

NON-LINEAR MICROLOCAL CUT-OFF FUNCTORS

ABSTRACT. We define the non-linear microlocal cut-off functors. Then we will explain its relation with microlocal cut-off functor defined by Kashiwara and Schapira, and prove a microlocal cut-off lemma for non-linear microlocal cut-off functors adapting new inputs from symplectic geometry.

0. INTRODUCTION

In the study of microlocal theory of sheaves, a crucial tool is the microlocal cut-off lemma of Kashiwara and Schapira (see [KS90, Proposition 5.2.3, Lemma 6.1.5], [Gui23, Chapter III] and [D'A96]), which constructs certain functors that enable us to cut-off the microsupport of sheaves functorially. On the other hand, in the study of symplectic topology, very similar functors was defined and studied for different purposes [Tam18, NS22, Kuo23].

Precisely, for a smooth manifold M , a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we will define the *non-linear microlocal cut-off functors* $L_U, L_Z, R_U, R_Z : \text{Sh}(M) \rightarrow \text{Sh}(M)$ together with natural transformations $L_U \rightarrow \text{id} \rightarrow L_Z, R_Z \rightarrow \text{id} \rightarrow R_U$ using adjoint data of the split Verdier sequence $\text{Sh}_Z(M) \rightarrow \text{Sh}(M) \rightarrow \text{Sh}(M; U)$. See Section 4 for more details.

The main goal of this article is first explaining precise relations of those functors with known microlocal cut-off functors of Kashiwara-Schapira, and then prove a non-linear microlocal cut-off lemma using the wrapping formula of L_Z introduced in [Kuo23]. Hence fore, most of results in this article are known or obvious to experts, but we did not find an affable reference.

Theorem (Non-linear microlocal cut-off lemma, Theorem 7.1 below.). *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have*

- (1) (a) $F \rightarrow L_Z(F)$ and $R_Z(F) \rightarrow F$ are isomorphisms if and only if $SS(F) \subset Z$.
 (b) $L_U(F) \rightarrow F$ is an isomorphism if and only if $F \in {}^\perp \text{Sh}_Z(M)$, and $F \rightarrow R_U(F)$ is an isomorphism if and only if $F \in \text{Sh}_Z(M)^\perp$.
- (2) (a) The morphisms $L_U(F) \rightarrow F$ and $F \rightarrow R_U(F)$ are isomorphisms on U .
 (b) If $0_M \subset Z$, the morphism $F \rightarrow L_Z(F)$ and $R_Z(F) \rightarrow F$ are isomorphisms on $\text{Int}(Z) \setminus 0_M$. Equivalently, we have $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$.

In fact, all of them except (2)-(b) follow directly from definition. Only (2)-(b) of the theorem is a non-trivial fact.

To explain why the theorem generalizes microlocal cut-off lemma in [KS90], we share explain the relation between non-linear microlocal cut-off functors with microlocal cut-off functors in [KS90]. Here, we follow notation of [Gui23, Section III.1] for microlocal cut-off functors.

We take $M = V$ to be a real vector space, $\lambda \subset V$ and $\gamma \subset V^*$ are pointed closed convex cones. The comparison is summarized in the following table, where the first column marks corresponding references, the last column marks their position in this article. Functors in the same columns are isomorphic with corresponding U and Z .

This article	Z	U	L_U	L_Z	R_U	R_Z	Definition 4.4
[Gui23, Section III.1]	$V \times \lambda^{\circ a}$		P'_λ	Q_λ	Q'_λ	P_λ	Proposition 6.1
[GS14, Section 3]		$V \times \text{Int}(\gamma^\circ)$	L_γ		R_γ		Example 6.3

TABLE 1. Known microlocal cut-off functors.

Based on this comparison, we see that the non-linear microlocal cut-off lemma recover [KS90, Proposition 5.2.3, Lemma 6.1.5], and [GS14, Proposition 3.17, 3.19, 3.20, 3.21.].

In the last section, we will discuss a relative case of the cut-off functors, and apply our results to Tamarkin categories.

Category convention. In this article, a category means an ∞ -category and we will emphasize 1-category we need.

We refer to [GR19, Chapter 1] for basics about higher algebra, and more details could be find in [Lur09, Lur17, Lur18].

Let Cat^{Ex} be the category of stable categories with exact functors between them. We denote Fun^{ex} the stable category of exact functors between stable categories.

We denote $\text{Pr}_{\text{st}}^{\text{L}}$ the category of presentable stable category and continuous functors. Lurie defines a closed symmetric monoidal structure on $\text{Pr}_{\text{st}}^{\text{L}}$, which is called the Lurie tensor product \otimes , and the internal hom is given by continuous functors Fun^{L} . For a symmetric monoidal presentable stable category \mathcal{C} (i.e. \mathcal{C} is a commutative algebra object in $\text{Pr}_{\text{st}}^{\text{L}}$), we denote the category of \mathcal{C} -module in $\text{Pr}_{\text{st}}^{\text{L}}$ by $\text{Pr}_{\text{st}}^{\text{L}}(\mathcal{C})$. Objects in $\text{Pr}_{\text{st}}^{\text{L}}(\mathcal{C})$ are called \mathcal{C} -linear categories, and morphisms in $\text{Pr}_{\text{st}}^{\text{L}}(\mathcal{C})$ are \mathcal{C} -module morphisms, i.e. \mathcal{C} -linear continuous functors (which automatically admit lax \mathcal{C} -linear right adjoints) such that its right adjoints are strict \mathcal{C} -linear. Lurie tensor product induces a relative tensor $\otimes_{\mathcal{C}}$ on $\text{Pr}_{\text{st}}^{\text{L}}(\mathcal{C})$, which is also closed with the internal hom $\text{Fun}_{\mathcal{C}}^{\text{L}}$ of \mathcal{C} -module morphisms.

In this article, we will fix a rigid monoidal compactly generated stable category $(\mathbf{k}, \otimes, 1)$. Here, rigidity means that the monoidal structure of \mathbf{k} restrict to a symmetric monoidal structure on the category of compact objects \mathbf{k}^c (in particular, the tensor unit 1 is compact) and all compact objects are dualizable. For the \mathbf{k} -relative Lurie tensor product, we will write $\otimes/\text{Fun}^{\text{L}}$ directly instead of $\otimes_{\mathbf{k}}/\text{Fun}_{\mathbf{k}}^{\text{L}}$. For example, for an E_{∞} ring-spectrum \mathbb{K} , we can take $\mathbf{k} = \text{Mod}_{\mathbb{K}}$, which is the category of \mathbb{K} -module. In particular, for the spherical spectrum \mathbb{S} , we have $\text{Pr}_{\text{st}}^{\text{L}} = \text{Pr}_{\text{st}}^{\text{L}}(\text{Mod}_{\mathbb{S}})$. When R is a discrete commutative ring, we abuse the notation $\text{Mod}_R = \text{Mod}_{HR}$ for the the Eilenberg–MacLane spectrum HR (c.f. [Lur17, 7.1.2.13]).

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1. SHEAVES AND KERNEL CALCULUS

For a \mathbf{k} -linear category \mathcal{C} and a topological space X , we denote the category of \mathcal{C} -value sheaves to be $\text{Sh}(X; \mathcal{C})$ [Lur09, 7.3.3.1], which is also \mathbf{k} -linear. It is explain by [Vol21] that, when \mathcal{C} is a \mathbf{k} -linear symmetric monoidal category (i.e. a commutative algebra object in $\text{Pr}_{\text{st}}^{\text{L}}(\mathbf{k})$), we have the \mathcal{C} -linear 6-functor formalism, and we also refer to [Sch22] for the 6-functors formalism. In particular, we denote $\text{Sh}(X; \mathbf{k})$ as $\text{Sh}(X)$.

For a locally closed inclusion $i : Z \subset X$ and $F \in \text{Sh}(X)$ we set

$$F_Z = i_! i^{-1} F, \quad \Gamma_Z F = i_* i^! F.$$

We denote the constant sheaf $1_X = a^* 1$ where a is the constant map $X \rightarrow \text{pt}$ and denote $1_Z = (1_X)_Z \in \text{Sh}(X)$ if there is no confusion.

Remark 1.1. It is explained in [Sch22, Proposition 7.1] that, if X is paracompact and has a finite covering dimension, we have that $\text{Sh}(X; \text{Mod}_R) \simeq D(\text{Sh}^{\heartsuit}(R_X))$ where $\text{Sh}^{\heartsuit}(R_X)$ denote the abelian category of sheaves of R -modules (c.f. [KS90]) and D stands for the ∞ -derived category. By similar arguments, we can verify that the ∞ -categorical 6-functors, passing to the homotopy category, coincide with unbounded derived functors defined by Spaltenstein [Spa88].

The main reason for us to use ∞ -category is the following classification of left adjoint functors, which is a combiation of a sequence of Lurie’s results. We refer to [KSZ23, Proposition 3.1] for a proof.

Proposition 1.2. *If H_1 is a locally compact Hausdorff space, then for all topological spaces H_2 , we have the equivalence of categories*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2))$$

*that sends $K \in \mathrm{Sh}(H_1 \times H_2)$ to the convolution functor $\Phi_K = [F \mapsto p_{2!}(K \otimes p_1^*F)]$, where $p_i : H_1 \times H_2 \rightarrow H_i$ are projections. We call K the kernel of Φ_K*

For locally compact Hausdorff spaces H_1, H_2, H_3 , and two convolution functors with kernel $F \in \mathrm{Sh}(H_1 \times H_2)$, $G \in \mathrm{Sh}(H_2 \times H_3)$, we have $\Phi_F \circ \Phi_G \simeq \Phi_{F \circ G}$ for

$$F \circ G := p_{13!}(p_{12}^*F \otimes p_{23}^*G) \in \mathrm{Sh}(H_1 \times H_3),$$

where $p_{ij} : H_1 \times H_2 \times H_3 \rightarrow H_i \times H_j$ are projections.

It is known that, taking right adjoint induces an equivalence of categories

$$\mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2), \mathrm{Sh}(H_1))^{op},$$

where Fun^R stands for right adjoint functors.

However, for any convolution functor $\Phi_K : \mathrm{Sh}(H_1) \rightarrow \mathrm{Sh}(H_2)$ with $K \in \mathrm{Sh}(H_1 \times H_2)$, there exists an obvious right adjoint functor¹

$$\Psi_K : \mathrm{Sh}(H_2) \rightarrow \mathrm{Sh}(H_1), F \mapsto p_{1*}\mathcal{H}om(K, p_2^!F),$$

such that $\Psi_F \circ \Psi_G \simeq \Psi_{G \circ F}$ for $F \in \mathrm{Sh}(H_1 \times H_2)$, $G \in \mathrm{Sh}(H_2 \times H_3)$.

Then we have the following equivalence of categories

Corollary 1.3. *Under the same condition of Proposition 1.2, we have*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2), \mathrm{Sh}(H_1))^{op}, K \mapsto \Psi_K.$$

Remark 1.4. Proposition 1.2 and Corollary 1.3 are only true on ∞ -category level. In classical theory of microlocal sheaves, many functors are built as triangulated convolution(nvolution) functors, which do not admit similar results, especially, we do not know if the kernel is unique in general.

2. MICROSUPPORT OF SHEAVES

From now, we assume M is a smooth manifold. Regarding on microlocal theory of sheaves in the ∞ -category setup, we remark that all arguments of [KS90] work well upon we have the non-characteristic deformation lemma [KS90, Proposition 2.7.2] for all sheaves. When \mathbf{k} is compactly generated, the non-characteristic deformation lemma is proven for all hypersheaves ([RS18]), then for all sheaves because of hypercompleteness for manifolds ([Lur09, 7.2.3.6, 7.2.1.12]).

Definition 2.1 ([KS90, Definition 5.1.2]). For $F \in \mathrm{Sh}(M)$ the microsupport of F is

$$SS(F) = \overline{\left\{ (\mathbf{q}, \mathbf{p}) \in T^*M : \begin{array}{l} \text{There is a } C^1\text{-function } f \text{ near } \mathbf{q} \text{ such that} \\ f(\mathbf{q}) = 0, df(\mathbf{q}) = \mathbf{p} \text{ and } (\Gamma_{\{f \geq 0\}}F)_{\mathbf{q}} \neq 0. \end{array} \right\}}.$$

By definition, $SS(F)$ is a conic closed subset of T^*M . There is a triangulated inequality for the microsupport: for a distinguished triangle $A \rightarrow B \rightarrow C \xrightarrow{+1}$, we have $SS(A) \subset SS(B) \cup SS(C)$.

We remark a further closure property of microsupport

Proposition 2.2 ([KS90, Exercise V.7]). *For a small collection of sheaves $F_\alpha \in \mathrm{Sh}(M)$ indexed by $\alpha \in A$, we have*

$$SS\left(\prod_{\alpha} F_{\alpha}\right) \cup SS\left(\bigoplus_{\alpha} F_{\alpha}\right) \subset \overline{\bigcup_{\alpha} SS(F_{\alpha})}.$$

¹We call it a nvolution functor since it is a dual of a convolution functor, while coconvolution should be the same with nvolution.

A proof can be find in [GV22b, Proposition 3.4.]. Importantly, the proof therein only use the non-characteristic deformation lemma and a geometric argument. In particular, it does not involve the microlocal cut-off lemma in [KS90, Section 5.2].

For $X_n \subset X$ with $n \in \mathbb{N}$, we set

$$\begin{aligned} \limsup_n X_n &= \bigcap_{N \geq 1} \overline{\bigcup_{n \geq N} X_n} \\ &= \{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for infinitely many } n, x_n \rightarrow x\}, \\ \liminf_n X_n &= \{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for all } n, x_n \rightarrow x\}. \end{aligned}$$

By definition, we have $\liminf_n X_n \subset \limsup_n X_n$. As a corollary of Proposition 2.2, we have

Corollary 2.3 ([GV22b, Proposition 6.26]). *For an functor $N(\mathbb{N}) \rightarrow \text{Sh}(X)$, we have*

$$SS(\varinjlim_n F_n) \subset \liminf_n SS(F_n).$$

Remark 2.4. We can also denote \mathbb{N} as a 1-truncated ∞ -category, then there exists a functor $\mathbb{N} \rightarrow N(\mathbb{N})$ which is cofinal. Therefore, we can compute the colimit using mapping telescope as in triangulated categories, and the proof of Corollary 2.3 is the same as references.

We need the following microsupport estimation of (co)nvolution functors:

Proposition 2.5 ([KL22, Lemma 7.8, Proposition 7.9]). *Let $K \in \text{Sh}(M \times M)$ with $SS(K) \subset A \times B$ for conic closed sets $A, B \subset T^*M$, and $F \in \text{Sh}(M)$, then we have*

$$SS(\Phi_K(F)) \subset B, \quad SS(\Psi_K(F)) \subset -A.$$

Remark 2.6. In the statement of [KL22, Proposition 7.9.], only the first statement is given. The second statement can be proven from [KL22, Lemma 7.8] using the same proof based on the same consideration.

3. VERDIER SEQUENCES

Now, we collect some facts about Verdier quotient for stable ∞ -categories, whose triangulated version are [KS06, Exercise 10.15]. All results here are collected from [BGT13, Section 5], [NS18, Section 1.3] and [CDH⁺, Appendix A].

Definition 3.1. Let $\iota : \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful stable subcategory of \mathcal{D} .

- (1) We say \mathcal{C} is thick if \mathcal{C} is stable under retraction.
- (2) Let $W = \{\text{Morphisms in } \mathcal{D} \text{ whose cofiber is in } \mathcal{C}\}$. We define the Verdier quotient $j : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C} := \mathcal{C}[W^{-1}]$ as the Dwyer-Kan localization.

Proposition 3.2. *Let $\iota : \mathcal{C} \rightarrow \mathcal{D}$ be a thick stable subcategory of \mathcal{D} , then we have*

- (1) \mathcal{D}/\mathcal{C} is a cofiber of ι in Cat^{Ex} . For all $\mathcal{E} \in \text{Cat}^{\text{Ex}}$, we have $\text{Fun}^{\text{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \xrightarrow{j^*} \text{Fun}^{\text{ex}}(\mathcal{D}, \mathcal{E})$ is fully faithful whose essential image consisting of those functors which send all objects of \mathcal{C} to 0. In particular, j is essential surjective.
- (2) The fiber of j is \mathcal{C} .
- (3) For $x, y \in \mathcal{D}$, we have

$$\text{Hom}_{\mathcal{D}/\mathcal{C}}(j(x), j(y)) \simeq \varinjlim_{f: \mathcal{C}/y} \text{Hom}_{\mathcal{D}}(x, \text{cofib}(f)) \simeq \varinjlim_{g: x \setminus \mathcal{C}} \text{Hom}_{\mathcal{D}}(\text{fib}(g), y),$$

where the colimit is filtered. I.e., we can compute Hom of the Verdier quotient using “roof”.

- (4) If further $\iota : \mathcal{C} \rightarrow \mathcal{D}$ is in $\text{Pr}_{\text{st}}^{\text{L}}(\mathbf{k})$, then j is in $\text{Pr}_{\text{st}}^{\text{L}}(\mathbf{k})$.

In this case, we call the sequence $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$ a *Verdier sequence*. We have the equivalence of triangulated categories $h(\mathcal{D}/\mathcal{C}) \simeq h\mathcal{D}/h\mathcal{C}$, where $h\mathcal{D}/h\mathcal{C}$ is the triangulated Verdier quotient.

Proposition 3.3. *For a Verdier sequence $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$, we have that ι admits left(right) adjoint if and only if j admits left(right) adjoints, the left(right) adjoint (if it exists) of j is fully-faithful, and the essential image of the left(right) adjoint is identified with the left(right) orthogonal complement ${}^\perp\mathcal{C}(\mathcal{C}^\perp)$:*

$$\begin{aligned} {}^\perp\mathcal{C} &:= \{x \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(x, y) \text{ is contractible for all } y \in \mathcal{C}\}, \\ \mathcal{C}^\perp &:= \{x \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(y, x) \text{ is contractible for all } y \in \mathcal{C}\}. \end{aligned}$$

Moreover, the left(right) adjoint of j induces an equivalence $\mathcal{D}/\mathcal{C} \xrightarrow{\cong} {}^\perp\mathcal{C}(\mathcal{C}^\perp)$.

In this case, we say the Verdier sequence is *left(right) split*, and say it is split if both left and right split. It is also proven in [CDH⁺, Proposition A.2.11] that $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$ is a split Verdier sequence is equivalent to say that $(\mathcal{C}, \mathcal{D}/\mathcal{C})$ is a recollement of \mathcal{D} in the sense of [Lur17, Definition A.8.1].

4. SPLIT VERDIER SEQUENCE FROM MICROLOCALIZATION

For a conic closed set $Z \subset T^*M$, we set $\text{Sh}_Z(M)$ as the full subcategory of $\text{Sh}(M)$ spanned by sheaves F with $SS(F) \subset Z$.

Proposition 4.1. *The category $\text{Sh}_Z(M)$ is a stable subcategory of $\text{Sh}(M)$ closed under small limits and colimits.*

In particular: 1) The inclusion $\iota : \text{Sh}_Z(M) \rightarrow \text{Sh}(M)$ admits both left and right adjoints. 2) $\text{Sh}_Z(M)$ is \mathbf{k} -linear.

Proof. By the triangulated inequality of microsupport, we only need to know $\text{Sh}_Z(M)$ is closed under small products and coproducts, which follows from Proposition 2.2.

The inclusion ι admits both adjoints by the adjoint functor theorem [Lur09, 5.5.2.9], and $\text{Sh}_Z(M)$ is presentable and \mathbf{k} -linear by [RS22] since ι is reflexive. \square

We set $U = T^*M \setminus Z$. Recall the following definition originally given in [KS90, Definition 6.1.1]

Definition 4.2. For the conic open set $U \subset T^*M$, we set

$$\text{Sh}(M; U) := \text{Sh}(M)/\text{Sh}_Z(M).$$

Remark 4.3. In [Gui12, NS22], Out of necessity to study microsheaves, we also denote $\text{Sh}(M; U)$ by $\mu sh^{pre}(U)$.

By Proposition 4.1 and Proposition 3.3, we have that

$$\text{Sh}_Z(M) \rightarrow \text{Sh}(M) \rightarrow \text{Sh}(M; U)$$

is a split Verdier sequence. Precisely, we write down the diagram of functors:

$$(4.1) \quad \begin{array}{ccccc} & \xleftarrow{\iota^*} & & \xleftarrow{j!} & \\ \text{Sh}_Z(M) & \xrightarrow{\iota} & \text{Sh}(M) & \xrightarrow{j} & \text{Sh}(M; U) \\ & \xleftarrow{\iota!} & & \xleftarrow{j_*} & \end{array}$$

Then we have adjunction pairs

$$L_U := j!j \dashv j_*j =: R_U, \quad L_Z := \iota^* \dashv \iota! =: R_Z,$$

and the unit/counits give us following fiber sequences of functors on $\text{Sh}(M)$

$$(4.2) \quad \begin{aligned} L_U &\rightarrow \text{id} \rightarrow L_Z, \\ R_Z &\rightarrow \text{id} \rightarrow R_U. \end{aligned}$$

By Proposition 1.2 and Corollary 1.3, there exists a fiber sequence in $\text{Sh}(M \times M)$

$$(4.3) \quad K_U \rightarrow 1_{\Delta_M} \rightarrow K_Z$$

that gives corresponding convolution/nvolution functors

$$(4.4) \quad L_U = \Phi_{K_U}, L_Z = \Phi_{K_Z}, R_U = \Psi_{K_U}, R_Z = \Psi_{K_Z}$$

and corresponding natural transformations. Therefore, in parctise, to construct those functors and fiber sequences between them, we only need to write down the fiber sequence (4.3).

Definition 4.4. We say the functors L_U, L_Z, R_U, R_Z *non-linear microlocal cut-off functors*. Corresponding kernels K_U, K_Z are called microlocal cut-off kernels (or microlocal kernel for short).

For any object in $\text{Sh}(M; U)$, triangulated inequality implies that $SS_U([F]) := SS(F) \cap U$ for any representative $F \in \text{Sh}(M)$ is well defined. In particular, we have $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for all $F \in \text{Sh}(M)$. We will recall this construction in Section 9.

Example 4.5. Take an open set $W \subset M$, and set $U = T^*W \subset T^*M$ with $Z = T^*M \setminus T^*W$. We naturally identifies $\text{Sh}_Z(M)$ with sheaves supported in $M \setminus W$ since $\pi_{T^*M}(SS(F)) = \text{supp} F$, and we can verify that the restriction functor $(\bullet)|_W : \text{Sh}(M) \rightarrow \text{Sh}(W)$ exhibits $\text{Sh}(W)$ as the Verdier quotient $\text{Sh}(M; T^*W)$. Then we have

$$L_U(F) = F_W, L_Z(F) = F_{M \setminus W}, R_U(F) = \Gamma_W(F), R_Z(F) = \Gamma_{M \setminus W}(F),$$

and can take microlocal kernels as following:

$$K_U = 1_{\Delta_W} \rightarrow 1_{\Delta_M} \rightarrow K_Z = 1_{\Delta_{M \setminus W}}.$$

Therefore, the fiber sequences (4.2) are exactly the excision sequence for sheaves. This example also motivates the notation of functors in Equation (4.1).

5. TENSOR PRODUCT AND FUNCTOR CLASSIFICATION

Next, we consider the relative version of the constructions. In particular, we compute some Lurie tensor products involved. All of them are taken from [KSZ23, Subsection 5.5], where corresponding results for Tamarkin categories are proven. For the reader's convenience, we present proofs here, which are essentially the same with *loc.cit.*

For a manifold N , conic closed sets $Z \subset T^*M$ and $X \subset T^*N$, and we set $U = T^*M \setminus Z$ and $V = T^*N \setminus X$. We consider the split Verdier sequence

$$\text{Sh}_{X \times Z}(N \times M) \rightarrow \text{Sh}(N \times M) \rightarrow \text{Sh}_{X \times U}(N \times M; T^*N \times U).$$

Then we would like to prove the following

Proposition 5.1. *We have isomorphism of fiber sequences of categories*

$$\begin{aligned} \text{Sh}_{X \times Z}(N \times M) &\rightarrow \text{Sh}_{X \times T^*M}(N \times M) \rightarrow \text{Sh}_{X \times U}(N \times M; T^*N \times U) \\ &\simeq \text{Sh}_X(N) \otimes (\text{Sh}_Z(M) \rightarrow \text{Sh}(M) \rightarrow \text{Sh}(M; U)). \end{aligned}$$

In particular, we have that $K_X \boxtimes K_Z \simeq K_{X \times Z}$.

Proof. The functor $- \mapsto \text{Sh}_X(N) \otimes -$ preserves colimits, so have the equivalence of Verdier sequences

$$\begin{aligned} \text{Sh}_X(N) \otimes (\text{Sh}_Z(M) \rightarrow \text{Sh}(M) \rightarrow \text{Sh}(M; U)) \\ \simeq \text{Sh}_X(N) \otimes \text{Sh}_Z(M) \rightarrow \text{Sh}_X(N) \otimes \text{Sh}(M) \rightarrow \text{Sh}_X(N) \otimes \text{Sh}(M; U). \end{aligned}$$

Therefore, under the natually identify $\text{Sh}(N) \otimes \text{Sh}(M) = \text{Sh}(N \times M)$ (see [Vol21, Proposition 2.30]), it remains to verify that the essential image of the functor

$$\text{Sh}_X(N) \otimes \text{Sh}_Z(M) \rightarrow \text{Sh}(N) \otimes \text{Sh}(M) = \text{Sh}(N \times M), \quad F \otimes G \mapsto F \boxtimes G$$

is $\text{Sh}_{X \times Z}(N \times M)$.

In fact, any object in $\text{Sh}_{X \times Z}(N \times M)$ can be written as a colimit of some $F_\alpha \boxtimes G_\alpha \in \text{Sh}(N \times M)$. We should show that the F_α can be chosen in $\text{Sh}_X(N)$ and G_α can be chosen to be in $\text{Sh}_Z(M)$. But since $\text{Sh}_{X \times Z}(N \times M)$ is stable under the $\Phi_{K_X \boxtimes K_Z}$, we have:

$$\varinjlim F_\alpha \boxtimes G_\alpha = \Phi_{K_X \boxtimes K_Z}(\varinjlim F_\alpha \boxtimes G_\alpha) = \varinjlim \Phi_{K_X \boxtimes K_Z}(F_\alpha \boxtimes G_\alpha) = \varinjlim \Phi_{K_X}(F_\alpha) \boxtimes \Phi_{K_Z}(G_\alpha).$$

This proof also identify $\Phi_{K_X \boxtimes K_Z}$ with $L_{X \times Z}$. Then the second statement follows. \square

Proposition 5.2. *We have an equivalence*

$$\mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N; V) \simeq \mathrm{Sh}(M \times N; U \times V).$$

In particular, we have that $K_U \boxtimes K_V \simeq K_{U \times V}$.

Proof. We fulfill the 9-diagram in Equation (9.1) by setting the manifold by $M \times N$, two conic closed sets are $Z \times T^*N \subset Z \times T^*N \cup T^*M \times X$. Noticed that, the 9-diagram follows directly from the functoriality of cofiber functor.

Therefore, we have the following equivalence

$$\mathrm{Sh}(M \times N; U \times V) \simeq \mathrm{Sh}(M \times N; U \times T^*N) / \mathrm{Sh}_{U \times X}(M \times N; U \times T^*N).$$

Now, we apply Proposition 5.1 twice to see that

$$\begin{aligned} \mathrm{Sh}(M \times N; U \times T^*N) &\simeq \mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N), \\ \mathrm{Sh}_{U \times X}(M \times N; U \times T^*N) &\simeq \mathrm{Sh}(M; U) \otimes \mathrm{Sh}_X(N). \end{aligned}$$

Moreover, one can check that

$$[\mathrm{Sh}_{U \times X}(M \times N; U \times T^*N) \hookrightarrow \mathrm{Sh}(M \times N; U \times T^*N)] = \mathrm{Sh}(M; U) \otimes [\mathrm{Sh}_X(N) \hookrightarrow \mathrm{Sh}(N)].$$

Consequently, we have

$$\mathrm{Sh}(M \times N; U \times T^*N) / \mathrm{Sh}_{U \times X}(M \times N; U \times T^*N) \simeq \mathrm{Sh}(M; U) \otimes \mathrm{Sh}(N; V). \quad \square$$

Last, let us state the functor classification results. Recall that $\mathcal{C} \in \mathrm{Pr}_{\mathrm{st}}^L(\mathbf{k})$ is dualizable if it is a dualizable object with respect to the symmetric monoidal structure defined by the Lurie tensor product, its dual is denoted by \mathcal{C}^\vee . Then a general fact is that if \mathcal{C} is dualizable, then $\mathcal{C}^\vee \otimes \mathcal{D} \simeq \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$. Recall the following fact

Lemma 5.3 ([KSZ23, Remark 3.7]). *The category $\mathrm{Sh}_Z(M)$ is dualizable with dual $\mathrm{Sh}_{-Z}(M)$. The category $\mathrm{Sh}(M; U)$ is dualizable with dual $\mathrm{Sh}(M; -U)$.*

Consequently, we have

Proposition 5.4. *We have equivalences*

$$\begin{aligned} \mathrm{Sh}_{-Z \times X}(M \times N) &\simeq \mathrm{Fun}^L(\mathrm{Sh}_Z(M), \mathrm{Sh}_X(N)) \simeq \mathrm{Fun}^R(\mathrm{Sh}_X(N), \mathrm{Sh}_Z(M))^{op}, \\ \mathrm{Sh}(M \times N; -U \times V) &\simeq \mathrm{Fun}^L(\mathrm{Sh}(M; U), \mathrm{Sh}(N; V)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(N; V), \mathrm{Sh}(M; U))^{op}, \end{aligned}$$

where kernels are sent to corresponding functors (which makes sense by Proposition 2.5).

We remark here that the first line of Proposition 5.4 was proven in [KL22] even though the statements therein include isotropic conditions. Their proofs for corresponding results here are enough without isotropic conditions.

6. KASHIWARA-SCHAPIRA MICROLOCAL CUT-OFF FUNCTORS

In this section, we study the microlocal cut-off functors defined by Kashiwara and Schapira. We will follow notation and formulation of [Gui23, Chapter III].

We assume $M = V$ is a real vector space of dimension n and we naturally identify $T^*V = V \times V^*$. A subset $\gamma \subset V$ is a cone if $\mathbb{R}_{>0}\gamma \subset \gamma$, we say γ is pointed if $0 \in \gamma$. We set $\gamma^a = -\gamma$. We say a cone γ is convex/closed if it is a convex/closed set, and is proper if $\gamma \cap \gamma^a = \{0\}$ (equivalently, γ contains no line). We define the dual cone γ° as

$$\gamma^\circ = \{l \in V^* : l(v) \geq 0, \forall v \in \gamma\}.$$

We also set

$$\tilde{\gamma} = \{(x, y) \in V^2 : x - y \in \gamma\}.$$

For a pointed closed cone $\gamma \subset V$, we define 4 functors

$$(6.1) \quad \begin{aligned} P_\gamma &: \mathrm{Sh}(V) \rightarrow \mathrm{Sh}(V), & F &\mapsto q_{2*}(1_{\tilde{\gamma}} \otimes q_1^* F), \\ Q_\gamma &: \mathrm{Sh}(V) \rightarrow \mathrm{Sh}(V), & F &\mapsto q_{2!}(\mathcal{H}om(1_{\tilde{\gamma}^a}, q_1^! F)), \\ P'_\gamma &: \mathrm{Sh}(V) \rightarrow \mathrm{Sh}(V), & F &\mapsto q_{2!}(\mathcal{H}om(1_{\tilde{\gamma}^a \setminus \Delta_V}[1], q_1^! F)), \\ Q'_\gamma &: \mathrm{Sh}(V) \rightarrow \mathrm{Sh}(V), & F &\mapsto q_{2*}(1_{\tilde{\gamma} \setminus \Delta_V}[1] \otimes q_1^* F). \end{aligned}$$

For $\gamma = \{0\}$, we have $\widetilde{\{0\}} = \Delta_V$ and $P_{\{0\}}(F) \simeq Q_{\{0\}}(F) \simeq F$. Using the fiber sequence $1_{\tilde{\gamma}} \rightarrow 1_{\Delta_V} \rightarrow 1_{\tilde{\gamma} \setminus \Delta_V}[1]$ (and the same with γ^a), we obtain the fiber sequence of functors

$$(6.2) \quad \begin{aligned} P'_\gamma &\rightarrow \mathrm{id} \rightarrow Q_\gamma, \\ P_\gamma &\rightarrow \mathrm{id} \rightarrow Q'_\gamma. \end{aligned}$$

Proposition 6.1. *Let γ be a pointed closed convex cone. For $Z = V \times \gamma^{\circ a}$ and $U = T^*M \setminus Z$, we have the isomorphism of functors*

$$L_U = P'_\gamma, L_Z = Q_\gamma, R_Z = P_\gamma, R_U = Q'_\gamma,$$

and corresponding natural transformations.

Remark 6.2. In the originally definition of [KS90, Gui23], functors $P_\gamma, Q_\gamma, P'_\gamma, Q'_\gamma$ are defined as triangulated functors. Here, we use the same formula to define those functors on ∞ -level, so those functors automatically descent to the same triangulated functors in *loc.cit* by Remark 1.1.

Proof. It is explained in [Gui23, Remark III.1.9] that, for the inclusion $\iota : \mathrm{Sh}_{V \times \gamma^{\circ a}}(V) \rightarrow \mathrm{Sh}(V)$, we have that $Q_\gamma = \iota^*$ and $P_\gamma = \iota^!$; and the natural transform induced by inclusions $1_{\Delta_V} \rightarrow 1_{\tilde{\gamma} \setminus \Delta_V}[1]$ and $1_{\tilde{\gamma}^a} \rightarrow 1_{\Delta_V}$ are unit/counit of corresponding adjunctions. Then the result follows from the fiber sequence of functors (4.2), (6.2). \square

By the virtue of Section 4, existence of cut-off functor appears as a purely categorical result. However, the main advantage of Equation (6.1) is that the kernel is explicitly written. One may observe that the kernels in Equation (6.1) are not written as Proposition 1.2 in a standard way. This is not a problem, it is explained in [Gui23, III.1.5] how to write down the kernel of P'_γ and Q_γ in the standard way, such that the fiber sequence $K_U \rightarrow 1_{\Delta_V} \rightarrow K_Z$ is given by the fiber sequence

$$K_U = D'(1_{\tilde{\gamma}^a \setminus \Delta_V})[n-1] \rightarrow 1_{\Delta_V} \rightarrow K_Z = D'(1_{\tilde{\gamma}^a})[n],$$

where $D'(F) = \mathcal{H}om(F, 1_{V^2})$.

Example 6.3. We consider a variant situation. Let $M = V$ be a real vector spaces. Take a pointed closed convex proper cone $\gamma \subset V^*$, set $U = V \times \mathrm{Int}(\gamma^\circ)$ and $Z = V \times V^* \setminus \mathrm{Int}(\gamma^\circ)$. The microlocal cut-off functor for this U is studied by Tamarkin [Tam18] and Guillermou-Schapira [GS14] (where $L_U(R_U)$ is called $L_\gamma(R_\gamma)$ respectively.) In particular, when $V = \mathbb{R}$ and $\gamma = [0, \infty)$, corresponding functors are called Tamarkin projectors.

The fiber sequence of kernels could be given by

$$K_U = 1_{\tilde{\gamma}} \rightarrow 1_{\Delta_V} \rightarrow K_V = 1_{\tilde{\gamma} \setminus \Delta_V}[1].$$

By Proposition 6.1, if we assume there exists a pointed closed convex cone λ such that $V^* \setminus \mathrm{Int}(\gamma^\circ) = \lambda^{\circ a}$, we have

$$L_U = P'_\lambda = L_\gamma, \quad R_U = Q'_\lambda = R_\gamma.$$

Noticed that we can show by either computation of dual functor D' or abstract uniqueness of kernel that $K_U = 1_{\tilde{\gamma}} \simeq D'(1_{\tilde{\gamma}^a \setminus \Delta_V})[n-1]$.

7. NON-LINEAR MICROLOCAL CUT-OFF LEMMA

Recall that we say a morphism $f : F \rightarrow G$ in $\text{Sh}(M)$ is an isomorphism on an open set U if f is an isomorphism in $\text{Sh}(M; U)$ (cf. [KS90, Definition 6.1.1]). Because $\text{Sh}_Z(M)$ is a thick subcategory of $\text{Sh}(M)$, we know that f is an isomorphism on U if and only if $SS(\text{cofib}(f)) \cap U = \emptyset$.

Theorem 7.1. *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have*

- (1) (a) $F \rightarrow L_Z(F)$ and $R_Z(F) \rightarrow F$ are isomorphisms if and only if $SS(F) \subset Z$.
- (b) $L_U(F) \rightarrow F$ is an isomorphism if and only if $F \in {}^\perp \text{Sh}_Z(M)$, and $F \rightarrow R_U(F)$ is an isomorphism if and only if $F \in \text{Sh}_Z(M)^\perp$.
- (a) The morphisms $L_U(F) \rightarrow F$ and $F \rightarrow R_U(F)$ are isomorphisms on U .
- (b) If $0_M \subset Z$, the morphism $F \rightarrow L_Z(F)$ and $R_Z(F) \rightarrow F$ are isomorphisms on $\text{Int}(Z) \setminus 0_M$. Equivalently, we have $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$.

Proof. (1)-(a) and (2)-(a) follows from definition of L_Z and L_U and fiber sequences (4.2). (1)-(b) follows from [CDH⁺, A.2.8]. For (2)-(b), we will prove in Proposition 8.5 that $SS(K_U) \subset (-\overline{U}) \times \overline{U} \cup 0_{M \times M}$ under the condition $0_M \subset Z$. Therefore, $SS(L_U(F)) \cup SS(R_U(F)) \subset \overline{U} \cup 0_M$ follows from Proposition 2.5. \square

Remark 7.2. (1) By virtue of Proposition 5.1, this result generalize easily to the relative case. (2) Based on results of Section 6, (1)-(a), (2)-(a) is [KS90, Proposition 5.2.3-(i), Lemma 6.1.5-(i)], (1)-(b) is explained in [Gui23, Section III.5], and (2)-(b) is essentially [KS90, Proposition 5.2.3-(ii), Lemma 6.1.5-(ii)]. Also, by Example 6.3, the theorem also recover [GS14, Proposition 3.17, 3.19, 3.20, 3.21.].

A simple application of the non-linear cut-off lemma is the follow.

For conic closed sets $Z \subset X$, we consider the category $\frac{\text{Sh}_X(M)}{\text{Sh}_Z(M)}$ as a full subcategory of $\text{Sh}(M; U)$. Then we can characterize its essential image.

Proposition 7.3. *If $0_M \subset Z$, then the essential image of $\frac{\text{Sh}_X(M)}{\text{Sh}_Z(M)}$ in $\text{Sh}(M; U)$ is $\text{Sh}_{X \cap U}(M; U)$, the full subcategory spanned by objects with $SS_U(F) \subset X \cap U$.*

Proof. Obviously, $\frac{\text{Sh}_X(M)}{\text{Sh}_Z(M)}$ is included in $\text{Sh}_{X \cap U}(M; U)$. We prove that this inclusion is equal. To do so, we only need to find a representative G of $[F]$ such that $SS(G) \subset X$.

For $[F] \in \text{Sh}_{X \cap U}(M; U)$, we take $G = j_!([F])$ as a canonical representative of $[F]$ in $\text{Sh}(M)$. Therefore, we have

$$SS(G) = SS(L_U(G)) \subset \overline{U} \cup 0_M.$$

Moreover, we have

$$SS_U([F]) = SS(G) \cap U \subset X \cap U.$$

Then we have

$$SS(G) \subset X \cap U \cup \partial U \cup 0_M \subset X \cup Z = X. \quad \square$$

Remark 7.4. In the following, we will abuse the notation $\text{Sh}_{X \cap U}(M; U)$ as a definition of $\frac{\text{Sh}_X(M)}{\text{Sh}_Z(M)}$. Here, we prove that in the case $0_M \subset Z$, it does not make any confusion.

8. WRAPPING FORMULA OF NON-LINEAR MICROLOCAL CUT-OFF FUNCTORS

The goal of those section is present a explicit formula of microlocal kernels using Guillermou-Kashiwara-Schapira sheaf quantization [GKS12]. We recall the results of *loc.cit.* here.

Let \dot{T}^*M be the complement of the zero section in T^*M , and, for subset $A \subset T^*M$, we set $\dot{A} = A \cap \dot{T}^*M$. In particular, we have the notion of $\dot{S}S(F)$ for $F \in \text{Sh}(M)$.

Let $(I, 0)$ be a pointed interval. Consider a C^∞ conic symplectic isotopy

$$\phi : I \times \dot{T}^*M \rightarrow \dot{T}^*M$$

which is the identity at $0 \in I$. Such an isotopy is always the Hamiltonian flow for a unique conic function $H : I \times \dot{T}^*M \rightarrow \mathbb{R}$ and we set $\phi = \phi_H$ when emphasize the Hamiltonian functions. At fixed $z \in I$, we have the graph of ϕ_z :

$$(8.1) \quad \Lambda_{\phi_z} := \{((q, -p), \phi_z(q, p)) : (q, p) \in \dot{T}^*M\} \subset \dot{T}^*M \times \dot{T}^*M \subset \dot{T}^*(M \times M).$$

As for any of Hamiltonian isotopy, we may consider the Lagrangian graph, which by definition is a Lagrangian subset $\Lambda_\phi \subset T^*I \times \dot{T}^*(M \times M)$ with the property that $\Lambda_{\phi_{z_0}}$ is the symplectic reduction of Λ_ϕ along $\{z = z_0\}$. It is given by the formula:

$$(8.2) \quad \Lambda_\phi := \{(z, -H(z, \phi_z(q, p)), (q, -p), \phi_z(q, p)) : z \in I, (q, p) \in \dot{T}^*Y\}$$

Theorem 8.1 ([GKS12, Theorem 3.7, Prop. 4.8]). *For ϕ as above, there is a sheaf $K = K(\phi) \in \text{Sh}(I \times Y^2)$ such that $\dot{S}S(K) = \Lambda_\phi$ and $K|_{\{0\} \times Y^2} \cong 1_{\Delta_Y}$. The pair $(K, K|_{\{0\} \times Y^2} \cong 1_{\Delta_Y})$ is unique up to unique isomorphism.*

Moreover, for isotopies $\phi_H, \phi_{H'}$ with $H' \leq H$, there's a map $K(\phi_{H'}) \rightarrow K(\phi_H)$. In particular, when $H \geq 0$, then there is a map $1_{I \times \Delta_Y} \rightarrow K(\phi_H)$.

Similar to Proposition 2.5, one can see that

$$(8.3) \quad \dot{S}S(\Phi_{K(\phi)|_z}(F)) = \phi_z(\dot{S}S(F)).$$

Remark 8.2. In [GKS12], only $\dot{S}S(K) \subset \Lambda_\phi$ is proven. However, as [Gui23, Remark II.1.2] explained, the equality follows from coisotropiness of microsupport.

Motivated by ideas of [Nad16, GPS18], it was shown in [Kuo23] that for any closed set $Z^\infty \subset S^*M$ and the conic closed set $Z = \mathbb{R}_{>0}Z^\infty \cup 0_M \subset T^*M$, the adjoints of the inclusion $\iota : \text{Sh}_Z(M) \rightarrow \text{Sh}(M)$ can be computed ‘by wrapping’. More precisely,

Theorem 8.3 ([Kuo23, Thm. 1.2]). *If H_n is any increasing sequence of positive compactly supported Hamiltonians supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, then the adjoints of ι could be computed by*

$$(8.4) \quad \iota^*F = \varinjlim (K(\phi_{H_n})|_1 \circ F), \quad \iota^!F = \varprojlim (K(\phi_{H_n})|_1 \circ F).$$

Moreover, the unit of the adjoint is given by the map $1_{\Delta_M} \rightarrow \varinjlim K(\phi_{H_n})|_1$ which is induced by the continuation map $1_{\Delta_M} \rightarrow K(\phi_{H_n})|_1$ defined by positivity of H_n .

Remark 8.4. In fact, as we explained in [KSZ23, Remark 6.5], the colimit in [Kuo23] is taken over an ∞ -categorical ‘wrapping category’, but one can compute the colimit by a cofinal sequence as explained in [Kuo23, Lemma 3.31].

Compose with the natural inclusion ι and combine with [Kuo23, Proposition 3.5], if we set $K^\circ(\phi_{H_n})|_1 = \text{cofib}(1_{\Delta_M} \rightarrow K(\phi_{H_n})|_1)$, the fiber sequence (4.3) can be taken as

$$K_U = \varinjlim (K^\circ(\phi_{H_n})|_1) \rightarrow 1_{\Delta_M} \rightarrow K_Z = \varinjlim (K(\phi_{H_n})|_1).$$

To complete the proof of Theorem 7.1 (2)-(b), we start from a microsupport estimation of $K^\circ((\phi_H)|_1)$. We also notice that the requirement of $0_M \subset Z$ comes from the wrapping formula.

Proposition 8.5. *For a Hamiltonian function H supported in $U^\infty = U/\mathbb{R}_{>0}$, we have*

$$\dot{S}S(K^\circ(\phi_H)|_1) \subset -U \times U.$$

In particular, we have

$$\dot{S}S(K_U) \subset -\overline{U} \times \overline{U}.$$

Proof. By the triangulated inequality, we have that

$$\dot{S}S(K^\circ(\phi_H)|_1) \subset \Lambda_{\phi_1} \cup \Lambda_{\text{id}} \setminus \Lambda_{\phi_1} \cap \Lambda_{\text{id}} \subset -U \times U.$$

The second statement follows from Corollary 2.3. \square

Proof of Theorem 7.1 (2)-(b). The estimation $\dot{S}S(L_U(F)) \cup \dot{S}S(R_U(F)) \subset \overline{U} \cup 0_M$ follows from Proposition 2.5, and Proposition 8.5. \square

9. RESULTS FOR PAIRS

In this section, we take consider the following construction: Take two conic closed set $Z \subset X \subset T^*M$. Let $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, then $V \subset U$.

Recall that $\text{Sh}_{X \cap U}(M; U) = \frac{\text{Sh}_X(M)}{\text{Sh}_Z(M)}$. We consider the following 9-diagram of \mathbf{k} -linear categories:

$$(9.1) \quad \begin{array}{ccccc} \text{Sh}_Z(M) & \hookrightarrow & \text{Sh}_X(M) & \twoheadrightarrow & \text{Sh}_{X \cap U}(M; U) \\ \parallel & & \downarrow & & \downarrow \\ \text{Sh}_Z(M) & \hookrightarrow & \text{Sh}(M) & \twoheadrightarrow & \text{Sh}(M; U) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \hookrightarrow & \text{Sh}(M; V) & \xrightarrow{\simeq} & \text{Sh}(M; U) / \text{Sh}_{X \cap U}(M; U) \end{array}$$

In this diagram, by the same argument with Proposition 4.1, we have that all vertical and horizontal sequences are split Verdier sequences in $\text{Pr}_{\text{st}}^L(\mathbf{k})$. We say a morphism $f : F \rightarrow G$ in $\text{Sh}(M; U)$ is an isomorphism on an open set $V \subset U$ if f is an isomorphism in $\text{Sh}(M; V)$. Because $\text{Sh}_{X \cap U}(M; U)$ is a thick subcategory of $\text{Sh}(M; U)$, we know that f is an isomorphism on V if and only if $SS_U(\text{cofib}(f)) \cap V = \emptyset$.

Here, we write down adjoint functors in the last column precisely

$$(9.2) \quad \begin{array}{ccc} & \begin{array}{c} \xleftarrow{\iota^*} \\ \xrightarrow{\iota^!} \end{array} & \\ \text{Sh}_{X \cap U}(M; V) & \xrightarrow{\iota} & \text{Sh}(M; U) \\ & \begin{array}{c} \xleftarrow{\iota^!} \\ \xrightarrow{\iota^*} \end{array} & \\ & \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \end{array} & \\ & \text{Sh}(M; V) & \end{array}$$

Then we have adjunction pairs

$$L_{(U,V)} := j_! j^* \dashv j_* j^* =: R_{(U,V)}, \quad L_{(X,Z)} := \iota^* \dashv \iota^! =: R_{(X,Z)},$$

and the unit/counits give us following fiber sequences of functors on $\text{Sh}(M; U)$

$$(9.3) \quad \begin{array}{c} L_{(U,V)} \rightarrow \text{id} \rightarrow L_{(X,Z)}, \\ R_{(X,Z)} \rightarrow \text{id} \rightarrow R_{(U,V)}. \end{array}$$

Therefore, we can state the following version of cut-off lemma

Theorem 9.1. *For conic closed sets $Z \subset X \subset T^*M$, and $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, we have*

- (1) (a) $F \rightarrow L_{(X,Z)}(F)$ and $R_{(X,Z)}(F) \rightarrow F$ are isomorphisms if and only if $SS_U(F) \subset X \cap U$.
- (b) $L_{(U,V)}(F) \rightarrow F$ is an isomorphism if and only if $F \in {}^\perp \text{Sh}_{X \cap U}(M; U)$, and $F \rightarrow R_{(U,V)}(F)$ is an isomorphism if and only if $F \in \text{Sh}_{X \cap U}(M; U)^\perp$.
- (a) The morphisms $L_{(U,V)}(F) \rightarrow F$ and $F \rightarrow R_{(U,V)}(F)$ are isomorphisms on $U \cap V$.
- (b) If $0_M \subset Z$, the morphism $F \rightarrow L_{(X,Z)}(F)$ and $R_{(X,Z)}(F) \rightarrow F$ are isomorphisms on $\text{Int}(X) \cap U$. Equivalently, we have $SS_U(L_{(U,V)}(F)) \cup SS_U(R_{(U,V)}(F)) \subset \bar{V} \cap U$. Here \bar{V} denotes the closure of V in T^*M .

Proof. Here, we noticed that $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for $[F] \in \text{Sh}(M; U)$. Then we apply the absolute version Theorem 7.1 to conclude. \square

Example 9.2. We consider the manifold $M \times \mathbb{R}_t$. We still take an closed set $Z \subset J^1 M = T^*M \times \mathbb{R}$ and $U = J^1 M \setminus Z$. (NOT necessarily conic!)

In stead, we consider their conification in $T^*(M \times \mathbb{R}_t) = T^*M \times \mathbb{R}_t \times \mathbb{R}_\tau$. We set

$$(9.4) \quad \begin{aligned} \Omega_{>0} &= \{\tau > 0\}, & \Omega_- &= \{\tau \leq 0\}. \\ \widetilde{U} &= \{(q, \tau p, t, \tau) : (q, p, t) \in U, \tau > 0\}, \\ \widetilde{Z} &= \{(q, \tau p, t, \tau) : (q, p, t) \in Z, \tau > 0\}, \\ \widetilde{Z}_- &= \widetilde{Z} \cup \Omega_-. \end{aligned}$$

Then $\Omega_- \subset \widetilde{Z}_-$ are two closed conic sets. In this case, the category

$$\mathrm{Sh}(M \times \mathbb{R}_t; \widetilde{U}) \simeq \mathrm{Sh}(M \times \mathbb{R}_t; \Omega_{>0}) \left/ \frac{\mathrm{Sh}_{\widetilde{Z}}(M \times \mathbb{R}_t)}{\mathrm{Sh}_{\Omega_-}(M \times \mathbb{R}_t)} \right.$$

is useful for contact topology of J^1M , since $\mathbb{R}_{>0}$ acts on $\Omega_{>0}$ freely, with the quotient $J^1M = T^*M \times \mathbb{R}_t$.

If we take $U = U_0 \times \mathbb{R}$ for an open set $U_0 \subset T^*M$ and $Z = Z_0 \times \mathbb{R}$ for an open set $U_0 \subset T^*M$, the category $\mathrm{Sh}(M \times \mathbb{R}_t; \widetilde{U})$ is exactly the so-called Tamarkin categories $\mathcal{T}(U_0)$ (cf. [KSZ23]). Tamarkin categories are first defined in [Tam18], and then are studied in [Vic13, Chi17, Ike19, AI20a, Zha20, AI22, GV22a, AGH⁺23, AIL23]. In this case, U admit a \mathbb{R} -action by translation along \mathbb{R}_t . Therefore, there exist an $\mathcal{T} = \mathcal{T}(\mathrm{pt})$ (which is an symmetric monoidal category [GS14, KSZ23]) action on $\mathcal{T}(U_0)$, which helps us understanding the action filtration from symplectic geometry. And functors

$$L_{U_0}^{\mathcal{T}} := L_{(\Omega_{>0}, \widetilde{U_0 \times \mathbb{R}})}, L_{Z_0}^{\mathcal{T}} := L_{(\widetilde{Z_0 \times \mathbb{R}}, \Omega_-)}$$

are actually \mathcal{T} -linear. By the \mathcal{T} -linear left adjoint functors classification (see [KSZ23, Proposition 5.12]), there exists a fiber sequences $K_{U_0}^{\mathcal{T}} \rightarrow 1_{\Delta_M}^{\mathcal{T}} \rightarrow K_{Z_0}^{\mathcal{T}}$ in $\mathcal{T}(T^*(M \times M)) = \mathrm{Sh}(M \times M; \mathcal{T})$ with

$$L_{U_0}^{\mathcal{T}} = \Phi_{K_{U_0}^{\mathcal{T}}}^{\mathcal{T}}, L_{Z_0}^{\mathcal{T}} = \Phi_{K_{Z_0}^{\mathcal{T}}}^{\mathcal{T}},$$

where $\Phi^{\mathcal{T}}$ means the convolution functor defined using \mathcal{T} -linear 6-operators. These functors are studied in [Tam15, Chi17, Zha21, Zha23, KSZ23], where corresponding kernels are denoted by $K_{U_0}^{\mathcal{T}} = P_{U_0}, K_{Z_0}^{\mathcal{T}} = Q_{U_0}$. In particular, we are free to use Theorem 9.1 for Tamarkin categories.

If the open set $U \subset J^1M$ admits a \mathbb{G} -action for a subgroup $\mathbb{G} \subset \mathbb{R}$ via translation, Asano, Ike and Kuwagaki introduce an equivariant version of sheaves to study corresponding categories, see [AI20b, IK23]. If \mathbb{G} is a discrete subgroup of \mathbb{R} , which means $\mathbb{G} \simeq \mathbb{Z}$, the equivariant version can be understood by the framework of this article by identifying $\mathbb{R}/\mathbb{G} = S^1$ and considering the category $\mathrm{Sh}(M \times S^1; \Omega_+)$ (which still makes sense.) However, if $\mathbb{G} = \mathbb{R}_a$ as a discrete group of the topological group \mathbb{R}_t , the situation becomes more complicated and we will not discuss related categories and cut-off lemma (i.e. Theorem 7.1) in this article (while we still believe it is true.)

10. SPLITTING

In this section, we always assume $0_M \subset Z$.

Lemma 10.1. *Suppose $a : A \rightarrow B$ and $c : C \rightarrow B$ are two arrows in a stable category, then we have*

$$\mathrm{cofib}((a, c)) \simeq \mathrm{cofib}([A \xrightarrow{a} B \rightarrow \mathrm{cofib}(c)]).$$

Proof. We set $w : A \xrightarrow{a} B \rightarrow \text{cofib}(c)$. Consider the following commutative diagram

$$\begin{array}{ccccc}
 C & \longrightarrow & A \oplus C & \xrightarrow{(a,c)} & B \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \xrightarrow{w} & \text{cofib}(c) \\
 & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \text{cofib}((a, c)).
 \end{array}$$

Then one can check that all squares in the diagram are pushout. Then the result follows. \square

Proposition 10.2. *Assume $\dot{S}S(L_Z(F)) \cup \dot{S}S(R_Z(F)) \subset \text{Int}(Z) \setminus 0_M$, then we have*

$$L_U(F) \oplus R_Z(F) \rightarrow F$$

*is isomorphic on \dot{T}^*M , and*

Proof. We apply the lemma to $a : L_U(F) \rightarrow F$ and $c : R_Z(F) \rightarrow F$. Then the cofiber of (a, c) is isomorphic to the cofiber of $w : R_Z(F) \rightarrow F \rightarrow L_Z(F)$, which is denoted by L . Therefore, we only need to check that $\dot{S}S(L) \subset 0_M$.

For $G = L_Z(F)$ or $G = R_Z(F)$, by the condition, we have

$$\dot{S}S(G) = \dot{S}S(F) \cap \text{Int}(Z) \subset \text{Int}(Z) \setminus 0_M$$

by Theorem 7.1 (2)-(b). It implies that $\dot{S}S(L) \subset \text{Int}(Z) \setminus 0_M$, but f is an isomorphism on $\text{Int}(Z) \setminus 0_M$. Then we have $\dot{S}S(L) \cap \text{Int}(Z) = \emptyset$. Therefore, we have $\dot{S}S(L) = \emptyset$. \square

Since $\dot{S}S(L_Z(F)) \cup \dot{S}S(R_Z(F)) \subset Z \setminus 0_M$ is always true by Theorem 7.1 (1)-(a), the condition only means that $[\dot{S}S(L_Z(F)) \cup \dot{S}S(R_Z(F))] \cap \partial Z = \emptyset$. In practise, we need find certain geometric criterion of this assumption. For example, if $Z = V \times \gamma^{oa}$ for a convex cone γ , [Gui23, Proposition III.1.10] gives a useful sufficient conditions (actually a better one than what we present here). And our proof here is just a formal repeat of the proof therein.

To achieve the target, we often

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