Power Allocation Problem

P = total power available at Tx > 0

x_i = power allocated to channel i $\geq 0, \forall i$

σ_i^2 = noise power on channel i $> 0, \forall i$

\# bits carried by channel i = $\log\left(1 + \frac{x_i}{\sigma_i^2}\right)$

Goal Maximize $\sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\sigma_i^2}\right)$

s.t. $x_i \geq 0, \forall i, \quad \sum_{i=1}^{n} x_i \leq P.$

Note Often functions that we are optimizing are only well-defined on subsets of $\mathbb{R}^n$. The subset of $\mathbb{R}^n$ for which $f: \mathbb{R}^n \to \mathbb{R}$ is well-defined is called the domain of f, denoted $\mathrm{dom}(f)$

Examples

- $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \log(x)$ $\quad \mathrm{dom}(f) = \{x : x > 0\}$
- $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \frac{1}{x}$ $\quad \mathrm{dom}(f) = \{x : x \neq 0\}$
- $f: \mathbb{R}^n \to \mathbb{R}$, $\quad f(x) = \sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\sigma_i^2}\right)$

$$\mathrm{dom}(f) = \left\{x : 1 + \frac{x_i}{\sigma_i^2} > 0, \ i = 1, \ldots, n\right\}$$

PAP Maximize $\sum_{i=1}^{n} \log\left(1+\frac{x_i}{\sigma_i^2}\right)$

s.t. $x_i \geq 0, \forall i, \quad \sum_{i=1}^{n} x_i \leq P.$

Define $f(x) = -\sum_{i=1}^{n} \log\left(1+\frac{x_i}{\sigma_i^2}\right)$

$g_i(x) = -x_i, \quad i = 1, \dots, n$

$g_0(x) = \sum_{i=1}^{n} x_i - P$

Then PAP can be written as:

$$\begin{aligned} &\underset{x \in \mathrm{dom}(f)}{\text{minimize}} \quad f(x) \\ &\text{s.t.} \quad g_i(x) \leq 0, \quad i = 0, 1, \dots, n \end{aligned}$$

$$L(x,\mu) = -\sum_{i=1}^{n} \log\left(1+\frac{x_i}{\sigma_i^2}\right) - \sum_{i=1}^{n} \mu_i x_i + \mu_0\left(\sum_{i=1}^{n} x_i - P\right)$$

$$\frac{\partial L(x,\mu)}{\partial x_i} = \frac{-1}{1+\frac{x_i}{\sigma_i^2}} \cdot \frac{1}{\sigma_i^2} - \mu_i + \mu_0$$

For $i = 1, \dots, n$, $\nabla g_i(x) = (0, \dots, 0, \underset{i^{th} \text{ position}}{-1}, 0, \dots, 0)$

$\nabla g_0(x) = (1, 1, \dots, 1)$

Therefore, $\nabla g_0(x), \nabla g_1(x), \dots, \nabla g_n(x)$ are l.d.

But constraints cannot all be active simultaneously!

Why? If $x_i = 0$, $i = 1, \dots, n$, then $\sum x_i \neq P > 0$

Also, any strict subset of $\nabla g_0(x), \dots, \nabla g_n(x)$ are l.i.

$\Rightarrow$ all feasible x are regular

KKT conditions: $\nabla_x L(x^*, \mu) = 0$

$$\mu_i \geq 0, \quad i = 0, 1, \dots, n$$

$$\mu_i x_i^* = 0, \quad i = 1, \dots, n$$

$$\mu_0 \left(\sum_{i=1}^{n} x_i^* - P \right) = 0$$

$$\nabla L(x^*, \mu) = 0 \Rightarrow -\frac{1}{1 + \frac{x_i^*}{\sigma_i^2}} \cdot \frac{1}{\sigma_i^2} - \mu_i + \mu_0 = 0, \quad i = 1, \dots, n$$

$$\Rightarrow \mu_i = \mu_0 - \frac{1}{\sigma_i^2 + x_i^*}, \quad i = 1, \dots, n$$

$$\mu_i \geq 0 \Rightarrow \mu_0 \geq \frac{1}{\sigma_i^2 + x_i^*}, \quad i = 1, \dots, n$$

$$> 0 \quad \text{since } x_i^* \geq 0.$$

Thus $\sum_{i=1}^{n} x_i^* = P$ (constraint is active)

For $i = 1, \dots, n$, if $\mu_i > 0$, then $x_i^* = 0$

$$\Rightarrow \mu_i = \mu_0 - \frac{1}{\sigma_i^2} > 0$$

$$\Rightarrow \mu_0 > \frac{1}{\sigma_i^2} \Rightarrow \frac{1}{\mu_0} < \sigma_i^2$$

Therefore, $\frac{1}{\mu_0} \geq \sigma_i^2 \Rightarrow \mu_i = 0$

$$\Rightarrow \mu_0 = \frac{1}{\sigma_i^2 + x_i^*}$$

$$\Rightarrow x_i^* = \frac{1}{\mu_0} - \sigma_i^2$$

Set $x_i^* = \left(\frac{1}{\mu_0} - \sigma_i^2\right)^+ = \max\{0, \frac{1}{\mu_0} - \sigma_i^2\}$

with μ_0 satisfying :

$$P = \sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n}\left(\frac{1}{\mu_0} - \sigma_i^2\right)^+ .$$

Check KKT Complementary slackness For $i=1, \dots, n$,

$$x_i^* = 0 \Rightarrow \frac{1}{\mu_0} - \sigma_i^2 \leq 0$$

$$\Rightarrow \mu_i = \mu_0 - \frac{1}{\sigma_i^2} \geq 0$$

$$x_i^* > 0 \Rightarrow x_i^* = \frac{1}{\mu_0} - \sigma_i^2$$

$$\Rightarrow \mu_i = \mu_0 - \frac{1}{\sigma_i^2 + x_i^*} = \mu_0 - \mu_0 = 0$$

Thus $x_i^* \mu_i = 0$

$$L(x, \mu) = -\sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\sigma_i^2}\right) - \sum_{i=1}^{n} \mu_i x_i + \mu_0\left(\sum_{i=1}^{n} x_i - P\right)$$

convex in x

$$\operatorname{dom}(L) = \operatorname{dom}(f) = \{x : 1 + \frac{x_i}{\sigma_i^2} > 0\}$$

Convex Open

Since $\nabla L(x^*, \mu) = 0$ and $x^* \in \operatorname{dom}(L)$,

$$L(x^*, \mu) = \min_{x \in \operatorname{dom}(f)} L(x, \mu)$$

By general sufficiency x^* is global min for PAP

Alternative Way to solve PAP

$$f(x) = -\sum_{i=1}^{n} \log\left(1+\frac{x_i}{\sigma_i^2}\right)$$

$$g_i(x) = -x_i, \quad i=1,\dots,n$$

$$g_0(x) = \sum_{i=1}^{n} x_i - P$$

$$\begin{aligned} &\underset{x\in \mathrm{dom}(f)}{\text{minimize}} && f(x) \\ &\text{s.t.} && g_i(x) \leq 0, \quad i=0,1,\dots,n \end{aligned}$$

Define $\mathcal{S} = \{x : g_i(x) \leq 0, i=1,\dots,n\}$.

$= \{x : x_i \geq 0, i=1,\dots,n\}$

Note that $\mathcal{S} \subset \mathrm{dom}(f) = \{x_i : 1+\frac{x_i}{\sigma_i^2} > 0\}$.

Therefore we can rewrite PAP as:

$$\begin{aligned} &\underset{x\in \mathcal{S}}{\text{minimize}} && f(x) \\ &\text{s.t.} && g_0(x) \leq 0 \end{aligned}$$

Only one inequality constraint.

$$L(x,\mu) = -\sum_{i=1}^{n} \log\left(1+\frac{x_i}{\sigma_i^2}\right) + \mu\left(\sum_{i=1}^{n} x_i - P\right)$$

$\mathcal{S}$ is a closed and convex set.

Using General Sufficiency to Solve PAP directly

x^*, μ need to satisfy:

$$\sum_{i=1}^{n} x_i^* \leq P$$

$$\mu \geq 0$$

$$\mu \left(\sum_{i=1}^{n} x_i - P \right) = 0$$

$$L(x^*, \mu) = \min_{x \in \mathcal{S}} L(x, \mu)$$

To find x^*, μ that satisfy conditions, assume first that $\mu > 0$ and see if this works.

$$L(x, \mu) = - \sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\sigma_i^2}\right) + \mu\left(\sum_{i=1}^{n} x_i - P\right)$$

Since L is convex in x, and $\mathcal{S}$ is closed, convex nec. and sufficient condition for x^* to minimize $L(x, \mu)$ over $\mathcal{S}$ is (see lec 10).

$$\nabla L(x^*, \mu)^T (x - x^*) \geq 0 \quad \forall\, x \in \mathcal{S}. \quad -(1)$$

To find x^* that satisfies (1), set

$$\nabla L(x^*, \mu) = 0$$

i.e. $$\frac{\partial}{\partial x_i} L(x^*, \mu) = \mu - \frac{1}{x_i^* + \sigma_i^2} = 0, \quad i = 1, \cdots, n$$

$$\Rightarrow \quad x_i^* = \frac{1}{\mu} - \sigma_i^2$$

But $x_i^* = \frac{1}{\mu} - \sigma_i^2 \geq 0$ only if $\frac{1}{\mu} \geq \sigma_i^2$

So set $x_i^* = \left(\frac{1}{\mu} - \sigma_i^2\right)^+$

To check that $x_i^* = \left(\frac{1}{\mu} - \sigma_i^2\right)^+, \ \forall i$ satisfies

$$\nabla L(x^*, \mu)^T (x - x^*) \geq 0 \quad \forall x \in \mathcal{S},$$

note that for i s.t. $\frac{1}{\mu} > \sigma_i^2$, $\frac{\partial}{\partial x_i} L(x^*, \mu) = 0$

For i s.t. $\frac{1}{\mu} \leq \sigma_i^2$, $x_i^* = 0$

$$\Rightarrow \frac{\partial}{\partial x_i} L(x^*, \mu) = \mu - \frac{1}{\sigma_i^2 + 0} \geq 0$$

Therefore,

$$\nabla L(x^*, \mu)^T (x - x^*) = \sum_{i:\ x_i^* = 0} \frac{\partial}{\partial x_i} L(x^*, \mu) \cdot (x_i - x_i^*)$$

$$= \sum_{i:\ x_i^* = 0} \underbrace{\frac{\partial}{\partial x_i} L(x^*, \mu)}_{\geq 0} \cdot \underbrace{x_i}_{\geq 0 \text{ on } \mathcal{S}}$$

$$\Rightarrow \nabla L(x^*, \mu)^T (x - x^*) \geq 0 \quad \forall x \in \mathcal{S}$$

$$\Rightarrow L(x^*, \mu) = \min_{x \in \mathcal{S}} L(x, \mu).$$

Only one remaining condition to check:

$$\mu\left(\sum_{i=1}^{n} x_i^* - P\right) = 0$$

Since we assumed $\mu > 0$, need to see if we can find μ s.t. $\sum_{i=1}^{n} x_i^* = P$.

I.e., we need to see if $\exists\ \mu > 0$ s.t.

$$\sum_{i=1}^{n}\left(\frac{1}{\mu} - \sigma_i^2\right)^+ = P \qquad -(2)$$

But as μ goes from 0 to ∞, LHS monotonically decreases from ∞ to 0.

$\Rightarrow\ \exists$ unique μ that satisfies (2).

$x_i^* = \left(\frac{1}{\mu} - \sigma_i^2\right)^+$ and μ satisfying (2)

satisfy the general sufficiency conditions:

$$\sum_{i=1}^{n} x_i^* \leq P$$

$$\mu \geq 0$$

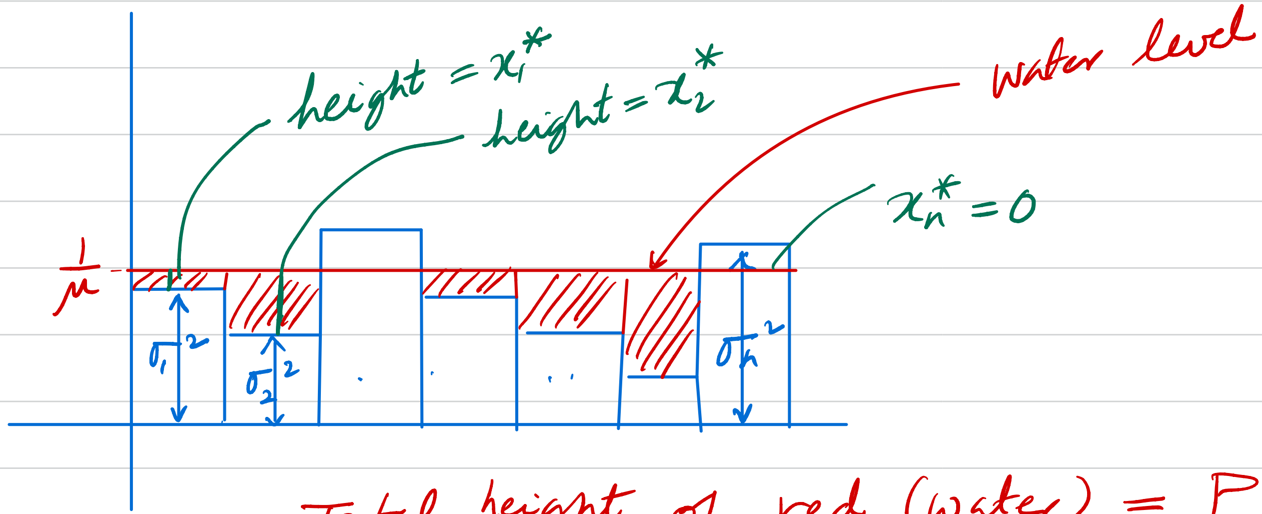
$$\mu\left(\sum_{i=1}^{n} x_i - P\right) = 0$$

$$L(x^*, \mu) = \min_{x \in \mathcal{A}} L(x, \mu)$$

$\Rightarrow\ x^*$ is a global min for PAP.

$$x_i^* = \left(\frac{1}{\mu} - \sigma_i^2\right)^+ \quad \text{and} \quad \sum_{i=1}^{n} \left(\frac{1}{\mu} - \sigma_i^2\right)^+ = P$$

Water-Filling Interpretation



Total height of red (water) $= P$.