Sheaves Techniques and Symplectic Geometry

On Algebra and Geometry of the Chiu-Tamarkin Complex

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Résumé. Dans cette thèse, nous étudions la structure algébrique et la géométrie du complexe de Chiu-Tamarkin, qui est un outil pour étudier la géométrie symplectique à l'aide de la théorie microlocale des faisceaux. Les principaux résultats de la thèse sont organisés en deux parties : structure et calcul.

Pour la partie structure, nous rappelons d'abord le complexe de Chiu-Tamarkin $\mathbb{Z}/\ell$-équivariant $C_{\ell,T}(U,\mathbb{K})$. Nous exposons une variante du produit de Yoneda. Il généralise le cup-produit habituel sur l'anneau de cohomologie d'une variété et le produit de Chas-Sullivan sur la topologie des cordes. Ensuite, nous définissons le complexe de Chiu-Tamarkin S^1-équivariant $C_T^{S^1}(U,\mathbb{K})$ en utilisant une structure cyclique sur le noyau microlocal. Enfin, nous construisons différentes capacités symplectiques associées à différentes versions du complexe de Chiu-Tamarkin. En particulier, la structure cyclique explique pourquoi nous avons besoin de la version $\mathbb{Z}/\ell$ pour la preuve du théorème de non-squeezing de contact.

Pour la partie calcul, nous présentons les calculs du complexe de Chiu-Tamarkin pour les domaines toriques convexes et les fibrés en disques unitaires. Pour les domaines toriques convexes, nous démontrons un théorème de structure qui nous aide à calculer les capacités. Le calcul implique que nos capacités sont les mêmes que les capacités de Gutt-Hutchings pour les domaines toriques convexes. Pour les fibrés en disques unitaires, nous prouvons un isomorphisme de Viterbo, qui est un isomorphisme d'algèbres entre le complexe de Chiu-Tamarkin du fibré de disques unitaires et la topologie des cordes de la base.

Mot-clés : Faisceaux microlocal, Complexe de Chiu-Tamarkin, Noyau microlocal, Capacités symplectiques, Théorème de non-écrasement, Topologie des cordes.

Classification MSC : 35A27, 53D35, 53D25, 57R17, 55N30, 55N31, 55N35, 55N45, 55N91, 55P48, 55P50.

Abstract. In this thesis, we study the algebra structure and geometry of the Chiu-Tamarkin complex, which is a tool to study symplectic geometry using the microlocal theory of sheaves. The main results of the thesis are organized into two parts: structure and computation.

On the structure part, we first review the $\mathbb{Z}/\ell$-equivariant Chiu-Tamarkin complex $C_{\ell,T}(U,\mathbb{K})$. We exhibit a variant of the Yoneda product. It generalizes the usual cup product on the cohomology ring of a manifold and the Chas-Sullivan product on the string topology. Next, we define the S^1-equivariant Chiu-Tamarkin complex $C_T^{S^1}(U,\mathbb{K})$ using a cyclic structure on the microlocal kernel. Finally, we construct different symplectic capacities associated with different versions of the Chiu-Tamarkin complex. In particular, the cyclic structure explains why we need $\mathbb{Z}/\ell$-version on the proof of the contact non-squeezing theorem.

On the computational part, we present the computations of the Chiu-Tamarkin complex for convex toric domains and unit disk bundles. For convex toric domains, we demonstrate a structure theorem that helps us to compute the capacities. The computation implies that our capacities are the same as the Gutt-Hutchings capacities for convex toric domains. For the unit disk bundle, we prove a Viterbo isomorphism, which is an algebra isomorphism between the Chiu-Tamarkin complex of the unit disk bundle and the string topology of the base.

Keywords: Microlocal sheaf, Chiu-Tamarkin complex, Microlocal kernel, Symplectic capacities, Non-squeezing theorem, String topology.

MSC classification: 35A27, 53D35, 53D25, 57R17, 55N30, 55N31, 55N35, 55N45, 55N91, 55P48, 55P50.

Let us be, first and above all, kind, then honest and then let us never forget each other!

—Fyodor Dostoevsky, The Brothers Karamazov

And stay curious.

Contents

Introduction (en Français)

1. Faisceaux microlocaux: ancien et nouveau.

L'analyse algébrique a été introduite par Sato dans les années 60. L'idée principale est d'utiliser des outils algébriques comme les faisceaux et les catégories pour étudier des problèmes d'analyse, notamment les équations aux dérivées partielles. En illustration de la philosophie de Sato, Kashiwara et Schapira ont introduit et développé la théorie microlocale des faisceaux dans [**KS82**, **KS83a**, **KS83b**, **KS90**]. En particulier, les applications de la théorie microlocale des faisceaux aux $\mathcal{D}-$modules nous présentent la puissance de l'analyse algébrique.

Les principales notions de la théorie des faisceaux microlocaux sont la microlocalisation de Sato et la notion de microsupport. Dans cette thèse, nous nous concentrerons principalement sur les applications de la notion de microsupport. Le microsupport $SS(F)$ détecte l'extensibilité locale des sections d'un faisceau F. On montre que, sur une variété complexe X, pour un $\mathcal{D}_X$-module cohérent $\mathcal{M}$, le microsupport $SS(Sol(\mathcal{M}))$ est le même que la variété caractéristique du $\mathcal{D}_X$-module $\mathcal{M}$. Précisément, un covecteur est dans le microsupport du faisceau de solutions $Sol(\mathcal{M})$ si les solutions locales près du covecteur peuvent être étendues près de la codirection, ce qui signifie exactement que la codirection est dans la variété caractéristique de $\mathcal{M}$. Nous pouvons également formuler l'extensibilité globale à l'aide du lemme de Morse microlocal (voir chapitre 1, Corollary 1.6). Un des points forts de la théorie est la nouvelle preuve du fait que les variétés caractéristiques d'un $\mathcal{D}$-module cohérent sont co-isotropes en utilisant le théorème de co-isotropie de Kashiwara-Schapira. D'autre part, il est prouvé dans [**KS90**] que $SS(F)$ est un sous-ensemble conique fermé et coisotrope de T^*X. Lorsque X est réellement analytique, $SS(F)$ est lagrangienne si et seulement si F est (faiblement) constructible.

Ce résultat illustre le rôle de la théorie des faisceaux microlocaux dans la géométrie symplectique du faisceau cotangent T^*X. En tant que nouvelle direction de la géométrie symplectique, les techniques de faisceaux sont en plein essor ces dernières années.

Faisceau	Géométrie
Les faisceaux constructibles sur X	Lagrangienne conique dans T^*X
Les faisceaux constructibles sur $X \times X$	Les correspondances lagrangiennes entre T^*X
Convolution des faisceaux	Composition des correspondances
Extension de zéro et restriction	Réduction Symplectique
Guillermou-Kashiwara-Schapira quantification[**GKS12**]	Action du groupe hamiltonien
Guillermou quantification[**Gui12**]/ Viterbo quantification[**Vit19**]	lagrangienne exact dans T^*X

TABLE 1. Une correspondance entre les faisceaux et la géométrie.

Commençons par les travaux de Tamarkin. Dans [**Tam18**], Tamarkin a développé la notion de catégorie de Tamarkin $\mathcal{D}(X)$. La catégorie de Tamarkin $\mathcal{D}(X)$ est un quotient de $D(X \times \mathbb{R}_t)$ par des faisceaux microsupportés négativement le long de t, et est isomorphe à une sous-catégorie triangulée complète de $D(X \times \mathbb{R}_t)$. Nous pouvons donc considérer ses objets comme des faisceaux sur $X \times \mathbb{R}_t$.

Le rôle de la variable t est double:

• Le microsupport $SS(F)$ est conique sous la dilatation des fibres cotangentes. Mais la plupart des problèmes en géométrie symplectique sur les fibrés cotangents ne sont pas coniques. Tamarkin suggère donc d'utiliser l'application de "déconification" $\rho(\mathbf{q}, \mathbf{p}, t, \tau) = (\mathbf{q}, \mathbf{p}/\tau)$.

Donc pour $A \subset T^*X$, nous avons un sous-ensemble conique $\rho^{-1}(A) \subset T^*X \times T^*_{\tau>0}\mathbb{R}_t$. De plus, si $F \in \mathcal{D}(X)$, on a automatiquement $SS(F) \subset \{\tau \geq 0\}$, et en pratique, on a souvent $SS(F) \cap \{\tau \leq 0\} \subset 0_{X\times\mathbb{R}}$. Par conséquent, une version du lemme de Morse microlocal montre que nous ne perdrons pas beaucoup d'information.

• D'autre part, l'application ρ se factorise par l'application de symplectisation q du fibré de 1-jets J^1X de manière tautologique :

$$
\begin{tikzcd}
T^*X \times T^*_{\tau>0}\mathbb{R}_t \arrow[r, "q"] \arrow[rr, bend right, "\rho"'] & J^1(X) = T^*X \times \mathbb{R}_t \arrow[r] & T^*X.
\end{tikzcd}
$$

Ainsi, la conicité provient en fait du processus de symplectisation, et la variable supplémentaire t, joue en fait le rôle d'action pour les lagrangiennes dans T^*X.

Maintenant, considérons les applications de translation

$$\mathrm{T}_c : X \times \mathbb{R}_t \to X \times \mathbb{R}_t,\ (\mathbf{q}, t) \mapsto (\mathbf{q}, t + c).$$

Microlocalement, les foncteurs T_{c*} quantifient le flot de Reeb $d\mathrm{T}_c$ de la forme de contact canonique $\alpha = dt + \mathbf{p}d\mathbf{q}$.

Il est crucial que, sur la catégorie de Tamarkin, nous ayons une transformation naturelle $\tau_c : \mathrm{Id} \Rightarrow \mathrm{T}_{c*}$ pour $c \geq 0$. La transformation naturelle n'existe pas dans $D(X \times \mathbb{R}_t)$ (mais elle existe dans une catégorie plus grande $D_{\tau \geq 0}(X \times \mathbb{R}_t)$, voir [**GS14**] pour plus de détails). Cette transformation naturelle peut nous aider à voir des barres finies dans le code-barres de $\mathrm{R}\pi_{X!}F$ pour $F \in \mathcal{D}(X)$.

Ainsi, au lieu de dire que les objets de la catégorie de Tamarkin quantifient des lagrangiennes dans des fibrés cotangents, il est préférable de dire que les objets de la catégorie Tamarkin quantifient des legendriennes dans des espaces de 1-jets.

A partir de ce point de vue, Tamarkin développe une nouvelle méthode pour étudier la déplacibilité dans [**Tam18**]. Les travaux de Tamarkin sont très influents. Asano et Ike ont développé la distance de persistance d'un point de vue quantitatif à partir des travaux de Tamarkin et ont commencé la recherche d'invariants numériques des faisceaux avec des applications sur l'énergie de déplacement symplectique, les immersions lagrangiennes rationnelles et la géométrie symplectique C^0 en [**AI20a, AI20b, AI22**]. Du côté des catégories, Biran, Cornea et Zhang ont développé la notion de catégorie de persistance triangulée dans [**BCZ21**], qui abstrait les structures catégoriques de la catégorie de Tamarkin.

En outre, il existe de nombreux travaux sur la géométrie symplectique qui sont basés sur la théorie des faisceaux microlocaux. Guillermou donne des preuves de la rigidité

C^0 de Gromov-Eliashberg, de la conjecture des 3 cuspides et du résultat d'Abouzaid et Kragh selon lequel les lagrangiennes exacts fermés dans les faisceaux cotangents sont homotopiquement équivalents à la section nulle (Voir [**Gui12**, **Gui13**, **Gui16**] et le mémoire [**Gui19**] sur ces sujets). Ike estime les intersections lagrangiennes exactes dans les fibrés cotangents (voir [**Ike19**]), et Li estime les cordes de Reeb dans les espaces des 1-jets (voir [**Li21a**]). Dans [**CG22**], Casals et Gao construisent une infinité de remplissages lagrangiens pour certains nœuds de tores legendriens en utilisant les espaces de modules des faisceaux comme invariants.

D'autre part, de nombreux travaux étudient la catégorie des faisceaux du point de vue de la catégorie de Fukaya. Cela commence par les travaux de Bondal-Ruan [**BdR**]. Voir également les travaux de Nadler et Zaslow sur la catégorie de Fukaya compacte [**NZ09**, **Nad09**], de Nadler [**Nad16**], et de Ganatra, Pardon, et Shende sur la catégorie de Fukaya enroulée [**GPS18a**].

2. Théorème de non-plongement de contact.

Le célèbre non plongement de Gromov a ouvert la porte à la géométrie symplectique moderne. Mais la correspondance de contact n'a pas été discutée avant les travaux pionniers d'Eilashberg-Kim-Poltervich : [**EKP06**].

Une tentative naïve est d'établir le problème de non plongement du contact dans le fibré des 1-jets $J^1\mathbb{R}^d = T^*\mathbb{R}^d \times \mathbb{R}_t$ équipé de la forme de contact $\alpha = dt + \mathbf{p}d\mathbf{q}$. Mais l'application de changement d'échelle $(\mathbf{q}, \mathbf{p}, z) \mapsto (r\mathbf{q}, r\mathbf{p}, r^2 t)$, qui est un contactomorphisme, envoie tout ensemble compact dans un petit voisinage arbitraire de l'origine lorsque r est suffisamment grand. Cette naturalité conforme de l'espace de 1-jets illustre qu'il est préférable d'étudier l'espace préquantifié $T^*\mathbb{R}^d \times S^1$, où S^1 est un cercle, équipé d'une forme de contact $\alpha = d\theta + \frac{1}{2}(\mathbf{q}d\mathbf{p} - \mathbf{p}d\mathbf{q})$. Mais il existe un contactomorphisme global $F_N : T^*\mathbb{R}^d \times S^1 \to T^*\mathbb{R}^d \times S^1$ défini comme suit : On utilise les coordonnées complexes $T^*\mathbb{R}^d \cong \mathbb{C}^d$, puis $F_N(z, \theta) \coloneqq (\nu(\theta)e^{2\pi N\theta}z, \theta)$, où $\nu(\theta) = (1 + N\pi|z|^2)^{-1/2}$. On peut calculer directement que F_N envoie toute boule dans un voisinage arbitrairement petit de $\{0\} \times S^1$ pour N assez grand. Cependant, nous remarquons que F_N n'est pas à support compact.

Une meilleure définition du plongement de contact est donc la suivante, proposée dans loc. cit.

DÉFINITION. [**EKP06**, p1636] Soit (V, α) une variété de contact. Soit $U_1, U_2 \subset V$ deux sous-ensembles ouverts. On dit que U_1 est plongeable dans U_2 s'il existe une isotopie de contact à support compact $\varphi_s : \overline{U_1} \to V$, $s \in [0,1]$, telle que $\varphi_0 = \mathrm{Id}$, et $\varphi_1(\overline{U_1}) \subset U_2$.

Un phénomène intéressant, qui n'apparaît pas dans la situation symplectique, est que la taille de la boule affecte la validité du plongement. Deux résultats concernant le plongement et le non plongement des boules de contact $B_{\pi R^2} \times S^1$ sont :

THÉORÈM. *(1)* [**EKP06**, Theorem 1.3] *Supposons que $d \geq 2$. Alors pour tout $0 < \pi r^2, \pi R^2 < 1$, on peut plonger la boule de contact $B_{\pi R^2} \times S^1$ dans $B_{\pi r^2} \times S^1$ quelle que soit la relation entre r et R.*

(2) [**EKP06**, Theorem 1.2] *S'il existe un entier $m \in [\pi r^2, \pi R^2]$, alors $B_{\pi R^2} \times S^1$ ne peut pas être plongé dans $B_{\pi r^2} \times S^1$.*

En ce qui concerne le phénomène à grande échelle, Eliashberg, Kim et Polterovich donnent une très bonne explication physique en utilisant le processus de quantification. Le seul cas restant concernant le non plongement de contact est le suivant : que se passe-t-il s'il existe un entier m tel que $m < \pi r^2 < \pi R^2 < m + 1$? Il est résolu par Chiu à l'aide de la théorie microlocale des faisceaux [**Chi17**], et par Fraser à l'aide de la technique des courbes J-holomorphes [**Fra16**] dans l'esprit de [**EKP06**]. Ils ont prouvé ce qui suit :

THÉORÈM ([**Chi17, Fra16**]). *Si $1 \leq \pi r^2 < \pi R^2$, alors $B_{\pi R^2} \times S^1$ ne peut pas être plongé dans $B_{\pi r^2} \times S^1$.*

L'outil de la preuve de Chiu est le complexe de Chiu-Tamarkin, qui est aussi notre principal sujet d'intérêt dans cette thèse. Rappelons la preuve de Chiu.

Pour la boule de contact $B_A \times S^1 \subset T^*\mathbb{R}^d \times S^1$, nous relevons d'abord $B_A \times S^1$ en un ensemble ouvert équivariant $B_A \times \mathbb{R} \subset J^1\mathbb{R}^d$. Nous pouvons alors étudier les deux catégories de faisceaux :

$$\mathcal{D}^c_{J^1\mathbb{R}^d\setminus B_A\times\mathbb{R}}(X) = \{F \in \mathcal{D}(X) : q(SS(F)) \subset J^1\mathbb{R}^d \setminus B_A \times \mathbb{R}\},$$
$$\mathcal{D}^c_{B_A\times\mathbb{R}}(X) = {}^\perp\mathcal{D}^c_{J^1\mathbb{R}^d\setminus B_A\times\mathbb{R}}(X), \text{ le complément orthogonal gauche de } \mathcal{D}^c_{J^1\mathbb{R}^d\setminus B_A\times\mathbb{R}}(X).$$

Chiu associe deux faisceaux $\mathscr{P}_{B_A\times S^1}, \mathscr{Q}_{B_A\times S^1} \in D(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^2)$ à un triangle distingué

$$\mathscr{P}_{B_A\times S^1} \to \mathbb{K}_{\Delta_{X^2}\times\{t_2\geq t_1\}} \to \mathscr{Q}_{B_A\times S^1} \xrightarrow{+1} .$$

On appelle $\mathscr{P}_{B_A\times S^1}, \mathscr{Q}_{B_A\times S^1}$ les noyaux microlocaux associés à $B_A \times S^1$.

En utilisant le noyau microlocal $\mathscr{P}_{B_A\times S^1}$, Chiu a introduit le complexe de Chiu-Tamarkin $\mathscr{C}_{\ell,n\ell}(B_A\times S^1, \mathbb{K}) \in D_{\mathbb{Z}/\ell}(\mathrm{pt})$, et une classe fondamentale $\eta^c_{\ell,n\ell}(B_A\times S^1, \mathbb{K}) \in H^0\mathscr{C}_{\ell,n\ell}(B_A\times S^1, \mathbb{K})$ qui est définie par le morphisme naturel $\mathscr{P}_{B_A\times S^1} \to \mathbb{K}_{\Delta_{X^2}\times\{t_2\geq t_1\}}$. Même si Chiu n'a pas mentionné l'existence de la classe fondamentale explicitement, elle apparaît déjà dans sa preuve du non-plongement de contact sous une forme différente.

Chiu a prouvé que $\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ et $\eta^c_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ sont naturels par rapport à l'inclusion $B_a \times S^1 \subset B_A \times S^1$, et invariants par contactomorphismes à support compact de $T^*\mathbb{R}^d \times S^1$. Par conséquent, $\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$, ou plus précisément $\eta^c_{\ell,n\ell}(B_A\times S^1, \mathbb{K})$, sont des obstructions pour le plongement de contact.

Un autre ingrédient est que $H^*\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ est un module sur $\mathbb{K}[u]$ si $car(\mathbb{K})|\ell$. Dans son article, Chiu utilise l'action $\mathbb{K}[u]$ sur $H^*\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ pour décomposer la classe fondamentale. Enfin, le résultat découle d'une comparaison attentive du degré de la classe fondamentale.

Géométriquement, une corde de Reeb de ∂B_A définit une corde de Reeb de $T^*\mathbb{R}^d \times S^1$, et de même qu'une corde de Reeb $\mathbb{Z}$-équivariante de $J^1\mathbb{R}^d$. Les actions A des cordes de Reeb sont des invariants numériques utiles pour le problème du non-plongement. Dans le cas symplectique, il est suffisant de définir l'invariance spectrale en utilisant uniquement les actions. Mais l'information sur l'indice des orbites de Reeb est perdue. Dans le cas de contact, un candidat naturel est la théorie S^1-équivariante, où l'information sur l'indice est fournie en utilisant l'action $\mathbb{K}[u]$. Mais dans ce cas, l'action est coupée en morceaux infinitésimaux comme $0 = A/\infty$, de sorte que nous ne pouvons pas lire les informations numériques efficaces pour le prolongement de contact.

La caractéristique principale de toutes les preuves du théorème du contact non plongement utilise une théorie $\mathbb{Z}/\ell$-équivariante, par exemple, le complexe de Chiu-Tamarkin. Pour le complexe de Chiu-Tamarkin des boules, l'idée la plus cruciale est que l'indice de l'orbite de Reeb est lié à son action. De plus, l'action de $\mathbb{Z}/\ell$ nous aide à trouver non seulement une orbite fermée mais aussi des points ℓ-périodiques du flot de Reeb. Les points périodiques peuvent nous aider à obtenir une action non nulle A/ℓ, qui est la longueur d'une corde de Reeb entre deux points ℓ-périodiques. De plus, l'indice est réalisé par l'action de $\mathbb{K}[u]$, ce qui nous aide à distinguer certaines orbites de Reeb. Ces deux caractéristiques peuvent nous aider à utiliser à la fois l'indice et l'action pour obtenir une obstruction pour le non-plongement de contact.

3. Aperçu de la thèse et principaux résultats

Dans cette thèse, nous voudrions comprendre systématiquement l'algèbre et la géométrie du complexe de Chiu-Tamarkin. Comme la preuve de Chiu pour le théorème de non-plongement de contact est convaincante mais mystèrieuse, on pense que de nombreuses structures algébriques se cachent derrière la définition du complexe de Chiu-Tamarkin. Le contenu de la thèse est une combinaison de l'article de l'auteur [**Zha21**] et de certains travaux en préparation. Pour être compréhensible, nous fournissons quelques préliminaires aux lecteurs.

La thèse est organisée en 3 points de vue : structures algébriques, informations numériques, calculs. Ils sont reliés les uns aux autres. Les structures algébriques nous aident à extraire des informations numériques significatives, et les calculs nous aident à revoir et vérifier nos structures.

Comme Chiu, nous partons d'un ensemble ouvert $U \subset T^*X$, et de $Z = T^*X \setminus U$. Considérons les deux catégories de faisceaux suivantes :

$$\begin{aligned} \mathcal{D}_Z(X) =& \{F \in \mathcal{D}(X) : \rho(SS(F) \cap \{\tau > 0\}) \subset Z\} \\ \mathcal{D}_U(X) =& {}^{\perp}\mathcal{D}_Z(X), \text{ le complément orthogonal gauche de } \mathcal{D}_Z(X). \end{aligned}$$

En nous basant sur l'idée de Chiu, nous prouvons que pour une classe d'ensembles ouverts, dynamiquement admissibles, il existe deux faisceaux $P_U, Q_U \in \mathcal{D}(X \times X)$, que nous appelons *noyaux microlocaux*, tels que les foncteurs de convolution qu'ils définissent sont des projecteurs sur ces deux catégories. La proposition suivante dit que la classe est suffisamment grande pour comprendre de nombreux exemples intéressants.

Proposition A ([**Zha21**])**.** *Les ensembles ouverts bornés, les domaines toriques et les faisceaux de disques unitaires sur des variétés complètes sont dynamiquement admissibles.*

La principale application des noyaux microlocaux est de définir le complexe de Chiu-Tamarkin $C_{\ell,T}(U, \mathbb{K})$, qui est mentionné implicitement dans [**Tam15**], et est écrit explicitement par Chiu dans [**Chi17**].

3.1. Complexe de Chiu-Tamarkin. Le complexe de Chiu-Tamarkin non équivariant avec le paramètre $T \geq 0$ est le complexe d'homologie

$$C_{1,T}(U, \mathbb{K}) = \mathrm{RHom}(P_U, \mathbb{K}_{\Delta_{X^2} \times \{T\}}).$$

De manière heuristique, le projecteur P_U enregistre certaines informations dynamiques du flot de Reeb de ∂U. Et le complexe de Chiu-Tamarkin non-équivariant nous donne des informations sur les orbites fermées de la dynamique de Reeb à bord. La signification géométrique du paramètre T est la borne d'action des orbites de Reeb.

En tant que projecteur, pour chaque $\ell \in \mathbb{N}$, le faisceau P_U satisfait à l'identité idempotente

$$P_U^{\star \ell} \xrightarrow{\cong} P_U \in \mathcal{D}(X^2).$$

Toutes nos constructions partent de l'identité idempotente.

La première application concerne la structure algébrique sur le complexe de Chiu-Tamarkin non-équivariant.

Quand $\ell = 2$. L'identité idempotente nous montre que le noyau P_U est une coalgèbre dans la catégorie monoïdale symétrique $(\mathcal{D}(X^2), \star)$. Comme sous-produit de la structure coalgébrique, nous étudions le produit de Yoneda décalé sur $\mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U)$, puis nous le formulons comme un cup-produit géométriquement défini sur l'homologie de Chiu-Tamarkin non équivariante $H^*C_{1,T}(U,\mathbb{K})$. Nous avons le théorème suivant

Théorèm B. *Le produit de Yoneda décalé sur* $\mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U)$ *est associatif, commutatif gradué et unital. Lorsque* X *est orientable, l'isomorphisme des espaces vectoriels* $\mathbb{K}$*,*

$$\mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U) \cong H^*C_{1,T}(U,\mathbb{K}),$$

identifie le produit de Yoneda décalé et le cup-produit.

Pour $U = T^*X$, nous avons l'isomorphisme $H^*C_{1,T}(T^*X,\mathbb{K}) \cong H^*(X,\mathbb{K})$ et le produit de Yoneda/cup-produit décalé est le même que le cup-produit habituel sur $H^*(X,\mathbb{K})$.

Ensuite, les identités idempotentes nous aident à comprendre la définition du complexe de Chiu-Tamarkin équivariant. En fait, en utilisant ces isomorphismes et les isomorphismes adjoints, nous obtenons un isomorphisme

$$\mathrm{RHom}(P_U, \mathbb{K}_{\Delta_{X^2}}) \cong \mathrm{RHom}\left(P_U^{\overset{L}{\boxtimes}\ell}, s_t^{\ell!}\tilde{\Delta}_{X*}\pi_{\underline{\mathbf{q}}}^!\mathbb{K}_T[-d]\right),$$

où $\tilde{\Delta}_X(\underline{\mathbf{q}},t) = (\mathbf{q}_n, \mathbf{q}_1, \mathbf{q}_1, \ldots, \mathbf{q}_{n-1}, \mathbf{q}_{n-1}, \mathbf{q}_n, t)$ est une diagonale tordue de X.

Une observation clé est que tant $P_U^{\overset{L}{\boxtimes}\ell}$ que $s_t^{\ell!}\tilde{\Delta}_{X*}\pi_{\underline{\mathbf{q}}}^! i_{T!}\mathbb{K}$ sont équivariants sous la permutation cyclique de $(X^2)^\ell$.

Ainsi, dans [Chi17], Chiu a défini le complexe de Chiu-Tamarkin $\mathbb{Z}/\ell$-équivariant comme étant le complexe hom équivariant

$$C_{\ell,T}(U,\mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_U^{\overset{L}{\boxtimes}\ell}, s_t^{\ell!}\tilde{\Delta}_{X*}\pi_{\underline{\mathbf{q}}}^!\mathbb{K}[-d]\right).$$

C'est la même chose que la définition originale de Chiu à un changement de degré canonique près.

En utilisant des idées similaires de la partie non-équivariante, nous espérons trouver l'information S^1-équivariante de la dynamique sur le bord. Mais $P_U^{\overset{L}{\boxtimes}\infty}$ n'a pas un sens, il n'est donc pas immédiat de construire une S^1-action en utilisant les noyaux microlocaux et le produit extérieur.

L'idée de Chiu est que, même si nous ne pouvons pas utiliser une théorie littéralement S^1-équivariante, nous pouvons lire des informations équivariantes pour chaque ℓ. Le succès de la preuve du théorème de non-plongement de contact implique que c'est une façon correcte d'utiliser l'information S^1-équivariante de la dynamique sur le bord. D'un autre côté, dans le calcul de Chiu pour les boules, et notre calcul pour les domaines toriques convexes, nous avons déjà observé que nous calculons la cohomologie $\mathbb{Z}/\ell$-équivariante d'un type d'homotopie qui admet une action de S^1, où le $\mathbb{Z}/\ell$ provient de la restriction de l'action de S^1.

Ainsi il est donc naturel de se demander: dans quel sens avons-nous une théorie S^1-équivariante ? Une des principales contributions de la thèse est une idée sur la construction de la S^1-structure. Revenons à l'identité idempotente $P_U^{\star\ell} \xrightarrow{\cong} P_U$. La définition du projecteur inclut un morphisme $P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$. Et l'isomorphisme idempotent est induit par

$$P_U^{\overset{L}{\boxtimes}\ell} \to P_U \overset{L}{\boxtimes} \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}^{\overset{L}{\boxtimes}\ell-1}.$$

La construction est similaire à la construction bar qui est commune dans l'algèbre homologique. D'autre part, nous avons déjà un opérateur de permutation cyclique induit par la permutation cyclique sur la base $(X^2)^\ell$. Une combinaison minutieuse de ces données devrait former l'ingrédient d'une structure algébrique de S^1 : un complexe (pré-co)cyclique.

Théorèm C. *Si $\mathbb{K}$ est un corps. Pour un ensemble ouvert admissible et $T \geq 0$. Nous pouvons construire une structure de complexe pré-cocyclique sur $F_\bullet^{S^1,+}(U,\mathbb{K})_T := (F_{n+1}^+(U,\mathbb{K})_T, d_i^+, t_n^+)_{n\in\mathbb{N}}$.*

Par conséquent, nous pouvons définir le complexe de Chiu-Tamarkin S^1-équivariant en prenant le complexe mixte associé du complexe pré-cocyclique.

Définition D. Pour un ensemble ouvert admissible et $T \geq 0$, on définit

$$C_T^{S^1,?}(U,\mathbb{K}) := \mathrm{RHom}_{\mathbb{K}[\epsilon]}(F_\bullet^{S^1,?}(U,\mathbb{K})_T, \mathbb{K}[-d]),\ ? \in \{\varnothing, +\},$$

où $\mathbb{K}[\epsilon]$ signifie que nous prenons le hom dans la catégorie des complexes mixtes.

3.2. Capacités. Jusqu'à présent, l'application la plus importante du complexe de Chiu-Tamarkin est la preuve du théorème de non-plongement de contact. En suivant l'idée de Chiu, nous disséquons la preuve de Chiu en plusieurs observations et constructions.

• Le rôle du nombre T : la signification géométrique de T est la borne d'action des orbites de Reeb dans ∂U. En particulier, en utilisant le complexe de Chiu-Tamarkin $C_{\ell,T}(U,\mathbb{K})$, l'orbite de Reeb est coupée en ℓ morceaux et l'action du groupe $\mathbb{Z}/\ell$ peut nous aider à observer l'action T/ℓ.

Dans la définition de Chiu, T n'apparaît pas dans la définition de manière explicite, puisqu'il travaille avec la préquantification $T^*X \times S^1$. Mais nous travaillons généralement avec le 1-jet J^1X, donc nous n'avons pas besoin de nous restreindre aux cordes de Reeb $\mathbb{Z}$-invariante, qui ont une action entière T. D'autre part, lorsque nous devons travailler dans $T^*X \times S^1$ comme Chiu, nous devons supposer que T/ℓ est un entier car une corde de Reeb non $\mathbb{Z}$-invariante de J^1X ne peut pas être une corde de Reeb de $T^*X \times S^1$, tandis qu'une corde de Reeb $\mathbb{Z}$-invariante de J^1X a une action entière.

• La classe fondamentale : Dans le cas non-équivariant, comme le noyau microlocal P_U est un objet de $\mathcal{D}(X \times X)$, il existe un morphisme naturel $\tau_T(P_U) \in Hom(P_U, \mathrm{T}_{T*}P_U) \cong H^0C_{1,T}(P_U,\mathbb{K})$. La définition classique de l'énergie d'un faisceau (voir Definition 2.5) nous aide à définir un invariant numérique associé à U. Elle est déjà utilisée par [**Zha20**] pour une preuve par faisceau du théorème de non-plongée de Gromov. D'autre part, la preuve de Chiu du théorème de non-plongement de contact nous suggère de définir une version équivariante de $\tau_T(P_U)$, qui est ce que nous avons appelé la classe fondamentale $\eta_{*,T}(U,\mathbb{K}) \in H^0C_{*,T}(U,\mathbb{K})$, où $* \in \{\ell, ^{S^1}\}$. Comme $\tau_T(P_U)$, il est défini en utilisant le relèvement équivariant du morphisme naturel $P_U \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$ et la structure de persistance du complexe de Chiu-Tamarkin.

• Indice : Si l'on considère la théorie équivariante, tant $H^*C_{\ell,T}(U,\mathbb{K})$ $(\ell \geq 2)$ que $H^*C_T^{S^1}(U,\mathbb{K})$ sont des modules sur $\mathbb{K}[u]$. L'action de $\mathbb{K}[u]$ est un outil de base pour suivre l'indice des orbites de Reeb.

En combinant tous ces ingrédients, on peut définir certaines capacités, tant dans le cas symplectique que dans le cas de contact.

Pour le cas $\mathbb{Z}/\ell$-équivariant, on désigne par p_ℓ le facteur premier minimal de ℓ, on considère

$$\mathrm{Spec}(U,k) \coloneqq \left\{ T \geq 0 : \begin{array}{l} \exists p \text{ premier de telle sorte que } \forall \ell \in \mathbb{N}_{\geq 2},\, p_\ell \geq p, \\ \exists \Lambda_\ell \in H^* C_{\ell,T}(U, \mathbb{F}_{p_\ell}),\, \eta_{\ell,T}(U, \mathbb{F}_{p_\ell}) = u^k \Lambda_\ell \end{array} \right\},$$

et

$$c_k(U) \coloneqq \inf \mathrm{Spec}(U,k) \in [0,+\infty].$$

Pour le cas S^1-équivariant, nous considérons

$$\overline{\mathrm{Spec}}(U,k) \coloneqq \left\{ T \geq 0 : \exists \gamma^{S^1} \in H^* C_T^{S^1}(U,\mathbb{Q}),\, \eta_T^{S^1}(U,\mathbb{Q}) = u^k \gamma^{S^1} \right\}$$

et

$$\bar{c}_k(U) \coloneqq \inf \overline{\mathrm{Spec}}(U,k)) \in [0,+\infty].$$

En général, si U n'est pas admissible, alors on définit c_k et $\bar{c}_k$ comme le supremum sur toutes les approximations admissibles. Le troisième résultat est

Théorèm E. *Les fonctions $c_k, \bar{c}_k : Open(T^*X) \to (0,\infty]$ sont des capacités symplectiques invariantes par isotopies hamiltoniennes globalement définies et à support compact.*

3.3. Fibré de préquantification. Dans le cas de la préquantification, comme nous l'avons discuté au début, la seule différence est que nous devons supposer que T/ℓ est un entier. Dans ses travaux, Chiu prend $T = n\ell$. Ici, nous pouvons supposer que T/ℓ est un entier positif quelconque (T est positif), mais ce n'est pas tout à fait nécessaire puisque les capacités capturent le T minimal sous certaines conditions. Ainsi, nous utilisons toujours l'hypothèse $T = n\ell$ au niveau de l'action. D'autre part nous relevons $U \subset T^*X \times S^1$ en $\widehat{U} \subset J^1X$.

Alors $\mathcal{D}^c_{J^1 \setminus \widehat{U}}(X)$ peut être défini en utilisant une version 1-jet du microsupport, et ensuite $\mathcal{D}^c_U(X)$ est toujours défini comme le complément orthogonal gauche. Alors la notion d'admissibilité et la notion de noyaux microlocaux peuvent être définies pour les ensembles ouverts $U \subset T^*X \times S^1$. Le complexe de Chiu-Tamarkin peut également être défini. Mais comme nous l'avons vu précédemment, nous devons prendre $T = n\ell$, et nous définissons le complexe de Chiu-Tamarkin $\mathscr{C}_{\ell,n\ell}(U, \mathbb{K})$ pour $U \subset T^*X \times S^1$. Ceci est également valable pour la classe fondamentale. De plus, l'invariance et la propriété functorielle sont valables dans ce cas. Ensuite, comme les capacités symplectiques, nous définissons

Définition F. Pour un ensemble ouvert admissible $U \subset T^*X \times S^1$, $k \in \mathbb{N}$, définissons

$$[\mathrm{Spec}](U,k) := \left\{ n\ell \in \mathbb{N}_{\geq 2} : \begin{array}{c} (n,\ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\ \exists p \text{ premier de telle sorte que } \forall \ell,\ p_\ell \geq p, \\ \exists \Lambda_\ell \in H^*\mathscr{C}_{\ell,n\ell}(U, \mathbb{F}_{p_\ell}),\ \eta^c_{\ell,n\ell}(U, \mathbb{F}_{p_\ell}) = u^k \Lambda_\ell \end{array} \right\},$$

et

$$[c]_k(U) := \min[\mathrm{Spec}](U,k) \in \mathbb{N}_{\geq 2}.$$

Pour un ensemble ouvert général U, nous prenons également

$$[c]_k(U) = \sup\{[c]_k(V) : V \subset U,\ V \text{ est admissible}\}.$$

Alors nous avons

Théorèm G. *Les fonctions $[c]_k : Open(T^*X \times S^1) \to \mathbb{N}_{\geq 2}$ sont une famille de capacités de "contact".*

Ici, remarquons à nouveau que la construction du complexe de Chiu-Tamarkin via une action de S^1 ne peut pas être définie dans le cas contact. Puisque la définition utilise toutes les informations des ℓ, il est impossible d'admettre $T/\ell \in \mathbb{N}$ pour tous les $\ell \in \mathbb{N}$.

3.4. Calcul du complexe de Chiu-Tamarkin. La dernière partie de la thèse consiste en deux calculs du complexe de Chiu-Tamarkin: domaines toriques convexes et fibrés en disques unitaires.

Domaines toriques convexes

L'action hamiltonienne standard du tore sur $\mathbb{C}^d_u = T^*\mathbb{R}^d_{\mathbf{q}}$ et son application moment sont les suivantes

$$z \cdot (u_1, \ldots, u_n) = (\exp(-2i\pi z_1)u_1, \ldots, \exp(-2i\pi z_d)u_d).$$

$$\mu(u_1, \ldots, u_n) = (\pi|u_1|^2, \ldots, \pi|u_d|^2).$$

Pour $\Omega \subset \mathbb{R}^d$ ouvert, nous appelons $X_\Omega \coloneqq \mu^{-1}(\Omega) \subset T^*\mathbb{R}^d$ un domaine torique (ouvert). Nous disons que X_Ω est un domaine torique convexe si $|\Omega| \coloneqq \{\zeta \in \mathbb{R}^d : (|\zeta_1|, \ldots, |\zeta_d|) \in \Omega\}$ est convexe. On dit que X_Ω est concave si $\mathbb{R}^d_{\zeta \geq 0} \setminus \Omega$ est convexe. Par exemple, les polydisques et les ellipsoïdes sont des domaines toriques convexes.

Après une application rapide de l'existence des noyaux microlocaux, nous obtenons un modèle de fonction génératrice du noyau microlocal pour les domaines toriques.

De plus, le modèle de fonction génératrice nous donne une formule claire pour $F_*(X_\Omega, \mathbb{K})$ pour les domaines toriques convexes, qui calcule une cohomologie équivariante d'une sphère.

Par conséquent, nous pouvons obtenir le théorème structurel du complexe de Chiu-Tamarkin pour les domaines toriques convexes. Pour être plus concis, nous énonçons une version plus simple du théorème.

Pour $T \geq 0$, nous définissons

$$\begin{aligned}
&\Omega^\circ_T \coloneqq \Omega^\circ \cap \{t = T\} = \{z \in \mathbb{R}^d : T + \langle z, \zeta \rangle \geq 0, \forall \zeta \in \Omega\}, \\
&\|\Omega^\circ_T\|_\infty = \max_{z \in \Omega^\circ_T} \|z\|_\infty.
\end{aligned}$$

Alors $\|\Omega^\circ_T\|_\infty = T\|\Omega^\circ_1\|_\infty$. On définit également

$$I(\Omega^\circ_T) = \max_{z \in \Omega^\circ_T} I(z), \qquad \text{avec } I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor \text{ for } z \in \mathbb{R}^d.$$

Théorèm H. *Pour un domaine torique convexe $X_\Omega \subsetneq T^*\mathbb{R}^d_{\boldsymbol{q}}$, et $\ell \in \mathbb{N}_{\geq 2}$:*

(1) Il existe une constante $C(\Omega)$ telle que si $0 \leq T < p_\ell C(\Omega)$, alors pour chaque $Z \in \Omega^\circ_T$, on peut trouver une décomposition de la classe fondamentale $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\gamma_{Z,\ell}$

pour un élément $\gamma_{Z,\ell} \in H^{-2I(Z)}C_{\ell,T}(X_{\Omega,\mathbb{F}_{p_\ell}})$*, et* $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ *est non nul. Le degré de cohomologie minimal de* $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ *est exactement* $-2I(\Omega_T^\circ)$.

(2) Si $T \geq 0$ *et pour tout corps* $\mathbb{K} \supset \mathbb{Q}$*, on a que pour chaque* $Z \in \Omega_T^\circ$ *on peut induire une décomposition de la classe fondamentale* $\eta_T^{S^1}(X_\Omega, \mathbb{K}) = u^{I(Z)}\gamma_Z^{S^1}$ *pour un élément* $\gamma_{Z,\ell} \in H^{-2I(Z)}C_T^{S^1}(X_{\Omega,\mathbb{F}_{p_\ell}})$*, et* $\eta_T^{S^1}(X_\Omega, \mathbb{K})$ *est non nul. Le degré de cohomologie minimal de* $H^*C_T^{S^1}(X_\Omega, \mathbb{K})$ *est exactement* $-2I(\Omega_T^\circ)$.

Comme corollaire, nous obtenons un calcul pour nos capacités.

Théorèm I. *Pour un domaine torique convexe* $X_\Omega \subsetneqq T^*V$*, on a*

$$c_k(X_\Omega) = \bar{c}_k(X_\Omega) = \inf\{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\} = c_k^{\mathrm{GH}}(X_\Omega),$$

où c_k^{GH} *est la capacité de Gutt-Hutchings.*

Fibré de disques unitaires

Pour le fibré de disques unitaires, nous montrons que $F_*(D^*X, \mathbb{K})$ calcule la cohomologie des espaces de lacets libres. Par conséquent, nous obtenons l'isomorphisme de Viterbo pour le complexe de Chiu-Tamarkin.

Théorèm J. *Pour une variété compacte* X*,* $T \in [0, \infty]$*, nous avons*

$$H^{-q}C_{\ell,T}(D^*X, \mathbb{K}) \cong H_{q+d}^{\mathbb{Z}/\ell}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

et

$$H^{-q}C_T^{S^1}(D^*X, \mathbb{K}) \cong H_{q+d}^{S^1}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

où $\mathcal{L}_{\leq T}X$ *est le sous-espace de l'espace des lacets libres constitué de lacets de longueur au plus égale à* T.

De plus, si l'on considère le cup-produit, on a

Théorèm K. *Pour une variété orientable compacte* X*, l'isomorphisme de Viterbo*

$$H^{-q}C_{1,T}(D^*X, \mathbb{K}) \cong H_{q+d}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

est un isomorphisme de $\mathbb{K}$-algèbres par rapport au cup-produit sur l'homologie de Chiu-Tamarkin et au produit de Chas-Sullivan sur la topologie des cordes.

4. Organisation de la thèse

Dans le chapitre 1, nous rassemblons les préliminaires sur la théorie des faisceaux microlocaux qui apparaissent dans la thèse. Nous commençons par la définition et des exemples du microsupport et des estimations de microsupport dont nous avons besoin. Nous présentons également des définitions et des exemples du calcul des noyaux, y compris la composition, la convolution et leurs estimations de microsupport. Enfin, nous rappelons la quantification des faisceaux de Guillermou-Kashiwara-Schapira.

Dans le chapitre 2, nous rappelons la notion de catégorie de Tamarkin. En particulier, nous introduisons la notion de noyaux microlocaux et présentons les propriétés de base des noyaux microlocaux. La partie la plus importante du chapitre est l'existence de noyaux microlocaux pour les ensembles ouverts dynamiquement admissibles.

Le chapitre 3 est une partie principale de la thèse. Nous rappelons la définition originale du complexe de Chiu-Tamarkin et nous développons la définition du complexe de Chiu-Tamarkin S^1-équivariant en utilisant une structure cyclique basée sur les propriétés de projecteur des noyaux microlocaux. Nous utilisons également les propriétés de projecteur pour définir un cup-produit sur un complexe de Chiu-Tamarkin non-équivariant, qui récupère le cup-produit habituel sur $H^*(X, \mathbb{K})$ lorsque $U = T^*X$. Nous étudions également l'invariance, les propriétés fonctorielles et la classe fondamentale du complexe de Chiu-Tamarkin. Ces ingrédients nous aident à définir les capacités symplectiques et de contact. L'idée de la définition des capacités est une généralisation de l'idée de Chiu sur le théorème de non-plongement de contact.

Le chapitre 4 est une autre partie principale de la thèse. Il consiste en deux calculs: un pour les domaines toriques convexes et un pour les faisceaux de disques unitaires sur les variétés compactes. Nous présenterons la preuve du théorème de structure du complexe de Chiu-Tamarkin pour les domaines toriques convexes et le calcul des capacités dans ces domaines. Pour les fibrés de disques unitaires, nous prouvons l'isomorphisme de

Viterbo du complexe de Chiu-Tamarkin et comparons le cup-produit et le produit de Chas-Sullivan dans le cas non-équivariant.

Nous discuterons des résultats et ferons quelques conjectures dans le chapitre 5.

En annexe, nous présentons un résumé sur la catégorie dérivée équivariante et l'homologie de Borel-Moore.

Introduction

1. Microlocal sheaves: old and new.

The algebraic analysis is introduced by Sato in the '60s. The main idea is to use algebraic tools like sheaves and categories to study problems in analysis, especially partial differential equations. As an illustration of Sato's philosophy, Kashiwara and Schapira introduced and developed the microlocal theory of sheaves in [**KS82**, **KS83a**, **KS83b**, **KS90**]. In particular, the applications of the microlocal theory of sheaves to $\mathcal{D}$-modules presents us the power of algebraic analysis.

The main notions of the microlocal sheaf theory are Sato's microlocalization and the notion of microsupport. In this thesis, we will mainly focus on applications of the notion of microsupport. The microsupport $SS(F)$ detects the local extendability of sections of a sheaf F. It is shown that, on a complex manifold X, for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, the microsupport $SS(Sol(\mathcal{M}))$ coincides with the characteristic variety of the $\mathcal{D}_X$-module $\mathcal{M}$. Precisely, a covector is in the microsupport of the solution sheaf $Sol(\mathcal{M})$ if the local solutions near the covector could be extended near the codirection, which exactly means that the codirection is in the characteristic variety of $\mathcal{M}$. We can also formulate global extendability using the microlocal Morse lemma (see chapter 1, Corollary 1.6). A highlight of the story is the new proof of the fact that the characteristic varieties of coherent $\mathcal{D}$-modules are co-isotropic using Kashiwara-Schapira's co-isotropic theorem. On the other hand, it is proved in [**KS90**] that $SS(F)$ is a closed conic and coisotropic subset of T^*X. When X is real analytic, $SS(F)$ is Lagrangian if and only if F is (weakly) constructible. This result illustrates the role of the microlocal sheaf theory in symplectic geometry of the cotangent bundle T^*X. As a new trend of symplectic geometry, sheaves techniques is shining in recent years.

Sheaf	Geometry
Constructible sheaves over X	conic Lagrangian in T^*X
Constructible sheaves over $X \times X$	Lagrangian correspondences between T^*X
Convolution of sheaves	Composition of correspondences
Zero extension and restriction	Symplectic Reduction
Guillermou-Kashiwara-Schapira quantization[GKS12]	Hamiltonian group action
Guillermou quantization[Gui12]/ Viterbo quantization[Vit19]	exact Lagrangian in T^*X

Table 2. A sheaf-geometry correspondence.

Let us start from the work of Tamarkin. In [Tam18], Tamarkin developed the notion of Tamarkin category $\mathcal{D}(X)$. The Tamarkin category $\mathcal{D}(X)$ is a quotient of $D(X \times \mathbb{R}_t)$ by sheaves microsupported negatively along t, and is isomorphic to a full triangulated subcategory of $D(X \times \mathbb{R}_t)$, so we can think of its objects as sheaves over $X \times \mathbb{R}_t$.

The role of the variable t is twofold:

• The microsupport $SS(F)$ is conic under the dilation on cotangent fibres. But most symplectic geometry problems on cotangent bundles are not conic. So, Tamarkin suggests using the cone map $\rho(\mathbf{q}, \mathbf{p}, t, \tau) = (\mathbf{q}, \mathbf{p}/\tau)$.

So for $A \subset T^*X$, we have a conic subset $\rho^{-1}(A) \subset T^*X \times T^*_{\tau>0}\mathbb{R}_t$. Moreover, if $F \in \mathcal{D}(X)$, we will automatically have $SS(F) \subset \{\tau \geq 0\}$, and it will be often the case in practice that $SS(F) \cap \{\tau \leq 0\} \subset 0_{X\times\mathbb{R}}$. Therefore a version of the microlocal Morse lemma shows that we will not lose much information.

• On the other hand, the cone map ρ factors through the symplectization map q of the 1-jet bundle J^1X tautologically:

$$\begin{tikzcd} T^*X \times T^*_{\tau>0}\mathbb{R}_t \arrow[r, "q"] \arrow[rr, bend right, "\rho"'] & J^1(X) = T^*X \times \mathbb{R}_t \arrow[r] & T^*X. \end{tikzcd}$$

So, the conicity actually comes from the symplectization process, and the extra variable t, in fact, play the role of action for Lagrangians in T^*X.

Now, consider the translation maps

$$\mathrm{T}_c : X \times \mathbb{R}_t \to X \times \mathbb{R}_t,\ (\mathbf{q}, t) \mapsto (\mathbf{q}, t + c).$$

Microlocally, the functors T_{c*} quantize the Reeb flow $d\mathrm{T}_c$ of the canonical contact form $\alpha = dt + \mathbf{p}d\mathbf{q}$.

It is crucial that, on the Tamarkin category, we have a natural transform $\tau_c : \mathrm{Id} \Rightarrow \mathrm{T}_{c*}$ for $c \geq 0$. The natural transform does not exist on $D(X \times \mathbb{R}_t)$ (but it exists on a larger category $D_{\tau \geq 0}(X \times \mathbb{R}_t)$, see [**GS14**] for more details). This natural transform can help us to see finite bars in the barcode of $\mathrm{R}\pi_{X!}F$ for $F \in \mathcal{D}(X)$.

So, instead of saying that objects in the Tamarkin category quantize Lagrangian in cotangent bundles, it is better to say objects in the Tamarkin category quantize Legendrian in 1 jet spaces.

Based on this point of view, Tamarkin develops a new method to study displacibility in [**Tam18**].

Tamarkin's works are so influential. Asano-Ike developed the persistence-like distance along a quantitative point of view on the work of Tamarkin and went into the numerical invariants research of sheaves with applications on symplectic displacement energy, rational Lagrangian immersions, and C^0-symplectic geometry in [**AI20a, AI20b, AI22**]. On the categorical side, Biran-Cornea-Zhang developed the notion of triangulated persistence category in [**BCZ21**], which abstract the categorical structures of the Tamarkin category.

Besides, there are many works on symplectic geometry that are based on the microlocal sheaf theory. Guillermou gives sheafy proofs of Gromov-Eliashberg C^0-rigidity, the 3-cusp conjecture, and of the result by Abouzaid and Kragh that closed exact Lagrangians in cotangent bundles are homotopically equivalent to the zero section. See [**Gui12, Gui13, Gui16**] and the survey [**Gui19**] about these topics. Ike estimates the exact Lagrangian intersections in the cotangent bundles (see [**Ike19**]), and Li estimates the Reeb chords in the 1-jet spaces (see [**Li21a**]). Casals and Gao construct infinitely many Lagrangian fillings for some Legendrian torus knots based using moduli spaces of sheaves as invariants in [**CG22**].

On the other hand, many works are studying the category of sheaves from the point of view of the Fukaya category. It is started from the work of Bondal-Ruan [**BdR**]. Also, see the work of Nadler and Zaslow on the compact Fukaya category [**NZ09**, **Nad09**]; and the work of Nadler[**Nad16**], and Ganatra, Pardon, and Shende on the wrapped Fukaya category[**GPS18a**].

2. Contact non-squeezing theorem.

The famous Gromov non-squeezing opened the door to modern symplectic geometry. But the contact correspondence was not discussed until the pioneering work of Eilashberg-Kim-Poltervich [**EKP06**].

A naive attempt is to setup the contact non-squeezing problem in the 1-jet bundle $J^1\mathbb{R}^d = T^*\mathbb{R}^d \times \mathbb{R}_t$ equipped with the contact form $\alpha = dt + \mathbf{p}d\mathbf{q}$. But the re-scaling map $(\mathbf{q}, \mathbf{p}, z) \mapsto (r\mathbf{q}, r\mathbf{p}, r^2 t)$, which is a contactomorphism, squeezes any compact set into an arbitrary small neighborhood of the origin when r is big enough. This conformal naturality of 1-jet space illustrates that it is better to study the prequantized space $T^*\mathbb{R}^d \times S^1$, where S^1 is a circle, equipped with a contact form $\alpha = d\theta + \frac{1}{2}(\mathbf{q}d\mathbf{p} - \mathbf{p}d\mathbf{q})$. But there is a global contactomorphism $F_N : T^*\mathbb{R}^d \times S^1 \to T^*\mathbb{R}^d \times S^1$ defined as follows: We use complex coordinates $T^*\mathbb{R}^d \cong \mathbb{C}^d$, and then $F_N(z, \theta) \coloneqq (\nu(\theta)e^{2\pi N\theta}z, \theta)$, where $\nu(\theta) = (1 + N\pi|z|^2)^{-1/2}$. One can compute directly that F_N is still embedding any ball into an arbitrarily small neighbourhood of $\{0\} \times S^1$ for N big enough. However, we notice that F_N is not compactly supported.

So a better definition of contact squeezing is the following proposed in loc. cit.

Definition. [**EKP06**, p1636] Let (V, α) be a contact manifold. If $U_1, U_2 \subset V$ are two open subsets, we say that U_1 is squeezed into U_2 if there exists a compactly support contact isotopy $\varphi_s : \overline{U_1} \to V$, $s \in [0, 1]$ such that $\varphi_0 = \mathrm{Id}$, and $\varphi_1(\overline{U_1}) \subset U_2$.

An interesting phenomenon, which does not appear in the symplectic situation, is the scale of the ball will affect the validity of the squeezing. Two results about both squeezing and non-squeezing of contact balls $B_{\pi R^2} \times S^1$ are:

Theorem. *(1)* [**EKP06**, Theorem 1.3] *Suppose $d \geq 2$. Then for all $0 < \pi r^2, \pi R^2 < 1$, one can squeeze the contact ball $B_{\pi R^2} \times S^1$ into $B_{\pi r^2} \times S^1$ whatever the relation between r and R is.*

(2) [**EKP06**, Theorem 1.2] *If there exists an integer $m \in [\pi r^2, \pi R^2]$, then $B_{\pi R^2} \times S^1$ cannot be squeezed into $B_{\pi r^2} \times S^1$.*

About the large scale phenomenon, Eliashberg, Kim, and Polterovich give a very nice physical explanation using the quantization process. Then the only case left about the contact non-squeezing is: what will happen if there is an integer m such that $m < \pi r^2 < \pi R^2 < m + 1$? It is solved by Chiu using the microlocal theory of sheaves [**Chi17**], and by Fraser using the technique of J-holomorphic curves [**Fra16**] in the spirit of [**EKP06**]. They proved the following:

Theorem ([**Chi17, Fra16**]). *If $1 \le \pi r^2 < \pi R^2$, then $B_{\pi R^2} \times S^1$ cannot be squeezed into $B_{\pi r^2} \times S^1$.*

The tool of Chiu's proof is the Chiu-Tamarkin complex, which is also our main focus on the thesis. Let us review Chiu's proof here.

For the contact ball $B_A \times S^1 \subset T^*\mathbb{R}^d \times S^1$, we first lift $B_A \times S^1$ to a $\mathbb{Z}$-invariant open set $B_A \times \mathbb{R} \subset J^1\mathbb{R}^d$. Then we can study the two categories of sheaves:

$$\begin{aligned}\mathcal{D}^c_{J^1\mathbb{R}^d \setminus B_A \times \mathbb{R}}(X) =& \{F \in \mathcal{D}(X) : q(SS(F)) \subset J^1\mathbb{R}^d \setminus B_A \times \mathbb{R}\},\\ \mathcal{D}^c_{B_A \times \mathbb{R}}(X) =& {}^{\perp}\mathcal{D}^c_{J^1\mathbb{R}^d \setminus B_A \times \mathbb{R}}(X), \text{ the left orthogonal complement of } \mathcal{D}^c_{J^1\mathbb{R}^d \setminus B_A \times \mathbb{R}}(X).\end{aligned}$$

Chiu associates two sheaves $\mathscr{P}_{B_A \times S^1}, \mathscr{Q}_{B_A \times S^1} \in D(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^2)$ together with a distinguished triangle

$$\mathscr{P}_{B_A \times S^1} \to \mathbb{K}_{\Delta_{X^2} \times \{t_2 \ge t_1\}} \to \mathscr{Q}_{B_A \times S^1} \xrightarrow{+1},$$

where $\mathscr{P}_{B_A \times S^1}, \mathscr{Q}_{B_A \times S^1}$ are called microlocal kernels associated with $B_A \times S^1$.

Using the microlocal kernel $\mathscr{P}_{B_A \times S^1}$, Chiu introduced the Chiu-Tamarkin complex $\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K}) \in D_{\mathbb{Z}/\ell}(\mathrm{pt})$, and a fundamental class $\eta^c_{\ell,n\ell}(B_A \times S^1, \mathbb{K}) \in H^0\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ that is defined by the natural morphism $\mathscr{P}_{B_A \times S^1} \to \mathbb{K}_{\Delta_{X^2} \times \{t_2 \ge t_1\}}$. Even though Chiu did not mention the existence of the fundamental class explicitly, it already appeared in his proof of the contact non-squeezing in a different form.

Chiu proved that both $\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ and $\eta^c_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ are natural with respect to inclusion $B_a \times S^1 \subset B_A \times S^1$, and invariant by compactly supported contactomorphisms of $T^*\mathbb{R}^d \times S^1$. Consequently, $\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$, or more specifically $\eta^c_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ are obstructions for contact embedding.

Another ingredient is that $H^*\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ is a module over $\mathbb{K}[u]$ if $char(\mathbb{K})|\ell$. In his paper, Chiu uses the $\mathbb{K}[u]$-action on $H^*\mathscr{C}_{\ell,n\ell}(B_A \times S^1, \mathbb{K})$ to decompose the fundamental class. Finally, the result follows from a careful comparison of degree of the fundamental class.

Geometrically, a Reeb chord of ∂B_A defines a Reeb chord of $T^*\mathbb{R}^d \times S^1$, and so does a $\mathbb{Z}$-invariant Reeb chord of $J^1\mathbb{R}^d$. Actions A of Reeb chords are useful numerical invariants for the non-squeezing problem. In the symplectic case, it is enough to define spectral invariance only using actions. But the index information of Reeb orbits is lost. In the contact case, a natural candidate is the S^1-equivariant theory, where the index information is provided using the $\mathbb{K}[u]$-action. But in this case, the action is divided into infinitesimal pieces as $0 = A/\infty$, so we can not read effective numerical information for contact embedding.

The main feature of all proofs of the contact non-squeezing theorem uses a $\mathbb{Z}/\ell$-equivariant theory, for example, the Chiu-Tamarkin complex. For the Chiu-Tamarkin complex of balls, the most crucial idea is that the index of Reeb orbit is related to its action. Moreover, the $\mathbb{Z}/\ell$-action help us find out not only a closed loop but also ℓ-periodic points of the Reeb flow. The periodic points can help us to obtain a non-zero action A/ℓ, which is the length of a Reeb chord between two ℓ-periodic points. Also, the index is realized by the $\mathbb{K}[u]$-action, which help us to distinguish some Reeb orbits. These two features can help us to use both index and action to derive an obstruction for contact non-squeezing.

3. Overview of the thesis and main results

In this thesis, we hope to systematically understand the algebra and geometry of the Chiu-Tamarkin complex. As Chiu's proof for the contact non-squeezing theorem is compelling but full of mystery, it is believed that many algebraic structures are behind the definition of the Chiu-Tamarkin complex. The content of the thesis is a combination of the author's paper [**Zha21**] and some works in preparation. To be comprehensible, we supple some preliminaries for readers.

The thesis is organised into 3 points of view: algebraic structures, numerical information, computations. They are consistent. Algebraic structures help us to extract meaningful numerical information, and computations help us to review and check our structures.

Like Chiu, let us start from an open set $U \subset T^*X$, and $Z = T^*X \setminus U$. Consider the following two categories of sheaves:

$$\begin{aligned}\mathcal{D}_Z(X) =& \{F \in \mathcal{D}(X) : \rho(SS(F) \cap \{\tau > 0\}) \subset Z\}\\ \mathcal{D}_U(X) =& {}^{\perp}\mathcal{D}_Z(X), \text{ the left orthogonal complement of } \mathcal{D}_Z(X).\end{aligned}$$

Based on the idea of Chiu, we prove that for a class of open sets, dynamically admissible open sets, there are two sheaves $P_U, Q_U \in \mathcal{D}(X \times X)$, that we call *microlocal kernels*, such that the convolution functors that they define are projectors to these two categories. The following proposition says the class is fruitful enough to include many interesting examples.

Proposition A ([**Zha21**]). *Bounded open sets, toric domains, and unit disk bundles over complete manifolds are dynamically admissible.*

The main application of microlocal kernels is to defined the Chiu-Tamarkin complex $C_{\ell,T}(U, \mathbb{K})$, which is mentioned implicitly in [**Tam15**], and written explicitly by Chiu in [**Chi17**].

3.1. Chiu-Tamarkin complex. The non-equivariant Chiu-Tamarkin complex with parameter $T \geq 0$ is the hom complex

$$C_{1,T}(U, \mathbb{K}) = \mathrm{RHom}(P_U, \mathbb{K}_{\Delta_{X^2} \times \{T\}}).$$

Heuristically, the projector P_U records some Reeb dynamical information of ∂U. And the non-equivariant Chiu-Tamarkin complex tells us information about closed orbits of the boundary Reeb dynamics. The geometry meaning of the parameter T is the action bound of Reeb orbits.

As a projector, for each $\ell \in \mathbb{N}$, the sheaf P_U satisfies the idempotent identity

$$P_U^{\star \ell} \xrightarrow{\cong} P_U \in \mathcal{D}(X^2).$$

All of our stories start from the idempotent identity.

The first application concerns the algebra structure on the non-equivariant Chiu-Tamarkin complex.

When $\ell = 2$. The idempotent identity shows us that the kernel P_U is a coalgebra in the symmetric monoidal category $(\mathcal{D}(X^2), \star)$. As a byproduct of the coalgebra structure, we study the shifted Yoneda product on $\mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U)$, and then formulate it as a geometric defined cup product on the non-equivariant Chiu-Tamarkin homology $H^*C_{1,T}(U, \mathbb{K})$. We have the following theorem

Theorem B. *The shifted Yoneda product on* $\mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U)$ *is associative, graded commutative and unital. When* X *is orientable, the isomorphism of* $\mathbb{K}$*-vector spaces,*

$$\mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U) \cong H^*C_{1,T}(U, \mathbb{K}),$$

identifies the shifted Yoneda product and the cup product.

For $U = T^*X$, we have the isomorphism $H^*C_{1,T}(T^*X, \mathbb{K}) \cong H^*(X, \mathbb{K})$ and the shifted Yoneda product/cup product is the same as the usual cup product on $H^*(X, \mathbb{K})$.

Next, the idempotent identities help us to understand the definition of the equivariant Chiu-Tamarkin complex. Actually, using these isomorphisms and adjoint isomorphisms, we obtain an isomorphism

$$\mathrm{RHom}(P_U, \mathbb{K}_{\Delta_{X^2}}) \cong \mathrm{RHom}\left(P_U^{\overset{L}{\boxtimes}\ell}, s_t^{\ell !}\tilde{\Delta}_{X*}\pi_{\underline{\mathbf{q}}}^!\mathbb{K}_T[-d]\right),$$

where $\tilde{\Delta}_X(\underline{\mathbf{q}}, t) = (\mathbf{q}_n, \mathbf{q}_1, \mathbf{q}_1, \dots, \mathbf{q}_{n-1}, \mathbf{q}_{n-1}, \mathbf{q}_n, t)$ is a twisted diagonal of X.

A key observation is that both $P_U^{\overset{L}{\boxtimes}\ell}$ and $s_t^{\ell !}\tilde{\Delta}_{X*}\pi_{\underline{\mathbf{q}}}^! i_{T!}\mathbb{K}$ are equivariant under the cyclic permutation of $(X^2)^\ell$.

So in [Chi17], Chiu defined the $\mathbb{Z}/\ell$-equivariant Chiu-Tamarkin complex to be the equivariant hom complex

$$C_{\ell,T}(U,\mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_U^{\overset{L}{\boxtimes}\ell}, s_t^{\ell!}\tilde{\Delta}_{X*}\pi_{\underline{\mathbf{q}}}^{!}\mathbb{K}[-d]\right).$$

It is the same as the original definition of Chiu up to a canonical degree shifting.

Using similar ideas of the non-equivariant part, we hope to know the S^1-equivariant information of the boundary dynamics. But $P_U^{\overset{L}{\boxtimes}\infty}$ does not make sense, so it is not direct to construct a S^1-action using microlocal kernels and exterior product.

The idea of Chiu is, even though we can not use a literally S^1-equivariant theory, we can read equivariant information for each ℓ. The success of proving the contact non-squeezing theorem implies that this is a correct way to utilize S^1-equivariant information of the boundary dynamics. On the other side, in Chiu's computation for balls, and our computation for convex toric domains, we already observed that we are computing $\mathbb{Z}/\ell$-equivariant cohomology of a homotopy type which admits a S^1-action, and the $\mathbb{Z}/\ell$ comes from the restriction of the S^1-action.

So, it is natural to ask: in what sense we have a S^1-equivariant theory? One main contribution of the thesis is an idea on the construction of the S^1-structure. Let us go back to the idempotent identity $P_U^{\star\ell} \xrightarrow{\cong} P_U$. The definition of projector includes a morphism $P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$. And the idempotent isomorphism is induced by

$$P_U^{\overset{L}{\boxtimes}\ell} \to P_U \overset{L}{\boxtimes} \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}^{\overset{L}{\boxtimes}\ell-1}.$$

The construction is similar to the bar construction that is common in homological algebra. On the other hand, we already have a cyclic permutation operator induced by the cyclic permutation on the base $(X^2)^\ell$. A careful combination of these data should form the ingredient for an algebraic S^1-structure: a (pre-co)cyclic complex.

Theorem C. *If $\mathbb{K}$ is a field. For an admissible open set, and $T \geq 0$. We can construct a pre-cocyclic complex structure on $F_\bullet^{S^1,+}(U,\mathbb{K})_T := ([F_{n+1}^+(U,\mathbb{K})]_T, d_i^+, t_n^+)_{n\in\mathbb{N}_0}$ for all $T \geq 0$.*

Consequently, we can define the S^1-equivariant Chiu-Tamarkin complex by taking the associated mixed complex of the pre-cocyclic complex.

Definition D. For an admissible open set, $T \geq 0$, we define

$$C_T^{S^1,?}(U,\mathbb{K}) \coloneqq \mathrm{RHom}_{\mathbb{K}[\epsilon]}(F_\bullet^{S^1,?}(U,\mathbb{K})_T, \mathbb{K}[-d]),\ ? \in \{\varnothing, +\},$$

where $\mathbb{K}[\epsilon]$ means that we take the hom in the category of mixed complex.

3.2. Capacities. So far, the most important application of the Chiu-Tamarkin complex is the proof of the contact non-squeezing theorem. Following the idea of Chiu, we dissect Chiu's proof into several observations and constructions.

• The role of the number T: the geometric meaning of T is the action bound of Reeb orbits in ∂U. In particular, when using the $\mathbb{Z}/\ell$ Chiu-Tamarkin complex $C_{\ell,T}(U,\mathbb{K})$, the Reeb orbit is divided into ℓ pieces and the $\mathbb{Z}/\ell$ group action can help us to observe the action T/ℓ.

At the definition of Chiu, T does not appear in the definition explicitly, since he works in the prequantization $T^*X \times S^1$. But we work generally in the 1-jet J^1X, then we do not need to restrict to $\mathbb{Z}$-invariant Reeb chord, which has an integer action T. On the other hand, when we need to work in $T^*X \times S^1$ like Chiu, we need to assume T/ℓ is an integer, because non $\mathbb{Z}$-invariant Reeb chord of J^1X can not be a Reeb chord of of $T^*X \times S^1$ while a $\mathbb{Z}$-invariant Reeb chord of J^1X has an integer action.

• The fundamental class: In the non-equivariant case, since the microlocal kernel P_U is an object of $\mathcal{D}(X \times X)$, there is a natural morphism $\tau_T(P_U) \in Hom(P_U, \mathrm{T}_{T*}P_U) \cong H^0C_{1,T}(P_U,\mathbb{K})$. The classical definition of sheaf energy (see Definition 2.5) help us to define a numerical invariant associated with U. It is already used by [**Zha20**] on a sheaf proof of Gromov's non-squeezing theorem. On the other hand, Chiu's proof for contact non-squeezing remind us to define an equivariant version of $\tau_T(P_U)$, which is what we called the fundamental class $\eta_{*,T}(U,\mathbb{K}) \in H^0C_{*,T}(U,\mathbb{K})$, for $* \in \{\ell, ^{S^1}\}$. Like $\tau_T(P_U)$, it is defined using the equivariant lifting of the natural morphism $P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$ and the persistence structure of the Chiu-Tamarkin complex.

• Index: Consider the equivariant theory, both $H^*C_{\ell,T}(U,\mathbb{K})$ ($\ell \geq 2$) and $H^*C_T^{S^1}(U,\mathbb{K})$ are modules over $\mathbb{K}[u]$. The $\mathbb{K}[u]$-action is a basic tool to track index of Reeb orbits.

Combining all these ingredients, one can define some capacities. Both in the symplectic case and in the contact case.

For the $\mathbb{Z}/\ell$-equivariant case. Denote p_ℓ the minimal prime factor of ℓ, we consider

$$\mathrm{Spec}(U,k) \coloneqq \left\{ T \geq 0 : \begin{array}{l} \exists p \text{ prime such that } \forall \ell \in \mathbb{N}_{\geq 2},\, p_\ell \geq p, \\ \exists \Lambda_\ell \in H^*C_{\ell,T}(U,\mathbb{F}_{p_\ell}),\, \eta_{\ell,T}(U,\mathbb{F}_{p_\ell}) = u^k \Lambda_\ell \end{array} \right\},$$

and

$$c_k(U) \coloneqq \inf \mathrm{Spec}(U,k) \in [0,+\infty].$$

For the S^1-equivariant case, we consider

$$\overline{\mathrm{Spec}}(U,k) \coloneqq \left\{ T \geq 0 : \exists \gamma^{S^1} \in H^*C_T^{S^1}(U,\mathbb{Q}),\, \eta_T^{S^1}(U,\mathbb{Q}) = u^k \gamma^{S^1} \right\}$$

and

$$\bar{c}_k(U) \coloneqq \inf \overline{\mathrm{Spec}}(U,k)) \in [0,+\infty].$$

In general, if U is not admissible, then we take define c_k and $\bar{c}_k$ by taking supremum over all admissible approximations. The third result is

Theorem E. *The functions $c_k, \bar{c}_k$: Open$(T^*X) \to (0,\infty]$ are symplectic capacities invariant under globally defined compactly supported Hamiltonian isotopies.*

3.3. Prequantization bundle. On the prequantization case, as we discussed at the beginning, the only difference is we need to assume T/ℓ to be an integer. In his work, Chiu requires that $T = \ell$. Here, we can allow T/ℓ to be any non-negative integer (T is non-negative), but this is not quite necessary since the capacities capture minimal T under some conditions. So, we still use the assumption $T = n\ell$ on the action level. On the other hand, for $U \subset T^*X \times S^1$, we lift it to $\widehat{U} \subset J^1X$.

Then, first, $\mathcal{D}^c_{J^1\setminus\widehat{U}}(X)$ could be defined using a 1-jet version of microsupport, and then $\mathcal{D}^c_U(X)$ is still defined as the left orthogonal complement. Then the notion of admissibility and the notion of microlocal kernels can be defined for open sets $U \subset T^*X \times S^1$. The

Chiu-Tamarkin complex can also be defined. But as we discussed before, we need to take $T = n\ell$, and we define the Chiu-Tamarkin complex $\mathscr{C}_{\ell,n\ell}(U,\mathbb{K})$ for $U \subset T^*X \times S^1$. The same also holds for the fundamental class. Also, the invariance and the functorial property hold in this case. Then, like symplectic capacities, we define

Definition F. For an admissible open set $U \subset T^*X \times S^1$, $k \in \mathbb{N}$, define

$$[\mathrm{Spec}](U,k) := \left\{ n\ell \in \mathbb{N}_{\geq 2} : \begin{array}{c} (n,\ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\ \exists p \text{ prime such that } \forall \ell,\ p_\ell \geq p, \\ \exists \Lambda_\ell \in H^*\mathscr{C}_{\ell,n\ell}(U,\mathbb{F}_{p_\ell}),\ \eta^c_{\ell,n\ell}(U,\mathbb{F}_{p_\ell}) = u^k\Lambda_\ell \end{array} \right\},$$

and

$$[c]_k(U) := \min[\mathrm{Spec}](U,k) \in \mathbb{N}_{\geq 2}.$$

For a general open set U, we also take

$$[c]_k(U) = \sup\{[c]_k(V) : V \subset U,\ V \text{ is admissible}\}.$$

Then we have

Theorem G. *The functions $[c]_k : Open(T^*X \times S^1) \to \mathbb{N}_{\geq 2}$ are a family of "contact" capacities.*

Here, let us remark again that the construction of the S^1 Chiu-Tamarkin complex can not be defined in the contact case. Since the definition use all ℓ information but it is impossible to allow $T/\ell \in \mathbb{N}$ for all $\ell \in \mathbb{N}$.

3.4. Computation of Chiu-Tamarkin complex. The last part of the thesis consists of two computations of the Chiu-Tamarkin complex: convex toric domains and unit disk bundles.

Convex toric domains

The standard Hamiltonian torus action on $\mathbb{C}^d_u = T^*\mathbb{R}^d_{\mathbf{q}}$ and its moment map are

$$z \cdot (u_1,\dots,u_n) = (\exp(-2i\pi z_1)u_1,\dots,\exp(-2i\pi z_d)u_d).$$

$$\mu(u_1,\dots,u_n) = (\pi|u_1|^2,\dots,\pi|u_d|^2).$$

For $\Omega \subset \mathbb{R}^d$ open, we call $X_\Omega \coloneqq \mu^{-1}(\Omega) \subset T^*\mathbb{R}^d$ an (open) toric domain. We say X_Ω is a convex toric domain if $|\Omega| \coloneqq \{\zeta \in \mathbb{R}^d : (|\zeta_1|, \dots, |\zeta_d|) \in \Omega\}$ is convex. We say X_Ω is concave if $\mathbb{R}^d_{\zeta \geq 0} \setminus \Omega$ is convex. For example, poly-discs and ellipsoids are convex toric domains.

After a quick application of the existence of microlocal kernels, we obtain a generating function model of the microlocal kernel for toric domains.

Moreover, the generating function model gives us a clear formula for $F_*(X_\Omega, \mathbb{K})$ for convex toric domains, which computes an equivariant cohomology of a sphere.

Therefore, we can obtain the structural theorem for the Chiu-Tamarkin complex for convex toric domains. To be shorter, we state a simpler version of the theorem.

For $T \geq 0$, we set

$$\begin{aligned} \Omega_T^\circ &\coloneqq \Omega^\circ \cap \{t = T\} = \{z \in \mathbb{R}^d : T + \langle z, \zeta \rangle \geq 0, \forall \zeta \in \Omega\}, \\ \|\Omega_T^\circ\|_\infty &= \max_{z \in \Omega_T^\circ} \|z\|_\infty. \end{aligned}$$

Then $\|\Omega_T^\circ\|_\infty = T\|\Omega_1^\circ\|_\infty$. We also set

$$I(\Omega_T^\circ) = \max_{z \in \Omega_T^\circ} I(z), \qquad \text{where } I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor \text{ for } z \in \mathbb{R}^d.$$

Theorem H. *For a convex toric domain $X_\Omega \subsetneq T^*\mathbb{R}^d_{\boldsymbol{q}}$, and $\ell \in \mathbb{N}_{\geq 2}$:*

(1) There is a constant $C(\Omega)$ such that if $0 \leq T < p_\ell C(\Omega)$, then we have that, for each $Z \in \Omega_T^\circ$, one can find a decomposition of the fundamental class $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\gamma_{Z,\ell}$ for an element $\gamma_{Z,\ell} \in H^{-2I(Z)}C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$, and $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is non-zero. The minimal cohomology degree of $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is exactly $-2I(\Omega_T^\circ)$.

(2) If $T \geq 0$ and for any field $\mathbb{K} \supset \mathbb{Q}$, we have that, for each $Z \in \Omega_T^\circ$, one can find a decomposition of the fundamental class $\eta_T^{S^1}(X_\Omega, \mathbb{K}) = u^{I(Z)}\gamma_Z^{S^1}$ for an element $\in H^{-2I(Z)}C_T^{S^1}(X_\Omega, \mathbb{F}_{p_\ell})$, and $\eta_T^{S^1}(X_\Omega, \mathbb{K})$ is non-zero. The minimal cohomology degree of $H^*C_T^{S^1}(X_\Omega, \mathbb{K})$ is exactly $-2I(\Omega_T^\circ)$.

As corollary, we obtain a computation for our capacities.

THEOREM I. *For a convex toric domain $X_\Omega \subsetneq T^*V$, we have*

$$c_k(X_\Omega) = \bar{c}_k(X_\Omega) = \inf\{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\} = c_k^{\mathrm{GH}}(X_\Omega),$$

where c_k^{GH} is the Gutt-Hutchings capacity.

Unit disk bundle

For the unit disk bundle, we show that $F_*(D^*X, \mathbb{K})$ compute the cohomology of free loop spaces. Therefore, we obtain the Viterbo isomorphism for the Chiu-Tamarkin complex.

THEOREM J. *For a compact manifold X, $T \in [0, \infty]$, we have*

$$H^{-q}C_{\ell,T}(D^*X, \mathbb{K}) \cong H_{q+d}^{\mathbb{Z}/\ell}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

and

$$H^{-q}C_T^{S^1}(D^*X, \mathbb{K}) \cong H_{q+d}^{S^1}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

where $\mathcal{L}_{\leq T}X$ is the subspace of the free loop space consists of loops of length at most T.

Moreover, if we consider the cup product, we have

THEOREM K. *For a compact orientable manifold X, the Viterbo isomorphism*

$$H^{-q}C_{1,T}(D^*X, \mathbb{K}) \cong H_{q+d}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

is an isomorphism of $\mathbb{K}$-algebras with respect to the cup product on the Chiu-Tamarkin homology and the Chas-Sullivan product on the string topology.

4. Organization of the thesis

In chapter 1, we collect preliminaries on the microlocal sheaf theory that appear in the thesis. We start with the definition and examples of the microsupport and microsupport estimates we need. We also present definitions and examples of kernel calculus, including composition, convolution and their microsupport estimates. Last, we will review the Guillermou-Kashiwara-Schapira sheaf quantization.

In chapter 2, we review the notion of the Tamarkin category. In particular, we introduce the notion of microlocal kernels and present the basic properties of microlocal kernels. The most important part of the chapter is the existence of microlocal kernels for dynamically admissible open sets.

Chapter 3 is one main part of the thesis. We review the original definition of the Chiu-Tamarkin complex and we develop the definition of the S^1-equivariant Chiu-Tamarkin complex using a cyclic structure based on projector properties of microlocal kernels. We also use the projector properties to define a cup product on a non-equivariant Chiu-Tamarkin complex, which recover the usual cup product on $H^*(X, \mathbb{K})$ when $U = T^*X$. We also study the invariance, functorial properties, and fundamental class for the Chiu-Tamarkin complex. These ingredients help us to define symplectic and contact capacities. The idea of the definition of the capacities is a generalization of Chiu's idea on contact non-squeezing theorem.

Chapter 4 is another main part of the thesis. It consists of two computations. One for convex toric domains and one for unit disk bundles over compact manifolds. We will present the proof of the structure theorem of the Chiu-Tamarkin complex for convex toric domains and the computation of capacities there. On the unit disk bundle, we prove the Viterbo isomorphism of the Chiu-Tamarkin complex and compare the cup product and the Chas-Sullivan product in the non-equivariant case.

We will discuss the results and make some conjectures in the Chapter 5.

In the appendix, we present a review on the equivariant derived category and the Borel-Moore homology.

Notations and conventions

In this short paragraph, we collect our general notations and conventions. We will not specify them in the later chapters.

The natural number $\mathbb{N}$ will start from 1, and $\mathbb{N}_0$ will denote $\mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$, we denote $[n]_0 = \{0, 1, \ldots, n\}$, and for $n \in \mathbb{N}$, we denote $[n] = \{1, \ldots, n\}$. For any $\ell \in \mathbb{N}_{\geq 2}$, we denote the minimal prime factor of ℓ by p_ℓ.

When we write G, we mean a group. The cyclic group of order n is denoted by $\mathbb{Z}/n$ (not only the group of mod n integers), and S^1 is the unit circle in $\mathbb{C}$. So, we treat $\mathbb{Z}/n$ as a subgroup of S^1 by thinking of $\mathbb{Z}/n$ as the set of all n^{th} root of units.

We use subscripts to represent elements in sets. For example, $a \in A$ is denoted by A_a. In some situations, we use underlined $\underline{a} = (a_1, \ldots, a_n)$ to record elements in Cartesian products A^n. The product set itself is denoted by $A^n_{\underline{a}}$. For the Cartesian product A^n, we define $\delta_{A^n} : A \to A^n$ to be the diagonal map and its image is denoted by Δ_{A^n} as well.

Projection maps of product spaces are always denoted by π, with a subscript that encode the fiber of the projection. For example, if there are two sets X_x and Y_y, two projections are

$$\pi_Y = \pi_y : X_x \times Y_y \to X_x, \quad \pi_X = \pi_x : X_x \times Y_y \to Y_y.$$

If we have a trivial vector bundle $X \times V_v$, its summation map is

$$\mathrm{id}_X \times s^n_V = \mathrm{id}_X \times s^n_v : X \times V^n \to X \times V,\, (x, v_1, \ldots, v_n) \mapsto (x, v_1 + \cdots + v_n).$$

In all cases, we will ignore the id_X and only use $s^n_V = s^n_v$ for simplicity.

Usually, for a manifold X, we use $\mathbf{q} \in X$ to represent both points and local coordinates of X. Correspondingly, the canonical Darboux coordinate of T^*X will be denoted by $(\mathbf{q}, \mathbf{p})$. Vector spaces that are considered as parameter spaces are an exception. For example, $\mathbb{R}_t$, its dual coordinate is denoted by $\tau \in (\mathbb{R}_t)^*$; for the time parameter $z \in \mathbb{R}^m$, its dual coordinate is denoted by $\zeta \in (\mathbb{R}^m_z)^*$.

In the cotangent bundle T^*X, we set $\dot{T}^*X = T^*X \setminus 0_X$. For $S \subset T^*X$, we set $\dot{S} = S \cap \dot{T}^*X$. We denote the bundle projection by $p_X : T^*X \to X$.

The 1-jet space of X is $J^1(X) = T^*X \times \mathbb{R}_t$, which is a contact manifold equipped with the contact form $\alpha = dt + \mathbf{p}d\mathbf{q}$. The symplectization of $J^1(X)$ is identified with $T^*X \times T^*_{\tau>0}\mathbb{R}_t = T^*X \times \mathbb{R}_t \times \mathbb{R}_{\tau>0}$, equipped with the symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q} + d\tau \wedge dt$. The symplectic reduction of $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ with respect to the hypersurface $\{\tau = 1\}$ is denoted by ρ, which is identified with

$$\rho : T^*X \times T^*_{\tau>0}\mathbb{R}_t \to T^*X,\ (\mathbf{q}, \mathbf{p}, t, \tau) \mapsto (\mathbf{q}, \mathbf{p}/\tau). \tag{0.1}$$

We call it the Tamarkin's cone map. The map ρ factors through the symplectization map q tautologically:

$$\begin{tikzcd} T^*X \times T^*_{\tau>0}\mathbb{R}_t \arrow[r, "q"] \arrow[rr, bend right, "\rho"] & J^1(X) \arrow[r] & T^*X. \end{tikzcd}$$

Let $f : X \to Y$ be a smooth map of manifolds. Then there is a diagram of cotangent maps:

$$\begin{tikzcd} T^*X \arrow[dr, "p_X"'] & X \times_Y T^*Y \arrow[l, "df^*"'] \arrow[r, "f_\pi"] \arrow[d, "p"] & T^*Y \arrow[d, "p_Y"] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \tag{0.2}$$

The coefficient ring $\mathbb{K}$ is a commutative Noetherian ring with finite global dimension. Usually, we will take $\mathbb{K}$ to be $\mathbb{Z}$ or a field.

For a manifold X, let us denote $D(X)$ the derived category of complexes of sheaves of $\mathbb{K}$-module over X. Let us remark that we do not specify the boundedness of complexes we used in general. But in most of our applications, the complexes are locally bounded in the sense that their restrictions on relatively compact open sets are bounded.

For a smooth map $f : X \to Y$, we have the standard six operations formalism: $\mathrm{R}\mathcal{H}om$, $\overset{L}{\otimes}$, $\mathrm{R}f_*$, f^{-1}, $\mathrm{R}f_!$, $f^!$. The relative dualizing complex is defined as $\omega_{X/Y} \coloneqq f^!\mathbb{K}_Y$. For $F \in D(X), G \in D(Y)$, we set $F\overset{L}{\boxtimes}G = \pi_Y^{-1}F\overset{L}{\otimes}\pi_X^{-1}G$.

For a locally closed inclusion $i : Z \subset X$ and $F \in D(X)$ we set

$$F|_Z = i^{-1}F, \quad F_Z = i_!i^{-1}F, \quad \mathrm{R}\Gamma_Z F = i_*i^!F.$$

For a sheaf $F \in D(X \times Y)$, and a locally closed subset $Z \subset Y$, we will write $F|_{X\times Z}$ by $F|_Z$ as well if there is no confusion.

For more details on the derived category and six operations, we refer to the first two chapters of [**KS90**].

CHAPTER 1

Microsupport and kernel calculus

In this chapter, I would like to review the notation and basic computations that will be utilized very often. In the first section, we will review the notion of microsupport and its functorial estimates, they play a fundamental role in the theory of the whole thesis. Next, we would like to discuss the calculus of kernels, in particular, the composition and convolution operations of sheaves. They are operations defined by the usual sheaf 6-operations, they share the idea of integral kernels in analysis. Finally, we review one particular class of kernels we need: The Guillermou-Kashiwara-Schapira sheaf quantization. They are kernels that correspond to Hamiltonian actions.

Most of the results in the chapter are well-known. In Subsection 1.3.1, we present a concrete formula of the sheaf quantization of the geodesic flow, which is known to experts but does not appear in the literature.

1.1. Microsupport of sheaves and functorial estimates

1.1.1. Reminder on the microsupport and its variants.

Definition 1.1 ([**KS90**, Definition 5.1.2])**.** For $F \in D(X)$ the microsupport of F is

$$SS(F) = \overline{\left\{ (\mathbf{q},\mathbf{p}) \in T^*X : \begin{array}{c} \text{There is a } C^1\text{-function } f \text{ near } \mathbf{q} \text{ such that} \\ f(\mathbf{q}) = 0,\, df(\mathbf{q}) = \mathbf{p} \text{ and } \left(\mathrm{R}\Gamma_{\{f\geq 0\}}F\right)_{\mathbf{q}} \neq 0. \end{array} \right\}}.$$

Let us look at some basic examples. They can be derived directly from the definition or by the functorial estimates in the Subsection 1.1.2.

Example 1.2. (1) If $\mathcal{L}$ is a non-zero locally constant sheaf on M, then $SS(\mathcal{L}) = 0_M$. Actually, we also have the converse direction:

Theorem 1.3 ([**KS90**, Theorem 5.4.5(ii)(c)]). *For $F \in D(X)$, we have the equivalence:*

$$SS(F) \subset 0_X \text{ if and only if } \forall k \in \mathbb{Z}, \mathcal{H}^k(F) \text{ are local systems.}$$

Consequently, we care more about $\dot{SS}(F) = SS(F) \cap \dot{T}^*X$.

(2) If $i : Z \subset X$ is a closed inclusion with smooth boundary, let $N_i^* = \{(\mathbf{q}, -\tau v(\mathbf{q})) : \mathbf{q} \in \partial D, \tau \geq 0\}$ be the interior conormal of ∂Z, where v is the exterior normal vector field on ∂Z. Then $\dot{SS}(\mathbb{K}_Z) = N_i^*$.

(3) If $j : U \subset X$ is an open inclusion with smooth boundary ∂U, let $N_e^* = \{(\mathbf{q}, sv(\mathbf{q})) : \mathbf{q} \in \partial D, \tau \geq 0\}$ be the exterior conormal of ∂U. Then $\dot{SS}(\mathbb{K}_U) = N_e^*$.

(4) If $i : S \subset X$ is a closed submanifold, then $SS(\mathbb{K}_S) = T_S^*X$ is the conormal bundle of S.

(5) Let $(X, \mathcal{O}_X)$ be a complex manifold, $\mathcal{M}$ be a coherent D-module, i.e. a module over the ring sheaf $\mathcal{D}_X = \mathrm{Der}(\mathcal{O}_X)$ of holomorphic differential operators. Let $F = R\mathcal{H}om(\mathcal{M}, \mathcal{O}_X)$ be the solution complex, then Kashiwara and Schapira show $SS(F) = char(\mathcal{M})$[**KS90**, Theorem 11.3.3], the characteristic of $\mathcal{M}$.

There are some basic properties of the microsupport:

- The microsupport is a conic closed subset of T^*X.
- $SS(F) \cap 0_X = p_X(SS(F)) = \mathrm{supp}(F)$ if we identify X with zero section 0_X, where p_X is the cotangent projection.
- The microsupport satisfies the triangular inequality: For a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, then for $a, b, c \in \{1, 2, 3\}$, we have

$$SS(F_a) \subset SS(F_b) \cup SS(F_c), \quad b \neq c,$$
$$SS(F_a) \Delta SS(F_b) \subset SS(F_c), \quad c \neq a, b.$$

The conicity is an issue since we want to consider general subsets of T^*X. We will use the Tamarkin's cone map ρ of (0.1) to resolve the conicity. This is important because most of symplectic geometric problems are non-conic. However, the cone map is only defined when $\tau > 0$ and it is helpful to introduce the Legendre microsupport and the sectional microsupport as follows: For sheaves $F \in D(X \times \mathbb{R}_t)$, we set

$$\begin{aligned} \mu s_L(F) &= q\left(SS(F) \cap \{\tau > 0\}\right) \subset J^1 X, \\ \mu s(F) &= \rho\left(SS(F) \cap \{\tau > 0\}\right) \subset T^* X. \end{aligned} \tag{1.1}$$

A direct consequence is that $\mu s_L(F)$ and $\mu s(F)$ are non-conic. However, $\mu s_L(F)$ and $\mu s(F)$ will lose $\tau \leq 0$ information. Usually, we will consider sheaves that satisfy $SS(F) \subset \{\tau \geq 0\}$ and it will be often the case, in practice, that $SS(F) \cap \{\tau \leq 0\} \subset 0_{X\times\mathbb{R}}$. So, the Theorem 1.3 shows that we will not lose much information.

1.1.2. Functorial estimate. The following functorial estimates are crucial and fundamental to the entire thesis. For the definitions of df^* and f_π, see (0.2).

Theorem 1.4 ([**KS90**, Theorem 5.4]). *Let $f : Y \to X$ be a C^∞ map of manifolds, $F \in D(Y), G \in D(X)$.*

(1) One has

$$\begin{aligned} &SS(F \overset{L}{\boxtimes} G) \subset SS(F) \times SS(G), \\ &SS(\mathrm{R}\mathcal{H}om(\pi_X^{-1}F, \pi_Y^{-1}G)) \subset (-SS(F)) \times SS(G). \end{aligned}$$

(2) Assume f is proper on $\operatorname{supp}(F)$, then $SS(\mathrm{R}f_!F) \subset f_\pi (df^*)^{-1}\left(SS(F)\right)$.

(3) Assume f is non-characteristic for $SS(G)$, i.e., if $(\boldsymbol{q}, \boldsymbol{p}) \in \dot{S}S(G)$ and $df^*_{\boldsymbol{q}}(\boldsymbol{p}) = 0$, then we have $\boldsymbol{p} = 0$. Then the morphism $f^{-1}G \overset{L}{\otimes} \omega_{X/Y} \to f^!G$ is an isomorphism, and $SS(f^{-1}G) \cup SS(f^!G) \subset df^* f_\pi^{-1}\left(SS(G)\right)$.

(4) Assume f is a submersion. Then $SS(F) \subset Y \times_X T^*X$ if and only if $\forall j \in \mathbb{Z}$, the sheaves $\mathcal{H}^j(F)$ are locally constant on the fibres of f.

Corollary 1.5. *Let $F_1, F_2 \in D(X)$.*

(1) Assume $SS(F_1) \cap (-SS(F_2)) \subset 0_X$, then $SS(F_1 \overset{L}{\otimes} F_2) \subset SS(F_1) + SS(F_2)$.

(2) Assume $SS(F_1) \cap SS(F_2) \subset 0_X$, then $SS(\mathrm{R}\mathcal{H}om(F_2, F_1)) \subset (-SS(F_2)) + SS(F_1)$.

Corollary 1.6. *For $F \in D(X)$, let $\phi : X \to \mathbb{R}$ be a C^1-function that is proper on* $\mathrm{supp}(F)$*. Let $a < b$ in $\mathbb{R}$ and assume $d\phi(x) \notin SS(F)$ for $a \leq \phi(x) < b$. Then the natural morphisms $\mathrm{R}\Gamma(\{\phi(x) < b\}, F) \to \mathrm{R}\Gamma(\{\phi(x) < a\}, F)$ and $\mathrm{R}\Gamma_{\{\phi(x)\geq b\}}(X, F) \to \Gamma_{\{\phi(x)\geq a\}}(X, F)$ are isomorphisms.*

The Corollary 1.6 is called the microlocal Morse lemma. In particular, if we take $F = \mathbb{K}_X$, we will get the cohomology version of the deformation lemma in classical Morse theory. For the non-proper pushforward, we have

THEOREM 1.7 ([**Tam18**, Corollary 3.4]). *Let V be an $\mathbb{R}$-vector space, $\pi_V : X \times V \to X$, and $\pi_V^{\#} : T^*X \times V \times V^* \to T^*V \times V^*$ be the corresponding projections, and $i : T^*X \to T^*V \times V^*$ be the inclusion. Then for $F \in D(X \times V)$, we have*

$$SS(\pi_{V!}F), SS(\pi_{V*}F) \subset i^{-1}\overline{\pi_V^{\#}(SS(F))}.$$

In this thesis, we will frequently use the equivariant derived category $D_G(X)$ for a G-space X, see appendix A for its definition.

Definition 1.8. For an object $F = (F_X, \overline{F}, \beta) \in D_G(X)$, where $F_X \in D(X)$, we *define* the microsupport of F to be $SS(F) \coloneqq SS(F_X)$.

This definition makes sense by (A.1), Theorem 1.4-(4), and Theorem 1.3.

1.2. Compositions and convolutions

The composition and convolution are the most important sheaf operations in the following geometric application. I will review the definitions and their microsupport estimates.

Let $X_i, i = 1, 2, 3, 4$ be smooth manifolds. In section, we will use the notation $X_{1234} = X_1 \times X_2 \times X_3 \times X_4$, $X_{123} = X_1 \times X_2 \times X_3$, $X_{jk} = X_j \times X_k$, $j, k = 1, 2, 3$. We will ignore some $\mathbf{q}$ (See notation and conventions) in the notation of projections to simplify.

1.2.1. Composition. Let's introduce composition first. Composition is a bifunctor:

$$\begin{aligned}\underset{X_2}{\circ}:&D(X_{12}) \times D(X_{23}) \to D(X_{13}),\\ &(F_1, F_2) \mapsto F_1 \circ F_2 = \mathrm{R}\pi_{2!}(\pi_3^{-1} F_1 \overset{L}{\otimes} \pi_1^{-1} F_2).\end{aligned} \tag{1.2}$$

Here, the projections are from X_{123} to X_{jk}. Sometimes, we also use subscript $\underset{\mathbf{q}_2}{\circ}$, $\mathbf{q}_2 \in X_2$. If there is no confusion, we could omit the subscript.

Example 1.9. As corollaries of the proper base change formula and the projection formula, one can show:

(1) For four manifolds X_i, $i = 1, 2, 3, 4$, $F_j \in D(X_{j,j+1})$, $j = 1, 2, 3$, we have

$$\left(F_1 \underset{X_2}{\circ} F_2\right) \underset{X_3}{\circ} F_3 \cong F_1 \underset{X_2}{\circ} \left(F_2 \underset{X_3}{\circ} F_3\right) \cong \mathrm{R}\pi_{23!}(\pi_{34}^{-1} F_1 \overset{L}{\otimes} \pi_{14}^{-1} F_2 \overset{L}{\otimes} \pi_{12}^{-1} F_3), \tag{1.3}$$

where π are projections.

(2) Let $X_1 = X_2 = X$, $X_3 = Y$. Let $\Delta_{X^2} \subset X \times X$ be the diagonal, then we have

$$\mathbb{K}_{\Delta_{X^2}} \underset{X}{\circ} F \cong F, \quad \text{for any } F \in D(X \times Y). \tag{1.4}$$

(3) If we identify products of X_1 and X_3 by $v : X_1 \times X_3 \to X_3 \times X_1, (\mathbf{q}_1, \mathbf{q}_3) \mapsto (\mathbf{q}_3, \mathbf{q}_1)$, then for $(F_1, F_2) \in D(X_{12}) \times D(X_{23})$, v induces a natural isomorphism

$$F_1 \underset{X_2}{\circ} F_2 \cong F_2 \underset{X_2}{\circ} F_1. \tag{1.5}$$

(4) Let $X_1 = X$, $X_2 = Y$, $X_3 = \{\mathrm{pt}\}$, $y \in Y$, and $\mathbb{K}_{\{y\}} \in D(Y)$ be the skyscraper sheaf. Then

$$F \underset{Y}{\circ} \mathbb{K}_{\{y\}} \cong F|_{X \times \{y\}}. \tag{1.6}$$

Remark 1.10. All isomorphisms above are natural. Moreover, we can verify some compatible conditions, which makes $\left(D(X \times X), \circ, \mathbb{K}_{\Delta_{X^2}}\right)$ a symmetric monoidal category. It can act on $D(X \times Y)$ by the composition.

To describe the effect of composition on the microsupport, we first introduce composition of sets. Suppose we have two correspondences, i.e. subsets, $\Lambda \subset T^*X_{12} \cong T^*X_1 \times T^*X_2$, $\Lambda' \subset T^*X_{23} \cong T^*X_2 \times T^*X_3$. Recall that $p_{X_{ij}} = p_{ij}$ is the cotangent projection to X_{ij}, then we set

$$\Lambda \circ \Lambda' := p_{13}(p_{12}^{-1}(-\Lambda) \cap p_{23}^{-1}(\Lambda')). \tag{1.7}$$

Then apply the estimate Theorem 1.4. We have that if

$$\begin{cases} a)\, \mathrm{supp}(F_1) \times_{X_2} \mathrm{supp}(F_2) \to X_{13} \text{ is proper, and} \\ b)\, \left(p_{12}^{-1}(-SS(F_1)) \cap p_{23}^{-1}(SS(F_2))\right) \bigcap (0_{X_1} \times T^*X_2 \times 0_{X_3}) \subset 0_{X_{123}}, \end{cases}$$

then we have

$$SS(F_1 \circ F_2) \subset SS(F_1) \circ SS(F_2). \tag{1.8}$$

It means that microsupport of a composition is bounded by the composition of microsupports.

The composition operation has a relative version. Let I be another manifold. Suppose $F_1 \in D(X_{12} \times I), F_2 \in D(X_{23} \times I)$, then we set

$$F_1 \circ_I F_2 := \mathrm{R}\pi_{2!}(\pi_3^{-1} F_1 \overset{L}{\otimes} \pi_1^{-1} F_2), \tag{1.9}$$

where the projections are from $X_{123} \times I$ to $X_{jk} \times I$. On the set level, we define the relative composition by

$$\Lambda \circ_I \Lambda' := r_2(r_{3^a}^{-1}(\Lambda) \cap r_1^{-1}(\Lambda')), \tag{1.10}$$

where the maps $T^*X_1 \times T^*X_2 \times T^*X_3 \times (T^*I \times_I T^*I) \to T^*X_i \times T^*X_j \times T^*I$ are given as follows:

$$\begin{aligned} r_{3^a} &: (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \zeta_1, \zeta_2) \mapsto (\mathbf{p}_1, -\mathbf{p}_2, \zeta_1), \\ r_1 &: (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \zeta_1, \zeta_2) \mapsto (\mathbf{p}_2, \mathbf{p}_3, \zeta_2), \\ r_2 &: (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \zeta_1, \zeta_2) \mapsto (\mathbf{p}_1, \mathbf{p}_3, \zeta_1 + \zeta_2). \end{aligned}$$

Similarly, we assume that

$$\begin{cases} a)\ \operatorname{supp}(F_1) \times_{X_2 \times I} \operatorname{supp}(F_2) \to X_{13} \times I \text{ is proper, and} \\ b)\ \left(r_{3^a I^a}^{-1}(\Lambda) \cap r_1^{-1}(\Lambda')\right) \bigcap (0_{X_1} \times T^*X_2 \times 0_{X_3} \times T^*I) \subset 0_{X_{123} \times I}. \end{cases}$$

where

$$r_{3^a I^a} : T^*X_1 \times T^*X_2 \times T^*X_3 \times T^*I \to T^*X_1 \times T^*X_2 \times T^*I, (\xi_1, \xi_2, \xi_3, u) \mapsto (\xi_1, -\xi_2, -u).$$

Then we have the following estimate:

$$SS(F_1 \circ_I F_2) \subset SS(F_1) \circ_I SS(F_2) \tag{1.11}$$

1.2.2. Convolution. Next, let us introduce the convolution.

The convolution over $\mathbb{R}$ is a bifunctor:

$$\begin{aligned} \underset{X_2}{\star} : D(X_{12} \times \mathbb{R}_{t_1}) \times & D(X_{23} \times \mathbb{R}_{t_2}) \to D(X_{13} \times \mathbb{R}_{t=t_1+t_2}), \\ (F_1, F_2) \mapsto & F_1 \underset{X_2}{\star} F_2 = \mathrm{R}s^2_{t!} \mathrm{R}\pi_{\mathbf{q}_2 !}(\pi^{-1}_{(\mathbf{q}_3, t_2)} F_1 \overset{L}{\otimes} \pi^{-1}_{(\mathbf{q}_1, t_1)} F_2). \end{aligned} \tag{1.12}$$

Sometimes, we also use subscript $\underset{\mathbf{q}_2}{\star}$, $\mathbf{q}_2 \in X_2$. For simplicity, we will omit the superscript and subscript if there is no confusion.

In particular, when X_2 is a point, we also use the notation $F_1 \boxast F_2$ to empathise.

Remark 1.11. (1) In some cases, convolution could be presented by composition on $X \times \mathbb{R}_t$. For example, if $F \in D(X^2 \times \mathbb{R}_{t_1})$ and $G \in D(X \times \mathbb{R}_{t_2})$. Let $m(t, t') = t - t' = t_1$, then we have

$$F \star G \cong (m^{-1}F) \circ G.$$

In fact, taking $t' = t_2$, we can prove the isomorphism by the proper base change and the projection formula since $s_t^2(t_1, t_2) = t$.

But convolution involves on spaces of lower dimension. Hence, we prefer to use convolution in this paper. More important, in geometric applications, the factor $\mathbb{R}_t$ will play the role of action. Then, convolutions are more helpful for us to look at action information.

(2) It is easy to generalize the convolution over $\mathbb{R}$ to convolution over a higher dimensional vector space V. This is unnecessary in this thesis, so we will not do it.

Example 1.12. Similarly to the case of composition, the proper base change formula and the projection formula show that

(1) For four manifolds X_i, $i = 1, 2, 3, 4$, $F_j \in D(X_{j,j+1} \times \mathbb{R})$, $j = 1, 2, 3$, $F_4 \in D(X_{34})$, we have the isomorphisms in $D(X_{14} \times \mathbb{R})$,

$$\begin{aligned}
&\left(F_1 \underset{X_2}{\star} F_2\right) \underset{X_3}{\star} F_3 \cong F_1 \underset{X_2}{\star} \left(F_2 \underset{X_3}{\star} F_3\right) \\
&\qquad \cong \mathrm{R}s^3_{t!}\mathrm{R}\pi_{(\mathbf{q}_2,\mathbf{q}_3)!}(\pi^{-1}_{(\mathbf{q}_3,\mathbf{q}_4,t_2,t_3)}F_1 \overset{L}{\otimes} \pi^{-1}_{(\mathbf{q}_1,\mathbf{q}_4,t_1,t_3)}F_2 \overset{L}{\otimes} \pi_{(\mathbf{q}_1,\mathbf{q}_2,t_1,t_2)}F_3), \\
&\left(F_1 \underset{X_2}{\star} F_2\right) \underset{X_3}{\circ} F_4 \cong F_1 \underset{X_2}{\star} \left(F_2 \underset{X_3}{\circ} F_4\right) \\
&\qquad \cong \mathrm{R}s^2_{t!}\mathrm{R}\pi_{(\mathbf{q}_2,\mathbf{q}_3)!}(\pi^{-1}_{(\mathbf{q}_3,\mathbf{q}_4,t_2)}F_1 \overset{L}{\otimes} \pi^{-1}_{(\mathbf{q}_1,\mathbf{q}_4,t_1)}F_2 \overset{L}{\otimes} \pi^{-1}_{(\mathbf{q}_1,\mathbf{q}_2,t_1,t_2)}F_4).
\end{aligned} \tag{1.13}$$

(2) Let $X_1 = X_2 = X$, $X_3 = Y$, and let $\Delta_{X^2} \subset X \times X$ be the diagonal. Then for any $F \in D(X \times Y \times \mathbb{R})$, we have

$$\mathbb{K}_{\Delta_{X^2} \times \{0\}} \underset{X}{\star} F \cong F. \tag{1.14}$$

(3) If we identify products of X_1 and X_3 by $v : X_1 \times X_3 \to X_3 \times X_1, (\mathbf{q}_1, \mathbf{q}_3) \mapsto (\mathbf{q}_3, \mathbf{q}_1)$, then for any $(F_1, F_2) \in D(X_{12} \times \mathbb{R}) \times D(X_{23} \times \mathbb{R})$, there is

$$F_1 \underset{X_2}{\star} F_2 \cong F_2 \underset{X_2}{\star} F_1. \tag{1.15}$$

(4) Let $X_1 = X$, $X_2 = Y$, $X_3 = \{\mathrm{pt}\}$, $y \in Y$, and let $\mathbb{K}_{\{(y,0)\}} \in D(Y \times \mathbb{R})$ be the skyscraper sheaf. Then for any $F \in D(X \times Y \times \mathbb{R})$

$$F \underset{Y}{\star} \mathbb{K}_{\{(y,0)\}} \cong F|_{X \times \{y\} \times \mathbb{R}}. \tag{1.16}$$

(5) Let $X_1 = X$, $X_2 = X_3 = \{\mathrm{pt}\}$, $c \in \mathbb{R}$, $\mathrm{T}_c : X \times \mathbb{R} \to X \times \mathbb{R}, (\mathbf{q}, t) \mapsto (\mathbf{q}, t + c)$. Then for any $F \in D(X \times \mathbb{R})$, we have

$$F \star \mathbb{K}_{\{c\}} \cong \mathrm{T}_{c*}F. \tag{1.17}$$

Remark 1.13. We also remark that all isomorphisms here are natural. Moreover, we can verify some compatible conditions, which makes $\left(D(X \times X \times \mathbb{R}), \star, \mathbb{K}_{\Delta_{X^2} \times \{0\}}\right)$ a symmetric monoidal category. This category acts on $D(X \times Y \times \mathbb{R})$ by convolution.

If we take $Y_1 = X_1 \times \mathbb{R}_1$, $Y_2 = X_2$, $Y_3 = X_3 \times \mathbb{R}_2$. Then we have

$$F_1 \star F_2 \cong \mathrm{R}s^2_{t!}(F_1 \circ_{Y_2} F_2).$$

Noticed that $ds_t^{2*}(\tau) = (\tau, \tau)$. Then $SS(\mathrm{R}s_{t!}F)$ is the symplectic reduction of $SS(F)$ with respect to $\{\tau_1 = \tau_2\}$ under some properness conditions. Then we can deduce the microsupport estimates of convolution using the microsupport estimates of composition.

1.3. The Guillermou-Kashiwara-Schapira sheaf quantization

As a sheaf pattern of Hamiltonian action, we introduce the Guillermou-Kashiwara-Schapira (GKS for short) sheaf quantization as a basic tool here, see [GKS12] for more details. It also provides many important composition kernels and convolution kernels. So far, most known kernels arise from GKS sheaf quantization. On the other hand, Li construct a Lagrangian cobordism functor in [Li21b], which isomorphic the convolution functor defined by GKS sheaf quantization.

Consider $\dot{T}^*Y$ as a symplectic manifold equipped with the Liouville symplectic form and with a $\mathbb{R}_{>0}$-action by dilation along the cotangent fibers. If $\varphi : \dot{T}^*Y \times I \to \dot{T}^*Y$ is a $\mathbb{R}_{>0}$-equivariant symplectic isotopy, one can show that it must be Hamiltonian with a $\mathbb{R}_{>0}$-equivariant Hamiltonian function H.

Consider its total graph $\Lambda_\varphi \subset \dot{T}^*(I \times Y^2)$:

$$\Lambda_\varphi \coloneqq \left\{(z, -H_z \circ \varphi_z(\mathbf{q}, \mathbf{p}), (\mathbf{q}, -\mathbf{p}), \varphi_z(\mathbf{q}, \mathbf{p})) : (\mathbf{q}, \mathbf{p}) \in \dot{T}^*Y, z \in I\right\}. \tag{1.18}$$

Besides, for each $z_0 \in I$, the graph of φ_{z_0} is the symplectic reduction of Λ_φ with respect to $\{z = z_0\}$

$$\Lambda_{\varphi_{z_0}} \coloneqq \left\{((\mathbf{q}, -\mathbf{p}), \varphi_{z_0}(\mathbf{q}, \mathbf{p})) : (\mathbf{q}, \mathbf{p}) \in \dot{T}^*Y\right\} \subset \dot{T}^*(Y^2). \tag{1.19}$$

Then Guillermou, Kashiwara, and Schapira proved the following theorem:

THEOREM 1.14 ([**GKS12**, Theorem 3.7]). *Using the above notation, we have a sheaf* $K = K(\varphi) \in D(I \times Y^2)$ *such that*

(1) $\dot{SS}(K) = \Lambda_\varphi$, *(2)* $K_0 = \mathbb{K}_{\Delta_{Y^2}}$, *where* $K_z = K|_{\{z\}\times Y^2}$.

If we set $K_z^{-1} = v^{-1}\mathrm{R}\mathcal{H}om(K_z, \omega_Y \overset{L}{\boxtimes} \mathbb{K}_Y)$, $v(\boldsymbol{q}_1, \boldsymbol{q}_2) = (\boldsymbol{q}_2, \boldsymbol{q}_1)$, $\boldsymbol{q}_1, \boldsymbol{q}_2 \in Y$, $z \in I$, *then*

a) $\mathrm{supp}(K) \rightrightarrows I \times Y$ *are both proper,*

b) $K_z \circ K_z^{-1} \cong K_z^{-1} \circ K_z \cong \mathbb{K}_{\Delta_{Y^2}}$,

c) K *is unique up to a unique isomorphism.*

Consequently, $F \mapsto K_z \circ F$, $D(Y) \to D(Y)$ *is an equivalence of categories for all* $z \in I$, *whose quasi inverse is* $K_z^{-1} \circ F$.

Example 1.15. Let us present the simplest example here. Take $Y = \mathbb{R}^d$ equipped with the standard norm. Consider the flow

$$\varphi_z(\mathbf{q}, \mathbf{p}) = (\mathbf{q} + z\mathbf{p}/\|\mathbf{p}\|, \mathbf{p}) : \dot{T}^*Y \to \dot{T}^*Y.$$

This is the flow associated with the Hamiltonian function $H(\mathbf{q}, \mathbf{p}) = \|\mathbf{p}\|$. The sheaf quantization K of the flow fits in a distinguished triangle:

$$\mathbb{K}_{\{(z,\mathbf{q}_1,\mathbf{q}_2):\{\|\mathbf{q}_1-\mathbf{q}_2\|<z\}}[d] \to K \to \mathbb{K}_{\{(z,\mathbf{q}_1,\mathbf{q}_2):\|\mathbf{q}_1-\mathbf{q}_2\|\leq -z\}} \xrightarrow{+1}.$$

Remark 1.16. (1) Microlocally, one can shows that $i_z : Y^2 \to I \times Y^2$ is non-characteristic for Λ_φ for all $z \in I$. Then, we have $\dot{SS}(K_z) = \Lambda_{\varphi_z}$, and for $F \in D(Y)$, we have

$$\dot{SS}(K_z \circ F) = \varphi_z^{-1}(\dot{SS}(F)). \tag{1.20}$$

It means that, geometrically, $K_z \circ$ acts as the Hamiltonian isotopy φ^{-1}.

(2) By the construction of the GKS quantization, we know that it is likely that there exists a topological space and a locally closed subset $Z \subset I \times Y^2 \times W$ such that $K \cong \mathrm{R}\pi_{W!}\mathbb{K}_Z$, when I is a finite interval.

Let us describe three situations where we will use the theorem.

1.3.1. Geodesic flows. Here, let us study the geodesic flow for a general complete Riemannian manifold, which generalizes the result in Example 1.15.

Assume (X, g) is a complete Riemannian manifold. Let us take the homogeneous Hamiltonian function $H(\mathbf{q}, \mathbf{p}) = |\mathbf{p}|_g$. Then the associated $\mathbb{R}_{>0}$-equivariant Hamiltonian flow is the (normalized) geodesic flow φ_z^{geo} (we identify T^*X and TX using the metric g). Now, applying the GKS theorem with $Y = X$, we know there is a sheaf $K_g \in D(\mathbb{R}_z \times X^2)$ that quantizes the geodesic flow. In the following we give an explicit formula of K_g.

We assume that the convex radius $r_{\mathrm{conv}}(X, g) > 2$, then the injective radius $r_{\mathrm{inj}}(X, g) > 4$. First of all, let us consider the small time z situation. Then [**PS21**] shows that, if we restrict to $I = (-2, 2) \subset \mathbb{R}_z$, we have

$$\mathbb{K}_{(z,\mathbf{q}_1,\mathbf{q}_2):\{d(\mathbf{q}_1,\mathbf{q}_2)<z\}} \overset{L}{\otimes} q_2^{-1}\omega_X \to K_g|_{(-2,2)_z} \to \mathbb{K}_{\{(z,\mathbf{q}_1,\mathbf{q}_2):d(\mathbf{q}_1,\mathbf{q}_2)\leq -z\}} \xrightarrow{+1} . \tag{1.21}$$

Our problem is to extent the formula to larger z. Since H is autonomous, on the flow level, we have $\varphi_{z_1+z_2} = \varphi_{z_1} \circ \varphi_{z_2}$. This is also true for the kernel $K_{g,z}$. In fact, for $z_1, z_2 \in \mathbb{R}$, we have $K_{g,z_1} \circ K_{g,z_2} \cong K_{g,z_1+z_2}$. In particular, for $N \in \mathbb{N}$, we have

$$K_{g,Nz} \cong K_{g,z} \circ \cdots \circ K_{g,z}.$$

Based on this slicewise formula, we first set $K_g^1 = K_g|_{(-2,2)_z}$, then let

$$K_g^N \cong r_N^{-1}(K_g^1 \circ_I \cdots \circ_I K_g^1), \tag{1.22}$$

where $r_N(z, \mathbf{q}_1, \mathbf{q}_2) = (z/N, \mathbf{q}_1, \mathbf{q}_2)$.

The microsupport estimate (1.11) shows

$$\dot{SS}(K_g^1 \circ_I \cdots \circ_I K_g^1) \subset \{(z, -N|p|_g, \mathbf{q}, -\mathbf{p}, \mathbf{q}', \mathbf{p}') : (\mathbf{q}', \mathbf{p}') = \varphi_{Nz}^{\mathrm{geo}}(\mathbf{q}, \mathbf{p}), z \in (-2, 2)\}.$$

Then,

$$\dot{SS}(K_g^N) \subset \{(z, -|p|_g, \mathbf{q}, -\mathbf{p}, \mathbf{q}', \mathbf{p}') : (\mathbf{q}', \mathbf{p}') = \varphi_z^{\mathrm{geo}}(\mathbf{q}, \mathbf{p}), z \in (-2N, 2N)\}.$$

Besides, it is direct to verify that $K_g^N|_{\{z=0\}} \cong \mathbb{K}_{\Delta_{X^2}}$. Then the uniqueness part of the Theorem 1.14 shows

$$K_g|_{(-2N,2N)} \cong K_g^N. \tag{1.23}$$

For example,

$$K_g|_{(-2N,0]} \cong K_{g,-}^N \cong \mathrm{R}\pi_{(\mathbf{q}_1,\dots,\mathbf{q}_{N-1})!}\mathbb{K}_{\mathcal{M}^N X}, \tag{1.24}$$

where

$$\mathcal{M}^N X = \{(z, \mathbf{q}_0, \dots, \mathbf{q}_N) : d(\mathbf{q}_i, \mathbf{q}_{i+1}) \leq -\frac{z}{N},\ i \in [N-1]_0,\ -2N < z \leq 0\}, \tag{1.25}$$

is the discrete Moore path space and $[N-1]_0 = \{0, 1, \dots, N-1\}$.

We will use this formula to prove the Viterbo isomorphism of the Chiu-Tamarkin complex in Section 4.2.

1.3.2. Compactly supported Hamiltonian flows. Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy. For $Y = X \times \mathbb{R}_t$, one can lift φ to $\widehat{\varphi} : I \times \dot{T}^*Y \to \dot{T}^*Y$. Specifically, we have the following:

Proposition 1.17 ([GKS12, Proposition A.6]). *Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy, whose Hamiltonian function is $H \in C^\infty(I \times T^*X)$.*

There is an $\mathbb{R}_{>0}$-equivariant Hamiltonian isotopy $\widehat{\varphi} : I \times \dot{T}^*Y \to \dot{T}^*Y$ such that:

a) $\widehat{H} = \tau H(-, \rho(-))$ *is a Hamiltonian function of $\widehat{\varphi}$.*

b) *The lifting $\widehat{\varphi}$ commutes with both the symplectization and the Tamarkin's cone map.*

c) *We can take*

$$\begin{aligned} &\widehat{\varphi}(z, \boldsymbol{q}, t, \boldsymbol{p}, \tau) = (\tau \cdot \varphi(z, \boldsymbol{q}, \boldsymbol{p}/\tau), t + u(z, \boldsymbol{q}, \boldsymbol{p}/\tau), \tau), && \tau \neq 0, \\ &\widehat{\varphi}(z, \boldsymbol{q}, t, \boldsymbol{p}, 0) = (\boldsymbol{q}, \boldsymbol{p}, t + v(z), 0), && \tau = 0, \end{aligned}$$

where $u \in C^\infty(I \times T^*X), v \in C^\infty(I)$. In fact, the proof shows that $u(z, \boldsymbol{q}, \boldsymbol{p}) = S_H(z, \boldsymbol{q}, \boldsymbol{p}) = \int_0^z [\alpha(X_{H_\lambda}) - H_\lambda] \circ \varphi_H^x(\boldsymbol{q}, \boldsymbol{p}) d\lambda$ is the symplectic action function.

We call this $\widehat{\varphi}$ or $\widehat{\varphi}_z$ the conification of φ.

Remark 1.18. (1) We notice that it is easy to lift φ to $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ without the compactly supported assumption, but this is not enough to apply the Guillermou-Kashiwara-Schapira theorem. If we want to lift φ to $\dot{T}^*(X \times \mathbb{R}_t)$, we need the compactly supported condition.

(2) Recently, Chiu propose a construction of sheaf quantization of the lifting of φ to $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ when H defines a flow which is complete and short-term separating. See [**Chi21**].

Now, applying Theorem 1.14 to $\widehat{\varphi}$, we obtain a sheaf $K(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}^2_t)$.

In our later application, we prefer to use only one t-variable, and using convolution. This is possible. Consider $m(t_1, t_2) = t_1 - t_2$, then [**Gui19**, Corollary 2.3.2] shows there is a unique $\mathcal{K}(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}_t)$ such that $K(\widehat{\varphi}) \cong m^{-1}\mathcal{K}(\widehat{\varphi})$, and $\mathcal{K}(\widehat{\varphi}) \cong \mathrm{R}m_!K(\widehat{\varphi})$. Then we can take $\mathcal{K}(\widehat{\varphi})$ as the sheaf quantization of φ. Moreover, since we have $SS(K(\widehat{\varphi})_z) = \Lambda_{\widehat{\varphi}_z}$, we will denote $K(\widehat{\varphi}_z)$ and $\mathcal{K}(\widehat{\varphi}_z)$ by $K(\widehat{\varphi})_z$ and $\mathcal{K}(\widehat{\varphi})_z$ respectively.

One can show that, for $F \in D(X \times \mathbb{R})$, we have $K(\widehat{\varphi}_z) \circ F \cong \mathcal{K}(\widehat{\varphi}_z) \star F$, see (1) of Remark 1.11.

By the commutativity of the lifting with symplectization, we have the following estimates for the Legendrian microsupport and sectional microsupport of $\mathcal{K}(\widehat{\varphi})$:

(1.26)

$$\mu s_L(\mathcal{K}(\widehat{\varphi})) \subset \{(z, -H(\mathbf{q},\mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q},\mathbf{p}), -S_H(z,\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in I \times T^*X\},$$
$$\mu s(\mathcal{K}(\widehat{\varphi})) \subset \{(z, -H(\mathbf{q},\mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in I \times T^*X\}.$$

From the point of view of (1.20), for $F \in D(X \times \mathbb{R})$, we have

$$\mu s(\mathcal{K}(\widehat{\varphi}_z) \star F) = \mu s(K(\widehat{\varphi}_z) \circ F) = \varphi_z^{-1}(\mu s(F)). \tag{1.27}$$

When $z = 0$, $K(\widehat{\varphi})|_{z=0} = \mathbb{K}_{\Delta_{(X\times\mathbb{R})^2}}$. Then $\mathcal{K}(\widehat{\varphi})|_{z=0} = \mathbb{K}_{\Delta_{X^2}\times\{0\}}$.

1.3.3. Contact isotopies on prequantized cotangent bundles. Let $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$ be a contact isotopy of $T^*X \times S^1$ with a contact Hamiltonian $H \in C^\infty(I \times T^*X \times S^1)$. One can lift φ to a $\mathbb{Z}$-equivariant contact isotopy φ' of

$J^1(X) = T^*X \times \mathbb{R}_t$, where $\mathbb{Z}$ acts by shifting t. Here, by $\mathbb{Z}$-equivariant, we mean that $J^1(\mathrm{T}_k)\varphi' = \varphi' J^1(\mathrm{T}_k)$ for $k \in \mathbb{Z}$, where $J^1(\mathrm{T}_k)(\mathbf{q}, \mathbf{p}, t) = (\mathbf{q}, \mathbf{p}, t + k)$.

Remark 1.19. In the symplectic case the Hamiltonian H does not depend on t, and does commute with T'_c for all real number c. In the contact case, usually φ' does not commute with T'_c for all real number c and merely commutes with T'_k for $k \in \mathbb{Z}$.

Then it is easy to lift φ' to the symplectization, $T^*X \times T^*_{\tau>0}\mathbb{R}_t$, of $J^1(X)$ to a $\mathbb{Z} \times \mathbb{R}_{>0}$ equivariant Hamiltonian isotopy $\widehat{\varphi'} : I \times T^*X \times T^*_{\tau>0}\mathbb{R}_t \to T^*X \times T^*_{\tau>0}\mathbb{R}_t$. Here, by $\mathbb{Z}$-equivariance, we mean that $d\mathrm{T}^*_k\widehat{\varphi'} = \widehat{\varphi'}d\mathrm{T}^*_k$ for $k \in \mathbb{Z}$, where $d\mathrm{T}^*_k(\mathbf{q}, \mathbf{p}, t, \tau) = (\mathbf{q}, \mathbf{p}, t + k, \tau)$ is the cotangent map of the shifting map $\mathrm{T}_k(\mathbf{q}, t) = (\mathbf{q}, t + k)$.

Similarly to the symplectic case, the compactly supported condition is necessary to extend $\widehat{\varphi'}$ to whole $\dot{T}^*(X \times \mathbb{R}_t)$.

In this case, we still take the sheaf quantization $K = K(\widehat{\varphi'}) \in D(I \times X^2 \times \mathbb{R}^2)$ of $\widehat{\varphi'}$ as sheaf quantization of φ. However the contact Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ will depend on the variable t, $K = K(\widehat{\varphi'})$ is not pulled back from $D(I \times X^2 \times \mathbb{R})$ by m. So, we will work with compositions rather than convolutions.

The $\mathbb{Z}$-equivariance is inherited by the sheaf $K(\widehat{\varphi'})$. Precisely, it means that

$$K(\widehat{\varphi'}) \circ \mathbb{K}_{\Delta_{X^2} \times \{(t,t+k):t\in\mathbb{R}\}} \cong \mathbb{K}_{\Delta_{X^2} \times \{(t,t+k):t\in\mathbb{R}\}} \circ K(\widehat{\varphi'}). \tag{1.28}$$

This is due to $\mathbb{K}_{\Delta_{X^2} \times \{(t,t+k):t\in\mathbb{R}\}} = \mathbb{K}_{\Gamma_{\mathrm{T}_k}}$ quantizes $d\mathrm{T}^*_k$, then we apply the uniqueness part of Theorem 1.14 to $\widehat{\varphi'} = d(\mathrm{T}_k^{-1})^*\widehat{\varphi'}d\mathrm{T}^*_k = d\mathrm{T}^*_{-k}\widehat{\varphi'}d\mathrm{T}^*_k$ to obtain the isomorphism (1.28).

CHAPTER 2

Tamarkin categories and microlocal projectors

In this chapter, we study the Tamarkin category and its open/closed subset version. For an open set $U \subset T^*X$ and its complement $Z = T^*X \backslash U$. We consider two categories and projector functors into them:

$$\begin{aligned}\mathcal{D}_Z(X) =& \{F \in \mathcal{D}(X) : \mu s(F) \subset Z\}\\ \mathcal{D}_U(X) =& {}^{\perp}\mathcal{D}_Z(X), \text{ the left orthogonal complement of } \mathcal{D}_Z(X).\end{aligned}$$

A decomposition of the diagonal as in Definition 2.9 will create microlocal kernels associated with the projector functors. Microlocal kernels are sheaves, so they are more accessible for computation than projector functors. They play the role of "matrix" of projector functors.

In the first two sections, we will review the notion of the Tamarkin category, Tamarkin category associated with open/closed subsets, and the definition of microlocal kernels. We explain some direct consequences of the existence of these kernels in Section 2. Finally, we will review Chiu's construction of microlocal kernels and some corollaries of the construction.

Before going into further discussion, let us review the notion of semi-orthogonal decomposition of a triangulated category.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a thick full triangulated subcategory of $\mathcal{T}$. The left semi-orthogonal of $\mathcal{C}$ is defined by

$$^{\perp}\mathcal{C} \coloneqq \{X \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(X, Y) = 0, \forall Y \in \mathcal{C}\}. \tag{2.1}$$

One can show that the following proposition holds, see [KS06, Chapter 4 and Exercise 10.15.].

Proposition 2.1. *Using the above notation, we have the following three equivalent properties:*

(1) The inclusion $\mathcal{C} \to \mathcal{T}$ admits a left adjoint functor $L : \mathcal{T} \to \mathcal{C}$.

(2) There is an isomorphism $\mathcal{T}/\mathcal{C} \xrightarrow{\cong} {}^{\perp}\mathcal{C}$, where $\mathcal{T}/\mathcal{C}$ is the Verdier localization.

(3) There are two functors $P, Q : \mathcal{T} \to \mathcal{T}$ such that $\forall X \in \mathcal{T}$, we have the distinguished triangle:

$$P(X) \to X \to Q(X) \xrightarrow{+1}$$

such that $P(X) \in {}^{\perp}\mathcal{C}$, and $Q(X) \in \mathcal{C}$.

In this situation, we say one of these data gives a left semi-orthogonal decomposition of $\mathcal{T}$. One can verify that if one of the conditions here is satisfied, then $P^2 \cong P$, and $Q^2 \cong Q$. P, Q are called a pair of projectors associated with $\mathcal{C}$.

2.1. Tamarkin categories

Now, let $\mathcal{T} = D(X \times \mathbb{R}_t)$, and $\mathcal{C} = \{F : SS(F) \subset \{\tau \leq 0\}\}$. The triangulated inequality of microsupport shows that $\mathcal{C}$ is a thick full triangulated subcategory of $\mathcal{T}$. Tamarkin constructs a pair of projectors associated to $\mathcal{C}$ given by convolution functors:

Theorem 2.2 ([**Tam18**]). *The functors $F \mapsto \mathbb{K}_{[0,\infty)} \star F$, $F \mapsto \mathbb{K}_{(0,\infty)}[1] \star F$ on $D(X \times \mathbb{R}_t)$ and the excision triangle,*

$$\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0 \to \mathbb{K}_{(0,\infty)}[1] \xrightarrow{+1},$$

give a left semi-orthogonal decomposition of $D(X \times \mathbb{R}_t)$ associated to $\mathcal{C}$. Namely, for $F \in D(X \times \mathbb{R}_t)$ we have the distinguished triangle

$$\mathbb{K}_{[0,\infty)} \star F \to F \to \mathbb{K}_{(0,\infty)}[1] \star F \xrightarrow{+1}, \tag{2.2}$$

with $\mathbb{K}_{[0,\infty)} \star F \in {}^{\perp}\mathcal{C}$, $\mathbb{K}_{(0,\infty)}[1] \star F \in \mathcal{C}$.

One can also see [**GS14**, Proposition 4.19] for a proof and some generalizations of the proposition.

Definition 2.3. We define the Tamarkin category as the following left semi-orthogonal complement:

$$\mathcal{D}(X) = {}^{\perp}\{F : SS(F) \subset \{\tau \leq 0\}\} \cong D(X \times \mathbb{R}) / \{F : SS(F) \subset \{\tau \leq 0\}\}.$$

By Proposition 2.1 and (2.2) , $F \in D(X \times \mathbb{R})$ is in $\mathcal{D}(X)$ if and only if

$$F \cong \mathbb{K}_{[0,\infty)} \underset{\mathrm{pt}}{\star} F \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \underset{X}{\star} F. \tag{2.3}$$

Consequently, the convolution functor $\mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \underset{X}{\star}$ of the Tamarkin category $\mathcal{D}(X)$ coincides with the identity functor.

Remark 2.4. (1) For $F \in \mathcal{D}(X)$, one can show $SS(F) \subset \{\tau \geq 0\}$ using functorial estimates of microsupport, see [**GS14**, Proposition 3.17]. In general, let $\gamma = (-\infty, 0]$. Then $\mathcal{D}(X)$ is a full-subcategory of $D(X \times \mathbb{R}_\gamma)$ where we equip $\mathbb{R}$ with the γ-topology in the sense of [**KS90**, Section 3.5]. The microlocal cut-off lemma, proposition 5.2.3 in loc.cit., shows that $D(X \times \mathbb{R}_\gamma)$ is equivalence to $\{F \in D(X \times \mathbb{R}) : SS(F) \subset \{\tau \geq 0\}\}$ and $\{F \in D(X \times \mathbb{R}) : \mathbb{K}_{[0,\infty)} \star_{np} F \cong F\}$ where $\mathbb{K}_{[0,\infty)} \star_{np} F \coloneqq Rs^2_{t*}(\mathbb{K}_{[0,\infty)} \overset{L}{\boxtimes} F)$. Objects in these equivalent categories are called γ-sheaves.

(2) If X admits a G action and we put the trivial action on $\mathbb{R}$, then we can define the equivariant Tamarkin category $\mathcal{D}_G(X) \subset D_G(X \times \mathbb{R})$ in the same way using the microsupport of equivariant sheaves defined in Definition 1.8. We also have the Tamarkin projector using the equivariant 6-operations. Then nothing needs to be changed here. The discussion also applies for equivariant γ-sheaves.

2.1.1. Functors of Tamarkin categories. Consider, the translation map $\mathrm{T}_c : X \times \mathbb{R} \to X \times \mathbb{R}$, $(x,t) \mapsto (x, t+c)$. Then the push forward map $\mathrm{T}_{c*} : D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$ is a family of endfunctors.

For $F \in D(X \times \mathbb{R}_\gamma)$, we have $\mathrm{T}_{c*} \cong - \star_{np} \mathbb{K}_{\{c\}} \cong - \star_{np} \mathbb{K}_{[c,\infty)}$ by Example 1.12(3). Therefore, when $c \geq 0$, we deduce a family of natural transforms $\tau_c : \mathrm{Id} \to \mathrm{T}_{c*}$ that is induced by the restriction map $\mathbb{K}_{[0,\infty)} \to \mathbb{K}_{[c,\infty)}$ on $D(X \times \mathbb{R}_\gamma)$. For a γ-open set

$U = U + \gamma$, the natural morphism $\tau_c(F)$ is induced by

$$\mathrm{R}\Gamma(U, F) \to \mathrm{R}\Gamma(U + c, F) \cong \mathrm{R}\Gamma(U, \mathrm{T}_{c*}F).$$

Therefore, T_{c*} and τ_c commute with the 6-operations and adjunctions on $D(X \times \mathbb{R}_\gamma)$. For example, if $f : X \to Y$, consider $f_\mathbb{R} = f \times \mathrm{Id}_\mathbb{R} : X \times \mathbb{R} \to Y \times \mathbb{R}$. Then $f_\mathbb{R}$ is a continuous map on $X \times \mathbb{R}_\gamma$. Then we have

$$\tau_c(\mathrm{R}f_{\mathbb{R}*}F) = \mathrm{R}f_{\mathbb{R}*}(\tau_c(F)) : \mathrm{R}f_{\mathbb{R}*}F \to \mathrm{T}_{c*}\mathrm{R}f_{\mathbb{R}*} \cong \mathrm{R}\overline{f}_*\mathrm{T}_{c*}F.$$

For $F \in \mathcal{D}(X)$, we also have $\mathrm{T}_{c*} \cong \mathbb{K}_{\{c\}} \star_{np} - \cong \mathbb{K}_{\{c\}} \star - \cong \mathbb{K}_{[c,\infty)} \star -$ by (2.3). Then the natural transform τ_c restricts to $\mathcal{D}(X)$. It also commutes with the 6-operations and adjunctions.

The discussion also applies if X admits a group action and f is an equivariant map.

Geometrically, the functor T_{c*} quantizes the Reeb flow of the canonical contact form of J^1X. It also indicates that the extra variable $\mathbb{R}_t$ for objects in the Tamarkin category $\mathcal{D}(X)$ is a kind of action, and the natural transform τ_c "moves" Reeb chords in a homological way. From this point of view, we define our first numerical invariant of sheaves.

Definition 2.5. For $F \in \mathcal{D}(X)$, the sheaf energy is defined to be:

$$e(F) = \inf\{c \geq 0 : \tau_c(F) = 0\}.$$

It means that when we move some Reeb chord of $\mu s(F)$ after action c, the chord will not intersect with itself. In particular, the smallest c corresponds to the Reeb action of the chord.

Asano and Ike define a translation distance and then displacement energy using a similar idea in [**AI20a**]. In particular, as a corollary of the theorem 4.18 in loc. cit., we have that if $\mu s(F)$ is compact, then

$$e(\mu s(F)) \geq e(F),$$

where the displacement energy e, for $A \subset T^*X$, is defined by

$$e(A) \coloneqq \inf \left\{ \|H\|_{\mathrm{Hofer}} : H \in C_c^\infty(I \times T^*X), A \cap \varphi_1^H(A) = \varnothing \right\}.$$

To build projectors we will follow Chiu's construction and use the Fourier-Sato transform, which is a sheaf-theoretic analogue of the Fourier transform. The Fourier-Sato transform defines a functor $D(V) \to D(V^*)$, where V is a real vector space and V^* is the dual of V. One can see [**KS90**, Section 3.7, Section 5.5] for more details. We want to mention that the Fourier-Sato transform gives an equivalence between $\mathbb{R}_{>0}$-equivariant sheaves on V and V^*. Tamarkin introduced a new version of the Fourier transform on the category $\mathcal{D}(V)$ which also works for non-$\mathbb{R}_{>0}$-equivariant sheaves. We call it the Fourier-Sato-Tamarkin transform. For the relation between the different versions of Fourier transforms, we refer to [**D'A13, Gao17a**].

Definition 2.6. Let $Leg(V) = \{(z, \zeta, t) : t + \langle z, \zeta \rangle \geq 0\} \subset V \times V^* \times \mathbb{R}_t$, we have that $\mathbb{K}_{Leg(V)} \in \mathcal{D}(V \times V^*)$. Then the Fourier-Sato-Tamarkin transform is defined as

$$\begin{aligned} &\mathrm{FT} : D(V_z \times \mathbb{R}) \to \mathcal{D}(V_\zeta^*), \\ &\mathrm{FT}(F) = \widehat{F} \coloneqq F \star \mathbb{K}_{Leg(V)}[\dim V]. \end{aligned}$$

One can see that the restriction of $\mathrm{FT}(F)$ on $\mathcal{D}(V_z)$ is an equivalence of categories in [**Tam18**, Theorem 3.5].

Sometimes, for $F \in D(V_\zeta)$, we will use the notation

$$\widehat{\widehat{F}} \coloneqq \mathbb{K}_{Leg(V)}[\dim V] \circ F, \tag{2.4}$$

here the composition is taken over V_ζ.

Geometrically, the set $Leg(V)$ is associated with the Legendre transform between V and V^*. The important thing for us is the microsupport estimate under the Fourier-Sato-Tamarkin transform. Combining the theorem 3.5 and the theorem 3.6 (and its proof) of [**Tam18**], we have

Theorem 2.7. *Let $\varphi\colon J^1V \to J^1V^*$ be the map $\varphi(z,\zeta,t) = (\zeta, -z, t - \langle z, \zeta\rangle)$, where we identify V^{**} with V naturally. Then for $F \in \mathcal{D}(V)$, then we have the microsupport relation:*

$$\mu s_L(\widehat{F}) = \varphi(\mu s_L(F)). \tag{2.5}$$

Proof. The original statement of [Tam18, Theorem 3.6] claim that $\mu s(\widehat{F}) \subset \varphi_0(\mu s(F))$, here $\varphi_0(z,\zeta) = (\zeta, -z)$. But the proof indicate that the inclusion can be lift to J^1V and φ, i.e.:

$$\mu s_L(\widehat{F}) \subset \varphi(\mu s_L(F)).$$

Moreover, the theorem 3.5 in loc. cit. shows that the Fourier transform $F \mapsto \widehat{F}$ has an inverse which is given by $G \mapsto \check{G} = G \star \mathbb{K}_{Leg'(V)}$ where $Leg'(V) = \{(\zeta, z, t) : t - \langle z, \zeta\rangle \geq 0\} \subset V^* \times V \times \mathbb{R}_t$. We also have an estimate

$$\mu s_L(\check{G}) \subset \varphi^{-1}(\mu s_L(G)).$$

Then the equal of (2.5) follows by taking $G = \widehat{F}$. $\square$

2.2. Tamarkin categories of subsets and microlocal projectors

In this section, we continue our discussion of the last section on the subset version of the Tamarkin category. Importantly, we will focus on *admissible* open sets (in the sense that the microlocal kernels of the microlocal projectors exist). We will then study some properties of microlocal kernels.

2.2.1. Tamarkin categories of open and closed sets. Now, we fix an *open* subset $U \subset T^*X$ and set $Z = T^*X \setminus U$. Then we define

Definition 2.8. For an open set $U \subset T^*X$ and the closed set $Z = T^*X \setminus U$, we define the full subcategories

$$\begin{aligned}
\mathcal{D}_Z(X) =& \{F \in \mathcal{D}(X) : \mu s(F) \subset Z\} \\
\mathcal{D}_U(X) =& {}^{\perp}\mathcal{D}_Z(X), \text{ the left orthogonal complement of } \mathcal{D}_Z(X).
\end{aligned}$$

Again, the triangulated inequality of microsupport shows that $\mathcal{D}_Z(X)$ is a thick full triangulated subcategory of $\mathcal{D}(X)$.

Now we have a diagram of inclusions

$$\mathcal{D}_Z(X) \hookrightarrow \mathcal{D}(X) \hookleftarrow \mathcal{D}_U(X) \tag{2.6}$$

In general, these two inclusions admit both left and right adjoint functors by the Brown representation theorem. An ∞-category version proof can be found in [**Kuo21**]. They are called *microlocal projectors*. However, we need more than just microlocal projector functors, we need that these projectors can be represented by convolution functors. This is the object of the next subsection.

2.2.2. Admissible open sets and microlocal projectors. Following Tamarkin, we are looking for convolution kernels that represent microlocal projector functors and give the corresponding semi-orthogonal decomposition.

Definition 2.9. We say U is $\mathbb{K}$-admissible if there is a distinguished triangle

$$P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U \xrightarrow{+1},$$

in $\mathcal{D}(X^2)$ such that the convolution functor $\star P_U$ is right adjoint to $\mathcal{D}_U(X) \hookrightarrow \mathcal{D}(X)$ and $\star Q_U$ is left adjoint to $\mathcal{D}_Z(X) \hookrightarrow \mathcal{D}(X)$, i.e.,

$$\mathcal{D}_Z(X) \xleftarrow{\star Q_U} \mathcal{D}(X) \xrightarrow{\star P_U} \mathcal{D}_U(X),$$

are two microlocal projectors. Such a pair of sheaves (P_U, Q_U) together with the distinguished triangle give an orthogonal decomposition of $\mathcal{D}(X)$ by Proposition 2.1. We call the pair (P_U, Q_U) *microlocal kernels* associated with U, and the distinguished triangle as the defining triangle of U.

We say U is admissible if U is $\mathbb{Z}$-admissible.

Remark 2.10. We define the $\mathbb{K}$-admissibility of U at the beginning. But the coefficient dependence seems redundant because all our existence results in the following work for all $\mathbb{K}$, especially for $\mathbb{K} = \mathbb{Z}$. Moreover, one can show that if U is admissible, then U is

$\mathbb{K}$-admissible for all $\mathbb{K}$ (by taking the tensor product $\mathbb{K}\overset{L}{\otimes}_{\mathbb{Z}}$ with kernels and then use the uniqueness). From this point of view, we do not emphasize the coefficient ring $\mathbb{K}$ for the kernels (P_U, Q_U). But we will see later that the $\mathbb{K}$ does affect the computation of the Chiu-Tamarkin complex.

We first study functoriality with respect to inclusions and the uniqueness of kernels. We start with some simple facts.

Lemma 2.11. *Suppose $U_1 \subset U_2$ is an inclusion between $\mathbb{K}$-admissible open subsets in T^*X and their defining triangles are*

$$P_{U_i} \xrightarrow{a_i} \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \xrightarrow{b_i} Q_{U_i} \xrightarrow{+1} .$$

(1) We have $Q_{U_2} \star P_{U_1} \cong 0$, and the natural morphism

$$a_2 \star P_{U_1} = [P_{U_2} \star P_{U_1} \to P_{U_1}],$$

is an isomorphism. In particular, we have $P_U \star P_U \cong P_U$ and $Q_U \star P_U \cong 0$ for any admissible open sets U.

(2) For any admissible open sets U and for all $F, G \in D(X^2 \times \mathbb{R})$, we have the isomorphism:

$$\mathrm{Hom}_{D(X^2\times\mathbb{R})}(F \star P_U, G \star P_U) \to \mathrm{Hom}_{D(X^2\times\mathbb{R})}(F \star P_U, G).$$

(3) For all $c \geq 0$, we have $\mathrm{RHom}(P_{U_1}, \mathrm{T}_{c*}(Q_{U_2})) \cong 0$ and

$$\mathrm{RHom}(P_{U_1}, \mathrm{T}_{c*}(a_2)) : \mathrm{RHom}(P_{U_1}, \mathrm{T}_{c*}(P_{U_2})) \cong \mathrm{RHom}(P_{U_1}, \mathbb{K}_{\Delta_{X^2}\times[c,\infty)}). \tag{2.7}$$

Proof. (1) First, applying $\star P_{U_1}$ to the defining triangle of V, we obtain

$$P_{U_2} \star P_{U_1} \xrightarrow{a_2\star P_{U_1}} P_{U_1} \xrightarrow{b_2\star P_{U_1}} Q_{U_2} \star P_{U_1} \xrightarrow{+1} .$$

Notice that, for all $x \in X$, and for all $K \in D(X \times X \times \mathbb{R})$ we have $\mathbb{K}_{\{x\}\times[0,\infty)} \star K \simeq K|_{\{x\}\times X\times[0,\infty)} \in \mathcal{D}(X)$.

Now, since $U_1 \subset U_2$, we have, for all $x \in X$,

$$\mu s(\mathbb{K}_{\{x\}\times[0,\infty)} \star Q_{U_2}) = \mu s(Q_{U_2}|_{\{x\}\times X\times\mathbb{R}}) \subset T^*X \setminus U_2 \subset T^*X \setminus U_1.$$

Consequently, we have $\mathbb{K}_{\{x\}\times[0,\infty)} \star Q_{U_2} \star P_{U_1} \cong (Q_{U_2} \star P_{U_1})|_{\{x\}\times X\times\mathbb{R}} = 0$ for all $x \in X$. Then we have $Q_{U_2} \star P_{U_1} = 0$ and $a_2 \star P_{U_1}$ is an isomorphism.

(2) Consider the functor $\widetilde{\mathcal{P}}(F) = F \star P_U : D(X^2 \times \mathbb{R}) \to D(X^2 \times \mathbb{R})$. Notice that, the functor $\widetilde{\mathcal{P}}$ has the same formula as the microlocal projector but they have different domains.

Using the morphism $a : P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$ and the isomorphisms $a \star P_U$, $P_U \star a$: $P_U \star P_U \cong P_U$, we have the natural transform of functors: $\varepsilon : \widetilde{\mathcal{P}} \to \mathrm{Id} = \mathrm{Id}_{D(X^2\times\mathbb{R})}$ and the natural isomorphisms $\widetilde{\mathcal{P}} \circ \varepsilon$, $\varepsilon \circ \widetilde{\mathcal{P}} : \widetilde{\mathcal{P}}^2 \to \widetilde{\mathcal{P}}$. Then $\widetilde{\mathcal{P}}$ defines a projector on $D(X^2 \times \mathbb{R})^{op}$ in the sense of [**KS06**, Definition 4.1.1].

So, we can apply the Proposition 4.1.3 in loc.cit. to conclude that, for any $F, G \in D(X^2 \times \mathbb{R})$, we have the isomorphism:

$$\mathrm{Hom}_{D(X^2\times\mathbb{R})}(F \star P_U, G \star P_U) \xrightarrow{-\circ G\star a} \mathrm{Hom}_{D(X^2\times\mathbb{R})}(F \star P_U, G).$$

(3) We take $U = U_1$, $F = \mathbb{K}_{\Delta_{X^2}\times\{0\}}$, and $G = \mathrm{T}_{c*}Q_{U_2}[d]$ for all $d \in \mathbb{Z}$. Then (2) implies that $\mathrm{RHom}(P_{U_1}, \mathrm{T}_{c*}(Q_{U_2})) \cong 0$ since $\mathbb{K}_{\Delta_{X^2}\times\{0\}} \star P_{U_1} \cong P_{U_1}$ and $\mathrm{T}_{c*}Q_{U_2}[d] \star P_{U_1} \cong \mathrm{T}_{c*}(Q_{U_2} \star P_{U_1})[d] \cong 0$. Next, applying $\mathrm{RHom}(P_{U_1}, -)$ to the defining triangle of U_2 (shifted by T_{c*}), we have that $\mathrm{RHom}(P_{U_1}, a_2)$ is an isomorphism.

$\square$

The functorial property of microlocal kernels is first proven in [**Chi17**, Theorem 4.7(2)] for the contact case, and the uniqueness appears in [**Zha20**, Section 4.6] for the symplectic case. Here, we prove a strong form of the functorial property of kernels, which ensures that the defining triangle is also functorial and unique.

Proposition 2.12. *For any inclusion $U_1 \subset U_2 \subset T^*X$ between $\mathbb{K}$-admissible open subsets and their defining triangles are*

$$P_{U_i} \xrightarrow{a_i} \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \xrightarrow{b_i} Q_{U_i} \xrightarrow{+1},$$

then we have a morphism between the defining triangles:

$$\begin{tikzcd} P_{U_1} \arrow[r, "a_1"] \arrow[d, "a"] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "b_1"] \arrow[d, "\mathrm{Id}"] & Q_{U_1} \arrow[r, "+1"] \arrow[d, "b"] & {} \\ P_{U_2} \arrow[r, "a_2"] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "b_2"] & Q_{U_2} \arrow[r, "+1"] & {} \end{tikzcd} .$$

These morphisms are natural with respect to inclusions of admissible open sets. In particular, when $U_1 = U_2$ (but P_{U_1} and P_{U_2} are a priori not the same), the morphism of the defining triangles is an isomorphism of distinguished triangles.

Proof. First of all, let us construct a and b such that $a_1 = a_2 a$ and $b_2 = b b_1$.

By (1) of the Lemma 2.11, we have an isomorphism $a_2 \star P_{U_1} : P_{U_2} \star P_{U_1} \cong P_{U_1}$. Then we obtain the commutative diagram below:

$$\begin{tikzcd} P_{U_1} \arrow[r, "a_1"] \arrow[dr, dashed, "a"] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \\ P_{U_2} \star P_{U_1} \arrow[u, "\cong"] \arrow[r, "P_{U_2}\star a_1"'] & P_{U_2}. \arrow[u, "a_2"'] \end{tikzcd}$$

The morphism $a : P_{U_1} \to P_{U_2}$ is the dashed arrow.

Similarly, we have an isomorphism $Q_{U_2} \star b_1$, which is given by:

$$Q_{U_2} \star P_{U_1} \to Q_{U_2} \xrightarrow{\cong} Q_{U_2} \star Q_{U_1} \xrightarrow{+1},$$

and the commutative diagram

$$\begin{tikzcd} \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "b_1"] \arrow[d, "b_2"] & Q_{U_1} \arrow[d, "b_2\star Q_{U_1}"] \arrow[dl, dashed, "b"] \\ Q_{U_2} \arrow[r, "\cong"] & Q_{U_2} \star Q_{U_1}. \end{tikzcd}$$

The morphism $b : Q_{U_1} \to Q_{U_2}$ is the dashed arrow.

One can verify the naturality of the morphisms a and b (see [**Chi17**, Theorem 4.7] for example). In particular, when $U_1 = U_2$, we have that a and b are isomorphisms.

Next, consider the following morphism of distinguished triangles constructed by TR3:

$$\begin{array}{ccccccc}
P_{U_1} & \xrightarrow{a_1} & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} & \xrightarrow{b_1} & Q_{U_1} & \xrightarrow{+1} & \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \psi} & & \\
P_{U_2} & \xrightarrow{a_2} & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} & \xrightarrow{b_2} & Q_{U_2} & \xrightarrow{+1} & .
\end{array}$$

Applying the functor $\mathrm{RHom}(-, Q_{U_2})$ to the defining triangle of U_1, we have the distinguished triangle as follows:

$$\mathrm{RHom}(Q_{U_1}, Q_{U_2}) \xrightarrow{-\circ b_1} \mathrm{RHom}(\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}, Q_{U_2}) \xrightarrow{-\circ a_1} \mathrm{RHom}(P_{U_1}, Q_{U_2}) \xrightarrow{+1} .$$

Then (3) of Lemma 2.11 shows that $\mathrm{RHom}(P_{U_1}, Q_{U_2}) \cong 0$ and $-\circ b_1$ is an isomorphism. Taking H^0, we have the isomorphism:

$$\mathrm{Hom}(Q_{U_1}, Q_{U_2}) \xrightarrow{-\circ b_1} \mathrm{Hom}(\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}, Q_{U_2}).$$

Finally, one conclude that $b = \psi$ since their image under the isomorphism $-\circ b_1$ is b_2.

Then we obtain the morphism of defining triangles that is functorial with respect to inclusion of admissible open sets. $\square$

Now we check that P_U, Q_U are determined locally around the projection of $U \times U$ to the base.

Proposition 2.13. *For an admissible open set $U \subset T^*X$ with the distinguished triangle:*

$$P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U \xrightarrow{+1},$$

let $X_U = p_X(U) \subset X$, which is an open set. Then

(1) $\mathrm{supp}(P_U) \subset \overline{X_U} \times X \times \mathbb{R}$ *and* $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}|_{(X\setminus X_U)\times X\times\mathbb{R}} \to Q_U|_{(X\setminus X_U)\times X\times\mathbb{R}}$ *is an isomorphism.*

(2) For any open set X' with $\overline{X_U} \subset X'$, the open set $U \subset T^*X$ is admissible in T^*X' by the distinguished triangle

$$P_U|_{X'^2\times\mathbb{R}} \to \mathbb{K}_{\Delta_{X'^2}\times[0,\infty)} \to Q_U|_{X'^2\times\mathbb{R}} \xrightarrow{+1} .$$

Proof. (1) The first part is a corollary of the observation: For $x \in X \setminus X_U$, the morphism, induced by convolution (see Example 1.12(4)),

$$\mathbb{K}_{\{x\}\times[0,\infty)} \to \mathbb{K}_{\{x\}\times[0,\infty)} \star Q_U \cong Q_U|_{\{x\}\times X\times\mathbb{R}},$$

is an isomorphism since $\mu S(\mathbb{K}_{\{x\}\times[0,\infty)}) = T^*_x X \subset T^*X \setminus U$.

Taking the stalk at $(y,t) \in X \times \mathbb{R}$, we have that, for all $(x,y,t) \in (X \setminus X_U) \times X \times \mathbb{R}$,

$$\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}|_{(x,y,t)} \to Q_U|_{(x,y,t)}$$

is an isomorphism. Then one concludes by the fact that a morphism is an isomorphism if it induces an isomorphism on every stalk.

(2) To be clear, we call $U' = U$ when we treat U as an open subset of T^*X', and set $j : X' \to X$ to be the inclusion map. For $F \in \mathcal{D}(X')$, then the first part and the proper base change induce a commutative diagram:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\cong"] & F \star (Q_U|_{X'^2\times\mathbb{R}}) \arrow[d, "\cong"] \\ j^{-1}j_!F \arrow[r] & j^{-1}(j_!F \star Q_U). \end{tikzcd}$$

On the other hand, for $F \in \mathcal{D}_{T^*X'\setminus U'}(X')$, we have $\mu s(j_!F) \subset T^*X \setminus U$ when $\overline{X_U} \subset X'$. In fact, we can not shrink X' to be X_U since $\mu s(j_!F)$ would change along $\partial X'$. But at least $\mu s(j_!F) \subset T^*X \setminus U$ is true in general.

Then we have

$$F \in \mathcal{D}_{T^*X'\setminus U'}(X') \iff F \to F \star (Q_U|_{X'^2\times\mathbb{R}}) \text{ is an isomorphism.}$$

Then we can take $Q_{U'} \cong Q_U|_{X'^2\times\mathbb{R}}$. Finally, we set $P_{U'}$ to be the cocone of

$$\mathbb{K}_{\Delta_{X'^2}\times[0,\infty)} \to Q_U|_{X'^2\times\mathbb{R}}.$$

However, an obvious candidate of the cocone is $P_U|_{X'^2\times\mathbb{R}}$. So, we have $P_{U'} \cong P_U|_{X'^2\times\mathbb{R}}$.

Finally, we can check that $(P_{U'}, Q_{U'})$ are microlocal kernels associated with U' since (P_U, Q_U) are microlocal kernels associated with U.

$\square$

2.3. Existence of microlocal kernels

Now, let us study the existence of admissible open sets $U \subset T^*X$. In general, we can take a smooth Hamiltonian function H such that $U = \{H < 1\}$. Our tools to construct kernels are sheaf quantizations and the Fourier-Sato-Tamarkin transform.

For our later application for toric domains, let us state our idea in a more general form. Suppose there is a Hamiltonian $\mathbb{R}_z^m$-action on T^*X, i.e., a symplectic action $\varphi : \mathbb{R}_z^m \times T^*X \to T^*X$ with a moment map $\mu : T^*X \to (\mathbb{R}_z^m)^* = \mathbb{R}_\zeta^m$. Let us consider U of the form $\mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}_\zeta^m$. We assume there is a *sheaf quantization* $\mathcal{K} \in D(\mathbb{R}_z^m \times X^2 \times \mathbb{R})$ associated with the Hamiltonian action in the sense:

$$(2.8) \qquad \begin{aligned} &\mathcal{K}|_{z=0} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}, \\ &\mu s(\mathcal{K}) \subset \{(z, -\mu(\mathbf{q},\mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in \mathbb{R}_z^m \times T^*X\}. \end{aligned}$$

Remark 2.14. One can see that when $m = 1$, it is exactly the single Hamiltonian situation $\mu = H$. Now, if we additionally assume that $\mu = H$ is compactly supported up to constant, then we can take $\mathcal{K} = \mathcal{K}(\widehat{\varphi^H}) \star \mathbb{K}_{[0,\infty)}$ as the sheaf quantization. Here, $\mathcal{K}(\widehat{\varphi^H})$ is introduced in Subsection 1.3.2.

In fact, as [**GKS12**, Remark 3.9] discussed, it exists for all $m \in \mathbb{N}$ and when the action is compactly supported.

The sheaf $\mathcal{K}$ lives over the z-variable space. Intuitively, if we want to restrict the microsupport of some sheaves into $\{\zeta \in \Omega\} \subset V_\zeta^*$, we need a sheaf transform to interchange z and ζ variables, which are dual to each other. Then we cut-off the support of the resulting sheaf in some way. This operation is classical in mechanics and thermodynamics, i.e. the Legendre transform. However, we have noticed that the sheaf correspondence of the Legendre transform is the Fourier-Sato(-Tamarkin) transform. Consequently, let us apply the Fourier-Sato-Tamarkin transform to the z-variable, i.e., $\widehat{\mathcal{K}} = \mathcal{K} \star \mathbb{K}_{Leg(\mathbb{R}_\zeta^m)}[m] \in \mathcal{D}(\mathbb{R}_\zeta^m \times X^2)$. So by (2.8) and (2.5), we have

$$(2.9) \qquad \mu s(\widehat{\mathcal{K}}) \subset \{(\mu(\mathbf{q},\mathbf{p}), z, \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in \mathbb{R}_z^m \times T^*X\}.$$

Then, we construct the kernels in the following way. Consider the excision triangle:

$$\mathbb{K}_{\Omega} \to \mathbb{K}_{\mathbb{R}^m_\zeta} \to \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

Composing the distinguished triangle with $\widehat{\mathcal{K}}$, we obtain a distinguished triangle in $\mathcal{D}(X^2)$:

$$\widehat{\mathcal{K}} \circ \mathbb{K}_{\Omega} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

By the associativity of convolutions and compositions (Example 1.12 (1)), we have $\widehat{\mathcal{K}} \circ F = (\mathcal{K} \star \mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m]) \circ F \cong \mathcal{K} \star (\mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \circ F)$. Recall that, we set $\widehat{\widehat{F}} = \mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \circ F$ in (2.4).

Since $\mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \circ \mathbb{K}_{\mathbb{R}^m_\zeta} = \mathbb{K}_{\{z=0,\, t \geq 0\}}$, we have

$$(\widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta}) \cong \mathcal{K} \star \mathbb{K}_{\{z=0,\, t \geq 0\}} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)},$$

where the last isomorphism comes from (2.8), i.e., $\mathcal{K}|_{z=0} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$. Therefore, we have the distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_{\Omega} \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

Proposition 2.15. *Let φ be a Hamiltonian $\mathbb{R}^m_z$-action on T^*X with a moment map $\mu : T^*X \to \mathbb{R}^m_\zeta$. We assume that there is a sheaf quantization $\mathcal{K} \in D(\mathbb{R}^m_z \times X^2 \times \mathbb{R}_t)$ of the Hamiltonian action in the sense of* (2.8)*. Then for an open subset $\Omega \subset \mathbb{R}^m_\zeta$ such that one of the following two conditions are correct:*

(1) ([**Chi17**, Theorem 3.1]*) For all $\zeta \in \Omega$, $\mu^{-1}(\zeta)$ is compact.*

(2) ([**Chi21**]*) The Hamiltonian H defines a complete and short-term separating flow.*

Then the open set $U = \mu^{-1}(\Omega) \subset T^*X$ is admissible.

More precisely, the pair of sheaves

$$P_U := \widehat{\mathcal{K}} \circ \mathbb{K}_{\Omega},\ Q_U := \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega}, \tag{2.10}$$

and the distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_{\Omega} \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1}, \tag{2.11}$$

provide the microlocal kernels of U and the semi-orthogonal decomposition.

Proof. Our construction is a straightforward generalization of Chiu's results [**Chi17, Chi21**].

Let us sketch the proof of Chiu for the convenience of readers.

Step 1: We prove that, for $F \in \mathcal{D}(X)$, we have $F \star Q_U \in \mathcal{D}_{T^*X \setminus U}(X)$. Let $\Omega^c = \mathbb{R}^m_\zeta \setminus \Omega$,

$$\pi = \pi_{(\zeta,\mathbf{q}_1)}, \quad \pi_1 = \pi_{(\zeta,\mathbf{q}_2,t_2)}, \quad \pi_2 = \pi_{(t_1)}.$$

They are maps with domain $\mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}$. By definition of convolution/composition (see Section 1.2), we have

$$F \star Q_U \cong F \star \widehat{\mathcal{K}} \circ \mathbb{K}_{\Omega^c} \cong \mathrm{R}\pi_! \mathrm{R}s^2_{t!}(\pi_1^{-1}F \overset{L}{\otimes} \pi_2^{-1}\widehat{\mathcal{K}} \overset{L}{\otimes} \mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}).$$

The microsupport estimate of $\mu s(\widehat{\mathcal{K}})$ can be found in (2.9). Then Theorem 1.4 shows

$$\begin{aligned} SS(\pi_1^{-1}F) \subset & \{(\zeta, 0, \mathbf{q}_1, \tau_1\mathbf{p}_1, \mathbf{q}_2, 0, t_1, \tau_1, t_2, 0) : \tau_1 \geq 0.\},\\ SS(\pi_2^{-1}\widehat{\mathcal{K}}) \subset & \{(\mu(\mathbf{q}_1, \mathbf{p}_1'), \tau_2 z, \mathbf{q}_1, -\tau_2\mathbf{p}_1', \tau_2\varphi_z(\mathbf{q}_1, \mathbf{p}_1'), t_1, 0, t_2, \tau_2) :\\ & \tau_2 \geq 0, (z, \mathbf{q}_1, \mathbf{p}_1') \in \mathbb{R}^m_z \times T^*X\} \cup 0_{\mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}},\\ SS(\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}) \subset & N^*(\Omega^c) \times 0_{X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}} \subset \Omega^c \times \mathbb{R}^m_z \times 0_{X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}. \end{aligned}$$

In the last estimate, $N^*(\Omega^c)$ is the conormal cone of Ω^c. See [**KS90**, Section 5.3] for the definition of conormal cone and the estimate. Actually, we only need the second inclusion which can be proven directly using $p_X(SS(F)) = \mathrm{supp}(F)$. But we still left the first inclusion here for the convenience of readers.

Since $SS(\pi_1^{-1}F) \cap [-SS(\pi_2^{-1}\widehat{\mathcal{K}})] \subset 0_{\mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}$, we have

$$\begin{aligned} & SS(\pi_1^{-1}F \overset{L}{\otimes} \pi_2^{-1}\widehat{\mathcal{K}}) = SS(\pi_1^{-1}F) + SS(\pi_2^{-1}\widehat{\mathcal{K}})\\ \subset & \{(\mu(\mathbf{q}_1, \mathbf{p}_1'), \tau_2 z, \mathbf{q}_1, \tau_1\mathbf{p}_1 - \tau_2\mathbf{p}_1', \tau_2\varphi_z(\mathbf{q}_1, \mathbf{p}_1'), t_1, \tau_1, t_2, \tau_2) :\\ & \tau_1, \tau_2 \geq 0, (z, \mathbf{q}_1, \mathbf{p}_1') \in \mathbb{R}^m_z \times T^*X\}. \end{aligned}$$

Then we have $SS(\pi_1^{-1}F\overset{L}{\otimes}\pi_2^{-1}\widehat{\mathcal{K}}) \cap [SS(\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}})] \subset 0_{\mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}$, and

$$\begin{aligned}
&SS(\pi_1^{-1}F\overset{L}{\otimes}\pi_2^{-1}\widehat{\mathcal{K}}\overset{L}{\otimes}\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}) \\
\subset & SS(\pi_1^{-1}F\overset{L}{\otimes}\pi_2^{-1}\widehat{\mathcal{K}}) + SS(\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}) \\
\subset & \{(\zeta, \tau_2 z + \sigma, \mathbf{q}_1, \tau_1\mathbf{p}_1 - \tau_2\mathbf{p}_1', \tau_2\varphi_z(\mathbf{q}_1, \mathbf{p}_1'), t_1, \tau_1, t_2, \tau_2) : \\
& \tau_1, \tau_2 \geq 0, (z, \mathbf{q}_1, \mathbf{p}_1') \in \mathbb{R}^m_z \times T^*X, \zeta = \mu(\mathbf{q}_1, \mathbf{p}_1'), (\zeta, \sigma) \in N^*(\Omega^c)\}.
\end{aligned}$$

The effect of $\mathrm{R}s^2_{t!}$ to microsupport is to make $\tau = \tau_1 = \tau_2$ and $t = t_1 + t_2$, then we have

$$\begin{aligned}
&SS(\mathrm{R}s^2_{t!}(\pi_1^{-1}F\overset{L}{\otimes}\pi_2^{-1}\widehat{\mathcal{K}}\overset{L}{\otimes}\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}})) \\
\subset & \{(\zeta, \tau z + \sigma, \mathbf{q}_1, \tau(\mathbf{p}_1 - \mathbf{p}_1'), \tau\varphi_z(\mathbf{q}_1, \mathbf{p}_1'), t_1 + t_2, \tau) : \\
& \tau \geq 0, (z, \mathbf{q}_1, \mathbf{p}_1') \in \mathbb{R}^m_z \times T^*X, \zeta = \mu(\mathbf{q}_1, \mathbf{p}_1'), (\zeta, \sigma) \in N^*(\Omega^c)\} =: Z.
\end{aligned}$$

Now, we need the estimate for non-proper pushforward Theorem 1.7. Notice that the theorem stated there works only when X is a vector space, and $\pi^\#$ is a well-defined map only when X is parallelizable. But the microsupport is a local notion. So we can take a Darboux chart and work locally, and here we use the notation $\pi^\#$ locally. It means that we forget $\mathbf{q}_1$ locally.

Then we have $(\mathbf{q}, \mathbf{p}, t, 1) \in SS(\mathrm{R}\pi_!\mathrm{R}s^2_{t!}(\pi_1^{-1}F\overset{L}{\otimes}\pi_2^{-1}\widehat{\mathcal{K}}\overset{L}{\otimes}\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}})) = SS(F \star Q_U)$ if

$$(-, 0, -, 0, \mathbf{q}, \mathbf{p}, t, 1) \in \overline{\pi^\#(SS(\mathrm{R}s^2_{t!}(\pi_1^{-1}F\overset{L}{\otimes}\pi_2^{-1}\widehat{\mathcal{K}}\overset{L}{\otimes}\mathbb{K}_{\Omega^c \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}})))} \subset \overline{\pi^\#(Z)},$$

where

$$\begin{aligned}
\pi^\#(Z) = & \{(\zeta, \tau z + \sigma, \tau(\mathbf{p}_1 - \mathbf{p}_1'), \tau\varphi_z(\mathbf{q}_1, \mathbf{p}_1'), t_1 + t_2, \tau) : \\
& \qquad \tau \geq 0, (z, \mathbf{q}_1, \mathbf{p}_1') \in \mathbb{R}^m_z \times T^*X, \zeta = \mu(\mathbf{q}_1, \mathbf{p}_1'), (\zeta, \sigma) \in N^*(\Omega^c)\}.
\end{aligned}$$

Then, by definition of closure, we have a sequence of $(\zeta^n, z^n, -, -, \varphi_{z^n}(\mathbf{q}^n, \mathbf{p}^n), t^n, \tau^n) \in \pi^\#(Z)$ such that

$$\begin{aligned}
&\tau^n\varphi_{z^n}(\mathbf{q}^n, \mathbf{p}^n) \to (\mathbf{q}, \mathbf{p}), \tau^n \to 1, \\
&\zeta^n = \mu(\mathbf{q}^n, \mathbf{p}^n),\ (\zeta^n, z^n) \in N^*(\Omega^c).
\end{aligned}$$

Now, as $\zeta^n = \mu(\mathbf{q}^n, \mathbf{p}^n) = \mu(\varphi_{z^n}(\mathbf{q}^n, \mathbf{p}^n)) \in \Omega^c$, we have

$$\mu(\varphi_{z^n}(\mathbf{q}^n, \mathbf{p}^n)) \in \Omega^c \to \mu(\mathbf{q}, \mathbf{p}) \in \Omega^c.$$

I.e. we have $(\mathbf{q}, \mathbf{p}) \in U = \mu^{-1}(\Omega^c)$, and $F \star Q_U \in \mathcal{D}_{T^*X \setminus U}(X)$.

Step 2: For $F \in \mathcal{D}_{T^*X\setminus U}(X)$, and $G \in \mathcal{D}(X)$, we prove $\mathrm{RHom}(G \star P_U, F) \cong 0$. Here, we refresh the notation. We set some projections and inclusions in the diagram below (and only valid in step 2).

With these maps and definition of composition/convolution, adjoint isomorphisms, proper base change and projection formula show that

$$\mathrm{RHom}(G \star P_U, F) \cong \mathrm{RHom}(j_1^{-1}\pi_1^{-1}G, \mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F)),$$

where $\sigma(\zeta, \mathbf{q}_1, \mathbf{q}_2, t_1, t_2) = (\mathbf{q}_2, t_1 + t_2)$.

$$\begin{tikzcd}
X_1 \times \mathbb{R}_{t_1} & \mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_2} & \\
\mathbb{R}^m_\zeta \times X_1 \times \mathbb{R}_{t_1} \arrow[u, "\pi_1"] & \mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2} \arrow[l, "\pi"'] \arrow[u, "\pi_2"'] \arrow[r, "\sigma"] & X_2 \times \mathbb{R}_t \\
\Omega \times X_1 \times \mathbb{R}_{t_1} \arrow[u, hook, "j_1"] & \Omega \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2} \arrow[l, "\pi"'] \arrow[u, hook, "j"'] &
\end{tikzcd}$$

Now, we can estimate the microsupport with the help of Theorem 1.4. We have

$$\begin{aligned} SS(\pi_2^{-1}\widehat{\mathcal{K}}) \subset \{&(\mu(\mathbf{q}_1, \mathbf{p}_1), \tau_1 z, \mathbf{q}_1, -\tau_1\mathbf{p}_1, \tau_1\varphi_z(\mathbf{q}_1, \mathbf{p}_1), t_1, 0, t_2, \tau_1) : \\ &\tau_1 \geq 0, (z, \mathbf{q}_1, \mathbf{p}_1) \in \mathbb{R}^m_z \times T^*X\} \cup 0_{\mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}. \end{aligned}$$

By $F \in \mathcal{D}_{T^*X\setminus U}(X)$, we have

$$SS(\sigma^! F) \subset \{(\zeta, 0, \mathbf{q}_1, 0, \mathbf{q}_2, \tau_2\mathbf{p}_2', t_1, \tau_2, t_2, \tau_2) : \tau_2 > 0, \mu(\mathbf{q}_2, \mathbf{p}_2') \in \Omega^c\} \cup \{\tau_2 = 0\}.$$

Then, $SS(\pi_2^{-1}\widehat{\mathcal{K}}) \cap SS(\sigma^! F) \subset 0_{\mathbb{R}^m_\zeta \times X_1 \times X_2 \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}}$, and we have

$$\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F) \subset [-SS(\pi_2^{-1}\widehat{\mathcal{K}})] + SS(\sigma^! F) \subset Z,$$

where

$$Z := \{(\mu(\mathbf{q}_1, \mathbf{p}_1), -\tau_1 z, \mathbf{q}_1, -\tau_1\mathbf{p}_1, (\mathbf{q}_2, \tau_2\mathbf{p}_2') - \tau_1\varphi_z(\mathbf{q}_1, \mathbf{p}_1), t_1, \tau_2, t_2, \tau_2 - \tau_1) :$$

$$\tau_1 \geq 0,\ \tau_2 > 0, (z, \mathbf{q}_1, \mathbf{p}_1) \in \mathbb{R}^m_z \times T^*X, \mu(\mathbf{q}_2, \mathbf{p}_2') \in \Omega^c\}$$
$$\cup\ T^*(\mathbb{R}^m_\zeta \times X_1 \times X_2) \times \{t_1, 0, t_2, -\tau_1 : \tau_1 \geq 0\}.$$

Claim: $\mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F) \in D_{\{\tau \leq 0\}}(\Omega \times X_1 \times \mathbb{R})$.

It means that $SS(\mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F)) \subset \{\tau \leq 0\}$. We prove this by contradiction. If it is false, there exists

$$v = (\zeta, z, \mathbf{q}_1, \mathbf{p}, t_1, 1) \in SS(\mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F)).$$

Like in the step 1, we need the estimate for non-proper pushforward Theorem 1.7. In the same way we use the notation $\pi^\#$ locally. Then we have

$$(\zeta, z, \mathbf{q}_1, \mathbf{p}, -, 0, t_1, 1, -, 0) \in \overline{\pi^\#(SS(j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F)))} \subset \overline{\pi^\#((dj)^{-1}Z)}.$$

So, we have a sequence of $v^n \in \pi^\#((dj)^{-1}Z)$ such that $v^n \to v$. In particular, $\tau_2^n \to 1$, $\tau_2^n - \tau_2^n \to 0$ show that $\tau_1^n, \tau_2^n \to 1$. Next, $\{(\mathbf{q}_2^n, \tau_2^n \mathbf{p}_2^{n\prime}) - \tau_1^n \varphi_z(\mathbf{q}_1^n, \mathbf{p}_1^n)\} \to 0$.

Under both conditions of the theorem, there is a subsequence of $\varphi_z(\mathbf{q}_1^n, \mathbf{p}_1^n)$ convergent to a finite limit $(\mathbf{q}, \mathbf{p})$. We assume the subsequence is the sequence itself. Combining with $\{(\mathbf{q}_2^n, \tau_2^n \mathbf{p}_2^{n\prime}) - \tau_1^n \varphi_z(\mathbf{q}_1^n, \mathbf{p}_1^n)\} \to 0$, one concludes that $(\mathbf{q}_2^n, \mathbf{p}_2^{n\prime})$ converges to $(\mathbf{q}, \mathbf{p})$ as well.

As $\zeta^n = \mu(\varphi_z(\mathbf{q}_1^n, \mathbf{p}_1^n)) = \mu(\mathbf{q}_1^n, \mathbf{p}_1^n) \in \Omega$. So, its limit $\mu(\mathbf{q}, \mathbf{p})$ is also in Ω. However, $\mu(\mathbf{q}_2^n, \mathbf{p}_2^{n\prime}) \in \Omega^c$, so its limit $\mu(\mathbf{q}, \mathbf{p})$ is also in Ω^c. Now, we get a contradiction.

Therefore, one concludes that

$$\mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F) \in D_{\{\tau \leq 0\}}(\Omega \times X_1 \times \mathbb{R}).$$

So, $\mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F) \star \mathbb{K}_{[0,\infty)} \cong 0$ in $\mathcal{D}(\Omega \times X_1)$.

Finally, it is direct to check that, as $G \in \mathcal{D}(X)$, we have $j_1^{-1}\pi_1^{-1}G \in \mathcal{D}(\Omega \times X_1)$.

Therefore, one concludes that

$$\begin{aligned}\mathrm{RHom}(G \star P_U, F) \cong& \mathrm{RHom}(j_1^{-1}\pi_1^{-1}G, \mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F))\\ \cong& \mathrm{RHom}(j_1^{-1}\pi_1^{-1}G, \mathrm{R}\pi_* j^{-1}\mathrm{R}\mathcal{H}om(\pi_2^{-1}\widehat{\mathcal{K}}, \sigma^! F) \star \mathbb{K}_{[0,\infty)}) \cong 0,\end{aligned}$$

since $\mathcal{D}(\Omega \times X_1)$ is a full subcategory of $D(\Omega \times X_1 \times \mathbb{R})$.

Finally, the result follows from the Proposition 2.1. $\square$

Definition 2.16. We say an admissible open set U is **dynamically** admissible if the microlocal kernels can be constructed using this recipe of Proposition 2.15 .

Remark 2.17. (1) For a dynamically admissible open set $U = \mu^{-1}(\Omega)$, we have $P_U \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega}$. Here, $\widehat{\mathbb{K}_\Omega}$ is a sheaf over $\mathbb{R}^m_z \times \mathbb{R}_t$. We denote $\pi_t(\mathrm{supp}(\widehat{\mathbb{K}_\Omega})) \subset \mathbb{R}^m_z$, then we have

$$P_U \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega} \cong \mathcal{K}_{X^2 \times \pi_t(\mathrm{supp}(\widehat{\mathbb{K}_\Omega})) \times \mathbb{R}} \star \widehat{\mathbb{K}_\Omega}.$$

For example, when $m = 1$, $\Omega = (-\infty, 1)$, and $U = \{H < 1\}$ for a Hamiltonian H, then we have $\widehat{\mathbb{K}_\Omega} \cong \mathbb{K}_{\{(z,t): -t \leq z \leq 0\}}$, and $\pi_t(\mathrm{supp}(\widehat{\mathbb{K}_\Omega})) = \{z \leq 0\}$. In this case, we have

$$P_U \cong \mathcal{K}_{\{z \leq 0\}} \star \mathbb{K}_{\{(z,t): -t \leq z \leq 0\}}.$$

(2) Due to Remark 1.16, when U is dynamically admissible, for a finite interval I, it is likely that there exists a topological space W and a locally closed set $Z \subset X^2 \times W \times I$ such that $(P_U)_I \cong \mathrm{R}\pi_{W!}\mathbb{K}_Z$. When the sheaf quantization K is a pushforward of a constant sheaf, P_U is a pushforward of a constant sheaf, since P_U is a composition of sheaves which are pushforward of constant sheaves.

2.3.1. Corollaries of the existence theorem.

Now let us state some corollaries of the existence theorem.

Proposition 2.18. *Bounded open sets are dynamically admissible.*

Proof. Let $U \subset T^*X$ be a bounded open set, we have $T^*X \setminus U$ is a closed subset of T^*X. Then there exists a smooth function $H : T^*X \to [0, 1]$ such that $U = \{H < 1\}$ and $T^*X \setminus U = \{H \geq 1\}$. Actually, we take a non negative function f such that $f^{-1}(0) = T^*X \setminus U$, see [Lee03, Theorem 2.29]. Then we take $H(x) = 1 - f(x)$.

Since U is bounded, the subsets $\{H = a\} \subset U$ with $a < 1$ are compact. Moreover dH has compact support. So we can take the GKS quantization $\mathcal{K}(\widehat{\varphi^H})$. Then the result follows from the Proposition 2.15 by taking $\Omega = (-\infty, 1)$. $\square$

In particular, using this construction, we can estimate the microsupport of the kernel P_U, in fact, we have

Corollary 2.19. *For a bounded open set U, the microlocal kernel P_U satisfies the following microsupport estimate:*

$$\begin{aligned}\mu s_L(P_U) \subset \{(\boldsymbol{q}, -\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}, 0) : H(\boldsymbol{q}, \boldsymbol{p}) \leq 1,\ \boldsymbol{q} \in \overline{X_U}\} \\ \cup \{(\boldsymbol{q}, -\boldsymbol{p}, \varphi_z^H(\boldsymbol{q}, \boldsymbol{p}), -\int_c \alpha) : H(\boldsymbol{q}, \boldsymbol{p}) = 1, z < 0,\ \boldsymbol{q} \in \overline{X_U}\},\end{aligned} \tag{2.12}$$

where c is the path $s \in [z, 0] \mapsto \varphi_s^H(\boldsymbol{q}, \boldsymbol{p})$ and $X_U = p_X(U)$ as the Proposition 2.13.

Proof. The formula (1.26) shows that

$$\mu s_L(\mathcal{K}(\widehat{\varphi^H})) \subset \{(z, -H(\mathbf{q}, \mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z^H(\mathbf{q}, \mathbf{p}), -S_H(z, \mathbf{q}, \mathbf{p})) : (z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\}.$$

So, we apply the microsupport estimate of the Fourier-Sato-Tamarkin transform (2.5), we have

$$\mu s_L(\widehat{\mathcal{K}}(\widehat{\varphi^H})) \subset \{(H(\mathbf{q}, \mathbf{p}), z, \mathbf{q}, -\mathbf{p}, \varphi_z^H(\mathbf{q}, \mathbf{p}), -S_H(z, \mathbf{q}, \mathbf{p}) - zH(\mathbf{q}, \mathbf{p})) : (z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\}.$$

For an autonomic Hamiltonian function H, we have

$$-S_H(z, \mathbf{q}, \mathbf{p}) = -\int_c (\alpha - H ds) = -\int_c \alpha + zH(\mathbf{q}, \mathbf{p}),$$

where c is the path $s \in [z, 0] \mapsto \varphi_s^H(\mathbf{q}, \mathbf{p})$.

So, the t-component of $\mu s_L(\widehat{\mathcal{K}}(\widehat{\varphi^H}))$ is $-\int_c \alpha$ for a path c as above.

Finally, the microsupport of $\mathbb{K}_{\{\zeta \leq 1\}}$ is

$$\{(\zeta, 0) : \zeta \leq 1\} \cup \{(1, z) : z < 0\} \subset \mathbb{R}_z \times \mathbb{R}_\zeta.$$

The functorial microsupport estimate of Theorem 1.4 implies that for $(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t) \in \mu s(P_U)$, we have either

$$\zeta = H(\mathbf{q}, \mathbf{p}) \leq 1,\ t = -\int_c \alpha = 0,$$

or,

$$\zeta = H(\mathbf{q}, \mathbf{p}) = 1,\ t = -\int_c \alpha,$$

for the path c: $s \in [z, 0] \mapsto \varphi_s^H(\mathbf{q}, \mathbf{p})$ with $z < 0$.

Finally, the support estimate follows directly from Proposition 2.13. $\square$

The second corollary here is about the kernel of products of open sets.

Proposition 2.20. *Suppose we have two dynamically admissible open sets $U_i \subset T^*X_i$ of the same type (i.e. satisfying the same condition of the Proposition 2.15), with two pairs of kernels (P_{U_i}, Q_{U_i}), $i = 1, 2$. Then $U_1 \times U_2$ is dynamically admissible and $P_{U_1 \times U_2} \cong P_{U_1} \boxtimes P_{U_2}$.*

Proof. By the assumption, we have two Hamiltonian functions $H_i \in C^\infty(T^*X_i)$ such that $U_i = \{H_i < 1\}$ and we associate with them two sheaf quantizations $\mathcal{K}_i$. Then

$$(P_{U_i}, Q_{U_i}) = (\widehat{\mathcal{K}}_i \circ \mathbb{K}_{(-\infty,1)}, \widehat{\mathcal{K}}_i \circ \mathbb{K}_{[1,\infty)}), \quad i = 1, 2.$$

Now, consider the product Hamiltonian $\mathbb{R}_z^2$-action on $T^*(X_1 \times X_2)$ whose moment map is $\mu = (H_1, H_2)$, and has same type with H_i. Then $\mathcal{K}_1 \boxtimes \mathcal{K}_2$ is a sheaf quantization of the Hamiltonian action in the sense of (2.8).

Observe that if we take $\Omega = \{\zeta = (\zeta_1, \zeta_2) : \zeta_1 < 1, \zeta_2 < 1\}$, then we have $U_1 \times U_2 = \mu^{-1}(\Omega)$. Consequently, the Proposition 2.15 implies that $U_1 \times U_2$ is admissible by the following distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta \times \{t \geq 0\}} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}_\zeta^2 \setminus \Omega} \xrightarrow{+1}.$$

Subsequently, let us compute $\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega$.

Recall

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega}.$$

Notice Ω is an open convex set. Therefore $\widehat{\mathbb{K}_\Omega} = \mathbb{K}_{\{(z,\zeta,t): t + z \cdot \zeta \geq 0\}}[2] \circ \mathbb{K}_\Omega$ is the constant sheaf $\mathbb{K}_{\Omega^\circ}$ supported on the polar cone Ω° of Ω, where

$$\Omega^\circ = \{(z, t) : t + z \cdot \zeta \geq 0, \forall \zeta \in \Omega\}.$$

In fact, $\mathbb{K}_{\{(z,\zeta,t): t + z \cdot \zeta \geq 0\}}[2] \circ \mathbb{K}_\Omega$ is the Fourier-Sato transform of the constant sheaf of the conification of Ω. The conification is a convex cone. The Fourier-Sato transform of a constant sheaf supported on a convex cone is the constant sheaf supported on the polar cone of the given cone. A direct computation shows that the polar cone of the conification of Ω is exactly Ω°. Then our computation follows.

In particular, when $\Omega = \{\zeta_1 < 1, \zeta_2 < 1\}$, we have $\Omega^\circ = \{(z,t) : z = (z_1, z_2), , z_1 \leq 0, z_2 \leq 0, t \geq -(z_1 + z_2) \geq 0\}$. Moreover, $\mathbb{K}_{\Omega^\circ} \cong \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1 \times \gamma_2})$, where $\gamma_i = \{(z_i, t) : t \geq -z_i \geq 0\}$.

Now we have

$$\begin{aligned} \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega &\cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega} \cong \mathcal{K} \star \mathbb{K}_{\Omega^\circ} \cong \mathcal{K} \star \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1 \times \gamma_2}) \\ &\cong (\mathcal{K}_1 \boxtimes \mathcal{K}_2) \star \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1 \times \gamma_2}) \cong (\mathcal{K}_1 \star \mathbb{K}_{\gamma_1}) \boxtimes (\mathcal{K}_2 \star \mathbb{K}_{\gamma_2}). \end{aligned}$$

Finally, noticing that $\mathbb{K}_{\{(z,t): t \geq -z \geq 0\}} \cong \mathbb{K}_{\{(z,\zeta,t): t + z\zeta \geq 0\}}[1] \circ \mathbb{K}_{(-\infty,1)}$, one can conclude that

$$P_{U_1 \times U_2} \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong (\mathcal{K}_1 \star \mathbb{K}_{\gamma_1}) \boxtimes (\mathcal{K}_2 \star \mathbb{K}_{\gamma_2}) \cong P_{U_1} \boxtimes P_{U_2}.$$

$\square$

CHAPTER 3

Chiu-Tamarkin complexes

This chapter is a main part of the thesis. We will review the definition of the Chiu-Tamarkin complex $C_{\ell,T}(U,\mathbb{K})$ and also define some variants, denoted $H^*C_{\ell,\infty}(U,\mathbb{K})$, $C_{\ell,(T,T']}(U,\mathbb{K})$ and $C^+_{\ell,T}(U,\mathbb{K})$. Two distinguished triangles related to them are also obtained. Usually, the Chiu-Tamarkin complex is negatively graded, so we sometime use the homology grading convention $H_q = H^{-q}$, and call the cohomology of the Chiu-Tamarkin complex the Chiu-Tamarkin homology.

We also study the functorial properties and invariance of the Chiu-Tamarkin complex. We define the fundamental class $\eta_{\ell,T}(U,\mathbb{K}) \in H^0C_{\ell,T}(U,\mathbb{K})$. It will be used to define capacities associated with U. Then we show that both $C_{\ell,T}(U,\mathbb{K})$ and $\eta_{\ell,T}(U,\mathbb{K})$ are natural with respect to T and U, and they are invariant under compactly supported Hamiltonian isotopies.

Next, we show that the coalgebra structure on P_U induced by the idempotent properties of P_U implies that the Ext-algebra $\mathrm{Ext}^*(P_U, \mathrm{T}_{c*}(P_U))$ equipped with the shifted Yoneda product is a graded commutative $\mathbb{K}$-algebra. We also present the Yoneda product as a cup product on the non-equivariant Chiu-Tamarkin cohomology $H^*C_{1,T}(U,\mathbb{K})$. This cup product is the usual cup product on X when $U = T^*X$, and we will see in the Subsection 4.2.3 that the cup product is the Chas-Sullivan product when $U = D^*X$ is the open disk bundle.

Next, we can present a cyclic structure on the Chiu-Tamarkin complex, which is a formal process to see the S^1 structure on the Chiu-Tamarkin complex. In particular, we define the S^1-equivariant Chiu-Tamarkin complex $C^{S^1,+}_T(U,\mathbb{K})$ and $C^{S^1}_T(U,\mathbb{K})$ over fields.

As an interlude, we develop some geometric understanding of microlocal kernels with the help of the notion of dynamical admissibility.

At the almost end of the chapter, we define some symplectic capacities: c, c_k, and $\bar{c}_k$ using different versions of Chiu-Tamarkin cohomology. They are also another main construction in this chapter.

Both of them extract delicately the ring action of the Ext-algebra on the fundamental class. The capacities c is defined using the non-equivariant Chiu-Tamarkin cohomology, which is a reformulation of the sheaf energy for the microlocal kernel P_U since the fundamental class $\eta_{1,T}(U,\mathbb{K})$ is equivalent with $\tau_T(P_U)$. For c_k, and $\bar{c}_k$, they are defined using the equivariant Chiu-Tamarkin homology by studying when $\eta_{?,T}(U,\mathbb{K})$ can be divided by u^k for $u \neq 0 \in A = \mathrm{Ext}^2_G(\mathbb{K},\mathbb{K})$. The restriction morphism $\mathrm{Ext}^*_G(\mathbb{K},\mathbb{K}) \to \mathrm{Ext}^*(\mathbb{K},\mathbb{K})$ maps u to 0 for fields, then we see that c_k, and $\bar{c}_k$ are non-equivariant generalization of the sheaf energy e. We will prove that all of c, c_k, and $\bar{c}_k$ are symplectic capacities and they can be represented by Reeb actions for the contact boundary.

Finally, we develop all the possible generalizations of the results in the chapter when we deal with the contact geometry of the prequantized contact manifold $T^*X \times S^1$. In particular, our cyclic structure helps to understand why we need $\mathbb{Z}/\ell$-theory on the contact non-squeezing theorem.

3.1. Chiu-Tamarkin complexes

Let $\mathbb{Z}/\ell$ be the finite cyclic group of order $\ell \in \mathbb{N}$, X be a smooth manifold of dimension d.

Now take an admissible open set $U \subset T^*X$, and let P_U be the kernel associated with U. The manifold $(X^2 \times \mathbb{R}_t)^\ell$ admits a $\mathbb{Z}/\ell$-action induced by the cyclic permutation of the ℓ factors. According to the appendix C, the object $P_U^{\overset{L}{\boxtimes}\ell}$ of $D((X^2 \times \mathbb{R}_t)^\ell)$ has a natural lift $St_D(P_U)$ as an object of the equivariant derived category $D_{\mathbb{Z}/\ell}((X^2 \times \mathbb{R}_t)^\ell)$, that we also denote, due to historically reason, by $P_U^{\overset{L}{\boxtimes}\ell}$. Then we have $P_U^{\boxplus\ell} = \mathrm{R}s^\ell_{t!}P_U^{\overset{L}{\boxtimes}\ell} \in D_{\mathbb{Z}/\ell}((X^2)^\ell \times \mathbb{R}_t)$.

Consider the $\mathbb{Z}/\ell$-equivariant maps

$$\pi_{\underline{\mathbf{q}}} : X^\ell \times \mathbb{R} \to \mathbb{R},$$

$$\tilde{\Delta}_X : X^\ell \times \mathbb{R} \to X^{2\ell} \times \mathbb{R},$$

$$\tilde{\Delta}_X(\mathbf{q}_1,\ldots,\mathbf{q}_\ell,t)=(\mathbf{q}_\ell,\mathbf{q}_1,\mathbf{q}_1,\ldots,\mathbf{q}_{\ell-1},\mathbf{q}_{\ell-1},\mathbf{q}_\ell,t),$$

$$i_T:\mathrm{pt}\to\mathbb{R},$$

where $\tilde{\Delta}_X$ is a twisted diagonal map of X.

There is an adjoint pair $(\alpha_{\ell,T,X},\beta_{\ell,T,X})$:

$$F\in D_{\mathbb{Z}/\ell}((X^2\times\mathbb{R}_t)^\ell)\ \underset{\beta_{\ell,T,X}}{\overset{\alpha_{\ell,T,X}}{\rightleftarrows}}\ D_{\mathbb{Z}/\ell}(\mathrm{pt})\ni G,$$

defined by:

$$\begin{aligned}\alpha_{\ell,T,X}(F)&=i_T^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\mathrm{R}s^\ell_{t!}\,(F)\,,\\ \beta_{\ell,T,X}(G)&=s_t^{\ell!}\tilde{\Delta}_{X*}\pi^!_{\underline{\mathbf{q}}}i_{T*}G.\end{aligned}\tag{3.1}$$

Now, we define a functor

$$F_{\ell,X}=\mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\mathrm{R}s^\ell_{t!}:D_{\mathbb{Z}/\ell}((X^2\times\mathbb{R}_t)^\ell)\to D_{\mathbb{Z}/\ell}(\mathbb{R}).\tag{3.2}$$

Then $\alpha_{\ell,T,X}=i_T^{-1}F_{\ell,X}$.

Similarly, we define another adjoint pair $(\alpha'_{\ell,T,X},\beta'_{\ell,T,X})$:

$$F\in D_{\mathbb{Z}/\ell}((X^2)^\ell\times\mathbb{R}_t)\ \underset{\beta'_{\ell,T,X}}{\overset{\alpha'_{\ell,T,X}}{\rightleftarrows}}\ D_{\mathbb{Z}/\ell}(\mathrm{pt})\ni G,$$

$$\begin{aligned}\alpha'_{\ell,T,X}(F)&=i_T^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\,(F)\\ \beta'_{\ell,T,X}(G)&=\tilde{\Delta}_{X*}\pi^!_{\underline{\mathbf{q}}}i_{T*}G.\end{aligned}\tag{3.3}$$

We also define a functor

$$F'_{\ell,X}=\mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}:D_{\mathbb{Z}/\ell}(X^{2\ell}\times\mathbb{R}_t)\to D_{\mathbb{Z}/\ell}(\mathbb{R}).\tag{3.4}$$

Then $\alpha'_{\ell,T,X}=i_T^{-1}F'_{\ell,X}$.

If there is no risk of confusion, we will also denote $(\alpha_{\ell,T,X},\beta_{\ell,T,X})=(\alpha_T,\beta_T)$ and $(\alpha'_{\ell,T,X},\beta'_{\ell,T,X})=(\alpha'_T,\beta'_T)$ for simplicity.

Remark 3.1. We will frequently use $\alpha_{\ell,T,X},\beta_{\ell,T,X}$ $(\alpha'_{\ell,T,X},\beta'_{\ell,T,X})$, and $F_{\ell,X}$ $(F'_{\ell,X})$ in the non-equivariant categories. We denote them by the same notation later.

Definition 3.2. With the notation above, we define an object of $D(\mathbb{K}[\mathbb{Z}/\ell]-\mathrm{Mod}) \simeq D_{\mathbb{Z}/\ell}(\mathrm{pt})$ that we call the Chiu-Tamarkin complex

$$\begin{aligned} C_{\ell,T}(U,\mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha_{\ell,T,X}(P_U^{\overset{L}{\boxtimes}\ell}), \mathbb{K}[-d]\right) \\ &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left((F_\ell(U,\mathbb{K}))_T, \mathbb{K}[-d]\right) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_U^{\overset{L}{\boxtimes}\ell}, \beta_{\ell,T,X}\mathbb{K}[-d]\right), \end{aligned}$$

where $F_\ell(U,\mathbb{K}) = F_{\ell,X}(P_U^{\overset{L}{\boxtimes}\ell}) = F'_{\ell,X}(P_U^{\star\ell})$.

We define the positive Chiu-Tamarkin complex, also in $D_{\mathbb{Z}/\ell}(\mathrm{pt})$

$$\begin{aligned} C^+_{\ell,T}(U,\mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha_{\ell,T,X}(Q_U^{\overset{L}{\boxtimes}\ell}), \mathbb{K}[1-d]\right) \\ &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left((F^+_\ell(U,\mathbb{K}))_T, \mathbb{K}[1-d]\right) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(Q_U^{\overset{L}{\boxtimes}\ell}, \beta_{\ell,T,X}\mathbb{K}[1-d]\right), \end{aligned}$$

where $F^+_\ell(U,\mathbb{K}) = F_{\ell,X}(Q_U^{\overset{L}{\boxtimes}\ell}) = F'_{\ell,X}(Q_U^{\star\ell})$.

We set $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})$, which is isomorphic to $H^*_{\mathbb{Z}/\ell}(B\mathbb{Z}/\ell.\mathbb{K})$ (see (B.2)). Then $H^*C_{\ell,T}(U,\mathbb{K})$ and $H^*C^+_{\ell,T}(U,\mathbb{K})$ are graded modules over $A \cong \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K}[-d],\mathbb{K}[-d])$ via the Yoneda product.

Usually, the Chiu-Tamarkin complex is negatively graded. Then the cohomology of the Chiu-Tamarkin complex should be think of a homology. So we will use the homology grading convention sometime, and call the cohomology of the Chiu-Tamarkin complex the Chiu-Tamarkin cohomology or homology, depending on the convention. It is defined for all versions of the Chiu-Tamarkin complex.

$$H_qC_?(U,\mathbb{K}) \coloneqq H^{-q}C_?(U,\mathbb{K}).$$

Remark 3.3. (1) The object $C_{\ell,T}(U,\mathbb{K})$ is mentioned by Tamarkin in [**Tam15**], and is defined explicitly by Chiu in [**Chi17**]. Our definition looks slightly different from the definition of Chiu. But one can check directly that, when X is orientable, $\beta_{\ell,T,X}\mathbb{K}[-d]$ is exactly the constant sheaf supported on the twisted diagonal with a degree shift depending only on ℓ and $\dim X$. So the complex $C_{\ell,T}(U,\mathbb{K})$ is essentially the same as what Chiu defined. Compare to $C_{\ell,T}(U,\mathbb{K})$, we will see later that $C^{+}_{\ell,T}(U,\mathbb{K})$ does not consist of information of constant loops insides X_U, but consist of information of constant loops outside of X_U.

(2) For later use we denote the adjoint isomorphism by:

$$\begin{aligned} N &: \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha_T(P_U^{\overset{L}{\boxtimes}\ell}), \mathbb{K}[-d]\right) \xrightarrow{\cong} \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_U^{\overset{L}{\boxtimes}\ell}, \beta_T\mathbb{K}[-d]\right). \\ N' &: \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha'_T(P_U^{\boxtimes\ell}), \mathbb{K}[-d]\right) \xrightarrow{\cong} \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_U^{\boxtimes\ell}, \beta'_T\mathbb{K}[-d]\right). \end{aligned} \tag{3.5}$$

(3) Let $X_U = p_X(U) \subset X$, then we have $(P_U)_{X_U\times X\times\mathbb{R}} \cong P_U$ by the Proposition 2.13. Then, one has

$$F_\ell(U,\mathbb{K}) = F_{\ell,X}(P_U^{\overset{L}{\boxtimes}\ell}) \cong F_{\ell,X_U}((P_U|_{X_U^2\times\mathbb{R}})^{\overset{L}{\boxtimes}\ell}).$$

That means, to obtain $C_{\ell,T}(U,\mathbb{K})$, we may restrict P_U to $X_U^2\times\mathbb{R}$.

On the contrary, $F^{+}_\ell(U,\mathbb{K})$ is defined using Q_U on whole $X^2\times\mathbb{R}$.

(4) Due to Remark 2.17 (2), when $P_U \cong \mathrm{R}\pi_!\mathbb{K}_Z$ for some locally closed set Z, the equivariant structure of $P_U^{\overset{L}{\boxtimes}\ell}$ is given in the way described in example C.3.

3.1.1. Persistence structure of Chiu-Tamarkin complexes. A persistence module M is a functor from $(\mathbb{R},\leq)$ to the category of R-modules. Equivalently, we have a family of R-modules M_T and a family of morphisms $M(T\leq T'): M_T\to M_{T'}$ such that $M(T\leq T'') = M(T'\leq T'')\circ M(T\leq T')$ and $M(T=T) = \mathrm{Id}_{M_T}$. In practice, we will assume $M_T = 0$ for $T << 0$. Here, we do not assume any regularity conditions as usual to simplify discussions. We refer the readers to [**PRSZ20**] for more about persistence modules.

Here, we will not completely follow this definition, and we will study a family of morphisms of complexes $\{M(T,T')\}_{T'\geq T}$ (we set $c = T'-T\geq 0$) in the equivariant derived category $D_{Z/\ell}(\mathrm{pt})$:

$$M(T,T+c): C_{\ell,T}(U,\mathbb{K}) \to C_{\ell,T+c}(U,\mathbb{K}),$$

with respect to the functor condition. Then we take cohomology H^q or H^* and put the trivial module at $T < 0$ to create persistence modules, over $R = \mathbb{K}$ or $R = A$, in the usual sense.

We study $\ell = 1$ first. Since $P_U \in \mathcal{D}(X^2)$, then there exists $\tau_c(P_U) : P_U \to \mathrm{T}_{c*}P_U$ for $c \geq 0$. Applying $\alpha_{1,X,T} \circ \mathrm{T}_{-c*}$, we have

$$F_1(U, \mathbb{K})_{T+c} \cong \alpha_{1,X,T}(\mathrm{T}_{-c*}P_U) \to F_1(U, \mathbb{K})_T.$$

Then, we have a family of morphisms for $c \geq 0$,

$$C_{1,T}(U, \mathbb{K}) \to C_{1,T+c}(U, \mathbb{K}). \tag{3.6}$$

For a general ℓ, we apply the same idea. Since $P_U \in \mathcal{D}(X^2)$, we have that $P_U^{\boxtimes \ell} \in \mathcal{D}_{\mathbb{Z}/\ell}(X^{2\ell})$. Therefore, for all $c \geq 0$, we have $\tau_c(P_U^{\boxtimes \ell}) : P_U^{\boxtimes \ell} \to \mathrm{T}_{c*}P_U^{\boxtimes \ell}$ in $\mathcal{D}_{\mathbb{Z}/\ell}(X^{2\ell})$ induced by $\tau_{c/\ell}(P_U) : P_U \to \mathrm{T}_{c/\ell *}P_U$. So, we obtain an equivariant morphism

$$F_\ell(U, \mathbb{K})_{T+c} = \alpha'_{\ell,X,T+c}(P_U^{\boxtimes \ell}) \cong \alpha'_{\ell,X,T}(\mathrm{T}_{-c*}P_U^{\boxtimes \ell}) \to \alpha'_{\ell,X,T}(P_U^{\boxtimes \ell}) \cong F_\ell(U, \mathbb{K})_T.$$

Then we have a family of morphisms for $c \geq 0$:

$$C_{\ell,T}(U, \mathbb{K}) \to C_{\ell,T+c}(U, \mathbb{K}). \tag{3.7}$$

On the other hand, we have a different way to obtain the persistence structure. We apply the Tamarkin projector in the equivariant case to obtain

$$\begin{aligned} C_{\ell,T}(U, \mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left(F_\ell(U, \mathbb{K})_T, \mathbb{K}[-d]\right) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(F_\ell(U, \mathbb{K}), \mathbb{K}_{\{T\}}[-d]\right) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(F_\ell(U, \mathbb{K}), \mathbb{K}_{[T,\infty)}[-d]\right). \end{aligned}$$

Then we also have a family of morphisms for $c \geq 0$:

$$C_{\ell,T}(U, \mathbb{K}) \to C_{\ell,T+c}(U, \mathbb{K}), \tag{3.8}$$

which is induced by $\mathbb{K}_{[T,\infty)} \to \mathbb{K}_{[T+c,\infty)}$.

Actually, these two ways are equivalent because τ_c commutes with the 6-operations and adjunctions as we discussed in Subsection 2.1.1.

The parameter T indicates the action of Reeb orbits and the translations T_{c*} quantize the Reeb flow. Therefore, we can define some Chiu-Tamarkin complexes of other action windows using a different cut-off of T.

Definition 3.4. When $T = \infty$, we define

$$H^*C_{\ell,\infty}(U,\mathbb{K}) := \varinjlim_{T\geq 0} H^*C_{\ell,T}(U,\mathbb{K}). \tag{3.9}$$

When we have an action window $(T, T']$, using the morphism $C_{\ell,T}(U,\mathbb{K}) \to C_{\ell,T'}(U,\mathbb{K})$, we define

$$\begin{aligned} C_{\ell,(T,T']}(U,\mathbb{K}) :=& \mathrm{RHom}_{\mathbb{Z}/\ell}\left(F_\ell(U,\mathbb{K}), \mathbb{K}_{[T,T')}[-d]\right) \\ \cong& cocone(C_{\ell,T}(U,\mathbb{K}) \to C_{\ell,T'}(U,\mathbb{K})) \end{aligned} \tag{3.10}$$

As an immediate consequence, we have a distinguished triangle for $0 \leq T < T'$, say:

$$C_{\ell,(T,T']}(U,\mathbb{K}) \to C_{\ell,T}(U,\mathbb{K}) \to C_{\ell,T'}(U,\mathbb{K}) \xrightarrow{+1}. \tag{3.11}$$

This is called the *action exact triangle* of the Chiu-Tamarkin complex.

The discussions here also apply to C^+, including versions for different actions, $H^*C^+_{\ell,\infty}(U,\mathbb{K})$, $C^+_{\ell,(T,T']}(U,\mathbb{K})$ and a similar distinguished triangle. One just need to replace P_U here by Q_U. Let us compute an example when $U = T^*X$. Recall $P_{T^*X} = \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$. So

$$\tilde{\Delta}^{-1}\left(P_{T^*X}^{\boxtimes \ell}\right) = \mathbb{K}_{\Delta_{X^\ell}\times[0,\infty)}.$$

Then

$$F_\ell(T^*X,\mathbb{K}) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\mathbb{K}_{\Delta_{X^\ell}\times[0,\infty)}) = E_{[0,\infty)},$$

where $E = R\Gamma_c(\Delta_{X^\ell},\mathbb{K})$, $E_{[0,\infty)}$ is the constant sheaf supported on $[0,\infty)$ and $\mathbb{Z}/\ell$ acts on $E = \mathrm{R}\Gamma_c(\Delta_{X^\ell},\mathbb{K}) \cong \mathrm{R}\Gamma_c(X,\mathbb{K})$ trivially. Since $\mathbb{Z}/\ell$ acts on E trivially, we have, by Poincaré-Verdier duality,

$$\begin{aligned} C_{\ell,T}(T^*X,\mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(E,\mathbb{K}[-d]) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})\overset{L}{\otimes}\mathrm{RHom}(E,\mathbb{K}[-d]) \\ \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})\overset{L}{\otimes}\mathrm{R}\Gamma(X, or_X) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})\overset{L}{\otimes}\mathrm{R}\Gamma_X(T^*X,\mathbb{K})[d]. \end{aligned} \tag{3.12}$$

Since $Q_{T^*X} \cong 0$, we have $F^+_\ell(T^*X, \mathbb{K}) = F_{\ell,X}(0) = 0$ and

$$C^+_{\ell,T}(T^*X, \mathbb{K}) = \mathrm{RHom}_{\mathbb{Z}/\ell}(0, \mathbb{K}_{[T,\infty)}[1]) \cong 0.$$

Also, for $0 \le T < T'$, we have

$$\begin{aligned} C_{\ell,(T,T']}(T^*X, \mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}(E_{[0,\infty)}, \mathbb{K}_{(T,T']}) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K}_{[0,\infty)}, \mathbb{K}_{(T,T']}) \overset{L}{\otimes} \mathrm{R}\Gamma(X, \mathbb{K}) \\ &\cong \mathrm{RHom}(\mathbb{K}_{[0,\infty)}, \mathbb{K}_{(T,T']}) \overset{L}{\otimes} \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K}) \overset{L}{\otimes} \mathrm{R}\Gamma(X, \mathbb{K}) \\ &\cong 0, \end{aligned}$$

where we use $\mathrm{RHom}(\mathbb{K}_{[0,\infty)}, \mathbb{K}_{(T,T']}) \cong \mathrm{RHom}(\mathbb{K}_{[0,\infty)\cap[T,T')}, \mathbb{K}_{\mathbb{R}}) \cong 0$, since $[0,\infty)\cap[T,T')$ is empty or half-closed. Finally, for $0 \le T < T' \le \infty$ and assuming $\mathbb{K}$ is a field, we have

$$\begin{aligned} & H^* C_{\ell,T}(T^*X, \mathbb{K}) \cong A \otimes H^{BM}_{d-*}(X, \mathbb{K}) \cong A \otimes H^{*+d}_X(T^*X, \mathbb{K}), \\ & H^* C^+_{\ell,T}(T^*X, \mathbb{K}) \cong H^* C_{\ell,(T,T']}(T^*X, \mathbb{K}) \cong 0. \end{aligned} \tag{3.13}$$

3.1.2. Functoriality and invariance.

THEOREM 3.5 (Theorem 4.7 of [Chi17]). *Let U, U_1, U_2 be admissible open sets and let $U_1 \overset{i}{\hookrightarrow} U_2$ be an inclusion. Then one has, for $T \ge 0$,*

(1) There is a morphism $C_{\ell,T}(U_2, \mathbb{K}) \xrightarrow{i^*} C_{\ell,T}(U_1, \mathbb{K})$, which is functorial with respect to inclusions of admissible open sets.

(2) For a compactly supported Hamiltonian isotopy $\varphi : T^*X \times I \to T^*X$, then there is an isomorphism, in the equivariant category, $\Phi_{z,\ell,T} : C_{\ell,T}(U, \mathbb{K}) \xrightarrow{\cong} C_{\ell,T}(\varphi_z(U), \mathbb{K})$, for all $z \in I$. The isomorphism $\Phi_{z,\ell,T}$ is functorial with respect to the restriction morphisms in (1). When $U = T^*X$, we have $\Phi_{z,\ell,T} = \mathrm{Id}$.

Taking into account the structure of $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K})$-modules, we have

Corollary 3.6. *Under the notation of Theorem 3.5, we have:*

(1) $H^*(i^*)$ is a morphism of A-modules.

(2) $H^*(\Phi_{z,\ell,T})$ is an isomorphism of A-modules.

For our later application, let us present a proof here. The notation is the same as in Theorem 3.5. But the proof for (2) is different from Chiu's original one.

Proof of Theorem 3.5: (1) Recall, Proposition 2.12 shows that we have a natural morphism $P_{U_1} \to P_{U_2}$. Then we apply F_ℓ to obtain

$$F_\ell(U_1, \mathbb{K}) \xrightarrow{F_\ell(i,\mathbb{K})} F_\ell(U_2, \mathbb{K}). \tag{3.14}$$

Then the first part follows by taking stalks over T.

(2) To prove the invariance we use the expression $C_{\ell,T}(U, \mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(P_U^{\boxtimes\ell}, \beta'_T\mathbb{K}[-d])$ given by the adjoint isomorphism N' in (3.5).

- We first relate P_U with $P_{\varphi_z(U)}$. By the results of [**GKS12**] recalled in Subsection 1.3.2, there exists $\mathcal{K} \in D(X^2 \times \mathbb{R}_t)$ (taking $\mathcal{K} = \mathcal{K}(\widehat{\varphi})_z^{-1}$) such that the convolution functor,

$$D(X \times \mathbb{R}_t) \to D(X \times \mathbb{R}_t), \quad F \mapsto \mathcal{K} \star F,$$

is an equivalence of categories and $\mu s_L(\mathcal{K} \star F) = \varphi_z(\mu s_L(F))$. Since $\widehat{\varphi}_z$ preserves τ of $T^*(X \times \mathbb{R}_t)$, this functor descends to the quotient $\mathcal{D}(X)$ and gives an auto-equivalence.

Since $\mathcal{K} \star -$ identifies $\mathcal{D}_{T^*X\setminus U}(X)$ with $\mathcal{D}_{T^*X\setminus\varphi_z(U)}(X)$, with inverse $\mathcal{K}^{-1} \star -$, we deduce that $F \mapsto \mathcal{K}^{-1} \star P_U \star \mathcal{K} \star F$ is the projector to $\mathcal{D}_{T^*X\setminus\varphi_z(U)}(X)$. By Proposition 2.12 we obtain $P_{\varphi_z(U)} \cong \mathcal{K}^{-1} \star P_U \star \mathcal{K}$. In particular, for $U = T^*X$, the isomorphism is realized by:

$$\mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \cong \mathcal{K}^{-1} \star \mathcal{K} \star \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \cong \mathcal{K}^{-1} \star \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \star \mathcal{K}.$$

Let us write $\mathcal{K}_\ell = \mathcal{K}^{\boxtimes\ell}$, $\mathcal{K}_\ell^{-1} = (\mathcal{K}^{-1})^{\boxtimes\ell}$. We remark that $\mathcal{K}_\ell$ has a natural lift in the equivariant category and that $\mathcal{K}_\ell$, $\mathcal{K}_\ell^{-1}$ are mutually inverse for the convolution. Hence $\mathcal{K}_\ell \star -$ is an equivalence and $\mathrm{RHom}_{\mathbb{Z}/\ell}(A, B) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathcal{K}_\ell \star A, \mathcal{K}_\ell \star B)$ for any $A, B \in D_{\mathbb{Z}/\ell}(X^{2l} \times \mathbb{R}_t)$. We denote by κ the auto-equivalence on $D_{\mathbb{Z}/\ell}(X^{2\ell} \times \mathbb{R}_t)$ induced by conjugation with $\mathcal{K}_\ell$:

$$\kappa(F) \coloneqq \mathcal{K}_\ell^{-1} \star F \star \mathcal{K}_\ell. \tag{3.15}$$

Then we have an isomorphism $P^{\boxtimes\ell}_{\varphi_z(U)} \cong \mathcal{K}_\ell^{-1} \star P^{\boxtimes\ell}_U \star \mathcal{K}_\ell = \kappa(P^{\boxtimes\ell}_U)$, and for $U = T^*X$, the isomorphism is realized by $\mathbb{K}^{\boxtimes\ell}_{\Delta_{X^2}\times[0,\infty)} \cong \mathcal{K}_\ell^{-1} \star \mathcal{K}_\ell \star \mathbb{K}^{\boxtimes\ell}_{\Delta_{X^2}\times[0,\infty)} \cong \mathcal{K}_\ell^{-1} \star \mathbb{K}^{\boxtimes\ell}_{\Delta_{X^2}\times[0,\infty)} \star \mathcal{K}_\ell$. Then the composition induces the isomorphism

$$\mathrm{RHom}_{\mathbb{Z}/\ell}(P^{\boxtimes\ell}_U, \beta'_T\mathbb{K}) \overset{\kappa}{\cong} \mathrm{RHom}_{\mathbb{Z}/\ell}(\kappa(P^{\boxtimes\ell}_U), \kappa(\beta'_T\mathbb{K})) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(P^{\boxtimes\ell}_{\varphi_z(U)}, \kappa(\beta'_T\mathbb{K})).$$

- Then it is enough to construct an isomorphism $\kappa(\beta'_T\mathbb{K}) = \mathcal{K}_\ell^{-1} \star \beta'_T\mathbb{K} \star \mathcal{K}_\ell \cong \beta'_T\mathbb{K}$. Compare to Chiu's original proof, we will construct the isomorphism explicitly.

Notice that $\beta'_T\mathbb{K}$ is, up to orientation and shift, the constant sheaf on the graph of the permutation map $f\colon X^\ell \to X^\ell$, $(\mathbf{q}_1, \dots, \mathbf{q}_\ell) \mapsto (\mathbf{q}_2, \dots, \mathbf{q}_\ell, \mathbf{q}_1)$. Set $Y = X^\ell$ and identify $Y^2 = (X^2)^\ell$ by $(\mathbf{q}^1_1, \dots, \mathbf{q}^1_\ell, \mathbf{q}^2_1, \dots, \mathbf{q}^2_\ell) \mapsto (\mathbf{q}^1_1, \mathbf{q}^2_1, \dots, \mathbf{q}^1_\ell, \mathbf{q}^2_\ell)$. Then, up to degree shifting, we have

$$\beta'_T\mathbb{K} \cong \mathbb{K}_{\Gamma_f\times\{T\}} \star E \cong E \star \mathbb{K}_{\Gamma_f\times\{T\}},$$

where $E = \delta_{Y^2!}(\omega_Y) \boxtimes \mathbb{K}_{\{T\}}$, with ω_Y the dualizing sheaf and δ_{Y^2} the usual diagonal embedding. In general, we have $E \star - \cong - \star E$.

Now we have the general fact $A \star \mathbb{K}_{\Gamma_g\times\{T\}} \cong (\mathrm{Id}_Y \times g \times \mathrm{T}_T)_!(A)$ for any A and any map g. This formula has the symmetric form $\mathbb{K}_{\Gamma'_g\times\{T\}} \star A \cong (g \times \mathrm{Id}_Y \times \mathrm{T}_T)_!(A)$ where Γ'_g is the switched graph $\Gamma'_g = \{(g(y), y) : y \in Y\}$. When g is invertible, we have $\Gamma_{g^{-1}} = \Gamma'_g$. So, we obtain

$$\mathcal{K}_\ell \star \beta'_T\mathbb{K} \cong \mathcal{K}_\ell \star \mathbb{K}_{\Gamma_f\times\{T\}} \star E \cong (\mathrm{Id}_Y \times f \times \mathrm{T}_T)_!(\mathcal{K}_\ell) \star E,$$

and

$$\beta'_T\mathbb{K} \star \mathcal{K}_\ell \cong E \star \mathbb{K}_{\Gamma_f\times\{T\}} \star \mathcal{K}_\ell = E \star \mathbb{K}_{\Gamma'_{f^{-1}}\times\{T\}} \star \mathcal{K}_\ell \cong E \star (f^{-1} \times \mathrm{Id}_Y \times \mathrm{T}_T)_!(\mathcal{K}_\ell).$$

But in coordinate $(X^2)^\ell$ we have $f \times f((\mathbf{q}^1_j, \mathbf{q}^2_j))_{j\in\mathbb{Z}/\ell} = ((\mathbf{q}^1_{j+1}, \mathbf{q}^2_{j+1}))_{j\in\mathbb{Z}/\ell}$. In other words $(f \times f)$ is the cyclic permutation of the X^2 factors in $(X^2)^\ell$. It is then clear that $(f \times f \times \mathrm{Id}_\mathbb{R})_!\mathcal{K}_\ell \cong \mathcal{K}_\ell$ (even in the equivariant category). Then we deduced that

$$\beta'_T\mathbb{K} \star \mathcal{K}_\ell \cong E \star (f^{-1} \times \mathrm{Id}_Y \times \mathrm{T}_T)_!(\mathcal{K}_\ell) \cong E \star (\mathrm{Id}_Y \times f \times \mathrm{T}_T)_!(\mathcal{K}_\ell) \cong \mathcal{K}_\ell \star \beta'_T\mathbb{K}.$$

Consequently, we have

$$\kappa(\beta'_T\mathbb{K}) = \mathcal{K}_\ell^{-1} \star \beta'_T\mathbb{K} \star \mathcal{K}_\ell \cong \mathcal{K}_\ell^{-1} \star \mathcal{K}_\ell \star \beta'_T\mathbb{K} \cong \beta'_T\mathbb{K}.$$

In summary, the $\Phi_{z,\ell,T}$ is defined as following. For any $f \in \mathrm{Ext}^*_{\mathbb{Z}/\ell}(P_U^{\boxtimes\ell}, \beta'_T\mathbb{K}[-d])$, we have

$$\Phi_{z,\ell,T}(f) : P^{\boxtimes\ell}_{\varphi_z(U)} \cong \kappa(P_U^{\boxtimes\ell}) \xrightarrow{\kappa(f)} \kappa(\beta'\mathbb{K}[-d]) \cong \beta'\mathbb{K}[-d].$$

The functoriality of $\Phi_{z,\ell,T}$ follows since κ is a functor.

For $U = T^*X$, the isomorphism $P^{\boxtimes\ell}_{T^*X} \cong \kappa(P^{\boxtimes\ell}_{T^*X})$ is induced by the natural isomorphism $P_{T^*X} \star \mathcal{K} \cong \mathcal{K} \star P_{T^*X}$. So does $\kappa(\beta'\mathbb{K}[-d]) \cong \beta'\mathbb{K}[-d]$. Then the induced isomorphism $\Phi_{z,\ell,T}(f)$ is the identity on the cohomology level. $\square$

Actually, the construction of $\kappa(\beta'\mathbb{K}) \cong \beta'\mathbb{K}$ is functorial with respect to $M = \mathbb{K}$. For general $M \in D_{\mathbb{Z}/\ell}(\mathrm{pt})$, we only need to replace $K = \delta_{Y^2!}(\pi_Y^!\mathbb{K})$ in the proof by $K(M) = \delta_{Y^2!}(\pi_Y^! M)$. Consequently, we can construct an isomorphism of functors

$$\Phi_{z,\ell,T}(-) : \mathrm{RHom}_{\mathbb{Z}/\ell}(F_\ell(U,\mathbb{K})_T, -) \xrightarrow{\cong} \mathrm{RHom}_{\mathbb{Z}/\ell}(F_\ell(\varphi(U),\mathbb{K})_T, -).$$

Now, let us take $M = F_\ell(\varphi(U),\mathbb{K})_T$. Then $\mathrm{Id}_{F_\ell(\varphi(U),\mathbb{K})_T}$ provide us an isomorphism

$$\Phi'_{z,\ell,T} \coloneqq \Phi^{-1}_{z,\ell,T}(\mathrm{Id}_{F_\ell(\varphi(U),\mathbb{K})_T}) = F_\ell(U,\mathbb{K})_T \to F_\ell(\varphi(U),\mathbb{K})_T.$$

In summary, we have

Proposition 3.7. *For a compactly supported Hamiltonian isotopy $\varphi : T^*X \times I \to T^*X$, then there is an isomorphism, in the equivariant category, $\Phi'_{z,\ell,T} : F_\ell(U,\mathbb{K})_T \to F_\ell(\varphi_z(U),\mathbb{K})_T$, for all $z \in I$.*

Remark 3.8. All results of this section are true for $H^*C_{\ell,\infty}(U,\mathbb{K})$, $C_{\ell,(a,b]}(U,\mathbb{K})$, and $C^+_{\ell,T}(U,\mathbb{K})$.

3.1.3. Fundamental class. In this subsection, let X be an oriented manifold of dimension d with a fixed orientation.

For $U \xhookrightarrow{i_U} T^*X$ an admissible open subset and $T \geq 0$, the Theorem 3.5-(1) shows that we have a morphism in the $\mathbb{Z}/\ell$-equivariant derived category:

$$C_{\ell,T}(T^*X, \mathbb{K}) \xrightarrow{i_U^*} C_{\ell,T}(U, \mathbb{K}),$$

and it induces a morphism on cohomology

$$H_{d-q}^{BM,\mathbb{Z}/\ell}(X, \mathbb{K}) \cong H^q C_{\ell,T}(T^*X, \mathbb{K}) \xrightarrow{i_U^*} H^q C_{\ell,T}(U, \mathbb{K}),$$

here the first isomorphism is given in (3.12). Since X is orientable, we have the equivariant fundamental class $[X]^{\mathbb{Z}/\ell}$ of X in $H_d^{BM,\mathbb{Z}/\ell}(X, \mathbb{K})$. (See appendix B.)

Definition 3.9. If X is orientable with a fixed orientation. For an admissible open set $U \xhookrightarrow{i_U} T^*X$, and $T \geq 0$, we define its fundamental class $\eta_T(U, \mathbb{K})$ as the image of $[X]^{\mathbb{Z}/\ell}$ under i_U^*. I.e. $\eta_T(U, \mathbb{K}) = i_U^*([X]^{\mathbb{Z}/\ell}) \in H^0 C_{\ell,T}(U, \mathbb{K})$.

Remark 3.10. (1) In the point of view of appendix C, we have $[X]^{\mathbb{Z}/\ell} = St_D([X])$, where $[X]$ is the non-equivariant fundamental class. Then we have $\eta_{\ell,T}(U, \mathbb{K}) = St_D(\eta_{1,T}(U, \mathbb{K}))$ since St_D commute with the 6-operations.

(2) For any open set $W \subset X$, under the restriction map $H_d^{BM,\mathbb{Z}/\ell}(X, \mathbb{K}) \to H_d^{BM,\mathbb{Z}/\ell}(W, \mathbb{K})$, we have $[X]^{\mathbb{Z}/\ell} \mapsto [W]^{\mathbb{Z}/\ell}$. For any W such that $U \subset T^*W \subset T^*X$, we have $\eta_T(U, \mathbb{K}) = i_U^*([X]^{\mathbb{Z}/\ell}) = i_U^*([W]^{\mathbb{Z}/\ell})$. In particular, one can take $W = X_U = p_X(U) \subset X$.

(3) By definition and the adjoint isomorphism, the fundamental class can be computed as the following composition:

$$(F_\ell(U, \mathbb{K}))_T \to (F_\ell(T^*X, \mathbb{K}))_T \cong \mathrm{R}\Gamma_c(X, \mathbb{K}) \xrightarrow{or} H^d \mathrm{R}\Gamma_c(X, \mathbb{K})[-d] \cong \mathbb{K}[-d]. \tag{3.16}$$

It is direct to see that the fundamental class is natural with respect to the morphism $C_{\ell,T}(U, \mathbb{K}) \to C_{\ell,T+c}(U, \mathbb{K})$, i.e.,

$$\eta_{\ell,T}(U, \mathbb{K}) \mapsto \eta_{\ell,T+c}(U, \mathbb{K}).$$

As a corollary of theorem 3.5, we have

Proposition 3.11. *(1) Let $U \subset V \subset T^*X$ be an inclusion of admissible open sets. Through the natural morphism*

$$H^0 C_{\ell,T}(V,\mathbb{K}) \to H^0 C_{\ell,T}(U,\mathbb{K})$$

we have

$$\eta_{\ell,T}(V,\mathbb{K}) \mapsto \eta_{\ell,T}(U,\mathbb{K}).$$

(2) Let $\varphi : T^*X \times I \to T^*X$ be a compactly supported Hamiltonian isotopy and U be an admissible open set. Recall the A-module isomorphism, defined in Theorem 3.5,

$$H^*(\Phi_{z,\ell,T}) : H^* C_{\ell,T}(U,\mathbb{K}) \xrightarrow{\cong} H^* C_{\ell,T}(\varphi_z(U),\mathbb{K}).$$

Then we have $H^0(\Phi_{z,\ell,T})(\eta_{\ell,T}(U,\mathbb{K})) = \eta_{\ell,T}(\varphi_z(U),\mathbb{K})$ for all $z \in I$.

3.1.4. Tautological triangle of Chiu-Tamarkin complex. For an admissible open set U with kernel P_U, Lemma 2.11 shows $P_U \star P_U \xrightarrow{\cong} P_U \in \mathcal{D}(X \times X)$. So, in general, we have $P_U^{\star\ell} \xrightarrow{\cong} P_U$ for $\ell \in \mathbb{N}$. Also, we have $Q_U \xrightarrow{\cong} Q_U^{\star\ell}$.

Now, let us describe $P_U^{\star\ell}$ more precisely. Consider

$$d : X^{\ell+1} \times \mathbb{R}_t \to X^{2\ell} \times \mathbb{R}_t, \quad d(\mathbf{q}_0,\dots,\mathbf{q}_\ell,t) = (\mathbf{q}_0,\mathbf{q}_1,\mathbf{q}_1,\dots,\mathbf{q}_{\ell-1},\mathbf{q}_{\ell-1},\mathbf{q}_\ell,t).$$

We have

$$H_1 \star H_2 \star \cdots H_\ell \cong \mathrm{R}\pi_{(\mathbf{q}_1,\dots,\mathbf{q}_{\ell-1})!} d^{-1} \mathrm{R}s^\ell_{t!}(H_1 \boxtimes H_2 \boxtimes \cdots \boxtimes H_\ell). \tag{3.17}$$

On the other hand, recall the proof of $P_U^{\star\ell} \xrightarrow{\cong} P_U$, which is given by $P_U^{\star\ell} \xrightarrow{\cong} \mathbb{K}^{\star(\ell-1)}_{\Delta_{X^2}\times[0,\infty)} \star P_U$. Then the isomorphism is obtained by applying $G = \mathrm{R}\pi_{(\mathbf{q}_1,\dots,\mathbf{q}_{\ell-1})!} d^{-1} \mathrm{R}s^\ell_{t!}$ to the following morphism

$$P_U^{\overset{L}{\boxtimes}\ell} \to \mathbb{K}^{\overset{L}{\boxtimes}(\ell-1)}_{\Delta_{X^2}\times[0,\infty)} \overset{L}{\boxtimes} P_U.$$

But the projection formula and the base change formula shows that

$$F_{1,X} \circ G \cong F_{\ell,X}.$$

Then we have the following isomorphism

$$F_{\ell,X}(P_U^{\overset{L}{\boxtimes}\ell})\cong F_{1,X}(P_U^{\star\ell}) \xrightarrow{\cong} F_{1,X}(P_U). \tag{3.18}$$

A similar result holds for Q_U, then we have

Proposition 3.12. *For an admissible open set U, we have isomorphisms of sheaves of $\mathbb{K}$-module:*

$$F_\ell(U,\mathbb{K}) \cong F_1(U,\mathbb{K}) \qquad F_1^+(U,\mathbb{K}) \cong F_\ell^+(U,\mathbb{K}).$$

Notice that, this isomorphism is not in the equivariant derived category. Actually, $F_1(U,\mathbb{K})$ is not an equivariant sheaves.

Recall that, if we apply the functor F_1 to the defining triangle of U, then we have a distinguished triangle

$$F_1(U,\mathbb{K}) \to \mathrm{R}\Gamma_c(X,\mathbb{K}) \to F_1^+(U,\mathbb{K}) \xrightarrow{+1} .$$

The proposition 3.12 shows that we can replace F_1 above by F_ℓ to obtain a distinguished triangle in the non-equivariant category. We would like to lift it to the equivariant category.

Proposition 3.13. *For an admissible open set U, we have a distinguished triangle in the equivariant derived category:*

$$F_\ell(U,\mathbb{K}) \to \mathrm{R}\Gamma_c(X,\mathbb{K}) \to F_\ell^+(U,\mathbb{K}) \xrightarrow{+1} . \tag{3.19}$$

Then a distinguished triangle for Chiu-Tamarkin complex follows:

$$\mathrm{R}\Gamma(X,\omega_X^{\mathbb{Z}/\ell}) \to C_{\ell,T}(U,\mathbb{K}) \to C_{\ell,T}^+(U,\mathbb{K}) \xrightarrow{+1} . \tag{3.20}$$

We call them the tautological exact triangles *for the Chiu-Tamarkin complex.*

Proof. Take the defining triangle of U, say:

$$P_U \xrightarrow{a} \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \xrightarrow{b} Q_U \xrightarrow{1},$$

then we have $ba = 0$.

By taking Steenrod operation and $\mathrm{R}s^{\ell}_{t!}$, we have two morphisms in the equivariant derived categories:

$$P_U^{\boxed{\star}\ell} \xrightarrow{a^{\boxed{\star}\ell}} \mathbb{K}^{\boxed{\star}\ell}_{\Delta_{X^2}\times[0,\infty)} \xrightarrow{b^{\boxed{\star}\ell}} Q_U^{\boxed{\star}\ell},$$

and their composition is 0. But the Steenrod operation is not a triangulated functor, so they can not form a distinguished triangle in an obvious way.

Let us take the cone of $a^{\boxed{\star}\ell}$, then we have a distinguished triangle in the equivariant category

$$P_U^{\boxed{\star}\ell} \xrightarrow{a^{\boxed{\star}\ell}} \mathbb{K}^{\boxed{\star}\ell}_{\Delta_{X^2}\times[0,\infty)} \xrightarrow{c^{\ell}} \mathcal{C}_\ell \xrightarrow{+1} . \tag{3.21}$$

Since $b^{\boxed{\star}\ell}a^{\boxed{\star}\ell} = 0$, by applying the cohomological functor $\mathrm{Hom}_{\mathbb{Z}/\ell}(-, Q_U^{\boxed{\star}\ell})$, we have a morphism $\psi : \mathcal{C}_\ell \to Q_U^{\boxed{\star}\ell}$ that fits into the commutative diagram in the equivariant category:

$$\begin{tikzcd}
P_U^{\boxed{\star}\ell} \arrow[r, "a^{\boxed{\star}\ell}"] & \mathbb{K}^{\boxed{\star}\ell}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "c^{\ell}"] \arrow[dr, "b^{\boxed{\star}\ell}"'] & \mathcal{C}_\ell \arrow[r, "+1"] \arrow[d, dashed, "\psi"] & {} \\
 & & Q_U^{\boxed{\star}\ell}. &
\end{tikzcd}$$

The tricky thing here is that we do not know if ψ is an isomorphism. But we can show that $F'_\ell(\psi)$ (see (3.4)) is an isomorphism in the equivariant category. Then the distinguished triangle (3.19) follows.

We will argue in the following steps. First, let us consider the forgetful functor $For : D_{\mathbb{Z}/\ell} \to D$. It is proven in Proposition A.2 that For is a conservative functor. Next, we will show that there exists an isomorphism $\phi : For(F'_\ell(\mathcal{C}_\ell)) \to For(F'_\ell(Q_U^{\boxed{\star}\ell}))$ such that $For(F'_\ell(\psi)) = \phi$ in the non-equivariant derived category D. So we omit the functor For in the following. The idea for the proof is more or less the same as the proof of Proposition 2.12. To the convenience of readers, we present details here.

Using the morphism $P_U^{\overset{L}{\boxtimes}\ell} \to \mathbb{K}^{\overset{L}{\boxtimes}\ell-1}_{\Delta_{X^2}\times[0,\infty)}\overset{L}{\boxtimes}P_U$, we embed (3.21) into the following diagram:

$$\begin{tikzcd}
P_U^{\boxast\ell} \arrow[r, "a^{\boxast\ell}"] \arrow[d] & \mathbb{K}^{\boxast\ell}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "c^\ell"] \arrow[d, "="] & \mathcal{C}_\ell \arrow[r, "+1"] & {} \\
\mathbb{K}^{\boxast\ell-1}_{\Delta_{X^2}\times[0,\infty)} \boxast P_U \arrow[r] & \mathbb{K}^{\boxast\ell}_{\Delta_{X^2}\times[0,\infty)} \arrow[r] & \mathbb{K}^{\boxast\ell-1}_{\Delta_{X^2}\times[0,\infty)} \boxast Q_U \arrow[r, "+1"] & {}
\end{tikzcd}$$

Applying $G' = \mathrm{R}\pi_{(\mathbf{q}_1,\ldots,\mathbf{q}_{\ell-1})!}d^{-1}$ (see (3.17)) to the diagram, we obtain a diagram below. To be precise, we explicitly give the morphisms for distinguished triangles.

$$\begin{tikzcd}
P_U^{\star\ell} \arrow[r, "a^{\star\ell}"] \arrow[d, "\cong"] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "G'(c^\ell)"] \arrow[d, "="] & G'(\mathcal{C}_\ell) \arrow[r, "e"] \arrow[d, dashed] & P_U^{\star\ell}[1] \arrow[d, "\cong"] \\
P_U \arrow[r, "a"] & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "b"] & Q_U \arrow[r, "d"] & P_U[1]
\end{tikzcd}$$

So TR3 shows that there exists an isomorphism $G'(\mathcal{C}_\ell) \to Q_U$ which fills the diagram into a commutative diagram. Then we take ψ' to be the composition:

$$G'(\mathcal{C}_\ell) \to Q_U \to Q_U^{\star\ell},$$

which is an isomorphism. Now, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \arrow[r, "G'(c^\ell)"] \arrow[d, "b"] & G'(\mathcal{C}_\ell) \arrow[d, "\psi'"] \arrow[dl] \\
Q_U \arrow[r] & Q_U^{\star\ell}.
\end{tikzcd}$$

So we obtain one factorization $b^{\star\ell} = \psi' G'(c^\ell)$.

On the other hand, applying G' to the factorization $b^{\boxast\ell} = \psi c^\ell$, we obtain another factorization $b^{\star\ell} = G'(\psi)G'(c^\ell)$.

Consequently, $(\psi' - G'(\psi))G'(c^\ell) = 0$ determines an element $u \in \mathrm{Hom}(P_U^{\star\ell}[1], Q_U^{\star\ell})$ such that

$$\psi' - G'(\psi) = ue.$$

However, we have

$$\mathrm{Hom}(P_U^{\star\ell}[1], Q_U^{\star\ell}) \cong \mathrm{Hom}(P_U[1], Q_U) \cong 0.$$

Then we have

$$\psi' = G'(\psi).$$

Consequently, we have

$$\phi = F_1(\psi') = F_1(G'(\psi)) = F'_\ell(\psi),$$

and $\phi = F_1(\psi')$ is an isomorphism since ψ' is an isomorphism. $\square$

3.2. Yoneda product and cup product

Let us study the Yoneda product on $\{\mathrm{Ext}^*(P_U, \mathrm{T}_{c*}P_U)\}_{\{c\geq 0\}}$. Then we will formulate the Yoneda product as a cup product on $H^*C_{1,c}(U, \mathbb{K})$ when X is orientable.

Definition 3.14. For $\alpha \in \mathrm{Ext}^a(P_U, \mathrm{T}_{A*}P_U)$, $\beta \in \mathrm{Ext}^b(P_U, \mathrm{T}_{B*}P_U)$, we define the shifted Yoneda product $\alpha \bullet \beta \in \mathrm{Ext}^{a+b}(P_U, \mathrm{T}_{(A+B)*}P_U)$ to be the composition:

$$P_U \xrightarrow{\beta} \mathrm{T}_{B*}P_U \xrightarrow{\mathrm{T}_{A*}\alpha} \mathrm{T}_{(A+B)*}P_U.$$

The shifted Yoneda product is a T_{c*} shifted version of the usual Yoneda product on $\mathrm{Ext}^a(P_U, P_U)$. It also appears in [**BCZ21**, Lemma 6.4.4]. We can see that it is unital and associative like the usual Yoneda product. But an interesting thing is that, since P_U is a coalgebra in the symmetric monoidal category $(\mathcal{D}(X^2), \star)$ by the kernel property (as will be seen in the proof), we have the following stronger property.

THEOREM 3.15. *The shifted Yoneda product is unital, associative and graded commutative.*

PROOF. First of all, the usual identity Id_{P_U} is the unit of the shifted Yoneda product. The associativity follows from the associativity of the usual Yoneda product and the functor identity $\mathrm{T}_{(A+B+C)*} \cong \mathrm{T}_{A*} \circ \mathrm{T}_{B*} \circ \mathrm{T}_{C*}$.

Let us prove the graded commutativity. As P_U is a projector, we have an isomorphism

$$\nu : P_U \star P_U \xrightarrow{\cong} P_U.$$

Now, let us define another product using ν. The convolution $\star$ is a bifunctor, for $\alpha \in \mathrm{Ext}^a(P_U, \mathrm{T}_{A*}P_U)$, $\beta \in \mathrm{Ext}^b(P_U, \mathrm{T}_{B*}P_U)$, then we have

$$\alpha \star \beta : P_U \star P_U \to \mathrm{T}_{A*}P_U \star \mathrm{T}_{B*}P_U \cong \mathrm{T}_{(A+B)*}(P_U \star P_U).$$

Then we define

$$\alpha \underline{\star} \beta = \mathrm{T}_{(A+B)*}(\nu) \circ (\alpha \star \beta) \circ \nu^{-1}.$$

We will apply the Eckmann-Hilton argument below to the shifted Yoneda product and the convolution to prove that

(1) : $\alpha \underline{\star} \beta = \alpha \bullet \beta$,

(2) : $\alpha \bullet \beta$ is graded commutative.

To apply the Eckmann-Hilton argument we first notice the following identities:

- $\mathrm{Id}_{P_U} \underline{\star} \alpha = \alpha \underline{\star} \mathrm{Id}_{P_U} = \alpha$,
- For $\alpha_i \in \mathrm{Ext}^{a_i}(P_U, \mathrm{T}_{A_i *} P_U)$, $i = 1, 2, 3, 4$, we have

$$(\alpha_1 \bullet \alpha_2) \underline{\star} (\alpha_3 \bullet \alpha_4) = (-1)^{a_2 a_3} (\alpha_1 \underline{\star} \alpha_3) \bullet (\alpha_2 \underline{\star} \alpha_4).$$

To prove them, let us notice that $\star$ is a bifunctor, and that the graded shifting [1] defines isomorphisms $\mathrm{Ext}^*(F, G[1]) \cong \mathrm{Ext}^*(F, G)[1] \cong \mathrm{Ext}^*(F[-1], G)$ such that the following diagram is anti-commutative for all F, G:

$$\begin{tikzcd}
\mathrm{Ext}^*(F[1], G[1]) \arrow[r] \arrow[d] & \mathrm{Ext}^*(F, G[1])[-1] \arrow[d] \\
\mathrm{Ext}^*(F[1], G)[1] \arrow[r] & \mathrm{Ext}^*(F, G).
\end{tikzcd}$$

Next, the Eckmann-Hilton argument uses the two above identities to conclude as follows:

$$\alpha_1 \underline{\star} \alpha_2 = (\alpha_1 \bullet \mathrm{Id}_{P_U}) \underline{\star} (\mathrm{Id}_{P_U} \bullet \alpha_2) = (\alpha_1 \underline{\star} \mathrm{Id}_{P_U}) \bullet (\mathrm{Id}_{P_U} \underline{\star} \alpha_2) = \alpha_1 \bullet \alpha_2$$

and

$$\alpha_1 \bullet \alpha_2 = \alpha_1 \underline{\star} \alpha_2 = (\mathrm{Id}_{P_U} \bullet \alpha_1) \underline{\star} (\alpha_2 \bullet \mathrm{Id}_{P_U}) = (-1)^{a_1 a_2} (\mathrm{Id}_{P_U} \underline{\star} \alpha_2) \bullet (\alpha_1 \underline{\star} \mathrm{Id}_{P_U}) = (-1)^{a_1 a_2} \alpha_2 \bullet \alpha_1.$$

$\square$

Now, let us recall that, when X is orientable, we have the following isomorphism (see (2.7)) of $\mathbb{K}$-modules:

$$\begin{aligned} \Theta : \mathrm{Ext}^*(P_U, \mathrm{T}_{T*}P_U) &\xrightarrow{\cong} \mathrm{Ext}^*(P_U, \mathbb{K}_{\Delta_{X^2}\times\{T\}}) \cong H^* C_{1,T}(U,\mathbb{K}) \\ [P_U \xrightarrow{\alpha} \mathrm{T}_{T*}P_U] &\mapsto \widetilde{\alpha} = [P_U \xrightarrow{\alpha} \mathrm{T}_{T*}P_U \to \mathbb{K}_{\Delta_{X^2}\times\{T\}}] \mapsto N(\widetilde{\alpha}), \end{aligned} \tag{3.22}$$

here we use the orientation of X to identify $\mathbb{K}_{\Delta_{X^2}\times\{T\}}$ with $\beta_{1,T,X}\mathbb{K}[-d]$ and the adjoint isomorphism N is defined in (3.5).

We would like to express $\widetilde{\alpha \bullet \beta} = \widetilde{\alpha \underline{\star} \beta}$ into a more geometric form in the non-equivariant Chiu-Tamarkin cohomology. Let us define a cup product on $H^* C_{1,T}(U,\mathbb{K})$.

Take $\Theta(\alpha) \in H^a C_{1,A}(U,\mathbb{K})$ and $\Theta(\beta) \in H^b C_{1,B}(U,\mathbb{K})$ that correspond to $\alpha, \beta \in \mathrm{Ext}^-(P_U, \mathrm{T}_{-*}P_U)$.

Then, we consider

$$F_2(\widetilde{\alpha} \overset{L}{\boxtimes} \widetilde{\beta}) : F_2(P_U \overset{L}{\boxtimes} P_U) \xrightarrow{\alpha \overset{L}{\boxtimes} \beta} \mathrm{T}_{(A+B)*}F_2(P_U \overset{L}{\boxtimes} P_U) \to \mathrm{T}_{(A+B)*}F_2(\mathbb{K}_{\Delta_{X^2}\times\{0\}} \overset{L}{\boxtimes} \mathbb{K}_{\Delta_{X^2}\times\{0\}}).$$

Recall Proposition 3.12, we take $\ell = 2$ here. We denote the isomorphism therein by:

$$s = s(U) : F_1(U,\mathbb{K}) \to F_2(U,\mathbb{K}).$$

Therefore, we have

$$\begin{tikzcd} F_2(P_U \overset{L}{\boxtimes} P_U) \arrow[r, "F_2(\alpha \overset{L}{\boxtimes} \beta)"] & \mathrm{T}_{(A+B)*}F_2(P_U \overset{L}{\boxtimes} P_U) \\ F_1(P_U) \arrow[u, "s(U)"] \arrow[r, "F_1(\alpha \underline{\star} \beta)"] & \mathrm{T}_{(A+B)*}F_1(P_U) \arrow[u, "\mathrm{T}_{(A+B)*}s(U)"'] \end{tikzcd}$$

On the other hand, consider the inclusion $U \subset T^*X$. Then, we apply the naturality of the morphism s to obtain

$$\begin{tikzcd} F_2(P_U \overset{L}{\boxtimes} P_U) \arrow[r] & F_2(\mathbb{K}_{\Delta_{X^2}\times\{0\}} \overset{L}{\boxtimes} \mathbb{K}_{\Delta_{X^2}\times\{0\}}) \\ F_1(P_U) \arrow[u, "s(U)"] \arrow[r, "\eta_0(U,\mathbb{K})"] & F_1(\mathbb{K}_{\Delta_{X^2}\times\{0\}}). \arrow[u, "s(T^*X)"'] \end{tikzcd}$$

Then, we apply $\mathrm{T}_{(A+B)*}$ to the second diagram, and put these two diagrams together to obtain the commutative diagram

$$\begin{tikzcd}
F_2(P_U\overset{L}{\boxtimes}P_U) \arrow[r, "F_2(\alpha\overset{L}{\boxtimes}\beta)"] & \mathrm{T}_{(A+B)*}F_2(P_U\overset{L}{\boxtimes}P_U) \arrow[r] & F_2(\mathbb{K}_{\Delta_{X^2}\times\{A\}}\overset{L}{\boxtimes}\mathbb{K}_{\Delta_{X^2}\times\{B\}}) \\
F_1(P_U) \arrow[u, "s(U)"] \arrow[r, "F_1(\alpha\underline{\star}\beta)"] & \mathrm{T}_{(A+B)*}F_1(P_U) \arrow[u, "\mathrm{T}_{(A+B)*}(s(U))"] \arrow[r, "\mathrm{T}_{(A+B)*}(\eta_0(U,\mathbb{K}))"] & F_1(\mathbb{K}_{\Delta_{X^2}\times\{A+B\}}) \arrow[u, "\mathrm{T}_{(A+B)*}(s(T^*X))"]
\end{tikzcd}$$

Now, the second row of the diagram is $F_1(\widetilde{\alpha\underline{\star}\beta})$, and then let us defined

Definition 3.16. Take $\alpha \in H^aC_{1,A}(U,\mathbb{K})$ and $\beta \in H^bC_{1,B}(U,\mathbb{K})$. We define the cup product to be

$$\alpha \cup \beta = \mathrm{T}_{(A+B)*}(s(T^*X))^{-1} \circ F_2(\Theta^{-1}(\alpha)\overset{L}{\boxtimes}\Theta^{-1}(\beta)) \circ s(U) \in H^{a+b}C_{1,A+B}(U,\mathbb{K}).$$

Then, our discussion here is summarized as

Proposition 3.17. *The isomorphism of* $\mathbb{K}$*-modules* Θ *in* (3.22) *is an isomorphism of* $\mathbb{K}$*-algebras.*

Compare to the Yoneda product, the cup product is more geometrical. Actually, consider the external tensor

$$\alpha\overset{L}{\boxtimes}\beta : P_U^{\overset{L}{\boxtimes}2} \to \mathbb{K}_{\Delta_{X^2}\times\{A\}\times\Delta_{X^2}\times\{B\}}[a+b].$$

Next, we apply the Gysin morphism associated to the pair $(\tilde{\Delta}, \Delta_{X^4})$.

$$e : \mathbb{K}_{\Delta_{X^2}\times\Delta_{X^2}\times\{(A,B)\}} \to \mathbb{K}_{\Delta_{X^4}\times\{(A,B)\}} \to \mathbb{K}_{\tilde{\Delta}\times\{(A,B)\}}[a+b+d],$$

where $\tilde{\Delta} = \{(\mathbf{q},\mathbf{q}',\mathbf{q}',\mathbf{q}) \in X^4 : \mathbf{q},\mathbf{q}' \in X\}$, and the first map is the natural restriction.

In fact, the Gysin morphism is given by composion with the Thom class, which is given by the following isomorphisms:

$$\begin{aligned}
&\mathrm{Ext}^d(\mathbb{K}_{\Delta_{X^4}\times\{(A,B)\}}, \mathbb{K}_{\tilde{\Delta}\times\{(A,B)\}}) \cong \mathrm{Ext}^d(\mathbb{K}_{\Delta_{X^4}}, \mathbb{K}_{\tilde{\Delta}}) \\
\cong &\mathrm{Ext}^d(\mathbb{K}_{\Delta_{X^2}}, \mathbb{K}_{X^2}) \cong H^d_{\Delta_{X^2}}(X^2,\mathbb{K}) \cong H^0(X,\mathbb{K}).
\end{aligned}$$

The last isomorphism comes from the excision and the Thom isomorphism. Then $1 \in H^0(X, \mathbb{K})$ correspond to the Thom class

$$\mathbb{K}_{\Delta_{X^4} \times \{(A,B)\}} \to \mathbb{K}_{\tilde{\Delta} \times \{(A,B)\}}[d].$$

Then, we obtain a class

$$e \circ (\alpha \overset{L}{\boxtimes} \beta) \in \operatorname{Ext}^{a+b+d}(P_U^{\overset{L}{\boxtimes} 2}, \mathbb{K}_{\tilde{\Delta} \times \{(A,B)\}}).$$

Finally, we apply $\mathrm{R}s^2_{t!}$ to $e \circ (\alpha \overset{L}{\boxtimes} \beta)$ to obtain

$$\mathrm{R}s^2_{t!}(e \circ (\alpha \overset{L}{\boxtimes} \beta)) \in \operatorname{Ext}^{a+b+d}(\mathrm{R}s^2_{t!} P_U^{\overset{L}{\boxtimes} 2}, \mathbb{K}_{\tilde{\Delta} \times \{A+B\}}).$$

Then, it is direct to use adjoint isomorphisms and the fact $\mathbb{K}_{\tilde{\Delta} \times \{A+B\}}[d] \cong \tilde{\Delta}_* \pi^!_{X^2} \mathbb{K}[-d]$ to see that $\mathrm{R}s^2_{t!}(e \circ (\alpha \overset{L}{\boxtimes} \beta))$ represents the cup product.

At the end, let us remark some corollaries and examples.

Remark 3.18. (1) Since $P_{T^*X} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$, we have that, for all $T \geq 0$,

$$H^* C_{1,T}(T^*X, \mathbb{K}) \cong \operatorname{Ext}^*(\mathbb{K}_{\Delta_{X^2} \times [0,\infty)}, \mathbb{K}_{\Delta_{X^2} \times [T,\infty)}) \cong \operatorname{Ext}^*(\mathbb{K}_X, \mathbb{K}_X) \cong H^*(X, \mathbb{K}).$$

It is known that the Yoneda product on $\operatorname{Ext}^*(\mathbb{K}_X, \mathbb{K}_X)$ is the usual cup product on $H^*(X, \mathbb{K})$, see [**Ive86**]. So, the cup product we defined here is the same as the usual cup product on $H^*(X, \mathbb{K})$. Moreover, the morphism

$$H^*(X, \mathbb{K}) \cong H^* C_{1,T}(T^*X, \mathbb{K}) \to H^* C_{1,T}(U, \mathbb{K}),$$

is a morphism of $\mathbb{K}$-algebra.

(2) Using the Yoneda product, we see that $H^* C_{1,T}(U, \mathbb{K})$ is a right module over $\operatorname{Ext}^*(P_U, P_U)$, and a left-module over $\operatorname{Ext}^*(\mathbb{K}_{\Delta_{X^2} \times [T,\infty)}, \mathbb{K}_{\Delta_{X^2} \times [T,\infty)}) \cong H^*(X, \mathbb{K})$.

(3) The cup product is compatible with the persistence structure on $H^* C_{1,T}(U, \mathbb{K})$. So, the cup product descents to a $\mathbb{K}$ -algebra structure on $H^* C_{1,\infty}(U, \mathbb{K})$.

3.3. Cyclic structure and S^1-equivariant Chiu-Tamarkin complex

In this section, we would like to present a formal discussion on the idea: The Chiu-Tamarkin complex studies a discrete approximation of the S^1 action. We will use the structure maps of microlocal kernels to give an algebraic S^1 action, i.e. mixed complex, on the $F_\ell(U,\mathbb{K})$ for all ℓ. Then we will define a S^1-equivariant Chiu-Tamarkin complex.

3.3.1. Cyclic structure. The idea of Proposition 3.12 can help us to define a pre-cyclic structure using $F_n(U,\mathbb{K})$ for all n. Basically, the morphism

$$P_U^{\overset{L}{\boxtimes} n} \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \overset{L}{\boxtimes} P_U^{\overset{L}{\boxtimes} n-1},$$

and its cyclic permutations induce some morphisms

$$d_i : F_{n+1}(U,\mathbb{K}) \to F_n(U,\mathbb{K})$$
$$t_n : F_n(U,\mathbb{K}) \to F_n(U,\mathbb{K}),$$

for $i \in [n]$. Moreover, they satisfy the relations:

$$d_i d_j = d_{j-1} d_i,\ i < j,$$
$$t_n^{n+1} = \mathrm{Id},$$

in $D(\mathbb{R})$. So we obtain a pre-cyclic object of $D(\mathbb{R})$. Similarly, we can define a pre-cocyclic object of $D(\mathbb{R})$ using Q_U. Taking the stalk at T, we will obtain a pre-(co)cyclic object of $D(\mathbb{K}-\mathrm{Mod})$.

A standard construction can help us define a mixed complex, in the sense of [**Kas87**], associated with a pre-(co)cyclic complex. However, the problem here is that what we have is a pre-cyclic object in $D(\mathbb{K}-\mathrm{Mod})$. Even though we take injective resolutions for every $F_n(U,\mathbb{K})_T$, the relations they satisfied are only true up to chain homotopy. So, in general, we can not easily obtain a pre-(co)cyclic complex in this way. It is probably possible to track all homotopies to define a ∞-version of pre-(co)cyclic complex, then it is possible to define a ∞-version of mixed complex. But this is beyond the scope of the discussion in the thesis and we will use a different approach based on classical tricks.

In this way, we are going to construct a strict pre-(co)cyclic complex using resolutions. Here, another difficulty is that if we consider $P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$, then we need to take an injective resolution $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to I$. But on the other hand, we need to use the fact that $\mathbb{K}_{\Delta_{X^2}} \otimes F \cong F_{\Delta_{X^2}}$ as chain complexes for a chain complex of sheaves F. This is not true for general I. Instead, we can take an injective resolution $Q_U \to Q$ and then take a chain morphism $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U \xrightarrow{qis} Q$. In this way, the representative of $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$ does not change. Moreover, to achieve that $Q^{\overset{L}{\boxtimes}\ell}$ is c-soft, we need to assume the resolution Q is flat. This is not true in general. So, it is better to assume $\mathbb{K}$ is a field. Then we can run the 6-operations without deriving many things. Consequently, we can construct a pre-cocyclic complex using only Q_U. Then we can define a mixed complex $F^{S^1,+}(U,\mathbb{K})_T$, and we will define $F^{S^1}(U,\mathbb{K})_T$ as the cocone of $\mathrm{R}\Gamma_c^{S^1}(X,\mathbb{K}) \to F^{S^1,+}(U,\mathbb{K})_T$.

First, let us recall some morphisms and define something new. Recall that we have

$$\pi_n = \pi_{\underline{\mathbf{q}}} : X^n \times \mathbb{R} \to \mathbb{R},$$
$$\tilde{\Delta}_n = \tilde{\Delta}_X : X^n \times \mathbb{R} \to (X^2)^n \times \mathbb{R},$$
$$\tilde{\Delta}_n(\mathbf{q}_1, \dots, \mathbf{q}_n, t) = (\mathbf{q}_n, \mathbf{q}_1, \mathbf{q}_1, \dots, \mathbf{q}_{n-1}, \mathbf{q}_{n-1}, \mathbf{q}_n, t),$$

here we will only emphasise the index n since X is fixed. Points in the i^{th}-copy of X^2 are called $(\mathbf{q}_i^1, \mathbf{q}_i^2)$, then the definition of $\tilde{\Delta}_n$ means requiring $\mathbf{q}_i^2 = \mathbf{q}_{i+1}^1 = \mathbf{q}_i$.

Besides, we define the partial diagonals $\delta_i : X^n \to X^{n+1}$,

$$\delta_i(\mathbf{q}_1, \dots, \mathbf{q}_n, t) = (\mathbf{q}_1, \dots, \mathbf{q}_{i+1}, \mathbf{q}_{i+1}, \dots, \mathbf{q}_n, t), i = 0, 1, \dots, n-1,$$
$$\delta_n(\mathbf{q}_1, \dots, \mathbf{q}_n, t) = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{q}_1, t),$$

and the cyclic permutation

$$\tau_n : X^{n+1} \times \mathbb{R} \to X^{n+1} \times \mathbb{R},\ (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{q}_{n+1}, t) \mapsto (\mathbf{q}_2, \mathbf{q}_3, \dots, \mathbf{q}_{n+1}, \mathbf{q}_1, t).$$

Then one has $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$, $\tau_n \delta_i = \delta_{i-1} \tau_{n-1}$ for $i \in [n]$ and $\tau_n \delta_0 = \delta_n$. So $(X^{n+1}, \delta_i, \tau_n)$ form a pre-cocyclic space.

Now, suppose U is admissible. Then we have a morphism $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U$ in $D(X^2 \times \mathbb{R})$. We can equip an injective model structure on the category of chain complex $C(X^2 \times$

$\mathbb{R}$). Fibrant objects of the model structure are K-injective and injective chain complexes of sheaves over $X^2 \times \mathbb{R}$. Then we can take a K-injective and injective resolution for Q_U, say $Q_U \to Q$. In particular, Q is a complex of c-soft sheaves over $X^2 \times \mathbb{R}$. We also take a flat and c-soft resolution of $\mathbb{K}_X$, say $\mathbb{K}_X \to I$ (for example the Godement resolution), where $I \in C^{\geq 0}(X)$. Then for $\delta = \delta_{X^2}$, we have $\mathbb{K}_{\Delta_{X^2} \times [0,\infty)} = \delta_! \mathbb{K}_X \boxtimes \mathbb{K}_{[0,\infty)} \to \delta_! I \boxtimes \mathbb{K}_{[0,\infty)}$ is a flat and c-soft (relative to X^2) resolution of $\mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$.

Since $Q_U \to Q$ is a K-injective resolution, we have the following isomorphisms of functors,

$$\operatorname{Hom}_D(-, Q_U) \cong \operatorname{Hom}_D(-, Q) \cong \operatorname{Hom}_K(-, Q), \tag{3.23}$$

where $D = D(X^2 \times \mathbb{R})$ and $K = K(X^2 \times \mathbb{R})$.

Now, we apply the isomorphisms (3.23) to the diagram

$$\mathbb{K}_{\Delta_{X^2} \times \{0\}} \leftarrow \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \xrightarrow{qis} \delta_! I \boxtimes \mathbb{K}_{[0,\infty)}.$$

The resulting diagram is a commutative diagram of isomorphisms since $Q \cong Q_U \in \mathcal{D}(X^2)$.

Consequently, we have

$$\operatorname{Hom}_D(\mathbb{K}_{\Delta_{X^2} \times [0,\infty)}, Q_U) \cong \operatorname{Hom}_K(\mathbb{K}_{\Delta_{X^2} \times \{0\}}, Q) \cong \operatorname{Hom}_K(\delta_! I \boxtimes \mathbb{K}_{[0,\infty)}, Q). \tag{3.24}$$

We may take two chain maps,

$$\mathbb{K}_{\Delta_{X^2} \times \{0\}} \to Q \leftarrow \delta_! I \boxtimes \mathbb{K}_{[0,\infty)},$$

representing $\mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to Q_U$ using the isomorphisms (3.24). In particular, we have the following commutative diagram of chain maps:

$$\begin{tikzcd} \mathbb{K}_{\Delta_{X^2} \times \{0\}} \arrow[r] & Q \\ \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \arrow[u] \arrow[r] \arrow[ur] & \delta_! I \boxtimes \mathbb{K}_{[0,\infty)} \arrow[u] \end{tikzcd} \tag{3.25}$$

Lemma 3.19. *Recall that $\mathbb{K}$ is a field. With above notation, we have an isomorphism of complexes of sheaves over $X^{n+1} \times \mathbb{R}$:*

$$\delta_{i!}\tilde{\Delta}_n^{-1} s_{t!}^n (Q^{\boxtimes n}) \cong \tilde{\Delta}_{n+1}^{-1}(s_{t!}^{n+1}([i]Q^{\boxtimes n+1})),$$

where $[i]Q^{\boxtimes n+1} = Q^{\boxtimes i+1} \boxtimes \mathbb{K}_{\Delta_{X^2} \times \{0\}} \boxtimes Q^{\boxtimes n-i-1}$ for $i = 0, 1, \ldots, n-1$ and $[n]Q^{\boxtimes n+1} = \mathbb{K}_{\Delta_{X^2} \times \{0\}} \boxtimes Q^{\boxtimes n}$. Moreover, the isomorphism represents the isomorphism

$$\mathrm{R}\delta_{i!}\tilde{\Delta}_n^{-1}\mathrm{R}s_{t!}^n(Q_U^{\boxtimes n}) \cong \tilde{\Delta}_{n+1}^{-1}(\mathrm{R}s_{t!}^{n+1}([i]Q_U^{\boxtimes n+1})),$$

in the derived category.

Proof. At the beginning, we assume $i = n$. Recall that we are working over a field $\mathbb{K}$, so we do not need to derive the tensor product.

First of all, Q is c-soft because Q is injective. By exercise II.14 of [**Bre97**], we have that $Q^{\boxtimes n}$ is c-soft. Precisely, the Künneth formula shows that if $U, V \subset X^2 \times \mathbb{R}$ are open, then $(Q^{\boxtimes 2})_{U \times V}$ is Γ_c-acyclic. Then a Mayer-Vietoris argument and the set of product open sets $\{U \times V\}$ is a base of the product topology of $(X^2 \times \mathbb{R})^2$ together show that $(Q^{\boxtimes 2})_W$ is Γ_c-acyclic for all open sets $W \subset (X^2 \times \mathbb{R})^2$. Then $Q^{\boxtimes 2}$ is c-soft. So, $Q^{\boxtimes n}$ is c-soft for both projection and summation.

Take $t' = t_1$, $t'' = t_2 + \cdots + t_n + t_{n+1}$, and $t = t' + t''$. Then we have a decomposition of $s_t^{n+1} = s_t^2 \circ (\mathrm{Id}_{t_0} \times s_t^n)$. In this way, we first have

$$s_{t!}^{n+1}(\mathbb{K}_{\Delta_{X^2} \times \{0\}} \boxtimes Q_U^{\boxtimes n}) \cong s_{t!}^2(\mathbb{K}_{\Delta_{X^2} \times \{0\}} \boxtimes s_{t!}^n Q_U^{\boxtimes n}) \cong \mathbb{K}_{\Delta_{X^2}} \boxtimes s_{t!}^n Q_U^{\boxtimes n}.$$

Here, since $\mathbb{K}$-vector spaces are flat, we can use the non-derived projection formula. Moreover, the non-derived proper base change is an isomorphism of functors. Then we can apply the Künneth isomorphism, which is a combination of the projection formula and the proper base change, to obtain the isomorphism of complexes. Moreover, $Q_U^{\boxtimes n}$ is s_t^n-c-soft, and $\mathbb{K}_{\Delta_{X^2} \times \{0\}} \boxtimes Q_U^{\boxtimes n}$ is $\mathrm{Id}_{t_1} \times s_t^n$-c-soft. The first isomorphism descents to the derived category. The last isomorphism also descents to the derived category since $\mathbb{K}_{\Delta_{X^2} \times \{0\}} \boxtimes s_{t!}^n Q_U^{\boxtimes n}$ is $s_{t!}^2$ c-soft. Now, we have

$$\tilde{\Delta}_{n+1}^{-1}(s_{t!}^{n+1}([i]Q_U^{\boxtimes n+1})) \cong \tilde{\Delta}_{n+1}^{-1}(\mathbb{K}_{\Delta_{X^2}} \boxtimes s_{t!}^n Q_U^{\boxtimes n}) \cong ((d')^{-1}\mathbb{K}_{\Delta_{X^2}}) \otimes (d^{-1} s_{t!}^n Q_U^{\boxtimes n}),$$

where $d(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{n+1}, t) = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_2, \ldots, \mathbf{q}_n, \mathbf{q}_n, \mathbf{q}_{n+1}, t)$ and $d'(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{n+1}, t) = (\mathbf{q}_{n+1}, \mathbf{q}_1)$. The second isomorphism comes from the commutation of inverse image and usual tensor.

Notice that the isomorphism $(d')^{-1}\mathbb{K}_{\Delta_{X^2}} = \delta_{n!}\mathbb{K}_{X^n \times \mathbb{R}}$ is an isomorphism in degree 0. Then we have

$$\delta_{n!}\mathbb{K}_{X^n \times \mathbb{R}} \otimes (d^{-1} s^n_{t!} Q_U^{\boxtimes n}) \cong \delta_{n!}((d\delta_n)^{-1} s^n_{t!} Q_U^{\boxtimes n}) \cong \delta_{n!}(\tilde{\Delta}_n^{-1} s^n_{t!} Q_U^{\boxtimes n}).$$

Here, we use the projection formula for δ_n, and this is an isomorphism of complexes.

For different i, we can apply τ_n several times times and use the relations $\tau_n \delta_i = \tau_{n-1}\delta_{i-1}$ for $i \in [n]$ and $\tau_n \delta_0 = \delta_{n-1}$. Then we conclude using the isomorphism $\tau_n^{-1} \tilde{\Delta}_n^{-1} s^n_{t!} Q_U^{\boxtimes n} \cong \tilde{\Delta}_n^{-1} s^n_{t!} Q_U^{\boxtimes n}$. □

With the notation of the lemma, we see that both sides of

$$\delta_{i!}\tilde{\Delta}_n^{-1} s^n_{t!}(Q^{\boxtimes n}) \cong \tilde{\Delta}_{n+1}^{-1}(s^{n+1}_{t!}([i] Q^{\boxtimes n+1})),$$

are π_{n+1}-c-soft complexes. Then one concludes that

$$\pi_{(n+1)!}(\tilde{\Delta}_{n+1}^{-1}(s^{n+1}_{t!}([i] Q^{\boxtimes n+1}))) \cong \pi_{n!}(\tilde{\Delta}_n^{-1} s^n_{t!}(Q^{\boxtimes n})),$$

and they represent $F_n^+(U, \mathbb{K})$ in the derived category by [**KS06**, Proposition 14.3.4] since both $\pi_{(n+1)!}$ and $\pi_{n!}$ have finite cohomological dimension. To simplify the notation, we set $F^+_{n,inj}(U, \mathbb{K}) = \pi_{n!}(\tilde{\Delta}_n^{-1} s^n_{t!}(Q^{\boxtimes n}))$.

Now, let us define the following morphisms

$$\begin{aligned} d_i^+ : & F^+_{n,inj}(U, \mathbb{K}) \cong \pi_{(n+1)!}(\tilde{\Delta}_{n+1}^{-1}(s^{n+1}_{t!}([i] Q^{\boxtimes n+1}))) \\ & \to \pi_{(n+1)!}(\tilde{\Delta}_{n+1}^{-1}(s^{n+1}_{t!}(Q^{\boxtimes n+1}))) = F^+_{n+1,inj}(U, \mathbb{K}), \end{aligned}$$

and

$$t_n^+ = (-1)^n \pi_{(n+1)!}(\tilde{\Delta}_{n+1}^{-1}(\tau_n^{-1})) : F^+_{n+1,inj}(U, \mathbb{K}) \to F^+_{n+1,inj}(U, \mathbb{K}).$$

They represent

$$d_i^+ : F_n^+(U, \mathbb{K}) \to F_{n+1}^+(U, \mathbb{K}),$$

which is induced by $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U$, and

$$t_n^+ = (-1)^n F_n(\tau_n^{-1}) : F_n^+(U,\mathbb{K}) \to F_n^+(U,\mathbb{K}),$$

in the derived category $D(\mathbb{R})$.

We can verify that d_i^+ and t_n^+ satisfy the following relations on chain level:

$$\begin{aligned} d_j^+ d_i^+ &= d_i^+ d_{j-1}^+, \quad i<j,\\ t_n^+ d_i^+ &= -d_{i-1}^+ t_{n-1}^+, \quad i\in[n],\\ t_n^+ d_0^+ &= (-1)^n d_n^+. \end{aligned}$$

Then we conclude

Proposition 3.20. *If $\mathbb{K}$ is a field, the data $(F_{n+1,inj}^+(U,\mathbb{K}), d_i^+, t_n^+)_{n\in\mathbb{N}_0})$ form a pre-cocyclic complex (with sign) of c-soft sheaves over $\mathbb{R}$.*

Taking stalks over $T \geq 0$, we obtain the a pre-cocyclic complex of $\mathbb{K}$-vector spaces $(F_{n+1,inj}^+(U,\mathbb{K})_T, d_i^+, t_n^+)_{n\in\mathbb{N}_0})$.

The construction shows that, if we take another resolution Q' of Q_U, the resulting pre-cocyclic complexes $(F_{n+1,inj}^+(U,\mathbb{K})_T, d_i^+, t_n^+)_{n\in\mathbb{N}_0})$ are homotopy equivalent with each other.

On the other hand, let us define a pre-cocyclic $\mathbb{K}$-module $(F_{n+1,inj}(T^*X,\mathbb{K})_T, d_i, t_n)_{n\in\mathbb{N}_0})$ as follows: We first set $F_{n+1,inj}(T^*X,\mathbb{K})_T = \left(\pi_{(n+1)!}(\tilde{\Delta}_{n+1}^{-1}(s_{t!}^{n+1}(\delta_! I \boxtimes \mathbb{K}_{[0,\infty)})^{\boxtimes n+1}))\right)_T$, which is isomorphic to $\left(\pi_{(n+1)!}(\tilde{\Delta}_{n+1}^{-1}((\delta_! I)^{\boxtimes n+1} \boxtimes \mathbb{K}_{[0,\infty)}))\right)_T$ as chain complexes. Now, a difference is that d_i is induced by $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to \delta_! I \boxtimes \mathbb{K}_{[0,\infty)}$. In this case, it is by construction that $(\delta_! I \boxtimes \mathbb{K}_{[0,\infty)})^{\boxtimes n+1}$ is $s_{t!}^{n+1}$-acyclic and π_{n+1}-c-soft. So, we would like to use $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$ rather than $\mathbb{K}_{\Delta_{X^2}\times\{0\}}$ to keep the correct direction of arrows. The definition of t_n is the same. So, we have that $F_{n+1,inj}(T^*X,\mathbb{K})_T$ represent $F_{n+1}(T^*X,\mathbb{K})_T$ in $D(\mathbb{R})$. Notice that I is flat for all ring $\mathbb{K}$ regardless requiring $\mathbb{K}$ to be a field. For compatibility with the definition of $F_{n,inj}^+$, we keep assuming $\mathbb{K}$ to be a field here.

The morphism $\delta_! I \boxtimes \mathbb{K}_{[0,\infty)} \to Q$ together with the diagram (3.25) induce a morphism of pre-cocyclic complexes:

$$(F_{n+1,inj}(T^*X,\mathbb{K})_T, d_i, t_n) \to (F^+_{n+1,inj}(U,\mathbb{K})_T, d_i^+, t_n^+),$$

that represents $F_{n+1}(T^*X,\mathbb{K})_T \to F^+_{n+1}(U,\mathbb{K})_T$ in $D(\mathbb{K}-\mathrm{Mod})$.

We will use them to construct two mixed complexes and a morphism of mixed complexes. To begin with, let us review the basic notions and constructions about mixed complexes.

3.3.2. Mixed complexes. Consider the dg-algebra $\mathbb{K}[\epsilon]$ where $|\epsilon| = -1$ and $\epsilon^2 = 0$. A dg-module over $\mathbb{K}[\epsilon]$ is a triple (M, b, B) where (M, b) is a cochain complex, $|b| = -|B| = 1$ and $b^2 = B^2 = Bb + bB = 0$, which is also called a mixed complex [**Kas87**]. The derived category of dg-module of $\mathbb{K}[\epsilon]$ is called the mixed derived category. For our later applications, we will use the dg derived category for dg algebras.

On the other hand, consider the dg-algebra $\mathbb{K}[u]$ where $|u| = 2$ and $d = 0$. Then we have the Koszul duality (see [**GKM98**]):

$$D_{dg}(\mathbb{K}[\epsilon]-\mathrm{Mod}) \cong D_{dg}(\mathbb{K}[u]-\mathrm{Mod}), \quad (M,b,B) \mapsto M^{hS^1} = (M[\![u]\!], \delta = b + uB),$$

where $M[\![u]\!]$ is the (u)-adic completion of $M \otimes \mathbb{K}[u]$. Using the following free resolution of $\mathbb{K}$,

$$\cdots \xrightarrow{\epsilon} \mathbb{K}[\epsilon] \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} \mathbb{K}[\epsilon] \xrightarrow{\epsilon} \mathbb{K}[\epsilon] \xrightarrow{\epsilon=0} \mathbb{K},$$

we have that $\mathrm{RHom}_{\mathbb{K}[\epsilon]}(\mathbb{K}, M) \cong M^{hS^1} = (M[\![u]\!], \delta = b + uB)$ is the homotopy fixed point of the mixed complex (M, b, B).

We can also define the homotopy orbit complex and the Tate complex, say

$$M_{hS^1} = M \overset{L}{\otimes}_{\mathbb{K}[\epsilon]} \mathbb{K} = (M((u))/uM[\![u]\!], \delta = b + uB),$$
$$M^{Tate} = M^{hS^1} \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}] = (M((u)), \delta = b + uB).$$

These three $\mathbb{K}[u]$ modules are related by the following distinguished triangle:

$$M^{hS^1} \to M^{Tate} \to M_{hS^1}[2] \xrightarrow{+1},$$

due to the following distingushed triangle:

$$\mathbb{K}[\![u]\!] \to \mathbb{K}((u)) \to \mathbb{K}((u))/\mathbb{K}[\![u]\!] \cong u^{-1}\mathbb{K}((u))/\mathbb{K}[\![u]\!] \xrightarrow{+1} .$$

Remark 3.21. In the literature, A_∞-modules of $\mathbb{K}[\epsilon]$ are called S^1-complexes or multi-complexes or ∞-mixed complexes, see [**BO16, DSV15, Gan19, Zha19**] and references therein, the mixed complex is call the strict S^1-complex. On algebra of the module theory over $\mathbb{K}[\epsilon]$, we will mainly follow the narrative of [**Gan19**] even though the author focus on A_∞-modules.

We also have the equivalence ([**BL94**])

$$D_{dg}(\mathbb{K}[u] - \mathrm{Mod}) \cong D_{S^1,dg}(\mathrm{pt}),$$

which treat the S^1-equivariant derived over point.

Now, I would like to explain how to equip a mixed complex from a pre-cocyclic chain complex, i.e. pre-cocyclic object valued in the abelian category $C(\mathbb{K} - \mathrm{Mod})$ of chain complex of $\mathbb{K}$-module. The idea is first found by Tsygan in [**Tsy83**] and independently by Loday-Quillen in [**LQ84**].

Now, take a pre-cocyclic complex $C = (C_p, d_i, t_p, \partial_p)$, where d_i are face maps, t_p are cyclic permutations, and ∂_p is the differential of the complex C_p. First, we define

$$d = \sum_{i=0}^{n} (-1)^i d_i, \quad d' = \sum_{i=0}^{n-1} (-1)^i d_i.$$

They satisfies $d^2 = (d')^2 = 0$, and $d\partial = \partial d$, $d'\partial = \partial d'$. So $(C_p^q, d, (-1)^{p+q+1}\partial)$ and $(C_p^q, d', (-1)^{p+q+1}\partial)$ are two bicomplexes.

Then consider the product totalization of these two bicomplexes. Specifically, we set $Tot(C)_n = \prod_{q+p=n} C_p^q$ and the differential $b = d + (-1)^{n+1}\partial$. Then the totalization is $(Tot(C), b = d + (-1)^{n+1}\partial)$. Similarly, we have $(Tot(C), b' = d' + (-1)^{n+1}\partial)$.

On the other hand, the cyclic permutation satisfies the relations $(1 - t)d = d'(1 - t)$. So we have a morphism of complexes:

$$(Tot(C), b = d + (-1)^{n+1}\partial) \to (Tot(C), b' = d' + (-1)^{n+1}\partial).$$

Now, taking the mapping cocone of this chain map, we get a complex $M(C) = (Tot(C) \oplus Tot(C)[-1], \bar{b})$. Also we define a degree -1 morphism $\overline{B} : M(C) \to M(C)$. Precisely, we have

$$\bar{b} = \begin{pmatrix} b & 0 \\ 1-t & -b' \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix},$$

where $N = 1 + t + \cdots + t^n$. It is direct to see that $\overline{B}^2 = \overline{B}\,\bar{b} + \bar{b}\,\overline{B} = 0$. So, we have that $M(C) = (M(C), \bar{b}, \overline{B})$ form a mixed complex.

If C comes from a cocyclic complex $C = (C_p, d_i, s_i, t, \partial)$ (satisfying some other relations about s_i), then we have $b's + sb' = \mathrm{Id}$ where $s = s_n$, and the following morphism is a homotopy equivalence of mixed complexes

$$\begin{pmatrix} 1 & s(1-t) \end{pmatrix}^t : (Tot(C), b, B) \to (M(C), \bar{b}, \overline{B}),$$

where $B = Ns(1-t)$. The mixed complex $(M(C), \bar{b}, \overline{B})$ is functorial with respect to morphisms of pre-cocyclic complexes, but $(Tot(C), b, B)$ is not. Moreover, if there is a quasi-isomorphism of pre-cocyclic complexes, then we have a quasi-morphism of the associated mixed complex.

3.3.3. S^1-equivariant Chiu-Tamarkin complex. Now, we apply the construction to the morphism of pre-cocyclic complexes

$$(F_{n+1,inj}(T^*X, \mathbb{K}), d_i, t_n) \to (F^+_{n+1,inj}(U, \mathbb{K})_T, d_i^+, t_n^+).$$

As we said, different choices of resolutions Q and I produce homotopy equivalences of associated mixed complexes. Different choices of morphism $\delta_! I \to Q$ produce a homotopy commutative diagram of mixed complexes. Consequently, we have

Proposition 3.22. *If $\mathbb{K}$ is a field. Then for an admissible open set U, the data*

$$F^{S^1,+}_\bullet(U, \mathbb{K})_T := M(F^+_{\bullet,inj}(U, \mathbb{K})_T, d_i^+, t_n^+),$$
$$F^{S^1}_\bullet(T^*X, \mathbb{K})_T := M(F_{\bullet,inj}(T^*X, \mathbb{K})_T, d_i, t_n),$$

define objects of $D_{dg}(\mathbb{K}[\epsilon] - \mathrm{Mod})$. The degree 0 closed morphism

$$F^{S^1}_\bullet(T^*X, \mathbb{K})_T \to F^{S^1,+}_\bullet(U, \mathbb{K})_T$$

defines an object in $\mathcal{M}or(D_{dg}(\mathbb{K}[\epsilon]-\mathrm{Mod}))$.

Presently, even though we can not define $F^{S^1}(U,\mathbb{K})$ for general U, the situation of $U=T^*X$ is quite simple as we presented. On the other hand, the defining triangle of kernel (P_U,Q_U), and consequently the tautological triangle ((3.20)), indicate that we should expect $F^{S^1}(U,\mathbb{K})$ to be the cocone of the morphism $F_\bullet^{S^1}(T^*X,\mathbb{K})_T\to F_\bullet^{S^1,+}(U,\mathbb{K})_T$. Therefore, we *define*

$$F_\bullet^{S^1}(U,\mathbb{K})_T\coloneqq cone(F_\bullet^{S^1}(T^*X,\mathbb{K})_T[-1]\to F_\bullet^{S^1,+}(U,\mathbb{K})_T[-1]).$$

Using the equivalence $D_{dg}(\mathbb{K}[\epsilon]-\mathrm{Mod})\cong D_{dg}(\mathbb{K}[u]-\mathrm{Mod})\cong D_{S^1,dg}(\mathrm{pt})$, we also *define* objects $F^{S^1,+}(U,\mathbb{K})_T$, $F^{S^1}(U,\mathbb{K})_T$ of $D_{S^1,dg}(\mathrm{pt})$ and we can also identify $F^{S^1}(T^*X,\mathbb{K})_T$ with $\mathrm{R}\Gamma_c(X,\mathbb{K}^{S^1})$ in $D_{S^1,dg}(\mathrm{pt})$, where we treat X as a trivial S^1-spaces.

Definition 3.23. If $\mathbb{K}$ is a field. For an admissible open set U, and $T\geq 0$, we define the S^1-equivariant Chiu-Tamarkin complex to be

$$\begin{aligned}C_T^{S^1,+}(U,\mathbb{K})\coloneqq&\mathrm{RHom}_{\mathbb{K}[\epsilon]}(F_\bullet^{S^1,+}(U,\mathbb{K})_T,\mathbb{K}[1-d])\cong\mathrm{RHom}_{S^1}(F^{S^1,+}(U,\mathbb{K})_T,\mathbb{K}[1-d]),\\ C_T^{S^1}(U,\mathbb{K})\coloneqq&\mathrm{RHom}_{\mathbb{K}[\epsilon]}(F_\bullet^{S^1}(U,\mathbb{K})_T,\mathbb{K}[-d])\cong\mathrm{RHom}_{S^1}(F^{S^1}(U,\mathbb{K})_T,\mathbb{K}[-d]),\end{aligned}$$

where we equip $\mathbb{K}$ with the trivial mixed complex structure, the $\mathrm{RHom}_{\mathbb{K}[\epsilon]}$ stands for the derived hom in the dg mixed derived category.

Remark 3.24. (1) In dg derived category $D_{dg}(\mathbb{K}[\epsilon]-\mathrm{Mod})$, the definition of $C_T^{S^1}(U,\mathbb{K})$ is canonical. If we work in the usual derived category, $C_T^{S^1}(U,\mathbb{K})$ is not canonical but its cohomology, what we really care, is well defined.

(2) In practice, we can find a good representation of $C_T^{S^1}(U,\mathbb{K})$ using topological resolution of $F_n(U,\mathbb{K})_T\cong\Gamma_c(W_n,\mathbb{K})$. In this case, we can take a resolution of $\mathbb{K}_{W_n}$ directly rather than take a resolution of Q_U, and then apply further manipulations like Lemma 3.19. In particular, we can define $C_T^{S^1}(U,\mathbb{K})$ over any coefficient ring $\mathbb{K}$ for this particular case. So far, all examples fit into this remark. But the author does not know how this idea works in general. See more in Remark 3.25.

Consequently, we have an exact sequence of mixed complexes:

$$0 \to F^{S^1,+}(U,\mathbb{K})_T[-1] \to F^{S^1}(U,\mathbb{K})_T \to \mathrm{R}\Gamma_c^{S^1}(X,\mathbb{K}) \to 0,$$

and the *tautological exact triangle* of S^1-equivariant Chiu-Tamarkin complex:

$$\mathrm{R}\Gamma(X,\omega_X^{S^1}) \to C_T^{S^1}(U,\mathbb{K}) \to C_T^{S^1,+}(U,\mathbb{K}) \xrightarrow{+1} . \tag{3.26}$$

The persistence module structure still exists here. But a little difference is we first use the persistence structure on $C_T^{S^1,+}(U,\mathbb{K})$, then we derive the persistence structure on $C_T^{S^1}(U,\mathbb{K})$. This is possible because the cone is functorial in the dg derived category. Then we can also define the $T=\infty$ version and action window version. By definition, we have $H^*C_T^{S^1}(U,\mathbb{K})$ and $H^*C_T^{S^1,+}(U,\mathbb{K})$ are left modules over $\mathrm{Ext}^*_{S^1}(\mathbb{K},\mathbb{K}) \cong \mathbb{K}[u]$. As we computed before, one has $H^*C_T^{S^1}(T^*X,\mathbb{K}) \cong H^{BM}_{d-*}(X,\mathbb{K}) \otimes \mathbb{K}[u]$ for all $T \in [0,\infty]$, with trivial $\mathbb{K}[u]$ action on $H^{BM}_{d-*}(X,\mathbb{K})$.

The functoriality and invariance are still true. Our proof of functoriality and invariance of F_n^+ work for all n. Then the discussion on well-definiteness also demonstrate the corresponding properties for $C_T^{S^1,+}(U,\mathbb{K})$ and $C_T^{S^1}(U,\mathbb{K})$.

The S^1-equivariant fundamental class $\eta_T^{S^1}(U,\mathbb{K}) \in H^0C^{S^1}(U,\mathbb{K})$ is defined as the image of $[X]^{S^1} \in H_d^{BM,S^1}(X,\mathbb{K})$ under the morphism

$$H_d^{BM,S^1}(X,\mathbb{K}) \cong H^0\mathrm{R}\Gamma(X,\omega_X^{S^1}) \to H^0C_T^{S^1}(U,\mathbb{K}),$$

which is induced by taking H^0 for the tautological distinguished triangle (3.26).

An important feature of cyclic cohomology is the Connes' long exact sequence, or the Gysin long exact sequence. Precisely, for trivial mixed complex $\mathbb{K}$, we have $\mathrm{Ext}^*_{\mathbb{K}[\epsilon]}(\mathbb{K},\mathbb{K}) \cong \mathbb{K}[u]$. Then there is a morphism $u : \mathbb{K} \to \mathbb{K}[2]$. One can embed it into the distinguished triangle

$$\mathbb{K}[\epsilon] \xrightarrow{\epsilon=0} \mathbb{K} \xrightarrow{u} \mathbb{K}[2] \xrightarrow{+1} .$$

For any mixed complex $M=(M,b,B)$, taking $\mathrm{RHom}_{\mathbb{K}[\epsilon]}(-,M)$, we have the distinguished triangle

$$M[\![u]\!]/uM[\![u]\!] \to M[\![u]\!] \xrightarrow{u} M[\![u]\!] \xrightarrow{+1},$$

of $\mathbb{K}[u]$-modules. Then it induces long exact sequence of $\mathbb{K}$-vector spaces:

$$\mathrm{Ext}^{p-2}_{\mathbb{K}[u]}(M[\![u]\!],\mathbb{K}) \xrightarrow{\cdot u} \mathrm{Ext}^{p}_{\mathbb{K}[u]}(M[\![u]\!],\mathbb{K}) \to \mathrm{Ext}^{p}_{\mathbb{K}[u]}(M[\![u]\!]/uM[\![u]\!],\mathbb{K}) \xrightarrow{+1} .$$

Using the equivalence $D_{dg}(\mathbb{K}[\epsilon]-\mathrm{Mod}) \cong D_{dg}(\mathbb{K}[u]-\mathrm{Mod})$, we have the following long exact sequence

$$\mathrm{Ext}^{p-2}_{\mathbb{K}[\epsilon]}(M,\mathbb{K}) \xrightarrow{\cdot u} \mathrm{Ext}^{p}_{\mathbb{K}[\epsilon]}(M,\mathbb{K}) \to \mathrm{Ext}^{p}_{\mathbb{K}}(M,\mathbb{K}) \xrightarrow{+1} .$$

For the third term, it is due to

$$\mathrm{Ext}^{p}_{\mathbb{K}[u]}(M[\![u]\!]/uM[\![u]\!],\mathbb{K}) \cong \mathrm{Ext}^{p}_{\mathbb{K}}(M,\mathbb{K}),$$

where we treat $M=(M,b)$ as a usual complex on the right hand side.

Now, we can take $M=F^{S^1,+}(U,\mathbb{K})$ and $M=F^{S^1}(U,\mathbb{K})$ to obtain the following long distinguished triangles:

$$H^{p-2}C^{S^1,+}_T(U,\mathbb{K}) \xrightarrow{\cdot u} H^pC^{S^1,+}_T(U,\mathbb{K}) \to \mathrm{Ext}^p_{\mathbb{K}}(F^{S^1,+}_\bullet(U,\mathbb{K})_T,\mathbb{K}) \xrightarrow{+1},$$
$$H^{p-2}C^{S^1}_T(U,\mathbb{K}) \xrightarrow{\cdot u} H^pC^{S^1}_T(U,\mathbb{K}) \to \mathrm{Ext}^p_{\mathbb{K}}(F^{S^1}_\bullet(U,\mathbb{K})_T,\mathbb{K}) \xrightarrow{+1} .$$

So, let us compute $\mathrm{Ext}^p_{\mathbb{K}}(F^{S^1,+}(U,\mathbb{K})_T,\mathbb{K})$ and $\mathrm{Ext}^p_{\mathbb{K}}(F^{S^1}(U,\mathbb{K})_T,\mathbb{K})$.

For $F^{S^1,+}_\bullet(U,\mathbb{K})_T$, the pre-cosimplicial complex structure is $(F^+_{n+1}(U,\mathbb{K})_T, d^+_i)$. Now, consider the cutoff pre-cosimplicial complex $G=(G_{n+1}(U,\mathbb{K})_T,0)$ such that $G_1(U,\mathbb{K})_T = F_1(U,\mathbb{K})_T$, $G_{n+1}(U,\mathbb{K})_T=0$ for $n\geq 1$. We have a cosimplicial map

$$F_\bullet(U,\mathbb{K})_T \to G_\bullet(U,\mathbb{K})_T.$$

We already know that all d^+_i are isomorphisms in the derived category of sheaves $D(\mathbb{R})$, then the induced map

$$(Tot(F), b+\partial) \to (Tot(G), b+\partial),$$

is a quasi isomorphism. However $G_\bullet(U,\mathbb{K})_T=(G_{n+1}(U,\mathbb{K})_T,0)$ is concentrated in the level $n=0$, we have

$$F^{S^1,+}_\bullet(U,\mathbb{K})_T \cong (Tot(G),b) = F^+_1(U,\mathbb{K}) \text{ in } D(\mathbb{K}-Mod).$$

For $F^{S^1}(U,\mathbb{K})_T$, by forgetting B, we have a distinguished triangle in $D(\mathbb{K}-\mathrm{Mod})$:

$$F^{S^1,+}(U,\mathbb{K})_T[-1] \to F^{S^1}(U,\mathbb{K})_T \to \mathrm{R}\Gamma_c(X,\mathbb{K}) \xrightarrow{+1}.$$

But we already know, in $D(\mathbb{K}-\mathrm{Mod})$, that $F^{S^1,+}(U,\mathbb{K})_T[-1] \cong F_1^+(U,\mathbb{K})_T[-1]$. Then we can conclude that $F^{S^1}(U,\mathbb{K})_T \cong F_1(U,\mathbb{K})_T$ in $D(\mathbb{K}-\mathrm{Mod})$ using the tautological triangle (3.19) of F_1.

So, we obtain the Connes' long exact sequence or the Gysin long exact sequence for S^1-equivariant Chiu-Tamarkin cohomology:

$$\begin{aligned} &H^{p-2}C_T^{S^1,+}(U,\mathbb{K}) \xrightarrow{\cdot u} H^p C_T^{S^1,+}(U,\mathbb{K}) \to H^p C_{1,T}^+(U,\mathbb{K}) \xrightarrow{+1}, \\ &H^{p-2}C_T^{S^1}(U,\mathbb{K}) \xrightarrow{\cdot u} H^p C_T^{S^1}(U,\mathbb{K}) \to H^p C_{1,T}(U,\mathbb{K}) \xrightarrow{+1}. \end{aligned} \tag{3.27}$$

It is direct to see that under the morphism $H^pC_T^{S^1}(U,\mathbb{K}) \to H^pC_{1,T}(U,\mathbb{K})$, the fundamental class $\eta_T^{S^1}(U,\mathbb{K})$ is mapped to $\eta_{1,T}(U,\mathbb{K})$.

For the subgroup $\mathbb{Z}/\ell \subset S^1$ with $\ell \geq 2$, we have the restriction functor:

$$Res_\ell : D_{S^1}(\mathrm{pt}) \to D_{\mathbb{Z}/\ell}(\mathrm{pt}).$$

Question: Do we have an isomorphism below?

$$Res_\ell(F^{S^1}(U,\mathbb{K})_T) \cong F_\ell(U,\mathbb{K})_T.$$

It would induce a morphism of Chiu-Tamarkin complex:

$$H^qC_T^{S^1}(U,\mathbb{K}) \to H^qC_{\ell,T}(U,\mathbb{K}).$$

It is a module homomorphism, with respect to the algebras $\mathrm{Ext}^*_{S^1}(\mathbb{K},\mathbb{K}) = \mathbb{K}[u]$ and $\mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})$. Under the morphism, $\eta_T^{S^1}(U,\mathbb{K})$ should be mapped to $\eta_{\ell,T}(U,\mathbb{K})$.

Remark 3.25. In practical computation in chapter 4, all S^1-structures have already been observed and realized by some homotopy types equipped with a S^1-action. In this case, we can define $F^{S^1}(U,\mathbb{K})_T$ directly, and it is quasi isomorphic to the cone definition of $F^{S^1}(U,\mathbb{K})_T$.

Specifically, let us remark on a typical situation we will face. For fixed T, we will find that

$$F_n(U,\mathbb{K})_T \cong \mathrm{R}\Gamma_c(W_n,\mathbb{K}),$$

where $W_n \subset Y^n$ is a sub CW-complex of a product manifold for a smooth manifold Y. Also, the face maps d_i are realized by diagonal maps Δ_i at the i^{th} position of Y^n, the cyclic permutation is realized by cyclic permutation of factors of Y^n. In this way, $(W_\bullet)$ will form a pre-cocyclic space.

Moreover, d_i are homotopy equivalence in practice, then we can reduce the computation of $H^*C_T^{S^1}(U,\mathbb{K})$ to a computation of cyclic cohomology of a pre-cyclic complex that comes from the cohomology of the pre-cocyclic space $(W_\bullet)$.

This pre-cyclic complex will compute (co)homology of the fat geometric-realization of the pre-cocyclic space $(W_\bullet)$. For a pre-cocyclic space, the method of Jones shows that the fat geometric realization $|W_\bullet|$ of $(W_\bullet)$ (as a simplicial space) is a S^1-space([**Jon87**]).

Another application for d_i being homotopy equivalence is that, under mild conditions (true for all known examples), we have

$$|W_\bullet| \cong |W_\ell| \cong |W_1|,$$

as S^1-spaces. Then our computation for $H^qC_T^{S^1}(U,\mathbb{K})$ will be reduced to compute the S^1-equivariant cohomology of $|W^1| \cong W^1$. Moreover, we observe that there exists a natural S^1-action on $|W_1| \cong W_1$ and the natural S^1-action is the same as the S^1-action on $|W_\bullet| \cong |W_1| \cong W_1$ induced by Jones' construction.

We also observe that the $\mathbb{Z}/\ell$-action, induced by cyclic permutation, on W_ℓ comes from the restriction of the S^1-action under the identification $|W_\ell| \cong |W_1|$. So, in this situation, the answer to the restriction isomorphism question is positive.

3.4. Geometry of $F_\ell(U,\mathbb{K})$

In this section we assume that U is dynamically admissible and we give a more accessible expression of the Chiu-Tamarkin complex using sheaf quantization. We then discuss the underlying geometry.

Following ideas of Chiu, we first compute $F_\ell(U,\mathbb{K}) \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}_X^{-1}\left(\mathrm{R}s^\ell_{t!}P_U^{\overset{L}{\boxtimes}\ell}\right)$ going back to the construction of P_U.

We recall that $\mathcal{K}$ is the sheaf quantization of a Hamiltonian $\mathbb{R}^m_z$ action on T^*X with a moment map μ, $\Omega \subset \mathbb{R}^m_\zeta$ and $U = \mu^{-1}(\Omega)$. Then we have $P_U \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega} \cong \mathcal{K}_{\pi_t(\mathrm{supp}(\widehat{\mathbb{K}_\Omega}))} \star \widehat{\mathbb{K}_\Omega}$, where the last isomorphism is given in Remark 2.17.

As a corollary of the proper base change and the projection formula, we have the following:

$$P_U^{\overset{L}{\boxtimes}\ell} \cong (\mathcal{K} \star \widehat{\mathbb{K}_\Omega})^{\overset{L}{\boxtimes}\ell} \cong \mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^2_{\mathbb{R}^\ell !}\left(\pi_{t_2}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right).$$

Next, we have

$$\begin{aligned}F_\ell(U,\mathbb{K}) &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}_X^{-1}\mathrm{R}s^\ell_{t!}\mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^2_{\mathbb{R}^\ell !}\left(\pi_{t_2}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right)\\ &\cong \mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^\ell_{t!}\mathrm{R}s^2_{\mathbb{R}^\ell !}\left(\pi_{t_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\right)\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right),\end{aligned}$$

where, $\underline{z} = (z_1,\dots,z_\ell) \in (\mathbb{R}^m)^\ell$, $t_i = (t_i^1,\dots,t_i^\ell) \in \mathbb{R}^\ell$ for $i = 1, 2$, and $t = (t^1,\dots,t^\ell) = s^2_{\mathbb{R}^\ell}(t_1,t_2)$. Now, let $z = z_1 + \cdots + z_\ell$ and take $t'_i = t_i^1 + \cdots + t_i^\ell$. Using this change of coordinate, we have the decomposition $\pi_{\underline{z}} = \pi_z s^\ell_z$ and $s^\ell_t s^2_{\mathbb{R}^\ell} = s^2_{t'}(s^\ell_{t_1} \times s^\ell_{t_2})$. Therefore, we obtain

$$\begin{aligned}F_\ell(U,\mathbb{K}) &\cong \mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^\ell_{t!}\mathrm{R}s^2_{\mathbb{R}^\ell !}\left(\pi_{t_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\right)\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right)\\ &\cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t'!}\mathrm{R}s^\ell_{z!}\mathrm{R}(s^\ell_{t_1}\times s^\ell_{t_2})_!\left(\pi_{t_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\right)\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right)\\ &\cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t'!}\mathrm{R}s^\ell_{z!}\left(\pi_{t'_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\star\ell}\right)\overset{L}{\otimes}\pi_{t'_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\star\ell}\right).\end{aligned} \tag{3.28}$$

The formula shows, as the construction itself, that we can consider separately the Hamiltonian action and the cut-off by Ω. Let us study the Hamiltonian action first. In view of (3.28), it is convenient to define

$$
\begin{aligned}
CL_\ell(\mathcal{K}) &:= \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell})) \in D_{\mathbb{Z}/\ell}((\mathbb{R}^m_z)^\ell \times \mathbb{R}_t), \\
\mathcal{CL}_\ell(\mathcal{K}) &:= \mathrm{R}s^\ell_{z*}\, CL_\ell(\mathcal{K}) \in D_{\mathbb{Z}/\ell}(\mathbb{R}^m_z \times \mathbb{R}_t).
\end{aligned} \tag{3.29}
$$

Here, in the definition, we use the same formula as the formula for $F_{\ell,X}$ (See (3.2)). But, here, as $\pi_{\underline{\mathbf{q}}}$ is projected to $\mathbb{R}^m_z \times \mathbb{R}$, we use a different notation $\mathcal{CL}_\ell(\mathcal{K})$.

The sheaves $CL_\ell(\mathcal{K})$ and $\mathcal{CL}_\ell(\mathcal{K})$ encode the cohomology information of a discrete Hamiltonian loop space. Precisely, we have

Proposition 3.26. *With the notation* (3.29) *we have*

(1) The sectional microsupport $\mu s(CL_\ell(\mathcal{K}))$, which is a subset of $T^*(\mathbb{R}^m_z)^\ell$, is contained in

$$
\left\{ (z_j, \zeta_j) : \begin{array}{l} \text{There exist } (\boldsymbol{q}_j, \boldsymbol{p}_j) \in T^*((X^2)^\ell) \text{ such that} \\ (\boldsymbol{q}_{j+1}, \boldsymbol{p}_{j+1}) = z_j \cdot (\boldsymbol{q}_j, \boldsymbol{p}_j),\ \zeta_j = -\mu(\boldsymbol{q}_j, \boldsymbol{p}_j)\ j \in \mathbb{Z}/\ell \end{array} \right\}
$$

(2) $CL_\ell(\mathcal{K}) \cong (s^\ell_z)^{-1}\mathrm{R}s^\ell_{z*}CL_\ell(\mathcal{K})$, $\mathcal{CL}_\ell(\mathcal{K}) \cong \mathrm{R}s^\ell_{z*}(s^\ell_z)^{-1}\mathcal{CL}_\ell(\mathcal{K})$.

Proof. (1) It follows directly from the functorial estimate of microsupport. First, the formula (1.26) shows that

$$
\mu s(\mathcal{K}^{\boxtimes\ell}) \subset \{(z_j, \zeta_j, \mathbf{q}_j, -\mathbf{p}_j, \mathbf{q}'_j, \mathbf{p}'_j) : (\mathbf{q}'_j, \mathbf{p}'_j) = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j),\ j \in \mathbb{Z}/\ell\}.
$$

The transpose derivative of $\widetilde{\Delta}$ is given by

$$
d\widetilde{\Delta}^*(\mathbf{q}_\ell, \mathbf{q}_1, \ldots, \mathbf{q}_{\ell-1}, \mathbf{q}_\ell; \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2\ell-1}, \mathbf{p}_{2\ell}) = (\mathbf{q}_1, \ldots, \mathbf{q}_\ell; \mathbf{p}_2 + \mathbf{p}_3, \ldots, \mathbf{p}_{2\ell} + \mathbf{p}_1).
$$

By the bound (1.4) (3), we deduce that $\mu s(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell}))$ is a subset of

$$
\left\{ (z_j, \zeta_j, \mathbf{q}''_j, \mathbf{p}''_j) : \begin{array}{l} \text{There exist } (\mathbf{q}_j, -\mathbf{p}_j, \mathbf{q}'_j, \mathbf{p}'_j) \in T^*((X^2)^{2\ell}) \text{ such that } (\mathbf{q}'_j, \mathbf{p}'_j) = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j), \\ \mathbf{q}''_j = \mathbf{q}'_j = \mathbf{q}_{j+1},\ \mathbf{p}''_j = \mathbf{p}'_j - \mathbf{p}_{j+1}, \zeta_j = -\mu(\mathbf{q}_j, \mathbf{p}_j)\ j \in \mathbb{Z}/\ell \end{array} \right\}.
$$

Finally, let us apply the non proper estimate Theorem 1.7.

The set $\pi^{\#}_{\underline{\mathbf{q}''}}(SS(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell})))$ comes from forgetting $\mathbf{q}''_j$ for all j from $SS(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell}))$. Then $(z_j, \zeta_j, t, 1) \in \mu s(CL_\ell(\mathcal{K}))$ if there exists a sequence $(z^n_j, \zeta^n_j, \mathbf{p}''^n_j) \in \pi^{\#}_{\underline{\mathbf{q}''}}(SS(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell})))$ such that $z^n_j \to z_j$, $\zeta^n_j \to \zeta_j$, and $\mathbf{p}''^n_j \to 0$.

On the other hand, the relations above imply that there exists $(\mathbf{q}_j^n, -\mathbf{p}_j^n, {\mathbf{q}_j}'^n, {\mathbf{p}_j}'^n) \in T^*((X^2)^\ell)$ such that $({\mathbf{q}_j}'^n, {\mathbf{p}_j}'^n) = z_j^n \cdot (\mathbf{q}_j^n, \mathbf{p}_j^n)$ and $\mathbf{q}_j''^n = \mathbf{q}_j'^n = \mathbf{q}_{j+1}^n$, $\mathbf{p}_j''^n = \mathbf{p}_j'^n - \mathbf{p}_{j+1}^n, \zeta_j^n = -\mu(\mathbf{q}_j^n, \mathbf{p}_j^n)$. So the continuity of the group action and the moment map shows that, after taking limit $n \to \infty$, we have $(\mathbf{q}_j', \mathbf{p}_j') = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j)$ and $\mathbf{q}_j'' = \mathbf{q}_j' = \mathbf{q}_{j+1}$, $0 = \mathbf{p}_j' - \mathbf{p}_{j+1}, \zeta_j = -\mu(\mathbf{q}_j, \mathbf{p}_j)$. Then we have that $\mu s(CL_\ell(\mathcal{K}))$ is contained in

$$\left\{ (z_j, \zeta_j) : \begin{array}{l} \text{There exist } (\mathbf{q}_j, -\mathbf{p}_j, \mathbf{q}_j', \mathbf{p}_j') \in T^*((X^2)^\ell) \text{ such that } (\mathbf{q}_j', \mathbf{p}_j') = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j), \\ \mathbf{q}_j'' = \mathbf{q}_j' = \mathbf{q}_{j+1},\, 0 = \mathbf{p}_j'' = \mathbf{p}_j' - \mathbf{p}_{j+1}, \zeta_j = -\mu(\mathbf{q}_j, \mathbf{p}_j)\, j \in \mathbb{Z}/\ell \end{array} \right\}.$$

Finally, we simplify the notation by reducing the variables with primes.

(2) If $(z_j, \zeta_j) \in \mu s(CL_\ell(\mathcal{K}))$, there exists $(\mathbf{q}_j, \mathbf{p}_j) \in T^*((X^2)^\ell)$ such that $(\mathbf{q}_{j+1}, \mathbf{p}_{j+1}) = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j)$. Therefore, the invariance of the moment map shows that

$$\zeta_{j+1} = \mu(\mathbf{q}_{j+1}, \mathbf{p}_{j+1}) = (\mathbf{q}_j, \mathbf{p}_j) = \zeta_j, \qquad j \in \mathbb{Z}/\ell.$$

Then, the isomorphism follows from [**KS90**, Proposition 5.4.5(ii)].

$\square$

Remark 3.27. If $\mathcal{K}$ is a sheaf quantization from a non-autonomous Hamiltonian function, then the microsupport estimate for $CL_\ell(\mathcal{K})$ is still true. But the second statement is not true in this case.

Now, using the projection formula, we can write the formula (3.28) as

$$F_\ell(U, \mathbb{K}) \cong \mathrm{R}\pi_{z!}\mathrm{R}s_{t!}^2 \left(\pi_{t_2}^{-1} \mathcal{CL}_\ell(\mathcal{K}) \overset{L}{\otimes} \pi_{t_1}^{-1} \mathrm{R}s_{z!}^\ell \widehat{\mathbb{K}_\Omega}^{\circledast \ell} \right). \tag{3.30}$$

Next, let us study $\mathrm{R}s_{z!}^\ell \widehat{\mathbb{K}_\Omega}^{\circledast \ell} \cong \mathrm{R}s_{(z,t_2)!}^\ell \widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes} \ell}$. First, with the help of [**D'A13**, Section 6, Appendix A], $\widehat{\mathbb{K}_\Omega}$ is the (inverse) Fourier-Sato transform $\widehat{\mathbb{K}_{\Omega'}}$ of $\mathbb{K}_{\Omega'}$, where $\Omega' = \{(\zeta, \tau) : \tau\zeta \in \Omega, \tau > 0\}$. Now, using the functorial properties of the Fourier-Sato transformation (see [**KS90**, Section 3.7]), and writing in the same way the two Fourier transforms, we have:

$$\mathrm{R}s_{z!}^\ell \widehat{\mathbb{K}_\Omega}^{\circledast \ell} \cong \mathrm{R}s_{(z,t_2)!}^\ell \widehat{\mathbb{K}_{\Omega'}}^{\overset{L}{\boxtimes} \ell} \cong \mathrm{R}s_{(z,t_2)!}^\ell \widehat{\mathbb{K}_{\Omega'}^{\overset{L}{\boxtimes} \ell}} \cong \mathrm{R}s_{(z,t_2)!}^\ell \widehat{\mathbb{K}_{\Omega'^\ell}} \cong \widehat{({}^t s_{(z,t_2)}^\ell)^{-1} \mathbb{K}_{\Omega'^\ell}}.$$

Since the transpose of the summation map $s^{\ell}_{(z,t_2)}$ is the diagonal map $\delta_{(z,t_2)^\ell}$, we conclude that

$$\mathrm{R}s^{\ell}_{z!}\widehat{\mathbb{K}_\Omega}^{\boxast \ell} \cong \mathrm{R}s^{\ell}_{(z,t_2)!}\widehat{\mathbb{K}_{\Omega'}}^{\overset{L}{\boxtimes}\ell} \cong \widehat{\delta^{-1}_{(z,t_2)^\ell}\mathbb{K}_{\Omega'^\ell}} \cong \widehat{\mathbb{K}_{\Omega'}} \cong \widehat{\mathbb{K}_\Omega}.$$

By the Steenrod's construction in appendix C, our external tensor power is in fact an object of the $\mathbb{Z}/\ell$-equivariant derived category. We need to mention that the Fourier transform (of any version) is a convolution functor defined by a kernel, which is a constant sheaf supported on a closed subset. So, on the product space, the Fourier transform is defined by a kernel that is a constant sheaf supported on a *product* of the same closed subsets. So, the kernel is a $\mathbb{Z}/\ell$-equivariant sheaf. Moreover, the Steenrod's construction is compatible with the Grothendieck 6-operations. So, the Fourier transform can be defined on the equivariant derived category. Finally, all maps here are $\mathbb{Z}/\ell$-equivariant with respect to cyclic permutation action and the formulas we used here are valid in the equivariant category. In conclusion, all identities here are true in the equivariant derived category.

Consequently, (3.30) could be read as

$$F_\ell(U,\mathbb{K}) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathcal{CL}_\ell(\mathcal{K})\overset{L}{\otimes}\pi^{-1}_{(\underline{\mathbf{q}},t_1)}\widehat{\mathbb{K}_\Omega}\right) \cong \mathrm{R}\pi_{z!}\left(\mathcal{CL}_\ell(\mathcal{K}) \star \widehat{\mathbb{K}_\Omega}\right). \tag{3.31}$$

From this formula, the study of $F_\ell(U,\mathbb{K})$ is reduced to understanding $\mathcal{CL}_\ell(\mathcal{K})$.

The $m = 1$ case is particularly useful for our applications. Now $\Omega = (-\infty, 1)$ and $\widehat{\mathbb{K}_\Omega} \cong \mathbb{K}_{\{(z,t):-t\leq z\leq 0\}}$. For $T \geq 0$, (3.31) shows

$$\alpha_{\ell,X,T}(P_U^{\overset{L}{\boxtimes}\ell}) \cong F_\ell(U,\mathbb{K})_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}_z \times \mathbb{R}^2_{(t_1,t_2)}; \left(\mathcal{CL}_\ell(\mathcal{K})\overset{L}{\boxtimes}\mathbb{K}_{\mathbb{R}_{t_2}}\right)_Z\right), \tag{3.32}$$

where $Z = \{(z,t_1,t_2) : t_1 + t_2 = T, -t_2 \leq z \leq 0\}$.

Again, using the formula (3.31), we obtain the following action spectrum estimate of the microsupport of $F_\ell(U,\mathbb{K})$ for dynamically admissible sets.

Lemma 3.28. *Let $U = \{H < 1\}$ be a dynamically admissible set defined by a Hamiltonian function H. If the boundary ∂U is a non-degenerated hypersurface of restricted* *contact type (RCT) given by $\partial U = \{H = 1\}$, then we have*

$$\mu s_L(F_\ell(U, \mathbb{K})) \subset \left\{ t \in \mathbb{R} : t = \left| \int_c \boldsymbol{p} d\boldsymbol{q} \right| \text{ for a closed orbit } c \text{ of } \varphi_z^H \text{ in } \partial U \right\}. \tag{3.33}$$

Actually, since $F_1(U, \mathbb{K}) \cong F_\ell(U, \mathbb{K})$ in $D(\mathbb{R})$ (by Proposition 3.12), we only verify the proposition for $F_1(U, \mathbb{K})$ (see Definition 1.8). This estimate is a corollary of (2.12), it also appears in [**Zha20**, formula 74].

Geometrically, we call the right hand side *the action spectrum* of the Reeb action in ∂U.

Proof. By assumption, we have $\overline{U} = \{H \leq 1\}$. Recall the microsupport bound (2.12), we have the estimate

$$\begin{aligned} \mu s_L(P_U) \subset & \{(\mathbf{q}, -\mathbf{p}, \mathbf{q}, \mathbf{p}, 0) : (\mathbf{q}, \mathbf{p}) \in \overline{U}\} \cup \\ & \qquad \{(\mathbf{q}, -\mathbf{p}, \mathbf{q}', \mathbf{p}', -\int_c \alpha) : (\mathbf{q}', \mathbf{p}') = \varphi_z^H(\mathbf{q}, \mathbf{p}) \in \partial U, z < 0\}, \end{aligned}$$

where c is the path $s \in [z, 0] \mapsto \varphi_s^H(\mathbf{q}, \mathbf{p})$.

Recall that $F_1(U, \mathbb{K}) \cong \mathrm{R}\pi_{\mathbf{q}!}(\Delta^{-1}(P_U))$, then we have

$$\begin{aligned} \mu s_L(\Delta^{-1}(P_U)) \subset & \{(\mathbf{q}, \mathbf{p}, 0) : H(\mathbf{q}, \mathbf{p}) \leq 1\} \cup \\ & \qquad \{(\mathbf{q}, \mathbf{p}, -\int_c \alpha) : (\mathbf{q}, \mathbf{p}) = \varphi_z^H(\mathbf{q}, \mathbf{p}) \in \partial U, z < 0\}. \end{aligned}$$

Then the curve c defined as above is a closed orbit of $\varphi_s^H(\mathbf{q}, \mathbf{p})$, i.e. $c(z) = c(0)$. Now, we need the non proper pushforward estimate Theorem 1.7.

We remark that $t = 0 \in \mu s_L(F_1(U, \mathbb{K}))$ since the constant orbits, which have action 0, occur here.

Now, for $t > 0$, if $(t, 1) \in SS(F_\ell(U, \mathbb{K}))$ then there exists $(\mathbf{q}^n, \mathbf{p}^n, t^n, \tau^n)$ and $z^n < 0$ such that $\tau^n \to 1$, $t^n \to t$, and $(\mathbf{q}^n, \mathbf{p}^n) = \varphi_{z^n} H(\mathbf{q}^n, \mathbf{p}^n)$. Let $c^n(s) = \varphi_s^H(s)$, $s \in [z_n, 0]$, then $t^n = -\int_{c^n} \mathbf{p} d\mathbf{q}$.

But the non-degenerate assumption of the hypersurface ∂U shows that the action spectrum of ∂U is discrete. So, $t^n = -\int_{c^n} \mathbf{p} d\mathbf{q} \to t$ means that $t^n = t$ for all n. On the other hand, for two orbits c_n and c_m, if they are different, i.e. assume $z^n \neq z^m$, we have

$$0 = \int_{c^n - c^m} \mathbf{p} d\mathbf{q} = \int_{z^n}^{z^m} \mathbf{p} d\mathbf{q}.$$

Therefore, it means that we get a non-constant closed Reeb orbit in ∂U which has 0 action. This is impossible since ∂U is of RCT. Therefore, all z^n are the same and we obtain the unique orbit $c = c_n$ for all n. The absolute value appears since $\int_c \mathbf{p} d\mathbf{q} < 0$ when $z = z_n < 0$. $\square$

For the S^1-equivariant theory, it does not make sense to talk about the microsupport estimate since we do not have a sheaf $F^{S^1}(U, \mathbb{K})$ over $\mathbb{R}$. But the estimate (3.33) shows that

Corollary 3.29. *Under the same condition of Lemma 3.28. For $(T, T'] \subset [0, \infty)$, if*

$$\left\{ t \in \mathbb{R} : t = \left| \int_c \boldsymbol{p} d\boldsymbol{q} \right| \text{ for a closed orbit } c \text{ of } \varphi_z^H \text{ in } \partial U \right\} \cap (T, T'] = \varnothing,$$

then the morphism,

$$F^{S^1}(U, \mathbb{K})_{T'} \to F^{S^1}(U, \mathbb{K})_T,$$

is an isomorphism in $D(\mathbb{K}[\epsilon] - \mathrm{Mod})$.

Proof. The estimate (3.33) shows that the morphism,

$$F_\ell(U, \mathbb{K})_{T'} \to F_\ell(U, \mathbb{K})_T,$$

is an isomorphism for all ℓ. Then the result follows because we use all F_ℓ to defines F^{S^1}. $\square$

So far, we find two different ways to understand $F_\ell(U, \mathbb{K})$. Initially, from the definition of $F_\ell(U, \mathbb{K})$, we first cut off the energy of a Hamiltonian isotopy up to Legendre transform to obtain the kernels and then use the functor α_T to obtain cohomology of some discrete loop space. On the other hand, the results of this section shows, we can study discrete loops of a Hamiltonian isotopy first, and then cut off energy up to Legendre transform. The result of the section clarifies that these two ways are the same. The second way is more direct than the first in many cases; we will see more about this point of view when doing computation for toric domains and the unit cotangent bundle.

3.5. Capacities

Now, for $\ell \in \mathbb{N}_{\geq 2}$, p_ℓ is the minimal prime factor of ℓ, and $\mathbb{F}_{p_\ell}$ is the finite field of order p_ℓ. The Yoneda algebra $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{F}_{p_\ell}, \mathbb{F}_{p_\ell})$ is isomorphic to $\mathbb{F}_{p_\ell}[u, \theta]$ (See (B.3)), where $|u| = 2$, $|\theta| = 1$, and $\theta^2 = ku$ ($k = 0$ if ℓ is odd and $k = \ell/2$ if ℓ is even).

Definition 3.30. For an admissible open set U and $k \in \mathbb{N}$ we define

$$\mathrm{Spec}(U, k) := \left\{ T \geq 0 : \begin{array}{l} \exists p \text{ prime such that } \forall \ell \in \mathbb{N}_{\geq 2},\, p_\ell \geq p, \\ \exists \Lambda_\ell \in H^* C_{\ell,T}(U, \mathbb{F}_{p_\ell}),\, \eta_{\ell,T}(U, \mathbb{F}_{p_\ell}) = u^k \Lambda_\ell \end{array} \right\},$$

and

$$c_k(U) := \inf \mathrm{Spec}(U, k) \in [0, +\infty]. \tag{3.34}$$

In general, if U is not admissible, then we define

$$c_k(U) = \sup\{c_k(V) : V \subset U,\, V \text{ is admissible}\}.$$

In the following, we will prove step by step that $(c_k)_{k\in\mathbb{N}}$ defines a sequence of non-trivial symplectic capacities.

Theorem 3.31. *The functions $c_k : Open(T^*X) \to [0, \infty]$ satisfy the following:*

(1) $c_k \leq c_{k+1}$ for all $k \in \mathbb{N}$.

(2) For two open sets $U_1 \subset U_2$, we have $c_k(U_1) \leq c_k(U_2)$.

(3) For a compactly supported Hamiltonian isotopy $\varphi_z : T^*X \to T^*X$, we have $c_k(U) = c_k(\varphi_z(U))$.

(4) Suppose $U = \{H < 1\}$ is admissible such that $\partial U = \{H = 1\}$ is a non-degenerated hypersurface of restricted contact type defined by a Hamiltonian function H. If $c_k(U) < \infty$, then $c_k(U)$ is represented by the action of a closed characteristic in the boundary ∂U.

(5) $c_k(U) > 0$ for all open sets U.

Proof. We can assume U is admissible; the general case follows directly. Then (1) is the direct consequences of the Definition 3.30 and (2), (3) are corollaries of the Proposition 3.11.

For (4), let $T = c_k(U)$. Suppose it is not given by the action of a closed characteristic.

By assumption, the boundary ∂U has non-degenerated Reeb dynamics, so there are only finitely many closed characteristics with action less than $2T$. So there is a small $\varepsilon > 0$ such that there is no action happening in $[T-\varepsilon, T+\varepsilon]$.

However, one has the following microsupport estimate (3.33):

$$\mu s_L(F_\ell(U,\mathbb{K})) \subset \left\{ t \in \mathbb{R} : t = \left| \int_c \mathbf{p} d\mathbf{q} \right| \text{ for some closed orbit } c \text{ of } \varphi_z^H \right\}.$$

Therefore $F_\ell(U,\mathbb{K})$ is constant on $[T-\varepsilon, T+\varepsilon]$. Consequently, $(F_\ell(U,\mathbb{K}))_{T-\varepsilon} \cong (F_\ell(U,\mathbb{K}))_T$, and then $\eta_{\ell,T-\varepsilon}(U,\mathbb{K}) = \eta_{\ell,T}(U,\mathbb{K})$ for all ℓ and all $\mathbb{K}$. In particular for $\mathbb{K} = \mathbb{F}_{p_\ell}$ for all $\ell \in \mathbb{N}_{\geq 2}$. Then we have that $c_k(U) \leq T - \varepsilon$, which gives a contradiction. So we have

$$c_k(U) \in \left\{ \left| \int_c \mathbf{p} d\mathbf{q} \right| : c \text{ is a closed orbit of } \varphi_z^H \right\}. \tag{3.35}$$

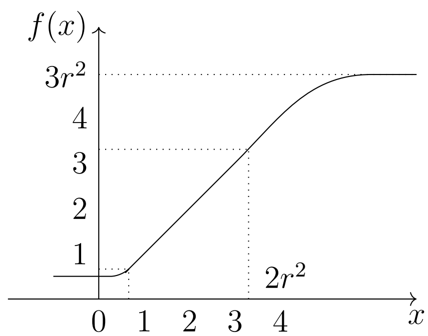
Finally, let us prove that the c_k's are positive. We will see, in Corollary 4.9, that for a ball B_a, one has $c_k(B_a) = \lceil k/d \rceil a$. In general, for any admissible open set U, for any $(\mathbf{q},\mathbf{p}) \in U$, we take a neighborhood $X' \cong \mathbb{R}^d$ of $\mathbf{q}$ and a small Darboux ball $B_a(\mathbf{q},\mathbf{p}) \subset U$ near $(\mathbf{q},\mathbf{p}) \in T^*X' \cong \mathbb{C}^d$. Then the invariance and the local property of fundamental classes show $c_k(U) \geq c_k(B_a(\mathbf{q},\mathbf{p})) = c_k(B_a) > 0$. □

Remark 3.32. We also see from $c_k(B_a) = \lceil k/d \rceil a$ that if U is a bounded open set (which is admissible by Proposition 2.18), then $c_k(U) < \infty$.

The property (4) is called representativity of capacity. For some class of capacities, representativity can imply the conformal property. But our capacities are not in this class. So, we present a proof of the conformal property of star-shaped domains in $T^*\mathbb{R}^d$.

Proposition 3.33. *For $X = \mathbb{R}^d$, if the bounded open set $U \subset T^*X$ has smooth boundary ∂U, and U is star-shaped (i.e. the radial vector field on $T^*\mathbb{R}^d = \mathbb{R}^{2d}$ is transverse to ∂U), then $c_k(rU) = r^2 c_k(U)$ for $r > 0$.*

Proof. Assume $r > 1$, otherwise we can study rU and replace r by $1/r$. When U is star-shaped, U can be defined as $U = \{H < 1\}$ for a unique function $H \in C^\infty(\mathbb{R}^{2d} \setminus \{0\})$, satisfying $H(cx) = c^2 H(x)$ for $c \geq 0$ (see [**GHR22**, Theorem 6.1]). The function H could be extended continuously to 0 by homogeneity, but the extension is not smooth at the origin. Besides H is not compactly supported up to a constant, so we can not directly use the GKS quantization. We choose a non-decreasing function $f(x) = 1/2$ for $x \leq 1/2$, $f(x) = x$ for $2/3 \leq x \leq 2r^2$, and $f(x) = 3r^2$ for $x \geq 4r^2$.



Now we take $\overline{H}(x) = f(H(x))$, which is smooth on $\mathbb{R}^{2d}$ and is constant outside $4r^2U$. Moreover, $U = \{\overline{H} < 1\}$, $rU = \{\overline{H} < r^2\}$.

Now, take the GKS quantization $\mathcal{K}(\overline{H})$, which exists since $\overline{H}$ is compactly supported up to a constant. Consider the map $i(z) = r^2 z$. One can estimate the microsupport to see that $i^{-1}\mathcal{K}(\overline{H})$ is the quantization of $r^{-2}\overline{H}$. Then applying our existence to $i^{-1}\mathcal{K}(\overline{H})$ and $\Omega = (-\infty, 1)$, we get the kernel $P_{\{\overline{H}<r^2\}} = P_{rU}$.

Then the projection formula shows us that

$$P_{rU} \cong i^{-1}\mathcal{K}(\overline{H}) \star \mathbb{K}_{\{(z,t):-t\leq z\leq 0\}} \cong \mathcal{K}(\overline{H}) \star \mathbb{K}_{\{(z,t):-t/r^2\leq z\leq 0\}}.$$

Then, after applying $\alpha_{\ell,r^2T,\mathbb{R}^d}$ (recall (3.1)), the formula (3.32) shows that we have an isomorphism

$$F_\ell(rU, \mathbb{K})_{r^2T} \cong F_\ell(U, \mathbb{K})_T.$$

Moreover, taking the convolution $\star\mathbb{K}_{\{(z,t):z=0,t\geq 0\}}$ shows that the fundamental class is respected under the isomorphism. So we have that

$$\eta_{\ell,r^2T}(rU,\mathbb{K}) = \eta_{\ell,T}(U,\mathbb{K}),$$

and

$$\mathrm{Spec}(U,k) \xrightarrow{\times r^2} \mathrm{Spec}(rU,k)$$

is a bijection. Then the result follows. $\square$

Next, we work on the S^1-equivariant version. We recall that $A = \mathrm{Ext}^*_{S^1}(\mathbb{K},\mathbb{K}) \cong \mathbb{K}[u]$ (see (B.3)) with $|u| = 2$. We can give a similar definition as c_k but easier.

Definition 3.34. For an admissible open set U and $k \in \mathbb{N}$ we define

$$\overline{\mathrm{Spec}}(U,k) := \left\{T \geq 0 : \exists \Lambda^{S^1} \in H^* C_T^{S^1}(U,\mathbb{Q}),\ \eta_T^{S^1}(U,\mathbb{Q}) = u^k \Lambda^{S^1}\right\}.$$

and

$$\bar{c}_k(U) := \inf \overline{\mathrm{Spec}}(U,k)) \in [0,+\infty]. \tag{3.36}$$

In general, if U is not admissible, then we define

$$\bar{c}_k(U) = \sup\{\bar{c}_k(V) : V \subset U,\ V \text{ is admissible}\}.$$

One can write down the following theorem without any modification of the proof for Theorem 3.31 and Proposition 3.33 (Notice that we should use Corollary 3.29 to replace (3.33)).

THEOREM 3.35. *The functions $\bar{c}_k : Open(T^*X) \to [0,\infty]$ satisfy the following:*

(1) $\bar{c}_k \leq \bar{c}_{k+1}$ for all $k \in \mathbb{N}$.

(2) For two open sets $U_1 \subset U_2$, then $\bar{c}_k(U_1) \leq \bar{c}_k(U_2)$.

(3) For a compactly supported Hamiltonian isotopy $\varphi : I \times T^*X \to T^*X$, we have $\bar{c}_k(U) = \bar{c}_k(\varphi_z(U))$.

(4) Suppose $U = \{H < 1\}$ is admissible such that $\partial U = \{H = 1\}$ is a non-degenerated hypersurface of restricted contact type defined by a Hamiltonian function H. If $\bar{c}_k(U) < \infty$, then $\bar{c}_k(U)$ is represented by the action of a closed characteristic in the boundary ∂U.

(5) $\bar{c}_k(U) > 0$ for all open sets U.

(6) If $X = \mathbb{R}^d$, and U is of star-shaped centered at the origin, then $\bar{c}_k(rU) = r^2\bar{c}_k(U)$ for $r > 0$.

We end this section with a quick discussion on the non-equivariant capacities. Recall that, P_U is an object of the Tamarkin category $\mathcal{D}(X^2)$. So, we can study its sheaf energy (see Definition 2.5), i.e.

$$e(P_U(\mathbb{Q})) = \inf\{c : \tau_c(P_U(\mathbb{Q})) = 0\}.$$

Under the isomorphism (3.22) and the definition of the fundamental class, $\tau_c(P_U(\mathbb{K}))$ is mapped to $\eta_{1,c}(U, \mathbb{K})$:

$$\eta_{1,c}(U, \mathbb{Q}) = \Theta(\tau_c(P_U(\mathbb{Q}))).$$

We can also defined for an admissible open set U

$$c(U) = e(P_U(\mathbb{Q})) = \inf\{c \geq 0 : \eta_{1,c}(U, \mathbb{Q}) = 0\}, \tag{3.37}$$

and, for a general open set U, $c(U) = \sup\{c(V) : V \subset U, V \text{ is admissible}\}$. So the capacities c_k and $\bar{c}_k$ are equivariant generalizations of the sheaf energy.

The proof of Theorem 3.31 and Proposition 3.33 still apply word by word and give:

THEOREM 3.36. *The function $c : Open(T^*X) \to [0, \infty]$ satisfies the following:*

(1) For two open sets $U_1 \subset U_2$, then $c(U_1) \leq c(U_2)$.

(2) For a compactly supported Hamiltonian isotopy $\varphi : I \times T^*X \to T^*X$, we have $c(U) = c(\varphi_z(U))$.

(3) Suppose $U = \{H < 1\}$ is admissible such that $\partial U = \{H = 1\}$ is a non-degenerated hypersurface of restricted contact type defined by a Hamiltonian function H. If $c(U) < \infty$, then $c(U)$ is represented by the action of a closed characteristic in the boundary ∂U.

(4) $c(U) > 0$ for all open sets U.

(5) If $X = \mathbb{R}^d$, and U is of star-shaped centered at the origin, then $c(rU) = r^2 c(U)$ for $r > 0$.

We have the following natural questions about the relation between the capacities introduced so far: do we have $c_k(U) = \bar{c}_k(U)$ and $c(U) = c_1(U) = \bar{c}_1(U)$?

Algebraically, in the $\mathbb{Z}/\ell$ or S^1 equivariant category we have a non-zero $u \in \mathrm{Ext}^2_G(\mathbb{K}, \mathbb{K})$ for many fields $\mathbb{K}$, and $\mathrm{Ext}^2(\mathbb{K}, \mathbb{K}) = 0$ for any field $\mathbb{K}$.

Under the restriction map of equivariant derived category, we find that $\eta_{\ell,T}$ and $\eta_T^{S^1}$ are mapped to $\eta_{1,T}(U, \mathbb{K})$. The class $u \in \mathrm{Ext}^2_G(\mathbb{K}, \mathbb{K})$ is mapped to 0. Consequently, if $\eta_{\ell,T}(U, \mathbb{K}) = u\Lambda$ or $\eta_T^{S^1}(U, \mathbb{K}) = u\Lambda$, then $\eta_{1,T}(U, \mathbb{K}) = 0 \in \mathrm{Ext}^0(\mathbb{K}, \mathbb{K})$. Therefore, we have $c_1(U) \geq c(U)$ and $\bar{c}_1(U) \geq c(U)$.

On the other hand, by the (3.27), we have $\eta_{1,T}(U, \mathbb{K}) = 0$ if and only if there exists $\Lambda \in H^{-2} C_T^{S^1}(U, \mathbb{K})$ such that $\eta_T^{S^1}(U, \mathbb{K}) = u\Lambda$. So, we have $\bar{c}_1(U) \leq c(U)$. Then we have

Proposition 3.37. *For an admissible domain U, we have*

$$\bar{c}_1(U) = c(U).$$

In general, we do not know if we have $c_k(U) = \bar{c}_k(U)$. But we will see later via the computation that equalities are true for convex toric domains (Theorem 4.8). Moreover, we guess the only difference comes from the algebraic property of cyclic homology.

Remark 3.38. Finally, let us remark about the computability of c_k. As $H^* C_{\ell,T}(U, \mathbb{K})$ is defined using P_U, which is an object in the derived category. Even though it is unique in the derived category, we can take different chain representatives of P_U. Therefore, to compute $c_k(U)$, we can choose particular chain representative of P_U. Usually, these chain representatives of P_U admit properties that are not so obvious from general existence results like Proposition 2.15, and Proposition 2.18.

3.6. Contact invariants

I will explain how the Chiu-Tamarkin complex works for the contact geometry of (contact) admissible open sets in the prequantized space $T^*X \times S^1$.

For any open set $U \subset T^*X \times S^1$, we can lift it to a $\mathbb{Z}$-invariant set $\widetilde{U} \subset J^1X$ in the sense $T'_k(\widetilde{U}) = \widetilde{U}$, where $T'_k(\mathbf{q}, \mathbf{p}, t) = (\mathbf{q}, \mathbf{p}, t+k)$ for $k \in \mathbb{Z}$. In this way, we can discuss sheaves microsupported in $J^1X \setminus \tilde{U}$. Then $\mathcal{D}_{J^1X\setminus\widetilde{U}}(X)$ and its left semi-orthogonal complement are all well-formulated. Specifically, for $Z = J^1X \setminus \widetilde{U}$, we define

$$\begin{aligned}\mathcal{D}^c_Z(X) =&\{F \in \mathcal{D}(X) : \mu s_L(F) \subset Z\},\\ \mathcal{D}^c_U(X) =&{}^{\perp}\mathcal{D}^c_Z(X),\ \text{the left orthogonal complement of } \mathcal{D}^c_Z(X).\end{aligned}$$

Same as the symplectic case, we can define the notion of admissibility and microlocal kernels. Be compatible with the Hamiltonian action of contact isotopy as we discussed in Section 1.3, we will use composition functors rather than convolution functors. On the other hand, in the symplectic case, we require that microlocal kernels are objects in the Tamarkin category. Now, we need a (2-variable) variant version of the Tamarkin category for contact microlocal kernels. Let $\mathscr{D}(X^2)$ be the full triangulated subcategory $\{F \in D(X^2 \times \mathbb{R}^2) : F \circ \mathbb{K}_{\{t_2 \geq t_1\}} \xrightarrow{\cong} F\}$ of $D(X^2 \times \mathbb{R}^2)$. Then we define

Definition 3.39. We say U is $\mathbb{K}$-*admissible* if there is a distinguished triangle

$$\mathscr{P}_U \to \mathbb{K}_{\Delta_{X^2} \times \{t_2 \geq t_1\}} \to \mathscr{Q}_U \xrightarrow{+1},$$

in $\mathscr{D}(X^2)$ such that the composition functor $\circ\mathscr{P}_U$ is right adjoint to $\mathcal{D}^c_U(X) \hookrightarrow \mathcal{D}(X)$ and $\circ\mathscr{Q}_U$ is left adjoint to $\mathcal{D}^c_Z(X) \hookrightarrow \mathcal{D}(X)$, i.e.,

$$\mathcal{D}^c_Z(X) \xleftarrow{\circ\mathscr{Q}_U} \mathcal{D}(X) \xrightarrow{\circ\mathscr{P}_U} \mathcal{D}^c_U(X),$$

are two microlocal projectors.

Such a pair of sheaves $(\mathscr{P}_U, \mathscr{Q}_U)$ together with the distinguished triangle give an orthogonal decomposition of $\mathcal{D}(X)$ by Proposition 2.1. We call the pair $(\mathscr{P}_U, \mathscr{Q}_U)$ *microlocal kernels* associated with $U \subset T^*X \times S^1$, and the distinguished triangle as the defining triangle of U.

We say U is *admissible* if U is $\mathbb{Z}$-admissible.

The uniqueness and functoriality has the same proof, just need to replace convolution by composition. We have the existence of kernels for the prequantized open set $W \times S^1$ where $W \subset T^*X$ is a symplectic admissible open set. Precisely, we have the following proposition.

Proposition 3.40. *If $W \subset T^*X$ is (symplectic) admissible by the following distinguished triangle:*

$$P_W \to \mathbb{K}_{\Delta_{X^2} \times \{t \geq 0\}} \to Q_W \xrightarrow{+1} .$$

Then $W \times S^1 \subset T^*X \times S^1$ is (contact) admissible by the following distinguished triangle:

$$\mathscr{P}_{W \times S^1} \to \mathbb{K}_{\Delta_{X^2} \times \{t_2 \geq t_1\}} \to \mathscr{Q}_{W \times S^1} \xrightarrow{+1},$$

where $\mathscr{P}_{W \times S^1} = m^{-1} P_W$, $\mathscr{Q}_{W \times S^1} = m^{-1} Q_W$ and $m(t_1, t_2) = t_2 - t_1$.

Notice that we have $\mathbb{K}_{\Delta_{X^2} \times \{t_2 \geq t_1\}} = m^{-1}\mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$.

Proof. The second distinguished triangle comes from applying m^{-1} to the first one and we have $m^{-1}F \in \mathscr{D}(X^2)$ for $F \in \mathcal{D}(X^2)$. On the other hand, as we mentioned in (1) of Remark 1.11, we have

$$F \star P_W \cong F \circ \mathscr{P}_{W \times S^1}, \quad F \star Q_W \cong F \circ \mathscr{Q}_{W \times S^1},$$

for $F \in \mathcal{D}(X)$. Finally, as $\widetilde{W \times S^1} = W \times \mathbb{R}$, we have that $\mu s_L(F) \subset J^1X \setminus \widetilde{W \times S^1}$ if and only if $\mu s(F) \subset T^*X \setminus W$. Then the result follows. $\square$

Now, we can define the contact Chiu-Tamarkin complex for admissible open sets $U \subset T^*X \times S^1$. As in the symplectic case, let us introduce the adjoint pair first:

$$F \in D_{\mathbb{Z}/\ell}((X^2 \times \mathbb{R}_t^2)^\ell) \underset{\beta^c_{\ell,T,X}}{\overset{\alpha^c_{\ell,T,X}}{\rightleftarrows}} D_{\mathbb{Z}/\ell}(\mathrm{pt}) \ni G,$$

defined by:

$$\begin{aligned} \alpha^c_{\ell,n,X}(F) &= (i^\ell_{n*})^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\mathrm{R}\widetilde{m}_!\,(F)\,, \\ \beta^c_{\ell,n,X}(G) &= \widetilde{m}^!\tilde{\Delta}_{X*}\pi^!_{\underline{\mathbf{q}}}i^\ell_{n*}G[-1], \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} &\widetilde{m} : (X^2 \times \mathbb{R}^2)^\ell \to X^{2\ell} \times \mathbb{R}^\ell, \\ &\widetilde{m}(\underline{\mathbf{q}}, t_1^1, t_1^2, \dots, t_\ell^1, t_\ell^2) = (\underline{\mathbf{q}}, t_\ell^2 - t_1^1, t_1^2 - t_2^1, \dots, t_{\ell-1}^2 - t_\ell^1); \\ &\widetilde{\Delta}_X : X^\ell \times \mathbb{R}^\ell \to X^{2\ell} \times \mathbb{R}^\ell, \\ &\widetilde{\Delta}_X(\mathbf{q}_1, \dots, \mathbf{q}_\ell, \underline{t}) = (\mathbf{q}_\ell, \mathbf{q}_1, \mathbf{q}_1, \dots, \mathbf{q}_{\ell-1}, \mathbf{q}_{\ell-1}, \mathbf{q}_n, \underline{t}); \\ &\pi_{\underline{\mathbf{q}}} : X^\ell \times \mathbb{R}^\ell \to \mathbb{R}^\ell; \\ &i^\ell_n(\mathrm{pt}) = (n, \dots, n) \in \mathbb{R}^\ell, \end{aligned}$$

where $\underline{\mathbf{q}} = (\mathbf{q}_1, \dots, \mathbf{q}_\ell)$ and $\underline{t} = (t_1, \dots, t_\ell)$.

Definition 3.41. With the notation above, for $\ell \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define the*contact Chiu-Tamarkin complex* as follows:

$$\begin{aligned} \mathscr{C}_{\ell,n\ell}(U, \mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha^c_{\ell,n,X}(\mathscr{P}_U^{\boxtimes\ell}), \mathbb{K}[-d]\right) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\mathscr{P}_U^{\boxtimes\ell}, \beta^c_{\ell,n,X}\mathbb{K}[-d]\right). \end{aligned}$$

Compare to the symplectic case, the parameter T is replaced by a discrete parameter $T = n\ell$. First, let us compare $\mathscr{C}_{\ell,n\ell}(W \times S^1, \mathbb{K})$ and $C_{\ell,n\ell}(W, \mathbb{K})$ if $W \subset T^*X$ is symplectic admissible. By Proposition 3.40, the prequantized open set $W \times S^1$ is contact admissible.

Proposition 3.42. *For a symplectic admissible open set $W \subset T^*X$, for $\ell \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have*

$$C_{\ell,n\ell}(W, \mathbb{K}) \cong \mathscr{C}_{\ell,n\ell}(W \times S^1, \mathbb{K}).$$

PROOF. Since $\mathscr{P}_{W\times S^1} \cong m^{-1}P_W$, we have

$$\mathscr{P}^{\boxtimes\ell}_{W\times S^1} \cong (m^\ell)^{-1}P^{\boxtimes}_W,$$

where $m^\ell(\underline{\mathbf{q}}, t_1^1, t_1^2, \ldots, t_\ell^1, t_\ell^2) = (\underline{\mathbf{q}}, t_1^2 - t_1^1, \ldots, t_\ell^2 - t_\ell^1)$. Then we have

$$\mathscr{C}_{\ell,n\ell}(W\times S^1, \mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_W^{\boxtimes\ell}, m_*^\ell \beta^c_{\ell,n,X}\mathbb{K}[-d]\right).$$

So, we only need to verify that

$$m_*^\ell \beta^c_{\ell,n,X}\mathbb{K} \cong \beta_{\ell,n\ell,X}\mathbb{K}.$$

By proper base change, we only need to assume $X = \mathrm{pt}$ and then show that $m_*^\ell \widetilde{m}^! i^\ell_{n*}\mathbb{K}[-1] \cong s_t^{\ell!} i_{n\ell*}\mathbb{K}$. Direct computation shows that both sides are isomorphic to $\mathbb{K}_{\{(t_1,\cdots,t_\ell):t_1+\cdots+t_\ell=n\ell\}}[\ell-1]$. □

On the other hand, the constrain $T/\ell \in \mathbb{N}_0$ is adapt to the problem of invariance. As the lifting of a contact isotopy is merely $\mathbb{Z}$-equivariant, the sheaf quantization will only be $\mathbb{Z}$-equivariant (see Remark 1.19). So our discussion on invariance for symplectic version does not applies directly. However a slight modification for the proof of the symplectic invariance works.

THEOREM 3.43 ([**Chi17**, Theorem 4.7]). *Let U, U_1, U_2 be contact admissible open sets and let $U_1 \xhookrightarrow{i} U_2$ be an inclusion. Then one has, for $\ell \in \mathbb{N}$, $n \in \mathbb{N}_0$,*

(1) There is a morphism $\mathscr{C}_{\ell,n\ell}(U_2, \mathbb{K}) \xrightarrow{i^*} \mathscr{C}_{\ell,n\ell}(U_1, \mathbb{K})$, which is natural with respect to inclusions of admissible open sets.

(2) For a compactly supported contact isotopy $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$,. We have an isomorphism, in the equivariant category, $\Phi^c_{z,\ell,n\ell} : \mathscr{C}_{\ell,n\ell}(U, \mathbb{K}) \xrightarrow{\cong} \mathscr{C}_{\ell,n\ell}(\varphi_z(U), \mathbb{K})$, for all $z \in I$. The isomorphism $\Phi^c_{z,\ell,n\ell}$ is functorial with respect to restriction morphisms in (1). When $U = T^*X \times S^1$, we have $\Phi^c_{z,\ell,n\ell} = \mathrm{Id}$.

The proof for (1) is the same as the symplectic case. Let us present the proof for invariance, which is slightly different from the symplectic one.

Proof of Theorem 3.43 (2). For the contact isotopy φ, we take the GKS quantization $K(\widehat{\varphi'})$ as we discussed in Section 1.3. Let $K = K(\widehat{\varphi'})_z^{-1}$, $K_\ell = K^{\boxtimes \ell}$ and $K_\ell^{-1} = (K^{-1})^{\boxtimes \ell}$.

Recall the proof of Theorem 3.5 (2). In the contact case, we still have an isomorphism

$$\mathscr{P}_{\varphi_z(U)} \cong K^{-1} \circ \mathscr{P}_U \circ K,$$

as well as the auto-equivalence $\kappa(F) := K_\ell^{-1} \circ F \circ K_\ell$ of $D_{\mathbb{Z}/\ell}((X \times \mathbb{R})^{2\ell})$.

So, we only need to construct an isomorphism

$$\kappa(\beta_n^c \mathbb{K}) = K_\ell^{-1} \circ \beta_n^c \mathbb{K} \circ K_\ell \cong \beta_n^c \mathbb{K}, \tag{3.39}$$

where $\beta_n^c = \beta_{\ell,n,X}^c$. As in Theorem 3.5, we only need to find an isomorphism

$$\beta_n^c \mathbb{K} \circ K_\ell \cong K_\ell \circ \beta_n^c \mathbb{K}.$$

To emphasize the difference between the contact case and the symplectic case, let us present the construction precisely. Let $W = X \times \mathbb{R}$, $f\colon W^\ell \to W^\ell$, $(w_1, \dots, w_\ell) \mapsto (w_2, \dots, w_\ell, w_1)$ where $w_i = (\mathbf{q}_i, t_i)$ and $\mathrm{T}_c^\ell : W^\ell \to W^\ell$, $(w_1, \dots, w_\ell) \mapsto (\mathrm{T}_c(w_1), \dots, \mathrm{T}_c(w_\ell))$, where $c \in \mathbb{R}$ and $\mathrm{T}_c(w_i) = \mathrm{T}_c(\mathbf{q}_i, t_i) = (\mathbf{q}_i, t_i + c)$. Set $Y = W^\ell$ and identify $Y^2 = (W^2)^\ell$ by $(w_1^1, \dots, w_\ell^1, w_1^2, \dots, w_\ell^2) \mapsto (w_1^1, w_1^2, \dots, w_\ell^1, w_\ell^2)$. Then $\beta_n^c \mathbb{K}$ is, up to orientation and shift, the constant sheaf on the graph of the composition $f \circ \mathrm{T}_n^\ell = \mathrm{T}_n^\ell \circ f$. Precisely, we have

$$\beta_n^c \mathbb{K} \cong \mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ E \cong E \circ \mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}},$$

where $E = \delta_{Y^2 !}(\omega_Y)$, with ω_Y the dualizing sheaf and δ_{Y^2} the usual diagonal embedding. The relation $f \circ \mathrm{T}_n^\ell = \mathrm{T}_n^\ell \circ f$ implies $\mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \cong \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ \mathbb{K}_{\Gamma_f}$. Moreover we have $E \circ - \cong - \circ E$.

Now we have the general fact $G \circ \mathbb{K}_{\Gamma_g} \cong (\mathrm{Id}_Y \times g)_!(G)$ for any G and any map g. This formula has the symmetric form $\mathbb{K}_{\Gamma'_g} \circ G \cong (g \times \mathrm{Id}_Y)_!(G)$ where Γ'_g is the switched graph $\Gamma'_g = \{(g(y), y) : y \in Y\}$. When g is invertible, we have $\Gamma_{g^{-1}} = \Gamma'_g$. So we obtain

$$K_\ell \circ \beta_n^c \mathbb{K} \cong K_\ell \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ \mathbb{K}_{\Gamma_f} \circ E \cong (\mathrm{Id}_Y \times f)_!(K_\ell \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}) \circ E,$$

and

$$\beta_n^c\mathbb{K}\circ K_\ell \cong E\circ \mathbb{K}_{\Gamma_f}\circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell = E\circ \mathbb{K}_{\Gamma'_{f^{-1}}}\circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell \cong E\circ (f^{-1}\times \mathrm{Id}_Y)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell).$$

Now, recall the GKS quantization $K(\widehat{\varphi'})$ satisfies the $\mathbb{Z}$-equivariant condition (1.28), so the restriction on z-slices, $K_\ell = K^{\boxtimes \ell}$, also satisfies

$$K\circ \mathbb{K}_{\Delta_{X^2}\times\{(t,t+n):t\in\mathbb{R}\}} \cong \mathbb{K}_{\Delta_{X^2}\times\{(t,t+n):t\in\mathbb{R}\}}\circ K.$$

Notice that $\Delta_{X^2}\times\{(t,t+n):t\in\mathbb{R}\} = \Gamma_{\mathrm{T}_n}$ where $\mathrm{T}_n(x,t)=(x,t+n)$. Then we have

$$K_\ell\circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \cong \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell.$$

Then we have

$$K_\ell\circ \beta_n^c\mathbb{K} \cong (\mathrm{Id}_Y\times f)_!(K_\ell\circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}})\circ E \cong E\circ (\mathrm{Id}_Y\times f)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell).$$

Then the isomorphism (3.39) follows from

$$(\mathrm{Id}_Y\times f)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell) \cong (f\times f)_!(f^{-1}\times \mathrm{Id}_Y)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell) \cong (f^{-1}\times \mathrm{Id}_Y)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}\circ K_\ell).$$

The isomorphism $\Phi^c_{z,\ell,n\ell}$ follows. □

Remark 3.44. The significant difference between $C_{\ell,T}$ and $\mathscr{C}_{\ell,n\ell}$ is that the definition of α^c also twist t variables while α only twist $\mathbf{q}$ variables. This is crucial for the contact invariance. However Proposition 3.42 shows that when we consider the admissible sets of the form $U\times S^1$ for $U\subset T^*X$, the Chiu-Tamarkin complex itself is not affected by the difference. This is helpful for our computations.

Now, we assume $\ell\in\mathbb{N}_{\geq 2}$ and $U\subset T^*X\times S^1$ is admissible. Then $H^*\mathscr{C}_{\ell,n\ell}(U,\mathbb{K})$ is a module of $A=\mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})$. For an orientable manifold X, the *fundamental class* $\eta^c_{\ell,n\ell}(U)$ is defined as the image of the fundamental class $[X]^{\mathbb{Z}/\ell}=[X]\otimes 1$ under the morphism $H_d^{BM}(X,\mathbb{K})\otimes \mathrm{Ext}^0_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})\cong H^0\mathscr{C}_{\ell,n\ell}(T^*X\times S^1,\mathbb{K}) \xrightarrow{i_U^*} H^0\mathscr{C}_{\ell,n\ell}(U,\mathbb{K})$. Similarly to Proposition 3.11, the Theorem 3.43 shows that the fundamental class is preserved under inclusion and contact isotopy.

For the definition of capacities, it is reasonable to require a discrete spectrum.

Definition 3.45. For an admissible open set $U \subset T^*X \times S^1$, $k \in \mathbb{N}$. Define

$$[\mathrm{Spec}](U,k) \coloneqq \left\{ n\ell \in \mathbb{N}_{\geq 2} : \begin{array}{l} (n,\ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\ \exists p \text{ prime such that } \forall \ell,\ p_\ell \geq p, \\ \exists \Lambda_\ell \in H^*\mathscr{C}_{\ell,n\ell}(U, \mathbb{F}_{p_\ell}),\ \eta^c_{\ell,n\ell}(U, \mathbb{F}_{p_\ell}) = u^k \Lambda_\ell \end{array} \right\},$$

and

$$[c]_k(U) \coloneqq \min[\mathrm{Spec}](U,k) \in \mathbb{N}_{\geq 2}.$$

For a general open set U, we also define

$$[c]_k(U) = \sup\{[c]_k(V) : V \subset U,\ V \text{ is admissible}\}.$$

Let us discuss the properties of $[c]_k$. The invariance and monotonicity are true with the same proof as in the symplectic case. The proof of representing property is invalid now. The positivity for open sets is obviously true by definition. However it is possible that $[c]_k$ is always 2, which is treated as the trivial situation here. To avoid this situation, we must address some restrictions on the size of domains. In fact, in the computation of $[c]_k(B_a \times S^1)$, we need to take $T = n\ell < ap_\ell$ to make sure $\eta^c_{\ell,n\ell}(B_a \times S^1, \mathbb{F}_{p_\ell})$ is non-zero. Here, the constraint is read as $a > n\ell/p_\ell$ for $\ell \in \mathbb{N}_{\geq 2}$ and $n \in \mathbb{N}$. In particular, for prime ℓ and $n = 1$, we have $a > 1$. So, we require $a > 1$ as a necessary size constraint for $B_a \times S^1$. This fits into the framework of [**EKP06**] that a small contact ball can be squeezed into smaller contact balls. Therefore, we define

Definition 3.46. For an open set $U \subset T^*X \times S^1$, we say it is *big* if there is a prequantized ball $B_a \times S^1 \subset U$ such that $a > 1$.

In summary, we organize our discussions as the following theorem. In the contact case, the spectrum sets could provide us more interesting obstructions. So we state results of spectrum sets as well.

Theorem 3.47. *The functions $[c]_k : Open(T^*X \times S^1) \to \mathbb{N}_{\geq 2}$ satisfy the following:*

(1) $[c]_k \leq [c]_{k+1}$ and $[\mathrm{Spec}](U,k+1) \subset [\mathrm{Spec}](U,k)$, for all $k \in \mathbb{N}$.

(2) For two open sets $U_1 \subset U_2$, then $[c]_k(U_1) \leq [c]_k(U_2)$ and $[\mathrm{Spec}](U_2,k) \subset [\mathrm{Spec}](U_1,k)$.

(3) For a compactly supported contact isotopy $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$, we have $[c]_k(U) = [c]_k(\varphi_z(U))$ and $[\mathrm{Spec}](U, k) = [\mathrm{Spec}](\varphi_z^H(U), k)$.

(4) If U is big, then it cannot happen that $[c]_k(U) = 2$ for all $k \in \mathbb{N}$.

At the end of the section, let us explain why we cannot define contact capacities for $T^*X \times S^1$ using the S^1-equivariant version of the Chiu-Tamarkin complex. To define the S^1-structure, we need to work on all F_ℓ with the persistence structure to define the capacities, $\bar{c}_k$ for example. To obtain the contactomorphism invariance on $T^*X \times S^1$, we must require $T/\ell \in \mathbb{N}_0$ for a fixed $T \geq 0$ and for all $\ell \in \mathbb{N}$. It only happen when $T = 0$. But as (3.33) shows, non-trivial geometric information appears only when $T > 0$.

Therefore, at least there is no obvious way to directly use the S^1-theory to define contact capacities. For the symplectic case, this problem will not happen, since for all $T \geq 0$, $T/\ell \geq 0$ is obvious. So the definition of $\bar{c}_k$ makes sense.

But the definition of $[c]_k$ is a sophisticated numerical analogy of the homological S^1-action, which does define contact capacities.

CHAPTER 4

Computing of the Chiu-Tamarkin complex

In this chapter, I would like to present some computational results.

The first one is about the toric domains. We construct the microlocal kernels of toric domains via the generating function model of Hamiltonian rotation given by Chiu. Then we use the generating function model to compute the Chiu-Tamarkin homology for convex toric domains. We will present a structure theorem, and then use it to compute the capacities we defined in Section 3.5.

The second example is the unit disk bundle. We will use the formula of the sheaf quantization for geodesic flow (see Subsection 1.3.1) to prove a Viterbo isomorphism of the Chiu-Tamarkin homology, which relates the Chiu-Tamarkin homology of the open disk bundle with the homology of the free loop space. Next, we will see that the cup product we defined in Section 3.2 is the Chas-Sullivan product.

4.1. Toric domains

In this section, we study toric domains. The 2-dimensional rotation $\varphi_z(u) = \exp(-2i\pi z)u$ on $\mathbb{C}_u$ is the Hamiltonian flow of the Hamiltonian function $H(u) = \pi|u|^2$. Here, we identify $\mathbb{C}_u$ with $T^*\mathbb{R}_q$ by $u = q + ip$.

Consider the product action of single 2-dimensional rotations given by

$$z \cdot (u_1, \dots, u_n) = (\exp(-2i\pi z_1)u_1, \dots, \exp(-2i\pi z_d)u_d).$$

This is a Hamiltonian action of $\mathbb{R}^d_z$, which is indeed a torus action, on $\mathbb{C}^d_u = T^*V$, where $V = \mathbb{R}^d_{\mathbf{q}}$ is a real vector space of dimension d, and $u = \mathbf{q} + i\mathbf{p}$. We call it the standard Hamiltonian torus action on $\mathbb{C}^d_u = T^*V$.

The moment map of the standard Hamiltonian torus action is

$$\mu : \mathbb{C}^d_u = T^*V \to (\mathbb{R}^d_z)^* = \mathbb{R}^d_\zeta, \quad \mu(u_1, \dots, u_n) = (\pi|u_1|^2, \dots, \pi|u_d|^2). \tag{4.1}$$

Definition 4.1. For an open set $\Omega \subset \mathbb{R}^d_\zeta$, we call $X_\Omega \coloneqq \mu^{-1}(\Omega) \subset T^*V$ an (open) toric domain. We say X_Ω is a convex toric domain if $|\Omega| \coloneqq \{\zeta \in \mathbb{R}^d : (|\zeta_1|, \ldots, |\zeta_d|) \in \Omega\}$ is convex. We say X_Ω is concave if $\mathbb{R}^d_{\zeta \geq 0} \setminus \Omega$ is convex.

Remark 4.2. Since the moment map μ has the image $\mathbb{R}^d_{\zeta\geq 0}$, the toric domain X_Ω is determined by $\Omega \cap \mathbb{R}^d_{\zeta\geq 0}$. So we have freedom to choose suitable Ω. For example, we always assume $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega$. If X_Ω is a convex or a concave toric domain, one can indeed take Ω to be convex or concave (in the usual sense) and satisfying the condition $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega$. (e.g. replace Ω by $\Omega - \mathbb{R}^d_{\zeta\geq 0}$).

For example, we can take a non-decreasing sequence $a = (a_1, \ldots, a_d)$ of positive real numbers, let $\Omega_{D(a)} = \{\zeta : \zeta_i < a_i, i \in [d]\}$ and $\Omega_{E(a)} = \{\zeta : \frac{\zeta_1}{a_1} + \cdots + \frac{\zeta_d}{a_d} < 1\}$. Then $X_{\Omega_{D(a)}} = D(a)$ is an open poly-disc and $X_{\Omega_{E(a)}} = E(a)$ is an open ellipsoid. Both of them are convex toric domains.

4.1.1. Generating function model for microlocal kernel of Toric domains. In [**Chi17**, Proposition 3.10], Chiu constructs a sheaf quantization of Hamiltonian rotation in all dimensions, particularly for the 2-dimensional φ_z, say $\mathcal{S} \in D(\mathbb{R}_z \times \mathbb{R}_{q_1} \times \mathbb{R}_{q_2} \times \mathbb{R}_t)$. This quantization possesses one more property than we stated for general sheaf quantizations (see (2.8)), namely

$$\mathcal{S} \cong \mathrm{R}\pi_{(q_2,\ldots,q_N)!}\mathbb{K}_{\{(z,q_1,\ldots,q_{N+1},t): t+\sum_{j=1}^N S_H(z/N, q_j, q_{j+1})\geq 0\}}, \tag{4.2}$$

where we identify q_{N+1} with q_2 after pushforward, N is big enough so that $z/N \in (-1/4, 0) \cup (0, 1/4)$, and S_H is the generating function of the Hamiltonian rotation:

$$S_H(z, q, q') = \frac{q^2 + {q'}^2}{2\tan(2\pi z)} - \frac{qq'}{\sin(2\pi z)}. \tag{4.3}$$

The formula (4.2) is essential when computing the Chiu-Tamarkin complexes for convex toric domains.

Let

$$\mathcal{T} \coloneqq \mathcal{S}^{\boxtimes d} = \mathrm{R}s^d_{t!}(\mathcal{S}^{\overset{L}{\boxtimes} d}) \in D(\mathbb{R}^d_z \times V_1 \times V_2 \times \mathbb{R}_t), \tag{4.4}$$

where $V_i = \mathbb{R}^d_{\mathbf{q}_i}$. The microsupport estimates show $\mathcal{T}$ is a sheaf quantization of the standard torus action in the sense of (2.8). As a corollary of Proposition 2.15, we have

Proposition 4.3. *A toric domains X_Ω is dynamically admissible by the distinguished triangle*

$$\widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{V^2} \times \{t \geq 0\}} \to \widehat{\mathcal{T}} \circ \mathbb{K}_{\mathbb{R}^d_\zeta \setminus \Omega} \xrightarrow{+1}, \tag{4.5}$$

and the pair of kernels

$$P_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_\Omega, \qquad Q_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_{\mathbb{R}^d_\zeta \setminus \Omega}. \tag{4.6}$$

This pair of microlocal kernels $(P_{X_\Omega}, Q_{X_\Omega})$ constructed from $\mathcal{T}$ is called the generating function models of the microlocal kernel associated to toric domains.

Actually, by the microsupport estimate of $\widehat{\mathcal{T}}$, see (2.9) for example. If $(\zeta, z, \mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t, \tau) \in \dot{SS}(\widehat{\mathcal{T}})$ then we have $\zeta = \mu(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^d_{\zeta \geq 0}$. So, if $\zeta \notin \mathbb{R}^d_{\zeta \geq 0}$ and $(\zeta, z, \mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t, \tau) \in SS(\widehat{\mathcal{T}})$, we have $(\mathbf{p}, \mathbf{p}', \tau) = 0$. Accordingly, for any $\zeta \notin \mathbb{R}^d_{\zeta \geq 0}$, we have $SS(\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')}) \subset \{\tau = 0\}$ by the microsupport estimate Theorem 1.4. So $\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')} \cong M_\mathbb{R}$ is a constant sheaf over $\mathbb{R}$ by Theorem 1.3 for some $M \in D(\mathbb{K}-\mathrm{Mod})$. But $\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')} \in \mathcal{D}(\mathrm{pt})$, and we have $\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')} \cong M_\mathbb{R} \cong M_\mathbb{R} \star \mathbb{K}_{[0,\infty)} \cong 0$. Therefore, we conclude that $\mathrm{supp}(\widehat{\mathcal{T}}) \subset \mathbb{R}^d_{\zeta \geq 0}$.

Consequently, the kernel P_{X_Ω} satisfies that

$$P_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \cong \mathrm{R}\pi_{\zeta !}(\widehat{\mathcal{T}} \overset{L}{\otimes} \mathbb{K}_{\Omega \times X^2 \times \mathbb{R}_t}) \cong \mathrm{R}\pi_{\zeta !}(\widehat{\mathcal{T}} \overset{L}{\otimes} \mathbb{K}_{(\Omega \cap \mathbb{R}^d_{\zeta \geq 0}) \times X^2 \times \mathbb{R}_t}), \tag{4.7}$$

which only depends on $\Omega \cap \mathbb{R}^d_{\zeta \geq 0}$. So, it is the same as Remark 4.2 that the notation P_{X_Ω} makes sense.

In general, it is complicated to compute the Fourier transform. However, with the help of associativity of composition and convolution (Example 1.12 (1)), we have

$$\widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \cong \mathcal{T} \star \widehat{\mathbb{K}_\Omega}. \tag{4.8}$$

When X_Ω is convex, we can take a suitable Ω, which is convex in the usual sense. Now, the Fourier transform $\widehat{\mathbb{K}_\Omega}$ is easy to compute. Actually, when X_Ω is convex, we have

$\widehat{\mathbb{K}_\Omega} \cong \mathbb{K}_{\Omega^\circ}$ by a similar argument with Proposition 2.20, where

$$\Omega^\circ = \{(z,t) : t + \langle z, \zeta\rangle \geq 0, \forall \zeta \in \Omega\}.$$

The assumption $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega$ shows $\Omega^\circ \subset \mathbb{R}^d_{z\leq 0} \times [0,\infty)$. Then we conclude that when X_Ω is a convex toric domain, we have

$$P_{X_\Omega} \cong \mathcal{T} \star \mathbb{K}_{\Omega^\circ}, \qquad F_\ell(X_\Omega, \mathbb{K}) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\mathcal{CL}_\ell(\mathcal{T}) \star \mathbb{K}_{\Omega^\circ}\right). \tag{4.9}$$

Example 4.4. Let $a = (a_1, \dots, a_d)$ be a non-decreasing sequence of positive real numbers.

(1) Suppose $\Omega_{D(a)} = \{\zeta : \zeta_i < a_i, i \in [d]\}$, then $X_{\Omega_{D(a)}} = D(a)$ is the open poly-disc. Let P_r be the kernel of the open disc $\{\pi|u|^2 < r\}$ in $\mathbb{C}$, then the Proposition 2.20 applies and $P_{D(a)} \cong P_{a_1} \boxtimes \cdots \boxtimes P_{a_d}$.

(2) Suppose $\Omega_{E(a)} = \{\zeta : \frac{\zeta_1}{a_1} + \cdots + \frac{\zeta_d}{a_d} < 1\}$, then $X_{\Omega_{E(a)}} = E(a)$ is the open ellipsoid, and $\Omega^\circ_{E(a)} = \{(z,t) : t \geq -a_1 z_1 = \cdots = -a_d z_d \geq 0\}$.

Let $i : \mathbb{R}_z \to \mathbb{R}^d_z$, $z \mapsto (a_1 z, \dots, a_d z)$, then $\mathbb{K}_{\Omega^\circ_{E(a)}} = \mathrm{R}(i \times \mathrm{Id}_\mathbb{R})_! \mathbb{K}_{\{t \geq -z \geq 0\}}$. Therefore, we have that

$$P_{E(a)} \cong \mathcal{T} \star \mathrm{R}(i \times \mathrm{Id}_\mathbb{R})_! \mathbb{K}_{\{t\geq -z\geq 0\}} \cong ((i \times \mathrm{Id}_\mathbb{R})^{-1}\mathcal{T}) \star \mathbb{K}_{\{t\geq -z\geq 0\}} \cong \widehat{(i \times \mathrm{Id}_\mathbb{R})^{-1}\mathcal{T}} \circ \mathbb{K}_{(-\infty,1)}.$$

Here we should be careful that, to obtain the second isomorphism, we need to use the explicit formula (4.4) and (4.2).

One can check directly that $(i \times \mathrm{Id}_\mathbb{R})^{-1}\mathcal{T}$ is the sheaf quantization of the diagonal Hamiltonian rotation $\varphi_z(u) = (e^{\frac{-2i\pi z}{a_1}} u_1, \dots, e^{\frac{-2i\pi z}{a_d}} u_d)$ in the sense of (2.8). In particular, when $a_1 = \cdots = a_d = \pi R^2 > 0$, the construction is the same as Chiu's for balls.

Remark 4.5. For the concave toric domain case, the Fourier transform $\widehat{\mathbb{K}_\Omega}$ is not as easy as the convex case (which is a complex only concentrated in degree 0). Actually, $\widehat{\mathbb{K}_\Omega}$ is represented by a complex of sheaves concentrated in cohomological degree $[0, d]$. Accordingly, the results in the next section cannot generalize directly to the concave situation.

For toric domains neither convex nor concave, the first example we can consider is an open annulus bounded by two concentric spheres. Then we can take $\Omega = \{x \in \mathbb{R}^d : a < \sum x_i < A\}$. In this case, when $T \geq 0$, we can only extract numerical information about the exterior sphere from $\widehat{\mathbb{K}_\Omega}$. Then we cannot know numerical information for the interior ball. Maybe it is a feature of the present definition of capacities, we expect more understanding to overcome this defect.

4.1.2. Chiu-Tamarkin complexes and Capacities of Convex Toric Domains. In this section, we focus on convex toric domains, i.e., $X_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set such that $\{(\zeta_1, \zeta_d) \in \mathbb{R}^d : (|\zeta_1|, \dots, |\zeta_d|) \in \Omega\}$ is convex. As we discussed in Remark 4.2, we could take a convex Ω such that $\mathbb{R}^d_{\zeta \leq 0} \subset \Omega$. Because of the identity (4.7), we see that such a choice of Ω does not affect the computation of Chiu-Tamarkin complex for X_Ω, we will see this feature again in the Remark 4.17.

One can verify that, under such conditions, the polar cone satisfies $\{O\} \times \mathbb{R}_{\geq 0} \subset \Omega^\circ \subset \mathbb{R}^d_{\leq 0} \times \mathbb{R}_{\geq 0}$, where $O \in \mathbb{R}^d$ is the origin. For $T \geq 0$, we set

$$\Omega^\circ_T \coloneqq \Omega^\circ \cap \{t = T\} = \{z \in \mathbb{R}^d : T + \langle z, \zeta \rangle \geq 0, \forall \zeta \in \Omega\}.$$

We also define the function $I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor$, $z \in \mathbb{R}^d$. For a subset $\Sigma \subset \mathbb{R}^d$, we define

$$\|\Sigma\|_\infty = \max_{z \in \Sigma} \|z\|_\infty \quad \text{and } I(\Sigma) = \max_{z \in \Sigma} I(z). \tag{4.10}$$

Then we have $\|\Omega^\circ_T\|_\infty = T\|\Omega^\circ_1\|_\infty$ for $T \geq 0$.

For $x, y \in \mathbb{R}^d$, the segment $\overline{xy}$ is defined as $\{tx + (1-t)y : t \in [0,1]\}$.

Here, we can state the structural theorem of Chiu-Tamarkin cohomology for a convex toric domain X_Ω. Since there are slight differences, I will repeat the $\mathbb{Z}/\ell$-version and the S^1-version:

THEOREM 4.6. *For a convex toric domain $X_\Omega \subsetneq T^*V$, and $\ell \in \mathbb{N}_{\geq 2}$*

(1) If $0 \leq T < p_\ell / \|\Omega^\circ_1\|_\infty$, we have

- *For each $Z \in \Omega_T^\circ$, the inclusion of the segment $\overline{OZ} \subset \Omega_T^\circ$ induces a decomposition of the fundamental class $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$ for a non-torsion element $\Lambda_{Z,\ell} \in H^{-2I(Z)}C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$. In particular, $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is non-zero.*
- *The minimal cohomology degree of $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is exactly $-2I(\Omega_T^\circ)$, i.e.,*

$$H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \cong H^{\geq -2I(\Omega_T^\circ)}C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}),$$

and

$$H^{-2I(\Omega_T^\circ)}C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \neq 0.$$

- *$H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is a finitely generated $\mathbb{F}_{p_\ell}[u]$-module. The free part is isomorphic to $A = \mathbb{F}_{p_\ell}[u, \theta]$, so $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is of rank 2 over $\mathbb{F}_{p_\ell}[u]$.*

The torsion part is given by $H^*C^+_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$, which is located in cohomology degree $[-2I(\Omega_T^\circ), -1]$. $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is torsion free when X_Ω is an open ellipsoid.

(2) If $T \geq 0$ and for any field $\mathbb{K} \supset \mathbb{Q}$, we have

- *For each $Z \in \Omega_T^\circ$, the inclusion of the segment $\overline{OZ} \subset \Omega_T^\circ$ induces a decomposition of the fundamental class $\eta_T^{S^1}(X_\Omega, \mathbb{K}) = u^{I(Z)}\Lambda_Z^{S^1}$ for a non-torsion element $\Lambda_Z^{S^1} \in H^{-2I(Z)}C_T^{S^1}(X_\Omega, \mathbb{K})$. In particular, $\eta_T^{S^1}(X_\Omega, \mathbb{K})$ is non-zero.*
- *The minimal cohomology degree of $H^*C_T^{S^1}(X_\Omega, \mathbb{K})$ is exactly $-2I(\Omega_T^\circ)$, i.e.,*

$$H^*C_T^{S^1}(X_\Omega, \mathbb{K}) \cong H^{\geq -2I(\Omega_T^\circ)}C_T^{S^1}(X_\Omega, \mathbb{K}),$$

and

$$H^{-2I(\Omega_T^\circ)}C_T^{S^1}(X_\Omega, \mathbb{K}) \neq 0.$$

- *$H^*C_T^{S^1}(X_\Omega, \mathbb{K})$ is a finitely generated $\mathbb{K}[u]$-module. The free part is isomorphic to $A = \mathbb{K}[u]$, so $H^*C_T^{S^1}(X_\Omega, \mathbb{K})$ is of rank 1 over $\mathbb{K}[u]$.*

The torsion part is given by $H^*C_T^{S^1,+}(X_\Omega, \mathbb{K})$, which is located in cohomology degree $[-2I(\Omega_T^\circ) + 1, -1]$. $H^*C_T^{S^1}(X_\Omega, \mathbb{K})$ is torsion free when X_Ω is an open ellipsoid.

Remark 4.7. (1) We will use the equivariant localization idea in the proof. The S^1 case is studied by Borel and Atiyah-Bott (See [**AB84, Bor16**]), while the $\mathbb{Z}/\ell$ case is studied by Quillen in a more general localization question about $\mathbb{Z}/\ell$-equivariant cohomology in [**Qui71**]. Quillen considered a localization with respect to $x = f(u, \theta) \in A$ for a polynomial f, which the generator θ appears. The result of Quillen shows again that, modulo torsion, $H^*C_{T,\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ is isomorphic to A as an A-module, which improves parts our result. However his result cannot provide us the information about minimal cohomology degree, that is why we provide this proof, which is essentially independent of the result of Quillen.

(2) We will see later in the proof that our computation is reduced to compute the equivariant homology of a S^1-space, then [**Vit97**, Appendix] applies and $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \cong H^*C_T^{S^1}(X_\Omega, \mathbb{Z}) \otimes \mathbb{F}_{p_\ell}[\theta]$ since $H^*C_T^{S^1}(X_\Omega, \mathbb{Z})$ is torsion-free as an abelian group if $0 \leq T < p_\ell / \|\Omega_1^\circ\|_\infty$. Notice here, the definition of $H^*C_T^{S^1}(X_\Omega, \mathbb{Z})$ should be careful due to the Remark 3.24.

Before proving Theorem 4.6, let us use it to compute the capacities $c_k(X_\Omega)$ and $\overline{c}_k(X_\Omega)$.

THEOREM 4.8. *For a convex toric domain $X_\Omega \subsetneq T^*V$, we have*

$$c_k(X_\Omega) = \overline{c}_k(X_\Omega) = \inf \{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\}.$$

PROOF. We only prove the c_k-case, the $\overline{c}_k$ is similar (actually easier). Let $S = \{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\}$, $L = \inf(S)$.

For $T \in S$, there is $Z \in \Omega_T^\circ$ such that $I(Z) = k$. Consider the closed inclusion of the segment $\overline{OZ} \subset \Omega_T^\circ$. We choose a prime p with $p > T\|\Omega_1^\circ\|_\infty$. Then for all $\ell \in \mathbb{N}_{\geq 2}$ with $p_\ell \geq p$, we have $p_\ell > T\|\Omega_1^\circ\|_\infty$, and the Theorem 4.6 shows that the closed inclusion induces a decomposition $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^k \Lambda_{Z,\ell}$. So $T \in \mathrm{Spec}(X_\Omega, k)$, and $L \geq c_k(X_\Omega)$.

Conversely, if $T \in \mathrm{Spec}(X_\Omega, k)$, there is a prime p such that for all $\ell \in \mathbb{N}_{\geq 2}$ with $p_\ell \geq p$ there is a $\Lambda_\ell \in H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ such that $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^k \Lambda_\ell$. Now, we can take a prime $\ell = p_\ell > p$ big enough such that $T < \ell/\|\Omega_1^\circ\|_\infty$, then $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$, and Λ_ℓ are non-zero. Hence we have an equation of degree: $0 = |\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})| = 2k + |\Lambda_\ell|$, which shows that $2k = -|\Lambda_\ell|$. Therefore, the Theorem 4.6 shows $2k = -|\Lambda_\ell| \leq 2I(\Omega_T^\circ)$. Hence $T \in S$, and $c_k(X_\Omega) \geq L$. $\square$

Here, let us test the result by the example of ellipsoids. They are all direct corollaries of the structure Theorem 4.6 and the computation Theorem 4.8.

Corollary 4.9. *Let $X_\Omega = E = E(a_1, \dots, a_d)$ be an ellipsoid and $\ell \in \mathbb{N}_{\geq 2}$. For $0 \leq T < p_\ell a_1$, set $Z(a) = (-T/a_1, \dots, -T/a_d)$. We have $H^*C_{\ell,T}(E, \mathbb{F}_{p_\ell}) \cong u^{-I(Z(a))}\mathbb{F}_{p_\ell}[u, \theta]$ and $H^*C_T^{S^1}(E, \mathbb{K}) \cong u^{-I(Z(a))}\mathbb{K}[u]$, the fundamental class is non-zero in all cases, and $c_k(E) = \bar{c}_k(E) = \min\{T \geq 0 : \sum_{i=1}^d \lfloor T/a_i \rfloor \geq k\}$. In particular, $c_k(B_a) = \bar{c}_k(B_a) = \lceil k/d \rceil a$.*

4.1.3. Cohomology sheaf $\mathcal{CL}_\ell(\mathcal{T})$ for the standard torus action. Recall the results of Subsection 4.1.1, and discussions in Section 3.4. It is necessary to study the cohomology sheaf $\mathcal{CL}_\ell(\mathcal{T})$ carefully. Recall that $\mathcal{T} = \mathcal{S}^{\boxtimes d} = \mathrm{R}s^d_{t!}(\mathcal{S}^{\overset{L}{\boxtimes} d})$, where $\mathcal{S}$ is the sheaf quantization of Hamiltonian rotation in dimension 2. Using the Künneth formula and the Proposition 3.26, we have

$$\begin{aligned}
\mathcal{CL}_\ell(\mathcal{T}) &\cong \mathrm{R}s^\ell_{\underline{z}*}\left(\left((s^\ell_{z_j})^{-1}\mathcal{CL}_\ell(\mathcal{S})\right)^{\boxtimes d}\right) \\
&\cong \mathrm{R}s^\ell_{\underline{z}*}\left((s^\ell_{\underline{z}})^{-1}\left(\mathcal{CL}_\ell(\mathcal{S})\right)^{\boxtimes d}\right) \\
&\cong \mathrm{R}s^\ell_{\underline{z}*}(s^\ell_{\underline{z}})^{-1}\left(\left(\mathcal{CL}_\ell(\mathcal{S})\right)^{\boxtimes d}\right) \\
&\cong \mathrm{R}s^d_{t!}\left(\mathcal{CL}_\ell(\mathcal{S})\right)^{\boxtimes d},
\end{aligned}$$

where $\underline{z} = (z_1, \dots, z_d)$. Moreover, an explicit formula for $\mathcal{CL}_\ell(\mathcal{S})$ is obtained by Chiu:

Proposition 4.10. ([Chi17, Formula (38)]) *For all fields $\mathbb{K}$, there is a (unique) sheaf $\mathcal{E}_\ell \in D_{\mathbb{Z}/\ell}(\mathbb{R}_z)$ such that we have an isomorphism in $D_{\mathbb{Z}/\ell}(\mathbb{R}_z \times \mathbb{R}_t)$*

$$\mathcal{CL}_\ell(\mathcal{S}) \cong \mathcal{E}_\ell \overset{L}{\boxtimes} \mathbb{K}_{[0,\infty)}. \tag{4.11}$$

Moreover, for any $N \in \mathbb{N}$,

$$\mathcal{E}_\ell|_{(-N\ell/4,0)} \cong \mathrm{R}\pi_{\underline{q}!}\mathbb{K}_{\mathcal{W}^N_\ell}, \tag{4.12}$$

with $\underline{q} = (q_1, \dots, q_{N\ell})$ and

$$\mathcal{W}^N_\ell = \{(z, q_1, \dots, q_{N\ell}) \in (-N\ell/4, 0) \times \mathbb{R}^{N\ell} : \sum_{k \in \mathbb{Z}/N\ell} S_H(z/N\ell, q_k, q_{k+1}) \geq 0\},$$

and

$$S_H(z, q_k, q_{k+1}) = \frac{q_k^2 + q_{k+1}^2}{2\tan(2\pi z)} - \frac{q_k q_{k+1}}{\sin(2\pi z)}.$$

The $\mathbb{Z}/\ell$-action on $\mathcal{E}_\ell$ is induced by the linear action $(q_k) \mapsto (q_{k-N})$ of $\mathbb{Z}/\ell$ on $\mathbb{R}^{N\ell}$, and $\mathbb{Z}/\ell$ acts trivially on $\mathbb{R}_z \times \mathbb{R}_t$.

A disadvantage for the formula (4.12) is that we don't know if the isomorphism can be extended to $z = 0$ since the right hand side is not defined for $z = 0$. But such an extension is necessary for our later computation. So, let us start from an extension of the isomorphism (4.12) to $z = 0$. Notice that $z/N\ell \in (-1/4, 0)$, so $\sin(2\pi z/N\ell) < 0$. One can rewrite $\mathcal{W}_\ell^N$ as follow:

$$\mathcal{W}_\ell^N = \left\{ (z, q_1, \ldots, q_{N\ell}) \in (-N\ell/4, 0) \times \mathbb{R}^{N\ell} : \cos(2\pi z/N\ell) \sum_{k \in \mathbb{Z}/N\ell} q_k^2 \le \sum_{k \in \mathbb{Z}/N\ell} q_k q_{k+1} \right\}.$$

Let us define

$$Q(z, q_1, \ldots, q_{N\ell}) \coloneqq \sum_{k \in \mathbb{Z}/N\ell\mathbb{Z}} \left(q_k q_{k+1} - \cos(2\pi z/N\ell) q_k^2 \right). \tag{4.13}$$

Since $Q(0, q_1, \ldots, q_{N\ell})$ is well defined, we can extend the definition of $\mathcal{W}_\ell^N$ (using the same notation) to

$$\mathcal{W}_\ell^N = \{(z, q_1, \ldots, q_{N\ell}) \in (-N\ell/4, 0] \times \mathbb{R}^{N\ell} : Q(z, q_1, \ldots, q_{N\ell}) \ge 0\}. \tag{4.14}$$

For our convenience, we also set, for $z \in (-N\ell/4, 0]$,

$$\mathcal{W}_\ell^N(z) = \{(q_1, \ldots, q_{N\ell}) \in \mathbb{R}^{N\ell} : Q(z, q_1, \ldots, q_{N\ell}) \ge 0\}. \tag{4.15}$$

The $\mathbb{Z}/\ell$-action on the extension is the same as the original one.

One can check that $\mathcal{CL}_\ell(\mathcal{S})|_{\{z=0\}} = \mathrm{R}\Gamma_c(\Delta_{\mathbb{R}^\ell}, \mathbb{K}_{\Delta_{\mathbb{R}^\ell}}) \overset{L}{\boxtimes} \mathbb{K}_{\{t \ge 0\}}$ since $\mathcal{S}|_{\{z=0\}} = \mathbb{K}_{\Delta_{\mathbb{R}^2}} \overset{L}{\boxtimes} \mathbb{K}_{\{t \ge 0\}}$. On the other hand, by the fundamental inequality, we have $\sum_k q_k^2 \ge \sum_k q_k q_{k+1}$, and it takes equality when $q_1 = \cdots = q_{N\ell}$. So

$$\mathcal{W}_\ell^N(0) = \{(q_1, \ldots, q_{N\ell}) \in \mathbb{R}^{N\ell} : q_1 = \cdots = q_{N\ell}\} = \Delta_{\mathbb{R}^{N\ell}}.$$

So, $\mathcal{CL}_\ell(\mathcal{S})|_{\{z=0\}} = \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(0), \mathbb{K}_{\mathcal{W}_\ell^N(0)}) \overset{L}{\boxtimes} \mathbb{K}_{\{t \ge 0\}}$.

Therefore, it is make sense to consider $\mathcal{E}'_\ell := \mathrm{R}\pi_{\underline{q}!} i_! \mathbb{K}_{\mathcal{W}_\ell^N} \in D_{\mathbb{Z}/\ell}((-N\ell/4, +\infty))$ where $i : (-N\ell/4, 0] \times \mathbb{R}^{N\ell} \subset (-N\ell/4, +\infty) \times \mathbb{R}^{N\ell}$ is the closed inclusion.

Lemma 4.11. *We have an equivariant isomorphism*

$$\mathcal{E}_\ell|_{(-N\ell/4,0]} \cong \mathcal{E}'_\ell|_{(-N\ell/4,0]} = \mathrm{R}\pi_{\underline{q}!}\mathbb{K}_{\mathcal{W}_\ell^N}.$$

Proof. First, since $\mathcal{W}_\ell^N$ is a closed set defined by a smooth function, the Example 1.2 (2) shows that $SS(i_!\mathbb{K}_{\mathcal{W}_\ell^N}) \subset \{\zeta \leq 0\}$. So the non-proper pushforward estimate Theorem 1.7 shows that $SS(\mathcal{E}'_\ell) \subset \{\zeta \leq 0\}$, and $SS((\mathcal{E}'_\ell)_{(-N\ell/4,0]}) \subset \{\zeta \leq 0\}$. On the other hand, we have $SS(\mathcal{CL}_\ell(\mathcal{S})) \subset \{\zeta \leq 0\}$. Hence $SS(\mathcal{E}_\ell) = SS(\mathcal{CL}_\ell(\mathcal{S})|_{t=0}) \subset \{\zeta \leq 0\}$ and then we also have $SS((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \subset \{\zeta \leq 0\}$.

Now, let j be the inclusion $(-N\ell/4, 0) \subset (-N\ell/4, \infty)$, consider the distinguished triangle

$$\mathrm{R}\Gamma_{\{z\geq 0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \to (\mathcal{E}_\ell)_{(-N\ell/4,0]} \to \mathrm{R}\Gamma_{(-N\ell/4,0)}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \xrightarrow{+1}.$$

We would like to show that $\mathrm{R}\Gamma_{\{z\geq 0\}}(\mathcal{E}_\ell)_{(-N\ell/4,0]} \cong 0$. In fact, by definition, we have $\mathrm{supp}(\mathrm{R}\Gamma_{\{z\geq 0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]})) \subset \{0\}$. So we only need to show that $(\mathrm{R}\Gamma_{\{z\geq 0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]}))_0 \cong 0$.

But on $(-N\ell/4, +\infty)$, the closed set $\{z \geq 0\}$ is defined by the function $f(z) = z$ and $\{f(z) \geq 0\}$. Therefore, by definition of microsupport, $(\mathrm{R}\Gamma_{\{z\geq 0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]}))_0 \cong 0$ if $df_0 = (0, 1) \notin SS((\mathcal{E}_\ell)_{(-N\ell/4,0]})$. This is true due to the microsupport estimate $SS((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \subset \{\zeta \leq 0\}$.

Consequently, we have an isomorphism $(\mathcal{E}_\ell)_{(-N\ell/4,0]} \cong \mathrm{R}\Gamma_{\{z<0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]})$. This isomorphism holds in the equivariant category since the corresponding morphism is an equivariant morphism.

Since the argument is purely microlocal, we can also conclude that $(\mathcal{E}'_\ell)_{(-N\ell/4,0]} \cong \mathrm{R}\Gamma_{\{z<0\}}((\mathcal{E}'_\ell)_{(-N\ell/4,0]})$.

On the other hand, the isomorphism (4.12) and our discussion on $\mathcal{W}_\ell^N$ show that $j^{-1}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \cong j^{-1}((\mathcal{E}'_\ell)_{(-N\ell/4,0]})$.

Therefore, the natural isomorphism $\mathrm{R}j_*j^{-1} \cong \mathrm{R}\Gamma_{\{z<0\}}$ shows that

$$(\mathcal{E}_\ell)_{(-N\ell/4,0]} \cong \mathrm{R}j_*j^{-1}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \cong \mathrm{R}j_*j^{-1}((\mathcal{E}'_\ell)_{(-N\ell/4,0]}) \cong (\mathcal{E}'_\ell)_{(-N\ell/4,0]} \,.$$

Finally, we conclude by restricting the isomorphism to $(-N\ell/4, 0]$ and the definition of $\mathcal{E}'_\ell$. □

Topology of $\mathcal{W}_\ell^N(z)$: By the Lemma 4.11, we know that $(\mathcal{E}_\ell)_z \cong \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(z), \mathbb{K})$ if $-N\ell/4 < z \leq 0$. So it is necessary to discuss the topology of $\mathcal{W}_\ell^N(z)$. For a fixed $z \in (-N\ell/4, 0]$, the function $Q_z(q_1, \ldots, q_{N\ell}) = Q(z, q_1, \ldots, q_{N\ell})$ is a quadratic form by (4.13). Therefore, it is easy to study the topology of $\mathcal{W}_\ell^N(z) = \{(q_1, \ldots, q_{N\ell}) \in \mathbb{R}^{N\ell} : Q_z \geq 0\}$. The matrix of Q_z is a circulant matrix

$$A_z = \begin{pmatrix} -\cos(\frac{2\pi z}{N\ell}) & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \frac{1}{2} & -\cos(\frac{2\pi z}{N\ell}) & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & -\cos(\frac{2\pi z}{N\ell}) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{2} & 0 & 0 & \cdots & -\cos(\frac{2\pi z}{N\ell}) \end{pmatrix}.$$

So one can diagonalize A_z unitarily using the discrete Fourier transform

$$(\omega^{(i-1)(j-1)})_{i,j=\mathbb{Z}/N\ell},$$

where ω is a primitive $N\ell^{th}$ root of unity. Therefore, the eigenvalues of A_z are

$$\lambda_k(z) = \mathrm{Re}\left(\exp\left(\frac{2\pi k\sqrt{-1}}{N\ell}\right)\right) - \cos\left(\frac{2\pi z}{N\ell}\right) = \cos\left(\frac{2\pi k}{N\ell}\right) - \cos\left(\frac{2\pi z}{N\ell}\right), \tag{4.16}$$

where $k \in \mathbb{Z}/N\ell$.

We always have $\lambda_0(z) = 1 - \cos\left(\frac{2\pi z}{N\ell}\right) \geq 0$. It is direct to see that $\lambda_k(z) = \lambda_{N\ell-k}(z)$ for $k = 1, \ldots, N\ell - 1$. So, for $k \geq 1$, we need to consider two situations:

If $N\ell$ is odd. For $k = 1, \ldots, (N\ell-1)/2$, $\lambda_k(z) \geq 0$ if $k \leq \lfloor -z \rfloor$. Therefore, in this case, A_z admits $\#\{k \in \mathbb{Z}/N\ell : \lambda_k \geq 0\} = 1 + 2\lfloor -z \rfloor$ non-negative eigenvalues.

If $N\ell$ is even. The eigenvalue $\lambda_{N\ell/2}(z) = -1 - \cos\left(\frac{2\pi z}{N\ell}\right) < 0$ since $z > -N\ell/4$. For $k = 1, \ldots, (N\ell/2) - 1$, $\lambda_k(z) \geq 0$ if $k \leq \lfloor -z \rfloor$. Therefore, in this case, A_z also admits $\#\{k \in \mathbb{Z}/N\ell : \lambda_k \geq 0\} = 1 + 2\lfloor -z \rfloor$ non-negative eigenvalues.

In any case, we have that A_z admits $\#\{k \in \mathbb{Z}/N\ell : \lambda_k \geq 0\} = 1 + 2\lfloor -z \rfloor$ non-negative eigenvalues.

Therefore, $\mathcal{W}_\ell^N(z) = \{Q_z \geq 0\}$ is a quadratic cone of index $1 + 2\lfloor -z \rfloor$. In particular, $\mathcal{W}_\ell^N(z) = \{Q_z \geq 0\}$ is properly homotopic to a vector space $\mathbb{R}^{1+2\lfloor -z \rfloor}$.

Now we can describe the non-equivariant structure of $\mathcal{E}_\ell|_{(-\infty,0]}$. Here, we forget its equivariant structure and use the same notation $\mathcal{E}_\ell|_{(-\infty,0]}$. In particular, $\mathcal{E}_\ell|_{(-\infty,0]} \cong \mathcal{E}_1|_{(-\infty,0]}$ non-equivariantly. Consider $\pi_{\underline{q}} : \mathcal{W}_\ell^N \to (-N\ell/4, 0]$ for $N\ell$ big enough, it restricts to a proper homotopical fiber bundle with fiber $\mathbb{R}^{1+2n}$ over each interval $(-n-1, -n]$ for $n \in \mathbb{N}_{\geq 0}$, and $n+1 < N\ell/4$. Therefore, we conclude that $\mathcal{E}_\ell|_{(-n-1,-n]} \cong \mathbb{K}_{(-n-1,-n]}[-1-2n]$. On the other hand, in the non-equivariant derived category, $\mathbb{K}_{(x,y]}$ and $\mathbb{K}_{(z,w]}[2]$ has no non-trivial extension if $\mathbb{K}$ is a field. Therefore, $(\mathcal{E}_\ell)_{(-n-1,-n]}$ has no non-trivial extension for different n. In conclusion, we have

Proposition 4.12. *For all fields $\mathbb{K}$ and for all $\ell \in \mathbb{N}$, we have the decomposition in the non-equivariant derived category $D((-\infty, 0])$:*

$$\mathcal{E}_\ell|_{(-\infty,0]} \cong \bigoplus_{n \in \mathbb{N}_{\geq 0}} \mathbb{K}_{(-n-1,-n]}[-1-2n].$$

To describe the equivariant information, more specifically, the $\mathbb{Z}/\ell$-action on $\mathcal{W}_\ell^N(z)$, it is better to consider the diagonal form of Q_z.

Let $x_k = (q_1, \ldots, q_{N\ell})(1, \omega^k, \omega^{2k}, \ldots, \omega^{(N\ell-1)k})^t \in \mathbb{C}$, $k \in \mathbb{Z}/N\ell$. They are coordinates after diagonalization using the discrete Fourier transform. As ω is a root of unity, we have that $x_k = \overline{x_{N\ell-k}}$. In particular, x_0 is a real number. Also recall that $\lambda_k(z) = \lambda_{N\ell-k}(z)$. Then the diagonal form of Q_z is

$$\begin{aligned} &Q_z(x_0, x_1, \ldots, x_{N\ell-1}) = \lambda_0(z)x_0^2 + \sum_{k=1}^{N\ell-1} \lambda_k(z)|x_k|^2, \\ &(x_0, x_1, \ldots, x_{N\ell-1}) \in \mathbb{R} \times \mathbb{C}^{N\ell-1}. \end{aligned} \tag{4.17}$$

Notice that the discrete Fourier transform that we applied is a complex linear transform, it is easier to work in complex coordinates. However, the constrains $x_k = \overline{x_{N\ell-k}}$ shows that actually we only have half independent complex coordinates, so the real dimension here is correct. But to our convenience in formulating the action, we still use the complex coordinates. We also need to discuss parity of $N\ell$. Since N is chosen arbitrarily, we can always assume N is odd. Then the parity of $N\ell$ is the parity of ℓ.

If ℓ is odd, then the diagonal form is

$$
\begin{aligned}
&Q_z(x_0, x_1, \ldots, x_{(N\ell-1)/2}) = \lambda_0(z)x_0^2 + 2\sum_{k=1}^{(N\ell-1)/2} \lambda_k(z)|x_k|^2, \\
&(x_0, x_1, \ldots, x_{(N\ell-1)/2}) \in \mathbb{R} \times \mathbb{C}^{(N\ell-1)/2} \cong \mathbb{R}^{N\ell}.
\end{aligned} \tag{4.18}
$$

If ℓ is even, then the diagonal form is

(4.19)

$$
\begin{aligned}
&Q_z(x_0, x_1, \ldots, x_{N\ell/2-1}, x_{N\ell/2}) = \lambda_0(z)x_0^2 + 2\sum_{k=1}^{N\ell/2-1} \lambda_k(z)|x_k|^2 + \lambda_{N\ell/2}(z)|x_{N\ell/2}|^2, \\
&(x_0, x_1, \ldots, x_{N\ell/2-1}, x_{N\ell/2}) \in \mathbb{R} \times \mathbb{C}^{N\ell/2-1} \times \mathbb{R} \cong \mathbb{R}^{N\ell}.
\end{aligned}
$$

Now, the action is easier to describe under the diagonal form. By definition of x_k, we have $x_k = \sum_{i\in\mathbb{Z}/N\ell} q_{i+1}\omega^{ik}$. The $\mathbb{Z}/\ell$-action is given by $(q_i) \mapsto (q_{i-N})$. Then we have

$$
x_k = \sum_{i\in\mathbb{Z}/N\ell} q_{i+1}\omega^{ik} \mapsto \sum_{i\in\mathbb{Z}/N\ell} q_{i+1-N}\omega^{ik} = \omega^{kN} \sum_{i\in\mathbb{Z}/N\ell} q_{i+1-N}\omega^{(i-N)k} = \omega^{kN}x_k.
$$

Therefore, the $\mathbb{Z}/\ell$-action on the diagonal form is as follows: if we take $\mu = \omega^N$ a primitive ℓ^{th} root of unity, then

$$
\mu \cdot (x_k) = (\mu^k x_k), \tag{4.20}
$$

where $k = 0, 1, \ldots, N\ell/2 - 1$ if ℓ is odd and $k = 0, 1, \ldots, N\ell/2$ if ℓ is even.

Consequently, the fixed point sets $\left(\mathcal{W}_\ell^N(z)\right)^{\mathbb{Z}/\ell}$ is again a quadratic cone, whose index is $1+2\lfloor -z/\ell \rfloor$. The diagonal $\Delta_{\mathbb{R}^{N\ell}}$ is given by $\{(x_0, 0, \ldots, 0) : x_0 \in \mathbb{R}\}$ under the diagonal form, it is a subset of $\left(\mathcal{W}_\ell^N(z)\right)^{\mathbb{Z}/\ell}$.

Under the diagonal form, we can also find the S^1-action, which is, for $\mu \in S^1$,

$$
\mu \cdot (x_k) = (\mu^k x_k), \tag{4.21}
$$

where $k = 0, 1, \ldots N\ell/2 - 1$ if ℓ is odd and $k = 0, 1, \ldots N\ell/2$ if ℓ is even.

Remark 4.13. We need to notice that when ℓ is even, the S^1-action formula is not valid since $x_{N\ell/2}$ is a real number but $\mu^{N\ell/2}x_{N\ell/2}$ is not necessarily real. However, this does not affect our eventual results because we know from the formula (4.16) that $\lambda_{N\ell/2}(z)$ is always negative. Then we can just, harmlessly, think of $x_{N\ell/2}$ as a complex variable, and extend the definition of Q_z and $\mathcal{W}_\ell^N(z)$ such that they are defined over $\mathbb{R} \times \mathbb{C}^{N\ell/2-1} \times \mathbb{C}$. This extension does not change the S^1-homotopy type of $\mathcal{W}_\ell^N(z)$.

Now the S^1 action is locally free and the fixed points set is just the diagonal $\{(x_0, 0, \ldots, 0) : x_0 \in \mathbb{R}\}$. In particular, the $\mathbb{Z}/\ell$-action is the restriction of the S^1-action.

Finally, we go back to the isomorphism of Proposition 4.12. Take $z' \le z \le 0$. As $SS(\mathcal{E}_\ell) \subset \{\zeta \le 0\}$, there is a natural map

$$(\mathcal{E}_\ell)_{z'} \cong \mathrm{R}\Gamma(\mathbb{R}, (\mathcal{E}_\ell)_{[z',0]}) \to \mathrm{R}\Gamma(\mathbb{R}, (\mathcal{E}_\ell)_{[z,0]}) \cong (\mathcal{E}_\ell)_z,$$

which is induced by

$$\mathrm{R}\Gamma_c(\mathbb{R}^{1+2\lfloor -z' \rfloor}, \mathbb{K}) \to \mathrm{R}\Gamma_c(\mathbb{R}^{1+2\lfloor -z \rfloor}, \mathbb{K}).$$

The decomposition, Proposition 4.12, tells us that the natural morphism is 0 in the non-equivariant category.

In the equivariant category, the action of $\mathbb{Z}/\ell$ on $\mathbb{R}^{1+2\lfloor -z \rfloor}$ is the restriction of a S^1-action. The S^1 acts trivially on $\mathbb{R}$, and acts on $\mathbb{R}^{2\lfloor -z \rfloor} \cong \mathbb{C}^{\lfloor -z \rfloor}$ via the weight $(1, 2, \ldots, \lfloor -z \rfloor)$. So, in the S^1-equivariant derived category, the morphism is given by the mod $\mathbb{K}$ reduction of the top Chern class for the vector bundle

$$\mathbb{R}^{1+2\lfloor -z' \rfloor} \times_{S^1} S^\infty \to \mathbb{R}^{1+2\lfloor -z \rfloor} \times_{S^1} S^\infty.$$

Therefore, the decomposition Proposition 4.12 is not necessarily a direct sum decomposition. But we can describe the extension class between two blocks.

For $\mathbb{K} = \mathbb{Z}$, the morphism is given by $(\lfloor -z' \rfloor!/\lfloor -z \rfloor!)u^{\lfloor -z' \rfloor - \lfloor -z \rfloor}$, which is non-zero. After restricting to the $\mathbb{Z}/\ell$-equivariant derived category, it lifts to the vector bundle

$$\mathbb{R}^{1+2\lfloor -z' \rfloor} \times_{\mathbb{Z}/\ell} S^\infty \to \mathbb{R}^{1+2\lfloor -z \rfloor} \times_{\mathbb{Z}/\ell} S^\infty.$$

So after suitable reduction in a finite field $\mathbb{K}$, for example if we require $\lfloor -z' \rfloor < \mathrm{char}(\mathbb{K})$, the morphism is non-zero as well.

The higher dimension ($d \geq 2$) case: Now, we start to discuss the higher dimension situation. We already know that $\mathcal{CL}_\ell(\mathcal{T}) \cong \mathcal{CL}_\ell(\mathcal{S})^{\boxtimes d}$. Then the Proposition 4.10 shows that

$$\mathcal{CL}_\ell(\mathcal{T}) \cong \mathcal{E}_\ell^{\overset{L}{\boxtimes} d} \overset{L}{\boxtimes} \mathbb{K}_{\{t \geq 0\}}. \tag{4.22}$$

As the decomposition indicated in Proposition 4.12, $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{\{z \leq 0\}}$ has a decomposition on $\{z \leq 0\}$ indexed by lattice points. Besides, we also have a topological description of $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{\{z \leq 0\}}$. Let us discuss the topological description first and then state the decomposition. As we have d copies of $\mathcal{E}_\ell$, it is convenient to denote $\mathbf{q} = (q^1, \ldots, q^d) \in \mathbb{R}^d =: V_{\mathbf{q}}$. Then the Lemma 4.11 shows us

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}\Big|_{(-N\ell/4, 0]^d} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\prod_{i=1} \mathcal{W}_{\ell,i}^N}, \tag{4.23}$$

where $\mathcal{W}_{\ell,i}^N$ means the ith copy of one $\mathcal{W}_\ell^N$, $i \in [d] = \{1, \ldots, d\}$ and $\underline{\mathbf{q}} = (\mathbf{q}_1, \ldots, \mathbf{q}_{N\ell})$. Also, let us define

$$\begin{aligned}
{}^d\mathcal{W}_\ell^N &:= \prod_{i=1}^d \mathcal{W}_{\ell,i}^N = \{(z, \mathbf{q}_1, \ldots, \mathbf{q}_{N\ell}) \in (-N\ell/4, 0]^d \times V^{N\ell} : Q_{z_i}(q_k^i) \geq 0,\, i \in [d]\},\\
{}^d\mathcal{W}_\ell^N(z) &:= \prod_{i=1}^d \mathcal{W}_{\ell,i}^N(z_i) = \{(\mathbf{q}_1, \ldots, \mathbf{q}_{N\ell}) \in V^{N\ell} : Q_{z_i}(q_k^i) \geq 0,\, i \in [d]\}.
\end{aligned}$$

Here $\mathbf{q}_k = (q_k^1, \ldots, q_k^d)$, $z = (z_1, \ldots, z_d)$. The group $\mathbb{Z}/\ell$ acts on each $\mathcal{W}_\ell^N$ via $(q_k^i) \mapsto (q_{k-N}^i)$. Therefore, $\mathbb{Z}/\ell$ acts diagonally on ${}^d\mathcal{W}_\ell^N$ via $(\mathbf{q}_k) \mapsto (\mathbf{q}_{k-N})$.

The diagonalization applies for each $i \in [d]$, and then on ${}^d\mathcal{W}_\ell^N(z)$. We set $\mathbf{x}_k = (x_k^1, \ldots, x_k^d)$ and $\mathbf{x}_k^i = (x_1^i, \ldots, x_{N\ell}^i)$, then the coordinates of ${}^d\mathcal{W}_\ell^N(z)$ after diagonalization are $(x_k^i) = (\mathbf{x}_k) = (\mathbf{x}^i)$, where $k = 0, 1, \ldots, (N\ell - 1)/2$ if ℓ is odd and $k = 0, 1, \ldots, N\ell/2$ if ℓ is even.

So for each $z = (z_1, \ldots, z_d) \in (-N\ell/4, 0]^d$, ${}^d\mathcal{W}_\ell^N(z)$ is a product of quadratic cones of indices $1 + 2\lfloor -z_i \rfloor$ respectively, and then ${}^d\mathcal{W}_\ell^N(z)$ is properly homotopic to a quadratic cone of index $d + 2I(z)$, where $I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor$. Therefore, ${}^d\mathcal{W}_\ell^N(z)$ is properly homotopic to $\mathbb{R}^{d+2I(z)}$, a refinement of this fact is proven in Lemma 4.19.

The fixed point sets $\left({}^d\mathcal{W}_\ell^N(z)\right)^{\mathbb{Z}/\ell}$ is also properly homotopic to a quadratic cone of index $d+2I(z/\ell)$. The diagonal $\Delta_{V^{N\ell}}$ is given by $\{(x_k^i) : \forall i,\, \forall k \neq 0,\, x_k^i = 0,\, x_0^i \in \mathbb{R}\}$ under the diagonal form, it is a subset of $\left({}^d\mathcal{W}_\ell^N(z)\right)^{\mathbb{Z}/\ell}$.

To be clear, let us set some higher dimensional interval notation. For $x, y \in \mathbb{R}^d$, we let $(x,y] = \prod_{i=1}^d (x_i, y_i]$ be the half-open cube from x to y. Similarly, we can define half-open cubes $[x,y)$, open cubes (x,y), and closed cubes $[x,y]$ in the same way. Recall that we use $\overline{xy}$ to denote the segment between x, y.

Also, recall $O \in \mathbb{R}^d$ is the origin, and we set $\mathbb{1} = (1, \dots, 1)$ and $e_i = (\delta_{ij})_{j=1}^d$ where δ_{ij} stand for the Kronecker symbol.

Then either our topology description of ${}^d\mathcal{W}_\ell^N$ or the decomposition result Proposition 4.12 shows that

Lemma 4.14. *For each $z \leq 0$, we have the equivariant isomorphism*

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_z \cong \mathrm{R}\Gamma_c(\mathbb{R}^{d+2I(z)}, \mathbb{K}) \cong \mathbb{K}[-d-2I(z)].$$

In the non-equivariant derived category, we have a decomposition as follows:

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{\{z \leq 0\}} \cong \bigoplus_{v \in \mathbb{N}_0^d} \mathbb{K}_{(-v-\mathbb{1}, -v]}[-d-2I(-v)].$$

In the equivariant derived category, for $z', z \in (-\infty, 0]^d$, if $z'_i \leq z_i$ for all $i \in [d]$, the natural morphism,

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{z'} \cong \mathbb{K}[-d-2I(z')] \to \mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_z \cong \mathbb{K}[-d-2I(z)],$$

is the mod-$\mathbb{K}$ reduction of the top Chern class of the vector bundle

$$\mathbb{R}^{d+2I(z')} \times_{S^1} ES^\infty \to \mathbb{R}^{d+2I(z)} \times_{S^1} S^\infty,$$

where S^1 acts on $\mathbb{R}^d$ trivially, and acts on $\mathbb{R}^{2I(z)}$ by the weight $((1, \dots, \lfloor -z_i \rfloor))_{i \in [d]}$.

Propagation and γ-topology Finally, let us describe a propagation phenomena of $\mathcal{E}_\ell$. It is simple but crucial for our later application. Notice that, for a given $z \in (-N\ell/4, 0]$, the map $z \mapsto \mathcal{W}_\ell^N(z)$ is a decreasing map with respect to the inclusion order. Microlocally, it means that $SS(\mathcal{E}_\ell) \subset \{\zeta \leq 0\}$, which is already known to us as a general fact from the microsupport estimate (by Corollary 1.6 for example). We have, for $z \leq 0$, that

$$(\mathcal{E}_\ell)_z \cong \mathrm{R}\Gamma_c(\mathbb{R}, (\mathcal{E}_\ell)_{[z,0]}) \cong \mathbb{K}[-1 - 2\lfloor -z \rfloor].$$

In higher dimension, the same thing still happens since we have $SS(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}) \subset \{\zeta_i \leq 0, i \in [d]\}$. Now, for $z \in (-\infty, 0]^d$, since $[z, O] = (\{z\} + [0, \infty)^d) \cap (-\infty, 0]^d$, we have

$$(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_z \cong \mathrm{R}\Gamma_c(\mathbb{R}, (\mathcal{E}_\ell)_{[z,O]}) \cong \mathbb{K}[-d - 2I(z)].$$

We can state a stronger result which follows directly from the microsupport estimate in the same way. We set $\gamma = (-\infty, 0]^d$, $\gamma^a = -\gamma = [0, \infty)^d$; and for any closed subset $\Sigma \subset \gamma$, we set $\Sigma_\gamma = (\Sigma + \gamma^a) \cap \gamma$. For example, $\{z\}_\gamma = [z, O]$ for $z \in \gamma$.

The notion above is related to the γ-topology, see [**KS90**, Section 3.5, Section 5.2] and [**KS18**] for more about the definition and sheaf theory related to γ-topology. A closed subset $Z \subset \mathbb{R}^d$ is γ-closed if $Z = Z + \gamma^a$. Now, let us consider the induced topology of the γ-topology on γ. Then the notation $\Sigma_\gamma = (\Sigma + \gamma^a) \cap \gamma$ is exactly the closure of the γ-topology for a closed set $\Sigma \subset \gamma$. So, for a closed subset $\Sigma \subset \gamma$, we say Σ_γ the γ-*closure* of Σ and we say Σ is γ-*closed* if $\Sigma_\gamma = \Sigma$.

Then, if Σ is compact and convex, we have that

$$\mathrm{R}\Gamma_c(\mathbb{R}^d_z, (\mathcal{E}_\ell^{\boxtimes d})_\Sigma) \cong \mathrm{R}\Gamma_c(\mathbb{R}^d_z, (\mathcal{E}_\ell^{\boxtimes d})_{\Sigma_\gamma}). \tag{4.24}$$

We can give a *proof* of (4.24) as follow: The microsupport $SS((\mathcal{E}_\ell^{\boxtimes d})_\gamma) \subset \mathbb{R}^d_z \times \gamma_\zeta$ together with the microlocal cut-off lemma [**KS90**, Proposition 5.2.3] shows that $(\mathcal{E}_\ell^{\boxtimes d})_\gamma$ is a γ^a-sheaf on $\mathbb{R}^d_z$, i.e. $(\mathcal{E}_\ell^{\boxtimes d})_\gamma$ is pullbacked from a sheaf on $\mathbb{R}^d_z$ equipped with the γ^a-topology. Then its global section over Σ is isomorphic to the global section over the γ-closure Σ_γ by [**KS90**, Proposition 3.5].

4.1.4. Proof of the Theorem 4.6. In this section, we will prove the structure theorem. We will only present the proof for the $\mathbb{Z}/\ell$ version. The proof for the S^1 version is given by minor modification which will be indicated in the proof. In this section, we assume $\mathbb{K}$ to be a field. So it is not necessary to derive the tensor product.

Idea and sketch of the proof: We present $(F_\ell(X_\Omega,\mathbb{K}))_T$ as $\mathrm{R}\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right)$, where we can apply the results in the Subsection 4.1.3. Now, consider the inclusion sequence $\{O\}\subset\overline{ZO}\subset\Omega_T^\circ$, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \arrow[r] \arrow[dr] & \mathrm{R}\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{\overline{ZO}}\right) \arrow[r, "\cong"] \arrow[d] & \mathbb{K}[-d-2I(Z)] \arrow[d, "k_Z u^{I(Z)}"] \\
 & \mathrm{R}\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_{O}\right) \arrow[r, "\cong"] & \mathbb{K}[-d]
\end{tikzcd}$$

By definition, the inclined arrow composed with the bottom isomorphism gives the fundamental class, and we call the upper horizontal arrow (up to constant) $\Lambda_{Z,\ell}$. The Lemma 4.14 shows that (up to a constant k_Z) the vertical morphism is $u^{I(Z)}$. Eventually, we absorb the constant into $\Lambda_{Z,\ell}$ since the constant is uniquely determined by Z and ℓ. The commutative diagram induces a decomposition $\eta_{\ell,T}(X_\Omega,\mathbb{F}_{p_\ell})=u^{I(Z)}\Lambda_{Z,\ell}$. In particular, the presence of $\Lambda_{Z,\ell}$ shows us the minimal cohomology degree is smaller than $-2I(Z)$ for all $Z\in\Omega_T^\circ$.

To achieve the non-torsioness, we need to prove that the fundamental class $\eta_{\ell,T}(U,\mathbb{K})$, a degree 0 morphism, is non-zero. We have two approaches. The easiest one is to take a small ball $B\subset U$, and then we apply the computation for balls (which can be derived directly from Lemma 4.14). The harder one is that we study its cocone, i.e. $F_\ell^+(X_\Omega,\mathbb{K})[-1]$ using (3.19), which is computed by homology of a union of finite dimensional manifolds.

I will discuss the harder approach since it provides us with more structural results, for example, rank and degree distribution of torsion elements. We will argue by a localization trick. In particular, we show that $H^*C_{\ell,T}(X_\Omega,\mathbb{F}_{p_\ell})$ is a finitely generated module whose free part is of rank 2. Then the argument also shows that torsion can not happen in non-negative degrees.

Finally, we study further the cocone of the fundamental class to show that the minimal cohomology degree is greater than $-2I(\Omega_T^\circ)$.

Therefore, our technical discussion will focus on the formula for $\mathrm{R}\Gamma_c\left(\mathbb{R}^d,(\mathcal{E}_\ell^{\boxtimes d})_W\right)$ for a locally closed set $W\subset\Omega_T^\circ$, and its minimal degree estimate. We will organize our arguments in the following way:

• We first compute $(F_\ell(X_\Omega,\mathbb{K}))_T$ using its isomorphism with $\mathrm{R}\Gamma_c\left(\mathbb{R}^d,({\mathcal{E}_\ell}^{\boxtimes d})_{\Omega_T^\circ}\right)$, where ${\mathcal{E}_\ell}^{\boxtimes d}$ is discussed in the last section. Consequently, we derive a similar formula for the cocone of the fundamental class, i.e. $F_\ell^+(X_\Omega,\mathbb{K})[-1]\cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d,({\mathcal{E}_\ell}^{\boxtimes d})_{\Omega_T^\circ\setminus O}\right)$. Then the result of the last section will reduce them to a cohomology computation of a topological space $\mathcal{W}_\ell^N(\Omega_T^\circ)$. We will realize the targets in Lemma 4.16.

• Recall the lattice decomposition (Proposition 4.12) of the sheaf $\mathcal{E}_\ell^{\boxtimes d}$, we hope to utilize the lattice description to obtain a minimal degree estimate for the cocone of the fundamental class. But a problem here is that we are computing cohomology of sheaves over Ω_T°. But Ω_T° is usually curved. So, our idea is to decompose Ω_T° into "almost cubes", which are introduced in Lemma 4.18. Next, we will study the proper homotopy type of $\mathcal{W}_\ell^N(\Omega_T^\circ)$ in case that Ω_T° is an almost cube. This is the Lemma 4.19.

• Finally, we use the computation for almost cubes as an induction step to obtain the minimal degree estimate in general. This is done using the Mayer–Vietoris sequence in Lemma 4.20. After that, we will finish the proof of Theorem 4.6.

Remark 4.15. A technical fact is that in the induction process of the minimal degree estimate, we have to deal with some sets that are not necessarily convex. But they are γ-closed. So, we will present the result for γ-closed set Σ, not only Ω_T°, from the beginning in the following.

Preliminary lemmas: For a convex toric domain X_Ω, we review the first paragraph of Subsection 4.1.2. We have $\Omega^\circ\subset\gamma^d\times[0,\infty)$. Then (3.31) and (4.22) show that

$$\begin{aligned}F_\ell(X_\Omega,\mathbb{K})&\cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left({\mathcal{E}_\ell}^{\boxtimes d}\boxtimes\mathbb{K}_{\{t_1\geq 0\}}\otimes\pi_{t_1}^{-1}\mathbb{K}_{\Omega^\circ}\right)\\&\cong \mathrm{R}\pi_{z!}\left[({\mathcal{E}_\ell}^{\boxtimes d}\boxtimes\mathbb{K}_{\{t\geq 0\}})_{\Omega^\circ}\right].\end{aligned}\tag{4.25}$$

Therefore, we conclude that

$$(F_\ell(X_\Omega,\mathbb{K}))_T\cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d,({\mathcal{E}_\ell}^{\boxtimes d})_{\Omega_T^\circ}\right).\tag{4.26}$$

In particular, for $X_{\mathbb{R}^d}=T^*V$, we have

$$(F_\ell(T^*V,\mathbb{K}))_T\cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d,({\mathcal{E}_\ell}^{\boxtimes d})_O\right)\cong\mathbb{K}[-d].$$

Then, by definition, the fundamental class is

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_O\right) \cong \mathbb{K}[-d].$$

For $Z \in \Omega_T^\circ$, we apply (4.24) for the segment $\Sigma = \overline{ZO}$, then we have (recall that $[Z, O]$ denotes a cube here)

$$\mathrm{R}\Gamma_c(\mathbb{R}_z^d, (\mathcal{E}_\ell^{\boxtimes \ell})_{\overline{ZO}}) \cong \mathrm{R}\Gamma_c(\mathbb{R}_z^d, (\mathcal{E}_\ell^{\boxtimes \ell})_{[Z,O]}) \cong \mathbb{K}[-d - 2I(Z)],$$

since $[Z, O] = \overline{ZO}_\gamma$. Now, we can embed the fundamental class into an excision triangle:

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ \setminus O}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \xrightarrow{\eta_{\ell,T}(X_\Omega,\mathbb{K})} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_O\right) \xrightarrow{+1}.$$

But by definition of $F_\ell^+(X_\Omega, \mathbb{K})$, we have $F_\ell^+(X_\Omega, \mathbb{K})[-1] \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ \setminus O}\right)$.

Both Ω_T° and $\overline{ZO}$ are compact convex. We would like to apply the isomorphism (4.23) to compute the cohomology of $\mathcal{E}_\ell^{\boxtimes d}$ in term of ${}^d\mathcal{W}_\ell^N$.

Assumption: For any compact subset $\Sigma \subset \gamma$, we will fix an odd integer $N = N(\Sigma) > 0$ and a positive number $\varepsilon > 0$ such that $\Sigma \subset [-N\ell/4 - \varepsilon, 0]^d$. The existence of N and ε is ensured by the compactness of Σ.

Lemma 4.16. *For a compact set $\Sigma \subset \gamma$ such that $\Sigma \cap [x, y]$ is empty or contractible for all $x \leq y$, $x, y \in \gamma$ (recall here, $[x, y]$ means the closed cube from x to y). We have*

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_\Sigma\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Sigma), \mathbb{K}\right), \tag{4.27}$$

where

$$\mathcal{W}_\ell^N(\Sigma) = \bigcup_{z \in \Sigma} {}^d\mathcal{W}_\ell^N(z) = \pi_z\left({}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})\right). \tag{4.28}$$

As $\Sigma = \Omega_T^\circ$ is convex for $T \geq 0$, we have, in particular

$$\begin{aligned} &(F_\ell(X_\Omega, \mathbb{K}))_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ}\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Omega_T^\circ), \mathbb{K}\right), \\ &(F_\ell^+(X_\Omega, \mathbb{K})[-1])_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Omega_T^\circ \setminus O)}\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Omega_T^\circ) \setminus \Delta_{V^{N\ell}}, \mathbb{K}\right). \end{aligned} \tag{4.29}$$

PROOF. For $N = N(\Sigma) > 0$ and $\varepsilon > 0$ such that $\Sigma \subset [-N\ell/4 - \varepsilon, 0]^d$, we have the isomorphism (4.23) $\mathcal{E}_\ell^{\boxtimes d}|_{[-N\ell/4-\varepsilon,0]^d} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N \cap([-N\ell/4-\varepsilon,0]^d \times V^{N\ell})}$, and then we obtain

$$\begin{aligned}\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_\Sigma\right) &\cong \mathrm{R}\pi_{z!}\left((\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N})_\Sigma\right)\\ &\cong \mathrm{R}\pi_{z!}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N\cap(\Sigma\times V^{N\ell})}\\ &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}\pi_{z!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N\cap(\Sigma\times V^{N\ell})}.\end{aligned}$$

Claim: When restricted to ${}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})$, the fibers of π_z are compact and contractible. Indeed, Chiu proved, in the Lemma 4.10 of [Chi17], that the fibers of the restriction of π_{z_i} on $\mathcal{W}_\ell^N \cap ([-N\ell/4 - \varepsilon, 0] \times \mathbb{R}^{N\ell})$ are closed intervals. Let us present the proof here. For fixed (q_k^i), $k \in \mathbb{R}^{N\ell}$, we have $(z_i, q_k^i) \in \mathcal{W}_\ell^N$ when $Q_{z_i} \geq 0$. It means that $-N\ell/4 - \varepsilon \leq z_i \leq 0$ and

$$\cos(2\pi z_i/N\ell)\sum_k (q_k^i)^2 \leq \sum_k q_k^i q_{k+1}^i,$$

where $k \in \mathbb{Z}/N\ell$. Now, $-N\ell/4 - \varepsilon \leq z_i \leq 0$ shows that $-\pi/2 \leq 2\pi z_i/N\ell \leq 0$. When $(q_k^i) = 0$, we obtain $-N\ell/4 - \varepsilon \leq z_i \leq 0$. When $(q_k^i) \neq 0$, we have

$$-\frac{N\ell}{4} - \varepsilon \leq z_i \leq -\arccos\left(\frac{\sum_k q_k^i q_{k+1}^i}{\sum_k (q_k^i)^2}\right).$$

So the fibers of the restriction of π_z on ${}^d\mathcal{W}_\ell^N \cap ([-N\ell/4-\varepsilon, 0] \times \mathbb{R}^{N\ell})$ are closed cubes. Hence, the fibers of the restriction of π_z on ${}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})$ are intersections of closed cubes and Σ, which are either empty or compact and contractible by assumption.

Consequently, the Vietoris-Begel theorem implies

$$\mathrm{R}\pi_{z!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N\cap(\Sigma\times V^{N\ell})} \cong \mathbb{K}_{\pi_z({}^d\mathcal{W}_\ell^N\cap(\Sigma\times V^{N\ell}))} = \mathbb{K}_{\mathcal{W}_\ell^N(\Sigma)}.$$

Therefore, $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_\Sigma\right) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\mathbb{K}_{\mathcal{W}_\ell^N(\Sigma)}) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Sigma), \mathbb{K}\right)$.

The last statement follows from the discussion above the lemma. □

Remark 4.17. The condition in the lemma is true for compact convex sets Σ. But for our last applications, we need to consider γ-closed sets. Recall that $\gamma = (-\infty, 0]^d$, $\gamma^a = -\gamma = [0, \infty)^d$. For a closed set $\Sigma \subset \gamma$, the γ-closure is defined as $\Sigma_\gamma = (\Sigma + \gamma^a) \cap \gamma$. We say Σ is γ-closed if $\Sigma_\gamma = \Sigma$. For example, $\gamma \setminus (\mathring{\gamma} + z)$ is γ-closed for $z \in \gamma$, and the intersection of two γ-closed sets is γ-closed. The γ-closed sets satisfy the condition of the Lemma 4.16. Indeed, for a closed cube $[x, y]$ with $x, y \in \gamma$, a γ-closed Σ, and any $z \in \Sigma \cap [x, y]$, the segment $\overline{xz} \subset \Sigma \cap [x, y]$ lies in $\Sigma \cap [x, y]$. Therefore, $\Sigma \cap [x, y]$ is star-shaped and then contractible.

As $z \mapsto \mathcal{W}(z)$ is a decreasing map, one can see that $\mathcal{W}_\ell^N(\Sigma) = \mathcal{W}_\ell^N(\Sigma_\gamma)$ for all compact subset $\Sigma \subset \gamma$. In particular, if Σ satisfies the condition of the Lemma 4.16, then the lemma implies that

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Sigma_\gamma}\right) \xrightarrow{\cong} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\boxtimes d})_{\Sigma}\right),$$

which can be seen as a generalization of (4.24) for compact sets satisfying the condition of Lemma 4.16 in the case of $\mathcal{E}_\ell^{\boxtimes d}$. Later, we will mainly focus on γ-closed sets Σ.

Now, to understand the cohomology of $\mathcal{W}_\ell^N(\Sigma)$ (see (4.28)), we start from a special case that Σ is an "almost closed cube", defined in Lemma 4.18 below. Let us recall some notation and introduce some new ones.

First, recall that, for $x, y \in \mathbb{R}^d$, we let $(x, y] = \prod_{i=1}^d (x_i, y_i]$ be the half-open cube from x to y. Similarly, we define open cubes and closed cubes in this way. Also, recall $O \in \mathbb{R}^d$ is the origin, and we set $\mathbb{1} = (1, \ldots, 1)$ and $e_i = (\delta_{ij})_{j=1}^d$ where δ_{ij} stands for the Kronecker symbol. For simplicity, we also denote $C_x = [x, x + \mathbb{1})$ for $x \in \mathbb{R}^d$.

Next, for a compact γ-closed set $\Sigma \subset \gamma$, we set

$$\begin{aligned} J_\Sigma &= (-\Sigma) \cap \mathbb{Z}_{\geq 0}^d = \{v \in \mathbb{Z}_{\geq 0}^d : (-\Sigma) \cap C_v \neq \varnothing\}, \\ \partial J_\Sigma &= \{v \in J_\Sigma : \forall i, v + e_i \notin J_\Sigma\} = \{v \in J_\Sigma : (-\Sigma) \cap (\overline{C_v} \setminus C_v) = \varnothing\}. \end{aligned}$$

The compactness of Σ shows that both J_Σ and ∂J_Σ are finite sets.

Lemma 4.18. *Let $\Sigma \subset \gamma$ be a compact γ-closed set. Then $\partial J_\Sigma = \{v\}$ for some $v \in \mathbb{Z}_{\geq 0}^d$ if and only if $[O, v] \subset -\Sigma \subset [O, v + \mathbb{1})$ for the same $v \in \mathbb{Z}_{\geq 0}^d$.*

We say that Σ is an almost cube if it satisfies these equivalent conditions.

Proof. When $[O,v] \subset -\Sigma \subset [O, v+\mathbb{1})$, taking the intersection with $\overline{C_w} \setminus C_w$, for all $w \in J_\Sigma$, we obtain

$$[O,v] \cap (\overline{C_w} \setminus C_w) \subset (-\Sigma) \cap (\overline{C_w} \setminus C_w) \subset [O, v+\mathbb{1}) \cap (\overline{C_w} \setminus C_w).$$

Then we can obtain $\partial J_\Sigma = \{v\}$ from that $[O,v] \cap (\overline{C_w} \setminus C_w) = \varnothing$ only when $v = w$.

Conversely, when $\partial J_\Sigma = \{v\}$, we have $-v \in \Sigma$. So $\Sigma = \Sigma_\gamma$ implies $[-v, O] = \{-v\}_\gamma \subset \Sigma$. Now, suppose $-\Sigma \not\subset [O, v+\mathbb{1})$, then there is a $z \in \Sigma$ such that $-z_i = v_i + 1$ for some $i \in [d]$. Therefore, $v + e_i \in J_\Sigma$. If $v + e_i \notin \partial J_\Sigma$, the argument repeats and there exists another $j \in [d]$ such that $v + e_i + e_j \in J_\Sigma$. We can continue until we obtain a index set I (with possible multiplicities) such that $v + \sum_I e_i \in \partial J_\Sigma$. Since J_Σ is a finite set, then the index set must be finite. But, $\partial J_\Sigma = \{v\}$, then $v + \sum_I e_i \notin \partial J_\Sigma$. Hence we get a contradiction. Then $-\Sigma \subset [O, v+\mathbb{1})$. □

Here, we are going to prove a refinement of the fact that ${}^d\mathcal{W}_\ell^N(-v)$ is properly homotopic to $\mathbb{R}^{d+2I(-v)}$ as noticed before Lemma 4.14.

Lemma 4.19. *For a compact γ-closed set $\Sigma \subset \gamma$ with $\partial J_\Sigma = \{v\}$, the subspace $\mathbb{R}^d \times \mathbb{C}^{I(-v)}$ is a strong deformation retract of $\mathcal{W}_\ell^N(\Sigma)$ under a proper deformation retraction. Moreover, $\Delta_{V^{N\ell}} \cong \mathbb{R}^d \times \{0\} \subset \mathbb{R}^d \times \mathbb{C}^{I(-v)}$ is invariant under the retraction.*

Proof. Here, we use the diagonal form of Q_z we introduced in (4.17). Then the coordinate system on $\left(\mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}\right)^d$ is $(x_k^i) = (\mathbf{x}_k) = (\mathbf{x}^i)$ with $\mathbf{x}_k = (x_k^1, \dots, x_k^d)$ and $\mathbf{x}^i = (x_0^i, \dots, x_k^i)$, where $i \in [d] = \{1, \dots, d\}$, $k = 0, 1, \dots (N\ell - 1)/2$ if ℓ is odd and $k = 0, 1, \dots N\ell/2$ if ℓ is even. For shortness, we only deal with the ℓ odd case. The ℓ even case has the same proof with minor corrections on the notation. Recall that

$$\mathcal{W}_\ell^N(\Sigma) = \left\{(\mathbf{x}^i) = (x_0^i, x_k^i) : \exists z \in \Sigma, \forall i,\, Q_{z_i}(\mathbf{x}^i) \geq 0\right\},$$

$$\Delta_{V^{N\ell}} = \left\{(\mathbf{x}^i) = (x_0^i, x_k^i) : \forall k \geq 1, i \in [d],\text{ such that } x_k^i = 0\right\}.$$

For $0 \leq m \leq (N\ell-1)/2$, $i \in [d]$, consider $h_m^i : \mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}} \times [0,1] \to \mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}$,

$$h_m^i(x_0^i, x_+^i, x_-^i, t) = h_{m,t}^i(x_0^i, x_+^i, x_-^i) = (x_0^i, x_+^i, t x_-^i),$$

where $x^i_+ = (x^i_1, \dots, x^i_m)$, $x^i_- = (x^i_{m+1}, x^i_{m+2}, \dots, x^i_{(N\ell-1)/2})$.

By assumption of the lemma, $v \in -\Sigma \subset [0, N\ell/4)^d$. Then we have $0 \le v_i < N\ell/4 \le (N\ell-1)/2$. Now, define $H_v : \left(\mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}\right)^d \times [0,1] \to \left(\mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}\right)^d$ by

$$H_{v,t} = h^1_{v_1,t} \times \cdots \times h^d_{v_d,t}.$$

Then we have $H_{v,1}$ is the identity map. Next, we have the following:

- $H_{v,t}(\mathcal{W}^N_\ell(\Sigma)) \subset \mathcal{W}^N_\ell(\Sigma)$. Indeed, $(\mathbf{x}^i) \in \mathcal{W}^N_\ell(\Sigma)$ implies there exists $z \in \Sigma$ such that for all $i \in [d]$, we have $Q_{z_i}(\mathbf{x}^i) \ge 0$. So, in the diagonal form (4.18), we have

$$\lambda_0(z_i)|x^i_0|^2 + 2\sum_{k=1}^{v_i} \lambda_k(z_i)|x^i_k|^2 \ge 2\sum_{k \ge v_i+1} (-\lambda_k(z_i))|x^i_k|^2.$$

Now $-\Sigma \subset [O, v + \mathbb{1})$ implies that $z_i < v_i + 1$ for all $i \in [d]$, hence $\lambda_k(z_i) < 0$ for $k \ge v_i + 1$ and for all $i \in [d]$. So

$$\lambda_0(z_i)|x^i_0|^2 + 2\sum_{k=1}^{v_i} \lambda_k(z_i)|x^i_k|^2 \ge 2\sum_{k \ge v_i+1} (-\lambda_k(z_i))|x^i_k|^2 \ge 2t^2 \sum_{k \ge v_i+1} (-\lambda_k(z_i))|x^i_k|^2,$$

i.e., $Q_{z_i}(h_{v_i,t}(\mathbf{x}^i)) \ge 0$ for all $i \in [d]$. Hence $H_{v,t}(\mathbf{x}^1, \dots, \mathbf{x}^d) \in \mathcal{W}^N_\ell(\Sigma)$.

- $H_v|_{\mathcal{W}^N_\ell(\Sigma)}$ is proper. Indeed, take $(\mathbf{x}^i) \in \mathcal{W}^N_\ell(\Sigma)$ such that $H_v(\mathbf{x}^i) \in [-R^2, R^2]^d$, we have $\sum_{k=0}^{v_i} |x^i_k|^2 + \sum_{k \ge v_i+1} |tx^i_k|^2 \le R^2$, for all $i \in [d]$.

Obviously, $\sum_{k=0}^{v_i} |x^i_k|^2 \le R^2$, for all $i \in [d]$, and

$$\begin{aligned}
2\max_{\substack{k=0,\dots,v_i\\ z\in\Sigma}} |\lambda_k(z_i)|R^2 &\ge \lambda_0(z_i)|x^i_0|^2 + 2\sum_{k=0}^{v_i} \lambda_k(z_i)|x^i_k|^2 \\
&\ge 2\sum_{k\ge v_i+1} (-\lambda_k(z_i))|x^i_k|^2 \\
&\ge 2\min_{\substack{k\ge v_i+1\\ z\in\Sigma}} |\lambda_k(z_i)| \sum_{k\ge v_i+1} |x^i_k|^2.
\end{aligned}$$

Since $\lambda_k(z_i) < 0$ for $k \ge v_i+1$, and $z \in \Sigma$, we have $\min_{\substack{k\ge v_i+1\\ z\in\Sigma}} |\lambda_k(z_i)| > 0$. Consequently,

$$\sum_{k\ge v_i+1} |x^i_k|^2 \le \frac{\max\limits_{\substack{k=1,\dots,v_i\\ z\in\Sigma}} |\lambda_k(z_i)|}{\min\limits_{\substack{k\ge v_i+1\\ z\in\Sigma}} |\lambda_k(z_i)|} R^2 =: KR^2.$$

It means that $\sum_{k=0}^{v_i} |x_k^i|^2 + \sum_{k\geq v_i+1} |x_k^i|^2 \leq (1+K)R^2$, for all $i \in [d]$, where $K = K(\Sigma)$ is a constant only depending on $\mathcal{W}_\ell^N(\Sigma)$.

So, we have shown that the pre-image of a bounded set under $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is bounded. It means that $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is proper.

Hence $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is a proper homotopy with $H_{v,1}|_{\mathcal{W}_\ell^N(\Sigma)} = \mathrm{Id}_{\mathcal{W}_\ell^N(\Sigma)}$

• $\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\} \subset \mathcal{W}_\ell^N(\Sigma)$. Let $(\mathbf{x}^i) \in \mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$. This means that for all $i \in [d]$, $\mathbf{x}^i = (x_0^i, x_+^i, x_-^i)$ satisfies $x_-^i = 0$. Since $z = -v \in \Sigma$, by assumption, it is enough to check that $Q_{-v_i}(\mathbf{x}^i) \geq 0$. Now $\lambda_k(-v_i) \geq 0$ for $k = 0, 1, \ldots, v_i$. Then for all $i \in [d]$, we have

$$Q_{-v_i}(x_0^i, x_+^i, 0) = \lambda_0(-v_i)|x_0^i|^2 + 2\sum_{k=1}^{v_i} \lambda_k(-v_i)|x_k^i|^2 \geq 0.$$

So $(\mathbf{x}^i) \in \mathcal{W}_\ell^N(\Sigma)$ and then $\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\} \subset \mathcal{W}_\ell^N(\Sigma)$.

• $H_{v,0}(\mathcal{W}_\ell^N(\Sigma)) \subset \mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$. Indeed for $(\mathbf{x}^i) \in \mathcal{W}_\ell^N(\Sigma)$, we have $h_{v_i,0}^i(\mathbf{x}^i) = (x_0^i, x_+^i, 0)$ for all $i \in [d]$. Then $H_{v,0}(\mathcal{W}_\ell^N(\Sigma)) \subset \mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$.

On the other hand, by definition of $h_{v_i}^i$, we have

$$h_{v_i}^i(x_0^i, x_+^i, 0, t) = (x_0^i, x_+^i, t0) = (x_0^i, x_+^i, 0).$$

So $H_{v,t}|_{\mathbb{R}^d\times\mathbb{C}^{I(-v)}\times\{0\}} = \mathrm{Id}_{\mathbb{R}^d\times\mathbb{C}^{I(-v)}\times\{0\}}$ for all $t \in [0,1]$. Therefore, $\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$ is a proper strong deformation retract of $\mathcal{W}_\ell^N(\Sigma)$ under $H_{v,t}|_{\mathcal{W}_\ell^N(\Sigma)}$.

□

Below, we will frequently use the equivariant global section functor $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell{}^{\boxtimes d})_W\right)$ for locally closed set $W \subset \gamma$. Then we denote it by

$$\Gamma\mathcal{E}(W) \coloneqq \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell{}^{\boxtimes d})_W\right), \tag{4.30}$$

for shortening the length of notation until the end of the subsection.

Lemma 4.20. *Let $\Sigma \subset \gamma$ be a compact γ-closed set. Let $\mathbb{K} = \mathbb{F}_{p_\ell}$ when $G = \mathbb{Z}/\ell$ and let $\mathbb{K}$ be a field of characteristic 0 when $G = S^1$. Recall the notation $I(\Sigma)$ at (4.10).*

We assume further that $\Sigma \subset (-p_\ell \mathbb{1}, O]$ *for the* $G = \mathbb{Z}/\ell$ *case. Then*

$$\mathrm{Ext}_G^{q-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{K}) \cong 0 \qquad \textit{if} \begin{cases} q \notin [-2I(\Sigma)+1, -1] & \textit{for } G = S^1, \\ q \notin [-2I(\Sigma), -1] & \textit{for } G = \mathbb{Z}/\ell. \end{cases}$$

The $\mathbb{K}$*-vector space* $\mathrm{Ext}_G^{q-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{K})$ *is finite dimensional.*

Proof. We proceed by induction on $|J_\Sigma|$. We notice that the maximum $I(\Sigma)$ can be achieved by some v since $\Sigma \cap \mathbb{Z}^d$ is finite. Moreover, if $v \in J_\Sigma$ satisfies $I(-v) = I(\Sigma)$, then $v \in \partial J_\Sigma$. We will use the excision distinguished triangle

$$\Gamma\mathcal{E}(\Sigma \setminus O) \to \Gamma\mathcal{E}(\Sigma) \xrightarrow{\eta_\Sigma} \Gamma\mathcal{E}(O) \xrightarrow{+1} . \tag{4.31}$$

If $|J_\Sigma| = 1$, that is $J_\Sigma = \{0\}$, then the Lemma 4.19 shows that η_Σ is an isomorphism in the derived category. Then $\Gamma\mathcal{E}(\Sigma \setminus O) \cong 0$ by (4.31) and the result follows.

Now, we suppose the result is true for all Σ' such that $|J_{\Sigma'}| < |J_\Sigma|$ and we distinguish the cases $|\partial J_\Sigma| = 1$ and $|\partial J_\Sigma| > 1$.

(1) If $\partial J_\Sigma = \{v\}$ is a singleton. The case $v = 0$ is already done and we assume $I(-v) > 0$. Then the excision sequence (4.31), the Lemma 4.16, and the Lemma 4.19 together show the isomorphisms in the equivariant derived category

$$\Gamma\mathcal{E}(\Sigma \setminus O) \cong \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(\Sigma) \setminus \Delta_{V^{N\ell}}, \mathbb{K}) \cong \mathrm{R}\Gamma(S^{2I(-v)-1}, \mathbb{K})[-d-1],$$

where the action of G on $S^{2I(-v)-1}$ is given in (4.20) for $G = \mathbb{Z}/\ell$ and (4.21) for $G = S^1$. We have

$$\begin{aligned} \mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{K}) &\cong \mathrm{Ext}_G^{*+1}\left(\mathrm{R}\Gamma(S^{2I(-v)-1}, \mathbb{K}), \mathbb{K}\right) \\ &\cong \mathrm{Ext}^{*+1}\left(\mathbb{K}_{S^{2I(-v)-1}}, \omega^!_{S^{2I(-v)-1}}\right) \\ &\cong H^G_{-*-1}(S^{2I(-v)-1}, \mathbb{K}), \end{aligned}$$

where we used the equivariant Poincaré duality, which holds since $S^{2I(-v)-1}$ is compact and orientable.

When $G = \mathbb{Z}/\ell$, under the assumption $\Sigma \subset (-p_\ell \mathbb{1}, O]$, the $\mathbb{Z}/\ell$-action is free by (4.20). Hence $H^G_{-*-1}(S^{2I(-v)-1}, \mathbb{K})$ computes the usual cohomology of the quotient. When $G = S^1$, the action is locally free by (4.21), i.e. all stabilizer are finite groups. Then, as we take a coefficient field of characteristic 0, we have that $H^G_{-*-1}(S^{2I(-v)-1}, \mathbb{K})$ also computes the usual cohomology of the quotient.

Let $Q_G = S^{2I(-v)-1}/G$, then $Q_{\mathbb{Z}/\ell}$ is a smooth lens space of dimension $2I(-v) - 1$ and Q_{S^1} is a weighted complex projective space of dimension $2I(-v) - 2$ and weight $(1, 2, \dots, I(-v) - 1)$.

Then, we have

$$H_q(Q_{S^1}) = \begin{cases} \mathbb{K}, & q \in [0, 2I(-v) - 2] \cap 2\mathbb{Z}, \\ 0, & q \notin [0, 2I(-v) - 2] \cap 2\mathbb{Z}, \end{cases}$$

and

$$H_q(Q_{\mathbb{Z}/\ell}) = \begin{cases} \mathbb{K}, & q \in [0, 2I(-v) - 1], \\ 0, & q \notin [0, 2I(-v) - 1]. \end{cases}$$

Converting to cohomology degree, we obtain: $\mathrm{Ext}^{*-d}_G(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{K})$ is concentrated in $[-2I(-v) + 1, -1] \cap (2\mathbb{Z} + 1)$ or $[-2I(-v), -1]$.

The proof of this part is independent of our induction, so it can be applied to the second case. Now, let us focus on $G = \mathbb{Z}/\ell$ case for simplicity.

(2) If $|\partial J_\Sigma| \geq 2$, take $v \in \partial J_\Sigma$ such that $I(-v) = I(\Sigma)$. Then we can take $1 > \epsilon > 0$ such that $\Sigma \cap (\gamma + (\epsilon \mathbb{1} - v)) \subset (-v, 0]$. This is possible due to the compactness of Σ. Let us define:

$$\begin{aligned} A &= [\Sigma \cap (\gamma + (\epsilon \mathbb{1} - v))]_\gamma, \\ B &= \Sigma \cap [\gamma \setminus (\mathring{\gamma} + (\epsilon \mathbb{1} - v))]. \end{aligned} \tag{4.32}$$

Then we have a closed covering $\Sigma = A \cup B$. Moreover, both A and B are compact γ-closed sets, then so is $A \cap B$.

Then we have the Mayer-Vietoris triangle,

$$\Gamma\mathcal{E}(\Sigma \setminus O) \to \Gamma\mathcal{E}(A \setminus O) \oplus \Gamma\mathcal{E}(B \setminus O) \to \Gamma\mathcal{E}((A \cap B) \setminus O) \xrightarrow{+1}.$$

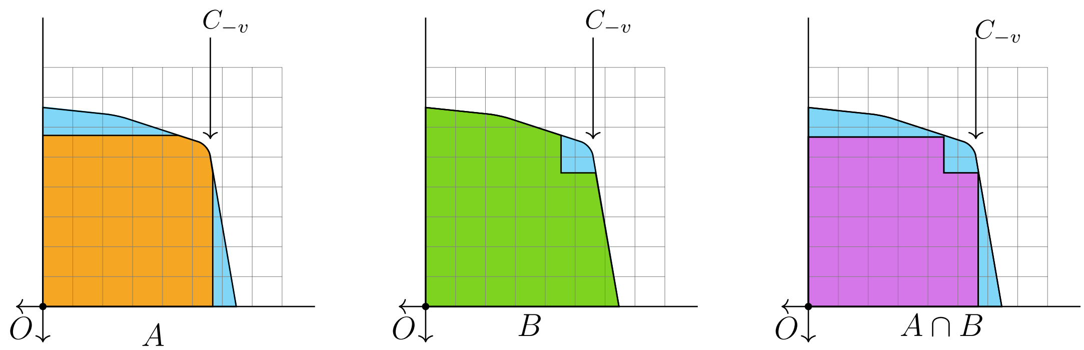


FIGURE 1. The picture illustrate the construction of A, B. Σ is the background blue set.

Next, we apply the $\mathrm{Ext}_G^{*-d}(-, \mathbb{K}) \cong \mathrm{Ext}_G^{*}(-, \mathbb{K}[-d])$ to obtain a long exact sequence

(4.33)

$$\mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}((A \cap B) \setminus O), \mathbb{K}) \to \begin{matrix} \mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(A \setminus O), \mathbb{K}) \\ \bigoplus \\ \mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(B \setminus O), \mathbb{K}) \end{matrix} \to \mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{K}) \xrightarrow{+1} .$$

By our construction (4.32), we have

- $|\partial J_A| = 1$. Hence we can apply the result of (1). So that $\mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(A \setminus O), \mathbb{K})$ is concentrated in $[-2I(A), -1] \subset [-2I(\Sigma), -1]$.
- $|J_B| < |J_\Sigma|$. We can use the induction hypothesis, hence $\mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(B \setminus O), \mathbb{K})$ is concentrated in $[-2I(B), -1] \subset [-2I(\Sigma), -1]$.
- $|J_{A\cap B}| < |J_\Sigma|$, since $J_{A\cap B} \subset J_A$ but $v \notin J_{A\cap B}$. Then we can use the induction hypothesis, that $\mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}((A \cap B) \setminus O), \mathbb{K})$ is concentrated in $[-2I(A \cap B), -1]$. Moreover, in J_A, v is the only lattice point such that $I(-v) = I(\Sigma)$, then for all $v' \in J_{A\cap B} \subset J_A \setminus \{v\}$, we have $I(-v') < I(\Sigma)$. Then $|I(A \cap B)| < I(\Sigma)$, and $[-2I(A \cap B), -1] \subset [-2I(\Sigma) + 2, -1]$.

Therefore, it follows from the sequence (4.33) that $\mathrm{Ext}_G^{*-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{K})$ is concentrated in $[-2I(\Sigma), -1]$.

For the case $G = S^1$, the argument is the same and we only need to be careful on the degree distribution. $\square$

Now, we are in the position to prove the Theorem 4.6.

Proof of Theorem 4.6. We only prove the $\mathbb{Z}/\ell$-case, the S^1 case being similar. So we take $\mathbb{K} = \mathbb{F}_{p_\ell}$. The equation (4.29) says that $(F_\ell(X_\Omega, \mathbb{F}_{p_\ell}))_T \cong \Gamma\mathcal{E}(\Omega_T^\circ)$. Now, consider the inclusion sequence $\{O\} \subset \overline{ZO} \subset \Omega_T^\circ$ of closed sets. Then we have a commutative diagram:

$$\begin{tikzcd}
\Gamma\mathcal{E}(\Omega_T^\circ) \arrow[r] \arrow[dr] & \Gamma\mathcal{E}(\overline{ZO}) \arrow[r, "\cong"] \arrow[d] & \mathbb{F}_{p_\ell}[-d-2I(Z)] \arrow[d, "k_Z u^{I(Z)}"] \\
& \Gamma\mathcal{E}(O) \arrow[r, "\cong"] & \mathbb{F}_{p_\ell}[-d]
\end{tikzcd}$$

By definition, the inclined arrow compose with the bottom isomorphism gives the fundamental class $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$. The terms $\Gamma\mathcal{E}(\overline{ZO})$ and $\Gamma\mathcal{E}(O)$ are computed using Lemma 4.14 and (4.24). Then loc. cit. also shows that the vertical morphism is $k_Z u^{I(Z)}$, where k_Z is a constant only depends on Z. We absorb the constant into the horizontal arrow, then we call it $\Lambda_{Z,\ell}$. Therefore, the commutative diagram induces a decomposition $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$.

Now, let us embed the fundamental class $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ into the excision triangle (the triangle (4.31) for $\Sigma = \Omega_T^\circ$)

$$\Gamma\mathcal{E}(\Omega_T^\circ \setminus O) \to \Gamma\mathcal{E}(\Omega_T^\circ) \xrightarrow{\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})} \Gamma\mathcal{E}(O) \xrightarrow{+1} . \tag{4.34}$$

The isomorphism $(F_\ell(X_\Omega, \mathbb{F}_{p_\ell}))_T \cong \Gamma\mathcal{E}(\Omega_T^\circ)$ and the distinguished triangle (3.19) show us that $(F_\ell^+(X_\Omega, \mathbb{F}_{p_\ell})[-1])_T \cong \Gamma\mathcal{E}(\Omega_T^\circ \setminus O)$.

So, after applying $\mathrm{RHom}_{\mathbb{Z}/\ell}(-, \mathbb{F}_{p_\ell}[-d])$, we get the tautological triangle (3.20) of Chiu-Tamarkin complex for X_Ω:

$$\mathrm{R}\Gamma(V, \omega_V^{\mathbb{Z}/\ell}) \to C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \xrightarrow{\mathrm{RHom}_{\mathbb{Z}/\ell}(\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}), \mathbb{F}_{p_\ell}[-d])} C_{\ell,T}^+(X_\Omega, \mathbb{F}_{p_\ell}) \xrightarrow{+1} . \tag{4.35}$$

Here $V \cong \mathbb{R}^d$ and it is equipped with the trivial $\mathbb{Z}/\ell$-action. Taking cohomology for the distinguished triangle, we get a long exact sequence of the Chiu-Tamarkin cohomology:

$$H^*_{\mathbb{Z}/\ell}(V, \mathbb{F}_{p_\ell}) \to H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \xrightarrow{\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}), \mathbb{F}_{p_\ell})} H^*C_{\ell,T}^+(X_\Omega, \mathbb{F}_{p_\ell}) \xrightarrow{+1} . \tag{4.36}$$

When $0 \leq T < p_\ell/\|\Omega_1^\circ\|_\infty$, we have $\Omega_T^\circ \subset Z_{p_\ell \mathbb{1}}$. Then we can apply the Lemma 4.20, $H^*C^+_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \cong \mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Omega_T^\circ \setminus O), \mathbb{F}_{p_\ell})$ is a finite dimensional graded $\mathbb{F}_{p_\ell}$ vector space which is concentrated in degrees $[-2I(\Omega_T^\circ), -1]$. Then, it is torsion as a $\mathbb{F}_{p_\ell}[u]$-module.

One the other hand, $H^*_{\mathbb{Z}/\ell}(V, \mathbb{F}_{p_\ell}) \cong \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{F}_{p_\ell}[-d], \mathbb{F}_{p_\ell}[-d]) \cong \mathbb{F}_{p_\ell}[u, \theta] = A$, where $|u| = 2$, $|\theta| = 1$, and $\theta^2 = ku$ ($k = 0$ if ℓ is odd and $k = \ell/2$ if ℓ is even), is concentrated in $[0, \infty)$.

Therefore, after tensoring with $\mathbb{F}_{p_\ell}((u))$, $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}), \mathbb{F}_{p_\ell}) \otimes_{\mathbb{F}_{p_\ell}[u]} \mathbb{F}_{p_\ell}((u))$ is an isomorphism of $\mathbb{F}_{p_\ell}((u))$-vector spaces. Then, we conclude that $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}), \mathbb{F}_{p_\ell}) \neq 0$ and so $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) \neq 0$.

Actually, it also shows that $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is a finitely generated $\mathbb{F}_{p_\ell}[u]$ module whose rank is 2, whose minimal degree is at least $-2I(\Omega_T^\circ)$ and torsion elements can only happen in degree $[-2I(\Omega_T^\circ), -1]$.

On the other hand, this estimate is sharp. Indeed, we take $Z \in \Omega_T^\circ$ such that $I(Z) = I(\Omega_T^\circ)$. Then the decomposition $\eta_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$ shows that we have a degree equation: $0 = 2|I(Z)| + |\Lambda_{Z,\ell}|$. Then $|\Lambda_{Z,\ell}| = -2I(\Omega_T^\circ)$ realizes the minimal degree $-2I(\Omega_T^\circ)$.

When X_Ω is an ellipsoid $E(a)$, let $Z = (-T/a_1, \ldots, -T/a_d)$, then $\Omega_T^\circ = \overline{ZO}$ is a segment. So, we can compute $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ directly from Lemma 4.14, and $\Lambda_{Z,\ell}$ we defined above is an isomorphism of A module. So $H^*C_{\ell,T}(X_\Omega, \mathbb{F}_{p_\ell})$ is torsion free as a $\mathbb{F}_{p_\ell}[u]$-module. $\square$

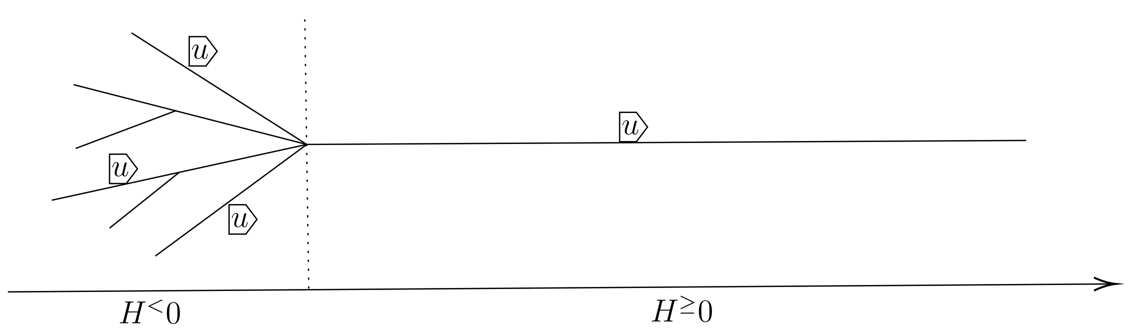


FIGURE 2. The shape of $\mathrm{Ext}^{*-d}_G(\Gamma\mathcal{E}(\Sigma), \mathbb{K})$

At the end of the section, let us make a description of the structure of $\operatorname{Ext}_G^{*-d}(\Gamma\mathcal{E}(\Sigma), \mathbb{K}[-d])$, where Σ is a γ-closed set, and $\Sigma \subset (-p_\ell \mathbb{1}, O]$.

Due to our structural theorem, we know that $\operatorname{Ext}_G^{*-d}(\Gamma\mathcal{E}(\Sigma), \mathbb{K})$ could be given as follow: For each $v \in \partial J_\Sigma$, we have a copy of $A = \mathbb{K}[u, \theta]$ or $\mathbb{K}[u]$, degree shifted by $-2I(-v)$, we can think of it as a flower, and treat the non-negative degree part as the stem of the flower.

Next, we can imagine we have many flowers. Then we tie their stems, so we have a beautiful bouquet now. That is $\operatorname{Ext}_G^{*-d}(\Gamma\mathcal{E}(\Sigma), \mathbb{K}[-d])$. Moreover, the $\mathbb{K}[u]$-module structure seems like a water-drop is draping down from the top of blooms into the earth (maybe never reach).

4.1.5. Contact toric domains. Finally, let us discuss the prequantized toric domains. I.e., $X_\Omega \times S^1$ for a symplectic toric domain X_Ω. We say $X_\Omega \times S^1$ is convex if X_Ω is convex.

Actually, we do not need to change the arguments much here, because we already set everything up well. Using Proposition 3.42, we only need to change the statement of the structural theorem slightly.

THEOREM 4.21. *For a big prequantized convex toric domain $X_\Omega \times S^1 \subset T^*V \times S^1$ (that means $\|\Omega_1^\circ\|_\infty < 1$, see Definition 3.46), $\ell \in \mathbb{N}_{\geq 2}$, $n \in \mathbb{N}_0$ and $n\ell \leq p_\ell / \|\Omega_1^\circ\|_\infty$. We have:*

- *For each $Z \in \Omega_{n\ell}^\circ$, the inclusion $\overline{ZO} \subset \Omega_{n\ell}^\circ$ induces a decomposition $\eta_{\ell,n\ell}^c(X_\Omega \times S^1, \mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$ for a non-torsion element $\Lambda_{Z,\ell} \in H^{-2I(Z)}\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$. In particular, $\eta_{\ell,n\ell}^c(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ is non-zero.*
- *The minimal cohomology degree of $H^*\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ is exactly $-2I(\Omega_{n\ell}^\circ)$, i.e.,*

$$H^*\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell}) \cong H^{\geq -2I(\Omega_{n\ell}^\circ)}\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$$

and

$$H^{-2I(\Omega_{n\ell}^\circ)}\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell}) \neq 0.$$

- *$H^*\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ is a finitely generated $\mathbb{F}_{p_\ell}[u]$-module. The free part is isomorphic to $A = \mathbb{F}_{p_\ell}[u, \theta]$, so $H^*\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ is of rank 2 over $\mathbb{F}_{p_\ell}[u]$.*

The torsion part is located in cohomology degree $[-2I(\Omega^\circ_{n\ell}), -1]$. $H^*\mathscr{C}_{\ell,n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ is torsion free when X_Ω is an open ellipsoid.

Theorem 4.22. *For a big prequantized convex toric domain $X_\Omega \times S^1 \subsetneq T^*V \times S^1$, we have:*

$$[c]_k(X_\Omega \times S^1) = \min\left\{n\ell \in \mathbb{N}_{\geq 2} : (n, \ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\, n\ell < \frac{p_\ell}{\|\Omega^\circ_1\|_\infty},\, \exists z \in \Omega^\circ_{n\ell}, I(z) \geq k\right\}$$
$$= \min\left\{n\ell \in \mathbb{N}_{\geq 2} : (n, \ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\, n\ell < \frac{p_\ell}{\|\Omega^\circ_1\|_\infty},\, n\ell \geq c_k(X_\Omega)\right\}.$$

The result is much weaker than the symplectic case, but it is still interesting. For example, when we consider ellipsoids, we have

$$[c]_k(E \times S^1) = \min\left\{n\ell \in \mathbb{N}_{\geq 2} : (n, \ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\, n\ell < p_\ell a_1,\, \sum_{i=1}^{d} \left\lfloor \frac{n\ell}{a_i} \right\rfloor \geq k\right\}$$
$$= \min\left\{n\ell \in \mathbb{N}_{\geq 2} : (n, \ell) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\, n\ell < p_\ell a_1,\, n\ell \geq c_k(E)\right\},$$

where $1 < a_1 \leq a_2 \leq \cdots \leq a_d$.

A more concrete example is as follow. Suppose $X_\Omega \times S^1 = E(3, 4) \times S^1$, we have:

k	1	2	3	4	5	6	7	8	9	10	11
c_k	3	4	6	8	9	12	12	15	16	18	20
$[c]_k$	3	4	6	10	10	13	13	15	17	19	22

4.2. Unit cotangent bundles and the Viterbo isomorphism

4.2.1. Microlocal kernel of unit disk bundles. Let us assume (X, g) is complete Riemannian manifold. The open unit codisk bundle is $D^*X = \{(\mathbf{q}, \mathbf{p}) : |\mathbf{p}|_g < 1\}$. Let us take $H(\mathbf{q}, \mathbf{p}) = |\mathbf{p}|_g$, then $D^*X = \{(\mathbf{q}, \mathbf{p}) : H < 1\}$.

However, H only defines a Hamiltonian function on $\dot{T}^*X$, and the associated Hamiltonian flow— the normalized geodesic flow, is defined on $\dot{T}^*X$. As H is homogeneous, we use the homogeneous version of the GKS sheaf quantization to assert that there is a sheaf $K \in D(\mathbb{R}_z \times X \times X)$ such that

$$\dot{SS}(K_g) = \Lambda_H, \tag{4.37}$$

the graph of φ_H.

We have already presented a precise description of K_g in the Subsection 1.3.1 under the condition $r_{\mathrm{conv}}(X, g) > 2$.

Now, take $\mathcal{K}_g \coloneqq K_g \overset{L}{\boxtimes} \mathbb{K}_{\{t \geq 0\}} \cong K_g \overset{L}{\boxtimes} \mathbb{K}_{\{0\}} \in \mathcal{D}(\mathbb{R}_z \times X \times X \times \mathbb{R}_t)$, we have

$$\mu s(\mathcal{K}_g) \subset \Lambda_H \sqcup 0_{\mathbb{R}_z \times X \times X}. \tag{4.38}$$

Directly, it seems that the microsupport condition of (2.8) is not satisfied on the zero section, since φ_H is only defined on $\dot{T}^*X$. But a feature of the geodesic flow is, if $(z, \zeta, q, p, q', p') \in \Lambda_H$, then $\zeta = -|p|_g = -|p'|_g$. So in case that one of them is non-zero, all of them are non-zero. Moreover, completeness of the metric g shows that the normalized geodesic flow is complete and short-term separating. Therefore, one can check carefully that the proof of Proposition 2.15 still works.

Let us compute P_{D^*X} here. For $\Omega = \{\zeta < 1\}$, we have $\widehat{\mathbb{K}_\Omega} = \mathbb{K}_{\{(z,t):-t \leq z \leq 0\}}$. Then we have

$$P_{D^*X} \cong \mathcal{K}_g \star \mathbb{K}_{\{(z,t):-t \leq z \leq 0\}} \cong K_g \circ \mathbb{K}_{\{(z,t):-t \leq z \leq 0\}}.$$

Now, if we restrict on $t \leq N$ for $N \in \mathbb{N}$, then we have

$$P_{D^*X}|_{\{t \leq N\}} \cong K_g \circ \mathbb{K}_{\{(z,t):-N \leq -t \leq z \leq 0\}} \cong (K_g)_{\{-N \leq z \leq 0\}} \circ \mathbb{K}_{\{(z,t):-N \leq -t \leq z \leq 0\}}.$$

Now, recall (1.24), we can take

$$K_g|_{(-2N,0]} \cong K_{g,-}^N \cong \mathrm{R}\pi_{(\mathbf{q}_1,\ldots,\mathbf{q}_{N-1})!}\mathbb{K}_{\mathcal{M}^N X},$$

where

$$\mathcal{M}^N X = \{(z, \mathbf{q}_0, \ldots, \mathbf{q}_N) : d(\mathbf{q}_i, \mathbf{q}_{i+1}) \leq -\frac{z}{N},\, i \in [N-1]_0,\, -2N < z \leq 0\},$$

is the discrete Moore path space and $[N-1]_0 = \{0, 1, \ldots, N-1\}$. Therefore, we have

$$P_{D^*X}|_{\{t \leq N\}} \cong \mathrm{R}\pi_{z!}\mathrm{R}\pi_{(\mathbf{q}_1,\ldots,\mathbf{q}_{N-1})!}\mathbb{K}_{\widetilde{\mathcal{M}}^N X} \cong \mathrm{R}\pi_{(\mathbf{q}_1,\ldots,\mathbf{q}_{N-1})!}\mathrm{R}\pi_{z!}\mathbb{K}_{\widetilde{\mathcal{M}}^N X},$$

where

$$\widetilde{\mathcal{M}}^N X = \{(z, \mathbf{q}_0, \ldots, \mathbf{q}_N, t) : d(\mathbf{q}_i, \mathbf{q}_{i+1}) \leq -\frac{z}{N},\, i \in [N-1]_0,\, -N \leq -t \leq z \leq 0\}.$$

Now, the restriction of the projection $\pi_z(z, \mathbf{q}_0, \ldots, \mathbf{q}_N, t) = (\mathbf{q}_0, \ldots, \mathbf{q}_N, t)$ on $\widetilde{\mathcal{M}}^N X$ is proper, and its fibers are closed intervals. Therefore, one can apply the Vietoris-Begle theorem to show that

$$\mathrm{R}\pi_{z!}\mathbb{K}_{\widetilde{\mathcal{M}}^N X} \cong \mathbb{K}_{\mathcal{M}_0^N X},$$

where

$$\begin{aligned}
\mathcal{M}_0^N X &\coloneqq \pi_{z!}(\widetilde{\mathcal{M}}^N X) \\
&= \{(\mathbf{q}_0, \ldots, \mathbf{q}_N, t) : d(\mathbf{q}_i, \mathbf{q}_{i+1}) \leq \frac{t}{N},\, i \in [N-1]_0,\, 0 \leq t \leq N\} \qquad (4.39)\\
&\subset \{(\mathbf{q}_0, \ldots, \mathbf{q}_N, t) : d(\mathbf{q}_i, \mathbf{q}_{i+1}) \leq 1,\, t \geq 0\}.
\end{aligned}$$

Therefore, we conclude that

Proposition 4.23. *For a complete Riemannian manifold (X, g), the microlocal kernel of its open unit disk bundle D^*X is given by*

$$P_{D^*X}|_{\{t \leq N\}} \cong \mathrm{R}\pi_{(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{N-1})!}\mathbb{K}_{\mathcal{M}_0^N X}.$$

4.2.2. The Viterbo isomorphism. Now, we are going to compute $F_\ell(D^*X, \mathbb{K})$. Recall (3.2) and Definition 3.2, we have

$$F_\ell(D^*X, \mathbb{K}) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}(P_{D^*X}^{\circledast \ell}).$$

By Proposition 4.23, we notice that P_{D^*X} is supported on $X \times X \times [0, \infty)$. Therefore, $F_\ell(D^*X, \mathbb{K})$ is supported on $[0, \infty)$.

Now, for $T \geq 0$, suppose $T \leq \ell N$ for some $N \in \mathbb{N}$, we have

$$F_\ell(D^*X, \mathbb{K})|_{[0,T]} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\mathrm{R}s_{t!}^\ell(P_{D^*X}|_{[0,T/\ell]})^{\overset{L}{\boxtimes}\ell} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}(P_{D^*X}|_{[0,T/\ell]})^{\circledast \ell}$$

Now, for $j \in \mathbb{Z}/\ell$, we apply the Proposition 4.23 to the j^{th} copy of $P_{D^*X}|_{[0,T/\ell]}^{\overset{L}{\boxtimes}\ell}$, which is the following sheaf

$$P_{D^*X}|_{\{t_j \leq T/\ell\}} \cong \mathrm{R}\pi_{(\mathbf{q}_1^j,\ldots,\mathbf{q}_{N-1}^j)!}\mathbb{K}_{(\mathcal{M}_0^N X)_j},$$

where $(\mathcal{M}_0^N X)_j$ is the j^{th}-copy of $\mathcal{M}_0^N X)$. In particular, we have $d(\mathbf{q}_i^j, \mathbf{q}_{i+1}^j) \leq 1$ for all i, j. Then we have

$$F_\ell(D^*X, \mathbb{K})|_{[0,T]} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\mathcal{L}_\ell^N X},$$

where

$$\mathcal{L}_\ell^N X := \left\{ (\underline{\mathbf{q}}, t) = (\mathbf{q}_i^j, t)_{i,j} : \begin{array}{c} d(\mathbf{q}_i^j, \mathbf{q}_{i+1}^j) \leq 1, \sum_{j=1}^{\ell} d(\mathbf{q}_i^j, \mathbf{q}_{i+1}^j) \leq t/N, \\ i \in [N-1]_0,\, j \in \mathbb{Z}/\ell,\, 0 \leq t \leq T \end{array} \right\}, \tag{4.40}$$

here we require $\mathbf{q}^j = \mathbf{q}_N^j = \mathbf{q}_0^{j+1}$. The space $\mathcal{L}_\ell^N X$ is a $\mathbb{Z}/\ell$-space and the action is given by $\sigma(t, \mathbf{q}_i^j)_{i,j} = (t, \mathbf{q}_i^{j+1})_{i,j}$, where σ is a generator of $\mathbb{Z}/\ell$. Therefore, we have

$$F_\ell(D^*X, \mathbb{K})|_T \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\mathcal{L}_{\ell,T}^N X},$$

where

$$\mathcal{L}_{\ell,T}^N X := \left\{ (\mathbf{q}_i^j)_{i,j} : \begin{array}{c} d(\mathbf{q}_i^j, \mathbf{q}_{i+1}^j) \leq 1, \sum_{j=1}^{\ell} d(\mathbf{q}_i^j, \mathbf{q}_{i+1}^j) \leq T/N, \\ i \in [N-1]_0,\, j \in \mathbb{Z}/\ell \end{array} \right\}, \tag{4.41}$$

is a $\mathbb{Z}/\ell$ space in the same way.

Now, consider the free loop space $\mathcal{L}X = Map(S^1, X)$. Here, the maps could be C^∞, L_1^2 or just C^0 maps. We equip $\mathcal{L}X$ with the compact-open topology. It is proven by Palais that no matter which model we choose, they are all homotopy equivalent with each other, see [**Pal68**, Theorem 13.14] or [**Kli83**, Theorem 1.2.10]. This is enough for us since we only study their homology groups. When we take L_1^2 maps, it is proven in [**Kli83**] that $\mathcal{L}X$ has a structure of Hilbert manifold; when we take C^∞ maps $\mathcal{L}X$ has a structure of Fréchet manifold. To be simpler, let us take C^∞ maps here. The length function

$$L : \mathcal{L}X \to,\, c \mapsto \int_c |\dot{c}|_g,$$

is a Morse function.

The free loop space $\mathcal{L}X$ is also a S^1-space, and the S^1-action is given by $(e^{i\theta} \cdot c)(t) = c(t+\theta)$. The length function is a S^1-invariant function. Let

$$\mathcal{L}_{\leq T}X = \{c \in \mathcal{L}X : L(c) \leq T\}.$$

The invariance of the length function shows $\mathcal{L}_{\leq T}X$ is also a S^1-space. In particular, $\mathcal{L}_{\leq T}X$ is restricted to a $\mathbb{Z}/\ell$-space. The $\mathbb{Z}/\ell$-action on $\mathcal{L}_{\leq T}X$ is given by $(\sigma \cdot c)(t) = c(t+1/\ell)$ (σ is the generator of $\mathbb{Z}/\ell$).

It follows the discussion of [**Mil63**], we can see that the closed subset $\mathcal{L}^N_{\ell,T}X$ is homeomorphic to a closed subset of the finite dimensional manifold X^N where $N = \lfloor T/2 \rfloor$. So, if we additionally assume that X is compact, then $\mathcal{L}^N_{\ell,T}X$ is compact.

It follows from the argument of loc. cit., that we have

Proposition 4.24. *The two spaces $\mathcal{L}^N_{\ell,T}X$ and $\mathcal{L}_{\leq T}X$ are $\mathbb{Z}/\ell$-equivariant homotopy equivalent.*

Proof. As $d(\mathbf{q}^j_i, \mathbf{q}^j_{i+1}) \leq 1$ for all i, j, then there are minimal geodesics $c^j_i : [0, 1/(N\ell)] \to X$ from $\mathbf{q}^j_i$ to $\mathbf{q}^j_{i+1}$, in particular, we have $L(c^j_i) = d(\mathbf{q}^j_i, \mathbf{q}^j_{i+1})$. Now, let $c^j : [0, 1/\ell] \to X$ be the concatenation of c^j_is, and $c : [0,1] \to X$ be the concatenation of c^js, then $c(\frac{i+1}{N\ell} + \frac{j-1}{\ell}) = \mathbf{q}^j_i$.

The condition $\sum_{j=1}^{\ell} d(\mathbf{q}^j_i, \mathbf{q}^j_{i+1}) \leq T/N$ makes sure $L(c) = \sum_{i,j} L(c^j_i) \leq \sum_i T/N = T$.

Then the piecewise geodesic map is given by $(\mathbf{q}^j_i) \mapsto c$. The actions here are $(\mathbf{q}^j_i) \mapsto (\mathbf{q}^{j+1}_i)$ and $c^j \mapsto c^{j+1}$. So the piecewise geodesic map is $\mathbb{Z}/\ell$-equivariant.

Finally, recall that the interpolation homotopy in [**Mil63**] can be taken piecewisely on each c^j. So we can take a $\mathbb{Z}/\ell$-equivariant homotopy equivalence between the identity map and the piecewise geodesic map. Then the proposition follows. $\square$

For the S^1-action. It is constructed by Abouzaid in [**Abo15**, Formula (11.79)] that there exists a homotopical S^1-action on $\mathcal{L}^N_{\ell,T}X$ which restricts to the $\mathbb{Z}/\ell$-action here, and the homotopy equivalence is S^1-equivariant. So, actually, the proposition provides us with a S^1-equivariant homotopy equivalent. Moreover, the cyclic structure of $F^{S^1}(D^*X,\mathbb{K})$ is compatible with Abouzaid's S^1-action.

Besides, since the action acts isometrically on $\mathcal{L}X$, and the isomorphism is compatible with the translation map along T. So we can conclude the following theorem.

Theorem 4.25. *For a compact manifold X, $T \in [0,\infty]$, we have*

$$H^q C_{\ell,T}(D^*X,\mathbb{K}) \cong H_{d-q}^{\mathbb{Z}/\ell}(\mathcal{L}_{\leq T}X,\mathbb{K}),$$

and we have

$$H^q C_T^{S^1}(D^*X,\mathbb{K}) \cong H_{d-q}^{S^1}(\mathcal{L}_{\leq T}X,\mathbb{K}).$$

Here the definition of $H^q C_T^{S^1}(D^*X,\mathbb{K})$ over any coefficient ring $\mathbb{K}$ follows from Remark 3.24.

As the $\mathbb{Z}/\ell$-action is exactly the restriction of the S^1-action on the free loop space, recall [**Vit97**, Appendix], we have

Corollary 4.26. *For $\ell \geq 3$, and any factor μ of ℓ, if $H_*^{S^1}(\mathcal{L}_{\leq T}X,\mathbb{Z})$ has no μ-torsion as an abelian group. Then for $T \in [0,\infty]$, we have*

$$H^q C_{\ell,T}(D^*X,\mathbb{Z}/\mu) \cong H^q C_{\ell,T}(D^*X,\mathbb{Z}) \overset{L}{\otimes} H^*(S^1,\mathbb{Z}/\mu) \cong H_{d-q}^{S^1}(\mathcal{L}_{\leq T}X,\mathbb{Z}) \overset{L}{\otimes} H^*(S^1,\mathbb{Z}/\mu).$$

An application for the Viterbo isomorphism is to compare the Chiu-Tamarkin cohomology of disk bundles with the symplectic cohomology of disk bundles, which is also known as the Viterbo isomorphism. The Viterbo isomorphism is first proposed by Viterbo in [**Vit99**], and is proven using generating function homology in [**Vit96**]. Later, Abbondandolo-Schwarz and Salamon-Weber prove it in different methods, see [**AS06, SW06**]. Then Kragh emphasizes the role of the Spin structure of the base in [**Kra18**]. See [**Abo15**] for a survey of the Viterbo isomorphism for symplectic cohomology.

Using the Viterbo isomorphism of the symplectic cohomology, we have:

Corollary 4.27. *For a compact simply connected spin manifold X. There are isomorphisms*

$$H^q C_{1,\infty}(D^*X,\mathbb{Z}) \cong H_{d-q}(\mathcal{L}X,\mathbb{Z}) \cong SH^q(\overline{D}^*X,\mathbb{Z}),$$
$$H^q C^{S^1}_{\infty}(D^*X,\mathbb{Z}) \cong H^{S^1}_{d-q}(\mathcal{L}X,\mathbb{Z}) \cong SH^q_{S^1}(\overline{D}^*X,\mathbb{Z}).$$

Also, when some local coefficients are considered, we can drop the Spin condition on the symplectic cohomology side (See [**Abo15, Kra18**]). But we do not know how local coefficients appear on the sheaf side.

4.2.3. Product structure. In this section, we will compute the product we constructed in Section 3.2. In particular, they are the same as the Chas-Sullivan product on the homology of free loop space. For the Chas-Sullivan product, we use the Thom collapse approach given in [**CJ02**] as the definition. We refer [**CHV06**] for more discussion on the string topology.

Let us review the notation of (4.41). In particular, let us denote

$$\mathcal{L}^N_T X = \mathcal{L}^N_{1,T} X = \{(\mathbf{q}_i)_i : d(\mathbf{q}_i,\mathbf{q}_{i+1}) \leq \min\{1, T/N\},\ i \in \mathbb{K}/N\}. \tag{4.42}$$

Then, when X is orientable and $0 \leq T \leq N$ for $N \in \mathbb{N}$, we have

$$H^{-q}C_{1,T}(D^*X,\mathbb{K}) \cong \mathrm{Ext}^{-q}(\mathbb{K}_{\mathcal{L}^N_T X}, \mathbb{K}_{X^N}[-d]). \tag{4.43}$$

Now, take $\alpha \in H^{-a}C_{1,A}(D^*X,\mathbb{K})$, and $\beta \in H^{-b}C_{1,B}(D^*X,\mathbb{K})$. Let us assume $0 \leq A, B \leq N$, then the identification (4.43) presents α and β as follow:

$$\alpha : \mathbb{K}_{\mathcal{L}^N_A X} \to \mathbb{K}_{X^N}[-a-d], \qquad \beta : \mathbb{K}_{\mathcal{L}^N_B X} \to \mathbb{K}_{X^N}[-b-d],$$

and $\alpha \overset{L}{\boxtimes} \beta$ corresponds to

$$\alpha \overset{L}{\boxtimes} \beta : \mathbb{K}_{\mathcal{L}^N_A X \times \mathcal{L}^N_B X} \to \mathbb{K}_{X^{2N}}[-a-b-2d].$$

Next, apply the collapsing map, we have

$$\mathbb{K}_{\mathcal{L}^N_A X \times \mathcal{L}^N_B X} \to \mathbb{K}_{X^{2N}}[-a-b-2d] \to \mathbb{K}_{X^{2N} \cap \{\mathbf{q}_N = \mathbf{q}_{2N}\}}[-a-b-2d].$$

It is a class in

$$\begin{aligned} &\mathrm{Ext}^{-a-b-2d}(\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X}, \mathbb{K}_{X^{2N}\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}}) \\ \cong &\mathrm{Ext}^{-a-b-2d}(\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}}, \mathbb{K}_{X^{2N}\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}}). \end{aligned}$$

Notice that, if we apply the piecewise geodesic map , we will see that the homotopy type of $\mathcal{L}_A^N X \times \mathcal{L}_B^N X \cap \{\mathbf{q}_N = \mathbf{q}_{2N}\}$ is the space of free composible loops:

$$\mathcal{F}_{A+B}X = \{(c_1, c_2) \in \mathcal{L}_{\leq A}X \times \mathcal{L}_{\leq B}X : c_1(0) = c_2(0)\}.$$

Next, we apply the Gysin map to the pair $(X^{2N}, X^{2N} \cap \{\mathbf{q}_N = \mathbf{q}_{2N}\})$, we have a class

$$\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}} \to \mathbb{K}_{X^{2N}\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}} \xrightarrow{e} \mathbb{K}_{X^{2N}}[d],$$

which corresponds to $e\circ\alpha\overset{L}{\boxtimes}\beta$. Under the piecewise geodesic construction, it corresponds to applying the Gysin map associated to the pair $(\mathcal{L}_{A+B}X, \mathcal{F}_{A+B}X)$.

Now, we have an element

$$e \circ \alpha\overset{L}{\boxtimes}\beta \in \mathrm{Ext}^{-a-b-d}(\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}}, \mathbb{K}_{X^{2N}}),$$

which defines a class, that we still call $e \circ \alpha\overset{L}{\boxtimes}\beta$, in $\mathrm{Ext}^{-a-b-d}(\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X}, \mathbb{K}_{X^{2N}})$ by composing the natural morphism $\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X} \to \mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X\cap\{\mathbf{q}_N=\mathbf{q}_{2N}\}}$ induced by the corresponding closed inclusion.

On the other hand, the construction of $s(D^*X)$ induces the isomorphism after applying adjoint isomorphisms:

$$s(D^*X): \quad \mathrm{Ext}^{-a-b-d}(\mathbb{K}_{\mathcal{L}_{A+B}^{2N}X}, \mathbb{K}_{X^{2N}}) \cong \mathrm{Ext}^{-a-b-d}(\mathbb{K}_{\mathcal{L}_A^N X\times\mathcal{L}_B^N X}, \mathbb{K}_{X^{2N}}),$$

which is the tautological map that turns a figure 8 type curve in X into a closed curve by forgetting the crossing point of 8.

So, we have

$$e \circ \alpha\overset{L}{\boxtimes}\beta \circ s(D^*X) \in \mathrm{Ext}^{-a-b-d}(\mathbb{K}_{\mathcal{L}_{A+B}^{2N}X}, \mathbb{K}_{X^{2N}}) \cong \mathrm{Ext}^{-a-b}(F_1(D^*X, \mathbb{K})_{A+B}, \mathbb{K}[-d]),$$

which represents $\alpha \cup \beta$.

Now, we track all steps and applies the piecewise geodesic construction in all steps, we see that $\alpha \cup \beta$ represent the Chas-Sullivan product of α, β in $H_{a+b+d}(\mathcal{L}_{\leq A+B}X, \mathbb{K})$.

Consequently, we proved that

THEOREM 4.28. *For a compact orientable manifold X, the Viterbo isomorphism*

$$H^q C_{1,T}(D^*X, \mathbb{K}) \cong H_{d-q}(\mathcal{L}_{\leq T}X, \mathbb{K}),$$

is an isomorphism of $\mathbb{K}$-algebras with respect to the cup product on the Chiu-Tamarkin homology and the Chas-Sullivan product on the string topology.

Interpolate the Viterbo isomorphism of symplectic cohomology and the comparison of products given by Abbondandolo-Schwarz in [**AS10**]. We conclude that

Corollary 4.29. *For a compact simply connected spin manifold X, we have an isomorphism of rings*

$$H^q C_{1,T}(D^*X, \mathbb{Z}) \cong SH^q(\overline{D}^* X, \mathbb{Z}),$$

where we equip the cup product on the Chiu-Tamarkin cohomology and the pair-of-pants product on symplectic cohomology.

CHAPTER 5

Discussion

Gutt and Hutchings constructed a sequence of capacities $(c_k^{\mathrm{GH}})_{k\in\mathbb{N}}$ in [**GH18**] based on the positive S^1-equivariant symplectic homology. They computed c_k^{GH} for convex toric domains and concave toric domains. For example, when X_Ω is convex, they showed that

$$(5.1)\quad c_k^{\mathrm{GH}}(X_\Omega) = \min\left\{\|v\|_\Omega^* : v\in\mathbb{N}^d, \sum_{i=1}^d v_i = k\right\} = \inf\left\{T\geq 0 : \exists z\in\Omega_T^\circ, I(z)\geq k\right\},$$

where $\|v\|_\Omega^* = \max\{\langle v,w\rangle : w\in\Omega\}$.

Therefore, $c_k(X_\Omega) = \bar{c}_k(X_\Omega) = c_k^{\mathrm{GH}}(X_\Omega)$ by (5.1) and Theorem 4.8.

On the other hand, one may ask how about the concave case. It is explained in Remark 4.5 that some technical issues exist. So we can not derive a clear structure theorem as Theorem 4.6, and then the computation of capacities is also not completely clear. But manual computation of some examples shows the coincidence with Gutt-Hutchings capacities is still true.

Based on the computation on the convex toric domains and concave toric domains, Gutt and Hutchings conjectured ([**GH18**, Conjecture 1.9]) that, for a bounded star-shaped domain U and for all $k\in\mathbb{N}$,

$$c_k^{\mathrm{EH}}(U) = c_k^{\mathrm{GH}}(U).$$

In fact, the result $c_1^{\mathrm{EH}}(U) = c_1^{\mathrm{GH}}(U) =$ Minimal action has been proven by Irie [**Iri19**] for convex bodies U. Comparing to our results, we hope the consistency could be extended to c_k and $\bar{c}_k$ as well.

Conjecture 5.1. *For a convex domain U and for all $k\in\mathbb{N}$, we have*

$$c_k^{EH}(U) = c_k^{GH}(U) = c_k(U) = \bar{c}_k(U).$$

We also hope for a cohomology level correspondence between the Chiu-Tamarkin cohomology and the symplectic cohomology.

Conjecture 5.2. *Let X be simply connected and spin, and let $U \subset T^*X$ be a bounded open set with contact boundary such that $\overline{U}$ is a Liouville domain with respect to the standard contact form induced from T^*X. Then we have an isomorphism of $\mathbb{Q}(u)$-vector space:*

$$SH^q_{S^1}(\overline{U}, \mathbb{Q}) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u) \to H^q C^{S^1}_{\infty}(U, \mathbb{Q}) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u)$$

and an isomorphism of $\mathbb{Q}$-algebra:

$$SH^q(\overline{U}, \mathbb{Q}) \to H^q C_{1,\infty}(U, \mathbb{Q}),$$

with respect to the pair of pants product and the cup product.

Also, we should have isomorphisms of filtered version on both sides.

The conjecture is already proved for disk bundle $U = D^*X$ in Theorem 4.25 and for convex toric domain X_Ω in Theorem 4.6. In particular, the convex toric domain case explains that why we need a tensor product $\otimes_{\mathbb{Q}[u]}\mathbb{Q}(u)$ because $SH^q_{S^1}(X_\Omega, \mathbb{Q})$ is free as $\mathbb{Q}[u]$ module while $H^q C^{S^1}_{\infty}(X_\Omega, \mathbb{Q})$ is not. So one may also ask, what is the corresponding $\mathbb{Q}[u]$-torsion part on the symplectic cohomology side.

Actually, by our construction for the S^1 equivariant Chiu-Tamarkin complex, we guess that it is a cyclic cohomology of the corresponding (dg)-Tamarkin category. I.e.

Conjecture 5.3. *For an admissible open set U, we have*

$$HC^*_T(\mathcal{D}_{U,dg}(X)) \cong H^* C^{S^1}_T(U, \mathbb{K}).$$

When the cyclic permutation is forgotten, we also hope an isomorphism

$$HH^*_T(\mathcal{D}_{U,dg}(X)) \cong H^* C_{1,T}(U, \mathbb{K}).$$

*Here, we hope there exists some filtered version of Hochschild/Cyclic cohomology HH^*_T/HC^*_T of a dg triangulated persistence category (see [**BCZ21**]), and $\mathcal{D}_{U,dg}(X)$ stands for the dg version of triangulated persistence category of the Tamarkin category.*

The conjecture is based on the following ideas:

• The similarity of the Chiu-Tamarkin complex and the symplectic homology has been observed by Zhang in [**Zha20**, Section 4.8].

• A similar idea has been realized for the non-equivariant case for the derived category of coherent sheaves over an algebraic variety V, see [**Kuz09**]. Kuznetsov shows that for an admissible subcategory $\mathcal{A}$ of $D^b(V)$, the derived category of coherent sheaves over an algebraic variety V, the Hochschild homology $HH_*(\mathcal{A})$ is isomorphic to $\mathrm{RHom}(P, P \otimes \omega_{V\times V/V}[\mathrm{dim}V])$, where P is the kernel of the projector associated with the admissible subcategory $\mathcal{A}$.

• Jones' work on equivariant homology [**Jon87**] studied the notion of cyclic space, which is an analog of simplicial space. Then he shows that we can construct a S^1-action on the geometric realization of a cyclic space. Subsequently, equivariant (co)homology is exactly a version of cyclic (co)homology. For the sheaf theory over cyclic space, a similar construction without the group action has been studied by Deligne, say the theory of simplicial sheaves (see [**Del74**]). The correspondence algebraic theory provides us with a point of view of the Hochschild cohomology. Then it goes back to the last paragraph.

• A comparison of the wrapped Fukaya category and the category of wrapped sheaves is established in [**GPS18a**] by Ganatra-Shende-Pardon. It is known that the S^1-equivariant/non-equi. symplectic cohomology are isomorphic to the Cyclic/Hochschild cohomology of wrapped Fukaya category for Weinstein manifolds (see [**CRGG17, Gao17b, GPS18b, Gan19**]). So, we hope that some variant of the Ganatra-Shende-Pardon's result and the conjecture Conjecture 5.3 combine to prove the Conjecture 5.2.

• Generating functions can also play a role. In this paper, we exhibit the existence of P_U using the sheaf quantization of Guillermou-Kashiwara-Schapira, whose construction is generating function theory natural[**GKS12**]. We can also see the construction of [**Tam18**] using generating functions. We also hope to know a construction of the Chiu-Tamarkin complex from the generating function, and then it is possible to deduce a correspondence between the generating function homology and the Chiu-Tamarkin homology.

On the other hand, Viterbo constructs a sheaf quantization of compact Lagrangian submanifold in cotangent bundles using the Lagrangian Floer theory [**Vit19**]. Hence, it is hopeful to construct a sheaf quantization using the Hamiltonian Floer theory, which is closer to the theory of symplectic cohomology. It is possible to construct a sheaf quantization of Hamiltonian isotopy using Floer theory. Then we hope that the uniqueness of the sheaf quantization of Hamiltonian isotopy can help us to identify it with the GKS construction. This is also a possible way to Conjecture 5.2.

APPENDIX A

Equivariant sheaves and equivariant derived categories

Here, we review basic notions about equivariant sheaves. We refer to [**BL94**] for all details about the general theory of equivariant sheaves and equivariant derived categories. On the other hand, to use the 6-operations on the unbounded derived category, we adapt the technical framework of [**KS06**] and [**SS16**]. All topological spaces are assumed to be locally contractible and paracompact to make sure the cohomology of constant sheaf is isomorphic to the singular cohomology.

A.1. Equivariant sheaves

For a topological space X with a G action $\rho : G \times X \to X$, a *G-equivariant sheaf* is a pair (F, θ) where $F \in Sh(X)$ and $\theta : \rho^{-1}F \cong \pi_G^{-1}F$ is an isomorphism of sheaves satisfying the cocycle conditions:

$$d_0^{-1}\theta \circ d_2^{-1}\theta = d_1^{-1}\theta, \quad s_0^{-1}\theta = \mathrm{Id}_F.$$

Here

$$\begin{aligned} d_0(g,h,x) &= (h, g^{-1}x), \\ d_1(g,h,x) &= (gh, x), \\ d_2(g,h,x) &= (g, x), \\ s_0(x) &= (e, x). \end{aligned}$$

A sheaf morphism between two G-equivariant sheaves is equivariant if it commutes with the θ's. We let $Sh_G(X)$ be the category of G-equivariant sheaves and equivariant sheaf morphisms. For example, when $X = \mathrm{pt}$ and G is discrete, $Sh_G(X) \simeq \mathbb{K}[G] - \mathrm{Mod}$, the category of all G-modules. The category of G-equivariant sheaves $Sh_G(X)$ is Abelian. Moreover, Grothendieck proved in [**Gro57**] that when G is discrete, $Sh_G(X)$ admits enough injective objects. Therefore, the derived category $D(Sh_G(X))$ makes sense, which is treated as a naive version of the equivariant derived category of sheaves.

A.2. Equivariant derived categories

For general topological groups, the naive version is not as good as we expected. A basic difference is the hom space $\mathrm{RHom}_{D(Sh_G(X))}(\mathbb{K}_X, \mathbb{K}_X)$ is not isomorphic to the equivariant cohomology of X, which is true for non-equivariant sheaf. A more serious problem is how to define the 6-operations.

To resolve these problems, we must use the equivariant derived category $D_G(X)$ defined by Burnstein-Lunts, where the expected isomorphism holds, and the correct 6-operations live.

Let us assume G is a compact Lie group in this article. Then there is a universal bundle EG and a classifying space BG, which are unique up to homotopy. Because G is a compact Lie group, there exists a family of finite dimensional approximations. Let us fix a model of EG and BG. That means $EG = \cup_m EG_m$ and $BG = \cup_m BG_m$ where $\pi_m : EG_m \to BG_m$ is a principal G-bundle and BG_m are compact manifolds with $\pi_i(EG_m) = 0$ for $m >> i$, and both of inclusions $EG_m \subset EG_{m+1}$ and $BG_m \subset BG_{m+1}$ are closed embeddings of submanifolds. We equip EG and BG with the weak topology.

Now, we have a diagram of topological spaces:

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X \times_G EG.$$

Definition A.1. An object $F \in D_G(X)$ is a triple $F = (F_X, \overline{F}, \beta_F)$, where $F_X \in D(X)$, $\overline{F} \in D(X \times_G EG)$, and $\beta_F : p^{-1}F_X \to q^{-1}\overline{F}$ is an isomorphism in $D(X \times EG)$. A morphism $\alpha : F \to H$ is a pair $(\alpha_X, \overline{\alpha})$ where $\alpha_X : F_X \to H_X$, $\overline{\alpha} : \overline{F} \to \overline{H}$, and a commutative diagram in $D(X \times EG)$:

$$\begin{tikzcd} p^{-1}F_X \arrow[r, "\beta_F"] \arrow[d, "p^{-1}\alpha_X"'] & q^{-1}\overline{F} \arrow[d, "q^{-1}\overline{\alpha}"] \\ p^{-1}H_X \arrow[r, "\beta_H"] & q^{-1}\overline{H}. \end{tikzcd}$$

For example, the equivariant constant sheaf is given by $\mathbb{K}_X^G = (\mathbb{K}_X, \mathbb{K}_{X\times_G EG}, \mathrm{Id}_{\mathbb{K}_{EG}})$.

Bernstein and Lunts constructed a triangulated structure and the 6-operations on the equivariant derived category.

We say $F \to F' \to F'' \xrightarrow{+1}$ is distinguished if $\overline{F} \to \overline{F'} \to \overline{F''} \xrightarrow{+1}$ is. Then one can check that we define a triangulated structure on $D_G(X)$.

One also can define the canonical t-structure of $D_G(X)$ in the same way as the non-equivariant case. One can prove that the heart of the t-structure is isomorphic to $Sh_G(X)$ (see [**BL94**, Appendix B]).

We have a forgetful functor $For : D_G(X) \to D(X)$ which is given by

$$F = (F_X, \overline{F}, \beta_F) \mapsto F_X.$$

The forgetful functor is a triangulated functor. We need the following fact in our main discussion.

Proposition A.2. *The forgetful functor $For : D_G(X) \to D(X)$ is a conservative functor. Precisely, it means that, for a morphism $\alpha : F \to H$ in $D_G(X)$, if $\alpha_X : F_X \to H_X$ is an isomorphism in $D(X)$, then α is an isomorphism.*

Proof. Taking the cone C of α, then we have a distinguished triangle

$$F \to H \to C \xrightarrow{+1} .$$

By definition, we have the distinguished triangle in $D(X \times_G EG)$:

$$\overline{F} \to \overline{H} \to \overline{C} \xrightarrow{+1} .$$

Then we have the isomorphism of distinguished triangle in $D(X \times EG)$:

$$\begin{tikzcd}
q^{-1}\overline{F} \arrow[r] \arrow[d, "\beta_F"] & q^{-1}\overline{H} \arrow[r] \arrow[d, "\beta_H"] & q^{-1}\overline{C} \arrow[r, "+1"] \arrow[d, "\beta_C"] & {} \\
p^{-1}F_X \arrow[r] & p^{-1}H_X \arrow[r] & p^{-1}C_X \arrow[r, "+1"] & {}
\end{tikzcd} .$$

Now, because α_X is an isomorphism, we have $C_X \cong 0$ and then $p^{-1}C_X \cong 0$. So we have $q^{-1}\overline{C} \cong 0$. Then we can conclude that $\overline{C} \cong 0$ since q is surjective. $\square$

Generally, suppose K is a Lie subgroup of G. Because G acts on EG freely, then so does K. In particular, we can take $EK = EG$. Now, we have a fibration

$$G/K \to X \times_K EG/K \xrightarrow{q_{G,K}} X \times_G EG.$$

Now, take $F = (F_X, \overline{F}, \beta_F) \in D_G(X)$, then $(F_X, q_{G,K}^{-1}\overline{F}, \beta_F)c$ defines an object of $D_K(X)$. It defines a triangulated functor, which is the restriction functor:

$$Res_K^G : D_G(X) \to D_K(X).$$

The forgetful functor is a particular case, where K is the trivial subgroup. This can be seen either from that EK could be taken as one point, or the following discussion about the quotient space functor.

The quotient space functor is $\mathrm{quo}^{-1} : D(X/G) \to D_G(X)$, $\mathrm{quo}^{-1}(F) = (q'^{-1}F, p'^{-1}F, \beta_F)$, where we have a commutative diagram:

$$\begin{array}{ccc} X \times EG & \xrightarrow{p} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ X \times_G EG & \xrightarrow{p'} & X/G, \end{array}$$

and β_F is the isomorphism $p^{-1}q'^{-1}F \cong q^{-1}p'^{-1}F$. When X is a free G-space, the quotient functor is an equivalence ([**BL94**, 2.2.1]).

On the other hand, we have another functor $D_G(X) \to D(X \times_G EG)$, $F \mapsto \overline{F}$. One can show that this functor is fully-faithful, and the essential image is

$$D(X \times_G EG|p) = \{\overline{F} : \exists F_X, \beta, \text{ such that} \beta : p^{-1}F_X \cong q^{-1}\overline{F}\}, \tag{A.1}$$

see [**BL94**, 2.3.2]. In particular, one has that $D_G(\mathrm{pt})$ is equivalent to the full subcategory of $D(BG)$ which consist of complexes of locally constant cohomology ([**BL94**, 2.7.2]).

For a G-map $f : X \to Y$, we define maps induced from f as follows:

$$
\begin{tikzcd}
X \arrow[d, "f"] & X \times EG \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\hat{f}"] & X \times_G EG \arrow[d, "\overline{f}"] \\
Y & Y \times EG \arrow[l, "p'"] \arrow[r, "q'"'] & Y \times_G EG.
\end{tikzcd}
$$

Now $F, F' \in D_G(X)$, $H \in D_G(Y)$ in the equivariant derived categories, we define the 6-operations:

$$
\begin{aligned}
&F \overset{L}{\otimes} F' = (F_X \overset{L}{\otimes} F'_X, \overline{F} \overset{L}{\otimes} \overline{F'}, \beta_F \overset{L}{\otimes} \beta_{F'}), \\
&\mathrm{R}\mathcal{H}om_G(F, F') = (\mathrm{R}\mathcal{H}om(F_X, F'_X), \mathrm{R}\mathcal{H}om(\overline{F}, \overline{F'}), \mathrm{R}\mathcal{H}om(\beta_F, \beta_{F'})), \\
&\mathrm{R}f_* F = (\mathrm{R}f_* F_X, \mathrm{R}\overline{f}_* \bar{F}, \mathrm{R}\hat{f}_* \beta_F), \\
&f^{-1} H = (f^{-1} H_X, \overline{f}^{-1} \overline{H}, \hat{f}^{-1} \beta_H)).
\end{aligned}
$$

For $f_!$ and $f^!$, we need to assume f is separated locally proper (it is true when X, Y are Hausdorff and locally compact) and $f_!$ has finite cohomological dimension. Then we set

$$
\begin{aligned}
&\mathrm{R}f_! F = (\mathrm{R}f_! F_X, \mathrm{R}\overline{f}_! \bar{F}, \mathrm{R}\hat{f}_! \beta_F), \\
&f^! H = (f^! H_X, \overline{f}^! \overline{H}, \hat{f}^! \beta_H).
\end{aligned}
$$

Then we have

Proposition A.3. *All properties of the 6-operations hold in the equivariant case under the condition that f is separated locally proper and $f_!$ has finite cohomological dimension.*

In particular, we have the equivariant Verdier duality: For $F, F' \in D_G(X)$, we have

$$
\mathrm{RHom}_G(\mathrm{R}f_! F, F') \cong \mathrm{RHom}(F, f^! F').
$$

Using the Verdier duality, we can construct another important equivariant sheaf: the dualizing complex. For the constant map $a_X : X \to \mathrm{pt}$, the non equivariant dualizing complex is $\omega_X = a_X^! \mathbb{K}$. So, we use the same formula $\omega_X = \omega_X^G = a_X^! \mathbb{K} \in D_G(X)$. Therefore, our definition of $a_X^!$ shows

$$
\omega_X = \omega_X^G = (a_X^! \mathbb{K}, \overline{a_X}^! \mathbb{K}_{BG}, \widehat{a_X}^! \mathrm{Id}_{\mathbb{K}_{EG}}).
$$

Remark A.4. In [**BL94**], the authors use the approximation EG_m to define these functors. The reason is, classically, the 6-operations and related propositions (especially the proper base change) are demonstrated for finite (cohomological) dimensional locally compact Hausdorff spaces. But usually, $X \times_G EG$ is not in this class.

However, in the framework of [**SS16**], the authors introduce a relative notion called separated locally proper maps, for which a proper base change formula is true. In particular, here our $\hat{f}$ and $\overline{f}$ are separated locally proper if f is, and $\hat{f}_!$ and $\overline{f}_!$ have finite cohomological dimension if $f_!$ has finite cohomological dimension. Consequently, we can provide those simpler formulas for the equivariant 6-operations, and they also work in the unbounded derived category.

On the other hand, suppose F is concentrated in an interval I. Then, for sufficient big m with $|I| \leq m$, we can replace EG and BG by EG_m and BG_m in the definition of the equivariant derived category and the 6-operations. This is close to the original approach of Bernstein-Lunts.

Then obviously, the 6-operations commute with the restriction functors, in particular the forgetful functor. So, in practice, we usually denote an object $F \in D_G(X)$ by the non-equivariant part F_X of F.

To compare the derived category of equivariant sheaves and the equivariant derived category, we start from a triangulated functor:

$$i : D(Sh_G(X)) \to D_G(X).$$

The definition is as follows: First, one can show that if G acts on Y freely, then the quotient map $q : Y \to Y/G$ induces a equivalence between $q^{-1} : Sh(Y/G) \to Sh_G(Y)$. Whose quasi inverse is given by $q^G : Sh_G(Y) \to Sh(Y/G)$, where F is mapped to the G-invariant subsheaf $q_*^G(F) = (q_*F)^G$.

Now, for $H \in D(Sh_G(X))$, we define

$$i(H) = (H, q_*^G(p^{-1}H), \beta_{i(H)}),$$

where

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X \times_G EG,$$

and $\beta_{i(H)}$ is the isomorphism that defines the quasi-inverse pair (q^{-1}, q_*^G).

One can show that i defines an equivalence between Abelian categories: $Sh_G(X)$ and the heart of $D_G(X)$. Hence, i is essentially surjective. However, in general, i is not fully-faithful. So it does not define an equivalence between triangulated categories.

But for discrete group, this is true.

Theorem A.5 ([**BL94**, Theorem 8.3.1]). *For a discrete group G, the triangulated functor i defined as before is an equivalence between $D(Sh_G(X))$ and $D_G(X)$.*

Under this identification, both the naive and advanced versions of equivariant sheaves are equivalent, i.e., $D(Sh_G(X)) \simeq D_G(X)$. In particular, $D(\mathbb{K}[G] - Mod) \simeq D_G(\mathrm{pt})$. So for us, $D(Sh_G(X))$ is enough for our applications. In practice, for a discrete group G, we always write everything in the usual derived category and run the machine of equivariant derived category implicitly. As a rule of convenience, we only write a lower subscript G for all possible places to indicate that we are working on a version of equivariant categories without mentioning which one we are really working on.

In this thesis, our main working example is $G = S^1$ and $K = \mathbb{Z}/\ell \subset G$ for $\ell \in \mathbb{N}$. Then, we take $EG = S^\infty$ and $EG_m = S^{2m+1}$. Since $\mathbb{Z}/\ell$ is discrete, we adapt the remark above: We work on $D_{\mathbb{Z}/\ell}(X)$ only when we need 6-operation. Otherwise we use $D(Sh_{\mathbb{Z}/\ell}(X))$ and treat objects there as complex of sheaves equipped with a $\mathbb{Z}/\ell$-action.

APPENDIX B

Equivariant cohomology, Borel-Moore homology, and equivariant Borel-Moore homology

In this appendix, we assume X is a locally contractible locally compact Hausdorff topological space (a manifold for example), and G is a compact Lie group as in the last section, we additionally assume that the finite dimensional approximations EG_m and BG_m are orientable.

B.1. Equivariant cohomology

Using the equivariant 6-operations, we have the notion of equivariant sheaf cohomology. Precisely, let $F \in D_G(X)$ and for $a_X : X \to \mathrm{pt}$ be the constant map.

The *equivariant cohomologies* are defined by

$$H^*_G(X, F) = \mathrm{R}^*\Gamma(BG, \mathrm{R}\overline{a_X}_*\overline{F}),$$
$$H^*_{c,G}(X, F) = \mathrm{R}^*\Gamma(BG, \mathrm{R}\overline{a_X}_!\overline{F}).$$

First, let us study the equivariant cohomology of the constant sheaf $\mathbb{K}_X = (\mathbb{K}_X, \mathbb{K}_{X\times_G EG}, \mathrm{Id})$.

$$\begin{aligned} H^*_G(X, \mathbb{K}_X) =& H^*\mathrm{R}\Gamma(BG, \mathrm{R}\overline{a_X}_*\mathbb{K}_{X\times_G EG}) \\ =& H^*\mathrm{R}a_{BG*}\mathrm{R}\overline{a_X}_*\mathbb{K}_{X\times_G EG} \\ =& H^*\mathrm{R}a_{X\times_G EG*}\mathbb{K}_{X\times_G EG} \\ =& H^*\mathrm{R}\Gamma(X\times_G EG, \mathbb{K}_{X\times_G EG}). \end{aligned}$$

Under our assumption of X, the last is the Borel equivariant cohomology of X. Directly, we have the expected isomorphism

(B.1) $$\mathrm{Ext}^*_G(\mathbb{K}_X, \mathbb{K}_X) \cong H^*_G(X, \mathbb{K}_X) \cong H^*_{G,Borel}(X, \mathbb{K}),$$

where the latter is the Borel equivariant cohomology of X.

In particular, when $X = \mathrm{pt}$ is a point, we have

$$\mathrm{Ext}^*_G(\mathbb{K},\mathbb{K}) \cong H^*_G(\mathrm{pt},\mathbb{K}) \cong H^*_{G,Borel}(\mathrm{pt},\mathbb{K}) \cong H^*(BG,\mathbb{K}). \tag{B.2}$$

For example,

$$\begin{aligned} &\mathrm{Ext}^*_{S^1}(\mathbb{Z},\mathbb{Z}) \cong H^*(\mathbb{C}P^\infty,\mathbb{Z}) \cong \mathbb{Z}[u], \\ &\mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K}) \cong H^*(L^\infty_\ell,\mathbb{K}) \cong \mathbb{K}[u,\theta], \end{aligned} \tag{B.3}$$

where $L^\infty_\ell = S^\infty/(\mathbb{Z}/\ell)$ is the infinite dimensional lens space, $\mathbb{K}$ is a finite field of $\mathrm{char}(\mathbb{K})|\ell$, $|u| = 2$, $|\theta| = 1$, and $\theta^2 = ku$ ($k = 0$ if ℓ is odd and $k = \ell/2$ otherwise). The second computation can be found in [**Hat02**, Example 3E.2, Exercise 3E.1].

B.2. Borel-Moore homology, and equivariant Borel-Moore homology

Definition B.1. The Borel-Moore homology of X is defined as

$$H^{BM}_i(X,\mathbb{K}) \coloneqq H^{-i}\mathrm{R}(X,\omega_X) \cong \mathrm{Ext}^{-i}\left(\mathrm{R}\Gamma_c(X,\mathbb{K}_X),\mathbb{K}\right).$$

When X is a G-space, we define the equivariant Borel-Moore homology to be:

$$H^{BM,G}_i(X,\mathbb{K}) \coloneqq H^{-i}\mathrm{R}(X,\omega^G_X) \cong \mathrm{Ext}^{-i}_G\left(\mathrm{R}\Gamma_c(X,\mathbb{K}_X),\mathbb{K}\right).$$

Then, because EG_m and BG_m are orientable, we have $\overline{a_X}^!\mathbb{K}_{BG_m} \cong \mathbb{K}_{BG_m}[-\dim BG_m]$. Consequently, there is

$$H^{BM,G}_i(X,\mathbb{K}) \cong H^{BM}_{i+\dim BG_m}(X \times_G EG_m,\mathbb{K}),$$

here we take $i << m$.

By the equivariant Verdier duality, we have the equivariant Poincaré duality when $\mathbb{K}$ is a field:

$$H^{BM,G}_i(X,\mathbb{K}) \cong H^i_G(X,\mathbb{K}),\ \forall i \in \mathbb{Z}.$$

Since X is locally contractible, we have

$$H^{BM}_i(X,\mathbb{K}) \cong H_i(S^{lf}_i(X,\mathbb{K})),$$

where $S_*^{lf}(X,\mathbb{K})$ is the chain complex of locally finite singular chain in X. This is also true for the equivariant version since EG_m and BG_m are manifolds.

Using the locally finite chain formulation, we have

$$H_i^{BM}(X,\mathbb{K}) \cong H_*(M, M \setminus X, \mathbb{K}),$$

where $j : X \to M$ is an (arbitrary) embedding and M is compact. In particular, one can take one point compactification $(M, M \setminus X) = (\hat{X}, \infty)$, and then the Borel-Moore homology computes the reduced homology of $\hat{X}$. Also, when X itself is compact, the Borel-Moore homology is the same as the singular homology.

When X is an orientable manifold of dimension d, one can show that $\omega_X \cong \mathbb{K}_X[-d]$ in both non-equivariant and equivariant derived categories. So, we have the following Poincaré duality.

$$H_G^i(X,\mathbb{K}) \cong H_{d-i}^{BM,G}(X,\mathbb{K}),\ H^i(X,\mathbb{K}) \cong H_{d-i}^{BM}(X,\mathbb{K}).$$

Therefore, we can assign a fundamental class from $(1,\ldots,1) \in H^0 \cong \mathbb{K}^{\pi_0(X)}$:

$$[X]^G \in H_d^{BM,G}(X,\mathbb{K}),\ [X] \in H_d^{BM}(X,\mathbb{K}).$$

Moreover, since the forgetful functor from equivariant to non-equivariant derived category commutes with the 6 operations, we have a forgetful map:

$$H_i^{BM,G}(X,\mathbb{K}) \to H_i^{BM}(X,\mathbb{K}),$$

which maps $[X]^G$ to $[X]$.

The Yoneda product, let us state the equivariant case only, here is

$$\mathrm{Ext}_G^i(\mathbb{K}_X,\mathbb{K}_X) \otimes \mathrm{Ext}_G^{-j}(\mathbb{K}_X,\omega_X) \to \mathrm{Ext}_G^{i-j}(\mathbb{K}_X,\omega_X).$$

Using the definition of Borel-Moore homology and (B.1), the product is tautologically:

$$H_G^i(X,\mathbb{K}) \otimes H_j^{BM,G}(X,\mathbb{K}) \to H_{j-i}^{BM,G}(X,\mathbb{K}).$$

Then the Yoneda product turns that $H_j^{BM,G}(X,\mathbb{K})$ into a graded module over $H_G^i(X,\mathbb{K})$.

One can verify, in the non-equivariant case, that if X is compact, then this module structure is given by the cap product of the singular chain (see [**Ive86**]). So we also call the module structure the cap product. Now, since we work on the equivariant case, $H^i_G(X, \mathbb{K})$ is an algebra over $H^i_G(\mathrm{pt}, \mathbb{K})$. So, $H^{BM,G}_j(X, \mathbb{K})$ is also a module over $H^i_G(\mathrm{pt}, \mathbb{K})$.

B.3. The restriction map

Now, suppose K is a Lie subgroup of G. Consider the restriction functor $Res^G_K : D_G(X) \to D_K(X)$. Then, it is direct to see that $Res^G_K(\mathbb{K}_X) = \mathbb{K}_X$ and $Res^G_K(\omega^G_X) = \omega^K_X$. Then the restriction functor induces the restriction map of equivariant cohomology and equivariant Borel-Moore holomogy:

$$H^*_G(X, \mathbb{K}) \to H^*_K(X, \mathbb{K}),$$
$$H^{BM,G}_i(X, \mathbb{K}) \to H^{BM,K}_i(X, \mathbb{K}).$$

In particular, these maps are induced by the fibration

$$G/K \to X \times_K EG/K \xrightarrow{q_{G,K}} X \times_G EG.$$

For example, for $G = S^1$ and $K = \mathbb{Z}/\ell$, the restriction map

$$\mathrm{Ext}^*_{S^1}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}[u] \to \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}[u.\theta]$$

is an embedding of algebras given by $u \mapsto u$, where $\mathbb{K}$ is a finite field with $\mathrm{char}(\mathbb{K})|\ell$, $|u| = 2$, $|\theta| = 1$,$\theta^2 = ku$.

APPENDIX C

Steenrod's construction for sheaves

In this chapter, I would like to explain Steenrod's construction in the framework of sheaves. Here, we follow the approach of Lonergan [**Lon21**, Section 2.2] without requiring that ℓ be prime. When ℓ is prime, Steenrod's construction admits more elegant properties, but we do not need it in the thesis, we refer to the loc. cit. for the readers.

Here, we denote by $C(X)$ the abelian category of complex of sheaves of $\mathbb{K}$-modules over X, where X is a topological space; and if Y is a $\mathbb{Z}/\ell$-space, we denote by $C_{\mathbb{Z}/\ell}(Y)$ the abelian category of complexes of $\mathbb{Z}/\ell$-equivariant sheaves (see Section A) of $\mathbb{K}$-modules over Y. The derived categories $D(X)$, $D_{\mathbb{Z}/\ell}(Y)$ are defined as usual.

Consider the ℓ-external tensor power functor

$$C(X) \xrightarrow{\boxtimes \ell} C(X^\ell).$$

The functor factor through, by the equivariant category, as

$$\boxtimes \ell : C(X) \xrightarrow{St_C^\ell} C_{\mathbb{Z}/\ell}(X^\ell) \to C(X^\ell).$$

Precisely, for $F^\bullet \in C(X)$, the external tensor power is the following complex:

$$(F^\bullet)^{\boxtimes \ell} = \left(\oplus_{n_1+\cdots+n_\ell=\bullet} F^{n_1} \boxtimes \cdots \boxtimes F^{n_\ell}\right)^\bullet.$$

Since $\mathbb{Z}/\ell$ is a discrete group, a $\mathbb{Z}/\ell$-equivariant sheaf is a sheaf with a $\mathbb{Z}/\ell$-action. So, we define a $\mathbb{Z}/\ell$-action on $(F^\bullet)^{\boxtimes \ell}$ in the following way: for the generator τ of $\mathbb{Z}/\ell$, there is an isomorphism of complex $\tau_* : (F^\bullet)^{\boxtimes \ell} \xrightarrow{\cong} \tau^{-1}(F^\bullet)^{\boxtimes \ell}$ defined degreewise by

$$\tau_* : F^{n_1} \boxtimes F^{n_2} \boxtimes \cdots \boxtimes F^{n_\ell} \xrightarrow{\cong} \tau^{-1} F^{n_2} \boxtimes \cdots \boxtimes F^{n_\ell} \boxtimes F^{n_1},$$

and twisted by $(-1)^{n_1(n_2+\cdot+n_\ell)}$ following the Koszul rule. In this way, we denote $(F^\bullet)^{\boxtimes \ell}$ together with the $\mathbb{Z}/\ell$-action by $St_C(F^\bullet)$. It is direct to see that for $f : F \to G$ in $C(X)$,

$f^{\boxtimes\ell}$ is a $\mathbb{Z}/\ell$-equivariant morphism. So St_C is a functor. Moreover, both St_C and the forgetful functor $C_{\mathbb{Z}/\ell}(X^\ell) \to C(X^\ell)$ preserve quasi isomorphisms. So, the universal property of derived category induces functors on derived categories:

$$\overset{L}{\boxtimes}\ell : D(X) \xrightarrow{St_D^\ell} D_{\mathbb{Z}/\ell}(X^\ell) \to D(X^\ell).$$

Since $St_D^\ell(F[1]) \cong St_D(F)[\ell]$, St_D^ℓ is not triangulated. Now St_D^ℓ is even not additive. It is proven in loc. cit., that

Proposition C.1. *For $f, g : F \to G$ in $D(X)$, there exists a non-equivariant morphism $h : F^{\overset{L}{\boxtimes}\ell} \to G^{\overset{L}{\boxtimes}\ell}$ such that*

$$St_D(f+g) - St_D(f) - St_D(g) : St_D(F) \to St_D(G),$$

equals

$$Av(h) = \sum_{x\in\mathbb{Z}/\ell} : St_D(F) \to St_D(G).$$

We can check on complex level that Steenrod's construction is compatible with the 6-operations.

Proposition C.2. *For a continuous map $u : X \to X'$, let $u^{\times\ell}$ be the Cartesian product of u, we have*

$$\mathrm{R}u_*^{\times\ell} St_D \cong St_D \mathrm{R}u_*,$$

$$(u^{\times\ell})^{-1} St_D \cong St_D u^{-1},$$

$$St_D(-)\overset{L}{\otimes} St_D(-) \cong St_D(-\overset{L}{\otimes}-),$$

$$\mathrm{R}\mathcal{H}om(St_D(-), St_D(-)) \cong St_D \mathrm{R}\mathcal{H}om(-,-).$$

When u is separated locally proper and $u_!$ has finite cohomological dimension, we have

$$\mathrm{R}u_!^{\times\ell} St_D \cong St_D \mathrm{R}u_!,$$

$$(u^{\times\ell})^! St_D \cong St_D u^!.$$

All of these morphisms commute with any and all adjunction morphisms of the 6-operations formalism.

Finally, we present an example that is essential for our applications.

Example C.3. Suppose $Z \subset X^2 \times Y$ is a locally closed subset, consider $F = \mathrm{R}\pi_{Y!}\mathbb{K}_Z \in D(X^2)$. We can take the singular cochain resolution for $\mathbb{K}_Z$, and then apply the Künneth isomorphism. The definition at complex level of Steenrod's construction shows that

$$F^{\overset{L}{\boxtimes}\ell} \cong \mathrm{R}\pi_{Y^\ell !}\mathbb{K}_{Z^\ell},$$

and the $\mathbb{Z}/\ell$-action is induced by the cyclic permutation on Z^ℓ. They together form $St_D(F)$.

Under the equivalence of Theorem A.5, the equivariant lifting of $St_D(F)$ is given by $(F^{\overset{L}{\boxtimes}\ell}, \overline{F^{\overset{L}{\boxtimes}\ell}}, \mathrm{Id})$, where

$$\overline{F^{\overset{L}{\boxtimes}\ell}} \cong \mathrm{R}\pi_{Y^\ell !}\mathbb{K}_{Z^\ell \times_{\mathbb{Z}/\ell} S^\infty}.$$

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Symbol

Index