Infinite Series

~ by Note Nexus

Sequence & Series

Sequence

A Sequence is a succession of numbers or terms formed according to some definite rule. The nth term in a sequence is denoted by u_n .
For example , if $u_n = 2n + 1$
then $u_1 = 3, u_2 = 5, u_3 = 7...$

If a sequence tends to a limit L, then we write:

$$\lim_{n \to \infty} (u_n) = L$$

Types of Sequence:

- Convergent Sequence: If the limit of a sequence tends to a finite value then the sequence is convergent.
    - eg: $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ..., \frac{1}{n^2}$ u_n

      $\lim_{n \to \infty} (u_n) = \lim_{n \to \infty} \frac{1}{n^2} = 0$ As the limit of the sequence tends to zero, it is a convergent sequence.

- Divergent Sequence: If the limit of a sequence does not tend to a finite value then the sequence is divergent.
    - eg: 3, 5, 7, ... , (2n+1) :- This is a divergent sequence as its limit tends to infinity.

- Bounded Sequence: It is a sequence whose terms all fall within a finite range.
    - eg:

- Monotonic Sequence: A sequence which is either increasing or decreasing is called as monotonic sequence.
    - Examples:
        - 2, 4, 6, 8, 10 ..... is a monotonic sequence
        - 0.1, 0.01, 0.001, ... is a monotonic sequence
        - 1, -1, 1, -1 ..... is not a monotonic sequence

Some Important Limits

(i) $\lim_{n \to \infty} x^n = 0$ if $x < 1$ and $\lim_{n \to \infty} x^n = \infty$ if $x > 1$

(ii) $\lim_{n \to \infty} \frac{x^n}{n!} = 0$ for all values of x

(iii) $\lim_{n \to \infty} \frac{\log n}{n} = 0$

(iv) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

(v) $\lim_{n \to \infty} (n)^{1/n} = 1$

(vi) $\lim_{n \to \infty} [n!]^{1/n} = \infty$

(vii) $\lim_{n \to \infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}$

(viii) $\lim_{n \to \infty} n\, x^n = 0$ if $x < 1$

(ix) $\lim_{n \to \infty} n^h = \infty$

(x) $\lim_{n \to \infty} \frac{1}{n^h} = 0$

(xi) $\lim_{x \to \infty} \left[\frac{a^x - 1}{x}\right] = \log a$ or $\lim_{n \to \infty} \frac{a^{1/n} - 1}{1/n} = \log a$

(xii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(xiii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Series

A series is the sum of sequence.

For example: 1 + 2 + 3 + 4 + 5 + ... is a sequence

If the number of terms is limited → Finite Series

If the number of terms is infinite → Infinite Series

An infinite Series is denoted by: $\sum_{n=1}^{\infty} u_n$

The sum of the first n terms is denoted by: S_n

Types of Series

Let the infinite series be: $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n + \infty$

$S_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n$

If S_n tend to finite number when n tends to ∞, then the series $\sum_{n=1}^{\infty} u_n$ is said to be convergent.

If S_n tend to infinity when n tends to ∞, then the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

If S_n does not tend to a unique limit the series $\sum_{n=1}^{\infty} u_n$ is called Oscillatory.

Properties of Infinite Series

- The nature of the infinite series does not change:
  - by multiplication of all terms by a constant term.
  - by addition or deletion of a finite number of term.

Example: *Prove that the following series:*

$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots$ *is convergent and find its sum.*

Here,
$$u_n = \frac{n+1}{(n+2)!} = \frac{n+2-1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!}$$
$$= \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$
$$S_n = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots$$
$$+ \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right) = \frac{1}{2!} - \frac{1}{(n+2)!}$$
$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[\frac{1}{2!} - \frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$\therefore$ Σu_n converges and its limit is $\frac{1}{2}$.

Example: *Discuss the nature of the series* $2 - 2 + 2 - 2 + 2 - \ldots \infty$.

Let
$$S_n = 2 - 2 + 2 - 2 + 2 - \ldots \infty$$
$$= 0 \text{ if } n \text{ is even}$$
$$= 2 \text{ if } n \text{ is odd.}$$

Hence, S_n does not tend to a unique limit, and, therefore, the given series is oscillatory.

Test the convergence of the following series:

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$$

Here, we have

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$$
$$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow$ Σu_n does not converge.

The given series is a series of + ve terms,

Hence by Cauchy fundamental test for divergence, the series is divergent.

Positive Term Series

If the majority terms in a series are Positive, then the series is called Positive Term Series.

Necessary Conditions for a Convergent Series

For every convergent series, $\sum u_n$,

$$\lim_{n\to\infty} (u_n) = 0$$

NOTE: The converse is not true.

There are certain tests to identify convergent or divergent series.

Cauchy's Fundamental Test For Divergence

If $\lim_{n\to\infty} (u_n) \neq 0$ then the series must be divergent.

Example: Test for the convergence of the series: $1 + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots\ldots + \infty$

Solution: Its u_n can be written as:

$$u_n = \frac{n}{n+1}$$

Neglecting the first term, as neglecting it will not effect the overall series as it is infinite.

$$\lim_{n\to\infty} (u_n) = \frac{n}{n(1 + 1/n)}$$

$$= \frac{1}{(1 + 1/n)}$$

$$= 1$$

As $\lim_{n\to\infty} (u_n) \neq 0$, Hence by Cauchy's Fundamental Test, the series is divergent.

p - Series

The series : $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots \infty$ is:

- convergent if $p > 1$
- divergent if $p \leq 1$

Comparison Test

If two positive term series, $\sum u_n$ & $\sum v_n$ be such that $|u_n| \leq |v_n|$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = k(\text{finite number})$$

then $\sum u_n$ & $\sum v_n$ converge or diverge together.

Example: Test for the convergence for: $\sum \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^2+2n+7}}$

Here,

$$u_n = \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^2+2n+7}}$$

$$u_n = \frac{n^{\frac{2}{3}}\sqrt[3]{3+\frac{1}{n^2}}}{n^{\frac{3}{4}}\sqrt[4]{4+\frac{2}{n}+\frac{7}{n^2}}}$$

$$u_n = \frac{\sqrt[3]{3+\frac{1}{n^2}}}{n^{\frac{1}{12}}\sqrt[4]{4+\frac{2}{n}+\frac{7}{n^2}}}$$

Let $\sum v_n = \sum \frac{1}{n^{\frac{1}{12}}}$, which is divergent by p series test ($p = 1/12 < 1$)

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim \left(\frac{\sqrt[3]{3+\frac{1}{n^2}}}{n^{\frac{1}{12}}\sqrt[4]{4+\frac{2}{n}+\frac{7}{n^2}}} \times n^{\frac{1}{12}} \right)$$

$$= \frac{3^{\frac{1}{3}}}{2^{\frac{1}{2}}} \quad \text{which is a finite number}$$

Hence the given series is divergent.

D'Alembert's Ratio Test

If $\sum u_n$ is a positive term series, such that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$ = k(finite number)

then

- the series is convergent if k < 1
- the series is divergent if k > 1
- and the test fails at k = 1

Example: Test for the convergence for: $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\ldots.\infty$

Here,

$$u_n=\frac{1.2.3.4.\ldots\ldots n}{3.5.7.9.\ldots\ldots(2n+1)}$$

$$u_{n+1}=\frac{1.2.3.4.\ldots\ldots n\,(n+1)}{3.5.7.9.\ldots\ldots(2n+1)\,(2n+3)}$$

Using D'Alembert's Ratio Test:

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)=\lim_{n\to\infty}\left(\left\{\frac{1.2.3\ldots n}{3.5.7.\ldots.(2n+1)\,(2n+3)}\right\}\left\{\frac{3.5.7\ldots(2n+1)}{1.2.3.\ldots.n}\right\}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{2n+3}\right)$$

$$=\frac{1}{2}$$

Since $\lim\limits_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)=\frac{1}{2}$, which is less than 1, therefore the given series is convergent.

Raabe's Test

If $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = k$

then

- the series is convergent if k > 1
- the series is divergent if k < 1
- and the test fails at k = 1

Example: Test for the convergence for: $\frac{1^2}{4^2} + \frac{5^2}{8^2} + \frac{9^2}{12^2} + \frac{13^2}{16^2}$

Here,

$$u_n = \frac{(4n-3)^2}{(4n)^2}$$

$$u_{n+1} = \frac{(4n+1)^2}{(4n+4)^2}$$

Using D'Alembert's Ratio Test:

$$\lim\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \frac{(4n+1)^2(4n)^2}{(4n+4)^2(4n-3)^2}$$

$$= 1$$

Since $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1$, therefore the Ratio Test fails.

Using Raabe's Test: $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = k$

$$= \lim_{n\to\infty} n\left(\frac{(4n+4)^2(4n-3)^2 - (4n)^2(4n+1)^2}{(4n)^2(4n+1)^2}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{-384n^2 - 96n + 144}{(4n)^2(4n+1)^2}\right)$$

$$= 0$$

Since $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = 0$, which is less than 1, therefore the given series is divergent.

Cauchy's Root Test

If $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty} (u_n)^{1/n} = k$

then

- the series is convergent if $k < 1$
- the series is divergent if $k > 1$
- and the test fails at $k = 1$

Example: Test for the convergence for: $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

Here,

$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n}\right)^{n}}\right)$$

$$= \frac{1}{e}$$

Since $\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{e}$, which is less than 1, therefore the given series is convergent.

Logarithmic Test

If $\sum u_n$ is a positive term series, such that $\lim\limits_{n\to\infty} n\left[\log \dfrac{u_n}{u_{n+1}}\right] = k$

then

- the series is convergent if k > 1
- the series is divergent if k < 1
- and the test fails at k = 1

Example 9.12. *Discuss the convergence of the series*

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \dots \infty$$

(P.T.U., 2008 ; Cochin, 2005 ; Rohtak, 2003)

Solution. Here $\dfrac{u_n}{u_{n+1}} = \dfrac{n^n x^n}{n!} \div \dfrac{(n+1)^{n+1}x^{n+1}}{(n+1)!} = \dfrac{n^n}{(n+1)^n x} = \dfrac{1}{(1+1/n)^n}\cdot\dfrac{1}{x}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{ex}.$$

Thus by Ratio test, the series converges for $x < 1/e$ and diverges for $x > 1/e$. But it fails for $x = 1/e$. Let us try the *log-test.*

Now $\dfrac{u_n}{u_{n+1}} = \dfrac{e}{(1+1/n)^n}$

$$\therefore \quad \log\frac{u_n}{u_{n+1}} = \log_e e - n\log\left(1+\frac{1}{n}\right) = 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right) = \frac{1}{2n} - \frac{1}{3n^2} + \dots$$

$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left(n\log\dfrac{u_n}{u_{n+1}}\right) = \dfrac{1}{2}$, which < 1. Thus by the *log-test*, the series diverges.

Hence the given series converges for $x < 1/e$ and diverges for $x \geq 1/e$.

Procedure to find the condition for convergence of a series

- Write u_n for a given series.
- Apply D'Alembert's Ratio Test on the given Series. Solve.
- If you get k = 1 in D'Alembert's Ratio Test, then use Raabe's Test to solve further.
- You will get your solution here definitely. If not then use the comparison test at last to solve.

Well-known Series

Power Series

A power series is an infinite series expressed in powers of (x−a). It's essentially a flexible framework to represent functions as infinite polynomials.

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

- a → center of the series
- c_n→ coefficients
- Converges only within a certain interval → radius & interval of convergence

Taylor Series

The Taylor series of a function f(x) about a point x=ax=ax=a represents the function as an infinite sum of its derivatives at a.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

General Formula:

$$f(x) = \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- If a=0, it's called the Maclaurin series
- Provides a local approximation of functions

Series for standard functions

- Exponential function $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ Valid for x = (-∞,∞)

- Sine Function $sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ Valid for x = (-∞,∞)

- Cosine Function $cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ Valid for x = (-∞,∞)

- Logarithmic Function $log\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ Valid for x = (-1, 1]