# 24300 HW4

Bin Yu

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## Question 1: Projection Matrices

### (1)

$$\mathbf{v} = \begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$P_{\|\mathbf{v}} = \frac{\mathbf{v}\mathbf{v}^\top}{\|\mathbf{v}\|^2}$$

First, compute the outer product $\mathbf{v}\mathbf{v}^\top$:

$$\mathbf{v}\mathbf{v}^\top = \begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 2 \end{pmatrix} = \begin{pmatrix} \left(\frac{3}{2}\right)^2 & \frac{3}{2} \times 2 \\ 2 \times \frac{3}{2} & 2^2 \end{pmatrix} = \begin{pmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{pmatrix}$$

$$\|\mathbf{v}\|^2 = \left(\frac{3}{2}\right)^2 + 2^2 = \frac{9}{4} + 4 = \frac{25}{4}$$

Thus, the orthogonal projection matrix is:

$$P_{\|\mathbf{v}} = \frac{1}{\frac{25}{4}} \begin{pmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{pmatrix} = \frac{4}{25} \begin{pmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

To verify $P_{\|\mathbf{v}}$ is an Orthogonal Projector, we need to verify two properties:

$P_{\|\mathbf{v}}^2 = P_{\|\mathbf{v}}$

$P_{\|\mathbf{v}}^\top = P_{\|\mathbf{v}}$

$$P_{\|\mathbf{v}}^2 = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{25} \times \frac{9}{25} + \frac{12}{25} \times \frac{12}{25} & \frac{9}{25} \times \frac{12}{25} + \frac{12}{25} \times \frac{16}{25} \\ \frac{12}{25} \times \frac{9}{25} + \frac{16}{25} \times \frac{12}{25} & \frac{12}{25} \times \frac{12}{25} + \frac{16}{25} \times \frac{16}{25} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{81}{625} + \frac{144}{625} & \frac{108}{625} + \frac{192}{625} \\ \frac{108}{625} + \frac{192}{625} & \frac{144}{625} + \frac{256}{625} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{225}{625} & \frac{300}{625} \\ \frac{300}{625} & \frac{400}{625} \end{pmatrix} = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} = P_{\|\mathbf{v}}$$

Thus, $P_{\|\mathbf{v}}^2 = P_{\|\mathbf{v}}$.

Also,

$$P_{\|\mathbf{v}}^\top = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}^\top = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} = P_{\|\mathbf{v}}$$

Thus, $P_{\|\mathbf{v}}^\top = P_{\|\mathbf{v}}$.

**Conclusion:**

$$P_{\|\mathbf{v}} = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$

Since $P_{\|\mathbf{v}}^2 = P_{\|\mathbf{v}}$ and $P_{\|\mathbf{v}}^\top = P_{\|\mathbf{v}}$, $P_{\|\mathbf{v}}$ is an orthogonal projection matrix.

## (2)

The projection of $\mathbf{w}$ onto $\mathbf{v}$ is given by:

$$\mathbf{w}_\| \mathbf{v} = P_{\|\mathbf{v}} \mathbf{w} = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{25} \times (-1) + \frac{12}{25} \times 3 \\ \frac{12}{25} \times (-1) + \frac{16}{25} \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{9}{25} + \frac{36}{25} \\ -\frac{12}{25} + \frac{48}{25} \end{pmatrix} = \begin{pmatrix} \frac{27}{25} \\ \frac{36}{25} \end{pmatrix}$$

$$\mathbf{w}_\perp \mathbf{v} = \mathbf{w} - \mathbf{w}_\| \mathbf{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{27}{25} \\ \frac{36}{25} \end{pmatrix} = \begin{pmatrix} -1 - \frac{27}{25} \\ 3 - \frac{36}{25} \end{pmatrix} = \begin{pmatrix} -\frac{52}{25} \\ \frac{39}{25} \end{pmatrix}$$

The orthogonal projection matrix $P$ onto the subspace orthogonal to $\mathbf{v}$ can be constructed using $\mathbf{w}_\perp$ as follows:

$$P = \frac{\mathbf{w}_\perp \mathbf{w}_\perp^\top}{\|\mathbf{w}_\perp\|^2}$$

$$\|\mathbf{w}_\perp\|^2 = \left(-\frac{52}{25}\right)^2 + \left(\frac{39}{25}\right)^2 = \frac{2704}{625} + \frac{1521}{625} = \frac{4225}{625} = \frac{169}{25}$$

$$\mathbf{w}_\perp \mathbf{w}_\perp^\top = \begin{pmatrix} -\frac{52}{25} \\ \frac{39}{25} \end{pmatrix} \begin{pmatrix} -\frac{52}{25} & \frac{39}{25} \end{pmatrix} = \begin{pmatrix} \frac{2704}{625} & -\frac{2028}{625} \\ -\frac{2028}{625} & \frac{1521}{625} \end{pmatrix}$$

Therefore,

$$P = \frac{1}{\frac{169}{25}} \begin{pmatrix} \frac{2704}{625} & -\frac{2028}{625} \\ -\frac{2028}{625} & \frac{1521}{625} \end{pmatrix} = \frac{25}{169} \begin{pmatrix} \frac{2704}{625} & -\frac{2028}{625} \\ -\frac{2028}{625} & \frac{1521}{625} \end{pmatrix}$$

$$\begin{aligned}
P_{11} &= \frac{25}{169} \times \frac{2704}{625} = \frac{2704 \times 25}{625 \times 169} = \frac{2704}{4225} = \frac{16}{25} \\
P_{12} &= \frac{25}{169} \times \left(-\frac{2028}{625}\right) = -\frac{2028 \times 25}{625 \times 169} = -\frac{2028}{4225} = -\frac{12}{25} \\
P_{21} &= P_{12} = -\frac{12}{25} \\
P_{22} &= \frac{25}{169} \times \frac{1521}{625} = \frac{1521 \times 25}{625 \times 169} = \frac{1521}{4225} = \frac{9}{25}
\end{aligned}$$

Therefore, the orthogonal projection matrix $P$ is:

$$P = \begin{pmatrix} \frac{16}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{9}{25} \end{pmatrix}$$

## (3)

**Given:**

- $Q$ is an $n \times n$ orthonormal matrix, meaning $Q^\top Q = QQ^\top = I_n$, where $I_n$ is the $n \times n$ identity matrix.
- $D$ is a diagonal matrix defined as:
$$D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$$
where $d_j = 0$ if $j \notin S$ and $d_j = 1$ if $j \in S$, for some subset $S \subseteq \{1, 2, \ldots, n\}$.

To show that the matrix $P = QDQ^\top$ defines an orthogonal projector by verifying that $P^2 = P$ and $P^\top = P$:

**Verify that** $P^2 = P$

$$P^2 = (QDQ^\top)(QDQ^\top)$$

$$P^2 = QD(Q^\top Q)DQ^\top$$

Since $Q$ is an orthonormal matrix, $Q^\top Q = I_n$:

$$P^2 = QDI_nDQ^\top = QD^2Q^\top$$

Since that $D$ is a diagonal matrix with entries $d_j$ being either 0 or 1, squaring $D$ leaves it unchanged (since $0^2 = 0$ and $1^2 = 1$):

$$D^2 = D$$

Thus:

$$P^2 = QDQ^\top = P$$

**Verify that** $P^\top = P$

$$P^\top = (QDQ^\top)^\top$$

Using the property of transpose for matrix products:

$$P^\top = (Q^\top)^\top D^\top Q^\top$$

Since $D$ is a diagonal matrix, $D^\top = D$, and $(Q^\top)^\top = Q$, therefore:

$$P^\top = (Q^\top)^\top D^\top Q^\top = QDQ^\top$$

Therefore,

$$P^\top = QDQ^\top = P$$

$P^\top = P$, satisfying the symmetry property.

## (4)

**Given:**

$$S = \{1, 2, 3\}, \quad n = 6$$

$$Q = [q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6]$$

where $q_j$ is the $j$-th column of $Q$.

Given $S = \{1, 2, 3\}$ and $n = 6$, the diagonal matrix $D$ is defined as:

$$D = \mathrm{diag}(1, 1, 1, 0, 0, 0)$$

Substitute $D$ into the expression for $P$:

$$P = QDQ^\top = Q \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Q^\top$$

Since $D$ has 1's only in the first three diagonal entries and 0's elsewhere, the multiplication $QD$ retains the first three columns of $Q$ and nullifies the rest. Therefore:

$$QD = [q_1 \quad q_2 \quad q_3 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}]$$

where $\mathbf{0}$ denotes a zero vector.

$$P = QDQ^\top = [q_1 \quad q_2 \quad q_3 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}]Q^\top = q_1 q_1^\top + q_2 q_2^\top + q_3 q_3^\top$$

Therefore,

$$P = q_1 q_1^\top + q_2 q_2^\top + q_3 q_3^\top$$

Each term $q_j q_j^\top$ represents an outer product of the $j$-th column vector of $Q$ with itself. The outer product $q_j q_j^\top$ is a matrix where each element $(i, k)$ is given by $(q_j)_i (q_j)_k$, where $(q_j)_i$ is the $i$-th component of $q_j$.

## Question 2: Projection and Projectors

### (1)

**Given Matrix:**

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \\ -2 & 5 \end{bmatrix}$$

Let $a_1$ and $a_2$ be the columns of $A$:

$$a_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix}$$

Start with $u_1 = a_1$.

$u_2 = a_2 - \text{proj}_{u_1}(a_2)$.

Compute the projection of $a_2$ onto $u_1$:

$$\text{proj}_{u_1}(a_2) = \frac{a_2^\top u_1}{u_1^\top u_1} u_1$$

$$u_1^\top u_1 = 2^2 + 2^2 + (-2)^2 = 12$$

$$a_2^\top u_1 = (-2)(2) + (-2)(2) + 5(-2) = -18$$

$$\text{proj}_{u_1}(a_2) = \frac{-18}{12} u_1 = \left(-\frac{3}{2}\right) u_1$$

$$u_2 = a_2 - \left(-\frac{3}{2} u_1\right) = a_2 + \frac{3}{2} u_1 = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Then we normalize $u_1$ and $u_2$ to get $q_1$ and $q_2$.

$$\|u_1\| = \sqrt{12} = 2\sqrt{3}, \quad \|u_2\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

**Collect $q_1$ and $q_2$ into matrix $Q$:**

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

## (2)

The orthogonal projector onto the range of $A$ is:

$$P_{\text{range}(A)} = QQ^\top$$

$QQ^\top$:

$$q_1 q_1^\top = \left( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \left( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$q_2 q_2^\top = \left( \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right) \left( \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \right) = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$P_{\text{range}(A)} = q_1 q_1^\top + q_2 q_2^\top = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$P_{\text{range}(A)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## (3)

The orthogonal projector onto the null space of $A^\top$ is:

$$P_{\text{null}(A^\top)} = I - P_{\text{range}(A)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{\text{null}(A^\top)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## (4)

**1.** Check Symmetry:

$$P_{\text{range}(A)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{\text{range}(A)}^\top = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{\text{null}(A^\top)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{\text{null}(A^\top)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{\text{range}(A)}^\top = P_{\text{range}(A)}, \quad P_{\text{null}(A^\top)}^\top = P_{\text{null}(A^\top)}$$

**2.** Check ($P^2 = P$):

$$P_{\text{range}(A)}^2 = P_{\text{range}(A)} P_{\text{range}(A)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_{\text{range}(A)}$$

$$P^2_{\text{null}(A^\top)} = P_{\text{null}(A^\top)} P_{\text{null}(A^\top)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = P_{\text{null}(A^\top)}$$

### (5)

Let:

$$b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

**Component in the Range of $A$:**

$$P_{\text{range}(A)} b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+2) \\ \frac{1}{2}(1+2) \\ 4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 4 \end{bmatrix}$$

**Component in the Null Space of $A^\top$:**

$$P_{\text{null}(A^\top)} b = b - P_{\text{range}(A)} b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1.5 \\ 4 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

The component of $b$ in the range of $A$ is $\begin{bmatrix} 1.5 \\ 1.5 \\ 4 \end{bmatrix}$.

The component of $b$ in the null space of $A^\top$ is $\begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}$.

Thus, $b$ is decomposed into:

$$b = P_{\text{range}(A)} b + P_{\text{null}(A^\top)} b$$

## Question 3: Gram-Schmidt and QR

Given the matrix:

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

## (1)

Denote the columns of $A$ as:

$$a_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

**Compute $q_1$ from $a_1$**

$r_{11} = \|a_1\|$:

$$r_{11} = \sqrt{2^2 + 2^2 + (-2)^2} = \sqrt{12} = 2\sqrt{3}$$

Normalize $a_1$ to get $q_1$:

$$q_1 = \frac{a_1}{r_{11}} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

**Compute $q_2$ from $a_2$**

$r_{12} = q_1^\top a_2$:

$$r_{12} = q_1^\top a_2 = \left( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \right) \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{3}}(-2 - 2 - 5) = -\frac{9}{\sqrt{3}} = -3\sqrt{3}$$

$u_2 = a_2 - r_{12} q_1$:

$$u_2 = a_2 - (-3\sqrt{3}) q_1 = a_2 + 3\sqrt{3} \left( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) = a_2 + 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$r_{22} = \|u_2\|$:

$$r_{22} = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

Normalize $u_2$ to get $q_2$:

$$q_2 = \frac{u_2}{r_{22}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

**Compute $q_3$ from $a_3$**

$r_{13} = q_1^\top a_3$:

$$r_{13} = q_1^\top a_3 = \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\right) \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{3}}(0+0+3) = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$r_{23} = q_2^\top a_3$:

$$r_{23} = q_2^\top a_3 = \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}\right) \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{6}}(0+0-6) = -\frac{6}{\sqrt{6}} = -\sqrt{6}$$

$u_3 = a_3 - r_{13}q_1 - r_{23}q_2$:

$$u_3 = a_3 - (\sqrt{3})q_1 - (-\sqrt{6})q_2 = a_3 - \sqrt{3}q_1 + \sqrt{6}q_2$$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-1+1 \\ 0-1+1 \\ -3+1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}$$

$r_{33} = \|u_3\|$:

$$r_{33} = \sqrt{0^2 + 0^2 + (-4)^2} = 4$$

Normalize $u_3$ to get $q_3$:

$$q_3 = \frac{u_3}{r_{33}} = \frac{1}{4} \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Since that $a_3$ can be expressed as a linear combination of $q_1$ and $q_2$ only. This indicates that the third vector is linearly dependent on the first two, and the rank of $A$ is 2.

Therefore, the columns of $A$ are linearly dependent, and we only need $q_1$ and $q_2$ for the orthonormal basis $Q$.

**Orthonormal Basis $Q$:**

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

## (2)

The coefficients $r_{ij}$ computed during the Gram-Schmidt process are:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & -3\sqrt{3} & \sqrt{3} \\ 0 & \sqrt{6} & -\sqrt{6} \end{bmatrix}$$

### (3)

$$QR = QR = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \end{bmatrix}$$

$$A_{:,1} = r_{11}q_1 = 2\sqrt{3} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

$$\begin{aligned} A_{:,2} &= r_{12}q_1 + r_{22}q_2 \\ &= (-3\sqrt{3}) \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \sqrt{6} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\ &= -3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$A_{:,2} = \begin{bmatrix} -3+1 \\ -3+1 \\ 3+2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix}$$

$$\begin{aligned} A_{:,3} &= r_{13}q_1 + r_{23}q_2 \\ &= \sqrt{3} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (-\sqrt{6}) \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$A_{:,3} = \begin{bmatrix} 1-1 \\ 1-1 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

Therefore,

$$A = QR = \begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ -2 & 5 & -3 \end{bmatrix}$$

## Question 4: Solving Least Squares 3 Ways

Given:

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ -2 & 5 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

## (1)

The normal equations for the least squares problem are:

$$A^\top A\,x = A^\top b$$

$$A^\top = \begin{bmatrix} 2 & 2 & -2 \\ -2 & -2 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

$$A^\top A = \begin{bmatrix} 12 & -18 & 6 \\ -18 & 33 & -15 \\ 6 & -15 & 9 \end{bmatrix}$$

- $(A^\top b)_1 = 2(1) + 2(2) + (-2)(4) = -2$
- $(A^\top b)_2 = (-2)(1) + (-2)(2) + 5(4) = 14$
- $(A^\top b)_3 = 0(1) + 0(2) + (-3)(4) = -12$

$$A^\top b = \begin{bmatrix} -2 \\ 14 \\ -12 \end{bmatrix}$$

The normal equations are:

$$\begin{bmatrix} 12 & -18 & 6 \\ -18 & 33 & -15 \\ 6 & -15 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \\ -12 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 12 & -18 & 6 & -2 \\ -18 & 33 & -15 & 14 \\ 6 & -15 & 9 & -12 \end{array}\right]$$

$$R_1 \leftarrow \frac{1}{6} R_1$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -\frac{1}{3} \\ -18 & 33 & -15 & 14 \\ 6 & -15 & 9 & -12 \end{array}\right]$$

$$R_2 \leftarrow R_2 + 9R_1$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -\frac{1}{3} \\ 0 & 6 & -6 & 11 \\ 0 & -6 & 6 & -11 \end{array}\right]$$

$$R_2 \leftarrow \frac{1}{6} R_2$$

$$R_3 \leftarrow R_3 + 6R_2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -\frac{1}{3} \\ 0 & 1 & -1 & \frac{11}{6} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Updated Row 3:

$$[0 \quad 0 \quad 0 \quad 0]$$

From Row 2:

$$x_2 - x_3 = \frac{11}{6} \quad \Rightarrow \quad x_3 = x_2 - \frac{11}{6}$$

From Row 1:

$$2x_1 - 3x_2 + x_3 = -\frac{1}{3}$$

Substitute $x_3 = x_2 - \frac{11}{6}$:

$$2x_1 - 3x_2 + x_2 - \frac{11}{6} = -\frac{1}{3}$$

$$2x_1 = 2x_2 + \frac{3}{2} \quad \Rightarrow \quad x_1 = x_2 + \frac{3}{4}$$

**Final Solution:**

$$x^* = \begin{bmatrix} \frac{3}{4} \\ 0 \\ -\frac{11}{6} \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $x_2$ is a free parameter.

## (2)

From previous results:

$$b_{\|\mathrm{range}(A)} = QQ^\top b = \begin{bmatrix} 1.5 \\ 1.5 \\ 4 \end{bmatrix}$$

To solve $Ax^* = b_{\|\mathrm{range}(A)}$, set up the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 1.5 \\ 2 & -2 & 0 & 1.5 \\ -2 & 5 & -3 & 4 \end{array}\right]$$

$$R_2 \leftarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 1.5 \\ 0 & 0 & 0 & 0 \\ -2 & 5 & -3 & 4 \end{array}\right]$$

From Row 1:

$$2x_1 - 2x_2 = 1.5$$

$$x_1 = x_2 + \frac{3}{4}$$

From Row 3:

$$-2x_1 + 5x_2 - 3x_3 = 4$$

Substitute $x_1 = x_2 + \frac{3}{4}$:

$$-2\left(x_2 + \frac{3}{4}\right) + 5x_2 - 3x_3 = 4$$

$$x_2 - x_3 = \frac{11}{6}$$

$$x_3 = x_2 - \frac{11}{6}$$

and,

$$x_1 = x_2 + \frac{3}{4}$$

Therefore, $x^*$is:

$$x^* = \begin{bmatrix} \frac{3}{4} \\ 0 \\ -\frac{11}{6} \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $x_2$ is a free parameter.

## (3)

Given that $A = QR$ and $Ax^* = P_{\text{range}(A)}b = QQ^\top b$, we have:

$$\begin{aligned} QRx^* &= QQ^\top b \\ Rx^* &= Q^\top b \end{aligned}$$

From previous results:

$$R = \begin{bmatrix} 2\sqrt{3} & -3\sqrt{3} & \sqrt{3} \\ 0 & \sqrt{6} & -\sqrt{6} \end{bmatrix}$$

Given:

$$Q^\top = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{aligned} (Q^\top b)_1 &= \frac{1}{\sqrt{3}} \times 1 + \frac{1}{\sqrt{3}} \times 2 + \left(-\frac{1}{\sqrt{3}}\right) \times 4 \\ &= \frac{1+2-4}{\sqrt{3}} \\ &= \frac{-1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} (Q^\top b)_2 &= \frac{1}{\sqrt{6}} \times 1 + \frac{1}{\sqrt{6}} \times 2 + \frac{2}{\sqrt{6}} \times 4 \\ &= \frac{1+2+8}{\sqrt{6}} \\ &= \frac{11}{\sqrt{6}} \end{aligned}$$

$$Q^\top b = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{11}{\sqrt{6}} \end{bmatrix}$$

We have the system:

$$\begin{cases} 2\sqrt{3}x_1 - 3\sqrt{3}x_2 + \sqrt{3}x_3 = -\dfrac{1}{\sqrt{3}} \\ \sqrt{6}x_2 - \sqrt{6}x_3 = \dfrac{11}{\sqrt{6}} \end{cases}$$

The Augmented Matrix

$$\left[\begin{array}{ccc|c} 2\sqrt{3} & -3\sqrt{3} & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{6} & -\sqrt{6} & \frac{11}{\sqrt{6}} \end{array}\right]$$

$$R_1 \leftarrow \frac{1}{\sqrt{3}} R_1$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -\frac{1}{3} \\ 0 & \sqrt{6} & -\sqrt{6} & \frac{11}{\sqrt{6}} \end{array}\right]$$

$$R_2 \leftarrow \frac{1}{\sqrt{6}} R_2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -\frac{1}{3} \\ 0 & 1 & -1 & \frac{11}{6} \end{array}\right]$$

From Row 2:

$$x_2 - x_3 = \frac{11}{6}$$

$$x_3 = x_2 - \frac{11}{6}$$

Substitute $x_3 = x_2 - \dfrac{11}{6}$ into Row 1:

$$2x_1 - 3x_2 + \left(x_2 - \frac{11}{6}\right) = -\frac{1}{3}$$

$$2x_1 - 2x_2 - \frac{11}{6} = -\frac{1}{3}$$

$$x_1 = x_2 + \frac{3}{4}$$

We have:

$$x_1 = x_2 + \frac{3}{4}$$
$$x_3 = x_2 - \frac{11}{6}$$

$$x^* = \begin{bmatrix} \frac{3}{4} \\ 0 \\ -\frac{11}{6} \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $x_2$ is a free parameter.