Assignment 4 (two pages)

Statistics 24400 (Autumn 2024)

Due on Gradescope, Tuesday, October 29 by 9 am

Requirements

Your answers should be typed or clearly written, started with your name, Assignment 4, STAT 24400;

saved as `Lastname-Firstname-hw4.pdf`, uploaded to Gradescope under P-set4, tag the pages for each question.

You may discuss approaches with others. However the assignment should be devised and written by yourself.

To get full credit, you must provide the reasoning of main steps of the derivation leading to your answer.

Reminder: **in-class midterm** Thursday Oct. 31; more details will be provided in class and on Ed Discussion.

Problem assignments (related sections in the text: 3.7, 4.3-4.4)

1. Suppose an urn contains 3 tickets numbered "1", 3 tickets numbered "2", 2 tickets numbered "3", and 1 ticket numbered "4". A student draws a ticket at random and notes the number, X_1. The student then returns the ticket to the urn, shakes it up, and draws again, noting the number, X_2.
   Let $Z = X_{(1)}$ = the minimum of X_1 and X_2.

   (a) Find the probability mass function (PMF) and the cumulative distribution function (CDF) of X_1, then graph both.

   (b) Find the probability distribution of Z (show the PMF in a table).

   (c) Find $\mathbb{E}(X_1), \mathbb{E}(X_2)$ and $\mathbb{E}(Z)$.

   (d) Find the bivariate probability mass function $p(x,z) = \mathbb{P}(X_1 = x, Z = z)$ of X_1 and Z (show the joint PMF in a table), and the covariance of X_1 and Z.

2. Suppose you flip a fair coin 10 times. Let X be the total number of times that you see the sequence HT. What is $\text{Var}(X)$? (Hint: recall from what you did in Assignment 3; think of X as a sum.)

3. Suppose that X and Y are independent Geometric(p) random variables and $W = \min\{X, Y\}$. Calculate

   $$\mathbb{P}(X = x \mid W = x)$$

   where x is some fixed positive integer.
   (Hint: it helps to consider various cases of possible values of X, Y relative to x.)

4. Suppose X and Y have joint density

   $$f(x,y) = \begin{cases} 2x + 2y - 4xy, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \textit{otherwise}. \end{cases}$$

   (a) Find $\mathbb{E}(Y \mid X)$.

   (b) Find $Var(Y \mid X)$.

5. A fair coin is tossed n times, and the number of Heads, N, is counted.
The coin is then tossed N more times. Let M be the additional Heads obtained.
Let $T = N + M$ be the total number of Heads generated by this process.

   (a) Derive

      i. $\mathbb{E}(T \mid N)$.

      ii. $\mathbb{E}(T)$.

   (b) Derive

      i. $Var(T \mid N)$.

      ii. $\mathbb{E}[Var(T \mid N)]$.

      iii. $Var(\mathbb{E}[T \mid N])$.

      iv. $Var(T)$.

6. Let A and B be independent Exponential(1) random variables.
Let C be a random sign, i.e. $\mathbb{P}(C = +1) = \mathbb{P}(C = -1) = 0.5$, and C is independent of A and B.
Define $X = A \cdot C$ and $Y = B \cdot C$.

   (a) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$. You may use the following facts: for an Exponential(λ) random variable, its expected value is $1/\lambda$ and its variance is $1/\lambda^2$.

   (b) Calculate $\mathrm{Cov}(X, Y)$.

   (c) Calculate $\mathbb{P}(X \leq t \mid Y \leq t)$, where $t > 0$ is a constant.

   (d) Plot $\mathbb{P}(X \leq t \mid Y \leq t)$, as a function of t and/or calculate its value for a few different values of t, and describe what you observe.

   (Details: Try to interpret the meaning of the plot, e.g., when t is large, when t is very small, and when $t \approx 0.5$.)