STAT 32950 Assignment 1

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Question 1

(a)

```
Command:
round(colMeans(ladyrun[,-1]), 2)
 100m     200m     400m     800m    1500m    3000m Marathon
 11.31    23.07    51.82     2.02     4.19     9.07   153.31
```

In this dataset the variables representing times (”100m”, ”200m”, ”400m”,”800m”, ”1500m”, ”3000m”, ”Marathon”) are numeric, so their means are meaningful, but the “Country” column is a categorical identifier and its mean is not meaningful.

(b)

Since column “Country” is non-numeric and not meaningful, we need to drop it when calculating the covariance and correlation matrix: The R commands are:

```
round(cov(ladyrun[,-1]),2)
round(cor(ladyrun[,-1]),2)
```

(c)

Since column “Country” is non-numeric and not meaningful, we need to drop it:

```
round(cor(ladyrun[,-1], method = "kendall"),2)
```

(d)

Since column “Country” is non-numeric and not meaningful, we need to drop it:

```
round(cor(ladyrun[,-1], method = "spearman"),2)
```

(e)

Since column "Country" is non-numeric and not meaningful, we need to drop it:

```
round(cor(log(ladyrun[,-1]), method = "pearson"),2)
round(cor(log(ladyrun[,-1]), method = "kendall"),2)
round(cor(log(ladyrun[,-1]), method = "spearman"),2)
```

Discussion: Rank correlations (Kendall's τ and Spearman's ρ) are invariant under any monotonic transformation since they are calculated using the relative ranks. Therefore, the Kendall and Spearman correlation matrices computed on the log-transformed data will be identical to those computed on the original data.

In contrast, the Pearson correlation measures linear association and is not invariant under nonlinear transformations (like the logarithm). Hence, the Pearson correlation matrix on log-transformed data might differ from that on the original data.

(f)

```
Command for eigenvalues:
round(eigen(cor(ladyrun[,-1]))$values, 2)
[1] 5.70 0.74 0.29 0.11 0.09 0.05 0.02

Command for eigenvectors:
eigen(cor(ladyrun[,-1]))$vectors
  [,1]       [,2]        [,3]        [,4]        [,5]       [,6]        [,7]
[1,] -0.3720342 -0.4575195 -0.14870245  0.52629124 -0.15450205  0.5677425  0.08348107
[2,] -0.3738784 -0.4801563 -0.07423786  0.11131548 -0.09164471 -0.7495258 -0.20389904
[3,] -0.3747904 -0.3314811  0.48724807 -0.50849863  0.45647911  0.1996520  0.07373480
[4,] -0.3949123  0.2210770  0.14789147 -0.37710528 -0.76947015  0.1212119 -0.15592393
[5,] -0.3956582  0.2305757 -0.42485979 -0.13992068  0.08162078 -0.1431547  0.75036695
[6,] -0.3834289  0.3180749 -0.47659266 -0.07501674  0.37659087  0.1401873 -0.59797109
[7,] -0.3490255  0.4970255  0.55267291  0.53351836  0.13707747 -0.1455350  0.03296996
```

(i) Sum of All Eigenvalues

Let R be the $n \times n$ sample correlation matrix. A property of any square matrix is that the sum of its eigenvalues is equal to the trace of the matrix. That is,

$$\mathrm{tr}(R) = \sum_{i=1}^{n} \lambda_i.$$

Since here our R is 7×7 a correlation matrix, all its diagonal entries are 1, so

$$\mathrm{tr}(R) = 7.$$

Thus, the sum of all eigenvalues is 7. The computed eigenvalues from R are:

$$5.70,\ 0.74,\ 0.29,\ 0.11,\ 0.09,\ 0.05,\ 0.02,$$

which add up to 7. This confirms the property that the sum of the eigenvalues equals the number of variables (the matrix dimension).

(ii) Dimension of Each Eigenvector

An eigenvector v corresponding to an eigenvalue λ of a matrix R satisfies the equation

$$Rv = \lambda v.$$

Since R is a 7×7 matrix, the vector v must be a 7-dimensional column vector (i.e., $v \in R^7$) in order for the matrix multiplication Rv to be defined. Thus, each eigenvector of R is a vector in R^7.

Question 2

(a)

We are given three eggs with the following markings:

- **Egg 1**: Blue number $B = 1$, Red number $R = 2$.
- **Egg 2**: Blue number $B = 3$, Red number $R = 4$.
- **Egg 3**: Blue number $B = 5$, Red number $R = 0$.

Each egg is chosen with probability $\frac{1}{3}$. The joint probability table for (R, B) is:

	$B = 1$	$B = 3$	$B = 5$	Row Total
$R = 2$	$\frac{1}{3}$	0	0	$\frac{1}{3}$
$R = 4$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
$R = 0$	0	0	$\frac{1}{3}$	$\frac{1}{3}$
Column Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

(b) Rule-I

Choosing Red:

- **Egg 1**: $R = 2$ vs. $B = 1 \Rightarrow$ win.
- **Egg 2**: $R = 4$ vs. $B = 3 \Rightarrow$ win.
- **Egg 3**: $R = 0$ vs. $B = 5 \Rightarrow$ lose.

Thus, the winning probability is

$$P(\text{win}|R) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

$$P(\text{win}|B) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Conclusion for Rule-I: Choose **red** with a winning probability of $\frac{2}{3}$.

(c) Rule-II

The marginal distributions are:

- Blue numbers: $1, 3, 5$ (each with probability $\frac{1}{3}$).
- Red numbers: $2, 4, 0$ (each with probability $\frac{1}{3}$).

Choosing Blue: Your number is B and the opponent's number is R. Compute $P(B > R)$:

- For $B = 1$: wins only if $R = 0 \Rightarrow 1$ win out of 3.
- For $B = 3$: wins if $R = 2$ or $R = 0 \Rightarrow 2$ wins out of 3.
- For $B = 5$: wins against all $R = 2, 4, 0 \Rightarrow 3$ wins out of 3.

The total number of outcomes is $3 \times 3 = 9$, and total wins are $1 + 2 + 3 = 6$. Therefore,

$$P(\text{win}|B) = \frac{6}{9} = \frac{2}{3}.$$

$$P(\text{win}|R) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Conclusion for Rule-II: Choose **blue** with a winning probability of $\frac{2}{3}$.

(d) Rule-III

Setup: As in Rule-II, you pick an egg and read your number, but this time the egg is *not* replaced. Your opponent then chooses from the remaining two eggs. We use conditional probabilities.

Choosing Blue (you get B, opponent gets R from one of the remaining eggs)

1. **If you pick Egg 1** ($B = 1$): Remaining eggs: Egg 2 ($R = 4$) and Egg 3 ($R = 0$).
   - If opponent picks Egg 2: $1 < 4$ (lose).
   - If opponent picks Egg 3: $1 > 0$ (win).

   $P(\text{win} \mid \text{Egg } 1) = \frac{1}{2}$.
2. **If you pick Egg 2** ($B = 3$): Remaining eggs: Egg 1 ($R = 2$) and Egg 3 ($R = 0$).
   - Both outcomes: $3 > 2$ and $3 > 0$ (win).

   $P(\text{win} \mid \text{Egg } 2) = 1$.
3. **If you pick Egg 3** ($B = 5$): Remaining eggs: Egg 1 ($R = 2$) and Egg 2 ($R = 4$).
   - Both outcomes: $5 > 2$ and $5 > 4$ (win).

   $P(\text{win} \mid \text{Egg } 3) = 1$.

Averaging over the three equally likely eggs:

$$P(\text{win}|B) = \sum P(\text{win})P(B=b) = \frac{1}{3}\left(\frac{1}{2}+1+1\right) = \frac{1}{3}\times\frac{5}{2} = \frac{5}{6}.$$

$$P(\text{win}|R) = \frac{1}{6}.$$

Conclusion for Rule-III: Choose **blue** since it gives a winning probability of $\frac{5}{6}$.

Question 3

(a)

Since the total probability must equal 1, summing all the nonzero entries gives

$$2c + 3c + 4c = 9c = 1 \quad \Longrightarrow \quad c = \frac{1}{9}.$$

Marginal PMF of X:

$$\begin{aligned}
f_X(1) &= P(X=1) = 2c = \frac{2}{9},\\
f_X(2) &= P(X=2) = 3c = \frac{3}{9} = \frac{1}{3},\\
f_X(3) &= P(X=3) = 4c = \frac{4}{9}.
\end{aligned}$$

Marginal PMF of Y:

$$\begin{aligned}
f_Y(1) &= P(Y=1) = 3c = \frac{1}{3},\\
f_Y(2) &= P(Y=2) = 3c = \frac{1}{3},\\
f_Y(3) &= P(Y=3) = 2c = \frac{2}{9},\\
f_Y(4) &= P(Y=4) = c = \frac{1}{9}.
\end{aligned}$$

(b)

The conditional probability is

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{f_X(x)}.$$

For $X=1$: Only $Y=1,2$ have nonzero probability:

$$P(Y=1 \mid X=1) = \frac{c}{2c} = \frac{1}{2}, \quad P(Y=2 \mid X=1) = \frac{c}{2c} = \frac{1}{2}.$$

Thus,

$$E(Y \mid X=1) = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = \frac{3}{2}.$$

For $X = 2$: Here $Y = 1, 2, 3$ are possible:

$$P(Y = y \mid X = 2) = \frac{c}{3c} = \frac{1}{3}, \quad y = 1, 2, 3.$$

Thus,

$$E(Y \mid X = 2) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = \frac{6}{3} = 2.$$

For $X = 3$: Here all $Y = 1, 2, 3, 4$ are possible:

$$P(Y = y \mid X = 3) = \frac{c}{4c} = \frac{1}{4}, \quad y = 1, 2, 3, 4.$$

Thus,

$$E(Y \mid X = 3) = \frac{1 + 2 + 3 + 4}{4} = \frac{10}{4} = \frac{5}{2}.$$

So, the conditional expectation function is:

$$g(1) = \frac{3}{2}, \quad g(2) = 2, \quad g(3) = \frac{5}{2}.$$

(c)

The conditional variance is given by

$$\mathrm{Var}(Y \mid X = x) = E(Y^2 \mid X = x) - \Big[E(Y \mid X = x)\Big]^2.$$

For $X = 1$:

$$E(Y^2 \mid X = 1) = 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = \frac{1 + 4}{2} = \frac{5}{2},$$
$$\mathrm{Var}(Y \mid X = 1) = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{5}{2} - \frac{9}{4} = \frac{10 - 9}{4} = \frac{1}{4}.$$

For $X = 2$:

$$E(Y^2 \mid X = 2) = 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{3} = \frac{1 + 4 + 9}{3} = \frac{14}{3},$$
$$\mathrm{Var}(Y \mid X = 2) = \frac{14}{3} - 2^2 = \frac{14}{3} - 4 = \frac{14 - 12}{3} = \frac{2}{3}.$$

For $X = 3$:

$$E(Y^2 \mid X = 3) = \frac{1^2 + 2^2 + 3^2 + 4^2}{4} = \frac{1 + 4 + 9 + 16}{4} = \frac{30}{4} = \frac{15}{2},$$
$$\mathrm{Var}(Y \mid X = 3) = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{15}{2} - \frac{25}{4} = \frac{30 - 25}{4} = \frac{5}{4}.$$

(d)

We have

$$E\big[E(Y \mid X)\big] = \sum_x E(Y \mid X = x)\, f_X(x).$$

Using the values of $E(Y \mid X)$ and the marginal PMF of X:

$$\begin{aligned} E\big[E(Y \mid X)\big] &= \frac{3}{2} \cdot \frac{2}{9} + 2 \cdot \frac{3}{9} + \frac{5}{2} \cdot \frac{4}{9} \\ &= \frac{3}{9} + \frac{6}{9} + \frac{10}{9} \\ &= \frac{19}{9}. \end{aligned}$$

Alternatively, using the marginal PMF of Y:

$$E(Y) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{1}{9} = \frac{19}{9}.$$

Thus, $E\big[E(Y \mid X)\big] = E(Y)$, which verifies the law of total expectation.

(e)

Let

$$E\big[E(Y \mid X)\big] = E[g(X)]$$

We have

$$g(1) = \frac{3}{2}, \quad g(2) = 2, \quad g(3) = \frac{5}{2}, \quad \text{and} \quad E[g(X)] = \frac{19}{9}.$$

$$\begin{aligned} E[g(X)^2] &= \left(\frac{3}{2}\right)^2 \frac{2}{9} + (2)^2 \frac{3}{9} + \left(\frac{5}{2}\right)^2 \frac{4}{9}. \\ &= \frac{1}{2} + \frac{4}{3} + \frac{25}{9}. \\ &= \frac{83}{18}. \end{aligned}$$

Thus,

$$\mathrm{Var}\left(E(Y \mid X)\right) = E[g(X)^2] - \Big[E(g(X))\Big]^2 = \frac{83}{18} - \left(\frac{19}{9}\right)^2.$$

$$\mathrm{Var}\left(E(Y \mid X)\right) = \frac{747 - 722}{162} = \frac{25}{162}.$$

Using the conditional variances from part (c):

$$\mathrm{Var}(Y \mid X = 1) = \frac{1}{4}, \quad \mathrm{Var}(Y \mid X = 2) = \frac{2}{3}, \quad \mathrm{Var}(Y \mid X = 3) = \frac{5}{4},$$

and the marginal PMF of X, we have:

$$E\big[\,\mathrm{Var}(Y \mid X)\big] = \frac{1}{4} \cdot \frac{2}{9} + \frac{2}{3} \cdot \frac{3}{9} + \frac{5}{4} \cdot \frac{4}{9}.$$

$$E\big[\,\mathrm{Var}(Y \mid X)\big] = \frac{1}{18} + \frac{2}{9} + \frac{5}{9} = \frac{1}{18} + \frac{4}{18} + \frac{10}{18} = \frac{15}{18} = \frac{5}{6}.$$

The law of total variance gives

$$\mathrm{Var}(Y) = \mathrm{Var}\left(E(Y \mid X)\right) + E\left[\mathrm{Var}(Y \mid X)\right].$$

Substitute the values:

$$\mathrm{Var}(Y) = \frac{25}{162} + \frac{5}{6}.$$

$$\mathrm{Var}(Y) = \frac{25 + 135}{162} = \frac{160}{162} = \frac{80}{81}.$$

This result can be verified by computing $E(Y)$ and $E(Y^2)$ from the marginal PMF of Y:

$$E(Y) = \frac{19}{9}, \quad E(Y^2) = \frac{49}{9}, \quad \text{and thus} \quad \mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{49}{9} - \frac{361}{81} = \frac{80}{81}.$$

Question 4

(a)

The total integral of the joint density over the positive quadrant equals 1:

$$\iint_{(x>0,y>0)} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \;=\; \int_0^\infty \int_0^\infty c\left(1+x+y\right)^{-3}\mathrm{d}x\,\mathrm{d}y \;=\; 1.$$

First compute the inner integral (fix $y > 0$ and integrate over x):

$$\int_0^\infty \left(1+x+y\right)^{-3}\mathrm{d}x.$$

Use the substitution $z = 1 + y + x$, so $\mathrm{d}z = \mathrm{d}x$, and when $x = 0$, $z = 1 + y$; as $x \to \infty$, $z \to \infty$. Hence

$$\int_{x=0}^\infty (1+x+y)^{-3}\,\mathrm{d}x \;=\; \int_{z=1+y}^\infty z^{-3}\,\mathrm{d}z \;=\; \left[-\tfrac{1}{2z^2}\right]_{z=1+y}^{z=\infty} \;=\; \frac{1}{2\,(1+y)^2}.$$

Thus the double integral becomes

$$\int_{y=0}^\infty \left[c\,\tfrac{1}{2\,(1+y)^2}\right]\mathrm{d}y \;=\; \frac{c}{2}\int_0^\infty \frac{1}{(1+y)^2}\,\mathrm{d}y.$$

$$\int_0^\infty \frac{1}{(1+y)^2}\,\mathrm{d}y = \left[-\frac{1}{1+y}\right]_0^\infty = 1.$$

Hence

$$\frac{c}{2}\times 1 \;=\; 1$$
$$c \;=\; 2.$$

(b)

With $c = 2$, the joint density is

$$f_{X,Y}(x,y) \;=\; 2\left(1+x+y\right)^{-3}, \quad x>0,\; y>0.$$

The marginal density of X is

$$f_X(x) \;=\; \int_0^\infty f_{X,Y}(x,y)\,\mathrm{d}y \;=\; \int_0^\infty 2\,(1+x+y)^{-3}\,\mathrm{d}y, \quad x>0.$$

Let $u = 1+x+y$. Then $\mathrm{d}u = \mathrm{d}y$, and as y goes from 0 to ∞, u goes from $(1+x)$ to ∞. So

$$\int_0^\infty (1+x+y)^{-3}\,\mathrm{d}y \;=\; \int_{u=1+x}^\infty u^{-3}\,\mathrm{d}u \;=\; \left[-\tfrac{1}{2u^2}\right]_{1+x}^\infty \;=\; \frac{1}{2(1+x)^2}.$$

Hence

$$f_X(x) \;=\; 2\times\frac{1}{2(1+x)^2} \;=\; \frac{1}{(1+x)^2}, \quad x>0.$$

(c)

For $x>0$, the conditional density is

$$f_{Y|X}(y\mid x) \;=\; \frac{f_{X,Y}(x,y)}{f_X(x)} \;=\; \frac{2\,(1+x+y)^{-3}}{\left[(1+x)^{-2}\right]} \;=\; 2\,(1+x+y)^{-3}\,(1+x)^2.$$

Therefore,

$$f_{Y|X}(y\mid x) \;=\; \frac{2(1+x)^2}{(1+x+y)^3}, \quad y>0.$$

(d)

$$E[Y] \;=\; \int_0^\infty\int_0^\infty y\left[2\,(1+x+y)^{-3}\right]\mathrm{d}x\,\mathrm{d}y.$$

Integrate over x:

$$\int_0^\infty (1+x+y)^{-3}\,\mathrm{d}x \;=\; \frac{1}{2(1+y)^2} \quad \text{(as before)}.$$

Thus

$$E[Y] \;=\; \int_0^\infty 2\,y\,\frac{1}{2(1+y)^2}\,\mathrm{d}y \;=\; \int_0^\infty \frac{y}{(1+y)^2}\,\mathrm{d}y.$$

We have

$$\begin{aligned}\int_0^\infty \frac{y}{(1+y)^2}\,\mathrm{d}y \;&=\; \int_0^\infty \left(\frac{1+y-1}{(1+y)^2}\right)\mathrm{d}y\\ &=\; \int_0^\infty \left(\frac{1+y}{(1+y)^2} - \frac{1}{(1+y)^2}\right)\mathrm{d}y\\ &=\; \int_0^\infty \frac{1}{1+y}\,\mathrm{d}y \;-\; \int_0^\infty \frac{1}{(1+y)^2}\,\mathrm{d}y.\end{aligned}$$

The second integral converges and the first integral $\int_0^\infty \frac{1}{1+y}\,\mathrm{d}y$ diverges to $+\infty$. Hence

$$E[Y] \;=\; +\infty.$$

The distribution is heavy-tailed in such a way that Y does not have a finite mean.

(e)

$$g(x) \;=\; E[Y \mid X = x] \;=\; \int_0^\infty y\, \frac{2(1+x)^2}{(1+x+y)^3}\, \mathrm{d}y, \quad x > 0.$$

Let $z = 1 + x + y$. Then $y = z - (1+x)$ and $\mathrm{d}y = \mathrm{d}z$. When $y = 0$, we have $z = 1 + x$. When $y \to \infty$, $z \to \infty$.

$$y\, \frac{2(1+x)^2}{(1+x+y)^3} \;=\; 2(1+x)^2\, \frac{z-(1+x)}{z^3} \;=\; 2(1+x)^2 \left(\frac{z}{z^3} - \frac{1+x}{z^3} \right) \;=\; 2(1+x)^2 \left(\frac{1}{z^2} - \frac{1+x}{z^3} \right).$$

$$g(x) \;=\; 2(1+x)^2 \int_{z=1+x}^{\infty} \left(\frac{1}{z^2} - \frac{1+x}{z^3} \right) \mathrm{d}z.$$

For each integral:

$$\int_{1+x}^{\infty} \frac{1}{z^2}\, \mathrm{d}z \;=\; \left[-\frac{1}{z} \right]_{1+x}^{\infty} \;=\; \frac{1}{1+x},$$

$$\int_{1+x}^{\infty} \frac{1}{z^3}\, \mathrm{d}z \;=\; \left[-\frac{1}{2z^2} \right]_{1+x}^{\infty} \;=\; \frac{1}{2(1+x)^2}.$$

Hence

$$g(x) \;=\; 2(1+x)^2 \left[\frac{1}{1+x} - (1+x)\, \frac{1}{2(1+x)^2} \right].$$

$$g(x) \;=\; 2(1+x) - (1+x) \;=\; 1+x$$

Thus

$$E[Y \mid X = x] \;= 1 + x, \quad x > 0.$$

Question 5

(a)

Let

$$C = AXB,$$

where

- X is a $p \times p$ random matrix,
- A is a non-random matrix of dimensions $k \times p$,
- B is a non-random matrix of dimensions $p \times r$.

(i) Dimensions of C

Since

$$A \text{ is } k \times p, \quad X \text{ is } p \times p, \quad \text{and} \quad B \text{ is } p \times r,$$

the product is defined as:

$$AX \text{ is } k \times p \quad \text{and} \quad (AX)B \text{ is } k \times r.$$

Thus, the matrix C has dimensions $k \times r$.

(ii) Expression for the (i, j)th Entry c_{ij} of C

Let the entries of A, X, and B be denoted by:

$$A = (a_{is})_{1 \le i \le k,\, 1 \le s \le p}, \quad X = (x_{st})_{1 \le s,t \le p}, \quad B = (b_{tj})_{1 \le t \le p,\, 1 \le j \le r}.$$

Then the (i, j)th entry of C is given by

$$c_{ij} = \big[AXB\big]_{ij} = \sum_{s=1}^{p} \left[a_{is} \left(\sum_{t=1}^{p} x_{st}\, b_{tj} \right) \right] = \sum_{s=1}^{p} \sum_{t=1}^{p} a_{is}\, x_{st}\, b_{tj}.$$

(iii) Showing that $E(C) = A\, E(X)\, B$

Since A and B are non-random matrices, by the linearity of expectation, we have for the (i, j)th entry:

$$\begin{aligned} [E(C)]_{ij} = E(c_{ij}) &= E\left(\sum_{s=1}^{p} \sum_{t=1}^{p} a_{is}\, x_{st}\, b_{tj} \right) \\ &= \sum_{s=1}^{p} \sum_{t=1}^{p} a_{is}\, E(x_{st})\, b_{tj}. \end{aligned}$$

since

$$\big[A\, E(X)\, B\big]_{ij} = \sum_{s=1}^{p} \sum_{t=1}^{p} a_{is}\, [E(X)]_{st}\, b_{tj}.$$

$\sum_{s=1}^{p} \sum_{t=1}^{p} a_{is}\, E(x_{st})\, b_{tj}$ is exactly the (i, j)th entry of the matrix product

$$A\, E(X)\, B,$$

Therefore,

$$E(C) = A\, E(X)\, B.$$

(b)

Let $Y = [Y_1, \ldots, Y_p]^T$ be a random vector in R^p with covariance matrix

$$\Sigma = \mathrm{Cov}(Y) = E\big[(Y - \mu)(Y - \mu)^T\big],$$

where $\mu = E(Y)$.

For any $a \in R^p$,

$$a^T \Sigma a = a^T E\big[(Y - \mu)(Y - \mu)^T\big] a.$$

Since expectation is linear, we can exchange the order:

$$a^T \Sigma a = E\Big[a^T (Y - \mu)(Y - \mu)^T a\Big].$$

Notice that $a^T (Y - \mu)$ is a scalar; hence,

$$a^T \Sigma a = E\Big[\big(a^T (Y - \mu)\big)^2\Big].$$

Since squares are always nonnegative,

$$\big(a^T (Y - \mu)\big)^2 \ge 0 \quad \text{almost surely,}$$

and therefore,

$$a^T \Sigma a = E\Big[\big(a^T (Y - \mu)\big)^2\Big] \ge 0.$$

Because this holds for all $a \in R^p$, the matrix Σ is positive semidefinite. Since Σ is symmetric and positive semidefinite, all its eigenvalues are real and nonnegative.

(c)

(i) Eigenvalues

Given:

$$A = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \rho \in (0, 1).$$

To find the eigenvalues of A, solve $\det(A - \lambda I) = 0$. Thus,

$$\det \begin{pmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) - \rho^2 = (1-\lambda)^2 - \rho^2 = \big[(1-\lambda) - \rho\big]\big[(1-\lambda) + \rho\big].$$

Hence the roots (eigenvalues) are

$$\lambda_1 = 1 + \rho, \quad \lambda_2 = 1 - \rho.$$

Since $\rho \in (0, 1)$, both λ_1 and λ_2 are positive.

(ii) Eigenvectors

For the eigenvalue $\lambda_1 = 1 + \rho$, we solve

$$\big(A - (1+\rho)I\big)v = 0.$$

$$A = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$A - (1+\rho)I = \begin{pmatrix} 1-(1+\rho) & \rho \\ \rho & 1-(1+\rho) \end{pmatrix} = \begin{pmatrix} -\rho & \rho \\ \rho & -\rho \end{pmatrix}.$$

Let

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$\begin{pmatrix} -\rho & \rho \\ \rho & -\rho \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\rho\, v_1 + \rho\, v_2 \\ \rho\, v_1 - \rho\, v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the equations:

$$-\rho\, v_1 + \rho\, v_2 = 0, \quad \rho\, v_1 - \rho\, v_2 = 0.$$

$$-v_1 + v_2 = 0$$

$$v_2 = v_1.$$

Thus, any vector of the form

$$v = \begin{pmatrix} t \\ t \end{pmatrix}, \quad t \neq 0,$$

is an eigenvector corresponding to λ_1. Choosing $t = 1$ gives

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To obtain a unit-length eigenvector, we normalize:

$$\|v\| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \text{so} \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For the eigenvalue $\lambda_2 = 1 - \rho$, we solve

$$\big(A - (1-\rho)I\big)v = 0.$$

Now,

$$A - (1-\rho)I = \begin{pmatrix} 1-(1-\rho) & \rho \\ \rho & 1-(1-\rho) \end{pmatrix} = \begin{pmatrix} \rho & \rho \\ \rho & \rho \end{pmatrix}.$$

Let

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \rho & \rho \\ \rho & \rho \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \rho v_1 + \rho v_2 \\ \rho v_1 + \rho v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This leads to

$$v_2 = -v_1.$$

Choosing $v_1 = 1$ gives the eigenvector

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Its norm is

$$\|v\| = \sqrt{1^2 + (-1)^2} = \sqrt{2},$$

so the normalized eigenvector is

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now check orthonormality:

$$v_1^T v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}(1-1) = 0.$$

Thus, v_1 and v_2 are orthonormal.

(iii) Spectral Decomposition $A = V\,\Lambda\,V^T$

Form the orthonormal matrix

$$V = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix}.$$

One can check that V is orthonormal, i.e. $V^T V = I$. Hence

$$A = V\,\Lambda\,V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(iv) Inverse of A

Since $\lambda_1 = 1+\rho$ and $\lambda_2 = 1-\rho$ are both nonzero (because $\rho \in (0,1)$), Λ is invertible with

$$\Lambda^{-1} = \begin{pmatrix} \frac{1}{1+\rho} & 0 \\ 0 & \frac{1}{1-\rho} \end{pmatrix}.$$

Thus

$$A^{-1} = V\,\Lambda^{-1}\,V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+\rho} & 0 \\ 0 & \frac{1}{1-\rho} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(v) Square Root of A

We want R such that $R^2 = A$. In spectral form, define

$$R = V\,\Lambda^{1/2}\,V^T, \quad \text{with} \quad \Lambda^{1/2} = \begin{pmatrix} \sqrt{1+\rho} & 0 \\ 0 & \sqrt{1-\rho} \end{pmatrix}.$$

Thus,

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{1+\rho} & 0 \\ 0 & \sqrt{1-\rho} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We could verify that:

$$R^2 = \left(V\,\Lambda^{1/2}\,V^T\right)\left(V\,\Lambda^{1/2}\,V^T\right) = V\,\Lambda^{1/2}\left(V^T V\right)\Lambda^{1/2}\,V^T = V\,\Lambda\,V^T = A.$$

(d)

(i) $\tau = 2\,P\big[(X_1 - X_2)(Y_1 - Y_2) > 0\big] - 1$

By continuity,

$$P\big((X_1 - X_2)(Y_1 - Y_2) = 0\big) = 0,$$

By definition of probability:

$$P\big((X_1 - X_2)(Y_1 - Y_2) > 0\big) \; + \; P\big((X_1 - X_2)(Y_1 - Y_2) < 0\big) \; = \; 1.$$

Let

$$p \; = \; P\big((X_1 - X_2)(Y_1 - Y_2) > 0\big).$$

Then

$$P\big((X_1 - X_2)(Y_1 - Y_2) < 0\big) = 1 - p,$$

$$\tau = p - (1-p) = 2p - 1 = 2\,P\big[(X_1 - X_2)(Y_1 - Y_2) > 0\big] - 1$$

(ii) $\tau = 4 \iint_{R^2} F(x,y)\,dF(x,y) \; - \; 1$

The event that the two pairs are in concordance, i.e.,

$$\{(X_1 - X_2)(Y_1 - Y_2) > 0\},$$

occurs in two cases:

1. Case 1 (Positive Concordance): $X_1 < X_2$ and $Y_1 < Y_2$.
2. Case 2 (Negative Concordance): $X_1 > X_2$ and $Y_1 > Y_2$.

By the symmetry of the distribution (since (X_1, Y_1) and (X_2, Y_2) are i.i.d.), the probabilities of these two cases are equal. Thus,

$$P\big[(X_1 - X_2)(Y_1 - Y_2) > 0\big] = P\big[X_1 < X_2,\, Y_1 < Y_2\big] + P\big[X_1 > X_2,\, Y_1 > Y_2\big] = 2\,P\big[X_1 < X_2,\, Y_1 < Y_2\big].$$

By (i), Kendall's τ is given by

$$\tau = 2\Big[2\,P\big(X_1 < X_2,\, Y_1 < Y_2\big)\Big] - 1 = 4\,P\big(X_1 < X_2,\, Y_1 < Y_2\big) - 1.$$

First, condition on the event $\{(X_2, Y_2) = (x, y)\}$. By the law of total probability,

$$P\big[X_1 < X_2,\ Y_1 < Y_2\big] = \iint_{R^2} P\big[X_1 < x,\ Y_1 < y\big]\ f_{X_2,Y_2}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Here,

- $f_{X_2,Y_2}(x, y)$ is the joint probability density function (PDF) of (X_2, Y_2).
- $P[X_1 < x,\ Y_1 < y]$ is the conditional probability that (X_1, Y_1) lies below (x, y), given that $(X_2, Y_2) = (x, y)$. Since the two pairs are independent, this conditional probability is simply the unconditional probability $P[X_1 < x,\ Y_1 < y]$.

Let

$$F_{X_1,Y_1}(x, y) = P\big[X_1 < x,\ Y_1 < y\big]$$

be the cumulative distribution function (CDF) of (X_1, Y_1). Because (X_1, Y_1) and (X_2, Y_2) have the same distribution,

$$f_{X_2,Y_2}(x, y) = f(x, y), \quad F_{X_1,Y_1}(x, y) = F(x, y),$$

where $f(x, y)$ and $F(x, y)$ are the common PDF and CDF, respectively. Substituting these into the integral, we obtain

$$P\big[X_1 < X_2,\ Y_1 < Y_2\big] = \iint_{R^2} F(x, y)\ f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since

$$\mathrm{d}F(x, y) \ = \ f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

$$\iint_{R^2} F(x, y)\ f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R^2} F(x, y)\,\mathrm{d}F(x, y).$$

Thus,

$$P\big[X_1 < X_2,\ Y_1 < Y_2\big] = \iint_{R^2} F(x, y)\,\mathrm{d}F(x, y).$$

$$\tau = 4\,P\big[X_1 < X_2,\ Y_1 < Y_2\big] - 1.$$

Thus,

$$\tau = 4\iint_{R^2} F(x, y)\,\mathrm{d}F(x, y) \ - \ 1.$$