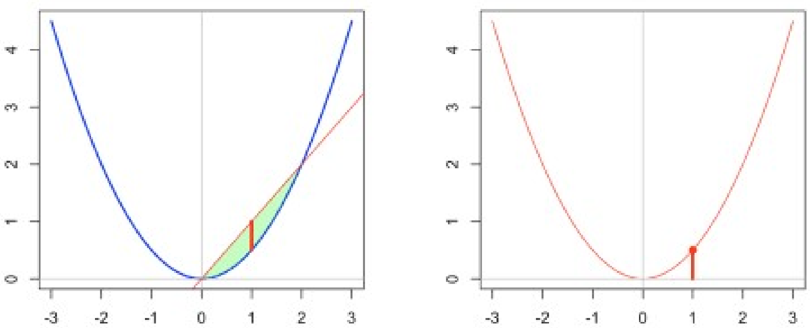
In the MAT7381 course (graduate course on regression models), we will talk about optimization, and a classical tool is the so-called conjugate. Given a function f:\mathbb{R}^p\to\mathbb{R} its conjugate is function f^{\star}:\mathbb{R}^p\to\mathbb{R} such that f^{\star}(\boldsymbol{y})=\max_{\boldsymbol{x}}\lbrace\boldsymbol{x}^\top\boldsymbol{y}-f(\boldsymbol{x})\rbraceso, long story short, f^{\star}(\boldsymbol{y}) is the maximum gap between the linear function \boldsymbol{x}^\top\boldsymbol{y} and f(\boldsymbol{x}).

Just to visualize, consider a simple parabolic function (in dimension 1) f(x)=x^2/2, then f^{\star}(\color{blue}{2}) is the maximum gap between the line x\mapsto\color{blue}{2}x and function f(x).

```
x = seq(-100,100,length=6001)
f = function(x) x^2/2
vf = Vectorize(f)(x)
fstar = function(y) max(y*x-vf)
vfstar = Vectorize(fstar)(x)
```

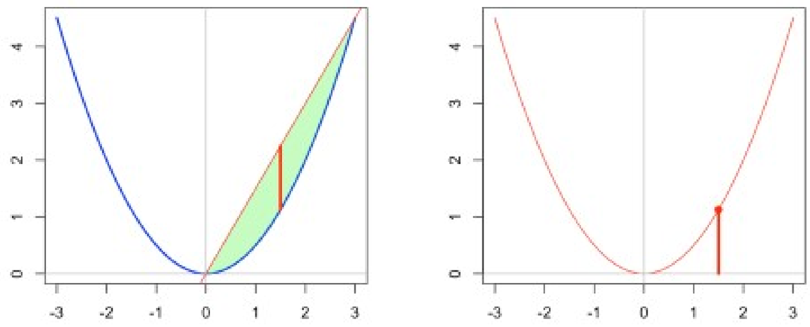
We can see it on the figure below.

```
viz = function(x0=1,YL=NA){
idx=which(abs(x)<=3) par(mfrow=c(1,2)) plot(x[idx],vf[idx],type="l",
xlab="",ylab="",col="blue",lwd=2) abline(h=0,col="grey") abline(v=0,col="grey")
idx2=which(x0*x>=vf)
polygon(c(x[idx2],rev(x[idx2])),c(vf[idx2],rev(x0*x[idx2])),
col=rgb(0,1,0,.3),border=NA)
abline(a=0,b=x0,col="red")
i=which.max(x0*x-vf)
segments(x[i],x0*x[i],x[i],f(x[i]),lwd=3,col="red")
if(is.na(YL)) YL=range(vfstar[idx])
plot(x[idx],vfstar[idx],type="l",xlab="",ylab="",col="red",lwd=1,ylim=YL)
abline(h=0,col="grey")
abline(v=0,col="grey")
segments(x0,0,x0,fstar(x0),lwd=3,col="red")
points(x0,fstar(x0),pch=19,col="red")
}
viz(1)
```



or

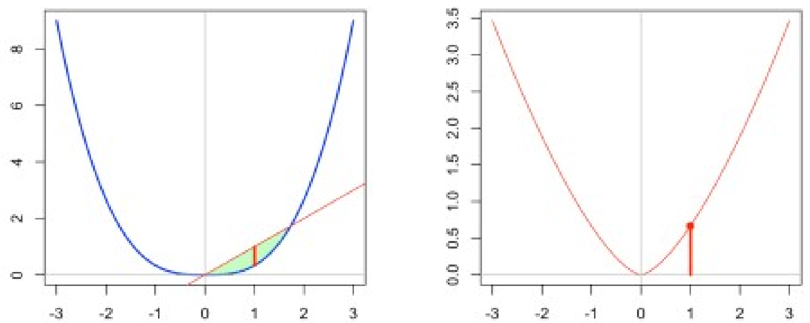
```
viz(1.5)
```



In that case, we can actually compute f^{\star}, since f^{\star}(y)=\max_{x}\lbrace xy-f(x)\rbrace=\max_{x}\lbrace xy-x^2/2\rbraceThe first order condition is here x^{\star}=y and thusf^{\star}(y)=\max_{x}\lbrace xy-x^2/2\rbrace=\lbrace x^{\star}y-(x^{\star})^2/2\rbrace=\lbrace y^2-y^2/2\rbrace=y^2/2And actually, that can be related to two results. The first one is to observe that f(\boldsymbol{x})=\|\boldsymbol{x}\|_2^2/2 and in that case f^{\star}(\boldsymbol{y})=\|\boldsymbol{y}\|_2^2/2 from the following general result : if f(\boldsymbol{x})=\|\boldsymbol{x}\|_p^p/p with p>1, where \|\cdot\|_p denotes the standard \ell_p norm, then f^{\star}(\boldsymbol{y})=\|\boldsymbol{y}\|_q^q/q where\frac{1}{p}+\frac{1}{q}=1The second one is the conjugate of a quadratic function. More specifically if f(\boldsymbol{x})=\boldsymbol{x}^{\top}\boldsymbol{Q}\boldsymbol{x}/2 for some definite positive matrix \boldsymbol{Q}, f^{\star}(\boldsymbol{y})=\boldsymbol{y}^{\top}\boldsymbol{Q}^{-1}\boldsymbol{y}/2. In our case, it was a univariate problem with \boldsymbol{Q}=1.

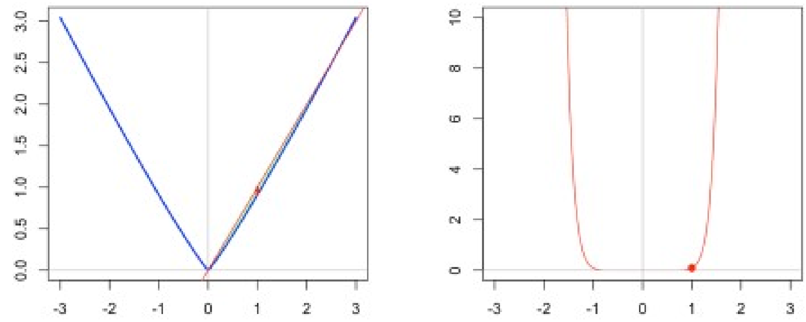
For the conjugate of the \ell_p norm, we can use the following code to visualize it

```
p = 3
f = function(x) abs(x)^p/p
vf = Vectorize(f)(x)
fstar = function(y) max(y*x-vf)
vfstar = Vectorize(fstar)(x)
viz(1.5)
```



or

```
p = 1.1
f = function(x) abs(x)^p/p
vf = Vectorize(f)(x)
fstar = function(y) max(y*x-vf)
vfstar = Vectorize(fstar)(x)
viz(1, YL=c(0,10))
```



Actually, in that case, we almost visualize that if f(x)=|x| then\displaystyle{f^{\star}\left(y\right)={\begin{cases}0,&\left|y\right|\leq 1\\\infty ,&\left|y\right|>1.\end{cases}}}

To conclude, another popular case, f(x)=\exp(x) then{\displaystyle f^{\star}\left(y\right)={\begin{cases}y\log(y)-y,&y>0\\0,&y=0\\\infty ,&y<0.\end{cases}}}[/latex]We can visualize that case below

```
f = function(x) exp(x)
vf = Vectorize(f)(x)
fstar = function(y) max(y*x-vf)
vfstar = Vectorize(fstar)(x)
viz(1,YL=c(-3,3))
```