24300 HW3

Bin Yu

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Question 1: Matrix Multiplication

To compute the product $C = AB$ where:

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \quad (4\times2 \text{ matrix})$$

$$B = \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 4 & -2 & -3 \end{bmatrix} \quad (2\times4 \text{ matrix})$$

Using the following three methods, show that the results are equivalent:

1. **Entry-wise**: $c_{ij} = \sum_{k=1}^{2} a_{ik}b_{kj}$
2. **Column-wise**: $C = [Ab_{:,1}, Ab_{:,2}, Ab_{:,3}, Ab_{:,4}]$ where $b_{:,j}$ is the j-th column of B.
3. **Sum of Outer Products**: $C = \sum_{k=1}^{2} A_{:,k} \otimes B_{k,:}$

Method 1: Entry-wise

Since we have:

$$c_{ij} = \sum_{k=1}^{2} a_{ik} \cdot b_{kj}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

$$
\begin{aligned}
c_{11} &= (1 \times 1) + (5 \times 0) = 1 + 0 = 1 \\
c_{12} &= (1 \times 0) + (5 \times 4) = 0 + 20 = 20 \\
c_{13} &= (1 \times 4) + (5 \times -2) = 4 - 10 = -6 \\
c_{14} &= (1 \times 2) + (5 \times -3) = 2 - 15 = -13 \\
\\
c_{21} &= (3 \times 1) + (0 \times 0) = 3 + 0 = 3 \\
c_{22} &= (3 \times 0) + (0 \times 4) = 0 + 0 = 0 \\
c_{23} &= (3 \times 4) + (0 \times -2) = 12 + 0 = 12 \\
c_{24} &= (3 \times 2) + (0 \times -3) = 6 + 0 = 6 \\
\\
c_{31} &= (2 \times 1) + (-1 \times 0) = 2 + 0 = 2 \\
c_{32} &= (2 \times 0) + (-1 \times 4) = 0 - 4 = -4 \\
c_{33} &= (2 \times 4) + (-1 \times -2) = 8 + 2 = 10 \\
c_{34} &= (2 \times 2) + (-1 \times -3) = 4 + 3 = 7 \\
\\
c_{41} &= (1 \times 1) + (0 \times 0) = 1 + 0 = 1 \\
c_{42} &= (1 \times 0) + (0 \times 4) = 0 + 0 = 0 \\
c_{43} &= (1 \times 4) + (0 \times -2) = 4 + 0 = 4 \\
c_{44} &= (1 \times 2) + (0 \times -3) = 2 + 0 = 2
\end{aligned}
$$

$$
C = \begin{bmatrix} 1 & 20 & -6 & -13 \\ 3 & 0 & 12 & 6 \\ 2 & -4 & 10 & 7 \\ 1 & 0 & 4 & 2 \end{bmatrix}
$$

Method 2: Column-wise

$$
C = \begin{bmatrix} Ab_{:,1} & Ab_{:,2} & Ab_{:,3} & Ab_{:,4} \end{bmatrix}
$$

$$
b_{:,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

$$
Ab_{:,1} = A \times b_{:,1} = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1 \times 1) + (5 \times 0) \\ (3 \times 1) + (0 \times 0) \\ (2 \times 1) + (-1 \times 0) \\ (1 \times 1) + (0 \times 0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}
$$

$$
b_{:,2} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}
$$

$$
Ab_{:,2} = A \times b_{:,2} = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} (1 \times 0) + (5 \times 4) \\ (3 \times 0) + (0 \times 4) \\ (2 \times 0) + (-1 \times 4) \\ (1 \times 0) + (0 \times 4) \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ -4 \\ 0 \end{bmatrix}
$$

$$b_{:,3} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$Ab_{:,3} = A \times b_{:,3} = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} (1 \times 4) + (5 \times -2) \\ (3 \times 4) + (0 \times -2) \\ (2 \times 4) + (-1 \times -2) \\ (1 \times 4) + (0 \times -2) \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \\ 10 \\ 4 \end{bmatrix}$$

$$b_{:,4} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$Ab_{:,4} = A \times b_{:,4} = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} (1 \times 2) + (5 \times -3) \\ (3 \times 2) + (0 \times -3) \\ (2 \times 2) + (-1 \times -3) \\ (1 \times 2) + (0 \times -3) \end{bmatrix} = \begin{bmatrix} -13 \\ 6 \\ 7 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} Ab_{:,1} & Ab_{:,2} & Ab_{:,3} & Ab_{:,4} \end{bmatrix} = \begin{bmatrix} 1 & 20 & -6 & -13 \\ 3 & 0 & 12 & 6 \\ 2 & -4 & 10 & 7 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

Method 3: Sum of Outer Products

The matrix product C can be expressed as the sum of outer products of the corresponding columns of A and rows of B:

$$C = \sum_{k=1}^{2} A_{:,k} \otimes B_{k,:}$$

where:

- $A_{:,k}$ is the k-th column of A (4×1 vector).
- $B_{k,:}$ is the k-th row of B (1×4 vector).

For $k = 1$

$$A_{:,1} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad B_{1,:} = \begin{bmatrix} 1 & 0 & 4 & 2 \end{bmatrix}$$

$$A_{:,1} \otimes B_{1,:} = \begin{bmatrix} 1 \times 1 & 1 \times 0 & 1 \times 4 & 1 \times 2 \\ 3 \times 1 & 3 \times 0 & 3 \times 4 & 3 \times 2 \\ 2 \times 1 & 2 \times 0 & 2 \times 4 & 2 \times 2 \\ 1 \times 1 & 1 \times 0 & 1 \times 4 & 1 \times 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 2 \\ 3 & 0 & 12 & 6 \\ 2 & 0 & 8 & 4 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

For $k = 2$

$$A_{:,2} = \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad B_{2,:} = \begin{bmatrix} 0 & 4 & -2 & -3 \end{bmatrix}$$

$$A_{:,2} \otimes B_{2,:} = \begin{bmatrix} 5 \times 0 & 5 \times 4 & 5 \times -2 & 5 \times -3 \\ 0 \times 0 & 0 \times 4 & 0 \times -2 & 0 \times -3 \\ -1 \times 0 & -1 \times 4 & -1 \times -2 & -1 \times -3 \\ 0 \times 0 & 0 \times 4 & 0 \times -2 & 0 \times -3 \end{bmatrix} = \begin{bmatrix} 0 & 20 & -10 & -15 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = (A_{:,1} \otimes B_{1,:}) + (A_{:,2} \otimes B_{2,:}) = \begin{bmatrix} 1 & 0 & 4 & 2 \\ 3 & 0 & 12 & 6 \\ 2 & 0 & 8 & 4 \\ 1 & 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 20 & -10 & -15 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+20 & 4-10 & 2-15 \\ 3+0 & 0+0 & 12+0 & 6+0 \\ 2+0 & 0-4 & 8+2 & 4+3 \\ 1+0 & 0+0 & 4+0 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 20 & -6 & -13 \\ 3 & 0 & 12 & 6 \\ 2 & -4 & 10 & 7 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

Thus, all three methods yield the same product matrix:

$$C = AB = \begin{bmatrix} 1 & 20 & -6 & -13 \\ 3 & 0 & 12 & 6 \\ 2 & -4 & 10 & 7 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

Therefore, the results of all three methods are equivalent for matrix multiplication.

Question 2: Gauss-Jordan and Inverses

Given the matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(1)

The augmented matrix by appending the identity matrix I to matrix A:

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

(2)

Row operations:

$$R_2 = \frac{1}{2}R_1 + R_2$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 = \frac{2}{3}R_2 + R_3$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

(3)

Scale each row to make the pivot (diagonal element) in each row is 1.

$$R_1 = \frac{1}{2}R_1$$
$$R_2 = \frac{2}{3}R_2$$
$$R_3 = \frac{3}{4}R_3$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

(4)

Perform the following row operations:

$$R_2 = R_2 + \frac{2}{3}R_3$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

$$R_1 = R_1 + \frac{1}{2}R_2$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

(5)

The augmented matrix now has the identity matrix on the left and the inverse A^{-1} on the right:

$$[I|A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

Therefore, the inverse of matrix A is:

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Verification

$$AA^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$(A \cdot A^{-1})_{11} = 2 \times \frac{3}{4} + (-1) \times \frac{1}{2} + 0 \times \frac{1}{4} = \frac{6}{4} - \frac{1}{2} + 0 = 1.5 - 0.5 = 1$$
$$(A \cdot A^{-1})_{12} = 2 \times \frac{1}{2} + (-1) \times 1 + 0 \times \frac{1}{2} = 1 - 1 + 0 = 0$$
$$(A \cdot A^{-1})_{13} = 2 \times \frac{1}{4} + (-1) \times \frac{1}{2} + 0 \times \frac{3}{4} = 0.5 - 0.5 + 0 = 0$$

$$(A \cdot A^{-1})_{21} = (-1) \times \frac{3}{4} + 2 \times \frac{1}{2} + (-1) \times \frac{1}{4} = -\frac{3}{4} + 1 - \frac{1}{4} = -1 + 1 = 0$$
$$(A \cdot A^{-1})_{22} = (-1) \times \frac{1}{2} + 2 \times 1 + (-1) \times \frac{1}{2} = -\frac{1}{2} + 2 - \frac{1}{2} = -1 + 2 = 1$$
$$(A \cdot A^{-1})_{23} = (-1) \times \frac{1}{4} + 2 \times \frac{1}{2} + (-1) \times \frac{3}{4} = -\frac{1}{4} + 1 - \frac{3}{4} = -1 + 1 = 0$$

$$(A \cdot A^{-1})_{31} = 0 \times \frac{3}{4} + (-1) \times \frac{1}{2} + 2 \times \frac{1}{4} = 0 - \frac{1}{2} + \frac{1}{2} = 0$$
$$(A \cdot A^{-1})_{32} = 0 \times \frac{1}{2} + (-1) \times 1 + 2 \times \frac{1}{2} = 0 - 1 + 1 = 0$$
$$(A \cdot A^{-1})_{33} = 0 \times \frac{1}{4} + (-1) \times \frac{1}{2} + 2 \times \frac{3}{4} = 0 - \frac{1}{2} + \frac{6}{4} = -0.5 + 1.5 = 1$$

Thus,

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Calculate each element of the product:

$$(A^{-1} \cdot A)_{11} = \frac{3}{4} \times 2 + \frac{1}{2} \times (-1) + \frac{1}{4} \times 0 = \frac{6}{4} - \frac{1}{2} + 0 = 1.5 - 0.5 = 1$$
$$(A^{-1} \cdot A)_{12} = \frac{3}{4} \times (-1) + \frac{1}{2} \times 2 + \frac{1}{4} \times (-1) = -\frac{3}{4} + 1 - \frac{1}{4} = -1 + 1 = 0$$
$$(A^{-1} \cdot A)_{13} = \frac{3}{4} \times 0 + \frac{1}{2} \times (-1) + \frac{1}{4} \times 2 = 0 - \frac{1}{2} + \frac{2}{4} = -0.5 + 0.5 = 0$$

$$(A^{-1} \cdot A)_{21} = \frac{1}{2} \times 2 + 1 \times (-1) + \frac{1}{2} \times 0 = 1 - 1 + 0 = 0$$
$$(A^{-1} \cdot A)_{22} = \frac{1}{2} \times (-1) + 1 \times 2 + \frac{1}{2} \times (-1) = -\frac{1}{2} + 2 - \frac{1}{2} = -1 + 2 = 1$$
$$(A^{-1} \cdot A)_{23} = \frac{1}{2} \times 0 + 1 \times (-1) + \frac{1}{2} \times 2 = 0 - 1 + 1 = 0$$

$$(A^{-1} \cdot A)_{31} = \frac{1}{4} \times 2 + \frac{1}{2} \times (-1) + \frac{3}{4} \times 0 = \frac{2}{4} - \frac{1}{2} + 0 = 0.5 - 0.5 = 0$$
$$(A^{-1} \cdot A)_{32} = \frac{1}{4} \times (-1) + \frac{1}{2} \times 2 + \frac{3}{4} \times (-1) = -\frac{1}{4} + 1 - \frac{3}{4} = -1 + 1 = 0$$
$$(A^{-1} \cdot A)_{33} = \frac{1}{4} \times 0 + \frac{1}{2} \times (-1) + \frac{3}{4} \times 2 = 0 - \frac{1}{2} + \frac{6}{4} = -0.5 + 1.5 = 1$$

$$A^{-1} \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, we have $AA^{-1} = I = A^{-1}A$.

Solving $Ax = b$ Using A^{-1}

Given that we are solving:

$$Ax = b$$

multiply both sides by A^{-1}:

$$A^{-1}Ax = A^{-1}b$$

By the definition of the inverse matrix, $A^{-1}A = I$, where I is the identity matrix. Therefore, the equation simplifies to:

$$Ix = A^{-1}b$$

Since multiplying any vector by the identity matrix I leaves it unchanged ($Ix = x$):

$$x = A^{-1}b$$

(6)

When we perform the reduction process to transform the augmented matrix $[A \mid I]$ into $[I \mid M]$, we are applying a sequence of elementary row operations to A. Each elementary row operation corresponds to multiplication on the left by an elementary matrix, which can be a permutation matrix (for row swaps), a scaling matrix (for multiplying a row by a nonzero scalar), or a shear matrix (for adding a multiple of one row to another).

Let $E_1, E_2, \ldots, E_k$ be the sequence of elementary matrices corresponding to the row operations performed. Then, the product of these matrices transforms A into I:

$$E_k E_{k-1} \dots E_1 A = I$$

This implies:

$$A^{-1} = E_k E_{k-1} \dots E_1$$

Because the product $E_k E_{k-1} \dots E_1$ is precisely the sequence of operations that inverts A.

When we augment A with I and apply the same sequence of row operations to I, we also multiply I by the same sequence of elementary matrices:

$$M = E_k E_{k-1} \dots E_1 I = A^{-1}$$

Therefore, the matrix M obtained after the reduction process is A^{-1}.

Thus, the reduction process produces the inverse because the sequence of elementary row operations (including multiplication by permutation matrices for row swaps) that reduces A to I simultaneously constructs A^{-1} on the augmented side.

Question 3: Reduced Row Echelon Form

Given matrix A:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 6 \\ 2 & -2 & 0 & 6 & -8 \\ 2 & 3 & 5 & 2 & 2 \end{bmatrix}$$

(1)

Row operations:

$$\begin{aligned} R2 &= R2 - R1 \\ R3 &= R3 - R1 \\ R4 &= R4 - R1 \end{aligned}$$

Matrix becomes:

$$\begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & -6 & -6 & 2 & -12 \\ 0 & -1 & -1 & -2 & -2 \end{bmatrix}$$

Next:

$$\begin{aligned} R3 &= R3 + 6 \times R2 \\ R4 &= R4 + R2 \end{aligned}$$

Calculations:

$$\begin{aligned}
R3 &= [0+6\times 0,\ -6+6\times 1,\ -6+6\times 1,\ 2+6\times 2,\ -12+6\times 2] \\
&= [0,\ 0,\ 0,\ 14,\ 0] \\
R4 &= [0+0,\ -1+1,\ -1+1,\ -2+2,\ -2+2] = [0,\ 0,\ 0,\ 0,\ 0]
\end{aligned}$$

Updated matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2)

Scale Row 1 and Row 3:

$$\begin{aligned}
R1 &= \frac{1}{2}R1 \\
R3 &= \frac{1}{14}R3
\end{aligned}$$

matrix becomes:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3)

Reduce the row from bottom to top:

$$\begin{aligned}
R1 &= R1 - 2\times R3 \\
R2 &= R2 - 2\times R3
\end{aligned}$$

Updated matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Next,

$$R1 = R1 - 2\times R2$$

Final matrix in RREF:

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & -2 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Therefore, the reduced row-echelon form of A is:

$$
R = \begin{bmatrix}
1 & 0 & 1 & 0 & -2 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Question 4:Bases for the Null-Space and Range

(1)

$$
R = \begin{bmatrix}
1 & 0 & 1 & 0 & -2 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The **pivot columns** are columns 1, 2, and 4.

The **free columns** are the remaining columns: 3 and 5.

Number of pivot columns (rank of A): $\text{rank}(A) = 3$

Number of free columns (nullity of A): $\text{nullity}(A) = 2$

Therefore:

- Pivot columns: Columns 1, 2, 4
- Free columns: Columns 3, 5
- Rank of A: 3
- Nullity of A: 2

(2)

Original matrix A:

$$
A = \begin{bmatrix}
2 & 4 & 6 & 4 & 4 \\
2 & 5 & 7 & 6 & 6 \\
2 & -2 & 0 & 6 & -8 \\
2 & 3 & 5 & 2 & 2
\end{bmatrix}
$$

Keep the pivot columns (columns 1, 2, 4) to form matrix C:

$$C = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 5 & 6 \\ 2 & -2 & 6 \\ 2 & 3 & 2 \end{bmatrix}$$

The columns of C span the **column space** (i.e. range) of A.

(3)

After removing the zero row from R, we have:

$$R' = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and:

$$C = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 5 & 6 \\ 2 & -2 & 6 \\ 2 & 3 & 2 \end{bmatrix}$$

Now, compute each column of CR' by multiplying C with each column of R'.

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C\mathbf{r}_1 = 1 \cdot \mathbf{c}_1 + 0 \cdot \mathbf{c}_2 + 0 \cdot \mathbf{c}_4 = \mathbf{c}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C\mathbf{r}_2 = 0 \cdot \mathbf{c}_1 + 1 \cdot \mathbf{c}_2 + 0 \cdot \mathbf{c}_4 = \mathbf{c}_2 = \begin{bmatrix} 4 \\ 5 \\ -2 \\ 3 \end{bmatrix}$$

$$\mathbf{r}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$C\mathbf{r}_3 = 1 \cdot \mathbf{c}_1 + 1 \cdot \mathbf{c}_2 + 0 \cdot \mathbf{c}_4 = \mathbf{c}_1 + \mathbf{c}_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 0 \\ 5 \end{bmatrix}$$

This matches the third column of A.

$$\mathbf{r}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C\mathbf{r}_4 = 0 \cdot \mathbf{c}_1 + 0 \cdot \mathbf{c}_2 + 1 \cdot \mathbf{c}_4 = \mathbf{c}_4 = \begin{bmatrix} 4 \\ 6 \\ 6 \\ 2 \end{bmatrix}$$

$$\mathbf{r}_5 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$C\mathbf{r}_5 = (-2) \cdot \mathbf{c}_1 + 2 \cdot \mathbf{c}_2 + 0 \cdot \mathbf{c}_4 = -2\mathbf{c}_1 + 2\mathbf{c}_2$$

$$= -2 \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ -4 \\ -4 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -8 \\ 2 \end{bmatrix}$$

This matches the fifth column of A.

By computing CR', we find:

$$CR' = \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 6 \\ 2 & -2 & 0 & 6 & -8 \\ 2 & 3 & 5 & 2 & 2 \end{bmatrix} = A$$

(4)

From calculations in part (3), observe:

Pivot Columns: The pivot columns of A (columns 1, 2, and 4) are directly included in C and correspond to identity vectors in R. When multiplied, they reproduce themselves:

$$A_{\text{col }1} = C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{c}_1$$

$$A_{\text{col }2} = C \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{c}_2$$

$$A_{\text{col }4} = C \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{c}_4$$

Free Columns: The free columns of A (columns 3 and 5) are linear combinations of the pivot columns, with coefficients specified by the nonzero entries in R:

- **Column 3:**
  - Coefficients from R:
$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$
  - Linear combination:
$$A_{\text{col }3} = 1 \cdot A_{\text{col }1} + 1 \cdot A_{\text{col }2}$$
  - Computation:
$$\begin{aligned} A_{\text{col }3} &= A_{\text{col }1} + A_{\text{col }2} \\ &= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ -2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 7 \\ 0 \\ 5 \end{bmatrix} \end{aligned}$$

    This matches $A_{\text{col }3}$.

- **Column 5:**
  - Coefficients from R:
$$\begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$
  - Linear combination:
$$A_{\text{col }5} = (-2) \cdot A_{\text{col }1} + 2 \cdot A_{\text{col }2}$$

- Computation:

$$\begin{aligned}
A_{\text{col }5} &= -2A_{\text{col }1} + 2A_{\text{col }2} \\
&= -2\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ -2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} -4 \\ -4 \\ -4 \\ -4 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \\ -4 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 6 \\ -8 \\ 2 \end{bmatrix}
\end{aligned}$$

  This matches $A_{\text{col }5}$.

Therefore, nonzero entries in each free column of R indicate how to combine the pivot columns to reconstruct the corresponding free column of A, which demonstrates that free columns can be expressed by the linear combinations of pivot columns, with coefficients provided by the nonzero entries in R.

(5)

From $Rx = 0$, we have:

$$\begin{cases} x_1 + x_3 - 2x_5 = 0 \\ x_2 + x_3 + 2x_5 = 0 \\ x_4 = 0 \end{cases}$$

Let x_3 and x_5 be free variables.

Express the other variables in terms of x_3 and x_5:

$$\begin{cases} x_1 = -x_3 + 2x_5 \\ x_2 = -x_3 - 2x_5 \\ x_4 = 0 \end{cases}$$

Therefore, the general solution to $Rx = 0$ is:

$$x = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where x_3 and x_5 are free variables.

(6)

Construct matrix N:

$$N = \begin{bmatrix} -1 & 2 \\ -1 & -2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The columns of N form a basis for the **null space** of A.

Question 5: Projections

Let:

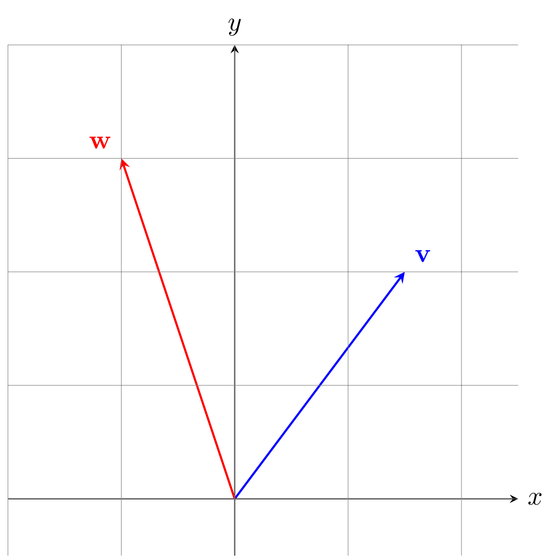
$$\mathbf{v} = \begin{bmatrix} \frac{3}{2} \\ 2 \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(1)

- $\mathbf{v}$ points to the coordinate $\left(\frac{3}{2}, 2\right)$.
- $\mathbf{w}$ points to the coordinate $(-1, 3)$.

Sketch of vectors $\mathbf{v}$ and $\mathbf{w}$:



(2)

The magnitudes (lengths) of vectors $\mathbf{v}$ and $\mathbf{w}$ are:

$$\|\mathbf{v}\| = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$$\|\mathbf{w}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$$

(3)

The projection of vector $\mathbf{w}$ onto vector $\mathbf{v}$ is given by:

$$\mathbf{w}_{\|\mathbf{v}} = \left(\frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2}\right)\mathbf{w}$$

$$\mathbf{v} = \begin{bmatrix} \frac{3}{2} \\ 2 \end{bmatrix}, \quad \mathbf{v}^T = \begin{bmatrix} \frac{3}{2} & 2 \end{bmatrix}$$

$$\mathbf{v}\mathbf{v}^T = \begin{bmatrix} \frac{3}{2} \\ 2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{bmatrix}$$

$$\|\mathbf{v}\|^2 = \left(\frac{3}{2}\right)^2 + 2^2 = \frac{9}{4} + 4 = \frac{25}{4}$$

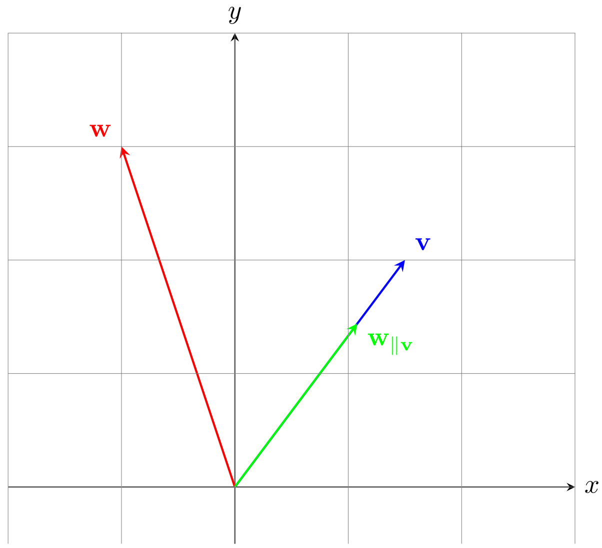
$$\mathbf{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\mathbf{w}_{\|\mathbf{v}} = \frac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2}\mathbf{w} = \frac{4}{25}\begin{bmatrix} \frac{9}{4} & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{4}{25}\begin{bmatrix} \frac{9}{4} \times (-1) + 3 \times 3 \\ 3 \times (-1) + 4 \times 3 \end{bmatrix} = \frac{4}{25}\begin{bmatrix} -\frac{9}{4} + 9 \\ -3 + 12 \end{bmatrix} = \frac{4}{25}\begin{bmatrix} \frac{27}{4} \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{27}{25} \\ \frac{36}{25} \end{bmatrix} = \begin{bmatrix} 1.08 \\ 1.44 \end{bmatrix}$$

Therefore,

$$\mathbf{w}_{\|\mathbf{v}} = \begin{bmatrix} \frac{27}{25} \\ \frac{36}{25} \end{bmatrix} = \begin{bmatrix} 1.08 \\ 1.44 \end{bmatrix}$$

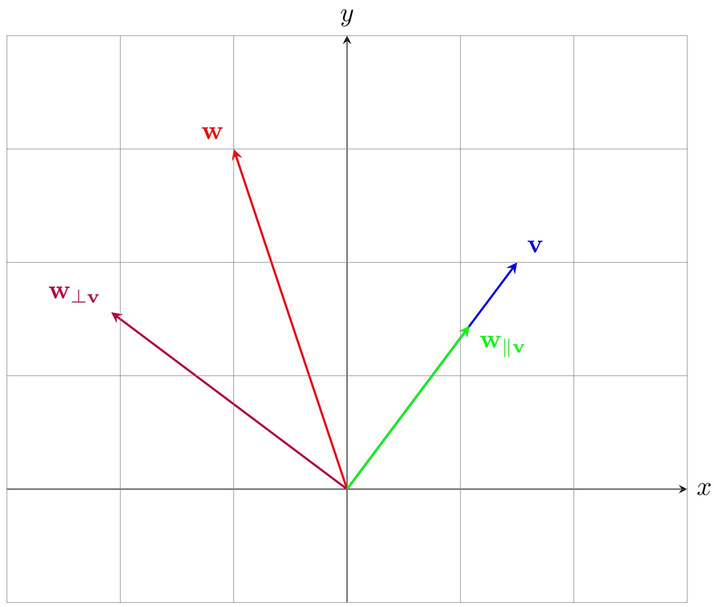
Updated Sketch with Projection:



(4)

$$\mathbf{w}_{\perp\mathbf{v}} = \mathbf{w} - \mathbf{w}_{\|\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1.08 \\ 1.44 \end{bmatrix} = \begin{bmatrix} -1 - 1.08 \\ 3 - 1.44 \end{bmatrix} = \begin{bmatrix} -2.08 \\ 1.56 \end{bmatrix}$$

Updated Sketch with Perpendicular Component:



(5)

To verify that $\mathbf{w}_{\perp\mathbf{v}}$ is orthogonal to $\mathbf{v}$, compute their dot product:

$$\mathbf{w}_{\perp\mathbf{v}} \cdot \mathbf{v} = (-2.08) \times \frac{3}{2} + 1.56 \times 2 = -3.12 + 3.12 = 0$$

Since the dot product is zero, $\mathbf{w}_{\perp\mathbf{v}}$ is orthogonal to $\mathbf{v}$.

(6)

$$\|\mathbf{w}_{\parallel \mathbf{v}}\| = \sqrt{1.08^2 + 1.44^2} = \sqrt{3.24} = 1.8$$
$$\|\mathbf{w}_{\perp \mathbf{v}}\| = \sqrt{(-2.08)^2 + 1.56^2} = \sqrt{6.76} = 2.6$$

$$\|\mathbf{w}_{\perp \mathbf{v}}\|^2 + \|\mathbf{w}_{\parallel \mathbf{v}}\|^2 = (2.6)^2 + (1.8)^2 = 6.76 + 3.24 = 10$$
$$\|\mathbf{w}\|^2 = (-1)^2 + 3^2 = 10$$

Thus,

$$\|\mathbf{w}_{\perp \mathbf{v}}\|^2 + \|\mathbf{w}_{\parallel \mathbf{v}}\|^2 = \|\mathbf{w}\|^2$$