24400 HW7

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Question 1

(a) Distribution of $\bar{X} + \bar{Z}$

$\bar{X}$ and $\bar{Z}$ are independent because X_i's and Z_j's are independent:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right), \quad \bar{Z} \sim N\left(0, \frac{1}{k}\right).$$

Since the sum of independent normal random variables is also normal:

$$\bar{X} + \bar{Z} \sim N\left(\mu + 0, \frac{\sigma^2}{n} + \frac{1}{k}\right) = N\left(\mu, \frac{\sigma^2}{n} + \frac{1}{k}\right).$$

Answer:

$$\bar{X} + \bar{Z} \sim N\left(\mu, \frac{\sigma^2}{n} + \frac{1}{k}\right).$$

(b) Distribution of $k\bar{Z}^2$

We have:

$$\bar{Z} \sim N\left(0, \frac{1}{k}\right).$$

Let $W = \sqrt{k}\bar{Z}$:

$$W \sim N(0, 1).$$

Therefore:

$$k\bar{Z}^2 = (\sqrt{k}\bar{Z})^2 = W^2.$$

Since $W \sim N(0, 1)$, W^2 follows a chi-squared distribution with 1 degree of freedom:

$$W^2 \sim \chi^2(1).$$

Answer:

$$k\bar{Z}^2 \sim \chi^2(1).$$

(c) Distribution of $\dfrac{Z_1}{\sqrt{Z_2^2}}$

Z_1 and Z_2 are independent standard normal variables:

$$Z_1 \sim N(0,1), \quad Z_2 \sim N(0,1).$$

Since:

$$\frac{Z}{\sqrt{V/n}} \sim t(n),$$

and we have $Z_2^2 \sim \chi^2(1)$, and $\sqrt{Z_2^2} = \sqrt{V/1}$

Therefore, the expression becomes:

$$\frac{Z_1}{\sqrt{Z_2^2}} = \frac{Z_1}{\sqrt{Z_2^2/1}} \sim t(1)$$

Answer:

$$\frac{Z_1}{\sqrt{Z_2^2}} \sim t(1).$$

(d) Distribution of $Z_1^2 + Z_2^2$

Since Z_1 and Z_2 are independent standard normal variables, the sum of their squares follows a chi-squared distribution with 2 degrees of freedom:

$$Z_1^2 + Z_2^2 \sim \chi^2(2).$$

Answer:

$$Z_1^2 + Z_2^2 \sim \chi^2(2).$$

(e) Distribution of $\sum_{i=1}^n \dfrac{(X_i - \mu)^2}{\sigma^2} + \sum_{j=1}^k (Z_j - \bar{Z})^2$

Since

$$\frac{X_i - \mu}{\sigma} \sim N(0,1).$$

Therefore, the sum:

$$\sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n).$$

For the Z_j's, since they are i.i.d. $N(0,1)$:

$$\sum_{j=1}^{k}(Z_j - \bar{Z})^2 \sim \chi^2(k-1).$$

Since the two sums are independent (because X_i's and Z_j's are independent), the total sum follows a chi-squared distribution with $n+k-1$ degrees of freedom:

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} + \sum_{j=1}^{k}(Z_j - \bar{Z})^2 \sim \chi^2(n+k-1).$$

Answer:

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} + \sum_{j=1}^{k}(Z_j - \bar{Z})^2 \sim \chi^2(n+k-1).$$

(f) Distribution of $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma\sqrt{\frac{1}{k}\sum_{j=1}^{k} Z_j^2}}$

$\bar{X}$ and $\sum_{j=1}^{k} Z_j^2$ are independent because they involve independent variables (X_i's and Z_j's).

First,

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1).$$

Also, since $Z_j \sim N(0,1)$, we have:

$$\sum_{j=1}^{k} Z_j^2 \sim \chi^2(k).$$

Therefore:

$$\sqrt{\frac{1}{k}\sum_{j=1}^{k} Z_j^2} = \sqrt{\frac{\chi^2(k)}{k}}.$$

Thus:

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma\sqrt{\frac{1}{k}\sum_{j=1}^{k} Z_j^2}} = \frac{Z}{\sqrt{\frac{\chi^2(k)}{k}}} \sim t(k).$$

Answer:

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma\sqrt{\frac{1}{k}\sum_{j=1}^{k} Z_j^2}} \sim t(k).$$

Question 2

We are asked for a 95% confidence interval, so:

$$1 - \alpha = 0.95$$

$$\alpha = 0.05$$

For $n - 1 = 19$ degrees of freedom and $\frac{\alpha}{2} = 0.025$:

$$t_{19,\,0.025} \approx 2.093.$$

Using the formula for $(1 - \alpha)$ confidence interval for μ

$$\mu \in \bar{X} \pm t_{n-1,\,\alpha/2} \cdot \left(\frac{S}{\sqrt{n}} \right),$$

Since:

- Sample size: $n = 20$
- Sample mean: $\bar{X} = 14.1$
- Sample variance: $S^2 = 9.9$
- Sample standard deviation: $S = \sqrt{S^2} = \sqrt{9.9} \approx 3.1464$

$$\mu \in 14.1 \pm 2.093 \cdot \left(\frac{3.1464}{\sqrt{20}} \right)$$

$$\mu = 14.1 \pm 1.473$$

Therefore, the 95% confidence interval for μ is:

$$(14.1 - 1.473,\ 14.1 + 1.473) = (12.627,\ 15.573)\,.$$

Question 3

(a)

Part (i)

Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ, we have:

$$E[\hat{\theta}_1] = \theta, \quad E[\hat{\theta}_2] = \theta.$$

Taking expectation:

$$E[\hat{\theta}_3] = cE[\hat{\theta}_1] + (1 - c)E[\hat{\theta}_2] = c\theta + (1 - c)\theta = \theta.$$

Therefore, $\hat{\theta}_3$ is an unbiased estimator of θ.

Part (ii)

Since $\hat{\theta}_3$ is unbiased:

$$MSE(\hat{\theta}_3) = (E[\hat{\theta}_3] - \theta)^2 + \text{Var}(\hat{\theta}_3) = \text{Var}(\hat{\theta}_3)$$

$$\text{Var}(\hat{\theta}_3) = \text{Var}\left(c\hat{\theta}_1 + (1-c)\hat{\theta}_2\right).$$

Given that $\hat{\theta}_1$ and $\hat{\theta}_2$ are uncorrelated:

$$\text{Var}(\hat{\theta}_3) = c^2\text{Var}(\hat{\theta}_1) + (1-c)^2\text{Var}(\hat{\theta}_2).$$

Let $V = \text{Var}(\hat{\theta}_2)$, so $\text{Var}(\hat{\theta}_1) = 2V$.

$$\text{Var}(\hat{\theta}_3) = c^2(2V) + (1-c)^2 V = \left(2c^2 + (1 - 2c + c^2)\right) V.$$

$$\text{Var}(\hat{\theta}_3) = \left(3c^2 - 2c + 1\right) V.$$

Take the derivative with respect to c and set it to zero:

$$\frac{d}{dc}\text{Var}(\hat{\theta}_3) = \frac{d}{dc}\left(3c^2 - 2c + 1\right) V = (6c - 2)V = 0.$$

$$c = \frac{1}{3}.$$

Therefore, the value of c that minimizes the MSE is $c = \dfrac{1}{3}$.

Part (iii):

$$\text{Var}(\hat{\theta}_3) = \left(3c^2 - 2c + 1\right) V.$$

If $\text{Var}(\hat{\theta}_3)$ is a better estimator, need to find c such that:

$$\text{Var}(\hat{\theta}_3) < \min\left\{\text{Var}(\hat{\theta}_1), \text{Var}(\hat{\theta}_2)\right\} = \min\{2V, V\} = V.$$

Therefore,

$$(3c^2 - 2c + 1)V < V$$

$$3c^2 - 2c < 0.$$

$$c(3c - 2) < 0.$$

The roots are at $c = 0$ and $c = \dfrac{2}{3}$. Since the parabola opens upwards, $3c^2 - 2c$ is negative between the roots. Therefore, the inequality holds for $c \in \left(0, \dfrac{2}{3}\right)$.

As a result, for all $c \in \left(0, \dfrac{2}{3}\right)$, $\hat{\theta}_3$ has a smaller MSE than both $\hat{\theta}_1$ and $\hat{\theta}_2$.

(b)

Part(i)

Let $X_1,\, X_2 \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ where $0 < p < 1$.

We are using these estimator for the sample mean and sample variance:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

For $n = 2$:

$$\bar{X} = \frac{X_1 + X_2}{2}, \quad S^2 = \frac{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2}{1}.$$

First, list all possible combinations of X_1 and X_2, and compute $\bar{X}$ and S^2 for each case.

X_1	X_2	$\bar{X}$	S^2	Probability
0	0	0	0	$(1-p)^2$
0	1	$\frac{1}{2}$	$\frac{1}{2}$	$p(1-p)$
1	0	$\frac{1}{2}$	$\frac{1}{2}$	$p(1-p)$
1	1	1	0	p^2

Therefore,

$$\begin{aligned} E[\bar{X}] &= 0 \cdot (1-p)^2 + \frac{1}{2} \cdot [p(1-p) + p(1-p)] + 1 \cdot p^2 \\ &= p. \end{aligned}$$

$$\begin{aligned} E[S^2] &= 0 \cdot (1-p)^2 + \frac{1}{2} \cdot [p(1-p) + p(1-p)] + 0 \cdot p^2 \\ &= 0 + \frac{1}{2} \cdot 2p(1-p) + 0 \\ &= p(1-p). \end{aligned}$$

$$\begin{aligned} E[\bar{X}S^2] &= 0 \cdot (1-p)^2 + \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot [p(1-p) + p(1-p)] + 0 \cdot p^2 \\ &= 0 + \left(\frac{1}{4} \cdot 2p(1-p)\right) + 0 \\ &= \frac{1}{2}p(1-p). \end{aligned}$$

$$\begin{aligned} \text{Cov}(\bar{X}, S^2) &= E[\bar{X}S^2] - E[\bar{X}]E[S^2] \\ &= \left(\frac{1}{2}p(1-p)\right) - (p \cdot p(1-p)) \\ &= p(1-p)\left(\frac{1}{2} - p\right). \end{aligned}$$

Therefore, the covariance $\mathrm{Cov}(\bar{X}, S^2)$ is zero when $p = \frac{1}{2}$. For $p \neq \frac{1}{2}$, the covariance is nonzero.

Part (ii)

We need to determine whether $\bar{X}$ and S^2 are independent.

$\bar{X}$	S^2	Outcomes	Probability
0	0	$(0,0)$	$(1-p)^2$
$\frac{1}{2}$	$\frac{1}{2}$	$(0,1),(1,0)$	$2p(1-p)$
1	0	$(1,1)$	p^2

Therefore, the joint probability distribution $P(\bar{X}, S^2)$ is:

- $P\left(\bar{X}=0,\ S^2=0\right) = (1-p)^2$
- $P\left(\bar{X}=\frac{1}{2},\ S^2=\frac{1}{2}\right) = 2p(1-p)$
- $P\left(\bar{X}=1,\ S^2=0\right) = p^2$

Calculate $P(\bar{X})$ and $P(S^2)$:

- $P\left(\bar{X}=0\right) = (1-p)^2$
- $P\left(\bar{X}=\frac{1}{2}\right) = 2p(1-p)$
- $P\left(\bar{X}=1\right) = p^2$

- $P\left(S^2=0\right) = (1-p)^2 + p^2$
- $P\left(S^2=\frac{1}{2}\right) = 2p(1-p)$

For $\bar{X}$ and S^2 to be independent:

$$P(\bar{X}=x,\ S^2=s^2) = P(\bar{X}=x) \cdot P(S^2=s^2).$$

$$P\left(\bar{X}=0,\ S^2=0\right) = (1-p)^2$$
$$P\left(\bar{X}=0\right) \cdot P\left(S^2=0\right) = (1-p)^2 \cdot \left((1-p)^2 + p^2\right)$$

Since $p \in (0,1)$, $(1-p)^2 + p^2 < 1$, so:

$$P\left(\bar{X}=0,\ S^2=0\right) \neq P\left(\bar{X}=0\right) \cdot P\left(S^2=0\right)$$

Therefore, $\bar{X}$ and S^2 are not independent for any value of p.

Question 4

(a)

The likelihood function based on the samples $X_1, \ldots, X_n$ is:

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta) = \prod_{i=1}^{n} (\theta+1) X_i^{\theta} = (\theta+1)^n \prod_{i=1}^{n} X_i^{\theta}.$$

The log-likelihood function is:

$$\ell(\theta) = \log L(\theta) = n \log(\theta+1) + \theta \sum_{i=1}^{n} \log X_i.$$

take the derivative of the log-likelihood with respect to θ and set it to zero:

$$\frac{d\ell}{d\theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n} \log X_i = 0.$$

$$\frac{n}{\theta+1} = -\sum_{i=1}^{n} \log X_i$$

$$\theta + 1 = -\frac{n}{\sum_{i=1}^{n} \log X_i}.$$

Thus, the MLE $\hat{\theta}$ is:

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log X_i} - 1.$$

(b)

For one observation X, the log-likelihood is:

$$\ell(\theta \mid X) = \log(\theta+1) + \theta \log X.$$

The first derivative is:

$$\frac{\partial \ell}{\partial \theta} = \frac{1}{\theta+1} + \log X.$$

The second derivative is:

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{1}{(\theta+1)^2}.$$

The Fisher information for one observation is:

$$I_1(\theta) = E\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] = E\left[\frac{1}{(\theta+1)^2}\right] = \frac{1}{(\theta+1)^2}$$

Since the observations are independent, the total Fisher information for n observations is:

$$I(\theta) = nI_1(\theta) = \frac{n}{(\theta+1)^2}.$$

(c)

For large n, the distribution of the MLE $\hat{\theta}$ is approximately normal:

Therefore:

$$\hat{\theta} \approx N\left(\theta_0,\ \frac{1}{I(\theta_0)}\right) = N\left(\theta_0,\ \frac{(\theta_0+1)^2}{n}\right).$$

Therefore, for large n, the MLE $\hat{\theta}$ is approximately normally distributed:

$$\hat{\theta} \sim N\left(\theta_0,\ \frac{(\theta_0+1)^2}{n}\right).$$

Question 5

(a)

The expected value of X is given by:

$$\mu = E(X) = \int_0^1 x\, f(x \mid \theta)\, dx = \int_0^1 x\, (\theta+1)x^\theta\, dx.$$

$$\mu = (\theta+1)\int_0^1 x^{\theta+1}\, dx.$$

$$\mu = (\theta+1)\int_0^1 x^{\theta+1}\, dx = (\theta+1)\left[\frac{x^{\theta+2}}{\theta+2}\right]_0^1$$

$$\mu = (\theta+1)\cdot\frac{1}{\theta+2} = \frac{\theta+1}{\theta+2}.$$

Therefore,

$$\mu = E(X) = \frac{\theta+1}{\theta+2}.$$

(b)

The method of moments estimator equates the sample mean $\bar{X}$ to the theoretical mean μ:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu = \frac{\theta+1}{\theta+2}.$$

$$\bar{X} = \frac{\theta+1}{\theta+2}.$$

$$\bar{X}(\theta+2) = \theta+1.$$

$$\theta(\bar{X}-1) = 1-2\bar{X}.$$

So,

$$\hat{\theta} = \frac{1 - 2\bar{X}}{\bar{X} - 1} = \frac{2\bar{X} - 1}{1 - \bar{X}}$$

Therefore, the method of moments estimator $\hat{\theta}$ is:

$$\hat{\theta} = \frac{2\bar{X} - 1}{1 - \bar{X}},$$

where $\bar{X} = \dfrac{1}{n} \sum_{i=1}^{n} X_i$.

Question 6

(a)

Given that $X_1, \ldots, X_n \sim N(0, \theta)$, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{X_i^2}{2\theta}\right)$$

The log-likelihood function is:

$$\ell(\theta) = \log L(\theta) = -\frac{n}{2} \log(2\pi\theta) - \frac{1}{2\theta} \sum_{i=1}^{n} X_i^2.$$

Take the derivative of the log-likelihood with respect to θ and set it to zero:

$$\frac{d\ell}{d\theta} = -\frac{n}{2} \cdot \frac{1}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 = 0.$$

$$-n\theta + \sum_{i=1}^{n} X_i^2 = 0.$$

$$n\theta = \sum_{i=1}^{n} X_i^2$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

The MLE of θ is:

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

(b)

The log-likelihood function for one observation X_i is:

$$\ell(\theta) = -\frac{1}{2}\log(2\pi\theta) - \frac{X_i^2}{2\theta}.$$

The first derivative with respect to θ is:

$$\frac{\partial \ell}{\partial \theta} = -\frac{1}{2\theta} + \frac{X_i^2}{2\theta^2}.$$

The second derivative is:

$$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{X_i^2}{\theta^3}.$$

The Fisher information for one observation is:

$$I_1(\theta) = E\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] = \left(-\frac{1}{2\theta^2} + \frac{1}{\theta^3}E[X_i^2]\right).$$

Since $E[X_i^2] = \theta$, we have:

$$I_1(\theta) = \left(-\frac{1}{2\theta^2} + \frac{\theta}{\theta^3}\right) = \left(-\frac{1}{2\theta^2} + \frac{1}{\theta^2}\right) = \frac{1}{2\theta^2}.$$

Therefore, the total Fisher information for n observations is:

$$I(\theta) = nI_1(\theta) = \frac{n}{2\theta^2}.$$

(c)

For large n, the MLE $\hat{\theta}$ is approximately normally distributed with mean θ_0 and variance equal to the inverse of the Fisher information at θ_0:

$$\hat{\theta} \approx N\left(\theta_0,\ \frac{1}{I(\theta_0)}\right) = N\left(\theta_0,\ \frac{2\theta_0^2}{n}\right).$$

Therefore, for large n, the MLE $\hat{\theta}$ is approximately normally distributed:

$$\hat{\theta} \sim N\left(\theta_0,\ \frac{2\theta_0^2}{n}\right).$$
