Stat 301

Def: $T = T(X)$ is sufficient iff $X \mid T$ does not depend on $\theta \in \Theta$

Def: S is minimal sufficient iff for every sufficient T,
S is a function of T. and S is sufficient.

① sub-family method.

Lemma: suppose $\Theta_0 \subseteq \Theta_1$, S is minimal sufficient for Θ_0, and sufficient for Θ_1, it is also minimal sufficient for Θ_1.

Theorem: for $(P_\theta : \theta \in \{\theta_0, \theta_1, \cdots, \theta_d\})$ with common support.

$$T(X) = \left(\frac{p_{\theta_1}}{p_{\theta_0}}(X), \cdots, \frac{p_{\theta_d}}{p_{\theta_0}}(X) \right)$$ is minimal sufficient.

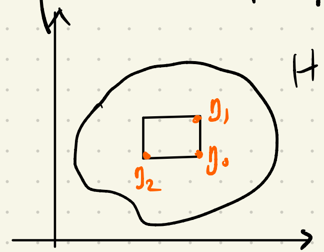
Theorem: minimal exponential family $\exp(\langle \eta, T(x)\rangle - A(\eta))\, h(x)$.

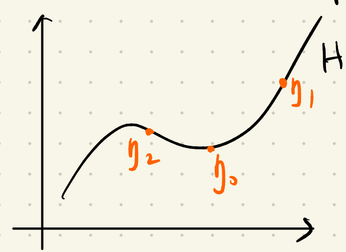
$\eta \in H \subseteq \mathbb{R}^d$, then $T(x) = (T_1(x), \dots, T_d(x))$ is minimal sufficient.

Proof: since exp. family is minimal, can find $\eta_0, \eta_1, \dots, \eta_d \in H$

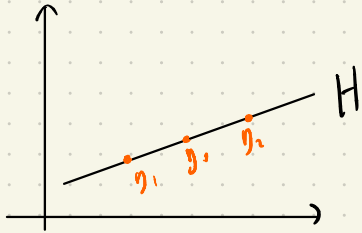
s.t. $\begin{pmatrix} (\eta_1 - \eta_0)^T \\ (\eta_2 - \eta_0)^T \\ \vdots \\ (\eta_d - \eta_0)^T \end{pmatrix} \in \mathbb{R}^{d\times d}$. has full rank.

Illustration of $d=2$. ① full rank exp. family ② curved exp. family





③ non-minimal exp. family



a sub family $\{\eta_0, \eta_1, \cdots, \eta_d\} \subseteq H$.

minimal sufficient statistic $\dfrac{p(x|\eta_j)}{p(x|\eta_0)}$, $j = 1, \cdots, d$.

$$\frac{p(x|\eta_j)}{p(x|\eta_0)} = \frac{\exp(\langle \eta_j, T(x)\rangle - A(\eta_j))}{\exp(\langle \eta_0, T(x)\rangle - A(\eta_0))} = \exp\left(\langle \eta_j - \eta_0, T(x)\rangle - A(\eta_j) + A(\eta_0)\right)$$

equivalent to

$$\begin{pmatrix} \langle \eta_1 - \eta_0, T(x)\rangle \\ \langle \eta_2 - \eta_0, T(x)\rangle \\ \vdots \\ \langle \eta_d - \eta_0, T(x)\rangle \end{pmatrix} = \begin{pmatrix} (\eta_1 - \eta_0)^T \\ (\eta_2 - \eta_0)^T \\ \vdots \\ (\eta_d - \eta_0)^T \end{pmatrix} T(x)$$

equivalent to $T(X) = \begin{pmatrix} T_1(x) \\ \vdots \\ T_d(x) \end{pmatrix}$, which is minimal sufficient $\square$

② completeness method. (removing all ancillary information)

e.g. $X_1, X_2 \overset{iid}{\sim} N(\theta, 1)$.

$T = (X_1, X_2)$ is sufficient but not minimal.

equivalent to $(\underbrace{X_1 - X_2}, X_1 + X_2)$.

$X_1 - X_2 \sim N(0, 2)$.

ancillary.

Def: $A = A(X)$ is ancillary iff its distribution does not depend on $\theta \in \Theta$, is first-order ancillary iff its expectation ($\mathbb{E}_\theta A(X)$)

does not depend on $\theta \in \Theta$.

Def: $T = T(X)$ is complete iff $\mathbb{E}_\theta f(T(X)) = 0 \ \forall \theta \in \Theta$

implies $f(T(X)) = 0$ a.s. $\forall \theta \in \Theta$

$$\left(\mathbb{P}_\theta \left(f(T(X)) = 0 \right) = 1, \ \forall \theta \in \Theta \right).$$

in words, no non-constant function of T is first-order ancillary.

$$\mathbb{E}_\theta f(T(X)) = c \Rightarrow f(T(X)) = c.$$

$$\mathbb{E}_\theta \left(f(T(X)) - c \right) \Rightarrow f(T(X)) - c = 0.$$

Theorem (Bahadur): T is sufficient & complete.

$\Rightarrow$ T is minimal sufficient.

Proof (sketch): assume minimal suff. statistic exists $U = U(X)$.

by def. $U = h(T)$.

want to show T is also a function of U.

define $g(u) = \mathbb{E}_\theta(T \mid U = u)$ is a function independent of θ

by sufficiency of U.

$$\mathbb{E}_\theta g(h(T)) = \mathbb{E}_\theta\, g(U) = \mathbb{E}_\theta\left(\mathbb{E}_\theta(T \mid U)\right) = \mathbb{E}_\theta T$$

$$\Rightarrow \mathbb{E}_\theta\left(g(h(T)) - T\right) = 0 \quad \forall \theta \in \Theta.$$

by completeness $g(h(T)) = T$ a.s. $\Rightarrow$ $g(U) = T$ a.s. $\square$

e.g. $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$. $\quad T = \sum_{i=1}^n X_i \sim \text{Binomial}(n, \theta)$.

suppose $\mathbb{E}_\theta f(T(X)) = 0$.

$$= \sum_{i=1}^n f(i) \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

$$= \sum_{i=1}^n f(i) \binom{n}{i} \left(\frac{\theta}{1-\theta}\right)^i (1-\theta)^n = 0 \quad \forall \theta \in (0,1)$$

$$\Rightarrow \quad \sum_{i=1}^n f(i) \binom{n}{i} \left(\frac{\theta}{1-\theta}\right)^i = 0 \quad \forall \theta \in (0,1)$$

Set $\beta = \frac{\theta}{1-\theta}$ $\qquad \sum_{i=1}^n f(i) \binom{n}{i} \beta^i = 0 \quad \forall \beta > 0$.

a degree-n polynomial has at most n roots.

$$\Rightarrow \quad f(i) \binom{n}{i} = 0 \quad \forall i = 1, \ldots, n \quad \Rightarrow \quad f(i) = 0 \quad \forall i = 1, \ldots, n.$$

T is complete. $\square$

e.g. $X_1, \cdots, X_n \overset{iid}{\sim} \mathrm{Unif}(0, \theta)$. $T = \max_{1 \le i \le n} X_i$

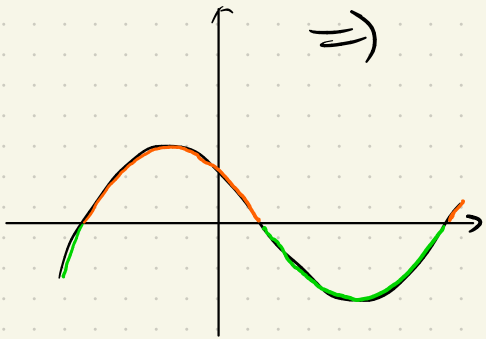
$$P(T \le t) = \prod_{i=1}^{n} P(X_i \le t) = \left(\frac{t}{\theta}\right)^n, \quad t \in (0, \theta).$$

$$p(t|\theta) = \frac{d}{dt} P(T \le t) = \theta^{-n} n t^{n-1}, \quad t \in (0, \theta).$$

suppose $E_\theta f(T(X)) = 0. \quad \forall \theta > 0.$

$$\Rightarrow \int_0^\theta f(t) \theta^{-n} n t^{n-1} dt = 0.$$

$$\Rightarrow \int_0^\theta t^{n-1} f(t) dt = 0 \quad \forall \theta > 0.$$



$f^+ = \max(f, 0)$

$f^- = \max(-f, 0)$

$f = f^+ - f^-$

$$\Rightarrow \quad \int_0^\theta t^{n-1} f^+(t)\,dt = \int_0^\theta t^{n-1} f^-(t)\,dt \quad \forall\, \theta > 0.$$

$$\Rightarrow \quad \int_{\theta_1}^{\theta_2} t^{n-1} f^+(t)\,dt = \int_{\theta_1}^{\theta_2} t^{n-1} f^-(t)\,dt \quad \forall\, 0 < \theta_1 < \theta_2.$$

$$\Rightarrow \quad \int_A t^{n-1} f^+(t)\,dt = \int_A t^{n-1} f^-(t)\,dt \quad \forall \text{ Borel } A.$$

$$\Rightarrow \quad t^{n-1} f^+(t) = t^{n-1} f^-(t). \quad \text{a.e.}$$

$$\Rightarrow \quad f = 0. \quad \text{a.e.}$$

$\Rightarrow$ T is complete.

e.g. $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, 1)$ $\quad T = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \sim N(\theta, 1)$.

suppose $E_\theta f(T(X)) = 0$, $\quad \forall\, \theta \in \mathbb{R}$.

$$\Rightarrow \int f(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}dx = 0. \qquad e^{-\frac{1}{2}x^2+x\theta}e^{-\frac{1}{2}\theta^2}.$$

$$\Rightarrow \int f(x)e^{-\frac{1}{2}x^2+x\theta}dx = 0 \qquad \forall\theta\in\mathbb{R}.$$

$$\Rightarrow f = f^+ - f^-, \qquad \int f^+(x)e^{-\frac{1}{2}x^2+x\theta}dx = \int f^-(x)e^{-\frac{1}{2}x^2+x\theta}dx.$$

$$\Rightarrow \text{take } \theta = 0, \qquad \int f^+(x)e^{-\frac{1}{2}x^2}dx = \int f^-(x)e^{-\frac{1}{2}x^2}dx.$$

$$\Rightarrow \frac{\int f^+(x)e^{-\frac{1}{2}x^2}e^{\theta x}dx}{\int f^+(x)e^{-\frac{1}{2}x^2}dx} = \frac{\int f^-(x)e^{-\frac{1}{2}x^2}e^{\theta x}dx}{\int f^-(x)e^{-\frac{1}{2}x^2}dx}.$$

(MGF).

$$\Rightarrow f^+ = f^- \text{ a.e.} \Rightarrow f = 0 \text{ a.e.} \Rightarrow T \text{ is complete.}$$

e.g. for full rank exponential family.

$$e^{\sum_{j=1}^{d} \eta_j T_j(x) - A(\eta)} h(x) \quad h \in H, \qquad T = (T_1(x), \cdots, T_d(x)) \text{ is complete.}$$

Theorem (Basu): T is completed sufficient.

A is ancillary $\Rightarrow$ $T \perp\!\!\!\perp A$.

Proof: want to show $\mathbb{P}_\theta(A \in B \mid T = t) = \mathbb{P}_\theta(A \in B)$. $\forall t$.

set $c = \mathbb{P}_\theta(A \in B)$ (does not depend on θ, b/c A is ancillary)

$g(t) = \mathbb{P}_\theta(A \in B \mid T = t)$ (does not depend on θ, b/c T is sufficient)

$$\mathbb{E}_\theta(g(T) - c) = \mathbb{E}_\theta \mathbb{P}_\theta(A \in B \mid T) - \mathbb{P}_\theta(A \in B)$$

$$= \mathbb{P}_\theta(A \in B) - \mathbb{P}_\theta(A \in B) = 0 \quad \forall \theta \in \Theta.$$

$\Rightarrow$ by completeness $g(t) = c$ a.s. $\square$

e.g. $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$.

$$\bar{X} \perp\!\!\!\perp \sum_{i=1}^{n} (X_i - \bar{X})^2.$$