Stat 301

$X \sim P_\theta \qquad (P_\theta : \theta \in \Theta)$.

$L(\hat\theta, \theta) \qquad R(\hat\theta, \theta) = \mathbb{E}_\theta L(\hat\theta, \theta) = \int L(\hat\theta(X), \theta)\, dP_\theta(x)$.

e.g. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. $\mu \in \mathbb{R}$. loss $(\hat\mu - \mu)^2$

Q: Is $\bar X$ minimax? $\qquad R(\bar X, \mu) = \mathbb{E}_\theta (\bar X - \mu)^2 = \dfrac{\sigma^2}{n}$.

Consider $\pi = N(0, \tau^2)$.

$$\pi(\mu | X) \propto \pi(\mu) \prod_{i=1}^n p(X_i | \mu) \propto e^{-\frac{\mu^2}{2\tau^2} - \frac{1}{2}\sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2}}$$

$$f(\mu) = \frac{\mu^2}{\tau^2} + \sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2}. \qquad f'(\mu) = \frac{2\mu}{\tau^2} + \frac{1}{\sigma^2}\sum_{i=1}^n 2(\mu - X_i) = 0.$$

$$\Leftrightarrow \quad \frac{\mu}{\tau^2} + \frac{n\mu}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^n X_i$$

$$\Rightarrow \hat{u} = \mathbb{E}(u|X) = \frac{\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}} = \frac{n/\sigma^2}{\frac{1}{\tau^2}+n/\sigma^2}\bar{X}$$

$$R(\hat{u},u) = Var(\hat{u}) + (\mathbb{E}\hat{u}-u)^2$$

$$= \left(\frac{n/\sigma^2}{1/\tau^2+n/\sigma^2}\right)^2\frac{\sigma^2}{n} + \left(\frac{1/\tau^2}{1/\tau^2+n/\sigma^2}\right)^2 u^2.$$

$$\int R(\hat{u},u)\pi(u)\,du = \left(\frac{n/\sigma^2}{1/\tau^2+n/\sigma^2}\right)^2\frac{\sigma^2}{n} + \left(\frac{1/\tau^2}{1/\tau^2+n/\sigma^2}\right)^2\tau^2$$

$$= \frac{1}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}.$$

to prove $\bar{X}$ is minimax. $\sup_{u\in\mathbb{R}} R(\bar{X},u) = \frac{\sigma^2}{n}.$

$$\forall\, \hat{\mu}, \quad \sup_{\mu \in \mathbb{R}} R(\hat{\mu}, \mu) \geq \int R(\hat{\mu}, \mu)\, \pi(\mu)\, d\mu. \qquad \pi \to N(0, \tau^2).$$

$$\geq \inf_{\hat{\mu}} \int R(\hat{\mu}, \mu)\, \pi(\mu)\, d\mu.$$

$$= \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}.$$

Letting $\tau^2 \to \infty$ from both sides, we get

$$\lim_{\tau^2 \to \infty} \sup_{\mu \in \mathbb{R}} R(\hat{\mu}, \mu) \geq \lim_{\tau^2 \to \infty} \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}$$

$$\Rightarrow \sup_{\mu \in \mathbb{R}} R(\hat{\mu}, \mu) \geq \frac{\sigma^2}{n} = R(\bar{X}, \mu)$$

$\square$

Theorem: $\exists \{\pi_m\}$ s.t.

$$\sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \lim_{m \to \infty} \inf_{\hat{\theta}} \int R(\hat{\theta}, \theta) \pi_m(\theta) d\theta .$$

then $\hat{\theta}$ is minimax.

Admissibility.

Def: $\hat{\theta}$ is inadmissible, if $\exists \tilde{\theta}$ s.t.

① $R(\tilde{\theta}, \theta) \leq R(\hat{\theta}, \theta) \quad \forall \theta \in \Theta$

② $R(\tilde{\theta}, \theta_0) < R(\hat{\theta}, \theta_0)$ for some θ_0.

$\hat{\theta}$ is admissible if it is not inadmissible.

Theorem: If $\hat{\theta}$ is Bayes, then it is admissible.

Proof: suppose $\hat{\theta}$ is inadmissible. $\exists\ \tilde{\theta}$ s.t.

① $R(\tilde{\theta},\theta) \le R(\hat{\theta},\theta) \quad \forall \theta \in \Theta$

② $R(\tilde{\theta},\theta_0) < R(\hat{\theta},\theta_0)$ for some $\theta_0 \in \Theta$.

$\Rightarrow$ $\exists$ an open set $\Theta_0 \ni \theta_0$ and $\varepsilon > 0$,

s.t. $R(\tilde{\theta},\theta) \le R(\hat{\theta},\theta) - \varepsilon \quad \forall \theta \in \Theta_0$.

$$\Rightarrow \int R(\tilde{\theta},\theta)\,\pi(\theta)\,d\theta = \int_{\Theta_0} R(\tilde{\theta},\theta)\,\pi(\theta)\,d\theta + \int_{\Theta_0^c} R(\tilde{\theta},\theta)\,\pi(\theta)\,d\theta$$

$$< \int_{\Theta_0} R(\hat{\theta},\theta)\,\pi(\theta)\,d\theta + \int_{\Theta_0^c} R(\hat{\theta},\theta)\,\pi(\theta)\,d\theta.$$

$$= \int R(\hat{\theta},\theta)\,\pi(\theta)\,d\theta$$

$\Rightarrow \hat{\theta}$ is not Bayes. $\square$

Complete class theorem (Brown, 1986)

$\hat{\theta}$ is admissible $\Rightarrow \exists \{\pi_m\}$ s.t. $\hat{\theta}_{\pi_m} \to \hat{\theta}$.

e.g. $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$ $\theta \in \mathbb{R}$ $(\hat{\theta}-\theta)^2$.

Q: Is $\bar{X}$ admissible? yes. Wald (1939).

Proof (Blyth's method): suppose $\hat{\theta} = \bar{X}$ is inadmissible.

$\exists\ \tilde{\theta}$ s.t. $\begin{cases} ① \ R(\tilde{\theta}, \theta) \le \frac{1}{n} & \forall \theta \in \mathbb{R}. \\ ② \ R(\tilde{\theta}, \theta_0) < \frac{1}{n} & \exists \theta_0 \in \mathbb{R}. \end{cases}$

$\exists\ a < b, \varepsilon > 0$ s.t. $(a,b) \ni \theta_0$ & $R(\tilde{\theta}, \theta) < \frac{1}{n} - \varepsilon, \ \forall \theta \in (a,b)$

Consider $\pi_m = N(0, m)$. $\int R(\hat{\theta}_{\pi_m}, \theta)\, \pi(\theta)\, d\theta = \dfrac{1}{n + \frac{1}{m}}$.

$$\frac{1}{n} - \int R(\hat{\theta}_{\pi_m}, \theta)\, \pi(\theta) d\theta = \frac{1}{n} - \frac{1}{n + \frac{1}{m}} = \frac{1}{n} \cdot \frac{\frac{1}{m}}{n + \frac{1}{m}} \asymp \frac{1}{m}.$$

$$\frac{1}{n} - \int R(\tilde{\theta}, \theta)\, \pi_m(\theta) d\theta$$

$$\int \frac{1}{n} \cdot \pi_m(\theta) d\theta = \int_{(a,b)} + \int_{(a,b)^c}$$

$$= \frac{1}{n} - \int_{(a,b)} R(\tilde{\theta}, \theta)\, \pi_m(\theta) d\theta - \int_{(a,b)^c} R(\tilde{\theta}, \theta)\, \pi_m(\theta)\, d\theta$$

$$= \int_{(a,b)} \left(\frac{1}{n} - R(\tilde{\theta}, \theta) \right) \pi_m(\theta) d\theta + \int_{(a,b)^c} \left(\frac{1}{n} - R(\tilde{\theta}, \theta) \right) \pi_m(\theta) d\theta.$$

$$\geq \varepsilon \int_{(a,b)} \pi_m(\theta) d\theta = \varepsilon\, \mathbb{P}\left(a < N(0, m) < b \right)$$

$$= \sum \mathbb{P}\left(\frac{a}{\sqrt{m}} < N(0,1) < \frac{b}{\sqrt{m}} \right).$$

$$= \sum \int_{a/\sqrt{m}}^{b/\sqrt{m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \asymp \frac{1}{\sqrt{m}}.$$

$\exists$ m sufficiently large s.t.

$$\frac{1}{n} - \int R(\tilde{\theta}, \theta) \pi_m(\theta) d\theta > \frac{1}{n} - \int R(\hat{\theta}_{\pi_m}, \theta) \pi_m(\theta) d\theta$$

$$\Leftrightarrow \int R(\tilde{\theta}, \theta) \pi_m(\theta) d\theta < \int R(\hat{\theta}_{\pi_m}, \theta) \pi_m(\theta) d\theta.$$

$\Rightarrow$ impossible. $\Rightarrow$ $\bar{X}$ is admissible $\square$

e.g. $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, I_2)$ $\qquad \|\hat{\theta} - \theta\|^2$

$\theta \in \mathbb{R}^2$

Q: is $\bar{X}$ admissible?

yes (proved by Stein).

e.g. $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, I_3)$ $\qquad \|\hat{\theta} - \theta\|^2$.

$\theta \in \mathbb{R}^3$

Q: is $\bar{X}$ admissible?

no. James – Stein estmator

$$\hat{\theta}_{JS} = \left(1 - \frac{p-2}{n\|\bar{X}\|^2}\right)\bar{X}$$

Theorem: $\mathbb{E}_\theta \|\hat{\theta}_{JS} - \theta\|^2 < \mathbb{E}_\theta \|\bar{X} - \theta\|^2 \quad \forall \theta \in \mathbb{R}^p$

for $p \geq 3$.

(Brown, 1971).