SOLUTIONS HW 1

1 Problem 1

1. f is continuous, $\mathcal{S}_1$ is compact (closed and bounded). Hence, according to Weierstrass Theorem, f achieves its min and max over $\mathcal{S}_1$.
2. Since $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$, f is coercive. Also, $\mathbb{R}^4$ is closed. Hence, by Corollary to Weierstrass Theorem, f achieves its min over $\mathbb{R}^4$, but not its max since $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$.
3. $\mathcal{S}_2$ is closed and f is coercive. Hence, by Corollary to Weierstrass Theorem, f achieves its min over $\mathcal{S}_2$, but not its max since $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$.

2 Problem 2

1. No. Consider $f(x) = (1-x)^3$. Note that for $x = 1$, we have $f'(1) = 0$, $f''(1) \geq 0$. However, $f(2) = -1 < 0 = f(1)$, which shows that $x = 1$ is not a local min.
2. $\nabla f(x) = 0 \Rightarrow [2(x_1 - 2x_2), -4(x_1 - 2x_2)]^T = 0$. The stationary points are the points of the line $\{(x_1, x_2) : x_1 = 2x_2\}$. Since $f(x_1, x_2) = (x_1 - 2x_2)^2 \geq 0$ and the zero value of f is attained by and only by the stationary points $\{(x_1, x_2) : x_1 = 2x_2\}$, all stationary points are global minima.

   An alternative way to solve this problem is to use the positive semidefiniteness of the Hessian matrix to show f is convex. Hence any stationary point is a global min.

3 Problem 3

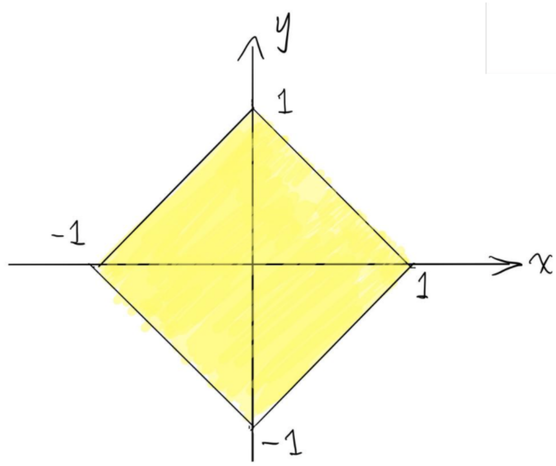
1. $f'(x) = 0 \Rightarrow x_1 = 0$, $x_2 = 2\sqrt{2}$, $x_3 = -2\sqrt{2}$ are the stationary points.
2. Since $f''(x_1) = -32 < 0$, x_1 is a local max. Since, $f''(x_2) = f(x_3) = 64 > 0$, the x_2, x_3 are a local minima.
3. Since, $f(\mathbf{x}) \to +\infty$ as $\|\mathbf{x}\| \to +\infty$ the global max does not exist. Function f is coercive and $\mathbb{R}$ is closed, thus by Corollary to Weierstrass Theorem a global min exists and since $f(x_2) = f(x_3) = 0$ both x_2, x_3 are global minima.

4 Problem 4

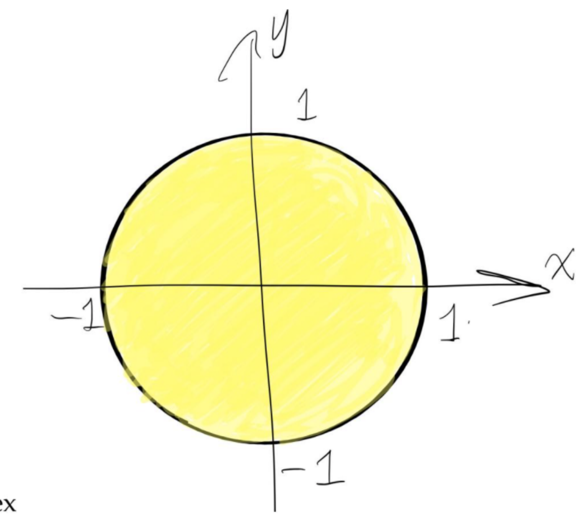
1. The eigenvalues are 0 and 5. Hence, A is PSD.
2. The eigenvalues are -1 and 3. Hence, B is indefinite.
3. $\det([4]) = 4 > 0$, $\det(A) = -5 < 0$. Hence, C is not PSD. We can use a similar argument to show C is not NSD. Hence C is indefinite.
4. The eigenvalues of D are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 4$. Hence, D is PD.
5. $\det([3]) = 3 > 0$, $\det \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} = 6 > 0$, $\det(-E) = 45 > 0$. Hence, $-E$ is PD, and E is ND.

5 Problem 5

1. Let us consider $f(x) = -x^2$ and $\alpha = -1$, then $\mathcal{S} = (-\infty, -1] \cup [1, +\infty)$. The $\mathcal{S}$ is not convex, because although $-1, 1 \in \mathcal{S}$, we have $(-1+1)/2 = 0 \notin \mathcal{S}$.



$p = 1$: convex



$p = 2$: convex

(2) We show the plot of the sets, and they are both convex.

Bonus (5pts): Mathematical Proof of the convexity.

Denote $\mathcal{S}_p = \{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$. For any $x, y \in \mathcal{S}_1$ and arbitrary $t \in [0, 1]$, we know $tx + (1-t)y \in \mathcal{S}_1$, since

$$
\begin{aligned}
\|tx + (1-t)y\|_1 = \sum_{i=1}^{n} |tx_i + (1-t)y_i| &\leq \sum_{i=1}^{n} t|x_i| + (1-t)|y_i| \\
&= t\sum_{i=1}^{n} |x_i| + (1-t)\sum_{i=1}^{n} |y_i| \\
&= t\|x\|_1 + (1-t)\|y\|_1 \\
&\leq t + (1-t) \leq 1
\end{aligned}
$$

where we used the triangle inequality. Thus, $\mathcal{S}_1$ is a convex set.

Similarly, any $x, y \in \mathcal{S}_2$ and arbitrary $t \in [0,1]$, we know $tx + (1-t)y \in \mathcal{S}_2$, since

$$\begin{aligned}
\|tx + (1-t)y\|_2^2 = \sum_{i=1}^{n} \|tx_i + (1-t)y_i\|_2^2 &\leq \sum_{i=1}^{n} t\|x_i\|_2^2 + (1-t)\|y_i\|_2^2 \\
&= t\sum_{i=1}^{n} \|x_i\|_2^2 + (1-t)\sum_{i=1}^{n} \|y_i\|_2^2 \\
&= t\|x\|_2^2 + (1-t)\|y\|_2^2 \\
&\leq t + (1-t) \leq 1 \ ,
\end{aligned}$$

which indicating

$$\|tx + (1-t)y\|_2 \leq 1 \ .$$

Thus $\mathcal{S}_2$ is a convex set.

3. We observe that f can be rewritten as

$$f(x_1, x_2, x_3) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2$$

Therefore, we can directly get

$$\nabla^2 f = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \ \forall x$$

Since $\det([4]) = 4 > 0$, $\det \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 16 > 0$ and $\det(\nabla^2 f) = -36 < 0$, we know $\nabla^2 f$ is not PSD. Since $\det(-[4]) = -4 < 0$, $\det \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = 16 > 0$ and $\det(-\nabla^2 f) = 36 > 0$, we know $\nabla^2 f$ is not NSD. Therefore, $\nabla^2 f$ is indefinite. We can conclude that f is neither convex nor concave.