = \Phi_{K(\phi_{H_\alpha})\boxtimes K(\phi_{G_\beta})}(F) \in \mathrm{Sh}(N \times \mathbb{R} \times M \times \mathbb{R})$, we have

$$SS(W) \subset \left\{ (q_1, z_1, p_1, \zeta_1, q_2, z_2, p_2, \zeta_2) : \begin{array}{l} \exists (q_1', p_1', q_2', p_2') \in SS(F),\ (q_i, p_i) = \phi_{z_i}^{H_\alpha}(q_i', p_i'), \\ \zeta_1 = -H_\alpha(q_1', p_1'), \zeta_2 = -G_\beta(q_2', p_2'). \end{array} \right\}.$$

Now, since $SS(F) \subset X \times Z$, we have that $SS(W) \subset T^*(N \times M) \times 0_{\mathbb{R}^2}$. Then by [KS90, Proposition 5.4.5], we have $F = W|_{z_1=z_2=0} \simeq W|_{z_1=z_2=1} = \Phi_{K(\phi_{H_\alpha})|_1 \boxtimes K(\phi_{G_\beta})|_1}(F)$. We can further verify that the isomorphism is commute with continuation map given by positivity of H_α, G_β (see Theorem 7.3). Then we taking limit with respect to (α, β) to see that $\Phi_K(F) \simeq F$.

Consequently, we have $K_{X\times Z} \simeq K = K_X \boxtimes K_Z$ by uniqueness of kernel. □

Corollary 8.4. *We have an equivalence of fiber sequences of categories*

$$\begin{aligned} &\mathrm{Sh}_{X\times Z}(N \times M) \to \mathrm{Sh}_{X\times T^*M}(N \times M) \to \mathrm{Sh}_{X\times U}(N \times M; T^*N \times U) \\ \simeq &\mathrm{Sh}_X(N) \otimes (\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)). \end{aligned}$$

Proof. The functor $- \mapsto \mathrm{Sh}_X(N) \otimes -$ preserves colimits, so we have the equivalence of Verdier sequences

$$\begin{aligned} &\mathrm{Sh}_X(N) \otimes (\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)) \\ \simeq &\mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \to \mathrm{Sh}_X(N) \otimes \mathrm{Sh}(M) \to \mathrm{Sh}_X(N) \otimes \mathrm{Sh}(M; U). \end{aligned}$$

Therefore, under the naturally identification $\mathrm{Sh}(N) \otimes \mathrm{Sh}(M) = \mathrm{Sh}(N \times M)$ (see [Vol21, Proposition 2.30]), it remains to verify that the essential image of the functor

$$\mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \to \mathrm{Sh}(N) \otimes \mathrm{Sh}(M) = \mathrm{Sh}(N \times M), \quad F \otimes G \mapsto F \boxtimes G$$

is $\mathrm{Sh}_{X\times Z}(N \times M)$.

In fact, any object F in $\mathrm{Sh}_{X\times Z}(N\times M)$ can be written as a colimit of some $F = \varinjlim F_\alpha \boxtimes G_\alpha \in \mathrm{Sh}(N \times M)$. We should show that the F_α can be chosen in $\mathrm{Sh}_X(N)$ and G_α can be chosen to be in $\mathrm{Sh}_Z(M)$. By definition of $K_{X\times Z}$, we have $F \simeq \Phi_{K_{X\times Z}}(F)$. As we already prove that $K_{X\times Z} \simeq K_X \boxtimes K_Z$, we have

$$F \simeq \Phi_{K_{X\times Z}}(\varinjlim F_\alpha \boxtimes G_\alpha) \simeq \varinjlim \Phi_{K_X \boxtimes K_Z}(F_\alpha \boxtimes G_\alpha) = \varinjlim \Phi_{K_X}(F_\alpha) \boxtimes \Phi_{K_Z}(G_\alpha). \qquad \square$$

Proposition 8.5. *We have an equivalence*

$$\mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N; V) \simeq \mathrm{Sh}(M \times N; U \times V).$$

In particular, we have that $K_U \boxtimes K_V \simeq K_{U\times V}$.

Proof. We fulfill the 9-diagram in Equation (9.1) by setting the manifold by $M \times N$, two conic closed sets are $Z \times T^*N \subset Z \times T^*N \cup T^*M \times X$. Noticed that, the 9-diagram follows directly from the colimits commute with each other. Therefore, we have the following equivalence

$$\mathrm{Sh}(M \times N; U \times V) \simeq \mathrm{Sh}(M \times N; U \times T^*N)/\mathrm{Sh}_{U\times X}(M \times N; U \times T^*N).$$

Now, we apply Corollary 8.4 twice to see that

$$\mathrm{Sh}(M \times N; U \times T^*N) \simeq \mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N),\ \mathrm{Sh}_{U\times X}(M \times N; U \times T^*N) \simeq \mathrm{Sh}(M; U) \otimes \mathrm{Sh}_X(N).$$

Moreover, one can check that

$$[\mathrm{Sh}_{U\times X}(M \times N; U \times T^*N) \hookrightarrow \mathrm{Sh}(M \times N; U \times T^*N)] = \mathrm{Sh}(M; U) \otimes [\mathrm{Sh}_X(N) \hookrightarrow \mathrm{Sh}(N)].$$

Consequently, we have

$$\mathrm{Sh}(M \times N; U \times T^*N)/\mathrm{Sh}_{U\times X}(M \times N; U \times T^*N) \simeq \mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N; V). \qquad \square$$

9. Results for pairs

In this section, we take consider the following construction: Take two conic closed set $Z \subset X \subset T^*M$. Let $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, then $V \subset U$.

Recall that $\mathrm{Sh}_{X\cap U}(M;U) = \frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$. We consider the following 9-diagram of $\mathbf{k}$-linear categories:

$$
\begin{tikzcd}
\mathrm{Sh}_Z(M) \arrow[r, hook] \arrow[d, equal] & \mathrm{Sh}_X(M) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Sh}_{X\cap U}(M;U) \arrow[d, hook] \\
\mathrm{Sh}_Z(M) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Sh}(M) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Sh}(M;U) \arrow[d, two heads] \\
0 \arrow[r, hook] & \mathrm{Sh}(M;V) \arrow[r, "\simeq"] & \mathrm{Sh}(M;U)\big/\mathrm{Sh}_{X\cap U}(M;U)
\end{tikzcd}
\tag{9.1}
$$

In this diagram, by the same argument with Proposition 4.1, we have that all vertical and horizontal sequences are split Verdier sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathbf{k})$. We say a morphism $f : F \to G$ in $\mathrm{Sh}(M;U)$ is an isomorphism on an open set $V \subset U$ if f is an isomorphism in $\mathrm{Sh}(M;V)$. Because $\mathrm{Sh}_{X\cap U}(M;U)$ is a thick subcategory of $\mathrm{Sh}(M;U)$, we know that f is an isomorphism on V if and only if $SS_U(\mathrm{cofib}\,(f)) \cap V = \varnothing$.

Here, we write down adjoint functors in the last column precisely

$$
\begin{tikzcd}
\mathrm{Sh}_{X\cap U}(M;V) \arrow[r, "\iota"] & \mathrm{Sh}(M;U) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j"] & \mathrm{Sh}(M;V). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}
\tag{9.2}
$$

Then we have adjunction pairs

$$L_{(U,V)} \coloneqq j_!j \dashv j_*j =: R_{(U,V)}, \qquad L_{(X,Z)} \coloneqq \iota\iota^* \dashv \iota\iota^! =: R_{(X,Z)},$$

and the unit/counits give us following fiber sequences of functors on $\mathrm{Sh}(M;U)$

$$
\begin{aligned}
L_{(U,V)} &\to \mathrm{id} \to L_{(X,Z)},\\
R_{(X,Z)} &\to \mathrm{id} \to R_{(U,V)}.
\end{aligned}
\tag{9.3}
$$

Remark 9.1. By Equation 0.1, we can also represent functors in Equation (9.3) by integral functors.

Therefore, we can state the following version of cut-off lemma

Theorem 9.2. *For conic closed sets $Z \subset X \subset T^*M$, and $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, we have*

(1) (a) $F \to L_{(X,Z)}(F)$ and $R_{(X,Z)}(F) \to F$ are isomorphisms if and only if $SS_U(F) \subset X \cap U$.
(b) $L_{(U,V)}(F) \to F$ is an isomorphism if and only if $F \in^\perp \mathrm{Sh}_{X\cap U}(M;U)$, and $F \to R_{(U,V)}(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_{X\cap U}(M;U)^\perp$.
(a) The morphisms $L_{(U,V)}(F) \to F$ and $F \to R_{(U,V)}(F)$ are isomorphisms on $U \cap V$.
(b) If $0_M \subset Z$, the morphism $F \to L_{(X,Z)}(F)$ and $R_{(X,Z)}(F) \to F$ are isomorphisms on $\mathrm{Int}(X) \cap U$. Equivalently, we have $SS_U(L_{(U,V)}(F)) \cup SS_U(R_{(U,V)}(F)) \subset \overline{V} \cap U$. Here $\overline{V}$ denotes the closure of V in T^*M.

Proof. Here, we noticed that $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for $[F] \in \mathrm{Sh}(M;U)$. Then we apply the absolute version Theorem 6.1 to conclude. ☐

Example 9.3. We consider the manifold $M \times \mathbb{R}_t$. We still take an closed set $Z \subset J^1M = T^*M \times \mathbb{R}$ and $U = J^1M \setminus Z$. (NOT necessarily conic!)

In stead, we consider their conification in $T^*(M\times\mathbb{R}_t)=T^*M\times\mathbb{R}_t\times\mathbb{R}_\tau$. We set

$$\begin{aligned}
&\Omega_{>0}=\{\tau>0\},\qquad \Omega_-=\{\tau\leq 0\}.\\
&\widetilde{U}=\{(q,\tau p,t,\tau):(q,p,t)\in U,\tau>0\},\\
&\widetilde{Z}=\{(q,\tau p,t,\tau):(q,p,t)\in Z,\tau>0\},\\
&\widetilde{Z}_-=\widetilde{Z}\cup\Omega_-.
\end{aligned}\tag{9.4}$$

Then $\Omega_-\subset\widetilde{Z}_-$ are two closed conic sets. In this case, the category

$$\mathrm{Sh}(M\times\mathbb{R}_t;\widetilde{U})\simeq\mathrm{Sh}(M\times\mathbb{R}_t;\Omega_{>0})\Big/\frac{\mathrm{Sh}_{\widetilde{Z}}(M\times\mathbb{R}_t)}{\mathrm{Sh}_{\Omega_-}(M\times\mathbb{R}_t)}$$

is useful for contact topology of J^1M, since $\mathbb{R}_{>0}$ acts on $\Omega_{>0}$ freely, with the quotient $J^1M=T^*M\times\mathbb{R}_t$.

If we take $U=U_0\times\mathbb{R}$ for an open set $U_0\subset T^*M$ and $Z=Z_0\times\mathbb{R}$ for an open set $U_0\subset T^*M$, the category $\mathrm{Sh}(M\times\mathbb{R}_t;\widetilde{U})$ is exactly the so-called Tamarkin categories $\mathcal{T}(U_0)$ (cf. [KSZ23]). Tamarkin categories are first defined in [Tam18], and then are studied in [Vic13, Chi17, Ike19, AI20a, Zha20, AI22, GV22a, AGH+23, AIL23]. In this case, U admits a $\mathbb{R}$-action by translation along $\mathbb{R}_t$. Therefore, there exist an $\mathcal{T}=\mathcal{T}(\mathrm{pt})$ (which is an symmetric monoidal category [GS14, KSZ23]) action on $\mathcal{T}(U_0)\coloneqq\mathrm{Sh}(M\times\mathbb{R}_t;\widetilde{U_0\times\mathbb{R}})$, which helps us understanding the action filtration from symplectic geometry. And functors

$$L^{\mathcal{T}}_{U_0}\coloneqq L_{(\Omega_{>0},\widetilde{U_0\times\mathbb{R}})},\ L^{\mathcal{T}}_{Z_0}\coloneqq L_{(\widetilde{Z_0\times\mathbb{R}}_-,\Omega_-)}$$

are actually $\mathcal{T}$-linear. By the $\mathcal{T}$-linear left adjoint functors classification (see [KSZ23, Proposition 5.12], which is an enriched version of Remark 9.1), there exists a fiber sequences $K^{\mathcal{T}}_{U_0}\to 1^{\mathcal{T}}_{\Delta_M}\to K^{\mathcal{T}}_{Z_0}$ in $\mathcal{T}(T^*(M\times M))=\mathrm{Sh}(M\times M;\mathcal{T})$ with

$$L^{\mathcal{T}}_{U_0}=\Phi^{\mathcal{T}}_{K^{\mathcal{T}}_{U_0}},\ L^{\mathcal{T}}_{Z_0}=\Phi^{\mathcal{T}}_{K^{\mathcal{T}}_{Z_0}},$$

where $\Phi^{\mathcal{T}}$ means the convolution functor defined using $\mathcal{T}$-linear 6-operators. These functors are studied in [Tam15, Chi17, Zha21, Zha23, KSZ23], where corresponding kernels are denoted by $K^{\mathcal{T}}_{U_0}=P_{U_0},K^{\mathcal{T}}_{Z_0}=Q_{U_0}$. In particular, we are free to use Theorem 9.2 for Tamarkin categories.

If the open set $U\subset J^1M$ admits a $\mathbb{G}$-action for a subgroup $\mathbb{G}\subset\mathbb{R}$ via translation, Asano, Ike and Kuwagaki introduce an equivariant version of sheaves to study corresponding categories, see [AI20b, IK23]. If $\mathbb{G}$ is a discrete subgroup of $\mathbb{R}$, which means $\mathbb{G}\simeq\mathbb{Z}$, the equivariant version can be understood by the framework of this article by identifying $\mathbb{R}/\mathbb{G}=S^1$ and considering the category $\mathrm{Sh}(M\times S^1;\Omega_+)$ (which still makes sense.) However, if $\mathbb{G}=\mathbb{R}_a$ as a discrete subgroup of the topological group $\mathbb{R}_t$, the situation becomes more complicated and we will not discuss related categories and cut-off lemma in this article (while we still believe it is true.)

10. Splitting

In this section, we always assume $0_M\subset Z$.

Lemma 10.1. *Suppose $a:A\to B$ and $c:C\to B$ are two arrows in a stable category, then we have*

$$\mathrm{cofib}\,((a,c))\simeq\mathrm{cofib}\left([A\xrightarrow{a}B\to\mathrm{cofib}\,(c)]\right).$$

Proof. We set $w : A \xrightarrow{a} B \to \operatorname{cofib}(c)$. Consider the following commutative diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \oplus C \arrow[r, "{(a,c)}"] \arrow[d] & B \arrow[d] \\
0 \arrow[r] & A \arrow[r, "w"] \arrow[d] & \operatorname{cofib}(c) \arrow[d] \\
& 0 \arrow[r] & \operatorname{cofib}((a,c)).
\end{tikzcd}$$

The result follows from that all squares in the diagram are pushout. . □

Proposition 10.2. *Assume $\dot{SS}(L_Z(F)) \cup \dot{SS}(R_Z(F)) \subset \operatorname{Int}(Z) \setminus 0_M$, then we have*

$$L_U(F) \oplus R_Z(F) \to F$$

is isomorphic on $\dot{T}^*M$, and

Proof. Wwe apply the lemma to $a : L_U(F) \to F$ and $c : R_Z(F) \to F$. Then the cofiber of (a, c) is isomorphic to the cofiber of $w : R_Z(F) \to F \to L_Z(F)$, which is denoted by L. Therefore, we only need to check that $SS(L) \subset 0_M$.

For $G = L_Z(F)$ or $G = R_Z(F)$, by the condition, we have

$$\dot{SS}(G) = \dot{SS}(F) \cap \operatorname{Int}(Z) \subset \operatorname{Int}(Z) \setminus 0_M$$

by Theorem 6.1 (2)-(b). It implies that $\dot{SS}(L) \subset \operatorname{Int}(Z) \setminus 0_M$, but f is an isomorphism on $\operatorname{Int}(Z) \setminus 0_M$. Then we have $\dot{SS}(L) \cap \operatorname{Int}(Z) = \varnothing$. Therefore, we have $\dot{SS}(L) = \varnothing$. □

Since $\dot{SS}(L_Z(F)) \cup \dot{SS}(R_Z(F)) \subset Z \setminus 0_M$ is always true by Theorem 6.1 (1)-(a), the condition only means that $[\dot{SS}(L_Z(F)) \cup \dot{SS}(R_Z(F))] \cap \dot{\partial} Z = \varnothing$. In practise, we need find certain geometric criterion of this assumption. For example, if $Z = V \times \gamma^{\circ a}$ for a convex cone γ, [Gui23, Proposition III.1.10] gives a useful sufficient conditions (actually a better one than what we present here). And our proof here is just a formal repeat of the proof therein.

References

[AGH+23] Tomohiro Asano, Stéphane Guillermou, Vincent Humilière, Yuichi Ike, and Claude Viterbo. *The γ-support as a micro-support*. Comptes Rendus. Mathématique, **volume 361**, 1333–1340 [2023]. doi: 10.5802/crmath.499.

[AI20a] Tomohiro Asano and Yuichi Ike. *Persistence-like distance on Tamarkin's category and symplectic displacement energy*. Journal of Symplectic Geometry, **volume 18**, no. 3, 613–649 [2020]. doi: 10.4310/jsg.2020.v18.n3.a1.

[AI20b] Tomohiro Asano and Yuichi Ike. *Sheaf quantization and intersection of rational Lagrangian immersions*. Annales l'Institut Fourier [2020]. doi: 10.5802/aif.3554.

[AI22] Tomohiro Asano and Yuichi Ike. *Completeness of derived interleaving distances and sheaf quantization of non-smooth objects* [2022]. arXiv:2201.02598.

[AIL23] Tomohiro Asano, Yuichi Ike, and Wenyuan Li. *Lagrangian Cobordism and Shadow Distance in Tamarkin Category* [2023]. arXiv:2312.14429.

[BGT13] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. *A universal characterization of higher algebraic K-theory*. Geometry & Topology, **volume 17**, no. 2, 733–838 [2013]. doi: 10.2140/gt.2013.17.733.

[CDH+] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity*. arXiv:2009.07224.

[Chi17] Sheng-Fu Chiu. *Non-squeezing property of contact balls*. Duke Mathematical Journal, **volume 166**, no. 4, 605–655 [2017]. doi: 10.1215/00127094-3715517.

[D'A96] Andrea D'Agnolo. *On the microlocal cut-off of sheaves*. Topological Methods in Nonlinear Analysis, **volume 8**, no. 1, 161 [1996]. ISSN 1230-3429. doi: 10.12775/tmna.1996.025.

[Efi24] Alexander I. Efimov. *K-theory and localizing invariants of large categories* [2024]. arXiv:2405.12169.

[GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*. Duke Mathematical Journal, **volume 161**, no. 2, 201–245 [2012]. doi: 10.1215/00127094-1507367.

[GPS18] Sheel Ganatra, John Pardon, and Vivek Shende. *Microlocal Morse theory of wrapped Fukaya categories* [2018]. arXiv: 1809.08807.

[GR19] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*, volume 221. American Mathematical Society [2019].

[GS14] Stéphane Guillermou and Pierre Schapira. *Microlocal theory of sheaves and Tamarkin's non Displaceability theorem*. In *Homological mirror symmetry and tropical geometry*, Springer, 43–85 [2014]. doi: 10.1007/978-3-319-06514-4_3.

[Gui12] Stéphane Guillermou. *Quantization of conic Lagrangian submanifolds of cotangent bundles* [2012]. arXiv:1212.5818.

[Gui23] Stéphane Guillermou. *Sheaves and symplectic geometry of cotangent bundles*, *Astérisque*, volume 440. Société mathématique de France [2023]. doi: 10.24033/ast.1199.

[GV22a] Stéphane Guillermou and Nicolas Vichery. *Viterbo's spectral bound conjecture for homogeneous spaces* [2022]. arXiv:2203.13700.

[GV22b] Stéphane Guillermou and Claude Viterbo. *The singular support of sheaves is γ-coisotropic* [2022]. arXiv:2203.12977.

[IK23] Yuichi Ike and Tatsuki Kuwagaki. *Microlocal categories over the Novikov ring I: cotangent bundles* [2023]. arXiv:2307.01561.

[Ike19] Yuichi Ike. *Compact Exact Lagrangian Intersections in Cotangent Bundles via Sheaf Quantization*. Publ. Res. Inst. Math. Sci., **volume 55**, no. 4, 737–778 [2019]. doi: 10.4171/prims/55-4-3.

[KL24] Christopher Kuo and Wenyuan Li. *Duality, Künneth formulae, and integral transforms in microlocal geometry* [2024]. arXiv:2405.15211.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds: With a short gistory. Les débuts de la théorie des faisceaux . By Christian Houzel*, volume 292. Springer [1990]. doi: 10.1007/978-3-662-02661-8.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332. Springer [2006]. doi: 10.1007/3-540-27950-4.

[KSZ23] Christopher Kuo, Vivek Shende, and Bingyu Zhang. *On the Hochschild cohomology of Tamarkin categories* [2023]. arXiv:2312.11447.

[Kuo23] Christopher Kuo. *Wrapped sheaves*. Advances in Mathematics, **volume 415**, 108882 [2023]. doi: 10.1016/j.aim.2023.108882.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press [2009]. doi: 10.1515/9781400830558.

[Lur17] Jacob Lurie. *Higher Algebra* [2017]. Preprint, available at https://www.math.ias.edu/ lurie/papers/HA.pdf.

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry* [2018]. Preprint, available at https://www.math.ias.edu/ lurie/papers/SAG-rootfile.pdf.

[Nad16] David Nadler. *Wrapped microlocal sheaves on pairs of pants* [2016]. arXiv preprint: 1604.00114.

[NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology.* Acta Mathematica, **volume 221**, no. 2, 203–409 [2018]. doi: 10.4310/acta.2018.v221.n2.a1.

[NS22] David Nadler and Vivek Shende. *Sheaf quantization in Weinstein symplectic manifolds* [2022]. arXiv: 2007.10154v3.

[RS18] Marco Robalo and Pierre Schapira. *A Lemma for microlocal sheaf theory in the ∞-Categorical Setting.* Publications of the Research Institute for Mathematical Sciences, **volume 54**, no. 2, 379–391 [2018]. doi: 10.4171/prims/54-2-5.

[RS22] Shaul Ragimov and Tomer M. Schlank. *The ∞-categorical reflection theorem and applications* [2022]. arXiv:2207.09244.

[Sch22] Peter Scholze. *Six-Functor Formalisms* [2022]. Lecture Notes.

[Spa88] Nicolas Spaltenstein. *Resolutions of unbounded complexes.* Compositio Mathematica, **volume 65**, no. 2, 121–154 [1988]. URL `http://eudml.org/doc/89885`.

[Tam15] Dmitry Tamarkin. *Microlocal Category* [2015]. arXiv:1511.08961.

[Tam18] Dmitry Tamarkin. *Microlocal condition for non-displaceability.* In Michael Hitrik, Dmitry Tamarkin, Boris Tsygan, and Steve Zelditch, editors, *Algebraic and Analytic Microlocal Analysis.* Springer [2018], 99–223. doi: 10.1007/978-3-030-01588-6_3.

[Vic13] Nicolas Vichery. *Homogénéisation symplectique et Applications de la théorie des faisceaux à la topologie symplectique.* Ph.D. thesis, Ecole polytechnique [2013]. URL `https://tel.archives-ouvertes.fr/pastel-00780016/`.

[Vol21] Marco Volpe. *The six operations in topology* [2021]. arXiv:2110.10212.

[Zha20] Jun Zhang. *Quantitative Tamarkin theory.* Springer [2020]. doi: 10.1007/978-3-030-37888-2.

[Zha21] Bingyu Zhang. *Capacities from the Chiu-Tamarkin complex.* Journal of Symplectic Geometry, to appear [2021]. doi: 10.48550/arXiv.2103.05143.

[Zha23] Bingyu Zhang. *Idempotence of microlocal kernels and the S^1-equivariant Chiu-Tamarkin invariant* [2023]. arXiv:2306.12316.

Bingyu Zhang
Centre for Quantum Mathematics, University of Southern Denmark
Campusvej 55, 5230 Odense, Denmark
Email: bingyuzhang@imada.sdu.dk