Union-Find

Algorithm : Design & Analysis

[10]

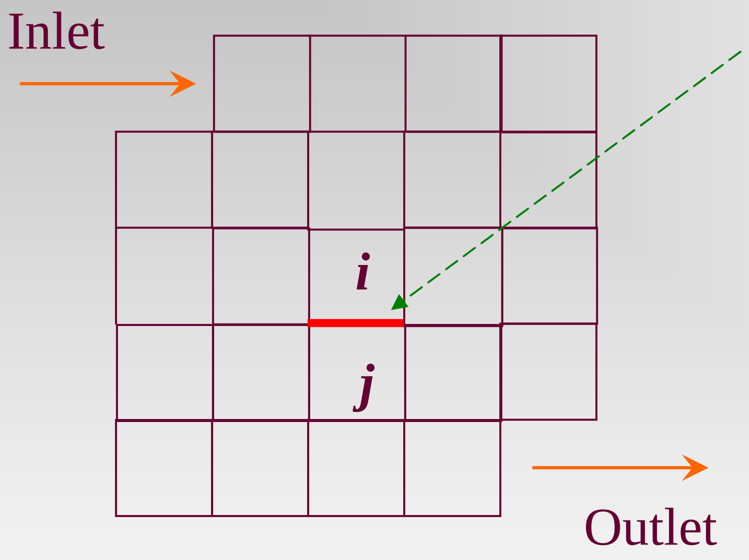
In the last class…

- Hashing
- Collision Handling for Hashing
  - Closed Address Hashing
  - Open Address Hashing
- Hash Functions
- Array Doubling and Amortized Analysis

Union-Find

- Dynamic Equivalence Relation
- Implementing Dynamic Set by Union-Find
  - Straight Union-Find
  - Making Shorter Tree by Weighted Union
  - Compressing Path by Compressing-Find
- Amortized Analysis of *wUnion-cFind*

Maze Creating: an Example



Selecting a wall to pull down randomly

If *i*,*j* are in same equivalence class, then select another wall to pull down, otherwise, joint the two classes into one.

The maze is complete when the inlet and outlet are in one equivalence class.

A More Serious Example

- Kruskal’s algorithm for MST(the minimum spaning tree).
- Greedy strategy: Select the edge not in the tree with the minimum weight, which will **NOT** result in a cycle with the edges having been selected.
- How to know **NO CYCLE**, however?

Dynamic Equivalence Relations

- Equivalence
  - reflexive, symmetric, transitive
  - equivalent classes forming a partition
- Dynamic equivalence relation
  - changing in the process of computation
  - **IS** instruction: *yes* or *no* (in the same equivalence class)
  - **MAKE** instruction: combining two equivalent classes, by relating two unrelated elements, and influencing the results of subsequent IS instructions.
  - Starting as equality relation

Implementation: How to Measure

- The number of basic operations for processing a sequence of *m* **MAKE** and/or **IS** instructions on a set *S* with *n* elements.
- An Example: *S*={1,2,3,4,5}
    - 0. [create] {{1}, {2}, {3}, {4}, {5}}
    - 1. **IS** 2≡4? No
    - 2. **IS** 3≡5? No
    - 3. **MAKE** 3≡5. {{1}, {2}, {3,5}, {4}}
    - 4. **MAKE** 2≡5. {{1}, {2,3,5}, {4}}
    - 5. **IS** 2≡3? Yes
    - 6. **MAKE** 4≡1. {{1,4}, {2,3,5}}
    - 7. **IS** 2≡4? No

Implementation: Choices

- Matrix (relation matrix)
  - Space in $\Theta(n^2)$, and worst-case cost in $\Theta(mn)$ (mainly for row copying for MAKE).
- Array (for equivalence class id.)
  - Space in $\Theta(n)$, and worst-case cost in $\Theta(mn)$ (mainly for search and change for MAKE).
- Union-Find
  - A object of type Union-Find is a collection of disjoint sets
  - There is no way to traverse through all the elements in one set.

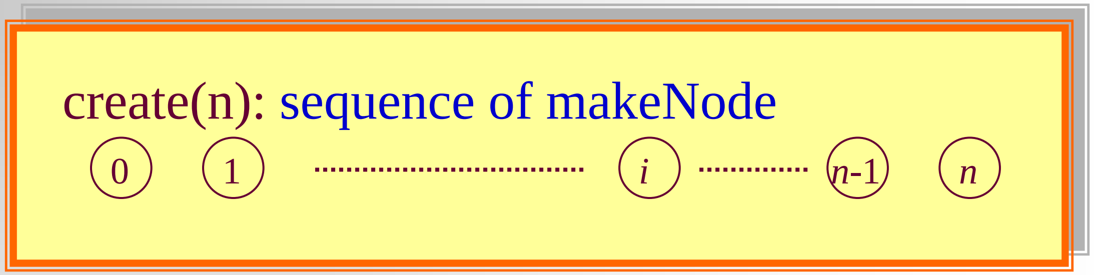
Union-Find ADT

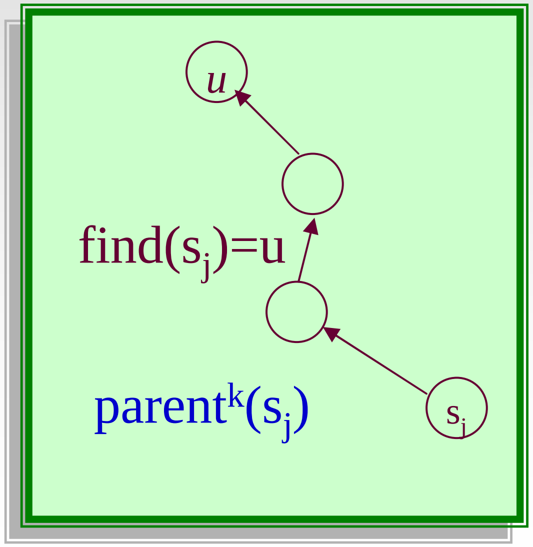
- Constructor: Union-Find create(**int** *n*)
  - sets=create(*n*) refers to a newly created group of sets {1}, {2}, ..., {*n*} (*n* singletons)
- Access Function: **int** find(UnionFind sets, *e*)
  - find(sets, *e*)=<*e*>
- Manipulation Procedures
  - **void** makeSet(UnionFind sets, **int** *e*)
  - **void** union(UnionFind sets, **int** *s*, **int** *t*)

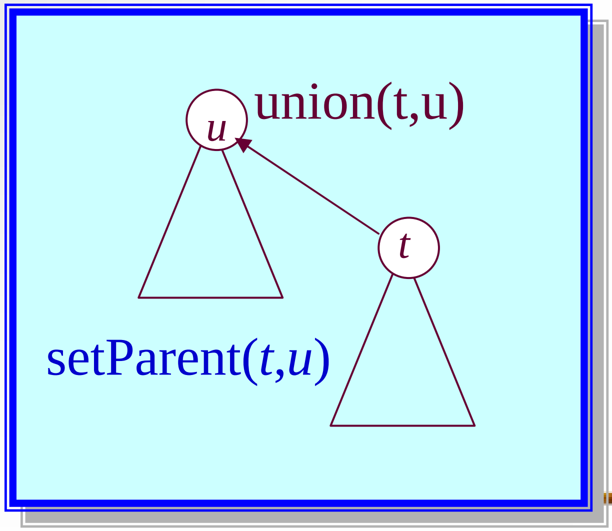
Implementing Dynamic Equivalence Using Union-Find (as inTree)

- **IS** $s_i \equiv s_j$:
  - t=find(s_i);
  - u=find(s_j);
  - (t==u)?
- **MAKE** $s_i \equiv s_j$:
  - t=find(s_i);
  - u=find(s_j);
  - union(t,u);

implementation by inTree







Union-Find Program

- A union-find program of length m is (a *create*(n) operation followed by) a sequence of m union and/or find operations interspersed in any order.
- A union-find program is considered an input, the object for which the analysis is conducted.
- The measure: number of accesses to the ***parent***
  - assignments: for union operations
  - lookups: for find operations

$\left.\begin{array}{l}\text{assignments}\\\text{lookups}\end{array}\right\}$ **link operation**

Worst-case Analysis for Union-Find Program

- Assuming each lookup/assignment take O(1).
- Each makeSet or union does one assignment, and each find does d+1 lookups, where d is the depth of the node.

```
1.     Union(1,2)
2.     Union(2,3)
          ⋮
n-1.  Union(n-1,n)
n.     Find(1)
          ⋮
m.    Find(1)
```

Example

The sequence of *Union* makes a chain of length n-1, which is the tree with the largest height

Find(1) needs n array lookups

operations done:

$n+(n-1)+(m-n+1)n$ $\quad \Theta(mn)$

Weighted Union: for Short Trees

- Weighted union: always have the tree with **fewer nodes** as subtree. (*wUnion*)

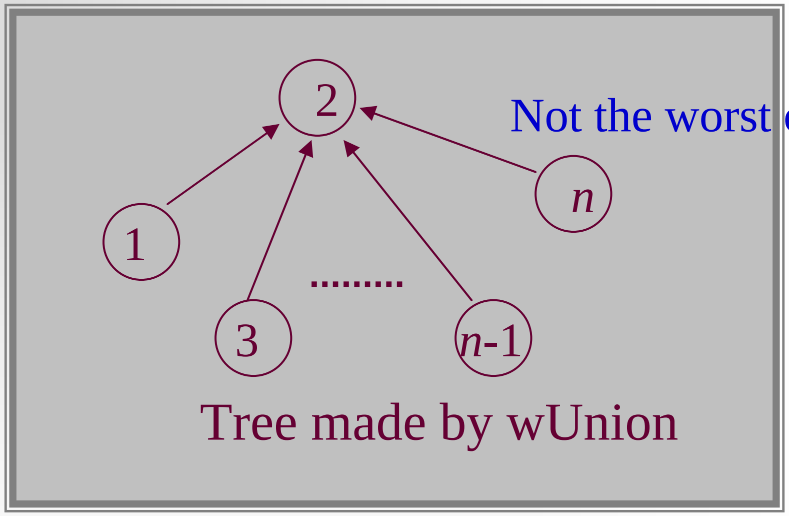
To keep the *Union* valid, each *Union* operation is replaced by:

t=find(i);

u=find(j);

union(t,u)

The order of (t,u) satisfying the requirement



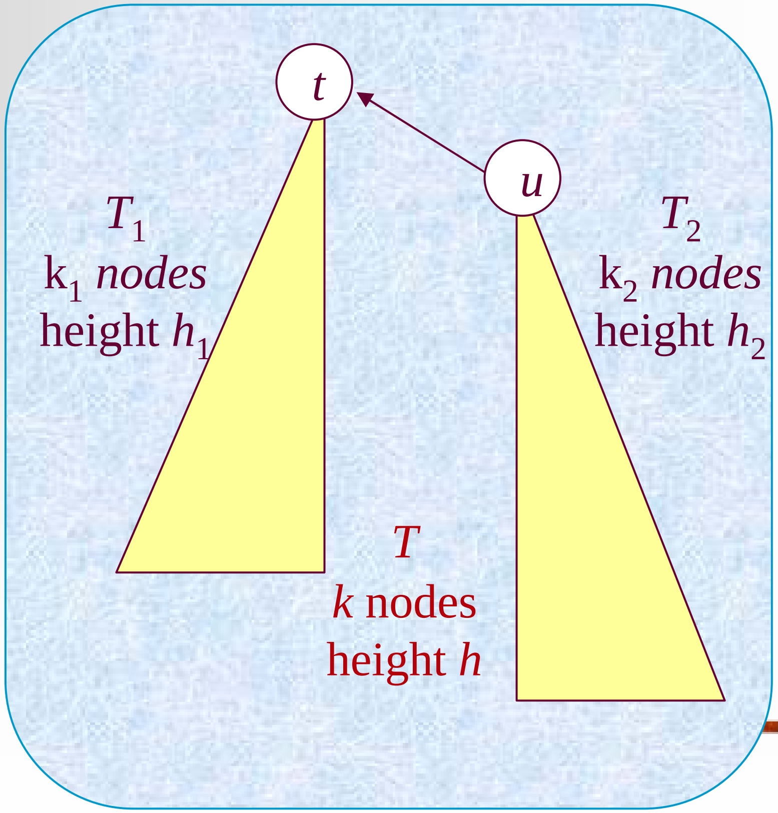
Tree made by wUnion

Cost for the program:

$n+3(n-1)+2(m-n+1)$

Upper Bound of Tree Height

- After any sequence of *Union* instructions, implemented by *wUnion*, any tree that has k nodes will have height at most $\lfloor \lg k \rfloor$
- Proof by induction on k:
  - base case: k=1, the height is 0.
  - by inductive hypothesis:
    - $h_1 \le \lfloor \lg k_1 \rfloor$, $h_2 \le \lfloor \lg k_2 \rfloor$
  - h=max(h1, h2+1), k=k1+k2
    - if $h = h_1$, $h \le \lfloor \lg k_1 \rfloor \le \lfloor \lg k \rfloor$
    - if $h = h_2 + 1$, note: $k_2 \le k/2$
      so, $h_2 + 1 \le \lfloor \lg k_2 \rfloor + 1 \le \lfloor \lg k \rfloor$

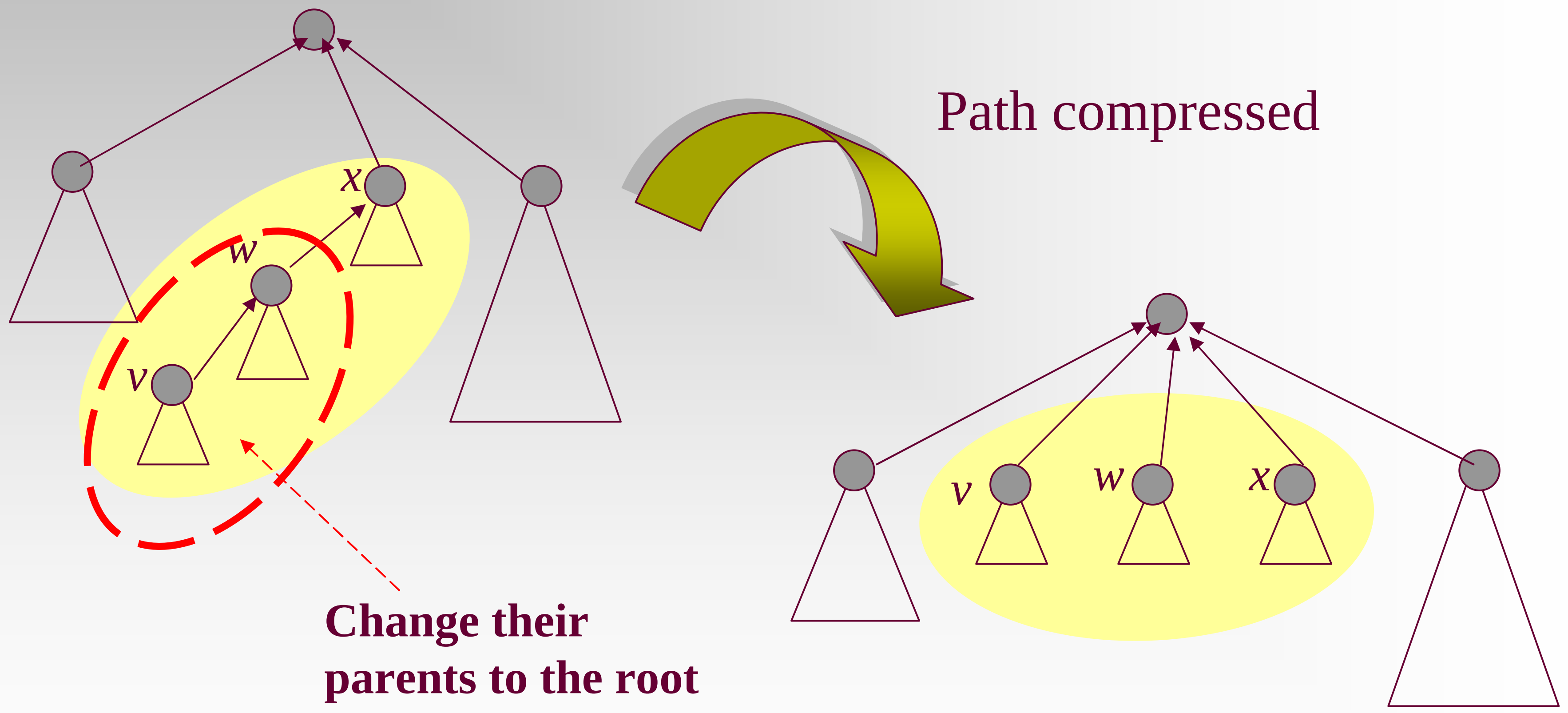


Upper Bound for Union-Find Program

- A Union-Find program of size m, on a set of n elements, performs **$\Theta(n+m\log n)$** link operations in the worst case if *wUnion* and straight *find* are used.
- Proof:
  - At most n-1 *wUnion* can be done, building a tree with height at most $\lfloor \lg n \rfloor$,
  - Then, each *find* costs at most $\lfloor \lg n \rfloor+1$.
  - Each *wUnion* costs in $O(1)$, so, the upper bound on the cost of any combination of m *wUnion/find* operations is the cost of m *find* operations, that is $m(\lfloor \lg n \rfloor+1) \in O(n+m\log n)$

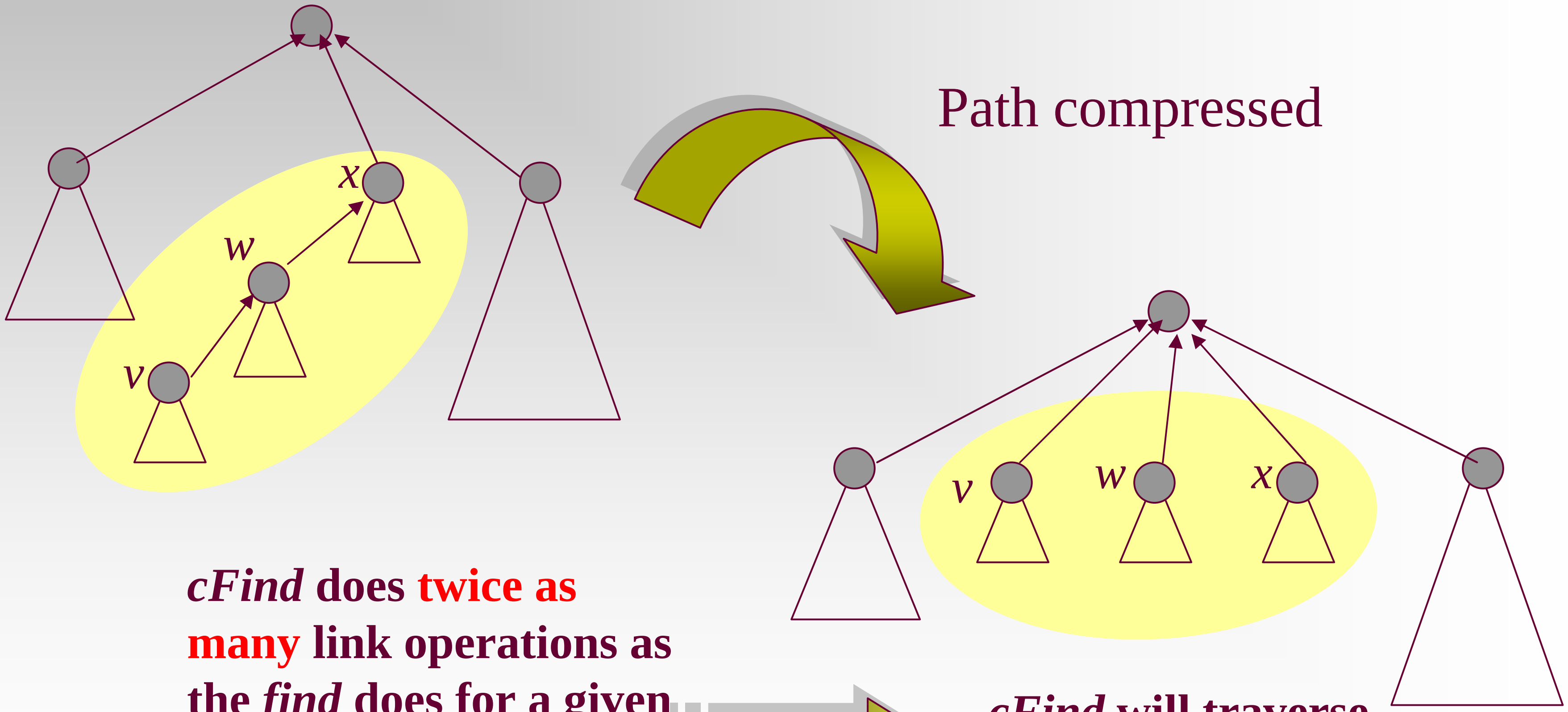
There do exist programs requiring $\Omega(n+m\log n)$ steps.

Path Compression



Change their parents to the root

Challenges for the Analysis



cFind does **twice as many** link operations as the *find* does for a given node in a given tree.

But…

cFind will traverse **shorter** paths

Analysis: the Basic Idea

- *c*Find may be an expensive operation, in the case that find(*i*) is executed and the node *i* has great depth.
- However, such *c*Find can be executed only for limited times, relative to other operations of lower cost.
- So, amortized analysis applys.

Co-Strength of *wUnion* and *cFind*

- The number of link operations done by a *Union-Find* program implemented with *wUnion* and *cFind*, of length *m* on a set of *n* elements is in $O((n+m)\lg^*(n))$ in the worst case.

- What's $\lg^*(n)$?
  - Define the function *H* as following:

$$\begin{cases} H(0)=1 \\ H(i)=2^{H(i-1)} \quad \text{for } i>0 \end{cases}$$

  - Then, $\lg^*(j)$ for $j\geq 1$ is defined as:

$$\lg^*(j)=\min\{\, k \mid H(k)\geq j \,\}$$

Definitions with a *Union-Find* Program *P*

- Forest *F*: the forest constructed by the sequence of *union* instructions in *P*, assuming:
  - *wUnion* is used;
  - the *finds* in the *P* are ignored
- Height of a node *v* in any tree: the height of the subtree rooted at *v*
- Rank of *v*: the height of *v* ***<u>in F</u>***

Note: *cFind* changes the height of a node, but the rank for any node is invariable.

Constraints on Ranks in *F*

- The upper bound of the number of nodes with rank *r* ($r \geq 0$) is $\frac{n}{2^r}$
  - Remember that the height of the tree built by *wUnion* is at most $\lfloor \lg n \rfloor$, which means the subtree of height *r* has at least 2^r nodes.
  - The subtrees with root at rank *r* are disjoint.
- There are at most $\lfloor \lg n \rfloor$ different ranks.
  - There are altogether n elements in S, that is, n nodes in F.

Increasing Sequence of Ranks

- The ranks of the nodes on a path from a leaf to a root of a tree in *F* form a strictly increasing sequence.
- When a *cFind* operation changes the parent of a node, the new parent has higher rank than the old parent of that node.
  - Note: the new parent was an ancestor of the previous parent.

A Function Growing Extremely Slowly

- Function H:

$$\begin{cases} H(0)=1 \\ H(i+1)=2^{H(i)} \end{cases}$$

that is: $H(k)=\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{k\text{ 2's}}$

Note:

H grows extremely fast:

$H(4)=2^{16}=65536$

$H(5)=2^{65536}$

- Function Log-star

lg*(j) is defined as the least i such that:

$H(i)\geq j$ for $j>0$

- Log-star grows extremely slowly

$$\lim_{n\to\infty}\frac{\lg^*(n)}{\log^{(p)} n}=0$$

p is any fixed nonnegative constant

For any x: $2^{16}\leq x\leq 2^{65536}-1$, lg*($x$)=5 !

Grouping Nodes by Ranks

- Node $v \in s_i$ ($i \geq 0$) iff. lg*(1+rank of v)=i
  - which means that: if node v is in group i, then

    $r_v \leq H(i)-1$, but not in group with smaller labels
- So,
  - Group 0: all nodes with rank 0
  - Group 1: all nodes with rank 1
  - Group 2: all nodes with rank 2 or 3
  - Group 3: all nodes with its rank in [4,15]
  - Group 4: all nodes with its rank in [16, 65535]
  - Group 5: all nodes with its rank in [65536, ???]

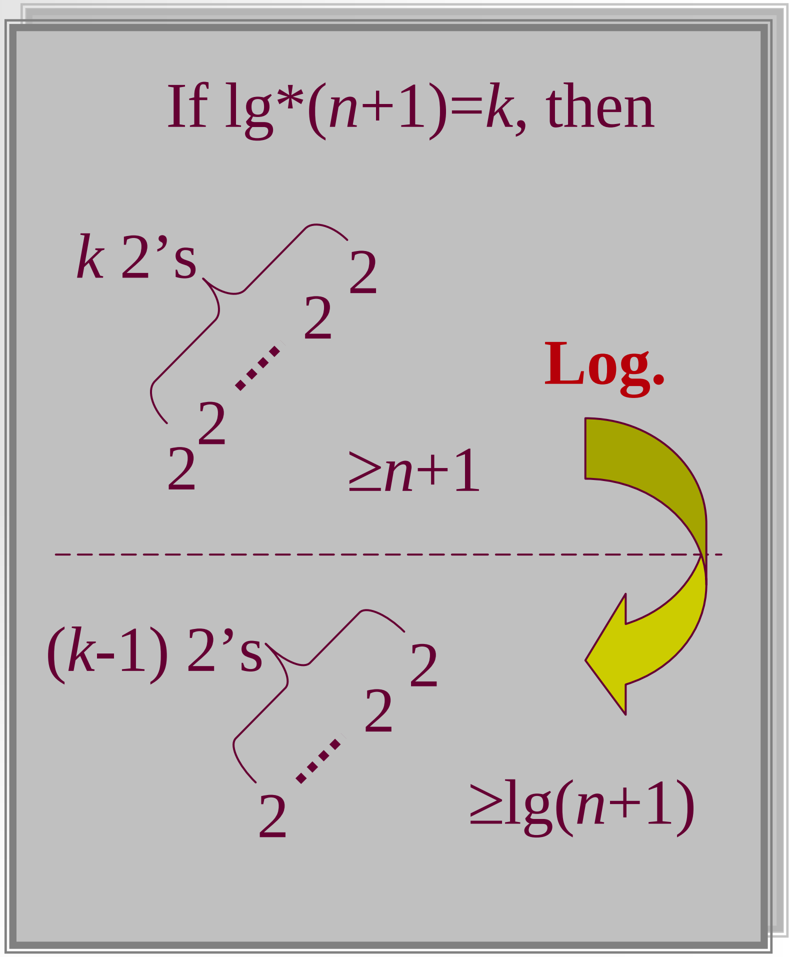
Group 5 exists only when n is at least 2^{65536}. What is that?

Very Few Groups

- **Node $v \in S_i$ ($i \geq 0$) iff.**

  $\lg^*(1+\text{rank of } v)=i$

- Upper bound of the number of distinct node groups is $\lg^*(n+1)$
  - The rank of any node in F is at most $\lfloor \lg n \rfloor$, so the largest group index is $\lg^*(1+\lfloor \lg n \rfloor)=\lg^*(\lceil \lg n+1 \rceil) = \lg^*(n+1)-1$


If $\lg^*(n+1)=k$, then

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k\ 2\text{'s}} \geq n+1$$

Log.

$$\underbrace{2^{\cdot^{\cdot^{\cdot^{2^{2}}}}}}_{(k-1)\ 2\text{'s}} \geq \lg(n+1)$$

Amortized Cost of *Union-Find*

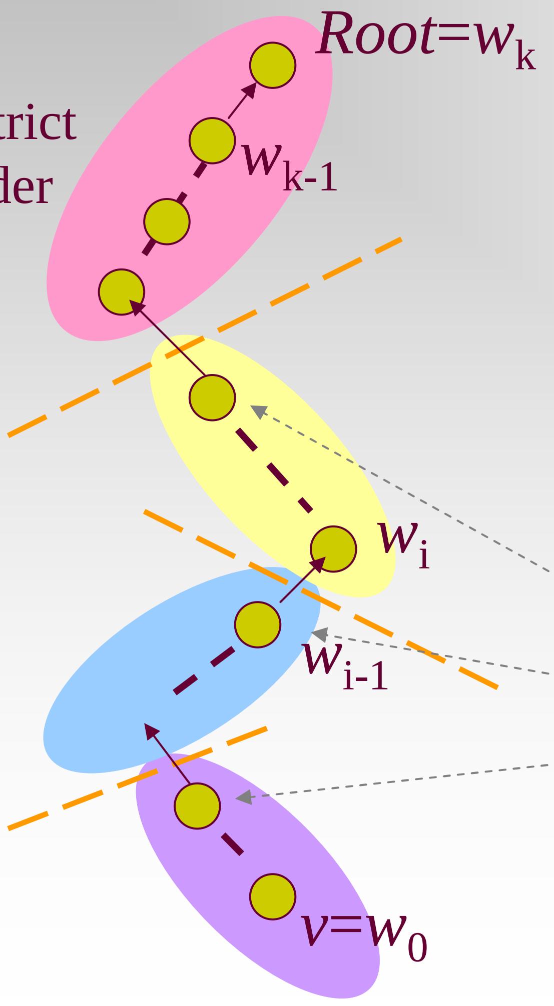
- Amortized Equation Recalled

amortized cost =
actual cost +
accounting cost

- The operations to be considered:
  - *n* makeSets
  - *m* union & find (with at most *n*-1 unions)

One Execution of *cFind*(w_0)

Groups in a strict increasing order



Group Boundary

Note: the ranks are not consecutive generally

Only when k=0,1, there is no parent change

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For one *cFind* operation, the actual cost is 2k Not 2(k+1)

Accounting cost is -2 for each pair of (w_{i-1}, w_i) for the the 2 nodes in ths **same group only**, which we call a **withdrawal**.

Amortizing Scheme for *wUnion-cFind*

- makeSet
  - Accounting cost is $4\lg^*(n+1)$
  - So, the amortized cost is $1+4\lg^*(n+1)$
- *wUnion*
  - Accounting cost is 0
  - So the amortized cost is 1
- *cFind*
  - Accounting cost is describes as in the previous page.
  - Amortized cost $\le 2k-2(\underline{(k-1)-(\lg^*(n+1)-1)})=2\lg^*(n+1)$ (Compare with the worst case cost of *cFind*, $2\lg n$)

Number of withdrawal

Validation of the Amortizing Scheme

- We must be assure that **the sum of the accounting costs is never negative**.
- The sum of the negative charges, incurred by *cFind*, does not exceed $4n\lg^*(n+1)$
  - We prove this by showing that at most $2n\lg^*(n+1)$ withdrawals on nodes occur during all the executions of *c*Find.

Key Idea in the Derivation

- For any node, the number of withdrawal will be less than the number of different ranks in the group it belong to
  - When a *cFind* changes the parent of a node, the new parent is always has higher rank than the old parent.
  - Once a node is assigned a new parent in a higher group, no more negative amortized cost will incurred for it again.
- The number of different ranks is limited within a group.

Derivation

a loose upper bound of ranks in a group

- The number of withdrawals for all $w \in S$ is:

$$\sum_{i=0}^{\lg^*(n+1)-1} H(i)\,(\text{number of nodes in group } i)$$

Note : number of nodes in group i is at most :

$$\sum_{r=H(i-1)}^{H(i)-1} \frac{n}{2^r} \le \frac{n}{2^{H(i-1)}} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{2n}{2^{H(I-1)}} = \frac{2n}{H(i)}$$

So ,

$$\sum_{i=0}^{\lg^*(n+1)-1} H(i) \frac{2n}{H(i)} = 2n \lg^*(n+1)$$

The Conclusion

- The number of link operations done by a *Union-Find* program implemented with *wUnion* and *cFind*, of length *m* on a set of *n* elements is in $O((n+m)\lg^*(n))$ in the worst case.
  - Note: since the sum of accounting cost is never negative, the actual cost is always not less than amortized cost. And, the upper bound of amortized cost is: $(n+m)(1+4\lg^*(n+1))$

Home Assignments

- 6.19
- 6.21
- 6.23
- 6.25-27