STAT 245 HW1 Solution

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Q1

The MGF for Poisson(λ) is $M(t) = \exp(\lambda(e^t - 1))$. By definition by MGF,

$$\begin{aligned} E[X^4] &= \frac{d^4}{dt^4} M(t)\Big|_{t=0} \\ &= \left(\lambda^4 e^{\lambda e^t - \lambda + 4t} + 6\lambda^3 e^{\lambda e^t - \lambda + 3t} + 7\lambda^2 e^{\lambda e^t - \lambda + 2t} + \lambda e^{\lambda e^t - \lambda + t}\right)\Big|_{t=0} \\ &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda. \end{aligned}$$

Q2

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} \mathrm{d}x = \frac{1}{\lambda}.$$

Q3

Exact:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Normal approximation: $\frac{X - np}{\sqrt{np(1-p)}} \approx Z$, equivalently $X \approx \sqrt{np(1-p)}Z + np$.

$$P(X = k) \approx P(|\sqrt{np(1-p)}Z + np - k| \leq 0.5) = P\left(\frac{k - 0.5 - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{k + 0.5 - np}{\sqrt{np(1-p)}}\right).$$

Possion Approximation: $\mathrm{Bin}(n,p) \approx \mathrm{Poisson}(\lambda)$ with $\lambda = np$.

$$P(X = k) = e^{-np}\frac{(np)^k}{k!}.$$

	Exact	Normal	Poisson
(a)	0.2269	0.2466	0.1890
(b)	0.0357	0.0353	0.0496
(c)	0.1842	0.2418	0.1839

(a): Normal approximation works better than Poisson approximation, but none of them gives very close approximation.

(b): Normal approximation works well because n is large and p is close to 0.5; or np is large.

(c): Poisson approximation works well because p is small.

Q4

(a) Log-likelihood of p is

$$l(p) = \sum_i x_i \log p + (n - \sum_i x_i) \log(1-p).$$

Take derivative and set it to 0,

$$\frac{\sum_i x_i}{p} - \frac{n - \sum_i x_i}{1-p} = 0 \Rightarrow \hat{p} = \bar{x}.$$

(b) $E[\hat{p}] = p, \mathrm{Var}(\hat{p}) = p(1-p)/n$.

$$\mathrm{MSE}(\hat{p}) = \mathrm{Var}(\hat{p}) + (\mathrm{Bias}(\hat{p}))^2 = p(1-p)/n + 0 = p(1-p)/n.$$

(c) By CLT, $\frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p(1-p)}} \to N(0,1)$, so

$$\sqrt{n}(\hat{p} - p) \to N(0, p(1-p)).$$

(d) Because $\frac{\sqrt{n}(\hat{p}-p)}{\sqrt{\hat{p}(1-\hat{p})}} \approx N(0,1)$, the Wald confidence-interval can be constructed as $\hat{p} \pm z_{0.975}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

(e) If we solve $\left|\frac{\sqrt{n}(\hat{p}-p)}{\sqrt{p(1-p)}}\right| \leq z_{0.975}$, we will get

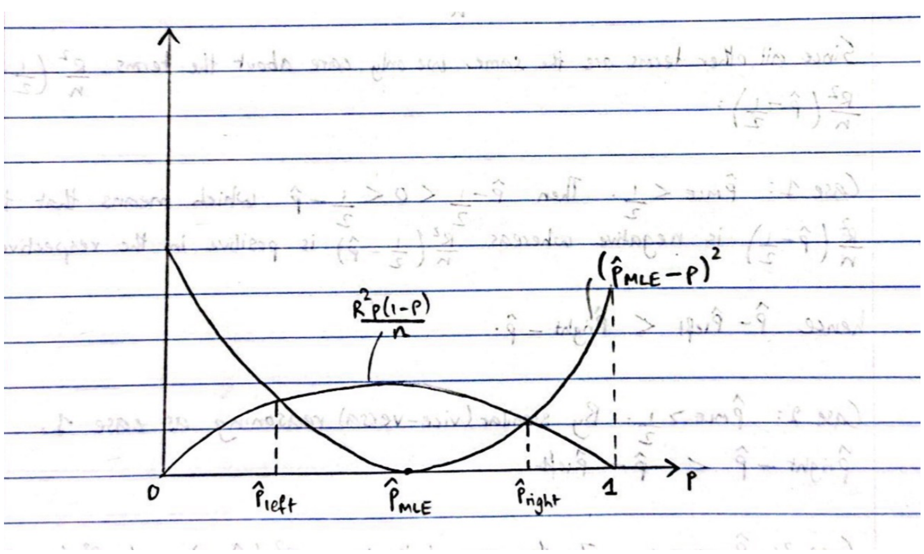
$$(1 + z_{0.975}^2/n)p^2 - (2\hat{p} + z_{0.975}^2/n)p + \hat{p}^2 \leq 0,$$

the roots are

$$\frac{\hat{p} + \frac{z_{0.975}^2}{2n} \pm z_{0.975}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{0.975}^2}{4n}}}{1 + z_{0.975}^2/n}$$

which give the Wilson's confidence interval.

(f) (By Courtesy, Adi Ramen)



1. $(\hat{p}_{\text{left}}, \hat{p}_{\text{right}})$ is the same as Wilson's confidence interval.
2. Both $\hat{p}_{\text{left}}, \hat{p}_{\text{right}}$ are in the interval $(0, 1)$.

Other reasonable observations are also acceptable.

(g) If $\hat{p} < 1/2$, then $\hat{p}_{\text{right}} - \hat{p} > \hat{p} - \hat{p}_{\text{left}}$.
If $\hat{p} = 1/2$, then $\hat{p}_{\text{right}} - \hat{p} = \hat{p} - \hat{p}_{\text{left}}$.
If $\hat{p} > 1/2$, then $\hat{p}_{\text{right}} - \hat{p} < \hat{p} - \hat{p}_{\text{left}}$.