24400 HW5

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Question 1

(a) $P(X > 2)$

$$z = \frac{2 - (-4)}{4} = \frac{6}{4} = 1.5$$
$$P(X > 2) = P(Z > 1.5) = 1 - \Phi(1.5)$$

$$\Phi(1.5) = 0.9332$$

$$P(X > 2) = 1 - 0.9332 = 0.0668$$

(b) $P(0 < X < 4)$

$$z_1 = \frac{0 - (-4)}{4} = 1.0$$

$$z_2 = \frac{4 - (-4)}{4} = 2.0$$

$$P(0 < X < 4) = \Phi(2.0) - \Phi(1.0)$$

$$\Phi(2.0) = 0.9772$$
$$\Phi(1.0) = 0.8413$$

$$P(0 < X < 4) = 0.9772 - 0.8413 = 0.1359$$

(c) $P(|X+3| \geq 3)$

$$|X+3| \geq 3 \iff X+3 \leq -3 \text{ or } X+3 \geq 3$$
$$X \leq -6 \text{ or } X \geq 0$$

For $X \leq -6$:

$$z_1 = \frac{-6-(-4)}{4} = \frac{-2}{4} = -0.5$$

For $X \geq 0$:

$$z_2 = \frac{0-(-4)}{4} = 1.0$$

$$P(X \leq -6) = \Phi(-0.5) = 1 - \Phi(0.5)$$

$$P(X \geq 0) = 1 - \Phi(1.0)$$

$$\Phi(0.5) = 0.6915$$
$$\Phi(1.0) = 0.8413$$

$$P(|X+3| \geq 3) = [1-0.6915] + [1-0.8413] = 0.3085 + 0.1587 = 0.4672$$

(d) $P(X \leq 0$ **or** $X \geq 3)$

For $X \leq 0$:

$$z_1 = \frac{0-(-4)}{4} = 1.0$$

For $X \geq 3$:

$$z_2 = \frac{3-(-4)}{4} = \frac{7}{4} = 1.75$$

$$P(X \leq 0) = \Phi(1.0)$$
$$P(X \geq 3) = 1 - \Phi(1.75)$$

$$\Phi(1.0) = 0.8413$$
$$\Phi(1.75) = 0.9599$$

$$P(X \leq 0 \text{ or } X \geq 3) = 0.8413 + [1 - 0.9599] = 0.8413 + 0.0401 = 0.8814$$

Question 2

(a)

Given that U follows a uniform distribution on $[0, c]$, its pdf is:

$$f_U(u) = \begin{cases} \frac{1}{c}, & 0 \leq u \leq c \\ 0, & \text{otherwise} \end{cases}$$

$$E(U^k) = \int_0^c u^k f_U(u)\, du = \int_0^c u^k \left(\frac{1}{c}\right) du = \frac{1}{c} \int_0^c u^k du$$

$$\int_0^c u^k du = \left[\frac{u^{k+1}}{k+1}\right]_0^c = \frac{c^{k+1}}{k+1}$$

$$E(U^k) = \frac{1}{c} \cdot \frac{c^{k+1}}{k+1} = \frac{c^k}{k+1}$$

$$E(U^k) = \frac{c^k}{k+1}$$

(b)

$$E(V^U) = E_U\left[E_V\left(V^U \mid U\right)\right] = E_U\left[E_V\left(V^U\right)\right]$$

Since V and U are independent, and $V \sim \text{Uniform}[0, 1]$, for a fixed U, the expected value $E_V(V^U)$ is:

$$E_V(V^U) = \int_0^1 V^U\, dV$$

Compute the inner integral:

$$E_V(V^U) = \int_0^1 V^U\, dV = \left[\frac{V^{U+1}}{U+1}\right]_0^1 = \frac{1^{U+1} - 0^{U+1}}{U+1} = \frac{1}{U+1}$$

$$E(V^U) = E_U\left(\frac{1}{U+1}\right) = \int_0^1 \frac{1}{U+1}\, dU$$

$$E(V^U) = \int_0^1 \frac{1}{U+1}\, dU = [\ln(U+1)]_0^1 = \ln(1+1) - \ln(0+1) = \ln 2 - \ln 1 = \ln 2$$

$$E(V^U) = \ln 2 \approx 0.6931$$

(c)

Given U, $W \mid U \sim \text{Uniform}[0, U]$.

$$E(W \mid U) = \frac{0+U}{2} = \frac{U}{2}$$
$$\text{Var}(W \mid U) = \frac{(U-0)^2}{12} = \frac{U^2}{12}$$

Therefore,

$$E(W) = E_U[E(W \mid U)] = E_U\left(\frac{U}{2}\right) = \frac{1}{2}E(U)$$

Since $U \sim \text{Uniform}[0, 1]$, $E(U) = \dfrac{0+1}{2} = \dfrac{1}{2}$.

$$E(W) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

For unconditional variance $\text{Var}(W)$, using the law of total variance:

$$\text{Var}(W) = E_U[\text{Var}(W \mid U)] + \text{Var}_U[E(W \mid U)]$$

$$E_U[\text{Var}(W \mid U) = E_U\left(\frac{U^2}{12}\right) = \frac{1}{12}E(U^2)$$

Since $U \sim \text{Uniform}[0, 1]$, $E(U^2) = \dfrac{1}{3}$.

$$E_U[\text{Var}(W \mid U)] = \frac{1}{12} \cdot \frac{1}{3} = \frac{1}{36}$$

and,

$$E(W \mid U) = \frac{U}{2}$$

$$\text{Var}_U\left(\frac{U}{2}\right) = \left(\frac{1}{2}\right)^2 \text{Var}(U) = \frac{1}{4} \cdot \text{Var}(U)$$

Since $\text{Var}(U) = \dfrac{(1-0)^2}{12} = \dfrac{1}{12}$.

$$\mathrm{Var}_U[E(W \mid U)] = \frac{1}{4} \cdot \frac{1}{12} = \frac{1}{48}$$

$$\mathrm{Var}(W) = \frac{1}{36} + \frac{1}{48} = \frac{4}{144} + \frac{3}{144} = \frac{7}{144}$$

Therefore,

$$E(W) = \frac{1}{4}, \quad \mathrm{Var}(W) = \frac{7}{144}$$

Question 3

(a) Find $E(Y \mid S)$ and $E(Y)$

When $S = 1$, a red marble is transferred from Box 1 to Box 2. Therefore, Box 2 now contains 11 red marbles and 16 marbles in total

Then draw $n = 4$ marbles without replacement. The number of red marbles Y follows a hypergeometric distribution with parameters $N = 16$, $K = 11$ (number of red marbles), and $n = 4$.

$$E(Y \mid S = 1) = n \cdot \frac{K}{N} = 4 \cdot \frac{11}{16} = \frac{44}{16} = \frac{11}{4}$$

When $S = 0$, a black marble is transferred. Box 2 now contains 10 red marbles and 16 marbles in total

$$E(Y \mid S = 0) = n \cdot \frac{K}{N} = 4 \cdot \frac{10}{16} = \frac{40}{16} = \frac{10}{4}$$

$$P(S = 1) = \frac{5}{11}$$

$$P(S = 0) = \frac{6}{11}$$

Then,

$$E(Y) = E[Y \mid S = 1] \cdot P(S = 1) + E[Y \mid S = 0] \cdot P(S = 0)$$

$$E(Y) = \frac{11}{4} \cdot \frac{5}{11} + \frac{10}{4} \cdot \frac{6}{11} = \frac{115}{44}$$

(b) Find $\text{Var}(Y \mid S)$ and $E[\text{Var}(Y \mid S)]$

Using the variance formula for the hypergeometric distribution:

$$\text{Var}(Y \mid S = 1) = n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1}$$

Substitute the values:

$$n = 4, \quad K = 11, \quad N = 16, \quad N - K = 5, \quad N - n = 12$$

$$\text{Var}(Y \mid S = 1) = 4 \cdot \frac{11}{16} \cdot \frac{5}{16} \cdot \frac{12}{15} = \frac{11}{16}$$

Similarly, with $K = 10$ and $N - K = 6$:

$$\text{Var}(Y \mid S = 0) = 4 \cdot \frac{10}{16} \cdot \frac{6}{16} \cdot \frac{12}{15} = \frac{3}{4}$$

$$E[\text{Var}(Y \mid S)] = \text{Var}(Y \mid S = 1) \cdot P(S = 1) + \text{Var}(Y \mid S = 0) \cdot P(S = 0)$$

$$E[\text{Var}(Y \mid S)] = \frac{11}{16} \cdot \frac{5}{11} + \frac{3}{4} \cdot \frac{6}{11} = \frac{55 + 72}{176} = \frac{127}{176}$$

Therefore,

$$\text{Var}(Y \mid S = 1) = \frac{11}{16}, \quad \text{Var}(Y \mid S = 0) = \frac{3}{4}, \quad E[\text{Var}(Y \mid S)] = \frac{127}{176}$$

(c) Find $\text{Var}(E[Y \mid S])$ and $\text{Var}(Y)$

Since we have:

$$E(Y) = E_S[E[Y \mid S]] = E[Y \mid S = 1] \cdot P(S = 1) + E[Y \mid S = 0] \cdot P(S = 0)$$

$$\text{Var}(E[Y \mid S]) = E_S\left[(E[Y \mid S])^2\right] - [E_S[E[Y \mid S]]]^2 = E_S\left[(E[Y \mid S])^2\right] - [E(Y)]^2$$

Calculate $E_S\left[(E[Y \mid S])^2\right]$:

$$E_S[(E[Y \mid S])^2] = (E[Y \mid S = 1])^2 \cdot P(S = 1) + (E[Y \mid S = 0])^2 \cdot P(S = 0)$$

$$= (\frac{11}{4})^2 \cdot \frac{5}{11} + (\frac{10}{4})^2 \cdot \frac{6}{11} = \frac{121}{16} \cdot \frac{5}{11} + \frac{25}{4} \cdot \frac{6}{11}$$

$$= \frac{1205}{176}$$

$$E(Y) = \frac{115}{44}, \quad [E(Y)]^2 = \left(\frac{115}{44}\right)^2 = \frac{13225}{1936}$$

$$\begin{aligned} \text{Var}(E[Y \mid S]) &= \frac{1205}{176} - \frac{13225}{1936} \\ &= \frac{15}{968} \end{aligned}$$

Since:

$$\text{Var}(Y) = E[\text{Var}(Y \mid S)] + \text{Var}(E[Y \mid S])$$

From part B:

$$E[\text{Var}(Y \mid S)] = \frac{127}{176}$$

$$\text{Var}(Y) = \frac{127}{176} + \frac{15}{968} = \frac{1397}{1936} + \frac{30}{1936} = \frac{1427}{1936}$$

Question 4

(a) Calculate $E(X_{(1)})$ and $E(X_{(2)})$ for $X_i \sim$ Uniform$[0, 1]$

We are given the pdfs of $X_{(1)}$ and $X_{(2)}$:

$$f_{X_{(1)}}(x) = 2(1-x), \quad f_{X_{(2)}}(x) = 2x, \quad x \in [0,1]$$

Compute $E(X_{(1)})$:

$$E(X_{(1)}) = \int_0^1 x \cdot f_{X_{(1)}}(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx$$

Compute the integral:

$$\begin{aligned} E(X_{(1)}) &= 2\int_0^1 x(1-x)\,dx = 2\int_0^1 (x - x^2)\,dx \\ &= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) \\ &= 2\left(\frac{3-2}{6}\right) = 2\left(\frac{1}{6}\right) = \frac{1}{3} \end{aligned}$$

Similarly, compute $E(X_{(2)})$:

$$
\begin{aligned}
E(X_{(2)}) &= \int_0^1 x \cdot f_{X_{(2)}}(x)\,dx = \int_0^1 x \cdot 2x\,dx \\
&= 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{3} - 0\right) = \frac{2}{3}
\end{aligned}
$$

Answer:

$$E(X_{(1)}) = \frac{1}{3}, \quad E(X_{(2)}) = \frac{2}{3}$$

(b) Calculate $E(X_{(1)})$ and $E(X_{(2)})$ for $X_i \sim$ Uniform$[a, b]$

Let $X_i \sim \text{Uniform}[a, b]$.

Consider a linear transformation of the Uniform$[0, 1]$ distribution:

$$Y_i = \frac{X_i - a}{b - a} \implies X_i = a + (b - a)Y_i$$

Then $Y_i \sim \text{Uniform}[0, 1]$. The order statistics transform accordingly:

$$X_{(1)} = a + (b - a)Y_{(1)}, \quad X_{(2)} = a + (b - a)Y_{(2)}$$

Therefore:

$$E(X_{(1)}) = a + (b - a)E(Y_{(1)}), \quad E(X_{(2)}) = a + (b - a)E(Y_{(2)})$$

Since

$$E(Y_{(1)}) = \frac{1}{3}, \quad E(Y_{(2)}) = \frac{2}{3}$$

Thus,

$$
\begin{aligned}
E(X_{(1)}) &= a + (b - a)\left(\frac{1}{3}\right) = \frac{2a + b}{3} \\
E(X_{(2)}) &= a + (b - a)\left(\frac{2}{3}\right) = \frac{a + 2b}{3}
\end{aligned}
$$

(c) Find an unbiased estimator of b as a function of X_1 and X_2

From part (b), we have:

$$E(X_{(2)}) = \frac{a + 2b}{3}, \quad E(X_{(1)}) = \frac{2a + b}{3}$$

Observing that:

$$2E(X_{(2)}) - E(X_{(1)}) = b$$

Thus,

$$E[2X_{(2)} - X_{(1)}] = b$$

Therefore, an unbiased estimator of b is:

$$g(X_1, X_2) = 2X_{(2)} - X_{(1)}$$

with:

$$E[2X_{(2)} - X_{(1)}] = b$$

Question 5

(a)

The hierarchical model:

$$\theta \sim \text{Uniform}\{0.25, 0.5, 0.75\}$$
$$X \mid \theta \sim \text{Bernoulli}(\theta)$$

(b)

$$P(X = 1) = \sum_{\theta} P(X = 1 \mid \theta) \cdot P(\theta)$$

Since $P(\theta) = \frac{1}{3}$ for each value of θ:

$$\begin{aligned} P(X = 1) &= (P(X = 1 \mid \theta = 0.25) \cdot P(\theta = 0.25)) \\ &\quad + (P(X = 1 \mid \theta = 0.5) \cdot P(\theta = 0.5)) \\ &\quad + (P(X = 1 \mid \theta = 0.75) \cdot P(\theta = 0.75)) \\ &= \left(0.25 \times \frac{1}{3}\right) + \left(0.5 \times \frac{1}{3}\right) + \left(0.75 \times \frac{1}{3}\right) \\ &= \frac{0.25 + 0.5 + 0.75}{3} = \frac{1}{2} \end{aligned}$$

$$P(X = 1) = \frac{1}{2}$$

Therefore,

$$P(X = 0) = \frac{1}{2}$$

(c)

According to Bayes' theorem:

$$P(\theta \mid X) = \frac{P(X \mid \theta)P(\theta)}{P(X)}$$

When $X = 0$:

$$P(\theta = 0.25 \mid X = 0) = \frac{P(X = 0 \mid \theta = 0.25)\, P(\theta = 0.25)}{P(X = 0)} = \frac{(1 - 0.25) \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(\theta = 0.5 \mid X = 0) = \frac{(1 - 0.5) \times \frac{1}{3}}{\frac{1}{2}} = \frac{0.5 \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$P(\theta = 0.75 \mid X = 0) = \frac{(1 - 0.75) \times \frac{1}{3}}{\frac{1}{2}} = \frac{0.25 \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{6}$$

When $X = 1$:

$$P(\theta = 0.25 \mid X = 1) = \frac{P(X = 1 \mid \theta = 0.25)\, P(\theta = 0.25)}{P(X = 1)} = \frac{0.25 \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{6}$$

$$P(\theta = 0.5 \mid X = 1) = \frac{0.5 \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$P(\theta = 0.75 \mid X = 1) = \frac{0.75 \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{2}$$

Therefore,

When $X = 0$:

$$\begin{aligned} P(\theta = 0.25 \mid X = 0) &= \frac{1}{2} \\ P(\theta = 0.5 \mid X = 0) &= \frac{1}{3} \\ P(\theta = 0.75 \mid X = 0) &= \frac{1}{6} \end{aligned}$$

When $X = 1$:

$$
\begin{aligned}
P(\theta = 0.25 \mid X = 1) &= \frac{1}{6} \\
P(\theta = 0.5 \mid X = 1) &= \frac{1}{3} \\
P(\theta = 0.75 \mid X = 1) &= \frac{1}{2}
\end{aligned}
$$

(d)

Using the posterior distribution obtained in part (c) when $X = 1$:

$$
\begin{aligned}
P(\theta = 0.25 \mid X = 1) &= \frac{1}{6} \\
P(\theta = 0.5 \mid X = 1) &= \frac{1}{3} \\
P(\theta = 0.75 \mid X = 1) &= \frac{1}{2}
\end{aligned}
$$

$$
P(Y = 1 \mid X = 1) = E_{\theta|X=1}[P(Y = 1 \mid \theta)] = E_{\theta|X=1}[\theta]
$$

$$
\begin{aligned}
E[\theta \mid X = 1] &= \left(\frac{1}{6} \times 0.25\right) + \left(\frac{1}{3} \times 0.5\right) + \left(\frac{1}{2} \times 0.75\right) \\
&= \frac{1}{24} + \frac{1}{6} + \frac{3}{8} \\
&= \frac{1}{24} + \frac{4}{24} + \frac{9}{24} = \frac{7}{12}
\end{aligned}
$$

Therefore,

$$
P(Y = 1 \mid X = 1) = \frac{7}{12}
$$