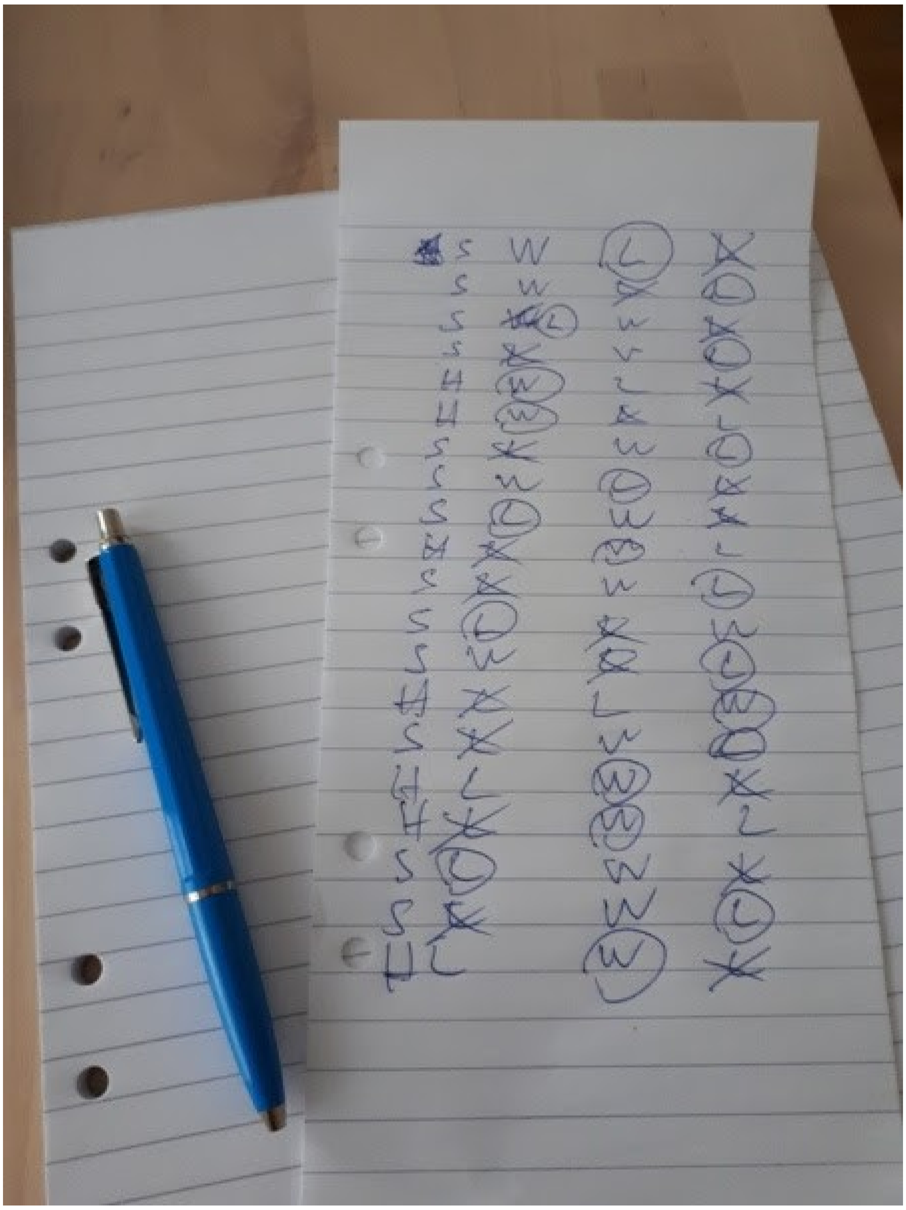
Like many others, I've never felt that the solution to the Monty Hall problem was intuitive, despite the fact that explanations of the correct solution are everywhere. I am not alone. Famously, columnist Marilyn vos Savant got droves of mail from people trying to school her after she had published the correct solution.

The problem goes like this: You are a contestant on a game show (based on a real game show hosted by Monty Hall, hence the name). The host presents you with three doors, one of which contains a prize — say, a goat — and the others are empty. After you've made your choice, the host opens one of the doors, showing that it is empty. You are now asked whether you would like to stick to your initial choice, or switch to the other door. The right thing to do is to switch, which gives you 2/3 probability of winning the goat. This can be demonstrated in a few different ways.

A goat is a great prize. Image: Casey Goat by Pete Markham (CC BY-SA 2.0)

So I sat down to do 20 physical Monty Hall simulations on paper. I shuffled three cards with the options, picked one, and then, playing the role of the host, took away one losing option, and noted down if switching or holding on to the first choice would have been the right thing to do. The results came out 13 out of 20 (65%) wins for the switching strategy, and 7 out of 20 (35%) for the holding strategy. Of course, the Monty Hall Truthers out there must question whether this demonstration in fact happened — it's too perfect, isn't it?



The outcome of the simulation is less important than the feeling that came over me as I was running it, though. As I was taking on the role of the host and preparing to take away one of the losing options, it started feeling self-evident that the important thing is whether the first choice is right. If the first choice is right, holding is the right strategy. If the first choice is wrong, switching is the right option. And the first choice, clearly, is only right 1/3 of the time.

In this case, it was helpful to take the game show host's perspective. Selvin (1975) discussed the solution to the problem in The American Statistician, and included a quote from Monty Hall himself:

> Monty Hall wrote and expressed that he was not "a student of statistics problems" but "the big hole in your argument is that once the first box is seen to be empty, the contestant cannot exchange his box." He continues to say, "Oh, and incidentally, after one [box] is seen to be empty, his chances are no longer 50/50 but remain what they were in the first place, one out of three. It just seems to the contestant that one box having been eliminated, he stands a better chance. Not so." I could not have said it better myself.

A generalised problem

Now, imagine the same problem with a number *d* number of doors, *w* number of prizes and *o* number of losing doors that are opened after the first choice is made. We assume that the losing doors are opened at random, and that switching entails picking one of the remaining doors at random. What is the probability of winning with the switching strategy?

The probability of picking the a door with or without a prize is:

$$\Pr(\text{pick right first}) = \frac{w}{d}$$

$$\Pr(\text{pick wrong first}) = 1 - \frac{w}{d}$$

If we picked a right door first, we have *w – 1* winning options left out of *d – o – 1* doors after the host opens *o* doors:

$$\Pr(\text{win}|\text{right first}) = \frac{w-1}{d-o-1}$$

If we picked the wrong door first, we have all the winning options left:

$$\Pr(\text{win}|\text{wrong first}) = \frac{w}{d-o-1}$$

Putting it all together:

$$\Pr(\text{win}|\text{switch}) = \Pr(\text{pick right first}) \cdot \Pr(\text{win}|\text{right first}) +$$
$$+ \Pr(\text{pick wrong first}) \cdot \Pr(\text{win}|\text{wrong first}) =$$
$$= \frac{w}{d}\frac{w-1}{d-o-1} + (1 - \frac{w}{d})\frac{w}{d-o-1}$$

As before, for the hold strategy, the probability of winning is the probability of getting it right the first time:

$$\Pr(\text{win}|\text{hold}) = \frac{w}{d}$$

With the original Monty Hall problem, *w = 1*, *d = 3* and *o = 1*, and therefore

$$\Pr(\text{win}|\text{switch}) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1$$

Selvin (1975) also present a generalisation due to Ferguson, where there are *n* options and *p* doors that are opened after the choice. That is, *w = 1*, *d = 3* and *o = 1*. Therefore,

$$\Pr(\text{win}|\text{switch}) = \frac{1}{n} \cdot 0 + (1 - \frac{1}{n})\frac{1}{n-p-1} = \frac{n-1}{n(n-p-1)}$$

which is Ferguson’s formula.

Finally, in Marilyn vos Savant’s column, she used this thought experiment to illustrate why switching is the right thing to do:

> Here’s a good way to visualize what happened. Suppose there are a million doors, and you pick door #1. Then the host, who knows what’s behind the doors and will always avoid the one with the prize, opens them all except door #777,777. You’d switch to that door pretty fast, wouldn’t you?

That is, *w = 1* still, $d = 10^6$ and $o = 10^6 - 2$.

$$\Pr(\text{win}|\text{switch}) = 1 - \frac{1}{10^6}$$

It turns out that the solution to the generalised problem is that it is always better to switch, as long as there is a prize, and as long as the host opens any doors. One can also generalise it to choosing sets of more than one door. This makes some intuitive sense: as long as the host takes opens some doors, taking away losing options, switching should enrich for prizes.