

```
In [3]: import math
import numpy as np
from matplotlib import pyplot as plt
import scipy.constants as constants
import scipy.optimize as opt
```

```
In [4]: def Himmelblau_2D(x, y):
        return (x**2 + y - 11)**2 + (x + y**2 - 7)**2

def nonsmooth_Himmelblau_2D(x, y):
    return np.abs(x**2 + y - 11) + np.abs(x + y**2 - 7)

def gradient_2D(x, y):
    dfdx = 2 * (x**2 + y - 11) * 2*x + 2 * (x + y**2 - 7)
    dfdy = 2 * (x**2 + y - 11) + 2 * (x + y**2 - 7) * 2*y
    return dfdx, dfdy

def Himmelblau(vec):
    x, y = vec
    return (x**2 + y - 11)**2 + (x + y**2 - 7)**2

def Himmelblau_gradient(vec):
    x, y = vec
    dfdx = 2 * (x**2 + y - 11) * 2*x + 2 * (x + y**2 - 7)
    dfdy = 2 * (x**2 + y - 11) + 2 * (x + y**2 - 7) * 2*y
    return np.array([dfdx, dfdy])

def Himmelblau_hessian(vec):
    x, y = vec
    H00 = 12*x**2 + 4*y - 42
    H01 = 4*x + 4*y
    H10 = H01
    H11 = 4*x + 12*y**2 - 26
    return np.array([[H00, H01],
                     [H10, H11]])
```

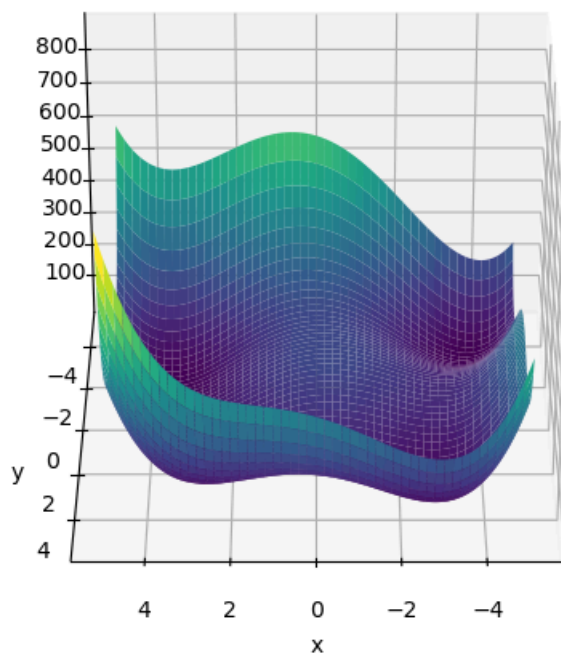
```
In [5]: x = np.linspace(-5, 5, 1000)
y = np.linspace(-5, 5, 1000)
X, Y = np.meshgrid(x, y)

Z = Himmelblau_2D(X, Y)

fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection="3d")
ax.plot_surface(X, Y, Z, cmap="viridis")

ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_title("Himmelblau's function")
ax.view_init(elev=30, azim=90)
plt.show()
```

Himmelblau's function



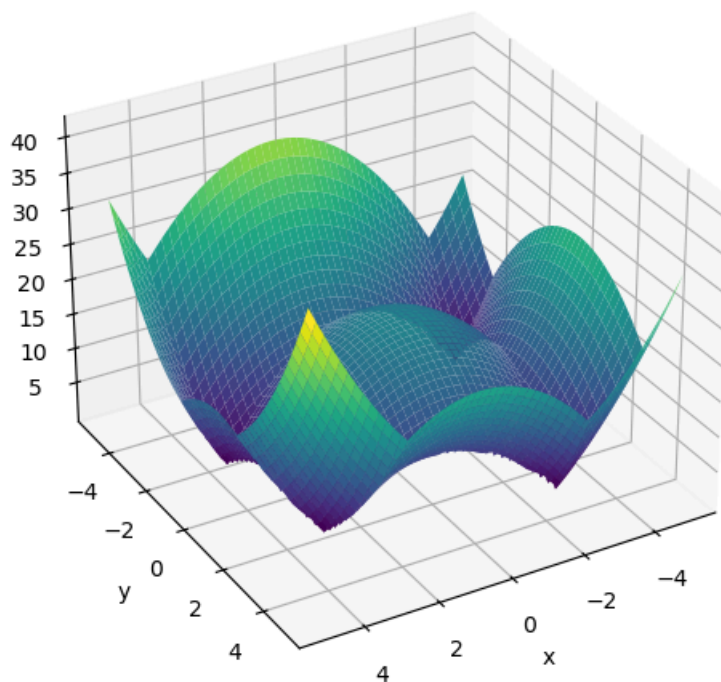
```
In [287]: x = np.linspace(-5, 5, 1000)
y = np.linspace(-5, 5, 1000)
X, Y = np.meshgrid(x, y)

Z = nonsmooth_Himmelblau_2D(X, Y)

fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection="3d")
ax.plot_surface(X, Y, Z, cmap="viridis")

ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_title("Nonsmooth Himmelblau's function")
ax.view_init(elev=30, azim=60)
plt.show()
```

Nonsmooth Himmelblau's function



## Demo 1: BFGS on smooth Himmelblau's Function

and comparison to simple gradient descent and Newton's method

Disclaimer: the line\_search step with zoom structure was borrowed from chapter 4 of Mathematical Optimization:

<https://indrag49.github.io/Numerical-Optimization/> (<https://indrag49.github.io/Numerical-Optimization/>).

Note: I realized that treating two variables became really hard to track due to repetitive names. I changed the function such that it takes in a single vector in the form of  $[x, y]$

```

In [7]: def line_search(f, grad, x, p, c1=1e-4, c2=0.9, alpha_init=1.0):
    alpha0 = 0
    alpha1 = alpha_init
    phi0 = f(x)
    dphi0 = np.dot(grad(x), p)

    phi_start = phi0

    counter = 0
    while(True):
        phi_alpha = f(x + alpha1 * p)
        g_alpha = grad(x + alpha1 * p)
        dphi_alpha = np.dot(g_alpha, p)

        if (phi_alpha > phi0 + c1 * alpha1 * dphi0) or (alpha0 > 0 and phi_alpha > phi_start):
            return zoom(f, grad, x, p, c1, c2, alpha_lo=alpha0, alpha_hi=alpha1, phi0=phi0, dphi0=dphi0)

        if dphi_alpha >= c2 * dphi0:
            return alpha1

        if dphi_alpha > 0:
            return zoom(f, grad, x, p, c1, c2, alpha_lo=alpha1, alpha_hi=alpha0, phi0=phi0, dphi0=dphi0)

        alpha0 = alpha1
        phi_prev = phi_alpha
        alpha1 *= 2.0
        counter += 1

    if counter >= 20:
        return alpha1

def zoom(f, grad, x, p, c1, c2, alpha_lo, alpha_hi, phi0, dphi0):
    # basically a bisection for alpha
    tolerance = 1e-9

    counter = 0
    while(True):
        alpha_mid = 0.5 * (alpha_lo + alpha_hi)

        if abs(alpha_hi - alpha_lo) < tolerance:
            return alpha_mid

        phi_amid = f(x + alpha_mid * p)
        phi_alo = f(x + alpha_lo * p)
        g_amid = grad(x + alpha_mid * p)
        dphi_amid = np.dot(g_amid, p)

        if (phi_amid > phi0 + c1 * alpha_mid * dphi0) or (phi_amid >= phi_alo):
            alpha_hi = alpha_mid
        else:
            if dphi_amid >= c2 * dphi0:
                return alpha_mid

            if dphi_amid * (alpha_hi - alpha_lo) >= 0:
                alpha_hi = alpha_lo

            alpha_lo = alpha_mid

        counter += 1

    if counter >= 20:
        return alpha_mid

```

```
In [417]: def bfgs(f, grad, x_init, tol=1e-12):
xs = []
x = np.array(x_init)
H = np.identity(2)

while(True):
    xs.append(x)
    grad_norm = np.linalg.norm(grad(x))

    if grad_norm < tol:
        return x, f(x), np.array(xs)

    g = grad(x)
    p = np.dot(-H, g)

    if np.dot(g, p) >= 0:
        p = -g

    alpha = line_search(f, grad, x, p)

    s = alpha * p
    x_new = x + s
    grad_new = grad(x_new)
    y = grad_new - g

    ys = np.dot(y, s)
    if ys > 1e-10:
        rho = 1.0 / ys
        I = np.identity(2)
        V = I - rho * np.outer(s, y)
        H = V.dot(H).dot(V.T) + rho * np.outer(s, s)

    x = x_new
```

```
In [475]: def newton(f, grad, hess, x_init, tol=1e-12):
xs = []
x = np.array(x_init)

while(True):
    xs.append(x)
    g = grad(x)
    if np.linalg.norm(g) < tol:
        return x, f(x), np.array(xs)

    H = hess(x)

    step = np.linalg.solve(H, -g)
    x = x + step
```

```
In [464]: def gradient_descent(f, grad, x_init, alpha=0.001, tol=1e-8):
xs = []
x = np.array(x_init)

while(True):
    xs.append(x)
    g = grad(x)
    if np.linalg.norm(g) < tol:
        return x, f(x), np.array(xs)

    x_new = x - alpha * g

    x = x_new
```

I apologize for the bad naming conventions here.  $x_i$  are supposed to be the starting points of each run.  $x_i$  is supposed to record the calculated minimum of a function. I ran into an issue where Newton's method starts at an already optimized point, then realized that it was due to  $x_i$  and  $x_i$  previously having the same name.

```
In [476]: x_0 = [4.3, 3]
x, minimum, history = bfgs(Himmelblau, Himmelblau_gradient, x_0)
x_newton, minimum_newton, history_newton = newton(Himmelblau, Himmelblau_gradient, Himmelblau_hessian, x_0)
x_gd, minimum_gd, history_gd = gradient_descent(Himmelblau, Himmelblau_gradient, x_0)
```

```
In [477]: x_1 = [4.2, 3]
x1, minimum1, history1 = bfgs(Himmelblau, Himmelblau_gradient, x_1)
x_newton1, minimum_newton1, history_newton1 = newton(Himmelblau, Himmelblau_gradient, Himmelblau_hessian, x_1)
```

```
In [478]: x_2 = [4, 3]
x2, minimum2, history2 = bfgs(Himmelblau, Himmelblau_gradient, x_2)
x_newton2, minimum_newton2, history_newton2 = newton(Himmelblau, Himmelblau_gradient, Himmelblau_hessian, x_2)
```

```
In [479]: x_3 = [1, -1]
x3, minimum3, history3 = bfgs(Himmelblau, Himmelblau_gradient, x_3)
x_newton3, minimum_newton3, history_newton3 = newton(Himmelblau, Himmelblau_gradient, Himmelblau_hessian, x_3)
```

```
In [480]: x_4 = [2, 0]
x4, minimum4, history4 = bfgs(Himmelblau, Himmelblau_gradient, x_4)
x_newton4, minimum_newton4, history_newton4 = newton(Himmelblau, Himmelblau_gradient, Himmelblau_hessian, x_4)
```

```
In [481]: history_newton
```

```
Out[481]: array([[4.3          , 3.          ],
 [3.40194858, 2.29075707],
 [3.05861144, 2.03324663],
 [3.00167182, 2.00020747],
 [3.00000156, 1.99999932],
 [3.          , 2.          ],
 [3.          , 2.          ]])
```

```

In [482]: x = np.linspace(-5, 5, 1000)
y = np.linspace(-5, 5, 1000)
X, Y = np.meshgrid(x, y)

Z = Himmelblau_2D(X, Y)

plt.figure(figsize=(5, 5))

contours = plt.contour(X, Y, Z, levels=40)
plt.clabel(contours, inline=True, fontsize=8)

plt.title('Himmelblau Contours')
plt.xlabel('x')
plt.ylabel('y')
plt.axis('equal')

zeros = np.array([[3, -2.805118, -3.779310, 3.584428], [2, 3.131312, -3.283186, -1.848126]])

plt.plot(history[:, 0], history[:, 1], c='b', label='BFGS (4.3, 3)')
plt.plot(history_newton[:, 0], history_newton[:, 1], c='g', label='Newton (4.3, 3)')
#plt.plot(history_gd[:, 0], history_gd[:, 1], label='Gradient Descent')

plt.plot(history1[:, 0], history1[:, 1], c='b', label='BFGS (4.2, 3)', ls='--')
plt.plot(history_newton1[:, 0], history_newton1[:, 1], c='g', label='Newton (4.2, 3)', ls='--')
#plt.plot(history_gd[:, 0], history_gd[:, 1], label='Gradient Descent')

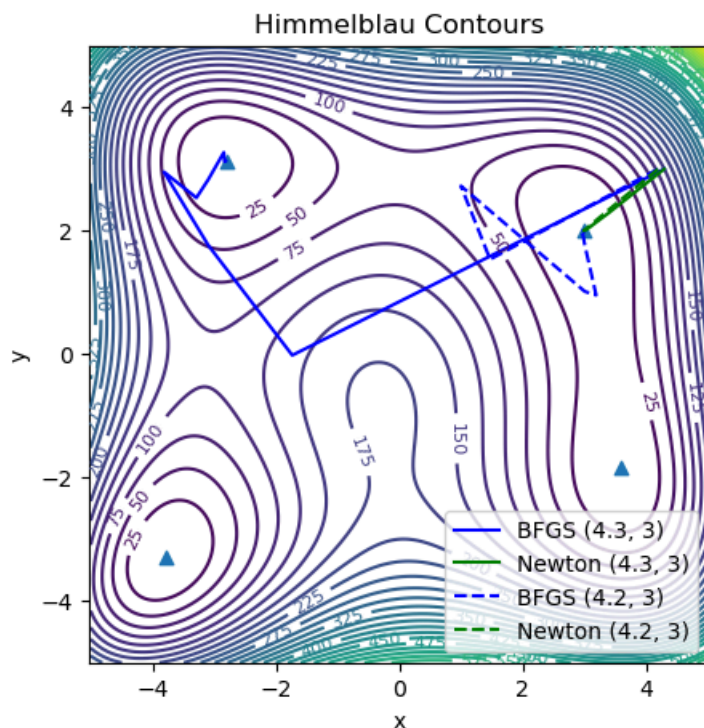
#plt.plot(history3[:, 0], history3[:, 1], c='b', label='BFGS (1, 3)', ls=':')
#plt.plot(history_newton3[:, 0], history_newton3[:, 1], c='g', label='Newton (1, 3)', ls=':')

#plt.plot(history4[:, 0], history4[:, 1], c='b', label='BFGS (2, 3)', ls='-.')
#plt.plot(history_newton4[:, 0], history_newton4[:, 1], c='g', label='Newton (2, 3)', ls='-.')

plt.legend()
plt.scatter(zeros[0, :], zeros[1, :], marker='^', label='Global minima')

```

Out[482]: <matplotlib.collections.PathCollection at 0x1f31437f950>



```

In [483]: def convergence(f, history):
    function_values = []
    for point in history:
        function_values.append(f(point))
    return np.array(function_values)

```

```
In [276]: bfgs_conv = convergence(Himmelblau, history)
newton_conv = convergence(Himmelblau, history_newton)
#gd_conv = convergence(Himmelblau, history_gd)
```

```
In [277]: bfgs_conv1 = convergence(Himmelblau, history1)
newton_conv1 = convergence(Himmelblau, history_newton1)
```

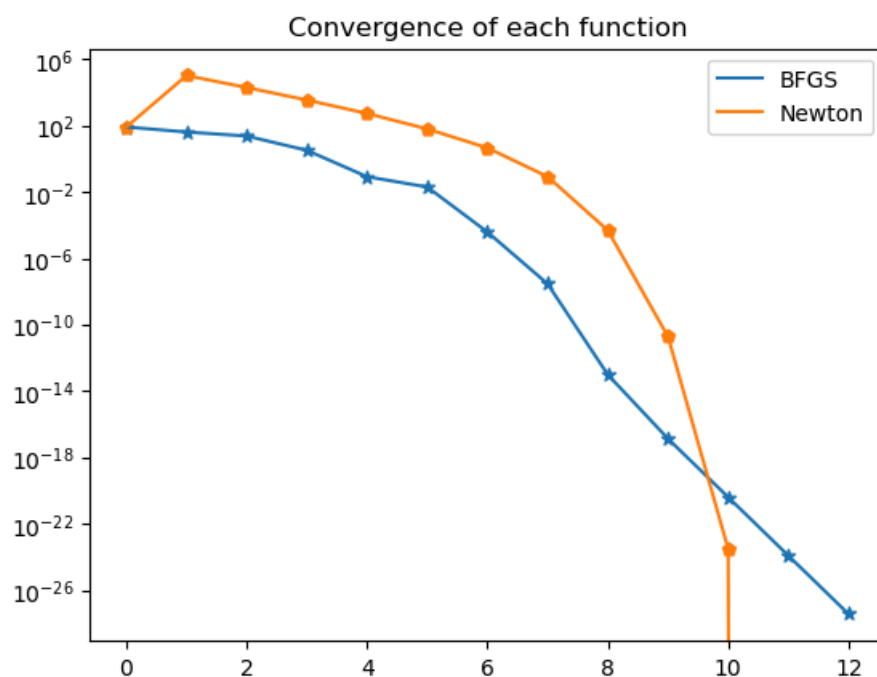
```
In [278]: bfgs_conv2 = convergence(Himmelblau, history2)
newton_conv2 = convergence(Himmelblau, history_newton2)
```

```
In [279]: bfgs_conv3 = convergence(Himmelblau, history3)
newton_conv3 = convergence(Himmelblau, history_newton3)
```

```
In [280]: bfgs_conv4 = convergence(Himmelblau, history4)
newton_conv4 = convergence(Himmelblau, history_newton4)
```

```
In [167]: plt.plot(np.arange(bfgs_conv.size), bfgs_conv, label='BFGS')
plt.scatter(np.arange(bfgs_conv.size), bfgs_conv, marker='*')
plt.plot(np.arange(newton_conv.size), newton_conv, label='Newton')
plt.scatter(np.arange(newton_conv.size), newton_conv, marker='p')
#plt.plot(gd_conv, label='Gradient Descent')
plt.yscale('log')
plt.title('Convergence of each function')
plt.legend()
```

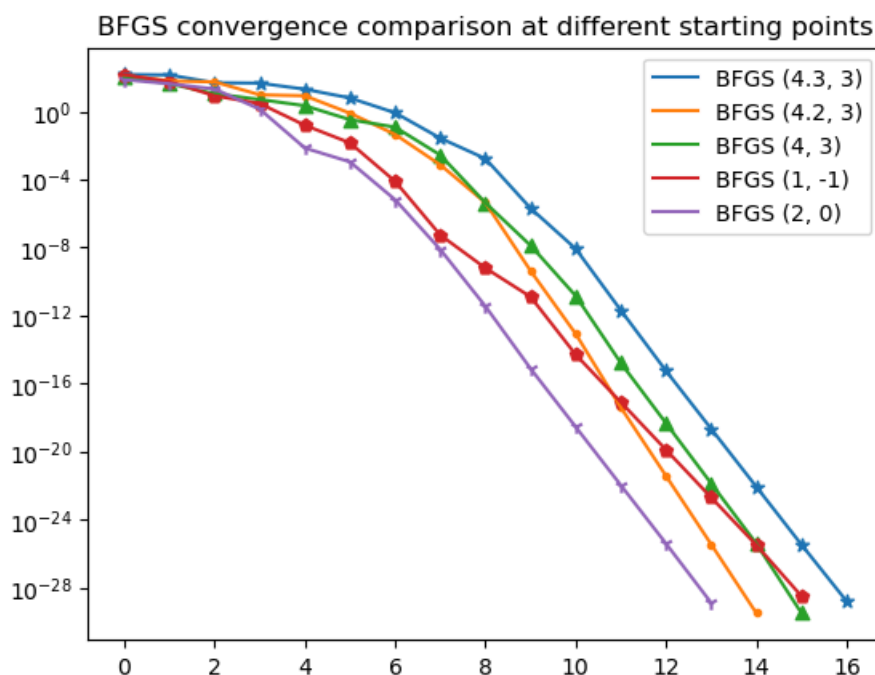
Out[167]: <matplotlib.legend.Legend at 0x1f2c3868ad0>





```
In [281]: plt.plot(np.arange(bfgs_conv.size), bfgs_conv, label='BFGS (4.3, 3)')
plt.scatter(np.arange(bfgs_conv.size), bfgs_conv, marker='*')
plt.plot(np.arange(bfgs_conv1.size), bfgs_conv1, label='BFGS (4.2, 3)')
plt.scatter(np.arange(bfgs_conv1.size), bfgs_conv1, marker='.')
plt.plot(np.arange(bfgs_conv2.size), bfgs_conv2, label='BFGS (4, 3)')
plt.scatter(np.arange(bfgs_conv2.size), bfgs_conv2, marker='^')
plt.plot(np.arange(bfgs_conv3.size), bfgs_conv3, label='BFGS (1, -1)')
plt.scatter(np.arange(bfgs_conv3.size), bfgs_conv3, marker='p')
plt.plot(np.arange(bfgs_conv4.size), bfgs_conv4, label='BFGS (2, 0)')
plt.scatter(np.arange(bfgs_conv4.size), bfgs_conv4, marker='1')
#plt.plot(gd_conv, label='Gradient Descent')
plt.yscale('log')
plt.title('BFGS convergence comparison at different starting points')
plt.legend()
```

Out[281]: <matplotlib.legend.Legend at 0x1f3095efe10>



## Rosenbrock function

This function only has one minimum at (1, 1)

```
In [16]: def Rosenbrock_2D(x, y):
return (1 - x)**2 + (y - x**2)**2

def Rosenbrock(vec):
x, y = vec
return (1 - x)**2 + (y - x**2)**2

def Rosenbrock_gradient(vec):
x, y = vec
dfdx = -2*(1-x) - 4*(y - x**2)
dfdy = 2*(y - x**2)
return np.array([dfdx, dfdy])

def Rosenbrock_hessian(vec):
x, y = vec
H00 = 2 - 4*y + 12*x**2
H10 = -4*x
H01 = -4*x
H11 = 2
return np.array([[H00, H01],
[H10, H11]])
```

```

In [222]: x = np.linspace(-1, 1.5, 1000)
y = np.linspace(-1, 1.5, 1000)
X, Y = np.meshgrid(x, y)

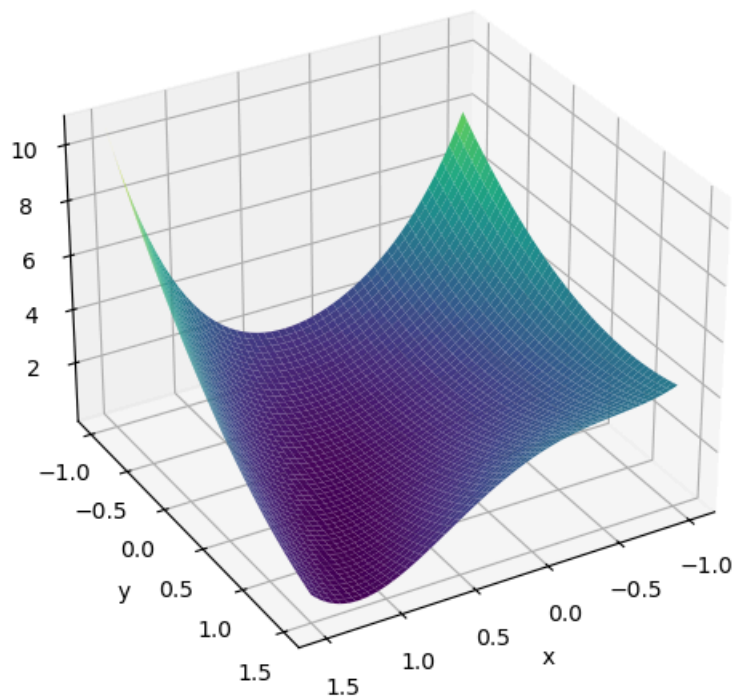
Z = Rosenbrock_2D(X, Y)

fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection="3d")
ax.plot_surface(X, Y, Z, cmap="viridis")

ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_title("Rosenbrock function")
ax.view_init(elev=30, azim=60)
plt.show()

```

Rosenbrock function



```

In [515]: x0 = [1.2, -0.5]
x, minimum, history = bfgs(Rosenbrock, Rosenbrock_gradient, x0)
x_newton, minimum_newton, history_newton = newton(Rosenbrock, Rosenbrock_gradient, Rosenbrock_hessian, x0)
x_gd, minimum_gd, history_gd = gradient_descent(Rosenbrock, Rosenbrock_gradient, x0)

```

```

In [516]: x = np.linspace(-1, 1.8, 400)
y = np.linspace(-1, 1.8, 400)
X, Y = np.meshgrid(x, y)

Z = Rosenbrock_2D(X, Y)

plt.figure(figsize=(5, 5))

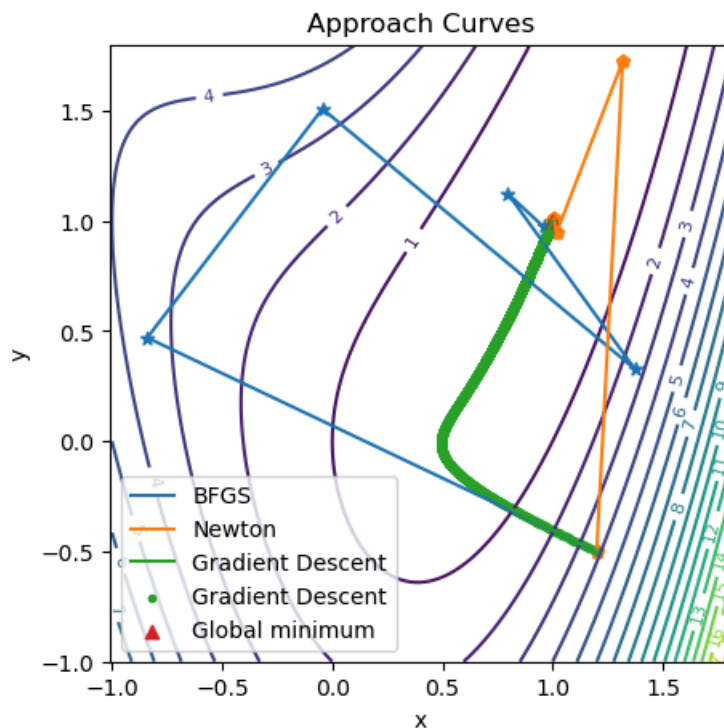
contours = plt.contour(X, Y, Z, levels=20)
plt.clabel(contours, inline=True, fontsize=8)

plt.title('Approach Curves')
plt.xlabel('x')
plt.ylabel('y')
plt.axis('equal')
#plt.scatter(1, 1, marker='^', label='Global minimum')

plt.plot(history[:, 0], history[:, 1], label='BFGS')
plt.scatter(history[:, 0], history[:, 1], marker='*')
plt.plot(history_newton[:, 0], history_newton[:, 1], label='Newton')
plt.scatter(history_newton[:, 0], history_newton[:, 1], marker='p')
plt.plot(history_gd[:, 0], history_gd[:, 1], label='Gradient Descent')
plt.scatter(history_gd[:, 0], history_gd[:, 1], marker='.', label='Gradient Descent')
plt.scatter(1, 1, marker='^', label='Global minimum')
plt.legend()

```

Out[516]: <matplotlib.legend.Legend at 0x1f32d167b90>



```

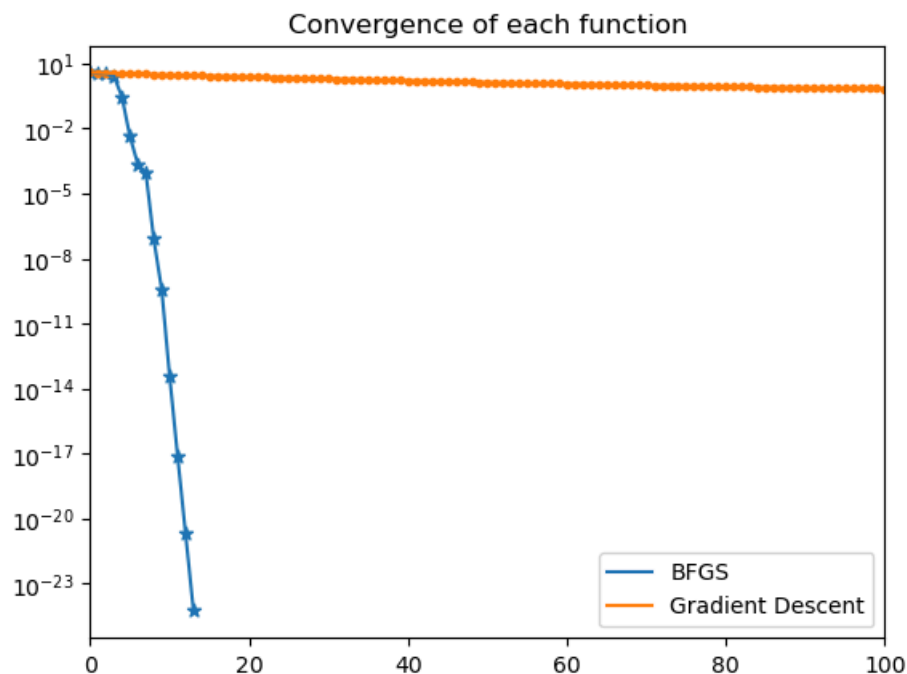
In [517]: bfgs_conv = convergence(Rosenbrock, history)
newton_conv = convergence(Rosenbrock, history_newton)
gd_conv = convergence(Rosenbrock, history_gd)

```

Comment out different convergence lines for different demos. If gradient descent is commented out, it shows comparison between Newton and BFGS. If Newton is commented out, it shows how slow GD is. gd = gradient descent.

```
In [518]: plt.plot(np.arange(bfgs_conv.size), bfgs_conv, label='BFGS')
plt.scatter(np.arange(bfgs_conv.size), bfgs_conv, marker='*')
#plt.plot(np.arange(newton_conv.size), newton_conv, label='Newton')
#plt.scatter(np.arange(newton_conv.size), newton_conv, marker='p')
plt.plot(gd_conv, label='Gradient Descent')
plt.scatter(np.arange(gd_conv.size), gd_conv, marker='.')
plt.yscale('log')
plt.xlim(left=0, right=100)
plt.title('Convergence of each function')
plt.legend()
```

Out[518]: <matplotlib.legend.Legend at 0x1f3093ef550>



## Nonsmooth Himmelblau Function

This function replaces the squares, and instead have "kinks" along certain lines.

```
In [520]: def Himmelblau_2D(x, y):
    return (x**2 + y - 11)**2 + (x + y**2 - 7)**2

def nonsmooth_Himmelblau_2D(x, y):
    return np.abs(x**2 + y - 11) + np.abs(x + y**2 - 7)

def gradient_2D(x, y):
    dfdx = 2 * (x**2 + y - 11) * 2*x + 2 * (x + y**2 - 7)
    dfdy = 2 * (x**2 + y - 11) + 2 * (x + y**2 - 7) * 2*y
    return dfdx, dfdy

def nonsmooth_Himmelblau(vec):
    x, y = vec
    return np.abs(x**2 + y - 11) + np.abs(x + y**2 - 7)

def nonsmooth_Himmelblau_gradient(vec):
    x, y = vec

    g1 = x**2 + y - 11
    g2 = x + y**2 - 7

    s1 = np.sign(g1)
    s2 = np.sign(g2)

    df_dx = 2*x*s1 + s2
    df_dy = s1 + 2*y*s2

    return np.array([df_dx, df_dy])

def nonsmooth_Himmelblau_hessian(vec):
    x, y = vec

    g1 = x**2 + y - 11
    g2 = x + y**2 - 7

    s1 = np.sign(g1)
    s2 = np.sign(g2)

    H00 = 2*s1
    H01 = 0
    H10 = 0
    H11 = 2*s2
    return np.array([[H00, H01],
                     [H10, H11]])
```

```
In [521]: x_0 = [2.5, 3]
x, minimum, history = bfgs(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, x_0)
x_newton, minimum_newton, history_newton = newton(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, nonsmooth_
#x_gd, minimum_gd, history_gd = gradient_descent(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, x_0)
```

```
In [522]: x_1 = [2.5, -2]
x1, minimum1, history1 = bfgs(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, x_1)
x_newton1, minimum_newton1, history_newton1 = newton(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, nonsmo
```

```
In [523]: x_2 = [-4, -1]
x2, minimum2, history2 = bfgs(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, x_2)
x_newton2, minimum_newton2, history_newton2 = newton(nonsmooth_Himmelblau, nonsmooth_Himmelblau_gradient, nonsmo
```

```

In [524]: x = np.linspace(-5, 5, 400)
y = np.linspace(-5, 5, 400)
X, Y = np.meshgrid(x, y)

Z = nonsmooth_Himmelblau_2D(X, Y)

plt.figure(figsize=(5, 5))

contours = plt.contour(X, Y, Z, levels=50)
plt.clabel(contours, inline=True, fontsize=8)

plt.title('Contours')
plt.xlabel('x')
plt.ylabel('y')
plt.axis('equal')

plt.plot(history[:, 0], history[:, 1], c='b', label='BFGS (2.5, 3.0)')
plt.scatter(history[:, 0], history[:, 1], c='b', marker='*')
plt.plot(history_newton[:, 0], history_newton[:, 1], c='g', label='Newton (2.5, 3.0)')
plt.scatter(history_newton[:, 0], history_newton[:, 1], c='g', marker='p')

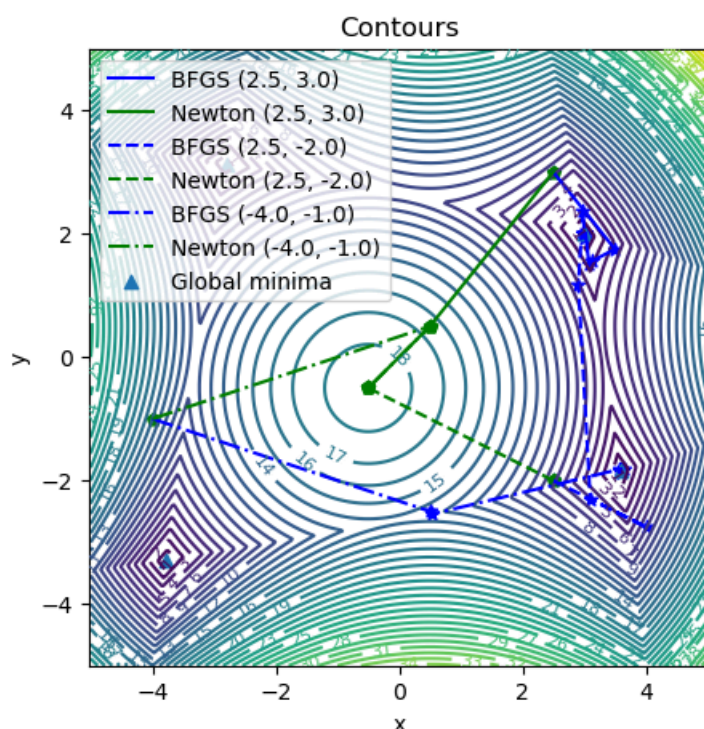
plt.plot(history1[:, 0], history1[:, 1], c='b', label='BFGS (2.5, -2.0)', ls='--')
plt.scatter(history1[:, 0], history1[:, 1], c='b', marker='*')
plt.plot(history_newton1[:, 0], history_newton1[:, 1], c='g', label='Newton (2.5, -2.0)', ls='--')
plt.scatter(history_newton1[:, 0], history_newton1[:, 1], c='g', marker='p')

plt.plot(history2[:, 0], history2[:, 1], c='b', label='BFGS (-4.0, -1.0)', ls='-.')
plt.scatter(history2[:, 0], history2[:, 1], c='b', marker='*')
plt.plot(history_newton2[:, 0], history_newton2[:, 1], c='g', label='Newton (-4.0, -1.0)', ls='-.')
plt.scatter(history_newton2[:, 0], history_newton2[:, 1], c='g', marker='p')
#plt.plot(history_gd[:, 0], history_gd[:, 1], label='Gradient Descent')

plt.scatter(zeros[0, :], zeros[1, :], marker='^', label='Global minima')
plt.legend()

```

Out[524]: <matplotlib.legend.Legend at 0x1f32d3e3b90>



```

In [525]: bfgs_conv = convergence(nonsmooth_Himmelblau, history)
newton_conv = convergence(nonsmooth_Himmelblau, history_newton)

bfgs_conv1 = convergence(nonsmooth_Himmelblau, history1)
newton_conv1 = convergence(nonsmooth_Himmelblau, history_newton1)

bfgs_conv2 = convergence(nonsmooth_Himmelblau, history2)
newton_conv2 = convergence(nonsmooth_Himmelblau, history_newton2)

```

Both BFGS and Newton's convergence here. I commented out differnt methods because of coloring.

```

In [530]: #plt.plot(np.arange(bfgs_conv.size), bfgs_conv, Label='BFGS (2.5, 3.0)')
#plt.scatter(np.arange(bfgs_conv.size), bfgs_conv, marker='*')
plt.plot(np.arange(newton_conv.size), newton_conv, label='Newton (2.5, 3.0)')
plt.scatter(np.arange(newton_conv.size), newton_conv, marker='*')

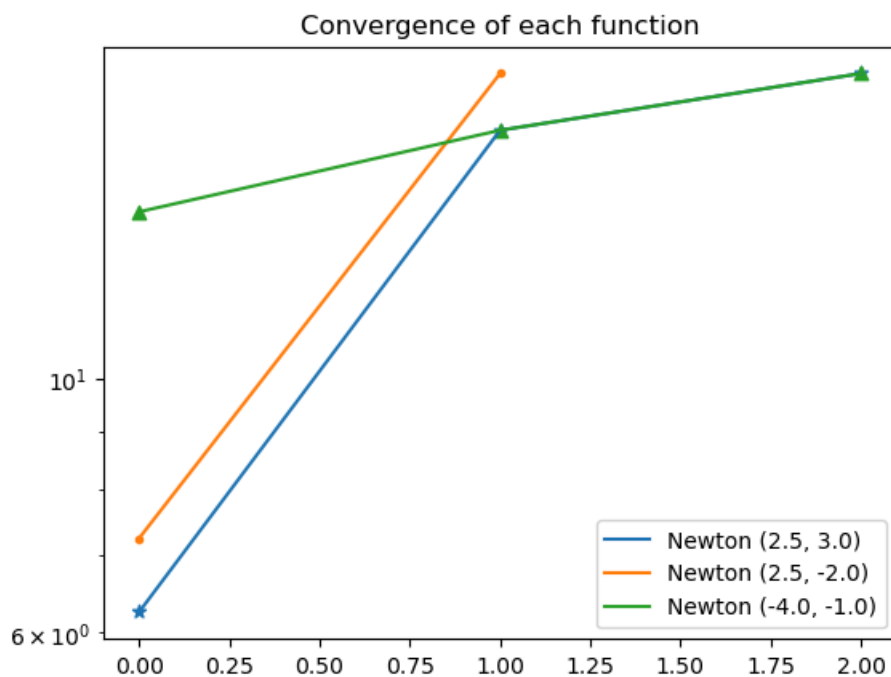
#plt.plot(np.arange(bfgs_conv1.size), bfgs_conv1, Label='BFGS (2.5, -2.0)')
#plt.scatter(np.arange(bfgs_conv1.size), bfgs_conv1, marker='.')
plt.plot(np.arange(newton_conv1.size), newton_conv1, label='Newton (2.5, -2.0)')
plt.scatter(np.arange(newton_conv1.size), newton_conv1, marker='.')

#plt.plot(np.arange(bfgs_conv2.size), bfgs_conv2, Label='BFGS (-4.0, -1.0)')
#plt.scatter(np.arange(bfgs_conv2.size), bfgs_conv2, marker='^')
plt.plot(np.arange(newton_conv2.size), newton_conv2, label='Newton (-4.0, -1.0)')
plt.scatter(np.arange(newton_conv2.size), newton_conv2, marker='^')

#plt.plot(gd_conv, Label='Gradient Descent')
plt.yscale('log')
plt.title('Convergence of each function')
plt.legend()

```

Out[530]: <matplotlib.legend.Legend at 0x1f2e25dfd50>



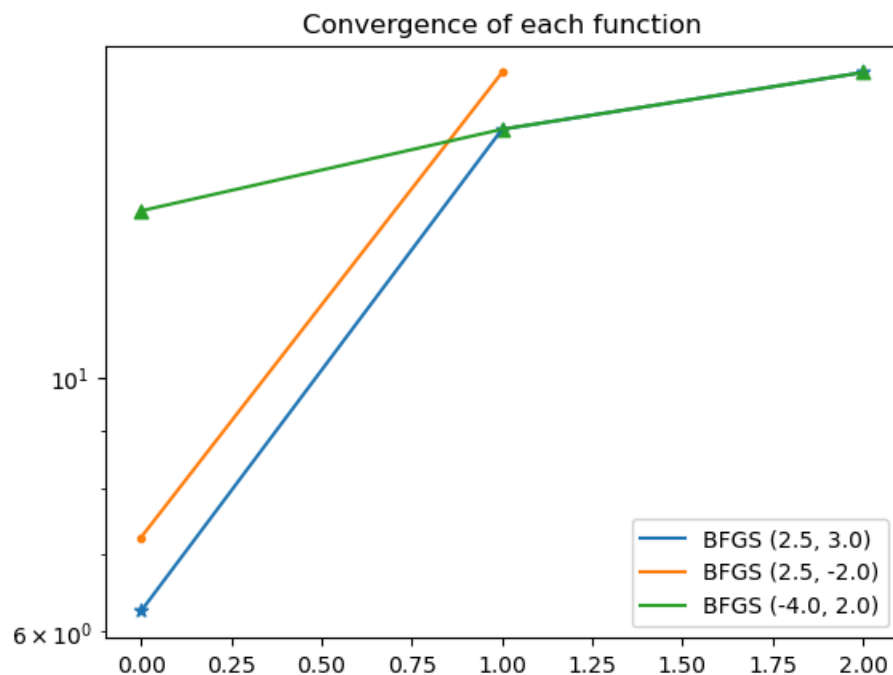
```

In [508]: plt.plot(np.arange(newton_conv.size), newton_conv, label='Newton (2.5, 3.0)')
plt.scatter(np.arange(newton_conv.size), newton_conv, marker='*')
plt.plot(np.arange(newton_conv1.size), newton_conv1, label='Newton (2.5, -2.0)')
plt.scatter(np.arange(newton_conv1.size), newton_conv1, marker='.')
plt.plot(np.arange(newton_conv2.size), newton_conv2, label='Newton (-4.0, 2.0)')
plt.scatter(np.arange(newton_conv2.size), newton_conv2, marker='^')

#plt.plot(gd_conv, label='Gradient Descent')
plt.yscale('log')
plt.title('Convergence of each function')
plt.legend()

```

Out[508]: <matplotlib.legend.Legend at 0x1f31d433b90>



## Ackley Function

This guy has a bunch of local minima and only one global minimum



```

In [27]: def ackley_2D(x, y, a=20, b=0.2, c=2*np.pi):
    return -a * np.exp(-b * np.sqrt(0.5 * (x**2 + y**2))) - np.exp(0.5 * (np.cos(c * x) + np.cos(c * y))) + np.e

def ackley(vec, a=20, b=0.2, c=2*np.pi):
    x, y = vec
    return -a * np.exp(-b * np.sqrt(0.5 * (x**2 + y**2))) - np.exp(0.5 * (np.cos(c * x) + np.cos(c * y))) + np.e

def ackley_grad(vec, a=20, b=0.2, c=2*np.pi):
    x, y = vec
    u = np.sqrt(0.5 * (x**2 + y**2))

    if u == 0.0:
        grad_T1_x = 0.0
        grad_T1_y = 0.0
    else:
        E1 = np.exp(-b * u)
        common = a * b * E1 / (2.0 * u) #  $ab e^{-bu} / (2u)$ 
        grad_T1_x = common * x
        grad_T1_y = common * y

    # Second term:  $T2 = -\exp((\cos(cx) + \cos(cy))/2)$ 
    #  $dT2/dx = (c/2) \sin(cx) * \exp((\cos(cx) + \cos(cy))/2)$ 
    #  $dT2/dy = (c/2) \sin(cy) * \exp((\cos(cx) + \cos(cy))/2)$ 
    E2 = np.exp(0.5 * (np.cos(c*x) + np.cos(c*y)))
    grad_T2_x = 0.5 * c * np.sin(c * x) * E2
    grad_T2_y = 0.5 * c * np.sin(c * y) * E2

    return np.array([grad_T1_x + grad_T2_x,
                     grad_T1_y + grad_T2_y])

```