

Barrier Method

Computational method to solve inequality constrained problems. Applies to problems of form:

$$\text{minimize } f(x)$$

$$\text{s.t. } x \in \mathcal{S} \text{ and } g_j(x) \leq 0, j=1, \dots, r$$

— P

where  $\mathcal{S}$  is closed set.

Barrier Function  $B(x)$  is a function that is continuous and  $\rightarrow \infty$  as any  $g_j(x) \rightarrow 0$ .

Examples  $B(x) = - \sum_{j=1}^r \ln(-g_j(x))$

$$B(x) = - \sum_{j=1}^r \frac{1}{g_j(x)}$$

Note that if  $g_j(x)$  is convex for all  $j$ , then both of these barrier functions are convex

In Barrier Method, choose sequence  $\{\varepsilon_k\}$

s.t.

$$0 < \varepsilon_{k+1} < \varepsilon_k, k=0, 1, \dots$$

and  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ .

Define feasible set  $\mathcal{F} = S \cap \{g_j(x) \leq 0, \forall j\}$   
 Note that  $\mathcal{F}$  is closed.

Let  $x^{(k)}$  be a solution to:

$$\begin{array}{l} \text{minimize } f(x) + \varepsilon_k B(x) \\ x \in \mathcal{F} \cap \text{dom}(B) \end{array} \quad (P_k)$$

Since  $B(x) \rightarrow \infty$  on the boundary of  $\mathcal{F}$ ,  
 $x^{(k)}$  must be an interior point of  $\mathcal{F}$

$$\Rightarrow \nabla f(x^{(k)}) + \varepsilon_k \nabla B(x^{(k)}) = 0$$

Therefore, if we have a initial point in the  
 interior of  $\mathcal{F}$ , we can choose step size of any  
 unconstrained GD method to stay in interior  
 of  $\mathcal{F}$  for all iterations and solve  $(P_k)$

As  $k \rightarrow \infty$ ,  $\varepsilon_k \rightarrow 0$ , and barrier becomes  
 inconsequential, and we expect  $x^{(k)}$  to  
 approach minimum of original problem  $(P)$

Proposition Every limit point  $\bar{x}$  of  $\{x^{(k)}\}$   
 is a global min of  $(P)$ .

Proof Let  $\bar{x} = \lim_{\substack{k \rightarrow \infty \\ k \in K}} x^{(k)}$ .

Since  $x^{(k)} \in \mathcal{F}$  for all  $k$ , and  $\mathcal{F}$  is closed,  
 $\bar{x} \in \mathcal{F}$ .

Suppose  $x^*$  is a global min of  $\textcircled{P}$  and  $x^*$  is in interior of  $\mathcal{F}$ , and  
 $f(x^*) < f(\bar{x})$ , i.e.  $\bar{x}$  is not global min for  $\textcircled{P}$ .

Then, by definition of  $x^{(k)}$ ,

$$f(x^{(k)}) + \sum_k B(x^{(k)}) \leq f(x^*) + \sum_k B(x^*)$$

Taking limit as  $k \rightarrow \infty$ ,  $k \in K$ , since  $|B(x^*)| < \infty$   
 $f(\bar{x}) + \lim_{\substack{k \rightarrow \infty \\ k \in K}} \sum_k B(x^{(k)}) \leq f(x^*) + 0$

If  $\bar{x}$  is in interior of  $\mathcal{F}$ , then  $|B(\bar{x})| < \infty$

$$\Rightarrow \lim_{\substack{k \rightarrow \infty \\ k \in K}} \sum_k B(x^{(k)}) = 0$$

If  $\bar{x}$  is on boundary of  $\mathcal{F}$ , then  $B(x^{(k)}) \rightarrow \infty$

$$\Rightarrow \lim_{\substack{k \rightarrow \infty \\ k \in K}} \sum_k B(x^{(k)}) \geq 0.$$

Therefore  $f(\bar{x}) \leq f(x^*) \Rightarrow \Leftarrow$   
*i.e.,*  $\bar{x}$  is a global min for  $\textcircled{P}$

If  $x^*$  is not in interior of  $\mathcal{F}$ , then we can assume that  $\exists$  an interior point  $\tilde{x}$  which can be made arbitrarily close to  $x^*$ . Proof holds using  $\tilde{x}$  in place of  $x^*$ .

## Barrier Method Applied to Linear Programs

$$\begin{aligned} & \min C^T x \\ \text{s.t. } & Ax \leq b \text{ and } x \geq 0. \end{aligned}$$

Linear Program (LP) in standard form

- LP's arise in many applications:
  - Agriculture - crops/quantity to maximize revenue
  - Transportation - airline routing, mail/package routing (travelling salesman problem)
  - Efficient Manufacturing
  - Optimizing power grid.
  - ...
- In this context, Barrier Method is also called Interior Point Method.
- Common technique for solving LP's is simplex method (Dantzig, 1947)
- Karmarkar (1984) showed that barrier based interior point methods can be much faster for large problems.

## Barrier Method Example

$$\text{minimize } f(x) = \frac{1}{2} (x_1^2 + x_2^2) \quad - (P)$$

$$\text{s.t. } x_1 \geq 2.$$

Solution using KKT conditions

$$g(x) = 2 - x_1.$$

$\nabla g(x) = (-1, 0) \Rightarrow$  all feasible  $x$  are regular

$$\nabla f(x) = (x_1, x_2)$$

$$L(x, \mu) = f(x) + \mu g(x)$$

$$\nabla L(x, \mu) = \nabla f(x) + \mu \nabla g(x) = (x_1 - \mu, x_2)$$

Case 1: Constraint inactive, i.e.  $\mu = 0$

$$\nabla L(x, \mu) = 0 \Rightarrow x = (0, 0)$$

But  $x = (0, 0)$  does not satisfy  $x_1 \geq 2 \Rightarrow$  infeasible

Case 2: constraint active

$$\nabla L(x, \mu) = 0 \Rightarrow x_1 - \mu = 0, x_2 = 0$$

$$g(x) = 0 \Rightarrow x_1 = 2$$

$\Rightarrow x^* = (2, 0), \mu = 2$  satisfies first-order KKT

$L(x, \mu)$  is strictly convex on  $\mathbb{R}^2$ .

$$\Rightarrow L(x^*, z) = \min_{x \in \mathbb{R}^2} L(x, z)$$

By general sufficiency condition  $x^* = (2, 0)$  is the global min for (P).

$$\text{minimize } f(x) = \frac{1}{2} (x_1^2 + x_2^2) \\ \text{s.t. } x_1 \geq 2. \quad - (P)$$

Logarithmic Barrier  $B(x) = -\ln(-g(x)) = -\ln(x_1 - 2)$

$$x^{(k)} \in \arg \min_{x \in \text{dom}(g^{(k)})} g^{(k)}(x)$$

$$g^{(k)}(x) = \frac{1}{2} (x_1^2 + x_2^2) - \varepsilon_k \ln(x_1 - 2)$$

$g^{(k)}(x)$  is convex in  $x$  over  $\text{dom}(g^{(k)}) = \{x : x_1 > 2\}$ .

$$\nabla g^{(k)}(x) = 0 \Rightarrow x_1 - \frac{\varepsilon_k}{x_1 - 2} = 0$$

$$\text{and } x_2 = 0$$

$$\text{i.e., } x_1^2 - 2x_1 - \varepsilon_k = 0$$

$$\Rightarrow x_1 = 1 \pm \sqrt{1 + \varepsilon_k}, \text{ But } x_1 < 0 \text{ not possible}$$

$$\Rightarrow x_1 = 1 + \sqrt{1 + \varepsilon_k}$$

Therefore,  $x^{(k)} = (1 + \sqrt{1 + \varepsilon_k}, 0)$

and as  $k \rightarrow \infty$ ,  $\varepsilon_k \rightarrow 0$  and  $x^{(k)} \rightarrow (2, 0) = x^*$ .

Penalty Method Computational method for solving:

minimize  $f(x)$

s.t.  $x \in \mathcal{S}$

$$h_i(x) = 0, i=1, \dots, m$$

— (P)

Algorithm

① choose an increasing positive sequence

$$\{c_k\} \text{ s.t. } c_k \rightarrow \infty \text{ as } k \rightarrow \infty.$$

② solve for  $x^{(k)}$  to :

minimize  $f(x) + c_k \|h(x)\|^2$

$x \in \mathcal{S}$

$$\text{Note: } \|h(x)\|^2 = \sum_{i=1}^m (h_i(x))^2$$

— (P<sub>k</sub>)

Result Every limit point  $\bar{x}$  of  $\{x^{(k)}\}$  is a global min of (P) if  $\mathcal{S}$  is closed.

Proof Let  $\bar{x} = \lim_{\substack{k \rightarrow \infty \\ x \in \mathcal{S}}} x^{(k)}$ .

$$f^* = \min_{\substack{x \in \mathcal{S} \\ h(x)=0}} f(x) = \min_{\substack{x \in \mathcal{S} \\ h(x)=0}} f(x) + c_k \|h(x)\|^2$$

$$\geq \min_{x \in \mathcal{S}} f(x) + c_k \|h(x)\|^2$$

$$= f(x^{(\infty)}) + c_k \|h(x^{(\infty)})\|^2$$

$$\Rightarrow c_k \|h(x^{(\infty)})\|^2 \leq f^* - f(x^{(k)}) \quad — (1)$$

By continuity of  $f$ ,  $\lim_{\substack{k \rightarrow \infty \\ k \in K}} f(x^{(k)}) = f(\bar{x})$

Thus, as  $k \rightarrow \infty$ ,  $k \in K$ , the RHS of (1) goes to  $f^* - f(\bar{x})$  which is finite.

Since  $c_k \rightarrow \infty$  as  $k \rightarrow \infty$ ,  $k \in K$ ,

$$\lim_{\substack{k \rightarrow \infty \\ k \in K}} \|h(x^{(k)})\|^2 = 0$$

By continuity of  $\|h(x)\|^2$ ,

$$\lim_{\substack{k \rightarrow \infty \\ k \in K}} \|h(x^{(k)})\|^2 = \|h(\bar{x})\|^2 = 0$$

Now, since  $S$  is closed, and  $x^{(k)} \in S$  for all  $k$ ,  $\bar{x} \in S$  as well.

Now from (1),

$$f^* - f(x^{(k)}) \geq c_k \|h(x^{(k)})\|^2 \geq 0$$

$$\Rightarrow f(x^{(k)}) \leq f^*$$

$$\Rightarrow \lim_{\substack{k \rightarrow \infty \\ k \in K}} f(x^{(k)}) \leq f^*$$

$$\Rightarrow f(\bar{x}) \leq f^*$$

But  $\bar{x} \in S$  and  $h(\bar{x})=0$ , which means  $\bar{x}$  is feasible and  $f(\bar{x}) \leq f^*$

$\Rightarrow \bar{x}$  is a global min for  $(P)$ .

Example minimize  $f(x) = x_1 + x_2 - 2x_1 x_2$   
 s.t.  $x_1 + x_2 - 2 = 0$

Easy to show using LMT that  $x^* = (1, 1)$  is global min.

$$\begin{aligned} g^{(k)}(x) &= f(x) + c_k \|h(x)\|^2 \\ &= x_1 + x_2 - 2x_1 x_2 + c_k (x_1 + x_2 - 2)^2 \end{aligned}$$

$$\nabla g^{(k)}(x) = \begin{bmatrix} 1 - 2x_2 + 2c_k(x_1 + x_2 - 2) \\ 1 - 2x_1 + 2c_k(x_1 + x_2 - 2) \end{bmatrix}$$

$$\nabla^2 g^{(k)}(x) = \begin{bmatrix} 2c_k & 2(c_k - 1) \\ 2(c_k - 1) & 2c_k \end{bmatrix} = 2 \begin{bmatrix} c_k & c_k - 1 \\ c_k - 1 & c_k \end{bmatrix}$$

$$\begin{aligned} \det(\nabla^2 g^{(k)}(x)) &= 2(c_k^2 - (c_k - 1)^2) \\ &= 4c_k - 2 \end{aligned}$$

If  $c_k > \frac{1}{2}$ , then  $\nabla^2 g^{(k)}(x) > 0$

$\Rightarrow x^{(k)}$  is solution to  $\nabla g^{(k)}(x) = 0$

$$2c_k x_1 + 2(c_k - 1)x_2 + 1 - 4c_k = 0$$

$$2c_k x_2 + 2(c_k - 1)x_1 + 1 - 4c_k = 0$$

$$\Rightarrow x_1^{(k)} = x_2^{(k)} = \frac{4c_k - 1}{4c_k - 2}$$

As  $k \rightarrow \infty$ ,  $c_k \rightarrow \infty$ , and  $x^{(k)} \rightarrow (1, 1) = x^*$