Lecture 1

Unifying the Analysis in Control and Optimization via Semidefinite Programs

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In this lecture, we review some stability analysis tools in the controls literature, and then tailor them to analyze the convergence rates of some simple optimization methods. Hopefully you will be convinced that there are similarities between the analysis problems in control and optimization, and hence it is not too surprising that some analysis tools developed in the controls field can be applied to study large-scale optimization.

1.1 Stability Analysis in Control

1.1.1 Autonomous Systems and Internal Stability

Possibly the simplest system in the controls literature is the following so-called linear autonomous system

$$x_{k+1} = Ax_k \tag{1.1}$$

Here we consider discrete-time systems, and x_k is the state at time step k. Given the initial condition x_0, then the sequence $\{x_k\}$ is completely determined by (1.1). One fundamental question control people usually ask is whether (1.1) is stable. The system (1.1) is internally stable if x_k converges to 0 given any arbitrary initial condition x_0. Notice (1.1) just states that we have $x_k = A^k x_0$. Therefore, it is straightforward to verify that (1.1) is stable if and only if the spectral radius of A is strictly less than 1. However, the spectral radius condition only works for such linear time-invariant (LTI)[1] system. It is hard to extend such conditions for time-varying or nonlinear systems. Alternatively, one can formulate necessary and sufficient stability conditions for (1.1) using semidefinite programs.

Theorem 1.1. *The system* (1.1) *is internally stable if and only if there exists a positive definite matrix P such that*

$$A^\mathsf{T} P A - P < 0 \tag{1.2}$$

Here the inequality holds in the definite sense (so what we really mean here is that the matrix $(A^\mathsf{T} P A - P)$ needs to be a negative definite matrix).

[1] This just means A is a constant matrix and does not change over time.

Proof: We will only show sufficiency since this direction can be generalized for time-varying or nonlinear systems. If (1.2) holds, then there exists a sufficiently small positive number $\varepsilon > 0$ such that $A^\mathsf{T} PA - P \leq -\varepsilon P$ which can be rewritten as

$$A^\mathsf{T} PA - (1-\varepsilon)P \leq 0.$$

Therefore we can left and right multiply both sides of the above inequality with x_k^T and x_k and obtain

$$(Ax_k)^\mathsf{T} P(Ax_k) - (1-\varepsilon)x_k^\mathsf{T} Px_k \leq 0$$

We have $x_{k+1} = Ax_k$ and the above inequality just states $x_{k+1}^\mathsf{T} Px_{k+1} \leq (1-\varepsilon)x_k^\mathsf{T} Px_k$. By induction, we have

$$x_k^\mathsf{T} Px_k \leq (1-\varepsilon)^k x_0^\mathsf{T} Px_0$$

Since P is positive definite, we have $x_k^\mathsf{T} Px_k \geq \lambda_{\min}(P)\|x_k\|^2$ where $\lambda_{\min}(P)$ is the smallest eigenvalue of P and is a positive number. Finally we have

$$\|x_k\|^2 \leq (1-\varepsilon)^k c \tag{1.3}$$

where $c = \frac{x_0^\mathsf{T} Px_0}{\lambda_{\min}(P)}$. We know $0 \leq 1-\varepsilon < 1$ and hence $\|x\|$ converges to 0 as k goes to ∞. We establish the internal stability of (1.1).

The proof for necessity relies on the LTI assumption and is omitted here. ■

How to use the condition (1.2)**?** The testing condition (1.2) leads to a semidefinite program (or equivalently linear matrix inequality) problem. Given A, the left side of (1.2) is linear in P. One just needs to search such positive definite P satisfying the matrix inequality condition in (1.2). Numerically this can be done using semidefinite programming solvers. In the controls field, many analysis and design conditions are formulated as linear matrix inequality (LMI) conditions, and (1.2) is one of the simplest. We will see more such LMI conditions later.

Lyapunov functions. The proof of Theorem 1.1 relies on constructing the Lyapunov function $V(x) = x^\mathsf{T} Px$. A physical interpretation of this function is that it measures how much energy is stored in the system. This function is nonnegative for all x and is zero at the $x = 0$ (which is the fixed point of (1.1)). In addition, it is radially unbounded. In the above proof, we have shown $V(x_{k+1}) \leq (1-\varepsilon)V(x_k)$. So we show the energy is decreased at every step and eventually the minimum energy is attained at the fixed point. Lyapunov arguments can be applied in many cases and provide a powerful unified framework for stability analysis. We will learn more about this approach later.

Advantages of (1.2)**.** It is emphasized that people do not really use (1.2) when testing the stability of (1.1). A more efficient approach is to look at the spectral radius of A directly. However, (1.2) can be extended to time-varying/nonlinear systems which one cannot apply the spectral radius arguments to analyze. For example, consider the so-called linear parameter-varying (LPV) system described by the following state space model:

$$x_{k+1} = A(\zeta_k)x_k \tag{1.4}$$

where the matrix A becomes a function of the scheduling parameter ζ_k. The parameter ζ can be measured at every step but we do not know them in advance. We also know how A depends on the value of ζ_k. Now we cannot come to a conclusion about the internal stability of (1.4) by just looking at the spectral radius of A for all ζ. However, the Lyapunov argument still works. We can show (1.4) is internally stable if there exists a positive definite matrix P such that

$$A(\zeta)^\mathsf{T} P A(\zeta) - (1-\varepsilon)P \leq 0, \ \forall \zeta \tag{1.5}$$

The proof is almost identical to the proof of Theorem 1.1. We left and right multiply both sides of the above inequality with x_k^T and x_k and obtain $V(x_{k+1}) \leq (1-\varepsilon)V(x_k)$ which immediately leads to the desired conclusion. Here we do not have necessity. If there is no solution for (1.5), it is still possible that (1.4) is internally stable and then less conservative conditions are required for further testing.

Convergence rate. Inequality (1.3) in the above proof actually gives an exponential convergence rate $\sqrt{1-\varepsilon}$ for $\|x_k\|$. A LTI system is either exponentially stable or is not stable. Actually one can modify the LMI condition (1.2) to test whether (1.1) converges at a given testing rate or not. If there exists a positive definite matrix P such that

$$A^\mathsf{T} P A - \rho^2 P \leq 0 \tag{1.6}$$

then the system (1.1) converges at the exponential rate ρ, i.e. $\|x_k\| \leq c\rho^k$ where c is a constant. The proof is based on a similar Lyapunov argument.

1.1.2 Taking Inputs into Accounts: Input-Output Gain

In the controls field, we study how inputs can be used to change the behavior of the system. Built upon the autonomous system model (1.1), now we introduce more general state-space models for dynamical systems. Let a dynamic system G be governed by a linear state-space model, which is described by the following recursive iteration:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1.7}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, and $D \in \mathbb{R}^{n_y \times n_u}$. At each step k, the variables x_k, u_k, and y_k are referred to as the state, input, and output of the system G. When the initial condition x_0 is given, the state $\{x_k\}$ and the output $\{y_k\}$ will be completely determined by the input sequence $\{u_k\}$.

Block diagram. In the controls field, block diagrams are widely applied. The input-output relationship of the dynamical system G can be described by the following block diagram.

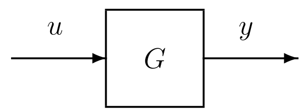

Figure 1.1. The Block-Diagram for a Dynamic System G

The above block diagram just states (u, y) satisfies $y = G(u)$ when one views the dynamical system G (with some fixed initial condition) as an input-output map.

Clearly one can set $u = 0$ and study the internal stability of the resultant autonomous system. We have already talked about this type of analysis. Another important question is how the input u_k will affect the output y_k. A useful tool for answering such questions is the following LMI condition.

Theorem 1.2. *If there exists a positive semidefinite matrix P such that*

$$\begin{bmatrix} A^\mathsf{T} PA - P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \leq 0 \tag{1.8}$$

then for any x_0 and arbitrary input sequence $\{u_k\}$, the system (1.7) *satisfies the following bound with any N*

$$\sum_{k=0}^{N} \|y_k\|^2 \leq \gamma^2 \sum_{k=0}^{N} \|u_k\|^2 + x_0^\mathsf{T} P x_0 \tag{1.9}$$

Proof: Based on the condition (1.8), we have

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} A^\mathsf{T} PA - P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq 0 \tag{1.10}$$

Notice we have $x_{k+1}^\mathsf{T} P x_{k+1} = (Ax_k + Bu_k)^\mathsf{T} P (Ax_k + Bu_k) = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} PA & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$. Therefore, we have

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} PA - P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} = x_{k+1}^\mathsf{T} P x_{k+1} - x_k^\mathsf{T} P x_k$$

Similarly, we have

$$\|y_k\|^2 - \gamma^2 \|u_k\|^2 = (Cx_k + Du_k)^\mathsf{T} (Cx_k + Du_k) - \gamma^2 \|u_k\|^2 = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

Consequently, (1.10) just leads to

$$x_{k+1}^\mathsf{T} P x_{k+1} - x_k^\mathsf{T} P x_k + \|y_k\|^2 - \gamma^2 \|u_k\|^2 \leq 0 \tag{1.11}$$

Since P is positive semidefinite, we know $x_{N+1}^\mathsf{T} P x_{N+1} \geq 0$. We can directly sum the above inequality form $k = 0$ to N to finish the proof of Theorem 1.2. ■

Interpretations of γ. The smaller γ is, the more stable G is subject to the input u. Therefore, γ is a measure for input-output stability. Many control problems including tracking and disturbance rejection can be formulated as optimization problems whose objectives are minimizing such input-output gain γ.

How to use the condition (1.8)**?** When (A, B, C, D) are given, the condition (1.8) is linear in P and γ^2. Therefore, minimizing γ^2 subject to the constraints (1.8) and $P \geq 0$ can also be done via semidefinite programs.

1.1.3 Analyze Nonlinearity: Stability of Feedback Interconnection

Another important object that has been extensively studied in the controls field is the feedback interconnection. For a dynamic jump system G and a mapping Δ, a feedback interconnection of G and Δ is shown in Figure 1.2 and denoted as $F_u(G, \Delta)$.

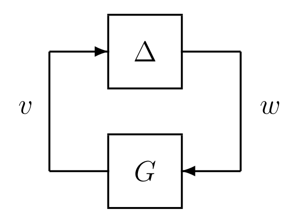

Figure 1.2. The Block-Diagram Representation for Feedback Interconnection $F_u(G, \Delta)$

The feedback interconnection states that v and w must satisfy $v = G(w)$ and $w = \Delta(v)$ simultaneously. For example, when G is an LTI system and Δ is a static nonlinearity, the feedback interconnection $F_u(G, \Delta)$ actually denotes the following recursive equations:

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k \\ v_k &= Cx_k + Dw_k \\ w_k &= \Delta(v_k) \end{aligned} \tag{1.12}$$

The first two equations in the above iterations state the fact $v = G(w)$, and the third equation enforces $w = \Delta(v)$.

Well-posedness. Clearly a basic question one should ask is whether there exists a pair of (v, w) satisfying $v = G(w)$ and $w = \Delta(v)$ simultaneously such that the feedback interconnection $F_u(G, \Delta)$ is well defined in the first place. This is the so-called well-posedness issue. When $D = 0$, the feedback system (1.12) is equivalent to $x_{k+1} = Ax_k + B\Delta(Cx_k)$. Given x_0, one can completely determine x_k using A, B, C, and Δ. Hence the feedback interconnection is well-posed in this case. When D is not a zero matrix, one need to prove well-posedness in a case-by-case manner. For simplicity, in this lecture we only consider the case where $D = 0$.

When $D = 0$, the system (1.12) is equivalent to a nonlinear autonomous system $x_{k+1} = Ax_k + B\Delta(Cx_k)$. Therefore, the sequences $\{x_k\}$, $\{w_k\}$, and $\{v_k\}$ will be completely determined given an initial condition x_0. It is more difficult to analyze the internal stability of the nonlinear system $x_{k+1} = Ax_k + B\Delta(Cx_k)$ than the linear autonomous system $x_{k+1} = Ax_k$. The nonlinear map Δ introduces some fundamental difficulty. If Δ is a linear function, then the nonlinear system $x_{k+1} = Ax_k + B\Delta(Cx_k)$ becomes linear and the internal stability analysis becomes easy. However, general Δ is hard to handle. For some types of nonlinearity, one can still modify the previous Lyapunov arguments to obtain similar stability conditions in the form of LMIs.

For example, if one knows Δ is a bounded operator and $\|\Delta(v_k)\| \leq \delta\|v_k\|$ for any v_k, then one can use the following LMI condition to test the internal stability of $F_u(G, \Delta)$.

Theorem 1.3. *Suppose Δ is a bounded operator and $\|\Delta(v_k)\| \leq \delta\|v_k\|$ for any v_k. If there exists a positive definite matrix P and a positive rate $0 < \rho < 1$ such that*

$$\begin{bmatrix} A^\mathsf{T}PA - \rho^2 P & A^\mathsf{T}PB \\ B^\mathsf{T}PA & B^\mathsf{T}PB \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} \delta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \leq 0 \tag{1.13}$$

then for any x_0, the feedback interconnection (1.12) satisfies $\|x_k\| \leq c\rho^k$ where c is some constant.

Proof: Based on the condition (1.13), we have

$$\begin{bmatrix} x_k \\ w_k \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} A^\mathsf{T}PA - \rho^2 P & A^\mathsf{T}PB \\ B^\mathsf{T}PA & B^\mathsf{T}PB \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} \delta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x_k \\ w_k \end{bmatrix} \leq 0 \tag{1.14}$$

Similarly as before, we have

$$\begin{bmatrix} x_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T}PA - \rho^2 P & A^\mathsf{T}PB \\ B^\mathsf{T}PA & B^\mathsf{T}PB \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} = x_{k+1}^\mathsf{T} P x_{k+1} - \rho^2 x_k^\mathsf{T} P x_k$$

We also have

$$\begin{aligned} -\|w_k\|^2 + \delta^2\|v_k\|^2 &= -\|w_k\|^2 + \delta^2 (Cx_k + Dw_k)^\mathsf{T}(Cx_k + Dw_k) \\ &= \begin{bmatrix} x_k \\ w_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} \delta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \end{aligned}$$

Consequently, (1.14) just leads to

$$x_{k+1}^\mathsf{T} P x_{k+1} - \rho^2 x_k^\mathsf{T} P x_k + \delta^2\|v_k\|^2 - \|w_k\|^2 \leq 0$$

Since $\|w_k\| = \|\Delta(v_k)\| \leq \delta\|v_k\|$, we know $\delta^2\|v_k\|^2 - \|w_k\|^2 \geq 0$, and the above inequality leads to $x_{k+1}^\mathsf{T} P x_{k+1} - \rho^2 x_k^\mathsf{T} P x_k \leq 0$. Since P is positive definite, we can immediately obtain the desired conclusion. ■

Again, when (A, B, C, D) and ρ^2 are given, the condition (1.13) is linear in P and can be solved as LMIs. The key idea in the above analysis is to replace the nonlinearity Δ with a bound $\|w_k\|^2 = \|\Delta(v_k)\|^2 \leq \delta^2\|v_k\|^2$ and then combine this bound with the linear state-space model of G to formulate an LMI condition.

Extensions. One can extend the above analysis to handle much more general systems. One can generalize the analysis for the cases where G is time-varying or even stochastic. One can also generalize the analysis for more general Δ including uncertainty and time delay. We will talk about these extensions later.

1.2 Analysis of Simple Optimization Algorithms

Many optimization methods for large-scale learning are first-order methods that can be viewed as special cases of the feedback system $F_u(G, \Delta)$. Therefore, it is not surprising that one can tailor the LMI tools in control for analysis of optimization methods. To give you a rough idea of what this course is about, we will present an example here. Consider the optimization problem

$$\min_{x \in \mathbb{R}^p} f(x)$$

where $f : \mathbb{R}^p \to \mathbb{R}$ is a differentiable function being L-smooth and m-strongly convex. A differentiable function $f : \mathbb{R}^p \to \mathbb{R}$ is L-smooth if for all $x, y \in \mathbb{R}^p$ the following inequality holds

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.$$

We say f is m-strongly convex (for some $m > 0$) if for all $x, y \in \mathbb{R}^p$ the following inequality holds

$$f(x) \geq f(y) + \nabla f(y)^\mathsf{T}(x - y) + \frac{m}{2}\|x - y\|^2.$$

A point $x^* \in \mathbb{R}^n$ is a global min of f if for all $x \in \mathbb{R}^n$ the following holds

$$f(x^*) \leq f(x).$$

When f is m-strongly convex, x^* is unique and satisfies $\nabla f(x^*) = 0$. To find x^*, a classical method is the gradient decent method which iterates as

$$x_{k+1} = x_k - \alpha \nabla f(x_k) \tag{1.15}$$

where α is a prescribed constant (called stepsize) one has to determine beforehand. One can choose any initial condition $x_0 \in \mathbb{R}^n$, and compute $x_1, x_2, \ldots, x_k, \ldots$. Here we assume given any x, one has the access to the first-order derivative information $\nabla f(x)$. Hence the gradient method is a first-order optimization method.

The gradient method has the advantage that it only requires the first-order derivative. In addition, when f is L-smooth and m-strongly convex, the gradient method is guaranteed to converge at a linear rate to the optimal point x^* as follows

$$\|x_k - x^*\| \leq \rho^k \|x_0 - x^*\| \tag{1.16}$$

where $0 \leq \rho < 1$. As mentioned before, in controls literature, the above convergence behavior is called exponential convergence. However, in optimization literature, the above convergence behavior is called linear convergence. The reason is that if one takes the log of $\rho^k \|x_0 - x^*\|$, one gets $k \log \rho + \log \|x_0 - x^*\|$, which is a linear function of k.

Clearly the smaller ρ is, the faster x_k converges to x^*. However, ρ cannot be arbitrarily small. This means the convergence speed of the algorithm depends on the parameter choice α and also the function properties (m and L).

The following theorem describes the dependence between ρ and (α, m, L).

Theorem 1.4. *Suppose f is L-smooth and m-strongly convex. Let x^* be the unique global min. Given a stepsize α, if there exists $0 < \rho < 1$ and $\lambda \geq 0$ such that*

$$\begin{bmatrix} 1-\rho^2 & -\alpha \\ -\alpha & \alpha^2 \end{bmatrix} + \lambda \begin{bmatrix} -2mL & m+L \\ m+L & -2 \end{bmatrix} \leq 0, \tag{1.17}$$

then the gradient method satisfies $\|x_k - x^*\| \leq \rho^k \|x_0 - x^*\|$.

The proof for the above theorem is almost identical to the proof of Theorem 1.3.

It relies on the so-called co-coercivity property which is guaranteed by L-smoothness and m-strong convexity. Specifically, when f is L-smooth and m-strongly convex, the following inequality holds for all $x, y \in \mathbb{R}^n$

$$(\nabla f(x) - \nabla f(y))^\mathsf{T}(x-y) \geq \frac{mL}{m+L}\|x-y\|^2 + \frac{1}{m+L}\|\nabla f(x) - \nabla f(y)\|^2. \tag{1.18}$$

The above inequality is equivalent to

$$\begin{bmatrix} x-y \\ \nabla f(x) - \nabla f(y) \end{bmatrix}^\mathsf{T} \begin{bmatrix} -2mLI_p & (m+L)I_p \\ (m+L)I_p & -2I_p \end{bmatrix} \begin{bmatrix} x-y \\ \nabla f(x) - \nabla f(y) \end{bmatrix} \geq 0$$

where the left side is in a quadratic form. The above inequality can be used to prove Theorem 1.4. We will cover the detailed proof of Theorem 1.4 in the next lecture. The basic idea is that the gradient method is just a special case of (1.12) with $A = I$, $B = -\alpha I$, $C = I$, and $\Delta = \nabla f$. Then it is not surprising that one can use similar Lyapunov arguments to obtain some convergence conditions for the resultant feedback system.

More importantly, many other optimization methods can also be recast in the form of (1.12) with well-chosen (A, B, C). We will see how to unify the analysis and design of optimization methods using control tools.