ECE586RL: MDPs and Reinforcement Learning **Spring 2020**

Solutions for Homework 1

1.

(a) P1 is irreducible and aperiodic. The chain has a unique stationary distribution that it always converges to.

P2 is not irreducible. P2 is aperiodic. The chain does not have a unique stationary distribution. Hence it is also impossible for the chain to always converge to a unique distribution.

P3 is irreducible but not aperiodic. The chain has a unique stationary distribution. But it does not always converge to this stationary distribution.

(b) Since Ax_k is known, then x_{k+1} will have a Gaussian distribution $\mathcal{N}(Ax_k, W)$. Next, we can compute the mean and covariance of x_{k+1} as

$$\begin{aligned}\mathbb{E}x_{k+1} &= A\mathbb{E}x_k \\ \mathbb{E}x_{k+1}x_{k+1}^{\mathsf{T}} &= A(\mathbb{E}x_k x_k^{\mathsf{T}})A^{\mathsf{T}} + W\end{aligned}$$

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(a) Denote the policy in the problem statement as μ. We can solve the Bellman equation. The state value function J_μ is just a vector. We can solve J_μ as $J_\mu = (I - \gamma P_\mu)^{-1}\bar{c}_\mu$. Here $\bar{c}_\mu(i) = 0.6c(i, a_1) + 0.4c(i, a_2)$. The (i, j)-th entry of P_μ is defined as $0.6P(j, i, a_1) + 0.4P(j, i, a_2)$ where $P(j, i, a) := P(s_{t+1} = j \| s_t = i, a_t = a)$.

We can also solve the Bellman equation for the Q-factor. Specifically, we have

$$Q(i, a) = c(s, a) + \gamma \sum_{j=1}^{n} P(j, i, a)[0.6Q(j, a_1) + 0.4Q(j, a_2)]$$

which is equivalent to another linear equation $Q_\mu = \hat{c}_\mu + \gamma M_\mu Q_\mu$ where the i-th row of M_μ is $\begin{bmatrix}0.6P(1, i, a_1) & 0.4P(1, i, a_2) & 0.6P(2, i, a_1) & 0.4P(2, i, a_2) \dots 0.6P(n, i, a_1) & 0.4P(n, i, a_2)\end{bmatrix}$, and $(Q_\mu, \hat{c}_\mu)$ can be calculated as

$$Q_\mu = \begin{bmatrix} Q(1, a_1) \\ Q(1, a_2) \\ Q(2, a_1) \\ Q(2, a_2) \\ \vdots \\ Q(n, a_1) \\ Q(n, a_2) \end{bmatrix} \quad , \quad \hat{c}_\mu = \begin{bmatrix} c(1, a_1) \\ c(1, a_2) \\ c(2, a_1) \\ c(2, a_2) \\ \vdots \\ c(n, a_1) \\ c(n, a_2) \end{bmatrix}$$

Then Q_μ can be calculated as $Q_\mu = (I - \gamma M_\mu)^{-1}\hat{c}_\mu$. When the transition model is unknown, one can apply Monte Carlo simulation or temporal difference learning to learn value functions directly.

(b) At every t, the action a_t is generated using the policy μ given in Problem 2(a). Then apply a_t and measure s_{t+1} and $c(s_t, a_t)$. Then update the Q-factor as

$$Q_{t+1}(s_t, a_t) = Q_t(s_t, a_t) + \alpha_t \left(c(s_t, a_t) + \gamma \max[Q_t(s_{t+1}, a_1), Q_t(s_{t+1}, a_2)] - Q_t(s_t, a_t)\right)$$

The size of the Q-table is $2n$. If $s_t = i$ and $a_t = a$, then we only update the $(2i + a - 1)$-th entry of the Q-table at step t.

(c) For SARSA, we need to specify an initial action a_0 (which can be generated arbitrarily). At every step t, apply the action a_t, and measure s_{t+1} and $c(s_t, a_t)$. Use Q_t to generate an ε-greedy policy and then use this policy to sample an action a_{t+1}. Then update the Q-factor as

$$Q_{t+1}(s_t, a_t) = Q_t(s_t, a_t) + \alpha_t \left(c(s_t, a_t) + \gamma Q_t(s_{t+1}, a_{t+1}) - Q_t(s_t, a_t)\right)$$

So at step $t \geq 1$, the action a_t is already generated using the ε-greedy policy based on Q_{t-1}.

We can see that Q-learning is off-policy in the sense that the choice of behavior policy can be independent of Q_t. In contrast, SARSA is on policy since the behavior policy is directly related to Q_t. Another difference is that in the update rules, Q-learning requires calculating $\max_{a'} Q_t(s_{t+1}, a')$ (which is equal to $\max[Q_t(s_{t+1}, a_1), Q_t(s_{t+1}, a_2)]$ in this problem) and SARSA directly applies $Q_t(s_{t+1}, a_t)$.

(d) Check Pages 3-5 of the pdf file at the following link:
https://uofi.app.box.com/s/sniit2g18p41rgdbmb2ccvkouy7z9hkv

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(a) Given a linear policy K, it is straightforward to use induction to show

$$V(x) = r_K + \quad x^\mathsf{T} \left(\sum_{t=0}^{\infty} \gamma^t ((A - BK)^\mathsf{T})^t (Q + K^\mathsf{T} R K)(A - BK)^t \right) x \tag{1}$$

where r_K is some extra term introduced by the noise w_t. Therefore, we can parameterize the value function as $x^\mathsf{T} P_K x + r_K$. Therefore, we have

$$V(x) = x^\mathsf{T}(Q + K^\mathsf{T} R K)x + \gamma \left(\mathbb{E}((A - BK)x + w)^\mathsf{T} P_K ((A - BK)x + w) + r_K \right) \tag{2}$$

Notice w is independent from x and has a zero mean, we have

$$\mathbb{E}((A - BK)x + w)^\mathsf{T} P_K ((A - BK)x + w) = x^\mathsf{T}(A - BK)^\mathsf{T} P_K (A - BK)x + \mathbb{E}(w^\mathsf{T} P_K w)$$

Notice that the left side of (2) is just $x^\mathsf{T} P_K x + r_K$. Hence (2) can be rewritten as

$$x^\mathsf{T} P_K x + r_K = x^\mathsf{T}(Q + K^\mathsf{T} R K)x + \gamma x^\mathsf{T}(A - BK)^\mathsf{T} P_K (A - BK)x + \gamma \mathbb{E}(w^\mathsf{T} P_K w) + \gamma r_K$$

To ensure that the quadratic functions on the left and right sides of the above equation are the same, the following have to be true:

$$\begin{aligned} x^\mathsf{T} P_K x &= x^\mathsf{T}(Q + K^\mathsf{T} R K)x + \gamma x^\mathsf{T}(A - BK)^\mathsf{T} P_K (A - BK)x \\ r_K &= \gamma \mathbb{E}(w^\mathsf{T} P_K w) + \gamma r_K \end{aligned}$$

Hence, the Bellman equation becomes

$$P_K = Q + K^\mathsf{T} R K + \gamma (A - BK)^\mathsf{T} P_K (A - BK)$$

and $r_K = \frac{\gamma}{1-\gamma}\mathbb{E}(w^\mathsf{T} P_K w) = \frac{\gamma}{1-\gamma}\operatorname{trace}(PW)$ where W is the covariance matrix of w_t.

For the $\mathcal{Q}$-function, we have

$$\begin{aligned} \mathcal{Q}(x,u) &= x^\top Q x + u^\top R u + \gamma \mathbb{E} V(Ax + Bu + w) \\ &= x^\top Q x + u^\top R u + \gamma \mathbb{E}(Ax + Bu + w)^\mathsf{T} P_K (Ax + Bu + w) + \gamma r_K \\ &= x^\top Q x + u^\top R u + \gamma (Ax + Bu)^\mathsf{T} P_K (Ax + Bu) + \gamma(\mathbb{E}(w^\mathsf{T} P_K w) + r_K) \\ &= \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q + \gamma A^\mathsf{T} P_K A & \gamma A^\mathsf{T} P_K B \\ \gamma B^\mathsf{T} P_K A & R + \gamma B^\mathsf{T} P_K B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + r_K \end{aligned}$$

We can also directly parameterize $\mathcal{Q}(x,u)$ as

$$\mathcal{Q}(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^\mathsf{T} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + r_K$$

Notice $V(x) = \mathcal{Q}(x, -Kx)$. Therefore, we can substitute this into the above equation to obtain the Bellman equation for $\mathcal{Q}$:

$$\mathcal{Q}(x,u) = x^\top Q x + u^\top R u + \gamma \mathbb{E}\mathcal{Q}\left(Ax + Bu + w, -K(Ax + Bu + w)\right)$$

which is equivalent to

$$\begin{aligned} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^\mathsf{T} & \mathcal{Q}_{22} \end{bmatrix} &= \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} + \gamma \begin{bmatrix} A & B \\ -KA & -KB \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^\mathsf{T} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} A & B \\ -KA & -KB \end{bmatrix} \\ r_K &= \gamma \mathbb{E} \begin{bmatrix} w \\ -Kw \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^\mathsf{T} & \mathcal{Q}_{22} \end{bmatrix} \begin{bmatrix} w \\ -Kw \end{bmatrix} + \gamma r_K \end{aligned}$$

(b) Optimal Bellman equation: Suppose the optimal state value function is $x^\mathsf{T} P x + r$. We have

$$\begin{aligned} x^\mathsf{T} P x + r &= \min_u (x^\mathsf{T} Q x + u^\mathsf{T} R u + \gamma \mathbb{E}(Ax + Bu + w)^\mathsf{T} P (Ax + Bu + w) + \gamma r) \\ &= \min_u (x^\mathsf{T} Q x + u^\mathsf{T} R u + \gamma (Ax + Bu)^\mathsf{T} P (Ax + Bu) + \gamma \mathbb{E} w^\mathsf{T} P w + \gamma r) \end{aligned}$$

Taking gradient of the function on the right side with respect to u leads to

$$u = -\gamma(R + \gamma B^\mathsf{T} P B)^{-1} B^\mathsf{T} P A x.$$

which can be substituted b back to get the following optimal Bellman equation:

$$P = Q + \gamma A^\mathsf{T} P A - \gamma^2 A^\mathsf{T} P B (R + \gamma B^\mathsf{T} P B)^{-1} B^\mathsf{T} P A$$

Then the optimal state-action value function can be calculated as

$$\mathcal{Q}^*(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q + \gamma A^\mathsf{T} P A & \gamma A^\mathsf{T} P B \\ \gamma B^\mathsf{T} P A & R + \gamma B^\mathsf{T} P B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{\gamma}{1-\gamma} \operatorname{trace}(PW)$$

(c) Policy iteration: The PI algorithm iterates as $K^{n+1} = \gamma(\gamma B^\mathsf{T} P^n B + R)^{-1} B^\mathsf{T} P^n A$ where P_n solves the Bellman equation $\gamma(A - BK^n)^\mathsf{T} P^n (A - BK^n) + Q + (K^n)^\mathsf{T} R K^n = P^n$. Another option is to evaluate $\mathcal{Q}$ for every step and then design a policy which is the greedy policy for $\mathcal{Q}$. Specifically, at every step n, we first solve the $\mathcal{Q}$ Bellman equation to obtain $(\mathcal{Q}^n_{11}, \mathcal{Q}^n_{12}, \mathcal{Q}^n_{22})$ (the policy evaluation step), and then update the policy as $K_{n+1} = -(\mathcal{Q}^n_{22})^{-1}(\mathcal{Q}^n_{12})^\mathsf{T}$ (the policy improvement step).

To estimate $\mathcal{Q}$-Factor from data, one can either use Monte Carlo simulation or LSTDQ (see the LQR note for the details).

(d) For SARSA, the initial action u_0 can be arbitrary. At every step n, apply the control action u_n and measure x_{n+1} and $c(x_n, u_n) = x_n^\mathsf{T} Q x_n + u_n^\mathsf{T} R u_n$. Choose u_{n+1} using the ε-greedy policy generated by $\mathcal{Q}_n(x,u) = \theta_n^\mathsf{T} \phi(x,u)$ where ϕ is the feature. Then update the weight vector as $\theta_{n+1} = \theta_n + \alpha_n \phi(x_n, u_n)(c(x_n, u_n) + \gamma \theta_n^\mathsf{T} \phi(x_{n+1}, u_{n+1}) - \theta_n^\mathsf{T} \phi(x_n, u_n))$. For every $n \geq 1$, u_n was sampled using the ε-greedy policy generated by $\theta_{n-1}^\mathsf{T} \phi(x,u)$. For Q-learning, we can choose any behavior policy that provides sufficient exploration. At every step n, sample u_n using the behavior policy and measure x_{n+1} and $c(x_n, u_n)$. Then update the weight vector as $\theta_{n+1} = \theta_n + \alpha_k \phi(x_n, u_n)(c(x_n, u_n) + \gamma \min_u \theta_n^\mathsf{T} \phi(x_{n+1}, u) - \theta_n^\mathsf{T} \phi(x_n, u_n))$. Again, $\mathcal{Q}$-learning is off-policy and the sampling can be done using any behavior policy providing sufficient exploration. SARSA is on-policy and sampling is done using the ε-greedy policy given by $\theta_n^\mathsf{T} \phi(x,u)$.

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(a) Popular options for Ψ_t:

- Monte Carlo estimation: $\sum_{t'=t}^{\infty} \gamma^{t'-t} c_{t'}$
- Baselined versions of Monte Carlo estimation: $\sum_{t'=t}^{\infty} (\gamma^{t'-t} c_{t'} - b(s_t))$
- State-action value function: $Q^\pi(x_t, u_t)$

- Advantage function: $A^\pi(x_t, u_t)$
- TD residual: $c_t + \gamma V^\pi(x_{t+1}) - V^\pi(x_t)$
- Generalized advantage estimation

To calculate the gradient, first notice K is a matrix. Hence $\nabla_\theta \log \pi_\theta$ is also a matrix. The (i, j)-th entry of this matrix is just

$$\frac{\partial \log \pi_\theta}{\partial K_{ij}} = -\sigma^{-1}(u_t^{(i)} + \sum_{j=1}^{n_x} K_{ij} x_t^{(j)}) x_t^{(j)}$$

More compactly, we can write $\nabla_\theta \log \pi_\theta(u_t|x_t) = -(\sigma I)^{-1}(u_t + Kx_t)x_t^\mathsf{T}$.

Now consider the case $u_t \sim \mathcal{N}(W^1 h(W^0 x_t), \sigma I)$. The derivative with respect to W^1 can be directly calculated as

$$\frac{\partial}{\partial W^1} \log \pi_\theta(u_t|x_t) = \sigma^{-1}(u_t + W^1 h(W^0 x_t))(h(W^0 x_t))^\mathsf{T}$$

The derivative with respect to W^0 requires a backpropagation step and can be calculated as

$$\frac{\partial}{\partial W^0} \log \pi_\theta(u_t|x_t) = (W^1 \operatorname{diag}(h'(W^0 x_t)))^\mathsf{T}(u_t - W^1 h(W^0 x_t))x_t^\mathsf{T}$$

where $\operatorname{diag}(h'(W^0 x_t))$ is a diagonal matrix whose (i, i)-th entry is equal to the i-th entry of the vector $h'(W^0 x_t)$.

(b)

$$\begin{aligned}
\mathbb{E}_{\varepsilon\sim\mathcal{N}(0,I)} \left(\lim_{\sigma\to 0} \frac{\mathcal{C}(K + \sigma\varepsilon) - \mathcal{C}(K)}{\sigma} \right) \varepsilon &= \mathbb{E}_{\varepsilon\sim\mathcal{N}(0,I)}(\varepsilon^\mathsf{T} \nabla \mathcal{C}(K))\varepsilon \\
&= \mathbb{E}_{\varepsilon\sim\mathcal{N}(0,I)} \varepsilon(\varepsilon^\mathsf{T} \nabla C(K)) \\
&= \mathbb{E}_{\varepsilon\sim\mathcal{N}(0,I)}(\varepsilon\varepsilon^\mathsf{T}) \nabla C(K) \\
&= \nabla C(K)
\end{aligned}$$

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A code will be provided on the course website. LSTD-Q works efficiently for (b), and approximate PI works efficiently for (c). See the code for how to setup behavior policy for efficient exploration.