

Linear Constraints

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & Ax = b \end{array} \quad \text{--- (P)}$$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$h_i(x) = a_i^T x - b_i, \quad i=1, \dots, m$$

$$\nabla h_i(x) = a_i$$

If  $a_i$ 's are l.i. (for which  $m$  must be  $\leq n$ )  
 Then all  $x \in \mathbb{R}^n$  are regular, and if  $x^*$  is a local min, then corresponding LM  $\lambda$  satisfies:

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla h_i(x^*) = 0$$

$$\text{i.e. } \sum_{i=1}^m \lambda_i a_i = -\nabla f(x^*)$$

$$\text{i.e. } A^T \lambda = -\nabla f(x^*)$$

Also,  $x^*$  satisfies:

$$Ax^* = b$$

If  $m = n$  then  $A$  is an  $n \times n$  matrix, and  
 $x^*$  is unique:

$$x^* = A^{-1}b$$

and this is the solution to (P) since this is  
 the only feasible choice for  $x$ .

If  $m < n$ ,  $A$  is an  $m \times n$  matrix with rank  $m$ ,  
then we can have infinitely many solutions to  $Ax^* = b$ .  
Need to solve for  $x^*$  and  $\lambda$  together using

$$\begin{aligned} A^T \lambda + \nabla f(x^*) &= 0 \\ Ax^* &= b \end{aligned} \quad \begin{array}{l} m+n \text{ equations} \\ m+n \text{ unknowns } \lambda, x^* \end{array}$$

If  $\nabla f(x^*)$  is linear in  $x^*$ , then solution can  
be unique. In general, we may have multiple  
solutions for  $x^*$  and corresponding  $\lambda$ .

Check second order sufficiency condition:

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

$$\nabla_x L(x, \lambda) = \nabla f(x) + \sum_{i=1}^m \lambda_i a_i$$

$$\nabla_x^2 L(x, \lambda) = \nabla^2 f(x)$$

Need to check:

$$\bar{z}^T \nabla^2 f(x^*) \bar{z} > 0$$

for all  $\bar{z} \neq 0$  s.t.

$$a_i^T \bar{z} = 0, i=1, \dots, m.$$

Example minimize  $-(x_1 x_2 + x_2 x_3 + x_1 x_3)$   
s.t.

$$\left. \begin{array}{l} x_1 + x_2 = 2 \\ x_2 + x_3 = 1 \end{array} \right\} Ax = b, A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$f(x) = -(x_1 x_2 + x_2 x_3 + x_1 x_3)$$

$$h_1(x) = x_1 + x_2 - 2$$

$$h_2(x) = x_2 + x_3 - 1$$

$a_1$

$a_2$

$$\nabla f(x) = - \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix}, \quad \nabla h_1(x) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \nabla h_2(x) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x^*, \lambda \text{ satisfy : } h_i(x) = 0, i=1,2, \quad \nabla f(x) + \sum_{i=1}^2 \lambda_i \nabla h_i(x) = 0$$

$$\left. \begin{array}{l} x_1^* + x_2^* = 2 \\ x_2^* + x_3^* = 1 \\ \lambda_1 - (x_2^* + x_3^*) = 0 \\ \lambda_1 + \lambda_2 - (x_1^* + x_3^*) = 0 \\ \lambda_2 - (x_1^* + x_2^*) = 0 \end{array} \right\} \Rightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ x_1^* + x_3^* = 3 \end{array}$$

$$\begin{aligned} 2(x_1^* + x_2^* + x_3^*) &= 3 + 2 + 1 = 6 \\ \Rightarrow x_1^* + x_2^* + x_3^* &= 3 \\ \Rightarrow x_3^* &= 1, \quad x_1^* = 2, \quad x_2^* = 0 \end{aligned}$$

Check second order suff. condition for  $x^* = (2, 0, 1)$ :

$$\nabla_{xx}^2 L(x, \lambda) = \nabla^2 f(x)$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\text{Consider principal minors : } |0| = 0 \quad \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

$\Rightarrow \nabla_{xx}^2 L(x^*, \lambda)$  not PD or even PSD.

But, we only need  $\bar{z}^\top \nabla_{xx}^2 L(x^*, \lambda) \bar{z} > 0$   
 for  $\bar{z} \neq 0$  s.t.  $a_i^\top \bar{z} = 0$ ,  $i=1,2$ .

$$a_1^\top \bar{z} = [1 \ 1 \ 0] \bar{z} = \bar{z}_1 + \bar{z}_2 = 0 \Rightarrow \bar{z}_1 = -\bar{z}_2$$

$$a_2^\top \bar{z} = [0 \ 1 \ 1] \bar{z} = \bar{z}_2 + \bar{z}_3 = 0 \Rightarrow \bar{z}_3 = -\bar{z}_2$$

$$\begin{aligned} \bar{z}^\top \nabla_{xx}^2 L(x^*, \lambda) \bar{z} &= [\bar{z}_1 \ \bar{z}_2 \ \bar{z}_3] \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \end{bmatrix} \\ &= [\bar{z}_1 \ \bar{z}_2 \ \bar{z}_3] \begin{bmatrix} -\bar{z}_2 - \bar{z}_3 \\ -\bar{z}_1 - \bar{z}_2 \\ -\bar{z}_2 - \bar{z}_3 \end{bmatrix} \\ &= -2 (\bar{z}_1 \bar{z}_2 + \bar{z}_2 \bar{z}_3 + \bar{z}_1 \bar{z}_3) \\ &= 2 \bar{z}_2^2 > 0 \quad \text{if } \bar{z} \neq 0 \end{aligned}$$

Thus  $x^* = (2, 0, 1)$  is a (strict) local min  
 of  $f(x)$  s.t.  $h_i(x) = 0$ ,  $i=1,2$ .

But is  $x^* = (2, 0, 1)$  a global min?

Does  $f$  have a global min under  $h(x) = 0$ ?

$$h(x) = 0 \Leftrightarrow x_1 = 2 - x_2 \quad x_3 = 1 - x_2$$

$$f(x) = -(x_1 x_2 + x_2 x_3 + x_1 x_3)$$

$$\begin{aligned} \text{on } h(x)=0 \rightarrow &= -((2-x_2)x_2 + (1-x_2)x_2 + (1-x_2)(2-x_2)) \\ &= -(2-x_2 + x_2 - x_2^2) = x_2^2 - 2 \end{aligned}$$

$\Rightarrow f$  is bercive on  $H = \{x : h(x) = 0\}$  which is closed

$\Rightarrow$  by corollary to WT,  $f$  achieves global min on  $H$

$\Rightarrow x^* = (2, 0, 1)$  is the global min of  $f(x)$  on  $H$ .

## Linear constraints with no regular $x$

$$\begin{array}{l} \text{minimize } f(x) \\ \text{s.t. } Ax = b \end{array} \quad A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad m \leq n \quad \underset{i=1, \dots, m}{a_i^T x = b_i}$$

Suppose  $a_1, a_2, \dots, a_m$  are not l.i.

Then  $\nabla h_i(x) = a_i \Rightarrow \{\nabla h_i(x)\}_{i=1}^m$  not l.i.  
 $\Rightarrow$  no regular  $x$ .

$a_1, a_2, \dots, a_m$  not l.i.  $\Rightarrow \text{rank}(A) = k < m$ .

Reorder the  $a_i$ 's so that  $a_1, a_2, \dots, a_k$  are l.i.

Then

$$a_i = \sum_{j=1}^k \alpha_j^{(i)} a_j, \quad i = k+1, \dots, m$$

Then for  $Ax = b$  to have at least one solution (consistency), we must have

$$b_i = \sum_{j=1}^k \alpha_j^{(i)} b_j, \quad i = k+1, \dots, m$$

Thus the constraints  $a_i^T x = b_i, i = k+1, \dots, m$   
 are redundant and can be discarded.

$$\begin{array}{ll} \text{minimize } f(x) & \equiv \text{minimize } f(x) \\ \text{s.t. } Ax = b & \text{s.t. } \tilde{A}x = \tilde{b} \end{array}$$

$$\text{with } \tilde{A} = \begin{bmatrix} a_1^T \\ \vdots \\ a_k^T \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}$$

$\Rightarrow$  L M's exist

But  $\tilde{A}, \tilde{b}$  may not be unique  $\Rightarrow \lambda$  may not be unique

Example minimize  $x_1^2 + x_2^2 + 2x_3^2$

$$\text{s.t. } x_1 + x_2 = 2$$

$$-x_2 + x_3 = 1$$

$$x_1 + x_3 = 3$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \text{rows are l.d.}$$

$$\nabla f(x) = [2x_1 \quad 2x_2 \quad 4x_3]^T$$

Choice 1  $\tilde{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \tilde{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x^*$ ,  $\lambda = (\lambda_1, \lambda_2)$  satisfy:

$$\tilde{A}^T \lambda = -\nabla f(x^*) \text{, i.e.,}$$

$$\left. \begin{array}{l} \lambda_1 = -2x_1^* \\ \lambda_1 - \lambda_2 = -2x_2^* \\ \lambda_2 = -4x_3^* \end{array} \right\} \Rightarrow \begin{array}{l} -2x_2^* = 4x_3^* - 2x_1^* \\ x_2^* = x_1^* - 2x_3^* \end{array}$$

and

$$\begin{array}{l} x_1^* + x_2^* = 2 \\ x_3^* - x_2^* = 1 \end{array}$$

$$\begin{array}{l} x_1^* + x_2^* = 2 \\ x_1^* - x_2^* - 2x_3^* = 0 \end{array} \Rightarrow \begin{array}{l} 2x_2^* + 2x_3^* = 2 \\ \Rightarrow x_2^* = 1 - x_3^* \end{array}$$

$$x_3^* - (1 - x_3^*) = 1 \Rightarrow 2x_3^* = 2 \Rightarrow x_3^* = 1$$
$$\Rightarrow x_2^* = 0$$

$$\Rightarrow x_1^* = 2$$

$$\lambda_1 = -4, \quad \lambda_2 = -4.$$

Choice 2

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$x^*$  and  $\lambda$  satisfy :

$$\left. \begin{array}{l} \lambda_1 + \lambda_2 = -2x_1^* \\ \lambda_1 = -2x_2^* \\ \lambda_2 = -4x_3^* \end{array} \right\} \Rightarrow x_1^* = x_2^* + 2x_3^*$$

$$x_1^* + x_2^* = 2$$

$$x_1^* + x_3^* = 3$$

$$\left. \begin{array}{l} x_1^* + x_2^* = 2 \\ x_1^* - x_2^* - 2x_3^* = 0 \end{array} \right\} \Rightarrow \begin{array}{l} 2x_1^* - 2x_3^* = 2 \\ x_1^* - x_3^* = 1 \end{array}$$

$$\left. \begin{array}{l} x_1^* + x_3^* = 3 \\ x_1^* - x_3^* = 1 \end{array} \right\} \Rightarrow \begin{array}{l} x_1^* = 2 \\ x_3^* = 1 \end{array}$$

$$\Rightarrow x_2^* = 0$$

So  $x^*$  is same as in choice 1,

$$\text{but } \lambda_1 = 0, \quad \lambda_2 = -4$$