…Throughout this article, one considers the balanced one-way ANOVA model with a random factor (`group`). The between standard deviation and the within standard deviation are denoted by σ_{b} and σ_{w} respectively. The grand mean is denoted by μ. The number of levels of the `group` factor is denoted by I and the number of individuals within each group is denoted by J. Thus the model is:

$$\mu_i \sim_{\text{iid}} \mathcal{N}(\mu, \sigma_{\mathrm{b}}^2), \, i = 1, \ldots, I \qquad (y_{i,j} \mid \mu_i) \sim_{\text{iid}} \mathcal{N}(\mu_i, \sigma_{\mathrm{w}}^2), \, j = 1, \ldots, J.$$

Using ‘rstanarm’ with the default priors

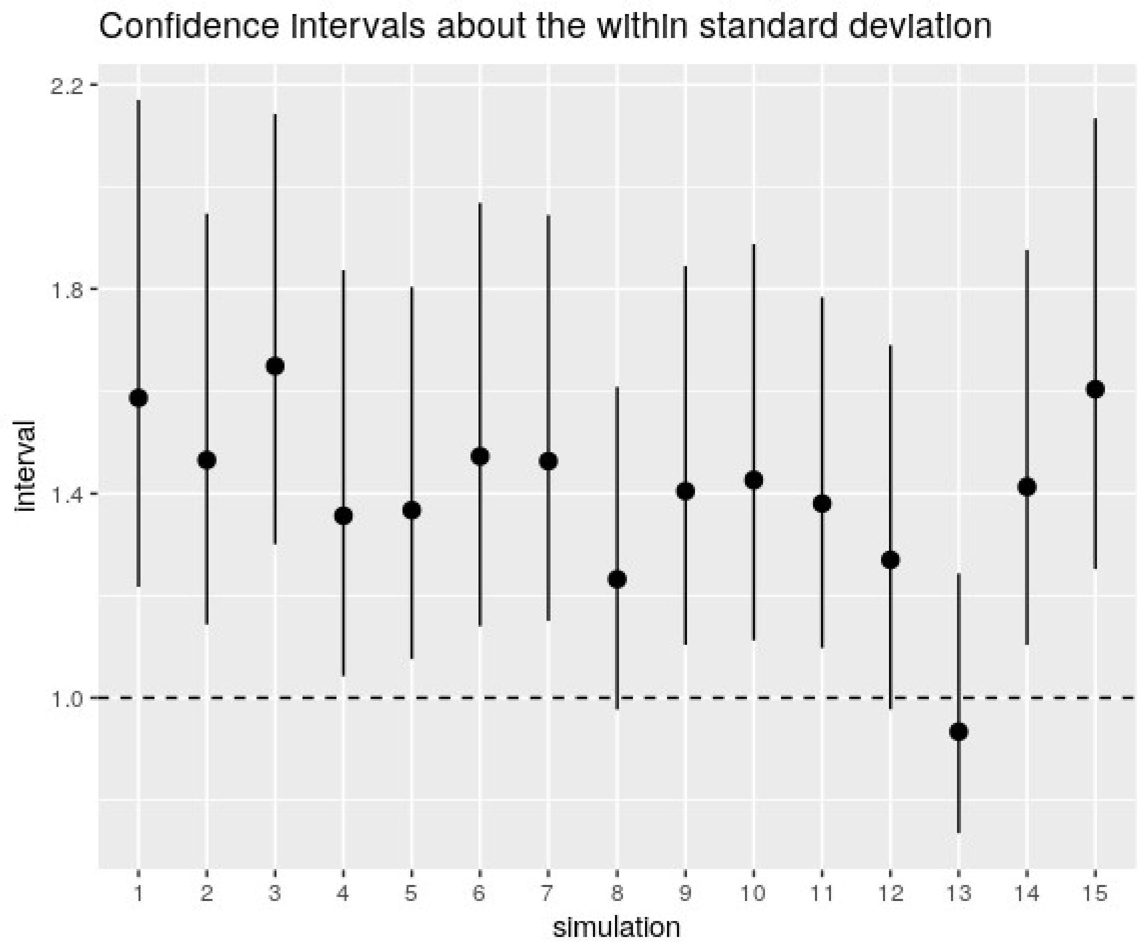
Below I fit the model with the ‘rstanarm’ package for fifteen simulated datasets with $I = 10$, $J = 5$, $\mu = 10000$, $\sigma_{\mathrm{b}} = 50$, $\sigma_{\mathrm{w}} = 1$. I assign a “vague” half-Cauchy prior distribution to σ_{w} and the other prior distributions are the default prior distributions of `stan_lmer`.

```
library(rstanarm)
options(mc.cores = parallel::detectCores())

mu           <- 10000
sigmaWithin  <- 1
ratio        <- 50
sigmaBetween <- sigmaWithin * ratio
I            <- 10L
J            <- 5L

nsims <- 15L

stanIntervals <- list( # to store the confidence intervals
  within = `colnames<-`(matrix(NA_real_, nrow = nsims, ncol = 3),
                        c("estimate", "lwr", "upr")),
  between = `colnames<-`(matrix(NA_real_, nrow = nsims, ncol = 3),
                        c("estimate", "lwr", "upr"))
)

set.seed(666L)

for(i in 1L:nsims){
```

```
  groupMeans <- rnorm(I, mu, sigmaBetween)
  y          <- c(
    vapply(groupMeans, function(gmean) rnorm(J, gmean, sigmaWithin),
numeric(J))
  )
  dat        <- data.frame(
    y = y,
    group = gl(I, J)
  )

  rstanarm <- stan_lmer(
    y ~ (1|group), data = dat, iter = 5000L,
    prior_aux = cauchy(0, 5)
  )

  pstrr <- as.data.frame( # extract posterior draws
    stan,
    pars = c(
      "(Intercept)",
      "sigma",
      "Sigma[group:(Intercept),(Intercept)]"
    )
  )
  names(pstrr)[2L:3L] <- c("sigma_error", "sigma_group")
  pstrr[["sigma_group"]] <- sqrt(pstrr[["sigma_group"]])

  x <- t(vapply(pstrr, quantile, numeric(3L), probs = c(50, 2.5,
97.5)/100))
  stanIntervals$within[i, ] <- x["sigma_error", ]
  stanIntervals$between[i, ] <- x["sigma_group", ]
}
```

Let’s plot the intervals now.

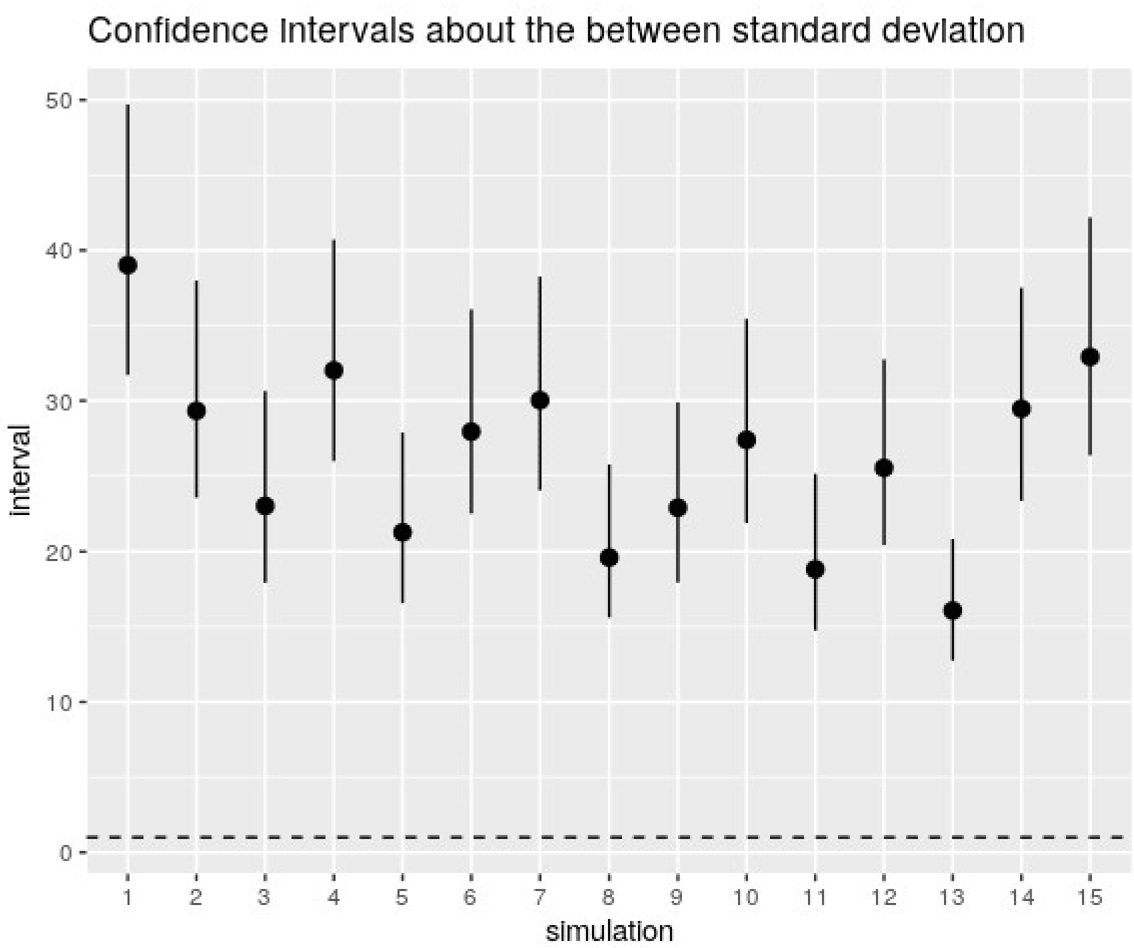
```
library(ggplot2)
stanWithin <- as.data.frame(stanIntervals[["within"]])
stanWithin[["simulation"]] <- factor(1L:nsims)
ggplot(
  stanWithin,
  aes(
    x = simulation, y = estimate, ymin = lwr, ymax = upr
  )
) +
  geom_pointrange() +
  ylab("interval") +
  geom_hline(yintercept = 1, linetype = "dashed") +
  ggtitle("Confidence intervals about the within standard deviation")
```



The horizontal line shows the value of
σ_{w}. As you can
see, the confidence intervals dramatically fail to catch this value.

And this is also the case for
σ_{b}:

```
stanBetween <- as.data.frame(stanIntervals[["between"]])
stanBetween[["simulation"]] <- factor(1L:nsims)
ggplot(
  stanBetween,
  aes(
    x = simulation, y = estimate, ymin = lwr, ymax = upr
  )
) +
  geom_pointrange() +
  ylab("interval") +
  geom_hline(yintercept = 1, linetype = "dashed") +
  ggtitle("Confidence intervals about the between standard deviation")
```



Resolving the issue

Why? The explanation is simple: `stan_lmer` assigns a unit exponential prior distribution to the between standard deviation, which is equal to \(50\).

So we have to change this prior distribution, and `stan_lmer` allows to use a Gamma distribution as the prior distribution of the between standard deviation. Its parameters `shape` and `scale` are settable in the `decov` function which is passed on to the `prior_covariance` option:

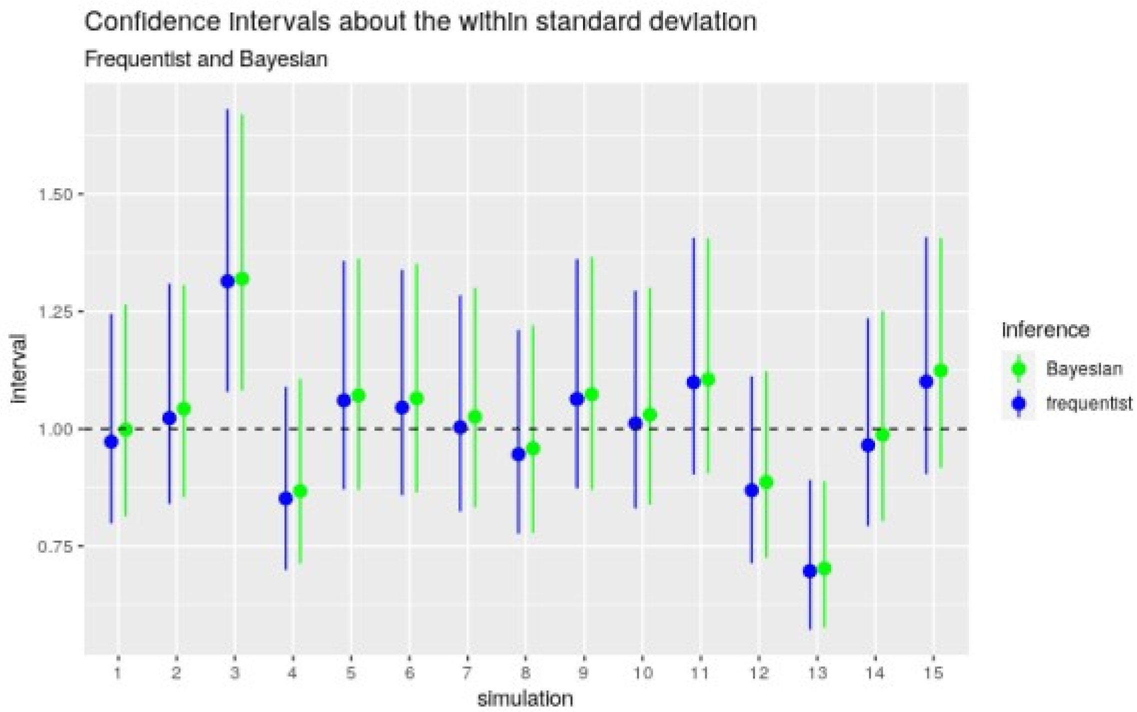
```
rstanarm <- stan_lmer(
  y ~ (1|group), data = dat, iter = 5000L,
  prior_aux = cauchy(0, 5),
  prior_covariance = decov(shape = 2, scale = 30)
)
```

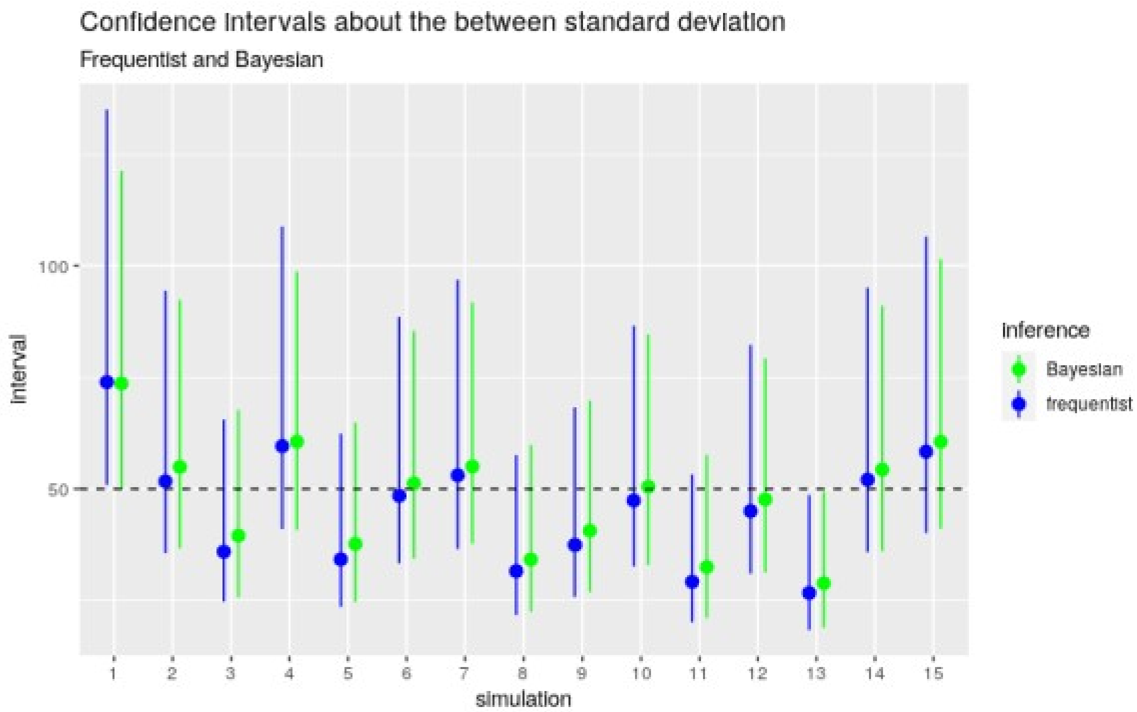
I choose the \(\textrm{Gamma}(\textrm{shape}=2, \textrm{scale=30})\) distribution because it has median \(\approx 50\) and is "vague" enough: its equi-tailed \(95\%\)-dispersion interval is \(\approx (7, 167)\).

☛

However it took me some time to pick up these parameters. I firstly tried a more dispersed Gamma distribution but `stan_lmer` returned a bunch of warnings and non-sensible results.

Below are the confidence intervals obtained with this Gamma prior distribution. I compare them with the frequentist intervals obtained with the 'AOV1R' package.





Quite good.

☛
I also noticed that the sampling was slower with this Gamma

distribution.

Try the generalized fiducial inference.

My new package ‘gfilmm’ allows to perform the *generalized fiducial inference* for any Gaussian linear mixed model with categorical random effects.

Fiducial inference and Bayesian inference have something in common: they are both based on a distribution representing the uncertainty about the parameters: the fiducial distribution and the posterior distribution, respectively.

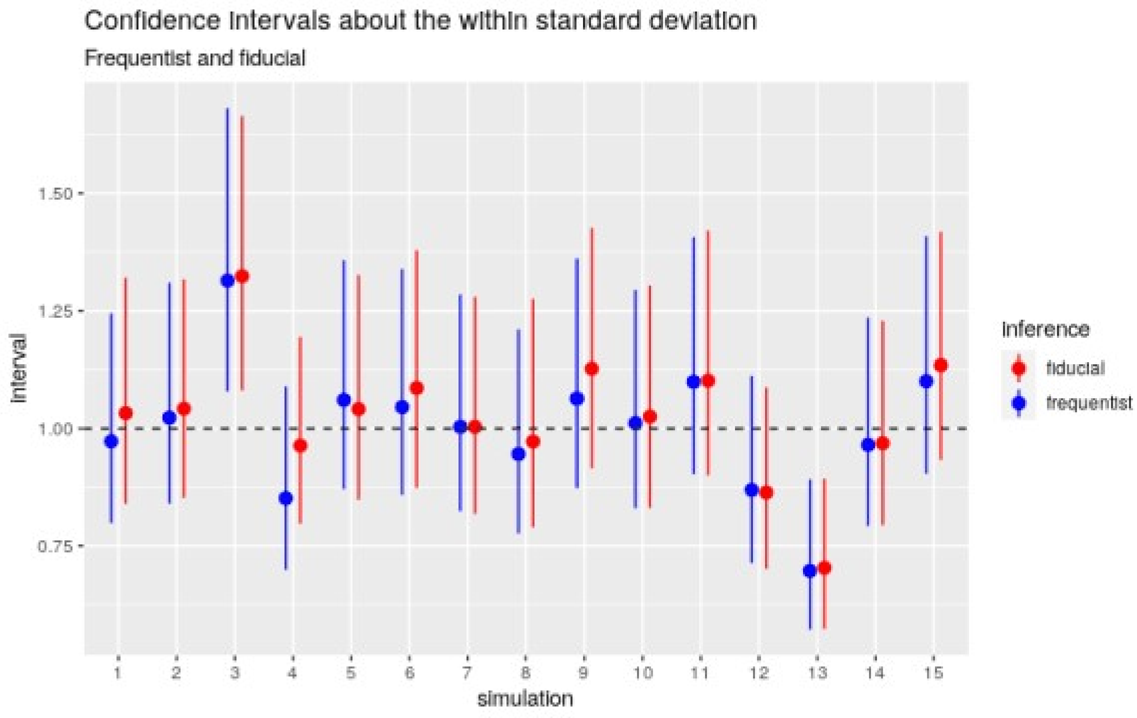
A notable difference between these two methods of inference is that *there’s no prior distribution in fiducial statistics*.

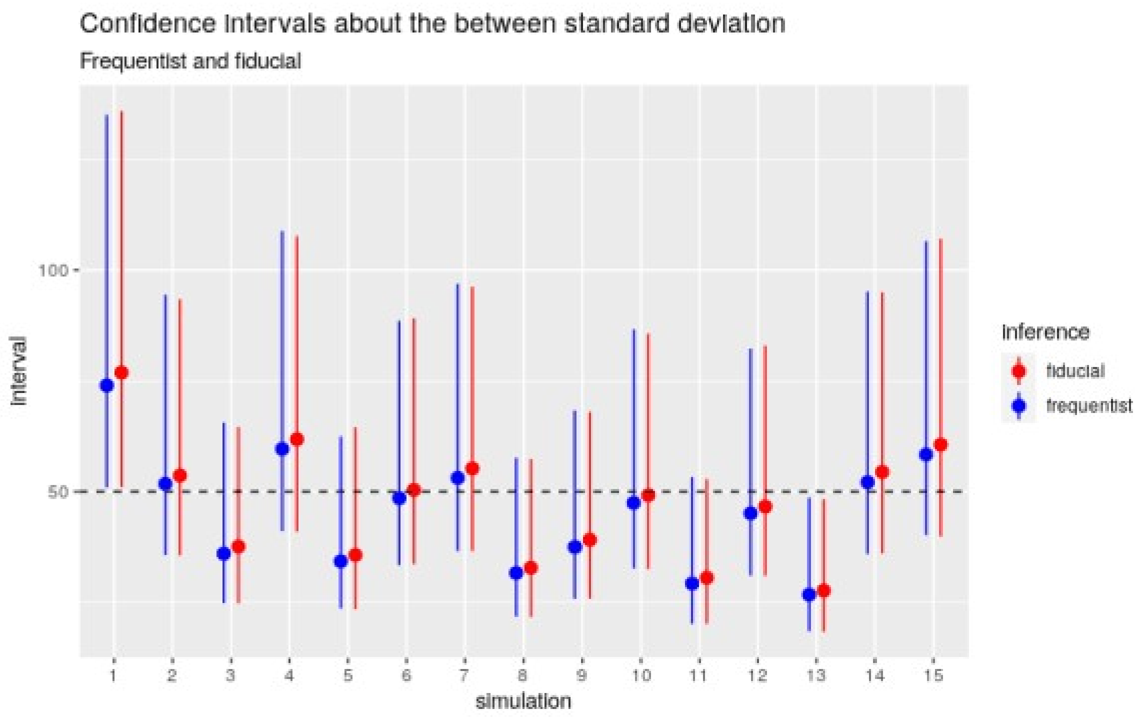
Here is how to run the fiducial sampler:

```
library(gfilmm)
fiducialSimulations <- gfilmm(
  y = ~ cbind(y - 0.01, y + 0.01), fixed = ~ 1, random = ~ group,
  data= dat, N = 10000L
)
```

Note the form of the response variable: `~ cbind(y - 0.01, y + 0.01)`. That’s because the generalized fiducial inference applies to interval data.

Below are the fiducial confidence intervals for the same simulated datasets as before.





Quite good too. And let me insist on this point:
there is no prior distribution, there is nothing to set, except the number of simulations. I implemented the algorithm (due to J. Cisewski and J. Hannig) in C++ and it takes less than 1000 lines of code.

Let’s increase the between standard deviation now.

```
ratio <- 1000
sigmaBetween <- ratio * sigmaWithin
set.seed(31415926L)
groupMeans <- rnorm(I, mu, sigmaBetween)
y          <- c(
  vapply(groupMeans, function(gmean) rnorm(J, gmean, sigmaWithin),
numeric(J))
)
dat        <- data.frame(
  y = y,
  group = gl(I, J)
)
library(AOV1R)
library(gfilmm)
aovfit <- aov1r(y ~ group, data = dat)
gf <- gfilmm(~ cbind(y-0.01, y+0.01), ~ 1, ~ group, data = dat, N =
5000L)
confint(aovfit)
##                estimate          lwr          upr
## grandMean 9783.4770335 9227.6673527 10339.286714
## within       0.9454425    0.7762205     1.209696
## between    776.9682432  534.4260219  1418.441282
## total      776.9688185  534.4268604  1418.441598
##
## attr(,"confidence level")
## [1] 0.95
```

```
## attr(,"standard deviations")
## [1] TRUE
```

```
gfiSummary(gf)
##                    mean      median         lwr          upr Pr(=0)
## (Intercept) 9786.579629 9787.161625 9247.718851 10344.743737     NA
## sigma_group  854.314055  807.650667  528.463101  1485.287694      0
## sigma_error    1.557918    1.543563    1.273913     1.931306      0
## attr(,"confidence level")
## [1] 0.95
```

The fiducial confidence interval about the within standard deviation does not match the frequentist interval, and does not catch the true value. Nothing to tinker with, except the number of simulations:

```
gf <- gfilmm(~ cbind(y-0.01, y+0.01), ~ 1, ~ group, data = dat, N =
30000L)
gfiSummary(gf)
##                     mean       median           lwr          upr
Pr(=0)
## (Intercept) 9780.0741293 9779.1304187 9230.1017080 10333.801655
NA
## sigma_group  847.3551078  805.8588214  536.0212551  1401.862597
0
## sigma_error    0.9540852    0.9455375    0.7762872     1.185648
0
## attr(,"confidence level")
## [1] 0.95
```

Now the fiducial intervals match the frequentist ones.

Epilogue

As you have seen, using the generalized fiducial inference is easy, easier than the Bayesian inference. The difficulty I mentioned regarding the Bayesian inference is not severe, but this is because the one-way ANOVA model with a random factor is the simplest Gaussian linear mixed model. Namely, it has only one between standard deviation. Things get more complicated for a mixed model with multiple random effects. With `rstanarm::stan_lmer`, one has to assign a Gamma prior distribution on the total between standard deviation, and then to specify a dispersion parameter of the between standard deviations.

Note

My package 'gfilmm' is already on CRAN (version `0.1.0`) but this version is not safe and there's a mistake in the algorithm. If you want to use this package now, install the development version:

```
remotes::install_github("stla/AOV1R", build_vignettes = TRUE) # soon on
CRAN
remotes::install_github("stla/gfilmm", build_vignettes = TRUE)
```