24300 HW6

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Question 1: LU decomposition

We have the matrix A:

$$A = \begin{bmatrix} 3 & 3 & 2 \\ -6 & -2 & -1 \\ 6 & 18 & 14 \end{bmatrix}$$

(1)

Initialize L and U:

$$L = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = A = \begin{bmatrix} 3 & 3 & 2 \\ -6 & -2 & -1 \\ 6 & 18 & 14 \end{bmatrix}$$

For $i = 1$, For $j = 2$:

$$l_{21} = \frac{u_{21}}{u_{11}} = \frac{-6}{3} = -2$$

$$U_2 = U_2 - l_{21}U_1 = U_2 - (-2)U_1 = U_2 + 2U_1$$

$$\begin{aligned} U_2 &= \begin{bmatrix} -6 & -2 & -1 \end{bmatrix} + 2\begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -6+6 & -2+6 & -1+4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 3 \end{bmatrix} \end{aligned}$$

For $j = 3$:

$$l_{31} = \frac{u_{31}}{u_{11}} = \frac{6}{3} = 2$$

$$U_3 = U_3 - l_{31}U_1 = U_3 - 2U_1$$

$$
\begin{aligned}
U_3 &= \begin{bmatrix} 6 & 18 & 14 \end{bmatrix} - 2\begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 6-6 & 18-6 & 14-4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 12 & 10 \end{bmatrix}
\end{aligned}
$$

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 12 & 10 \end{bmatrix}
$$

For $i = 2$, $j = 3$:

$$
l_{32} = \frac{u_{32}}{u_{22}} = \frac{12}{4} = 3
$$

$$
U_3 = U_3 - l_{32}U_2 = U_3 - 3U_2
$$

$$
\begin{aligned}
U_3 &= \begin{bmatrix} 0 & 12 & 10 \end{bmatrix} - 3\begin{bmatrix} 0 & 4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 0-0 & 12-12 & 10-9 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Therefore,

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}
$$

Since $\det(A) = \det(L) \times \det(U)$ and $\det(L) = 1$, because L is a lower triangular matrix with ones on the diagonal:

$$
\det(U) = u_{11} \times u_{22} \times u_{33} = 3 \times 4 \times 1 = 12
$$

Therefore,

$$
\det(A) = \det(L) \times \det(U) = 1 \times 12 = 12
$$

$$
\det(A) = 12
$$

Question 2: Solving for Eigenvalues and Eigenvectors

Consider the matrices:

$$
A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
$$

(1)

For matrix A:

$$\det(A) = (1)(4) - (-1)(2) = 4 + 2 = 6$$

For matrix B:

Since B is an upper triangular matrix, the determinant is the product of its diagonal entries:

$$\det(B) = (3)(1)(0) = 0$$

(2)

For matrix A:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial is:

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I) &= (1-\lambda)(4-\lambda) - (-1)(2) \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

For matrix B:

$$B - \lambda I = \begin{bmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix}$$

The characteristic polynomial is:

$$p_B(\lambda) = \det(B - \lambda I) = (3-\lambda)(1-\lambda)(-\lambda)$$

(3)

For matrix A:

$$p_A(\lambda) = \lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$

Eigenvalues are:

$$\lambda = 2, \quad \lambda = 3$$

For matrix B:

$$p_B(\lambda) = -\lambda(3-\lambda)(1-\lambda) = 0$$

Eigenvalues are:

$$\lambda = 0, \quad \lambda = 1, \quad \lambda = 3$$

(4)

For matrix A:

$$\det(A) = 6, \quad \text{Product of eigenvalues} = 2 \times 3 = 6$$

For matrix B:

$$\det(B) = 0, \quad \text{Product of eigenvalues} = 0 \times 1 \times 3 = 0$$

(5)

Matrix A

For $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 1-2 & -1 \\ 2 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

Set up the equation $(A - 2I)\mathbf{v} = \mathbf{0}$:

$$\begin{cases} -1x - 1y = 0 \\ 2x + 2y = 0 \end{cases}$$

$$y = -x$$

Eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|\mathbf{v}_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Normalized eigenvector:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 3$*:*

$$A - 3I = \begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

Set up the equation $(A - 3I)\mathbf{v} = \mathbf{0}$:

$$\begin{cases} -2x - 1y = 0 \\ 2x + 1y = 0 \end{cases}$$

$$y = -2x$$

Eigenvector:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\|\mathbf{v}_2\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Normalized eigenvector:

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Matrix B

For $\lambda = 0$*:*

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Set up the equations:

$$\begin{cases} 3x + 4y + 2z = 0 \\ y + 2z = 0 \\ 0 = 0 \end{cases}$$

From the second equation:

$$y = -2z$$

Substitute into the first equation:

$$x = 2z$$

Eigenvector:

$$\mathbf{v}_3 = \begin{bmatrix} 2z \\ -2z \\ z \end{bmatrix} = z \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Choose $z = 1$:

$$\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\|\mathbf{v}_3\| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

Normalized eigenvector:

$$\mathbf{u}_3 = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda = 1$:

$$B - I = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Set up the equations:

$$\begin{cases} 2x + 4y + 2z = 0 \\ 2z = 0 \\ -z = 0 \end{cases}$$

From the last equation:

$$z = 0$$

From the first equation:

$$x = -2y$$

Eigenvector:

$$\mathbf{v}_4 = \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix}$$

Choose $y = 1$:

$$\mathbf{v}_4 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\|\mathbf{v}_4\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

Normalized eigenvector:

$$\mathbf{u}_4 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$:

$$B - 3I = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Set up the equations:

$$\begin{cases} 4y + 2z = 0 \\ -2y + 2z = 0 \\ -3z = 0 \end{cases}$$

From the last equation:

$$z = 0$$

From the first equation:

$$4y = 0 \implies y = 0$$

The variable x is free.

Eigenvector:

$$\mathbf{v}_5 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

Choose $x = 1$:

$$\mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Normalized eigenvector:

$$\mathbf{u}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Summary:

- Determinants:
$$\det(A) = 6, \quad \det(B) = 0$$
- Eigenvalues of A:
$$\lambda_1 = 2, \quad \lambda_2 = 3$$
- Eigenvectors of A:
$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
- Eigenvalues of B:
$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3$$
- Eigenvectors of B:
$$\mathbf{u}_3 = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Question 3: Matrix Powers and Matrix Functions

(1)

Since $A = V\Lambda V^{-1}$, we have:

$$\begin{aligned} A^2 = A \cdot A = (V\Lambda V^{-1})(V\Lambda V^{-1}) &= V\Lambda V^{-1} V\Lambda V^{-1} \\ &= V\Lambda(V^{-1}V)\Lambda V^{-1} = V\Lambda I \Lambda V^{-1} = V\Lambda^2 V^{-1}. \end{aligned}$$

For A^3:

$$\begin{aligned} A^3 = A^2 \cdot A = (V\Lambda^2 V^{-1})(V\Lambda V^{-1}) &= V\Lambda^2 V^{-1} V\Lambda V^{-1} \\ &= V\Lambda^2(V^{-1}V)\Lambda V^{-1} = V\Lambda^2 I \Lambda V^{-1} = V\Lambda^3 V^{-1}. \end{aligned}$$

By induction, assume $A^k = V\Lambda^k V^{-1}$ holds for some $k \geq 1$. Then:

$$\begin{aligned} A^{k+1} = A^k \cdot A = (V\Lambda^k V^{-1})(V\Lambda V^{-1}) &= V\Lambda^k V^{-1} V\Lambda V^{-1} \\ &= V\Lambda^k(V^{-1}V)\Lambda V^{-1} = V\Lambda^k I \Lambda V^{-1} = V\Lambda^{k+1} V^{-1}. \end{aligned}$$

(2)

If $|\lambda_j| > 1$, then as k diverges:

$$|\lambda_j^k| = |\lambda_j|^k \to \infty.$$

If $|\lambda_j| < 1$, then as k diverges:

$$|\lambda_j^k| = |\lambda_j|^k \to 0.$$

Therefore, when $|\lambda_j| > 1$, the magnitude $|\lambda_j^k|$ grows without bound as k increases. When $|\lambda_j| < 1$, the magnitude $|\lambda_j^k|$ decreases towards zero as k diverges.

(3)

$$A = \frac{1}{3}\begin{bmatrix} 1 & 8 \\ 4 & 5 \end{bmatrix}.$$

$$A = \begin{bmatrix} \frac{1}{3} & \frac{8}{3} \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix}.$$

$$\begin{aligned} \det(A - \lambda I) &= \left(\frac{1}{3} - \lambda\right)\left(\frac{5}{3} - \lambda\right) - \left(\frac{8}{3} \cdot \frac{4}{3}\right) \\ &= \left(\frac{1}{3} - \lambda\right)\left(\frac{5}{3} - \lambda\right) - \frac{32}{9} \\ \det(A - \lambda I) &= \left(\frac{5}{9} - 2\lambda + \lambda^2\right) - \frac{32}{9} = \lambda^2 - 2\lambda - 3. \end{aligned}$$

Solve for eigenvalues:

$$\lambda^2 - 2\lambda - 3 = 0 \implies (\lambda - 3)(\lambda + 1) = 0.$$

Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.

For $\lambda = 3$:

$$A - 3I = \begin{bmatrix} \frac{1}{3} - 3 & \frac{8}{3} \\ \frac{4}{3} & \frac{5}{3} - 3 \end{bmatrix} = \begin{bmatrix} -\frac{8}{3} & \frac{8}{3} \\ \frac{4}{3} & -\frac{4}{3} \end{bmatrix}.$$

The equation $(A - 3I)\mathbf{v} = \mathbf{0}$ yields:

$$\begin{cases} -\frac{8}{3}x + \frac{8}{3}y = 0, \\ \frac{4}{3}x - \frac{4}{3}y = 0. \end{cases}$$

$$-x + y = 0 \implies y = x.$$

Therefore, an eigenvector corresponding to $\lambda = 3$ is:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -1$:

$$A - (-1)I = A + I = \begin{bmatrix} \frac{1}{3} + 1 & \frac{8}{3} \\ \frac{4}{3} & \frac{5}{3} + 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{8}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix}.$$

The equation $(A + I)\mathbf{v} = \mathbf{0}$ yields:

$$\frac{4}{3}x + \frac{8}{3}y = 0 \implies x = -2y.$$

Therefore, an eigenvector corresponding to $\lambda = -1$ is:

$$\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Thus,

$$V = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\det(V) = (1)(1) - (-2)(1) = 1 + 2 = 3,$$

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Thus, $A = V\Lambda V^{-1}$, where,

$$V = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad V^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

(4)

Since $A = V\Lambda V^{-1}$, we have:

$$e^A = V e^{\Lambda} V^{-1},$$

where:

$$e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} = \begin{bmatrix} e^3 & 0 \\ 0 & e^{-1} \end{bmatrix}.$$

Therefore:

$$e^A = Ve^\Lambda V^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^3 & 0 \\ 0 & e^{-1} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

$$Ve^\Lambda = \begin{bmatrix} 1 \cdot e^3 + (-2) \cdot 0 & 1 \cdot 0 + (-2) \cdot e^{-1} \\ 1 \cdot e^3 + 1 \cdot 0 & 1 \cdot 0 + 1 \cdot e^{-1} \end{bmatrix} = \begin{bmatrix} e^3 & -2e^{-1} \\ e^3 & e^{-1} \end{bmatrix}.$$

$$e^A = \frac{1}{3} \begin{bmatrix} e^3 & -2e^{-1} \\ e^3 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^3(1) + (-2e^{-1})(-1) & e^3(2) + (-2e^{-1})(1) \\ e^3(1) + e^{-1}(-1) & e^3(2) + e^{-1}(1) \end{bmatrix}.$$

$$e^A = \frac{1}{3} \begin{bmatrix} e^3 + 2e^{-1} & 2e^3 - 2e^{-1} \\ e^3 - e^{-1} & 2e^3 + e^{-1} \end{bmatrix}.$$

Justification of the Computation of e^A

The exponential of a matrix A is defined via the power series expansion:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Since the series is convergent, we can use the properties of matrix functions:

$$e^A = Ve^\Lambda V^{-1},$$

where e^Λ is the diagonal matrix:

$$e^\Lambda = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}.$$

(5)

For any scalar x:

$$x^0 = 1.$$

Using the property that $A^k = V\Lambda^k V^{-1}$ for any integer k, we have:

$$A^0 = V\Lambda^0 V^{-1}.$$

and,

$$\Lambda^0 = \mathrm{diag}(\lambda_1^0, \lambda_2^0, \ldots, \lambda_n^0).$$

For each eigenvalue λ_j, we have $\lambda_j^0 = 1$, then:

$$\Lambda^0 = \mathrm{diag}(1, 1, \ldots, 1) = I,$$

$$A^0 = VIV^{-1} = VV^{-1} = I,$$

since VV^{-1} is the identity matrix.

Question 4: Computing the Singular Value Decomposition (SVD) using Eigenvalues

(1)

$$A = \begin{bmatrix} \frac{9}{5}\frac{1}{\sqrt{2}} & 1 & 1 & \frac{12}{5}\frac{1}{\sqrt{2}} \\ \frac{9}{5}\frac{1}{\sqrt{2}} & -1 & -1 & \frac{12}{5}\frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$A^T = \begin{bmatrix} \frac{9}{5}\frac{1}{\sqrt{2}} & \frac{9}{5}\frac{1}{\sqrt{2}} \\ 1 & -1 \\ 1 & -1 \\ \frac{12}{5}\frac{1}{\sqrt{2}} & \frac{12}{5}\frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$\begin{aligned}
(AA^T)_{11} &= A_{11}A_{11} + A_{12}A_{12} + A_{13}A_{13} + A_{14}A_{14} \\
&= \left(\frac{9}{5}\cdot\frac{1}{\sqrt{2}}\right)^2 + (1)^2 + (1)^2 + \left(\frac{12}{5}\cdot\frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{81}{50} + 1 + 1 + \frac{72}{25} \\
&= \frac{325}{50} \\
&= \frac{13}{2}.
\end{aligned}$$

$$\begin{aligned}
(AA^T)_{12} &= A_{11}A_{21} + A_{12}A_{22} + A_{13}A_{23} + A_{14}A_{24} \\
&= \left(\frac{9}{5}\cdot\frac{1}{\sqrt{2}}\right)\left(\frac{9}{5}\cdot\frac{1}{\sqrt{2}}\right) + (1)(-1) + (1)(-1) + \left(\frac{12}{5}\cdot\frac{1}{\sqrt{2}}\right)\left(\frac{12}{5}\cdot\frac{1}{\sqrt{2}}\right) \\
&= \left(\frac{81}{25}\cdot\frac{1}{2}\right) - 1 - 1 + \left(\frac{144}{25}\cdot\frac{1}{2}\right) \\
&= \frac{125}{50} \\
&= \frac{5}{2}.
\end{aligned}$$

Since AA^T is symmetric, $(AA^T)_{21} = (AA^T)_{12} = \frac{5}{2}$.

$$
\begin{aligned}
(AA^T)_{22} &= A_{21}A_{21} + A_{22}A_{22} + A_{23}A_{23} + A_{24}A_{24} \\
&= \left(\frac{9}{5} \cdot \frac{1}{\sqrt{2}}\right)^2 + (-1)^2 + (-1)^2 + \left(\frac{12}{5} \cdot \frac{1}{\sqrt{2}}\right)^2 \\
&= \left(\frac{81}{25} \cdot \frac{1}{2}\right) + 1 + 1 + \left(\frac{144}{25} \cdot \frac{1}{2}\right) \\
&= \frac{325}{50} \\
&= \frac{13}{2}.
\end{aligned}
$$

Therefore, the matrix AA^T is:

$$
AA^T = \begin{bmatrix} \frac{13}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} \end{bmatrix}.
$$

(2)

The characteristic equation of AA^T is:

$$
\det(AA^T - \lambda I) = 0.
$$

$$
\det \begin{bmatrix} \frac{13}{2} - \lambda & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} - \lambda \end{bmatrix} = \left(\frac{13}{2} - \lambda\right)^2 - \left(\frac{5}{2}\right)^2 = 0.
$$

$$
\begin{gathered}
\left(\frac{13}{2} - \lambda\right)^2 - \left(\frac{5}{2}\right)^2 = 0. \\
\frac{169}{4} - 13\lambda + \lambda^2 - \frac{25}{4} = 0. \\
\lambda^2 - 13\lambda + 36 = 0.
\end{gathered}
$$

Solve for λ:

$$
\lambda = \frac{13 \pm \sqrt{13^2 - 4 \times 1 \times 36}}{2} = \frac{13 \pm \sqrt{169 - 144}}{2} = \frac{13 \pm 5}{2}.
$$

Therefore, the eigenvalues are:

$$
\lambda_1 = \frac{13 + 5}{2} = 9, \quad \lambda_2 = \frac{13 - 5}{2} = 4.
$$

The singular values are the positive square roots of the eigenvalues:

$$
\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{4} = 2.
$$

(3)

For $\lambda = 9$:

$(AA^T - 9I)\mathbf{u}_1 = \mathbf{0}$:

$$\begin{bmatrix} \frac{13}{2} - 9 & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} - 9 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & -\frac{5}{2} \end{bmatrix}.$$

$$\begin{cases} -\frac{5}{2}u_{11} + \frac{5}{2}u_{12} = 0, \\ \frac{5}{2}u_{11} - \frac{5}{2}u_{12} = 0. \end{cases}$$

Simplify:

$$u_{12} = u_{11}.$$

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\|\mathbf{u}_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Normalized eigenvector:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 4$:

$(AA^T - 4I)\mathbf{u}_2 = \mathbf{0}$:

$$\begin{bmatrix} \frac{13}{2} - 4 & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} - 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix}.$$

This leads to:

$$\begin{cases} \frac{5}{2}u_{21} + \frac{5}{2}u_{22} = 0, \\ \frac{5}{2}u_{21} + \frac{5}{2}u_{22} = 0. \end{cases}$$

Simplify:

$$u_{22} = -u_{21}.$$

$$u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u}_2\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Normalized $\mathbf{u}_2$:

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The matrix U is:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Verify that $U^T U = I$:

$$U^T U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(4)

$$V = \begin{bmatrix} \frac{3}{5} & 0 & 0 & \frac{4}{5} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{4}{5} & 0 & 0 & -\frac{3}{5} \end{bmatrix}.$$

Verify Normalization:

Compute the norm of each column $\mathbf{v}_j$:

1. Column 1:

$$\|\mathbf{v}_1\| = \sqrt{\left(\frac{3}{5}\right)^2 + 0^2 + 0^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1.$$

2. Column 2:

$$\|\mathbf{v}_2\| = \sqrt{0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1.$$

3. Column 3:

$$\|\mathbf{v}_3\| = \sqrt{0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + 0^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1.$$

4. Column 4:

$$\|\mathbf{v}_4\| = \sqrt{\left(\frac{4}{5}\right)^2 + 0^2 + 0^2 + \left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1.$$

Verify Orthogonality:

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 &= v_{11}v_{21} + v_{12}v_{22} + v_{13}v_{23} + v_{14}v_{24} \\
&= \left(\frac{3}{5} \times 0\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + \left(\frac{4}{5} \times 0\right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_3 &= v_{11}v_{31} + v_{12}v_{32} + v_{13}v_{33} + v_{14}v_{34} \\
&= \left(\frac{3}{5} \times 0\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + \left(0 \times -\frac{1}{\sqrt{2}}\right) + \left(\frac{4}{5} \times 0\right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_4 &= v_{11}v_{41} + v_{12}v_{42} + v_{13}v_{43} + v_{14}v_{44} \\
&= \left(\frac{3}{5} \times \frac{4}{5}\right) + (0 \times 0) + (0 \times 0) + \left(\frac{4}{5} \times -\frac{3}{5}\right) \\
&= \frac{12}{25} + 0 + 0 - \frac{12}{25} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_2 \cdot \mathbf{v}_3 &= v_{21}v_{31} + v_{22}v_{32} + v_{23}v_{33} + v_{24}v_{34} \\
&= (0 \times 0) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}}\right) + (0 \times 0) \\
&= 0 + \frac{1}{2} - \frac{1}{2} + 0 \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_2 \cdot \mathbf{v}_4 &= v_{21}v_{41} + v_{22}v_{42} + v_{23}v_{43} + v_{24}v_{44} \\
&= \left(0 \times \frac{4}{5}\right) + \left(\frac{1}{\sqrt{2}} \times 0\right) + \left(\frac{1}{\sqrt{2}} \times 0\right) + \left(0 \times -\frac{3}{5}\right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_3 \cdot \mathbf{v}_4 &= v_{31}v_{41} + v_{32}v_{42} + v_{33}v_{43} + v_{34}v_{44} \\
&= \left(0 \times \frac{4}{5}\right) + \left(\frac{1}{\sqrt{2}} \times 0\right) + \left(-\frac{1}{\sqrt{2}} \times 0\right) + \left(0 \times -\frac{3}{5}\right) \\
&= 0.
\end{aligned}$$

Thus, all columns of V are orthogonal and normalized.

(5)

The diagonal matrix Σ is:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

$$U\Sigma = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 & 0 \\ \frac{3}{\sqrt{2}} & -\frac{2}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$$

$A = (U\Sigma)V^T$:

$$A = \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 & 0 \\ \frac{3}{\sqrt{2}} & -\frac{2}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & 0 & 0 & \frac{4}{5} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{4}{5} & 0 & 0 & -\frac{3}{5} \end{bmatrix}.$$

$$\begin{aligned} A_{11} &= \left(\frac{3}{\sqrt{2}} \times \frac{3}{5}\right) + \left(\frac{2}{\sqrt{2}} \times 0\right) + (0 \times 0) + \left(0 \times \frac{4}{5}\right) \\ &= \frac{9}{5} \cdot \frac{1}{\sqrt{2}}. \end{aligned}$$

$$\begin{aligned} A_{12} &= \left(\frac{3}{\sqrt{2}} \times 0\right) + \left(\frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + (0 \times 0) \\ &= 0 + \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 + 0 \\ &= \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ &= 1. \end{aligned}$$

$$\begin{aligned} A_{13} &= \left(\frac{3}{\sqrt{2}} \times 0\right) + \left(\frac{2}{\sqrt{2}} \times \left(\frac{1}{\sqrt{2}}\right)\right) + \left(0 \times -\frac{1}{\sqrt{2}}\right) + (0 \times 0) \\ &= 0 + \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 + 0 \\ &= 1. \end{aligned}$$

$$\begin{aligned} A_{14} &= \left(\frac{3}{\sqrt{2}} \times \frac{4}{5}\right) + \left(\frac{2}{\sqrt{2}} \times 0\right) + (0 \times 0) + \left(0 \times \left(-\frac{3}{5}\right)\right) \\ &= \frac{12}{5\sqrt{2}} \\ &= \frac{12}{5} \cdot \frac{1}{\sqrt{2}}. \end{aligned}$$

$$\begin{aligned} A_{21} &= \left(\frac{3}{\sqrt{2}} \times \frac{3}{5}\right) + \left(-\frac{2}{\sqrt{2}} \times 0\right) + (0 \times 0) + \left(0 \times \frac{4}{5}\right) \\ &= \frac{9}{5} \cdot \frac{1}{\sqrt{2}}. \end{aligned}$$

$$
\begin{aligned}
A_{22} &= \left(\frac{3}{\sqrt{2}} \times 0\right) + \left(-\frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + (0 \times 0) \\
&= 0 - \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 + 0 \\
&= -1.
\end{aligned}
$$

$$
\begin{aligned}
A_{23} &= \left(\frac{3}{\sqrt{2}} \times 0\right) + \left(\frac{2}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right)\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + (0 \times 0) \\
&= 0 - \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 + 0 \\
&= -1.
\end{aligned}
$$

$$
\begin{aligned}
A_{24} &= \left(\frac{3}{\sqrt{2}} \times \frac{4}{5}\right) + \left(-\frac{2}{\sqrt{2}} \times 0\right) + (0 \times 0) + \left(0 \times \left(-\frac{3}{5}\right)\right) \\
&= \frac{12}{5\sqrt{2}} \\
&= \frac{12}{5} \cdot \frac{1}{\sqrt{2}}.
\end{aligned}
$$

Therefore, the matrix A is:

$$
A = \begin{bmatrix} \frac{9}{5} \cdot \frac{1}{\sqrt{2}} & 1 & 1 & \frac{12}{5} \cdot \frac{1}{\sqrt{2}} \\ \frac{9}{5} \cdot \frac{1}{\sqrt{2}} & -1 & -1 & \frac{12}{5} \cdot \frac{1}{\sqrt{2}} \end{bmatrix}.
$$

which is the same as the original matrix A