24400 HW4

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Question 1

(a)

The probability mass function (PMF) of X_1 is given by:

$$P(X_1 = x) = \begin{cases} \frac{1}{3} & \text{if } x = 1 \\ \frac{1}{3} & \text{if } x = 2 \\ \frac{2}{9} & \text{if } x = 3 \\ \frac{1}{9} & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

The cumulative distribution function (CDF) of X_1 is:

$$F_{X_1}(x) = P(X_1 \leq x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{3} & \text{if } 1 \leq x < 2 \\ \frac{2}{3} & \text{if } 2 \leq x < 3 \\ \frac{8}{9} & \text{if } 3 \leq x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$$

Graphs of the PMF and CDF of X_1

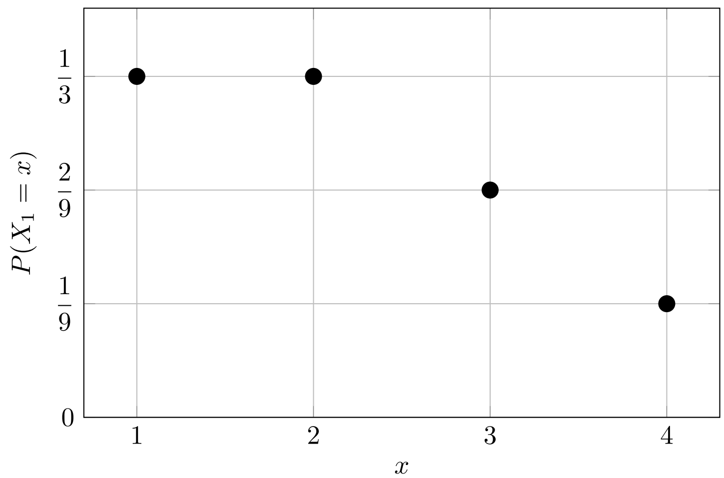

Figure 1: Probability Mass Function (PMF) of X_1

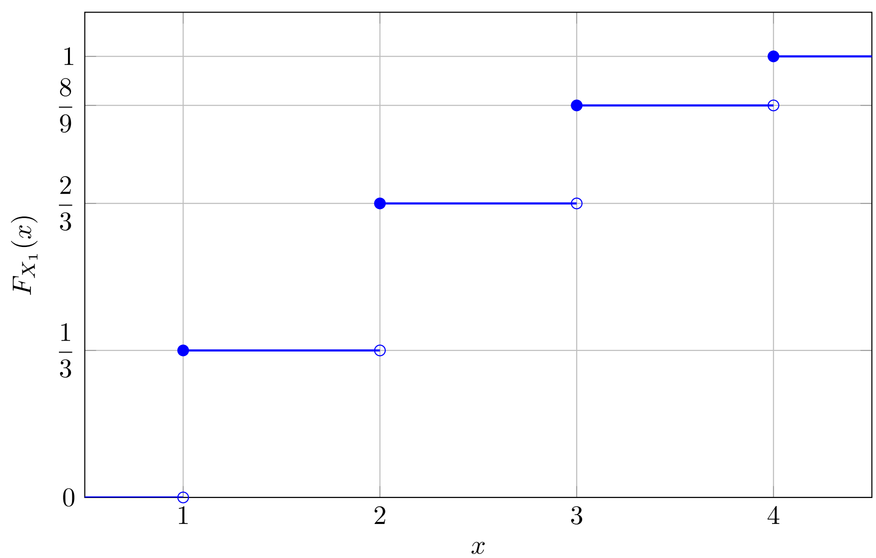

Figure 2: Cumulative Distribution Function (CDF) of X_1

(b)

Since X_1 and X_2 are independent and in same distribution, we have:

$$P(Z \geq z) = P(X_1 \geq z \text{ and } X_2 \geq z) = [P(X_1 \geq z)]^2$$

Therefore,

$$P(Z = z) = P(Z \geq z) - P(Z \geq z + 1) = [P(X_1 \geq z)]^2 - [P(X_1 \geq z + 1)]^2$$

- For $z = 1$:

$$\begin{aligned} P(Z = 1) &= P(Z \geq 1) - P(Z \geq 2) \\ &= [P(X_1 \geq 1)]^2 - [P(X_1 \geq 2)]^2 \\ &= (1)^2 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9} \end{aligned}$$

- For $z = 2$:

$$\begin{aligned}P(Z=2) &= P(Z \geq 2) - P(Z \geq 3) \\ &= \left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}\end{aligned}$$

- For $z = 3$:

$$\begin{aligned}P(Z=3) &= P(Z \geq 3) - P(Z \geq 4) \\ &= \left(\frac{1}{3}\right)^2 - \left(\frac{1}{9}\right)^2 = \frac{1}{9} - \frac{1}{81} = \frac{9}{81} - \frac{1}{81} = \frac{8}{81}\end{aligned}$$

- For $z = 4$:

$$\begin{aligned}P(Z=4) &= P(Z \geq 4) - P(Z \geq 5) \\ &= \left(\frac{1}{9}\right)^2 - 0 = \frac{1}{81}\end{aligned}$$

The PMF of Z is thus:

$$P(Z=z) = \begin{cases} \frac{5}{9} & \text{if } z = 1 \\ \frac{1}{3} & \text{if } z = 2 \\ \frac{8}{81} & \text{if } z = 3 \\ \frac{1}{81} & \text{if } z = 4 \\ 0 & \text{otherwise} \end{cases}$$

Presented in a table:

z	$P(Z = z)$
1	$\frac{5}{9}$
2	$\frac{1}{3}$
3	$\frac{8}{81}$
4	$\frac{1}{81}$

(c)

Since X_1 and X_2 are identically distributed:

$$E(X_1) = E(X_2) = \sum_x x \cdot P(X_1 = x) = \left(1 \cdot \frac{1}{3}\right) + \left(2 \cdot \frac{1}{3}\right) + \left(3 \cdot \frac{2}{9}\right) + \left(4 \cdot \frac{1}{9}\right) = \frac{19}{9}$$

Expected value of Z:

$$E(Z) = \sum_z z \cdot P(Z = z) = \left(1 \cdot \frac{5}{9}\right) + \left(2 \cdot \frac{1}{3}\right) + \left(3 \cdot \frac{8}{81}\right) + \left(4 \cdot \frac{1}{81}\right) = \frac{127}{81}$$

(d)

Since $Z = \min(X_1, X_2)$, we need to compute the conditional probabilities for each possible x and z.

For $x = 1$:

- Possible values of z: $z = 1$ (since $Z \leq x$)

$$P(Z = 1 \mid X_1 = 1) = 1$$

For $x = 2$:

- $z = 1$:

$$P(Z = 1 \mid X_1 = 2) = P(X_2 = 1) = \frac{1}{3}$$

- $z = 2$:

$$P(Z = 2 \mid X_1 = 2) = 1 - P(X_2 = 1) = \frac{2}{3}$$

For $x = 3$:

- $z = 1$:

$$P(Z = 1 \mid X_1 = 3) = P(X_2 = 1) = \frac{1}{3}$$

- $z = 2$:

$$P(Z = 2 \mid X_1 = 3) = P(X_2 = 2) = \frac{1}{3}$$

- $z = 3$:

$$P(Z = 3 \mid X_1 = 3) = 1 - P(X_2 = 1) - P(X_2 = 2) = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

For $x = 4$:

- $z = 1$:

$$P(Z = 1 \mid X_1 = 4) = P(X_2 = 1) = \frac{1}{3}$$

- $z = 2$:

$$P(Z = 2 \mid X_1 = 4) = P(X_2 = 2) = \frac{1}{3}$$

- $z = 3$:

$$P(Z = 3 \mid X_1 = 4) = P(X_2 = 3) = \frac{2}{9}$$

- $z = 4$:

$$P(Z = 4 \mid X_1 = 4) = P(X_2 = 4) = \frac{1}{9}$$

Now using the formula:

$$p(x, z) = P(X_1 = x) \times P(Z = z \mid X_1 = x)$$

For $x = 1$:

$$p(1, 1) = \frac{1}{3} \times 1 = \frac{1}{3}$$

For $x = 2$**:**

$$p(2,1) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$
$$p(2,2) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$$

For $x = 3$**:**

$$p(3,1) = \frac{2}{9} \times \frac{1}{3} = \frac{2}{27}$$
$$p(3,2) = \frac{2}{9} \times \frac{1}{3} = \frac{2}{27}$$
$$p(3,3) = \frac{2}{9} \times \frac{1}{3} = \frac{2}{27}$$

For $x = 4$**:**

$$p(4,1) = \frac{1}{9} \times \frac{1}{3} = \frac{1}{27}$$
$$p(4,2) = \frac{1}{9} \times \frac{1}{3} = \frac{1}{27}$$
$$p(4,3) = \frac{1}{9} \times \frac{2}{9} = \frac{2}{81}$$
$$p(4,4) = \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$$

Joint Probability Mass Function Table

$x \backslash z$	1	2	3	4
1	$\frac{1}{3}$	0	0	0
2	$\frac{1}{9}$	$\frac{2}{9}$	0	0
3	$\frac{2}{27}$	$\frac{2}{27}$	$\frac{2}{27}$	0
4	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{2}{81}$	$\frac{1}{81}$

To find the covariance, we first compute $E[X_1 Z]$, calculate $xzp(x,z)$ for all combinations of x and z:

For $x = 1$**:**

$$(1)(1) \cdot p(1,1) = 1 \cdot \frac{1}{3} = \frac{1}{3}$$

For $x = 2$**:**

$$(2)(1) \cdot p(2,1) = 2 \cdot \frac{1}{9} = \frac{2}{9}$$
$$(2)(2) \cdot p(2,2) = 4 \cdot \frac{2}{9} = \frac{8}{9}$$

For $x = 3$**:**

$$(3)(1) \cdot p(3,1) = 3 \cdot \frac{2}{27} = \frac{6}{27} = \frac{2}{9}$$
$$(3)(2) \cdot p(3,2) = 6 \cdot \frac{2}{27} = \frac{12}{27} = \frac{4}{9}$$
$$(3)(3) \cdot p(3,3) = 9 \cdot \frac{2}{27} = \frac{18}{27} = \frac{6}{9} = \frac{2}{3}$$

For $x = 4$:

$$\begin{aligned}
(4)(1) \cdot p(4,1) &= 4 \cdot \frac{1}{27} = \frac{4}{27} \\
(4)(2) \cdot p(4,2) &= 8 \cdot \frac{1}{27} = \frac{8}{27} \\
(4)(3) \cdot p(4,3) &= 12 \cdot \frac{2}{81} = \frac{24}{81} = \frac{8}{27} \\
(4)(4) \cdot p(4,4) &= 16 \cdot \frac{1}{81} = \frac{16}{81}
\end{aligned}$$

$$\begin{aligned}
E[X_1 Z] &= \left(\frac{1}{3}\right) + \left(\frac{2}{9} + \frac{8}{9}\right) + \left(\frac{2}{9} + \frac{4}{9} + \frac{2}{3}\right) + \left(\frac{4}{27} + \frac{8}{27} + \frac{8}{27} + \frac{16}{81}\right) \\
&= \frac{1}{3} + \frac{10}{9} + \frac{12}{9} + \left(\frac{20}{27} + \frac{16}{81}\right) \\
&= \frac{301}{81}
\end{aligned}$$

$$E[X_1]E[Z] = \frac{19}{9} \times \frac{127}{81} = \frac{19 \times 127}{9 \times 81} = \frac{2413}{729}$$

Thus,

$$\begin{aligned}
\mathrm{Cov}(X_1, Z) &= E[X_1 Z] - E[X_1]E[Z] \\
&= \frac{301}{81} - \frac{2413}{729} \\
&= \frac{2709}{729} - \frac{2413}{729} \\
&= \frac{296}{729}
\end{aligned}$$

$$\mathrm{Cov}(X_1, Z) = \frac{296}{729}$$

Question 2

Let $n = 10$ (the number of coin flips).

For positions $i = 1$ to $n - 1$, define the indicator random variables:

$$I_i = \begin{cases} 1, & \text{if the sequence at positions } i \text{ and } i+1 \text{ is HT,} \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$X = \sum_{i=1}^{n-1} I_i.$$

Since each flip is independent, we have:

$$P(I_i = 1) = P(\text{flip } i \text{ is H and flip } i+1 \text{ is T}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

$$E[I_i] = \frac{1}{4}.$$

$$\mathrm{Var}(I_i) = E[I_i^2] - [E[I_i]]^2 = E[I_i] - [E[I_i]]^2 = \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}.$$

For $i \neq j$, the covariance between I_i and I_j depends on whether the positions overlap.

When $|i-j| \geq 2$, the pairs $(i, i+1)$ and $(j, j+1)$ do not share any flips. I_i and I_j are independent:

$$\mathrm{Cov}(I_i, I_j) = 0.$$

When $|i-j| = 1$, the indicators I_i and I_{i+1} share flip $i+1$). They are not independent.

$$\mathrm{Cov}(I_i, I_{i+1}) = E[I_i I_{i+1}] - E[I_i]E[I_{i+1}].$$
$$E[I_i I_{i+1}] = P(I_i = 1, I_{i+1} = 1).$$

So we need the probability that both $I_i = 1$ and $I_{i+1} = 1$.

If $I_i = 1$, we must have H on place i and T on place $i+1$, and $I_i + 1 = 1$ only if H on place $i+1$ and T on place $i+2$, which shows that $I_i I_{i+1} = 1$ never occurs. Therefore:

$$E[I_i I_{i+1}] = 0.$$

$$\mathrm{Cov}(I_i, I_{i+1}) = 0 - \left(\frac{1}{4} \cdot \frac{1}{4}\right) = -\frac{1}{16}.$$

$$\mathrm{Var}(X) = \sum_{i=1}^{n-1} \mathrm{Var}(I_i) + 2\left(\sum_{\substack{1 \leq i < j \leq n-1 \\ |i-j|=1}} \mathrm{Cov}(I_i, I_j) + \sum_{\substack{1 \leq i < j \leq n-1 \\ |i-j|\geq 2}} \mathrm{Cov}(I_i, I_j)\right).$$

Since $\mathrm{Cov}(I_i, I_j) = 0$ when $|i-j| \geq 2$, we have:

$$\mathrm{Var}(X) = \sum_{i=1}^{n-1} \mathrm{Var}(I_i) + 2\sum_{i=1}^{n-2} \mathrm{Cov}(I_i, I_{i+1}) + 2 \sum_{\substack{1 \leq i < j \leq n-1 \\ |i-j|\geq 2}} 0.$$

Therefore:

$$\begin{aligned} \mathrm{Var}(X) &= (n-1)\,\mathrm{Var}(I_i) + 2(n-2)\,\mathrm{Cov}(I_i, I_{i+1}) \\ &= 9\left(\frac{3}{16}\right) + 2 \cdot 8\left(-\frac{1}{16}\right) \\ &= \frac{27}{16} - \frac{16}{16} = \frac{11}{16} \end{aligned}$$

Therefore,

$$\mathrm{Var}(X) = \frac{11}{16}$$

Question 3

The probability can be calculated by:

$$P(X = x \mid W = x) = \frac{P(X = x, W = x)}{P(W = x)}.$$

Since $W = x$ and $X = x$, this means $X = x$ and $Y \geq x$, Thus, we can write:

$$P(X = x, W = x) = P(X = x, Y \geq x).$$

Given that X and Y are independent, we have:

$$P(X = x, Y \geq x) = P(X = x) \cdot P(Y \geq x).$$

For a Geometric(p) random variable:

$$P(X = x) = (1 - p)^{x-1} \cdot p,$$

and $Y \geq x$ means failing x-1 times before x trials:

$$P(Y \geq x) = (1 - p)^{x-1}.$$

Therefore,

$$P(X = x, Y \geq x) = (1 - p)^{x-1} \cdot p \cdot (1 - p)^{x-1} = (1 - p)^{2x-2} \cdot p.$$

The event $W = x$ occurs if either $X = x$ and $Y \geq x$, or $Y = x$ and $X \geq x$, and subtract the case where both $X = x$ and $Y = x$ (to avoid double-counting). Thus,

$$P(W = x) = P(X = x, Y \geq x) + P(Y = x, X \geq x) - P(X = x, Y = x).$$

$$P(W = x) = (1 - p)^{2x-2} \cdot p + (1 - p)^{2x-2} \cdot p - (1 - p)^{2x-2} \cdot p^2.$$

$$= (1 - p)^{2x-2} \cdot (2p - p^2).$$

Therefore,

$$P(X = x \mid W = x) = \frac{(1 - p)^{2x-2} \cdot p}{(1 - p)^{2x-2} \cdot (2p - p^2)} = \frac{p}{2p - p^2} = \frac{1}{2 - p}$$

So,

$$P(X = x \mid W = x) = \frac{1}{2 - p}$$

Question 4

(a)

Since:

$$E(Y \mid X = x) = \int_0^1 y \cdot f_{Y|X}(y|x)\, dy$$

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}$$

The marginal density function $f_X(x)$ is given by:

$$f_X(x) = \int_0^1 f(x, y)\, dy = \int_0^1 [2x + 2y - 4xy]\, dy.$$

$$
\begin{aligned}
f_X(x) &= \int_0^1 2x\,dy + \int_0^1 2y\,dy - \int_0^1 4xy\,dy \\
&= 2x\left(y\Big|_0^1\right) + 2\left(\frac{y^2}{2}\Big|_0^1\right) - 4x\left(\frac{y^2}{2}\Big|_0^1\right) \\
&= 2x(1-0) + 2\left(\frac{1}{2}-0\right) - 4x\left(\frac{1}{2}-0\right) \\
&= 2x + 1 - 2x \\
&= 1.
\end{aligned}
$$

Therefore, for $0 \le x \le 1$, the marginal density function is $f_X(x) = 1$.

The conditional density function is:

$$
f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = f(x,y) = 2x + 2y - 4xy, \quad 0 \le y \le 1.
$$

The conditional expectation is:

$$
\begin{aligned}
E(Y \mid X = x) &= \int_0^1 y \cdot f_{Y|X}(y|x)\,dy \\
&= \int_0^1 y(2x + 2y - 4xy)\,dy \\
&= \int_0^1 (2xy + 2y^2 - 4xy^2)\,dy \\
&= 2x\int_0^1 y\,dy + 2\int_0^1 y^2\,dy - 4x\int_0^1 y^2\,dy \\
&= 2x\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) - 4x\left(\frac{1}{3}\right) \\
&= x + \frac{2}{3} - \frac{4x}{3} \\
&= \frac{2-x}{3}.
\end{aligned}
$$

Therefore,

$$
E(Y \mid X = x) = \frac{2-x}{3}.
$$

(b)

$$
\mathrm{Var}(Y \mid X = x) = E(Y^2 \mid X = x) - [E(Y \mid X = x)]^2
$$

$$
\begin{aligned}
E(Y^2 \mid X = x) &= \int_0^1 y^2 \cdot f_{Y|X}(y|x)\, dy \\
&= \int_0^1 y^2[2x + 2y - 4xy]\, dy \\
&= \int_0^1 [2xy^2 + 2y^3 - 4xy^3]\, dy \\
&= 2x \int_0^1 y^2\, dy + 2 \int_0^1 y^3\, dy - 4x \int_0^1 y^3\, dy \\
&= 2x \left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right) - 4x\left(\frac{1}{4}\right) \\
&= \frac{2x}{3} + \frac{1}{2} - x \\
&= \frac{1}{2} - \frac{x}{3}.
\end{aligned}
$$

The conditional variance:

$$
\begin{aligned}
\mathrm{Var}(Y \mid X = x) &= E(Y^2 \mid X = x) - [E(Y \mid X = x)]^2 \\
&= \left(\frac{1}{2} - \frac{x}{3}\right) - \left(\frac{2-x}{3}\right)^2 \\
&= \left(\frac{1}{2} - \frac{x}{3}\right) - \left(\frac{(2-x)^2}{9}\right) \\
&= \left(\frac{1}{2} - \frac{x}{3}\right) - \left(\frac{4 - 4x + x^2}{9}\right) \\
&= \frac{9 - 6x}{18} - \frac{8 - 8x + 2x^2}{18} \\
&= \frac{1 + 2x - 2x^2}{18}.
\end{aligned}
$$

Thus,

$$
\mathrm{Var}(Y \mid X = x) = \frac{1 + 2x - 2x^2}{18}.
$$

Question 5

(a)

i. $E(T \mid N)$

Given N, $M \sim \text{Binomial}(N, 0.5)$.

Therefore, $E(M \mid N) = 0.5 \times N$

The conditional expectation is:

$$
E(T \mid N) = E(N + M \mid N) = N + E(M \mid N) = N + N \times 0.5 = N + \frac{N}{2} = \frac{3N}{2}.
$$

$$
E(T \mid N) = \frac{3N}{2}
$$

ii. $E(T)$

Using the law of total expectation:

$$E(T) = E\left[E(T \mid N)\right] = E\left(\frac{3N}{2}\right) = \frac{3}{2}E(N).$$

Since $N \sim \text{Binomial}(n, 0.5)$:

$$E(N) = n \times 0.5 = \frac{n}{2}.$$

Therefore,

$$E(T) = \frac{3}{2} \times \frac{n}{2} = \frac{3n}{4}.$$

$$E(T) = \frac{3n}{4}$$

(b)

i. $\text{Var}(T \mid N)$

Given N, M is binomially distributed with parameters N and $p = 0.5$. Therefore,

$$\begin{aligned}
\text{Var}(T \mid N) &= \text{Var}(N + M \mid N) \\
&= \text{Var}(N \mid N) + \text{Var}(M \mid N) + 2\ \text{Cov}(N, M \mid N)
\end{aligned}$$

Since N is known when conditioned on itself, it is a constant, and thus its variance is zero.

$$\text{Var}(N \mid N) = 0.$$

Given N, N is a constant, so the covariance between a constant and any random variable is zero:

$$\text{Cov}(N, M \mid N) = 0.$$

Therefore,

$$\begin{aligned}
\text{Var}(N \mid N) + \text{Var}(M \mid N) + 2\ \text{Cov}(N, M \mid N) &= 0 + \text{Var}(M \mid N) + 2 \times 0 \\
&= \text{Var}(M \mid N) \\
&= N \times 0.5 \times (1 - 0.5) \\
&= \frac{N}{4}.
\end{aligned}$$

Therefore,

$$\text{Var}(T \mid N) = \frac{N}{4}$$

ii. $E[\text{Var}(T \mid N)]$

$$E[\text{Var}(T \mid N)] = E\left(\frac{N}{4}\right) = \frac{1}{4}E(N) = \frac{1}{4} \times \frac{n}{2} = \frac{n}{8}.$$

Thus,

$$E[\text{Var}(T \mid N)] = \frac{n}{8}$$

iii. $\text{Var}(E[T \mid N])$

We have:

$$\text{Var}(E[T \mid N]) = \text{Var}\left(\frac{3N}{2}\right) = \left(\frac{3}{2}\right)^2 \text{Var}(N) = \frac{9}{4}\,\text{Var}(N).$$

Since $N \sim \text{Binomial}(n, 0.5)$, its variance is:

$$\text{Var}(N) = n \times 0.5 \times (1 - 0.5) = n \times \frac{1}{4} = \frac{n}{4}.$$

Therefore,

$$\text{Var}(E[T \mid N]) = \frac{9}{4} \times \frac{n}{4} = \frac{9n}{16}.$$

$$\text{Var}(E[T \mid N]) = \frac{9n}{16}$$

iv. $\text{Var}(T)$

Using the law of total variance:

$$\text{Var}(T) = E[\text{Var}(T \mid N)] + \text{Var}(E[T \mid N]).$$

Substitute the results from parts (ii) and (iii):

$$\text{Var}(T) = \frac{n}{8} + \frac{9n}{16} = n\left(\frac{2}{16} + \frac{9}{16}\right) = \frac{11n}{16}.$$

$$\text{Var}(T) = \frac{11n}{16}$$

Question 6

(a)

Given:

- A and B are independent Exponential(1) random variables.

- C is independent of A and B, with $P(C = +1) = P(C = -1) = 0.5$.

- $X = A \cdot C$

- $Y = B \cdot C$

Since C is independent of A and B, and $E[C] = (+1)(0.5) + (-1)(0.5) = 0$, we have:

$$E(A) = E(Y) = 1$$

$$E(X) = E[A \cdot C] = E[A] \cdot E[C] = 1 \cdot 0 = 0$$

$$E(Y) = E[B \cdot C] = E[B] \cdot E[C] = 1 \cdot 0 = 0$$

Since $C^2 = 1$:

$$\text{Var}(X) = \text{Var}(A \cdot C) = E[(X)^2] - [E(X)]^2 = E[(A \cdot C)^2] - [E(X)]^2 = E[A^2 \cdot C^2] - 0 = E[A^2]$$

Given that for $A \sim$ Exponential(1):

$$\text{Var}(A) = \frac{1}{1^2} = 1, E[A] = 1$$

And:

$$E[A^2] = \text{Var}(A) + [E(A)]^2 = 1 + 1^2 = 2$$

Therefore:

$$\text{Var}(X) = E[A^2] = 2$$

Similarly:

$$\text{Var}(Y) = 2$$

Thus,

- $E(X) = 0$

- $E(Y) = 0$

- $\text{Var}(X) = 2$

- $\text{Var}(Y) = 2$

(b)

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = E[XY] - 0 = E[XY]$$

$$\begin{aligned} E[XY] &= E[A \cdot C \cdot B \cdot C] = E[ABC^2] = E[AB] \cdot E[C^2] \\ &= E[AB] \cdot 1 \\ &= E[AB] \end{aligned}$$

Since A and B are independent:

$$E[AB] = E[A]E[B] = 1 \cdot 1 = 1$$

Therefore:

$$\text{Cov}(X, Y) = E[XY] = 1$$

$$\text{Cov}(X, Y) = 1$$

(c)

Since $t > 0$, we first should consider the possible values of C:

If $C = 1$:

$$X = A$$
$$Y = B$$

Both X and Y are positive.

If $C = -1$:

$$X = -A$$
$$Y = -B$$

Both X and Y are negative.

$$\begin{aligned} P(X \le t, Y \le t) &= P(C = 1)P(A \le t, B \le t) + P(C = -1)P(-A \le t, -B \le t) \\ &= 0.5 \cdot P(A \le t, B \le t) + 0.5 \cdot P(-A \le t, -B \le t) \end{aligned}$$

For $t > 0$, since $A \ge 0$, $-A \le t$ always holds. Therefore:

$$P(-A \le t) = 1, \quad P(-B \le t) = 1$$

So:

$$P(-A \le t, -B \le t) = 1$$

For $P(A \leq t, B \leq t)$, since A and B are independent:

$$P(A \leq t, B \leq t) = P(A \leq t) \cdot P(B \leq t) = [1 - e^{-t}]^2$$

Therefore:

$$\begin{aligned} P(X \leq t, Y \leq t) &= 0.5[1 - e^{-t}]^2 + 0.5 \cdot 1 \\ &= 0.5[1 - 2e^{-t} + e^{-2t}] + 0.5 \\ &= 0.5 - e^{-t} + 0.5e^{-2t} + 0.5 \\ &= 1 - e^{-t} + 0.5e^{-2t} \end{aligned}$$

$P(Y \leq t)$:

$$\begin{aligned} P(Y \leq t) &= P(C = +1)P(B \leq t) + P(C = -1)P(-B \leq t) \\ &= 0.5[1 - e^{-t}] + 0.5 \cdot 1 \\ &= 0.5[1 - e^{-t} + 1] \\ &= 1 - 0.5e^{-t} \end{aligned}$$

Thus:

$$P(X \leq t \mid Y \leq t) = \frac{P(X \leq t, Y \leq t)}{P(Y \leq t)} = \frac{1 - e^{-t} + \frac{1}{2}e^{-2t}}{1 - \frac{1}{2}e^{-t}}$$

So:

$$P(X \leq t \mid Y \leq t) = \frac{1 - e^{-t} + \frac{1}{2}e^{-2t}}{1 - \frac{1}{2}e^{-t}}$$

(d)

We are given:

$$P(X \leq t \mid Y \leq t) = \frac{1 - e^{-t} + \frac{1}{2}e^{-2t}}{1 - \frac{1}{2}e^{-t}}$$

Compute the Derivative $P'(t)$

Let:

$$P(t) = \frac{N(t)}{D(t)}$$

where:

$$N(t) = 1 - e^{-t} + \frac{1}{2}e^{-2t}$$

$$D(t) = 1 - \frac{1}{2}e^{-t}$$

$$N'(t) = \frac{d}{dt}\left(1 - e^{-t} + \frac{1}{2}e^{-2t}\right) = e^{-t} - e^{-2t}$$
$$D'(t) = \frac{d}{dt}\left(1 - \frac{1}{2}e^{-t}\right) = \frac{1}{2}e^{-t}$$

Using the quotient rule:

$$P'(t) = \frac{N'(t)D(t) - N(t)D'(t)}{[D(t)]^2}$$

$$\begin{aligned}
P'(t) &= \frac{\left(e^{-t} - e^{-2t}\right)\left(1 - \frac{1}{2}e^{-t}\right) - \left(1 - e^{-t} + \frac{1}{2}e^{-2t}\right)\left(\frac{1}{2}e^{-t}\right)}{\left(1 - \frac{1}{2}e^{-t}\right)^2} \\
&= \frac{\left(e^{-t} - e^{-2t}\right)\left(1 - \frac{1}{2}e^{-t}\right) - \frac{1}{2}e^{-t}\left(1 - e^{-t} + \frac{1}{2}e^{-2t}\right)}{\left(1 - \frac{1}{2}e^{-t}\right)^2} \\
&= \frac{e^{-t}\left(1 - \frac{1}{2}e^{-t}\right) - e^{-2t}\left(1 - \frac{1}{2}e^{-t}\right) - \frac{1}{2}e^{-t}\left(1 - e^{-t} + \frac{1}{2}e^{-2t}\right)}{\left(1 - \frac{1}{2}e^{-t}\right)^2}
\end{aligned}$$

$$\begin{aligned}
\text{Numerator} &= \left[e^{-t} - \frac{1}{2}e^{-2t}\right] - \left[e^{-2t} - \frac{1}{2}e^{-3t}\right] - \left[\frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-3t}\right] \\
&= \left(e^{-t} - \frac{1}{2}e^{-2t} - e^{-2t} + \frac{1}{2}e^{-3t}\right) - \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-3t}\right) \\
&= \left(e^{-t} - \frac{3}{2}e^{-2t} + \frac{1}{2}e^{-3t}\right) - \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-3t}\right) \\
&= e^{-t} - \frac{3}{2}e^{-2t} + \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{4}e^{-3t} \\
&= \left(e^{-t} - \frac{1}{2}e^{-t}\right) + \left(-\frac{3}{2}e^{-2t} + \frac{1}{2}e^{-2t}\right) + \left(\frac{1}{2}e^{-3t} - \frac{1}{4}e^{-3t}\right) \\
&= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{4}e^{-3t}
\end{aligned}$$

Therefore, the derivative is:

$$P'(t) = \frac{\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{4}e^{-3t}}{\left(1 - \frac{1}{2}e^{-t}\right)^2}$$

setting $P'(t) = 0$

$$\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{4}e^{-3t} = 0$$

$$\begin{aligned} 4e^{3t}\left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{4}e^{-3t}\right) &= 0 \\ \Rightarrow 4e^{3t}\left(\frac{1}{2}e^{-t}\right) - 4e^{3t}\left(e^{-2t}\right) + 4e^{3t}\left(\frac{1}{4}e^{-3t}\right) &= 0 \\ \Rightarrow 2e^{2t} - 4e^{t} + 1 &= 0 \end{aligned}$$

Let $u = e^t$ (since $e^t > 0$):

$$2u^2 - 4u + 1 = 0$$

$$u = \frac{4 \pm \sqrt{16-8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = 1 \pm \frac{\sqrt{2}}{2}$$

- $u = 1 + \frac{\sqrt{2}}{2} \implies t = \ln\left(1 + \frac{\sqrt{2}}{2}\right) \approx 0.531$
- $u = 1 - \frac{\sqrt{2}}{2} \implies t = \ln\left(1 - \frac{\sqrt{2}}{2}\right)$ (invalid since $u > 0$ but $1 - \frac{\sqrt{2}}{2} < 0$)

Therefore,

$$t^* = \ln\left(1 + \frac{\sqrt{2}}{2}\right) \approx 0.531$$

Now, evaluate $P(t)$ at key points:

- As $t \to 0^+$:
$$P(0^+) = \frac{1 - 1 + \frac{1}{2}\cdot 1}{1 - \frac{1}{2}\cdot 1} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$
- At $t = t^* \approx 0.531$:
$$P(t^*) = \frac{1 - e^{-t^*} + \frac{1}{2}e^{-2t^*}}{1 - \frac{1}{2}e^{-t^*}} \approx 0.827$$
- As $t \to \infty$:
$$P(\infty) = \lim_{t\to\infty} \frac{1 - e^{-t} + \frac{1}{2}e^{-2t}}{1 - \frac{1}{2}e^{-t}} = \frac{1 - 0 + 0}{1 - 0} = 1$$

Since $P(t)$ decreases from 1 to approximately 0.827 at $t \approx 0.531$, and then increases back to 1, the critical point corresponds to a **minimum**.

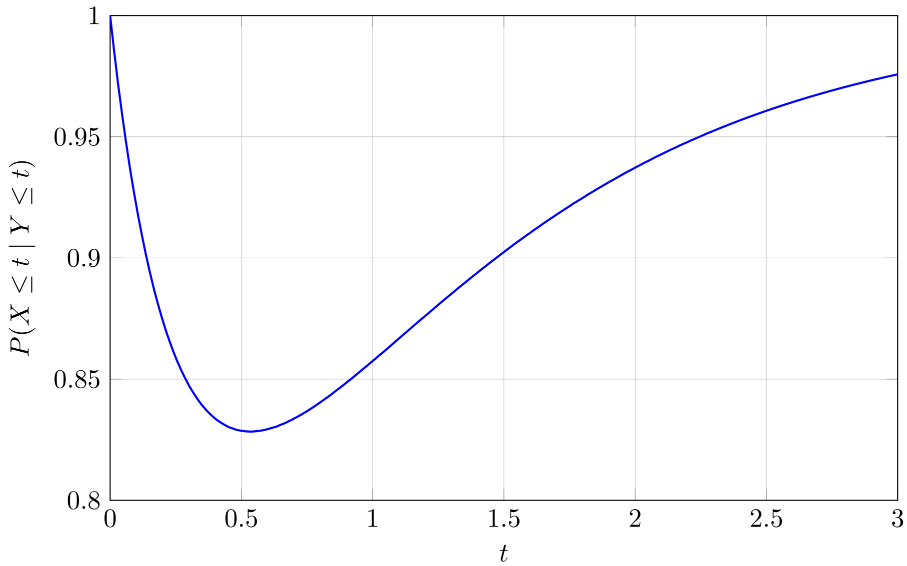

Figure 3: Plot of $P(X \leq t \mid Y \leq t)$ vs. t

Interpretation of $P(X \leq t \mid Y \leq t)$

1. **When $t \to 0^+$:**

   - **Observation:** $P(X \leq t \mid Y \leq t) \to 1$.
   - **Explanation:** For very small positive t, both X and Y are likely to be only negative (which is always true when $C = -1$, due to the Exponential distributions will only have positive numbers and shared random sign C. The conditional probability is close to 1 because if $Y \leq t$ (i.e., Y is very small or negative), it's almost certain that $X \leq t$ as well due to the shared sign.

2. **At $t \approx 0.531$:**

   - **Observation:** $P(X \leq t \mid Y \leq t)$ reaches a minimum value of approximately 0.827.
   - **Explanation:** At this value of t, there's the lowest likelihood that $X \leq t$ given $Y \leq t$, which indicates the point where the dependence between X and Y due to the shared random sign C has the least effect on the conditional probability.

3. **When $t \to \infty$:**

   - **Observation:** $P(X \leq t \mid Y \leq t) \to 1$.
   - **Explanation:** For large t, both X and Y are likely to be less than t, regardless of the value of C. Given $Y \leq t$ (which is almost certain for large t), the probability that $X \leq t$ is also almost certain.