Homework 3

1. Suppose we have i.i.d. $X_1, ..., X_n \sim N(\mu, \sigma^2)$. From the class, we know that $\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$. Use this fact, together with the mean and variance calculation from the last homework, and answer the following questions.

   (a) Consider an estimator $\hat{\sigma}^2_c = c \sum_{i=1}^n (X_i - \bar{X})^2$. For which c is $\hat{\sigma}^2_c$ the MLE?

   (b) Find the mean and variance of $\hat{\sigma}^2_c$. For which c is $\hat{\sigma}^2_c$ unbiased?

   (c) Find the MSE $\mathbb{E}[\hat{\sigma}^2_c - \sigma^2]^2$. For which c is the MSE minimized?

2. Suppose we have i.i.d. $Z_1, Z_2 \sim N(0, 1)$. In the class, we learned that $\frac{Z_1}{|Z_2|}$ is distributed by Cauchy. What is the distribution of $\frac{Z_1}{Z_2}$?

3. Consider random (column) vectors $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^q$. The covariance between the two vectors is defined by $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)^T] \in \mathbb{R}^{p \times q}$. For matrices $A \in \mathbb{R}^{d \times p}$ and $B \in \mathbb{R}^{k \times q}$, show $\mathrm{Cov}(AX, BY) = A\mathrm{Cov}(X, Y)B^T$.

4. Consider $Z \sim N(0, 1)$ and we know that Z^2 is distributed by χ^2_1.

   (a) Show $\mathbb{P}(Z^2 \le t) = 2\mathbb{P}(Z \le \sqrt{t}) - 1$.

   (b) The density of $N(0, 1)$ is given by $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$. Calculate the density of χ^2_1 by $\frac{d}{dt}\mathbb{P}(Z^2 \le t)$. Check Wikipedia to see if your answer is correct.

5. Consider i.i.d. random variables $Z_1, ..., Z_n \sim N(0, 1)$. According to the class, it can be compactly written as an n-dimensional column vector $Z \sim N(0, I_n)$. Let $\mathbb{1}_n$ be an n-dimensional column vector with all entries 1. Then, $\mathbb{1}_n \mathbb{1}_n^T$ is an $n \times n$ matrix of all entries 1. Define $P = \frac{1}{n}\mathbb{1}_n \mathbb{1}_n^T$.

   (a) Show $P^2 = P$, $(I_n - P)^2 = (I_n - P)$, and $(I_n - P)P = 0$.

   (b) Show $\sum_{i=1}^n (Z_i - \bar{Z})^2 = \|(I_n - P)Z\|^2$ and $\bar{Z} = \frac{1}{n}\mathbb{1}_n^T PZ$.

   (c) Show $\|(I_n - P)Z\|^2$ and $\frac{1}{n}\mathbb{1}_n^T PZ$ are independent by computing the covariance $\mathrm{Cov}((I_n - P)Z, PZ)$. This is a matrix proof of Conclusion 3 of the theorem we learned in the class.