Assignment 2 (two pages)

Statistics 24400 (Autumn 2024)

Due on Gradescope, Tuesday, Oct. 15 by 9 am

Requirements

Your answers should be typed or clearly written, started with your name, Assignment 2, STAT 24400;
saved as `Lastname-Firstname-hw2.pdf`, and uploaded to Gradescope under P-set2.
Make sure to tag the page numbers for each question.
You may discuss approaches with others. However the assignment should be devised and written by yourself.
To get full credit, you must provide the reasoning of main steps of the derivation leading to your answer.

Problem assignments (related sections in the text: 2.1-2.3, 4.1)

1. Let X be a continuous random variable with density function $f(x) = \begin{cases} 0, & \text{if } x \le 0, \\ x, & \text{if } 0 < x < 1, \\ c/x^3, & \text{if } 1 \le x < \infty. \end{cases}$

   (a) Find the value of c so that $f(x)$ is a probability density function (PDF).

   (b) Find the cumulative distribution function (CDF).

   (c) Calculate $\mathbb{P}(0.5 \le X \le 1.5)$.

2. Let X be a continuous random variable with CDF $F(x) = \mathbb{P}(X \le x) = \begin{cases} 0, & \text{if } x < \theta, \\ 1 - cx^{-k}, & \text{if } x \ge \theta, \end{cases}$

   where $\theta > 0, k > 0$ are fixed (but unknown) parameters.

   (a) Find the value of c (as a function of θ and k) so that $F(x)$ is a cumulative distribution function (CDF). (Hint: F is continuous.)

   (b) Find $\mathbb{P}(X > 3\theta)$.

   (c) Find the probability density function $f(x)$ of X. (Note: Clearly state the domain of f.)

3. Calculate $\mathbb{P}(X \text{ is even})$ in each setting below. Show your calculation and explain your reasoning. Express your answer as a fraction (e.g. 3/5).

   (a) $X \sim \text{Geometric}(0.8)$ (i.e., X is a discrete r.v. with Geometric distribution with parameter $p = 0.8$).

   (b) $X \sim \text{Binomial}(101, 0.5)$.

4. You have n random number generators, where the ith one draws a number uniformly at random from the interval $[0, t_i]$, for some constant $t_i \in (0, \infty)$.

   Let X_i be the number drawn by the ith random number generator.

   (a) What is the expected value of the sum, $S = X_1 + \cdots + X_n$?
   (Your answer will be in terms of $t_1, \ldots, t_n$.)

   (b) What is the expected value of Y, which counts how many of the X_i's are ≤ 1?
   (Your answer will also be in terms of $t_1, \ldots, t_n$.)

5. Let $X \sim \text{Uniform}[0,1]$ (i.e., X is a continuous r.v. with uniform distribution on $[0,1]$).

   (a) Calculate the density of the ratio $R = \frac{X}{1-X}$.

   (b) Calculate the density of the product $Y = X(1-X)$.

6. Let the random variable T be the time until some event occurs (e.g. time until an atom decays, time until next rainfall, etc). Suppose this T is a continuous random variable supported on $[0, \infty)$.

   The <u>hazard rate</u> for T is defined as

$$h(t) = \frac{f(t)}{1 - F(t)},$$

   where f and F are the density and CDF for the distribution of T.

   On an intuitive level, this is the chance that the event will occur in the very near future, given that it has not yet occurred. Hazard rate is a function of time since it can rise or fall as time goes on.

   (a) Calculate the hazard rate $h(t)$ if $T \sim \text{Exponential}(\lambda)$.

   (b) Now suppose that T follows a Weibull distribution with shape $k > 0$ and scale $\alpha > 0$, which is supported on $[0, \infty)$ and defined by the CDF $F(t) = 1 - e^{-(t/\alpha)^k}$ on that interval.
   Calculate the density $f(t)$, and the hazard rate function $h(t)$, for this Weibull distribution.

   (c) For the Weibull distribution, for which values of k and α is $h(t)$ decreasing over time, increasing over time, or constant over time?

   (d) i. One common shape for the hazard function is a monotone increase, where the hazard rate is low initially and grows higher over time. Give a real-life example for what T could measure that would likely have this type of hazard rate function, and explain.

      ii. Finally, a common shape for the hazard function is a "U" shape, where the hazard rate is high initially, goes down to a low rate for a long time, and then rises again later on. Give a real-life example for what T could measure that would likely have this type of hazard rate function, and explain.