Homework 3 Solutions

1-(a). The log-likelihood is

$$\ell(\mu, \sigma^2) = -\frac{n}{2}\log(\sigma^2) - \sum_{i=1}^{n}\frac{(X_i-\mu)^2}{2\sigma^2} - \frac{n}{2}\log(2\pi).$$

By solving $\frac{\partial \ell}{\partial \mu} = 0$, we have $\hat{\mu} = \bar{X}$. By solving $\frac{\partial \ell}{\partial \sigma^2} = 0$, we have

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Hence, $\hat{\sigma}_c^2$ with $c = \frac{1}{n}$ yields the MLE.

1-(b). As $\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$, we have

$$\mathbb{E}\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \sigma^2(n-1) \quad \text{and} \quad \mathrm{Var}\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = 2\sigma^4(n-1).$$

Hence, $\mathbb{E}\hat{\sigma}_c^2 = c\sigma^2(n-1)$ and $\mathrm{Var}(\hat{\sigma}_c^2) = 2c^2\sigma^4(n-1)$. Hence, $\hat{\sigma}_c^2$ with $c = \frac{1}{n-1}$ is unbiased for σ^2.

1-(c). The MSE is

$$\mathbb{E}[(\hat{\sigma}_c^2 - \sigma^2)^2] = \mathrm{Var}(\hat{\sigma}_c^2) + (\mathbb{E}\hat{\sigma}_c^2 - \sigma^2)^2 = 2c^2\sigma^4(n-1) + \sigma^4(c(n-1)-1)^2.$$

Note that

$$2(n-1)c^2 + (c(n-1)-1)^2 = (n^2-1)c^2 - 2(n-1)c + 1,$$

which is minimized by $c = \frac{1}{n+1}$.

2. Observe that

$$\begin{aligned}\mathbb{P}\left(\frac{Z_1}{|Z_2|} \le t\right) &= \mathbb{P}\left(\frac{Z_1}{|Z_2|} \le t \,|\, Z_2 > 0\right)\mathbb{P}(Z_2 > 0) + \mathbb{P}\left(\frac{Z_1}{|Z_2|} \le t \,|\, Z_2 < 0\right)\mathbb{P}(Z_2 < 0)\\ &= \mathbb{P}\left(\frac{Z_1}{Z_2} \le t \,|\, Z_2 > 0\right)\mathbb{P}(Z_2 > 0) + \mathbb{P}\left(-\frac{Z_1}{Z_2} \le t \,|\, Z_2 < 0\right)\mathbb{P}(Z_2 < 0).\end{aligned}$$

By the independence and symmetry of Z_1 and Z_2, notice that the conditional distribution of $\frac{Z_1}{Z_2}$ given Z_2 must coincide with that of $-\frac{Z_1}{Z_2}$ given Z_2, and thus

$$\mathbb{P}\left(-\frac{Z_1}{Z_2} \le t \,|\, Z_2 < 0\right) = \mathbb{P}\left(\frac{Z_1}{Z_2} \le t \,|\, Z_2 < 0\right).$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(\frac{Z_1}{|Z_2|} \le t\right) &= \mathbb{P}\left(\frac{Z_1}{Z_2} \le t \,|\, Z_2 > 0\right)\mathbb{P}(Z_2 > 0) + \mathbb{P}\left(\frac{Z_1}{Z_2} \le t \,|\, Z_2 < 0\right)\mathbb{P}(Z_2 < 0)\\ &= \mathbb{P}\left(\frac{Z_1}{Z_2} \le t\right).\end{aligned}$$

Hence, $\frac{Z_1}{Z_2}$ and $\frac{Z_1}{|Z_2|}$ have the same distribution, and thus $\frac{Z_1}{Z_2}$ is also distributed by Cauchy.

3. By definition,

$$\mathrm{Cov}(AX, BY) = \mathbb{E}[(AX - \mathbb{E}[AX])(BY - \mathbb{E}[BY])^T].$$

By the linearity of the expectation, we have

$$\begin{aligned}(AX - \mathbb{E}[AX])(BY - \mathbb{E}[BY])^T &= A(X - \mathbb{E}X)(B(Y - \mathbb{E}Y))^T \\ &= A(X - \mathbb{E}X)(Y - \mathbb{E}Y)^T B^T.\end{aligned}$$

By the linearity of the expectation again,

$$\mathrm{Cov}(AX, BY) = A\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)^T]B^T = A\mathrm{Cov}(X, Y)B^T.$$

4-(a). For $t \geq 0$, we have $z^2 \leq t$ if and only if $-\sqrt{t} \leq z \leq \sqrt{t}$. Hence,

$$\mathbb{P}(Z^2 \leq t) = \mathbb{P}(-\sqrt{t} \leq Z \leq \sqrt{t}) = 2\mathbb{P}(0 \leq Z \leq \sqrt{t}),$$

where the last equality follows from the symmetry of $Z \sim N(0, 1)$. Since

$$\mathbb{P}(0 \leq Z \leq \sqrt{t}) = \mathbb{P}(Z \leq \sqrt{t}) - \mathbb{P}(Z < 0) = \mathbb{P}(Z \leq \sqrt{t}) - \frac{1}{2},$$

we have

$$\mathbb{P}(Z^2 \leq t) = 2\mathbb{P}(Z \leq \sqrt{t}) - 1.$$

4-(b). Let Φ be the cumulative distribution function of $Z \sim N(0, 1)$ so that $\Phi' = \phi$. By the above derivation,

$$\mathbb{P}(Z^2 \leq t) = 2\Phi(\sqrt{t}) - 1.$$

Hence,

$$\frac{d}{dt}\mathbb{P}(Z^2 \leq t) = 2\frac{d}{dt}\Phi(\sqrt{t}) = 2\phi(\sqrt{t})\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{2\pi t}}e^{-\frac{t}{2}}.$$

Compare with https://en.wikipedia.org/wiki/Chi-squared_distribution.

5-(a). By definition,

$$P^2 = \frac{1}{n^2}\mathbb{1}_n\mathbb{1}_n^T\mathbb{1}_n\mathbb{1}_n^T = \frac{n}{n^2}\mathbb{1}_n\mathbb{1}_n^T = P.$$

Accordingly,

$$(I_n - P)^2 = I_n - 2P + P^2 = I_n - P.$$

Also,

$$(I_n - P)P = P - P^2 = 0.$$

5-(b). Note that

$$PZ = \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^T Z = \bar{Z}\mathbb{1}_n.$$

Since $\sum_{i=1}^n (Z_i - \bar{Z})^2$ is the squared norm of the difference of the two vectors Z and $\bar{Z}\mathbb{1}_n$, we have

$$\sum_{i=1}^n (Z_i - \bar{Z}) = \|Z - \bar{Z}\mathbb{1}_n\|^2 = \|(I_n - P)Z\|^2.$$

Also, from $PZ = \bar{Z}\mathbb{1}_n$, we have

$$\frac{1}{n}\mathbb{1}_n^T PZ = \frac{\bar{Z}}{n}\mathbb{1}_n^T\mathbb{1}_n = \bar{Z}.$$

5-(c). By Q3, we have

$$\mathrm{Cov}((I_n - P)Z, PZ) = (I_n - P)\mathrm{Cov}(Z, Z)P^T = (I_n - P)I_nP = (I_n - P)P = 0,$$

where we use $\mathrm{Cov}(Z, Z) = I_n$ and $P = P^T$.