Homework 1

When solving the problems below as well as future homework problems, please give detailed derivations and arguments in order to receive credit. Check the Canvas announcement by David Chen for submission instruction.

1. Let X be a random variable distributed by Poisson(λ). Find $\mathbb{E}(X^4)$.

2. Let X be a random variable with an exponential distribution with density $p(x) = \lambda e^{-\lambda x}$ for all $x > 0$. Find $\mathbb{E}(X)$.

3. Let X be distributed according to a Binomial(n, p) distribution. We are interested in the probability $\mathbb{P}(X = k)$ for

   (a) $n = 7$, $p = 0.3$, $k = 3$;

   (b) $n = 40$, $p = 0.4$, $k = 11$;

   (c) $n = 400$, $p = 0.0025$, $k = 2$.

   In each of these cases determine the exact Binomial probability, an approximation using the normal distribution, and an approximation based on the Poisson distribution (you may use R or python). In each case comment on the accuracy of the approximation. If approximations are good explain why.

4. Consider $X_1, ..., X_n \overset{iid}{\sim}$ Bernoulli(p).

   (a) Find the MLE of p.

   (b) Find the mean, variance and MSE of $\widehat{p}_{\mathsf{MLE}}$. (MSE is defined by $\mathbb{E}(\widehat{p}_{\mathsf{MLE}} - p)^2$.)

   (c) Find the asymptotic distribution of $\widehat{p}_{\mathsf{MLE}}$.

   (d) Construct Wald's confidence interval with 95% confidence.

   (e) Construct Wilson's confidence interval with 95% confidence.

   (f) For Wilson's method, draw the picture that I did in the class (by hand or by R or python). For example, you may want to plot the functions $(\widehat{p}_{\mathsf{MLE}} - p)^2$ and $\frac{R^2 p(1-p)}{n}$, and explain the properties of $\widehat{p}_{\mathsf{left}}$ and $\widehat{p}_{\mathsf{right}}$.

   (g) From your picture, between $\widehat{p}_{\mathsf{right}} - \widehat{p}_{\mathsf{MLE}}$ and $\widehat{p}_{\mathsf{MLE}} - \widehat{p}_{\mathsf{left}}$, which one is larger? You may discuss the three cases separately: $\widehat{p}_{\mathsf{MLE}} < 1/2$, $\widehat{p}_{\mathsf{MLE}} = 1/2$ and $\widehat{p}_{\mathsf{MLE}} > 1/2$.