Exercise

1. Find the distributions:
   (a) $\sum_{i=1}^n X_i^2/\sigma^2$, where $X_1, ..., X_n \sim N(0, \sigma^2)$ independently.
   (b) $\sum_{i=1}^n (X_i - \bar{X})^2/\sigma^2$, where $X_1, ..., X_n \sim N(\mu, \sigma^2)$ independently.
   (c) $\sum_{i=1}^n (X_i - \bar{X})^2/\sigma^2$, where $X_1, ..., X_n \sim N(0, \sigma^2)$ independently.
   (d) $\sum_{i=1}^n (X_i - Y_i)^2/(2\sigma^2)$, where $X_i, Y_i \sim N(\mu_i, \sigma^2)$ independently for $i = 1, ..., n$.
   (e) $\sum_{i=1}^n (X_i - \bar{X})^2/\sigma^2 + \sum_{i=1}^m (Y_i - \bar{Y})^2/\sigma^2$, where $X_1, ..., X_n \sim N(\mu, \sigma^2)$ and $Y_1, ..., Y_n \sim N(\theta, \sigma^2)$ independently.
   (f) $\frac{(X-Y)^2}{2} + \frac{(X+Y)^2}{2}$, where $X, Y \sim N(0, 1)$ independently.

2. Construct confidence intervals using Wilson's method:
   (a) CI for p with independent observations $X_1, ..., X_n \sim \text{Bernoulli}(p)$.
   (b) CI for λ with independent observations $X_1, ..., X_n \sim \text{Poisson}(\lambda)$.
   (c) CI for σ^2 with independent observations $X_1, ..., X_n \sim N(\mu, \sigma^2)$.
   (d) CI for σ^2 with $y \sim N(X\beta, \sigma^2 I_n)$.

3. For the same set of problems above, construct confidence intervals using variance stabilizing transform.

4. Consider independent observations $y_i \sim N(\beta x_i, \sigma^2)$ for $i = 1, ..., n$.
   (a) Find the MLE of β.
   (b) Find the distribution of the MLE $\hat{\beta}$.
   (c) Show $\hat{\beta}$ and $\sum_{i=1}^n (y_i - \hat{\beta} x_i)^2$ are independent.
   (d) Find the distribution of $\sum_{i=1}^n (y_i - \hat{\beta} x_i)^2/\sigma^2$.
   (e) Construct a confidence interval for $\hat{\beta}$ using t-distribution.
   (f) Consider prior distribution $\beta \sim N(0, \tau^2)$, find the posterior distribution of $\beta | y_1, ..., y_n$.
   (g) Find the posterior mean.

5. Review linear regression, both the univariate and the multivariate case: derivation of MLE, distribution of MLE, the corresponding theorems about chi-squared and t distributions.

6. Review all homework and midterm problems.