Intro to discrete random variables (part 1)

Lecture 2a (STAT 24400 F24)

Intro to random variables - Definition

After observing the outcome of an random experiment or a random process, we might want to quantify some aspect of what we have observed.

Formally, a **random variable** (r. v.) is a function from the sample space Ω to the real numbers. It assigns a numerical value to each possible outcome.

$$X : \quad \Omega \quad \longrightarrow \quad \mathbb{R} = (-\infty, \infty)$$

Example of r.v. on simple finite sample space

Example

Roll a fair die. If it's a 6 you win \$10, otherwise lose \$1.

Define random variable X to be the net gain.

- Sample space $\Omega = \{\underbrace{1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ 5}_{\text{maps to } X=-1}, \underbrace{6}_{\text{maps to } X=10}\}$
- Since it's a fair die, we know the probability measure: Each outcome has probability $\frac{1}{6}$.

Outcome	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
X	-1	-1	-1	-1	-1	10

Possible values of X are -1 and 10.

We can calculate $\mathbb{P}(X = -1) = \frac{5}{6}$ and $\mathbb{P}(X = 10) = \frac{1}{6}$, the distribution of X.

Example of r.v. on sample space of short-sequences

Example

Record sequence of three rolls of a fair die. Let $X = \#$ of 2's in the sequence.

- Sample space $\Omega = \{\ \underbrace{1{,}1{,}1}_{\text{maps to } X=0}\ ,\ \underbrace{1{,}1{,}2}_{\text{maps to } X=1}\ ,\ \underbrace{1{,}1{,}3}_{\text{maps to } X=0}\ ,\ \ldots.\ ,\ \underbrace{6{,}6{,}6}_{\text{maps to } X=0}\ \}$
- Since it's a fair die, we know the probability distribution: Each outcome has probability $\frac{1}{6^3}$

Outcome	1,1,1	1,1,2	1,1,3	...
Probability	$\frac{1}{6^3}$	$\frac{1}{6^3}$	$\frac{1}{6^3}$	...
X	0	1	0	...

Possible values of X are 0, 1, 2, and 3.

Using the given probability measure on Ω,
we can calculate $\mathbb{P}(X = x)$ for each $x = 0, 1, 2, 3$ (exercise).

Probability laws from sample space to r.v.'s

From probability measure on Ω to probability distribution of r.v. X

For a random variable X, we might be interested in probabilities that it takes a certain value or lies in a certain range.

For example, if we ask

"What is the probability that X is higher than 1?", written as $\mathbb{P}(X > 1)$,

we can equivalently consider the event

$$A = \{\text{all outcomes in } \Omega \text{ for which } X \text{ is higher than } 1\},$$

which is a subset of Ω.

Given a probability measure on Ω, we can then calculate $\mathbb{P}(A)$, which leads to $\mathbb{P}(X > 1)$.

Discrete random variables

A **discrete random variable** is a random variable with finitely many, or countably-infinitely many, possible values.

- **Probability function**: (probability mass function, PMF, pmf, or frequency function)

  The probability that X will be equal to some particular value x

  $$p(x) = \mathbb{P}(X = x)$$

  Sometimes written as $p_X(\cdot)$ rather than $p(\cdot)$ to emphasize it's the distribution of X.

- **Cumulative distribution function** (CDF, or cdf):

  The probability that X will be less than or equal to some particular value x

  $$F(x) = \mathbb{P}(X \le x)$$

  Sometimes written as $F_X(\cdot)$ rather than $F(\cdot)$ to emphasize it's the distribution of X.

Discrete distributions

For a random variable X,

what does it mean to ask "what is the **distribution** of X"?

We want sufficient information to characterize the random behavior of X:

For a discrete r.v., this can be the PMF, or CDF, or some other description, that completely quantify the probability pattern of the outcome.

The distributions of common r.v.'s are given designated names (with parameters).

Example of PMF and conditional probability

Example Suppose you have four 6-sided dice:
one has 1 red side, one has 2 red sides, one has 3 red sides, and one has 4 red sides.

You choose one die at random, and roll two times.

Q1. What is the distribution of the # of times you roll a red?

Let $X = \#$ red rolls. Possible values are 0, 1, 2.
By the law of total probability,

$$\mathbb{P}(X = x) = \sum_{i=1}^{4} \mathbb{P}(\mathrm{die}_i)\, \mathbb{P}(X = x \mid \mathrm{die}_i)$$

- $\mathbb{P}(X = 2) = \frac{1}{4} \cdot \left(\frac{1}{6}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{6}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{6}\right)^2 + \frac{1}{4} \cdot \left(\frac{4}{6}\right)^2 = \frac{5}{24} = 0.208$
- $\mathbb{P}(X = 0) = \frac{1}{4} \cdot \left(\frac{5}{6}\right)^2 + \frac{1}{4} \cdot \left(\frac{4}{6}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{6}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{6}\right)^2 = \frac{9}{24} = 0.375$
- $\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 2) - \mathbb{P}(X = 0) = \frac{10}{24} = 0.417$
  which is the same as calculating the probability of $X = 1$ directly (exercise).

Example of PMF and conditional probability (cont.)

Q2. What is the probability that you chose the die with 4 red sides, given that you rolled two reds?

Let A be the event that we chose the die with 4 red sides,
B be the event that we rolled two reds.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A) \cdot \mathbb{P}(B \mid A)}{\mathbb{P}(B)} = \frac{\frac{1}{4} \cdot \left(\frac{4}{6}\right)^2}{\frac{5}{24}} = \frac{8}{15} = 0.533$$

Q3. What is the probability that you chose the die with 1 red side, given that you rolled two reds?

Let C be the event that we chose the die with 1 red side,

$$\mathbb{P}(C \mid B) = \frac{\mathbb{P}(C \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(C) \cdot \mathbb{P}(B \mid C)}{\mathbb{P}(B)} = \frac{\frac{1}{4} \cdot \left(\frac{1}{6}\right)^2}{\frac{5}{24}} = \frac{1}{8 \times 15} = 0.033$$

Remarks

The example shows

- The usefulness of
  - conditional probabilities
  - the law of total probability (Q1)
  - Bayes' formula (Q2, Q3)
- The relationship between values of the random variable and the outcomes/events in the sample space Ω
- The transfer of the probability measure on Ω to the probability distribution of the random variable
- The implicit existence of underlying sample space for any random variable (always there)

Equivalence of discrete probability distributions

If X and Y are discrete random variables that have the same PMF, i.e.,

$$\mathbb{P}(X = x) = \mathbb{P}(Y = x) \quad \text{for any value } x \in \mathbb{R}$$

we say that X and Y have the same distribution.

X and Y then share the same probabilistic properties:
same CDF, same average, etc.

Comments on r.v.'s with the same distribution

Caution:

"X and Y have the same distribution" is not the same as "$X = Y$"!

Example: Roll a fair die 3 times. Let $X = \#$ of 1's and $Y = \#$ of 6's.

- Then X and Y have the same distribution, and
  X and Y have the sample possible values.
  For example, $\mathbb{P}(X = 3) = \mathbb{P}(Y = 3) = \frac{1}{6^3}$
- But it's not true that $X = Y$

Remark

It is possible that X and Y have the same distribution even if they're not defined on the same sample space.

Example: X = number of evens after 10 rolls of a fair die
Y = number of Heads after 10 flips of a fair coin