Random variables & distributions – Part 2

Lecture 4a (STAT 24400 F24)

Functions of a continuous random variable

Recall: If X is a discrete random variable, $Y = g(X)$, then Y is a discrete r.v.

If X is a continuous random variable with density $f_X(x)$, and $Y = g(X)$, then what is the distribution of Y?

Depending on the function g, Y might be discrete or continuous or mixed, e.g.,

- $Y = X + 2 \rightsquigarrow$ continuous
- $Y = (X$ rounded to the nearest integer$) \rightsquigarrow$ discrete
- $Y = (X$ truncated to the interval $[0, 10]) \rightsquigarrow$ mixed

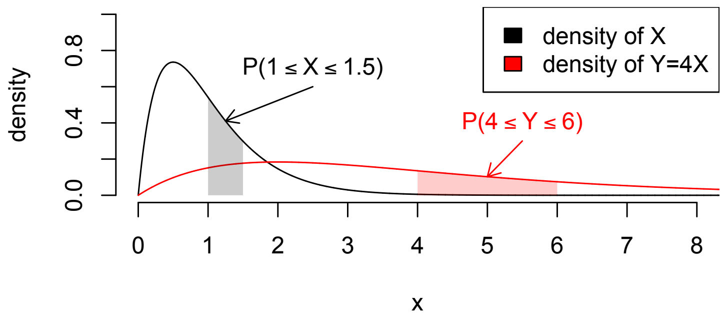
Special case: linear transformations

Suppose g is a change of units, $g(x) = a \cdot x$ for some $a > 0$.

Then $Y = g(X) = a \cdot X$ is a continuous r.v. with density

$$f_Y(y) = f_X(y/a) \cdot 1/a$$

Why do we need to rescale by $1/a$?



Special case: linear transformations

Suppose for some $a \neq 0$ and some $b \in \mathbb{R}$,

$$g(x) = a \cdot x + b$$

Then $Y = g(X) = a \cdot X + b$ is a continuous r.v. with density

$$f_Y(y) = f_X\left(\frac{y - b}{a}\right) \cdot \frac{1}{|a|}$$

The support for Y is induced by the support of X.

Linear transformation for specific distributions

- If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ with $a \neq 0$,
  then $Y \sim N(a\mu + b, a^2\sigma^2)$
- If $X \sim \text{Uniform}[a, b]$ and $Y = cX + d$ with $c > 0$ (or $c < 0$),
  then $Y \sim \text{Uniform}[ca + d, cb + d]$ (or $Y \sim \text{Uniform}[cb + d, ca + d]$)
- If $X \sim \text{Exponential}(\lambda)$ and $Y = aX$ for $a > 0$,
  then $Y \sim \text{Exponential}(\lambda/a)$
- If $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y = aX$ for $a > 0$,
  then $Y \sim \text{Gamma}(\alpha, \lambda/a)$

Special case: differentiable and strictly monotone

Suppose g is differentiable & strictly monotone.

Then $Y = g(X)$ is a continuous r.v. with density

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

Why?

Case 1: g is strictly increasing, thus $x = g^{-1}(y)$ exists,

$$\begin{aligned}\mathbb{P}(a < Y < b) &= \mathbb{P}(a < g(X) < b) = \mathbb{P}(g^{-1}(a) < X < g^{-1}(b)) \\ &= \int_{x=g^{-1}(a)}^{g^{-1}(b)} f_X(x)\, dx \underset{\substack{\uparrow \\ \text{change of variables}}}{=} \int_{y=a}^{b} f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| dy\end{aligned}$$

Case 2: a similar calculation if g is strictly decreasing

Example: differentiable and strictly monotone

Example $X \sim \text{Exponential}(\lambda)$ and $Y = 1/X$.

$f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$

$g(x) = 1/x \rightsquigarrow g^{-1}(y) = 1/y \rightsquigarrow \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{y^2}$

Calculate density of Y:

$$f_Y(y) = f_X(1/y) \cdot \frac{1}{y^2} = \lambda e^{-\lambda/y} \cdot \frac{1}{y^2}$$

where $y > 0$.

Beyond special cases

What if the special cases don't apply?

For example, what if g is non-monotone or non-differentiable?

- We can still calculate the CDF
  $$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \cdots$$
- Often we may need to consider piecewise intervals,
  where the transformation is monotone over each interval.
- Express $F_Y(y)$ in terms of F_X's,
  then use $f_Y = F_Y'$ to derive the density function.

Beyond special cases (example)

Example $X \sim \text{Exponential}(1)$ and $Y = |X - 1|$.

Calculate the CDF: $F_Y(y) = 0$ for $y < 0$. Now for $y \geq 0$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(|X-1| \leq y) = \mathbb{P}(1-y \leq X \leq 1+y) = F_X(1+y) - F_X(1-y)$$

Recalling that $F_X(x) = 1 - e^{-x}$ for $x > 0$,

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ e^{-1+y} - e^{-1-y}, & 0 \leq y \leq 1, \\ 1 - e^{-1-y}, & y > 1. \end{cases}$$

Taking derivative to obtain $f_Y = F_Y'$:

$$f_Y(y) = \begin{cases} 0, & y < 0, \\ e^{-1+y} + e^{-1-y}, & 0 \leq y \leq 1, \\ e^{-1-y}, & y > 1. \end{cases}$$

(Notes: The values at endpoints of continuous intervals are by one-sided limit, not unique.)

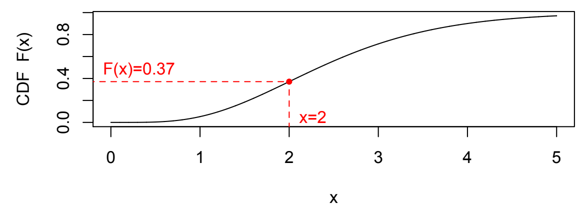
Transforming to a uniform

Suppose X is a continuous r.v. with CDF F, and $Y = F(X)$.
What is the distribution of Y?

For any $p \in (0,1)$, there is some x with $F(x) = p$ (since X continuous).

$$F_Y(p) = \mathbb{P}(Y \leq p) = \mathbb{P}(F(X) \leq p) \underset{\uparrow}{=} \mathbb{P}(F(X) \leq F(x)) = \mathbb{P}(X \leq x) = F(x) = p$$

choose some x so that $F(x) = p$



$\Rightarrow$ $Y \sim \text{Uniform}[0,1]$ (since its CDF is the Uniform CDF)

Transforming from a uniform

Let F be the CDF for some continuous distribution,
and let $U \sim \text{Uniform}[0,1]$ and $X = F^{-1}(U)$.

let's assume F strictly increasing

What is the distribution of X?

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x) \underset{\uparrow}{=} \mathbb{P}(U \leq F(x)) \underset{\uparrow}{=} F(x)$$

since F is strictly increasing   since U is Uniform$[0,1]$

$\Rightarrow$ X has CDF equal to F

Another proof

Another way to prove for the distribution of $X = F^{-1}(U)$:

Recall the density transformation formula for $x = g(u)$,

$$f_X(x) = f_U(g^{-1}(x)) \cdot \left| \frac{d}{dx} g^{-1}(x) \right|$$

Since F^{-1} is differentiable and strictly monotone, we can apply the formula with $g = F^{-1}, g^{-1} = (F^{-1})^{-1} = F$, and $U \sim \mathit{Uniform}[0,1]$ with $f_U(u) = \mathbb{1}_{\{[0,1]\}}$.

$$\begin{aligned} f_X(x) &= f_U((F^{-1})^{-1}(x)) \cdot \left| \frac{d}{dx} (F^{-1})^{-1}(x) \right| \\ &= f_U(F(x)) \cdot F'(x) = 1 \cdot F'(x) = F'(x) \end{aligned}$$

So X has density F', which means X has CDF F.