SOLUTIONS HW 2

1 Problem 1

1. We have
$$\nabla f = [2x_1 + 2\frac{1-\epsilon}{1+\epsilon}x_2, 2x_2 + 2\frac{1-\epsilon}{1+\epsilon}x_1]^T, \quad \text{and} \quad \nabla^2 f = \begin{pmatrix} 2 & 2\frac{1-\epsilon}{1+\epsilon} \\ 2\frac{1-\epsilon}{1+\epsilon} & 2 \end{pmatrix} \tag{1}$$
Since $0 < (1-\epsilon)/(1+\epsilon) < 1$ we have $\nabla^2 f \succ 0$, the unique minimizer is the solution of $\nabla f = 0$ which is $x_1 = x_2 = 0$.

2. We must have
$$\begin{pmatrix} 2-m & 2\frac{1-\epsilon}{1+\epsilon} \\ 2\frac{1-\epsilon}{1+\epsilon} & 2-m \end{pmatrix} \succeq 0, \quad \begin{pmatrix} M-2 & -2\frac{1-\epsilon}{1+\epsilon} \\ -2\frac{1-\epsilon}{1+\epsilon} & M-2 \end{pmatrix} \succeq 0 \tag{2}$$
or equivalently
$$2-m \geq 0, \quad (2-m)^2 - \left(2\frac{1-\epsilon}{1+\epsilon}\right)^2 \geq 0 \quad \text{and} \quad M-2 \geq 0, \quad (M-2)^2 - \left(2\frac{1-\epsilon}{1+\epsilon}\right)^2 \geq 0 \tag{3}$$
The largest possible m is $2 - 2\frac{1-\epsilon}{1+\epsilon}$ and the smallest possible M is $2 + 2\frac{1-\epsilon}{1+\epsilon}$. Hence, $\kappa = M/m = 1/\epsilon$

3. As $\epsilon \to 0$, it holds $\kappa = 1/\epsilon \to \infty$. Thus, we should expect gradient descent to converge slower.