Happy Pi Day! I don't encounter π very much in my area of statistics, so this post might seem a little forced… In this post, I'm going to show one way to estimate π.

The starting point is the integral identity

$$4 \int_0^1 \sqrt{1-x^2}dx = \pi.$$

There are two ways to see why this identity is true. The first is that the integral is simply computing the area of a quarter-circle with unit radius. The second is by explicitly evaluating the integral:

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \left[\frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right)\right]_0^1 \\ &= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] \\ &= \frac{\pi}{4}.\end{aligned}$$

If $X \sim \text{Unif}[0,1]$, then the integral identity means that

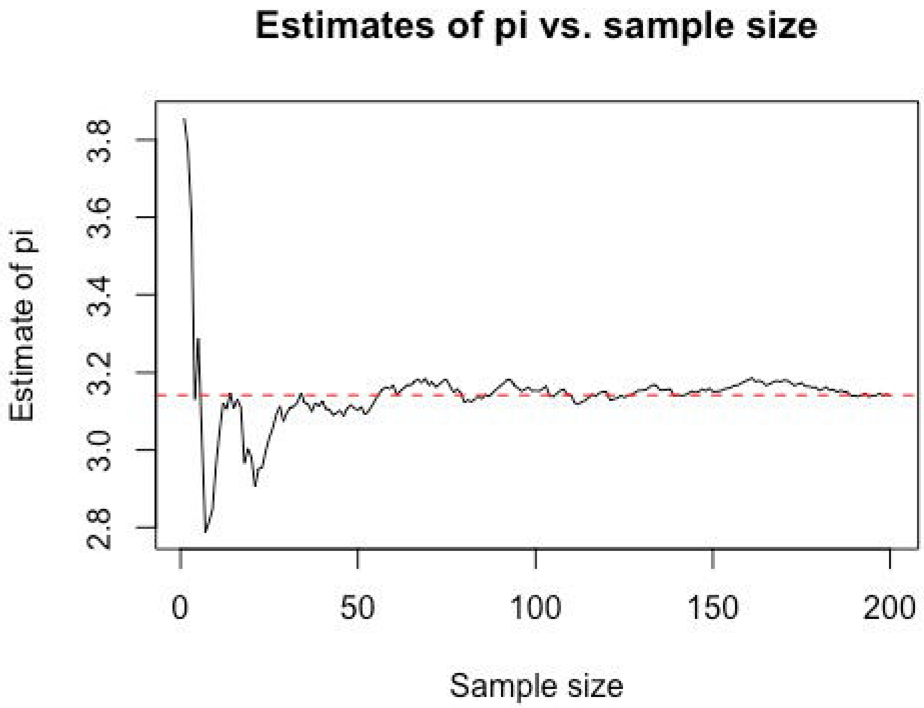
$$\mathbb{E}\left[\sqrt{1-X^2}\right] = \pi.$$

Hence, if we take i.i.d. draws $X_1, \dots, X_n$ from the uniform distribution on $[0,1]$, it is reasonable to expect that

$$\frac{1}{n}\sum_{i=1}^n \sqrt{1-X_i^2} \approx \pi.$$

The code below shows how well this estimation procedure does for one run as the sample size goes from 1 to 200:

```
set.seed(1)
N <- 200
x <- runif(N)
samples <- 4 * sqrt(1 - x^2)
estimates <- cumsum(samples) / 1:N
plot(1:N, estimates, type = "l",
     xlab = "Sample size", ylab = "Estimate of pi",
     main = "Estimates of pi vs. sample size")
abline(h = pi, col ="red", lty = 2)
```



The next plot shows the relative error on the y-axis instead (the red dotted line represents 1% relative error):

```
rel_error <- abs(estimates - pi) / pi * 100
plot(1:N, rel_error, type = "l",
xlab = "Sample size", ylab = "Relative error (%)",
main = "Relative error vs. sample size")
abline(h = 0)
abline(h = 1, col = "red", lty = 2)
```