STAT 24300 - Numerical Linear Algebra
Assignment 7: Spectral Linear Algebra

Question 1: Matrix norms

In class, we defined the matrix norm of a matrix $A_{m\times n}$ as

$$\|A\| = \max_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}\,.$$

Prove that $\|A\| = \sigma_1$, where σ_1 is the largest singular value of A.

Hint: First prove that $\|Qx\| = \|x\|$ for any orthonormal matrix Q.

Question 2: Frobenious norm and SVD

Show that the Frobenious norm $\|A\|^2_{Fro} = \sum_{j=1}^r \sigma_j^2$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ denote the singular values of the matrix $A_{m\times n}$ whose rank is r.

Hint: First prove that $\|A\|^2_{Fro} = \text{Trace}(A^\top A)$. Then show that $\text{Trace}(AB) = \text{Trace}(BA)$ for matrices $A_{m\times n}, B_{n\times m}$.

Question 3: SVD and fundamental subspaces

Suppose $A_{m\times n} = U\Sigma V^\top$ is an SVD of A. Denote the rank of A by r. Prove that

1. range($A^\top$)=span($v_1, \cdots, v_r$), i.e, the span of the first r columns of V.
2. null($A^\top$)=span($u_{r+1}, \cdots, u_m$), i.e, the span of the last $m - r$ columns of U.

Question 4: Least squares

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

1. Find the pseudo-inverse of A.
2. Find the minimium norm solution to the least squares problem

   $$\min_{x\in\mathbb{R}^3} \|Ax - b\|\,,$$

   where A is as above and

   $$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

Question 5: Stability and conditioning

Suppose

$$A = \frac{1}{25}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & 1/\beta \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}, \quad \beta > 1$$

1. Compute the SVD of A.

2. Solve for x such that $Ax = b$ when $b = \frac{1}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Call the solution as x_1.

3. Solve for x such that $Ax = b$ when $b = \frac{1}{25}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Call the solution as x_2.

4. Compare your answers in parts 2 and 3, for instance by computing the relative norm $\|x_1 - x_2\|/\|x_1\|$. Explain the behaviour of the solutions as β varies.