ECE 490: Introduction to Optimization **Fall 2018**

Solutions for Homework 6

1. We write the Lagrangian and study the KKT conditions:

$$\begin{aligned} L(x,\mu) &= (x-a)^2 + (y-b)^2 + xy + \mu_1(-x) + \mu_2(x-1) + \mu_3(-y) + \mu_4(y-1) \\ \nabla_x L(x,y,\mu) &= 2(x-a) + y - \mu_1 + \mu_2 \\ \nabla_y L(x,y,\mu) &= 2(y-b) + x - \mu_3 + \mu_4 \end{aligned}$$

In addition, the Hessian of L with respect to x and y is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0$.

Case 1: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0 \to x = \frac{4a}{3} - \frac{2b}{3}, y = \frac{4b}{3} - \frac{2a}{3}$ *: A local min if both values are in* $[0,1]$

Case 2: $\mu_1 \neq 0 \to \mu_1 = b - 2a, x = 0, y = b$ *: A local min if* $0 \leq b \leq 1$ *and* $b > 2a$

Case 3: $\mu_2 \neq 0 \to \mu_2 = 2a - b - \frac{3}{2}, x = 1, y = b - \frac{1}{2}$ *: A local min if* $0.5 \leq b \leq 1.5$ *and* $2a - b - \frac{3}{2} > 0$

Case 4: $\mu_3 \neq 0 \to \mu_3 = a - 2b, x = a, y = 0$ *: A local min if* $0 \leq a \leq 1$ *and* $a > 2b$

Case 5: $\mu_4 \neq 0 \to \mu_4 = 2b - a - \frac{3}{2}, x = a - \frac{1}{2}, y = 1$ *: A local min if* $0.5 \leq a \leq 1.5$ *and* $4b - 2a > 3$

Case 6: $\mu_1, \mu_3 \neq 0 \to \mu_1 = -2a, \mu_3 = -2b, x = 0, y = 0$ *: A local min if* $a, b < 0$

Case 7: $\mu_1, \mu_4 \neq 0 \to \mu_1 = 1 - 2a, \mu_4 = 2(b-1), x = 0, y = 1$ *: A local min if* $a < 0.5$ *and* $b > 1$

Case 8: $\mu_2, \mu_3 \neq 0 \to \mu_2 = 2(a-1), \mu_3 = 1 - 2b, x = 1, y = 0$ *: A local min if* $a > 1$ *and* $b < 0.5$

Case 9: $\mu_2, \mu_4 \neq 0 \to \mu_2 = 2a - 3, \mu_4 = 2b - 3, x = 1, y = 1$ *: A local min if* $a, b > 1.5$

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2. We write the Lagrangian and apply KKT conditions:
a) $L(x,\mu) = 2x^2 + 1 + \mu(x-1)(x-5)$.

$\nabla_x L(x,\mu) = 4x + 2\mu x - 6\mu$

Case 1 : $\mu = 0 \to x = 0$ *(Not satisfying the constraint)*

Case 2 : $\mu \neq 0$ and $x = 1$ $(\mu = 1, \nabla^2_{xx} L = 4 + 2\mu = 6 > 0)$ *This is a minimum for* $\mu = 1$

Case 3 : $\mu \neq 0$ and $x = 5$ *Not a minimum,* $\mu = -5 < 0$

Hence the optimal point is $x = 1$ and the solution for this primal problem is $2x^2 + 1 = 3$.

b) Given $\mu \geq 0$, L is a strongly-convex quadratic function and we can directly minimize it as follows.

$$L(x, \mu) = 2x^2 + 1 + \mu(x-1)(x-5)$$
$$\nabla_x L(x, \mu) = 0 \rightarrow x = \frac{3\mu}{2+\mu}$$

So the dual problem is

$$\max_{\mu \geq 0} \left\{ \frac{18\mu^2}{(2+\mu)^2} + 1 + \mu \left(\frac{3\mu}{(2+\mu)} - 1 \right) \left(\frac{3\mu}{(2+\mu)} - 5 \right) \right\}$$

We can simplify the dual function under the condition $\mu \geq 0$ and the dual problem becomes

$$\max_{\mu \geq 0} \left\{ -4\mu + 19 - \frac{36}{\mu+2} \right\}$$

To solve the dual problem, we just notice $-4\mu + 19 - \frac{36}{\mu+2} = -4(\mu+2) - \frac{36}{\mu+2} + 27 \leq -24 + 27 = 3$. Here we apply the fact $a + b \geq 2\sqrt{ab}$ for any positive a and b. The equality holds when $4(\mu+2) = \frac{36}{\mu+2}$. Since $\mu \geq 0$, we have $\mu = 1$ in this case. Now it is obvious that strong duality holds for this problem.

3.

$$L(x, \mu) = x^T Q x + \mu^T (Ax - b)$$
$$\nabla_x L = 2Qx + A^\mathsf{T} \mu = 0$$
$$\mu \geq 0$$
$$x = -\frac{1}{2} Q^{-1} A^\mathsf{T} \mu$$

Therefore, the dual problem is

$$\max_{\mu \geq 0} \left\{ \frac{\mu^T A Q^{-1} A^T \mu}{4} + \mu^T \left(-\frac{A Q^{-1} A^T \mu}{2} - b \right) \right\}$$

which can be further simplified as

$$\max_{\mu \geq 0} \left\{ -\frac{1}{4} \mu^T A Q^{-1} A^T \mu - b^\mathsf{T} \mu \right\}$$

4.

$$\begin{aligned} L(P, Y) &= \mathrm{trace} \left(Y \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - YX \right) \\ &= -\mathrm{trace}(YX) + trace \left(Y \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \right) \end{aligned}$$

Now denote $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^\intercal & Y_{22} \end{bmatrix} \geq 0$. We have

$$trace\left(Y \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix}\right) = \mathrm{trace}((Y_{11}A^\intercal + AY_{11} + BY_{12}^\intercal + Y_{12}B^\intercal)P)$$

Therefore, the dual function can be computed as

$$D(Y) = \min_P L(P, Y) = \begin{cases} -\,\mathrm{trace}(YX) & \text{if } Y_{11}A^\intercal + AY_{11} + BY_{12}^\intercal + Y_{12}B^\intercal = 0 \\ -\infty & \text{Otherwise} \end{cases} \tag{1}$$

Therefore, the dual problem for the given SDP is

$$\begin{array}{ll} \text{maximize} & -\,\mathrm{trace}(YX) \\ \text{subject to} & Y_{11}A^\intercal + AY_{11} + BY_{12}^\intercal + Y_{12}B^\intercal = 0 \\ & Y \geq 0 \end{array}$$