STAT 24300 - Numerical Linear Algebra
Assignment 3: Matrix Operations, RREF and Projections

Question 1: Matrix Multiplication

Compute the product $C = AB$ where A is $m \times n$, B is $n \times p$, and:

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 4 & -2 & -3 \end{bmatrix}$$

using the following three methods and show that the results are equivalent.

1. **Entry-wise:** $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ (entries are an inner product of rows of A with columns of B).

2. **Column-wise:** $C = [Ab_{:,1}, Ab_{:,2}, \ldots, Ab_{:,p}]$ where $b_{:,j}$ is the j^{th} column of B.

3. **Sum of outer products:** The outer product between two vectors u and v accepts a column vector, multiplies it by a row vector, and produces a matrix. Recall that an inner product accepts a row vector, multiplies by a column, and returns a scalar. Specifically, the outer product between a column and row vector is:

$$uv^\intercal = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \begin{bmatrix} u_1, u_2, \ldots, u_n \end{bmatrix} = \begin{bmatrix} v_1 u_1 & v_1 u_2 & \ldots & v_1 u_n \\ v_2 u_1 & v_2 u_2 & \ldots & v_2 u_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m u_1 & v_m u_2 & \ldots & v_m u_n \end{bmatrix} \tag{1}$$

   That is, $[vu^\intercal]_{ij} = v_i u_j$. Then, the outer product of two vectors is a matrix whose columns are all proportional to the column vector, and whose rows are all proportional to the row vector. It turns out that we can compute matrix matrix products as a sum of the following outer-products:

$$C = \sum_{k=1}^{n} \left(k^{th} \text{ column of } A\right) \left(k^{th} \text{ row of } B\right) = \sum_{k=1}^{n} \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1}, b_{k2}, \ldots, b_{kp} \end{bmatrix} \tag{2}$$

(Note: the matrix multiplication rule encompasses the product between row vectors $\times$ column vectors, row vectors $\times$ matrices, and matrices $\times$ column vectors, and matrices $\times$ matrices).

Question 2: Gauss-Jordan and Inverses

Let:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

1. Set up the augmented matrix $[A|I]$ where I is the three-by-three identity.

2. Row reduce (working top to bottom) the augmented matrix until A is upper triangular. Remember to apply your operations to both sides of the augmented system.

3. Scale all the rows so that the pivots (diagonal entries) are equal to 1.

4. Row reduce (working bottom to top) to eliminate all the nonzero entries above the pivots. Remember to apply your operations to both sides of the augmented system.

5. Show that the augmented block of the matrix left over from the reduction process is A^{-1}. That is, call the matrix produced by reducing I, M, then confirm that $MA = I = AM$. Given a vector b, what is the solution x such that $Ax = b$ in terms of b and A^{-1}?

6. Explain as best you can why this reduction process produces the inverse. (Please type your answer if you prefer to explain using sentences)

Question 3: Reduced Row Echelon Form

Let:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 6 \\ 2 & -2 & 0 & 6 & -8 \\ 2 & 3 & 5 & 2 & 2 \end{bmatrix}.$$

Then, the row-reduced echelon form of A, $R = \text{rref}(A)$ is a reduced form of A that comes from trying to reduce A to the identity using row operations. The idea is to use the same steps from Gauss-Jordan elimination (row reduce top to bottom, scale the rows by the pivot value, then reduce bottom to top) without augmenting the matrix, and with less ambitious objectives.

When we did row reduction in the past we aimed to arrive at an upper triangular matrix with nonzero entries on the diagonal. This asks too much of singular or rectangular matrices. Instead, let's aim for echelon form.

In echelon form, the nonzero entries of the matrix form a staircase pattern. The "steps" in the staircase are the pivots (first nonzero entries in each row). The pivots must move left to right, so that the pivot in row $j > i$ appears in column $j > i$ (all pivots beneath row i appear to the right of the pivot in row i). The pivots must also be the last nonzero entries in their columns when reading top to bottom.

Once we have reduced A to echelon form we can scale the rows as before, then cancel the columns above each pivot via row reduction working up from the bottom. The resulting matrix is $R = \text{rref}(A)$.

1. Working top to bottom, row reduce A until it is in echelon form. Do not swap rows.

2. Scale the rows so that the pivots all equal 1.

3. Working bottom to top, use row reduction to eliminate all nonzero entries above each pivot. The resulting matrix is R.

Question 4: Bases for the Null-Space and Range

Let:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 6 \\ 2 & -2 & 0 & 6 & -8 \\ 2 & 3 & 5 & 2 & 2 \end{bmatrix}.$$

Let R be the reduced row echelon form of A from question 4. Let's use R to find a basis for the null-space and range of A.

1. Identify the free columns and pivot columns in R. Count the number of free and pivot columns. What is the rank and nullity of A?

2. Remove all columns of A that correspond to free columns of R (that is, leave only the pivot columns). Call this matrix C. What fundamental subspace do the columns of C span?

3. Check that $CR = A$ if we remove the rows of R that contain only zeros.

4. Explain how the multiplication CR produces the pivot columns of A. Use your answer in part 3 to show that the nonzero entries of each free column of R specify linear combinations of the pivot columns of A that add up to the corresponding free column of A. (Please type your answer if you prefer to explain using sentences)

5. Write down the homogeneous equation $Rx = 0$. Introduce a free variable for each column. Then, solve the homogeneous equation $Rx = 0$ and express your solution as a linear combination of vectors, one for each free column, scaled by the corresponding free variable.

6. Make a matrix N storing the column vectors you found in part 5. What are the columns of N a basis for?

Question 5: Projections

Let:

$$v = \begin{bmatrix} 3/2 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

1. Carefully sketch v and w

2. Compute the length of v and w

3. Compute the projection of w onto v and add it to your sketch. Label the projection $w_{\|v}$.

4. Compute $w_{\perp v} = w - w_{\|v}$ and add it to your sketch.

5. Show that $w_{\perp v}$ is orthogonal to v.

6. Compute the length of $w_{\perp v}$ and show that $\|w_{\perp v}\|^2 + \|w_{\|v}\|^2 = \|w\|^2$.