Homework 2

1. Suppose we have i.i.d. $X_1, ..., X_n \sim N(\mu, 1)$. To construct a 95%-confidence interval for μ with length at most 0.5, what is the smallest sample size n that you need?

2. Suppose $\hat{\theta}$ is an estimator of θ. Its mean squared error (MSE) is defined as $\mathbb{E}(\hat{\theta} - \theta)^2$. Show $\mathbb{E}(\hat{\theta} - \theta)^2 = \text{var}(\hat{\theta}) + (\mathbb{E}\hat{\theta} - \theta)^2$. This is referred to as the variance bias decomposition.

3. A detector counts the number of particles emitted from a radioactive source over the course of 10-second intervals. For 180 such 10-second intervals, the following counts were observed:

Count	# intervals
0	23
1	77
2	34
3	26
4	13
5	7

This table states, for example, that in 34 of the 10-second intervals a count of 2 was recorded. Sometimes, however, the detector did not function properly and recorded counts over intervals of length 20 seconds. This happened 20 times and the recorded counts are

Count	# intervals
0	2
1	4
2	9
3	5

Assume a Poisson process model for the particle emission process. Let $\lambda > 0$ (time unit = 1 sec.) be the unknown rate of the Poisson process.

   (a) Formulate an appropriate likelihood function for the described scenario and derive the maximum likelihood estimator $\hat{\lambda}$ of the rate λ. Compute $\hat{\lambda}$ for the above data.

   (b) What approximation to the distribution of $\hat{\lambda}$ does the central limit theorem suggest? (Note that the sum of all 200 counts has a Poisson distribution. What is its parameter?)

4. Answer the following questions.

   (a) For $Z \sim N(0, 1)$, what are the values of $\mathbb{E}(Z^2)$ and $\mathbb{E}(Z^4)$? Check Wikipedia and directly write down the answer.

   (b) For i.i.d. $Z_1, \cdots, Z_n \sim N(0, 1)$, the distribution of $Z_1^2 + \cdots + Z_n^2$ is called a chi-squared distribution with degrees of freedom n, denoted by χ_n^2. For $Y \sim \chi_n^2$, calculate the mean and variance of Y using your answer to the previous question.