

KKT Theorem (Necessary conditions)

Let x^* be local min. of (ICP) and assume that x^* is regular, i.e. $\nabla h_i(x^*)$, $i=1, \dots, m$, and $\nabla g_j(x^*)$, $j \in A(x^*)$, are linearly independent. Then

\exists unique Lagrange multipliers $\lambda = \lambda_1, \dots, \lambda_m$ and $\mu = \mu_1, \dots, \mu_r$. s.t. :

$$\nabla_x L(x^*, \lambda, \mu) = 0$$

$$M_j \geq 0, \quad j=1, \dots, r$$

$$M_j = 0 \quad \nabla g_j(x^*) = M_j g_j(x^*) = 0, \quad \forall j \in A(x^*)$$

Furthermore, under twice diff. cont.,

$$y^T \nabla_{xx}^2 L(x^*, \lambda, \mu) y \geq 0$$

$$\forall y \in \mathbb{R}^n \text{ s.t. } \nabla h_i(x^*)^T y = 0, \quad i=1, \dots, m, \quad \nabla g_j(x^*)^T y = 0 \quad \forall j \in A(x^*)$$

Sufficient conditions suppose x^* , λ , μ satisfy the first order necc. conditions above, and in addition

$$M_j > 0 \quad \forall j \in A(x^*)$$

and

$$y^T \nabla_{xx}^2 L(x^*, \lambda, \mu) y > 0$$

$$\forall y \neq 0 \text{ s.t. } \nabla h_i(x^*)^T y = 0 \text{ for } i=1, \dots, m$$

$$\nabla g_j(x^*)^T y = 0 \quad \forall j \in A(x^*)$$

Then x^* is a (strict) local min. of (ICP)

Example minimize $2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$

s.t. $x_1^2 + x_2^2 \leq 5$
 $3x_1 + x_2 \leq 6$

$$f(x) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$$

$$g_1(x) = x_1^2 + x_2^2 - 5, \quad g_2(x) = 3x_1 + x_2 - 6$$

$$\nabla f(x) = \begin{bmatrix} 4x_1 + 2x_2 - 10 \\ 2x_1 + 2x_2 - 10 \end{bmatrix}, \quad \nabla g_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \quad \nabla g_2(x) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

KKT First-order NCC. Conditions: $g_1(x^*) \leq 0, g_2(x^*) \leq 0$,
assuming x^* regular,

$$\nabla L(x^*, \mu) = \nabla f(x^*) + \mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) = 0$$

$$\mu_1 \geq 0, \quad \mu_2 \geq 0$$

$$\mu_1 g_1(x^*) = 0, \quad \mu_2 g_2(x^*) = 0$$

Don't know before-hand which constraints are active
So need to check all possibilities.

Case 1 (1 inactive, 2 inactive) : $\mu_1 = \mu_2 = 0$.

(No need to check for regularity).

$$\nabla f(x) = 0 \Rightarrow 4x_1 + 2x_2 - 10 = 0$$

$$2x_1 + 2x_2 - 10 = 0$$

$$\Rightarrow 2x_1 + x_2 = 5$$

$$x_1 + x_2 = 5$$

$$\Rightarrow x_1 = 0, \quad x_2 = 5$$

Drop * on
 x for convenience

But $x_1^2 + x_2^2 = 25 \neq 5 \Rightarrow$ not feasible

Case 2 (1 inactive, 2 active), i.e. $\mu_1 = 0$

$$\nabla g_2(x) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \text{all feasible } x \text{ are regular}$$

$$\nabla L(x, \mu) = \nabla f(x) + \mu_2 \nabla g_2(x) = 0$$

$$\Rightarrow \begin{cases} 4x_1 + 2x_2 - 10 + 3\mu_2 = 0 \\ 2x_1 + 2x_2 - 10 + \mu_2 = 0 \end{cases} \quad \begin{array}{l} \text{Solving we get.} \\ \mu_2 = 2/5 \\ x_1 = 2/5 \\ x_2 = 24/5 \end{array}$$

$$g_2(x) = 0 \Rightarrow 3x_1 + x_2 - 6 = 0 \quad \begin{array}{l} \\ \\ \mu_2 = -2/5 \end{array}$$

But $\mu_2 < 0$ not allowed \Rightarrow solution invalid

Case 3 (1 active, 2 inactive), i.e., $\mu_2 = 0$.

$$\nabla g_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Rightarrow \text{all } x \neq 0 \text{ are regular.}$$

$$\nabla L(x, \mu) = \nabla f(x) + \mu_1 \nabla g_1(x) = 0$$

$$\Rightarrow 4x_1 + 2x_2 - 10 + 2\mu_1 x_1 = 0$$

$$2x_1 + 2x_2 - 10 + 2\mu_1 x_2 = 0$$

$$g_1(x) = 0 \Rightarrow x_1^2 + x_2^2 - 5 = 0$$

Solving these 3 equations on Matlab yields

$$x_1^* = 1, x_2^* = 2, \mu_1 = 1$$

as only real solution

Check: $x^* = (1, 2)$ is regular (since $\neq 0$)

$$g_1(x^*) = 1 + 4 - 5 = 0$$

$$g_2(x^*) = 3 + 2 - 6 = -1 < 0$$

$x^* = (1, 2), \mu = (1, 0)$ satisfy 1st order KKT.

Case 4 (1 active, 2 active)

$$\nabla g_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \text{ and } \nabla g_2(x) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ are l.i.,}$$

as long as $x \neq 0, x_1 \neq 3x_2$.

But $x=0$ does not satisfy $g_1(x)=0$

$$\text{and } x_1 = 3x_2, g_2(x)=0 \Rightarrow 10x_2 = 6 \Rightarrow x_2 = \frac{3}{5}, x_1 = \frac{9}{5}$$

$$\text{and } \left(\frac{9}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{90}{25} < 5 \text{ does not satisfy } g_1(x)=0$$

\Rightarrow all feasible x in this case are regular

$$\nabla L(x, \mu) = \nabla f(x) + \mu_1 \nabla g_1(x) + \mu_2 \nabla g_2(x) = 0$$

$$\Rightarrow 4x_1 + 2x_2 - 10 + 2\mu_1 x_1 + 3\mu_2 = 0$$

$$2x_1 + 2x_2 - 10 + 2\mu_1 x_2 + \mu_2 = 0$$

$$g_1(x) = 0 \Rightarrow x_1^2 + x_2^2 - 5 = 0$$

$$g_2(x) = 0 \Rightarrow 3x_1 + x_2 - 6 = 0$$

Solving these 4 equations on Matlab yields:

$$x_1^* = 2.2, x_2^* = -0.5, \mu_1 = -2.4, \mu_2 = 4.2 \quad -(1)$$

$$\text{or } x_1^* = 1.4, x_2^* = 1.7, \mu_1 = 1.4, \mu_2 = -1.0 \quad -(2)$$

(1) not valid since $\mu_1 < 0$, (2) not valid since $\mu_2 < 0$

Thus only candidate for local min is from Case 3:

$$x^* = (1, 2), \mu = (1, 0).$$

Is $x^* = (1, 2)$ a local min? check sufficiency

$$f(x) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$$

$$g_1(x) = x_1^2 + x_2^2 - 5, \quad g_2(x) = 3x_1 + x_2 - 6$$

$$\nabla f(x) = \begin{bmatrix} 4x_1 + 2x_2 - 10 \\ 2x_1 + 2x_2 - 10 \end{bmatrix}, \quad \nabla g_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \quad \nabla g_2(x) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

For $x^* = (1, 2)$, $\mu = (1, 0)$,

$$1) \quad \nabla^2 L(x^*, \mu) = \nabla^2 f(x^*) + 1 \cdot \nabla^2 g_1(x^*) + 0$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} > 0$$

2) active constraint has $\mu_i > 0$

$\Rightarrow x^*$ is a local min. by suff. condition

Is x^* a global min for ICP?

constraint set: $\mathcal{C} = \{x : g_1(x) \leq 0, g_2(x) \leq 0\}$

$$= \{x : g_1(x) \leq 0\} \cap \{x : g_2(x) \leq 0\}$$

$$= \underbrace{\{x : x_1^2 + x_2^2 \leq 5\}}_{\text{compact}} \cap \underbrace{\{x : 3x_1 + x_2 \leq 6\}}_{\text{closed}}$$

$\Rightarrow \mathcal{C}$ is compact

\Rightarrow By WT global min exists and $= x^*$.

General Sufficiency Condition

minimize $f(x)$
 s.t. $x \in \mathcal{S} \leftarrow$ possible additional constraints.

$$\begin{aligned} h_i(x) = 0, \quad i=1, \dots, m \\ g_j(x) \leq 0, \quad j=1, \dots, r \end{aligned} \quad - (P)$$

Theorem Let

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x)$$

Suppose (x^*, λ, μ) satisfy:

$$h_i(x^*) = 0, \quad i=1, \dots, m$$

$$g_j(x^*) \leq 0, \quad j=1, \dots, r$$

$$\mu_j \geq 0, \quad j=1, \dots, r$$

$$\mu_j g_j(x^*) = 0, \quad j=1, \dots, r$$

and

$$L(x^*, \lambda, \mu) = \min_{x \in \mathcal{S}} L(x, \lambda, \mu)$$

Then, x^* is a global min for (P) (linear+const)

If \mathcal{S} is a convex set, and f is convex, h_i 's are affine, g_j 's are convex over \mathcal{S} , then $L(x, \lambda, \mu)$ is convex $\Rightarrow \nabla L(x^*, \lambda, \mu) = 0$ is sufficient for x^* to be global min for (P) .

Application of General Sufficiency Condition to example:

$$\begin{array}{ll} \text{minimize} & 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \\ x \in \mathbb{R}^2 & \text{s.t. } x_1^2 + x_2^2 \leq 5 \\ & \quad 3x_1 + x_2 \leq 6 \end{array} \quad \text{--- (P)}$$

$$f(x) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$$

$$g_1(x) = x_1^2 + x_2^2 - 5, \quad g_2(x) = 3x_1 + x_2 - 6$$

$$L(x, \mu) = f(x) + \mu_1 g_1(x) + \mu_2 g_2(x)$$

For $x^* = (1, 2)$, $\mu = (1, 0)$,

$L(x, \mu) = f(x) + g_1(x)$ is convex in x

$$\text{and } \nabla L(x^*, \mu) = 0$$

$$\Rightarrow L(x^*, \mu) = \min_{x \in \mathbb{R}^2} L(x, \mu)$$

Thus, by general suff. condition,

x^* is global min for (P)

Example

$$\begin{array}{ll} \text{maximize} & y^T x \\ \text{s.t.} & x^T Q x \leq 1 \\ & y \neq 0 \end{array} \quad Q > 0 \quad -(P)$$

$$\text{Let } f(x) = -y^T x, \quad g(x) = x^T Q x - 1$$

$$(P) \quad \equiv \quad \text{minimize } f(x) \quad \text{s.t. } g(x) \leq 0.$$

$$\nabla f(x) = -y \quad \text{and} \quad \nabla g(x) = 2Qx$$

\Rightarrow all $x \neq 0$ are regular.

$$\text{First-order KKT: } L(x, \mu) = f(x) + \mu g(x)$$

$$\underbrace{\nabla_x L(x^*, \mu)}_{} = 0, \quad \mu \geq 0, \quad \mu g(x^*) = 0$$

$$\hookrightarrow \Rightarrow -y + 2\mu Qx^* = 0 \Rightarrow y = 2\mu Qx^* \quad (1)$$

$$\mu g(x^*) = 0 \Rightarrow \mu (x^{*T} Q x^* - 1) = 0 \quad — (2)$$

$$y \neq 0 \Rightarrow \text{by (1), } x^* \neq 0, \quad \mu \neq 0$$

$$\Rightarrow \text{by (2), } x^{*T} Q x^* = 1$$

$$\Rightarrow y^T x^* = 2\mu x^{*T} Q x^* = 2\mu$$

$$\Rightarrow x^* = \frac{Q^{-1}y}{2\mu}$$

$$\mu = \frac{y^T x^*}{2} = \frac{y^T Q^{-1}y}{4\mu}$$

$$\Rightarrow 4\mu^2 = y^T Q^{-1}y \quad \text{and} \quad \mu = \pm \frac{\sqrt{y^T Q^{-1}y}}{2}$$

$$\text{Since we need } \mu > 0, \quad \mu = \frac{1}{2} \sqrt{y^T Q^{-1}y}$$

$$x^* = \frac{Q^{-1}y}{\sqrt{y^T Q^{-1} y}}, \mu = \frac{1}{2} \sqrt{y^T Q^{-1} y} \text{ unique}$$

Candidate for local min.

Sufficiency $L(x, \mu) = -y^T x + \mu(n^T Q x - 1)$

$$\nabla L(x, \mu) = -y + 2\mu Q x$$

$$\nabla^2 L(x^*, \mu) = 2\mu Q > 0$$

$\Rightarrow x^*$ is unique local min.

constraint set is compact $\Rightarrow x^*$ is global min by WT.

Alternatively, $L(x, \mu)$ is convex in x , and

$$\nabla L(x^*, \mu) = 0$$

$$\Rightarrow L(x^*, \mu) = \min_{x \in \mathbb{R}^n} L(x, \mu)$$

$\Rightarrow x^*$ is global min by general diff. condition

Maximum value of $y^T x$

$$y^T x^* = \frac{y^T Q^{-1} y}{\sqrt{y^T Q^{-1} y}} = \sqrt{y^T Q^{-1} y}$$