Conditional expectation & conditional variance

(part 2)

Lecture 8b (STAT 24400 F24)

Law of total variance

Law of total variance:

$$\mathrm{Var}(Y) = \mathbb{E}[\mathrm{Var}(Y \mid X)] + \mathrm{Var}[\mathbb{E}(Y \mid X)]$$

Intuitively, when we relate Y with another random variable X,

if Y has high variance, it comes from one of two sources:

- Either Y is highly variable even if you already know the value of X
- Or if not, then expected value of Y must change a lot as you vary X

Law of total variance (an inequality)

In particular, the law of total variance implies

$$\mathbb{E}[\mathrm{Var}(Y \mid X)] \leq \mathrm{Var}(Y)$$

- On average, after conditioning on X we have less variability in Y (as compared to variability without conditioning)
- However, this is on average — it's possible that $\mathrm{Var}(Y \mid X = x) > \mathrm{Var}(Y)$ for some values x

Law of total variance (proof)

Proof:

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \\
&= \mathbb{E}[\mathbb{E}(Y^2 \mid X)] - \mathbb{E}[\mathbb{E}(Y \mid X)]^2 \quad \leftarrow \text{Tower Law} \\
&= \mathbb{E}[\mathbb{E}(Y^2 \mid X)] - \mathbb{E}[\mathbb{E}(Y \mid X)^2] + \mathbb{E}[\mathbb{E}(Y \mid X)^2] - \mathbb{E}[\mathbb{E}(Y \mid X)]^2 \\
&= \mathbb{E}[\mathbb{E}(Y^2 \mid X) - \mathbb{E}(Y \mid X)^2] + \mathbb{E}[\mathbb{E}(Y \mid X)^2] - \mathbb{E}[\mathbb{E}(Y \mid X)]^2 \\
&= \mathbb{E}(\quad \mathrm{Var}(Y \mid X) \quad) + \quad \mathrm{Var}(\mathbb{E}(Y \mid X)) \quad \leftarrow \text{def. of variance}
\end{aligned}$$

Examples (use law of total variance)

Example 1. You play the following game: at each round, you toss a coin.

- If it's Heads, you roll a die and win \$1 if you rolled a 6
- If it's Tails, the game ends

What is the variance of the amount of money you win?

X = total \$ won, Y = # rounds played.

We will apply the law of total variance:

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}[\mathbb{E}(X \mid Y)]$$

Examples (use law of total variance, cont.)

We know that $X \mid Y = y \sim \text{Binomial}(y-1, \frac{1}{6})$, and so:

$$\mathbb{E}(X \mid Y = y) = (y-1)\cdot\frac{1}{6}, \quad \mathrm{Var}(X \mid Y = y) = (y-1)\cdot\frac{1}{6}\cdot\frac{5}{6}.$$

We also know that marginally $Y \sim \text{Geometric}(\frac{1}{2})$, and so

$$\mathbb{E}(Y) = 2, \quad \mathrm{Var}(Y) = 2.$$

Law of total variance:

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}(\mathrm{Var}(X \mid Y)) + \mathrm{Var}(\mathbb{E}(X \mid Y))\\ &= \mathbb{E}\left((Y-1)\cdot\frac{1}{6}\cdot\frac{5}{6}\right) + \mathrm{Var}\left((Y-1)\cdot\frac{1}{6}\right) = \frac{5}{36} + \frac{2}{36} = \frac{7}{36}.\end{aligned}$$

Examples (use law of total expectation/variance)

Example 2. Suppose a betting game has the following rules:

- Each ticket costs \$1
- For each ticket, with probability $\frac{1}{5}$ you win \$4, otherwise you win nothing

Suppose you buy two tickets.
Then, with any money you win, you buy additional tickets.
After these two rounds, you stop.

What is the expectation and variance of X, your total earnings?

Examples (use law of total expectation/variance cont.)

For a single ticket:

$$\mathbb{E}(\text{earnings}) = \tfrac{1}{5}\cdot 4 + \tfrac{4}{5}\cdot 0 = 0.8$$

$$\mathbb{E}(\text{earnings}^2) = \tfrac{1}{5}\cdot 4^2 + \tfrac{4}{5}\cdot 0^2 = 3.2$$

$$\Rightarrow \quad \mathrm{Var}(\text{earnings}) = 3.2 - 0.8^2 = 2.56$$

Examples (use law of total expectation/variance cont.)

Let Y = \$ after first round
= \$ from 1st ticket + \$ from 2nd ticket
So,

$$\mathbb{E}(Y) = \mathbb{E}(\text{\$ from 1st ticket}) + \mathbb{E}(\text{\$ from 2nd ticket}) = 0.8 + 0.8 = 1.6$$

$$\text{Var}(Y) = \text{Var}(\text{\$ from 1st ticket}) + \text{Var}(\text{\$ from 2nd ticket}) = 2.56 + 2.56 = 5.12$$

since the tickets' outcomes are independent

Examples (use law of total expectation/variance cont.)

Given $Y = y > 0$ (second round)

X = \$ after second round
= \$ from 1st ticket + ... + \$ from yth ticket

Then

$$\mathbb{E}(X \mid Y = y) = \mathbb{E}(\text{\$ from 1st ticket}) + \cdots + \mathbb{E}(\text{\$ from } y\text{th ticket}) = y \cdot 0.8$$

$$\text{Var}(X \mid Y = y) = \text{Var}(\text{\$ from 1st ticket}) + \cdots + \text{Var}(\text{\$ from } y\text{th ticket}) = y \cdot 2.56$$

since the tickets' outcomes are independent

And, if $Y = 0$ then $X = 0$, so the formulas above work for $y = 0$ as well.

Examples (use law of total expectation/variance cont.)

Apply the tower law for expectations,

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(0.8Y) = 0.8\mathbb{E}(Y) = 0.8 \cdot 1.6 = 1.28$$

Apply the law of total variance:

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}(\text{Var}(X \mid Y)) + \text{Var}(\mathbb{E}(X \mid Y)) \\ &= \mathbb{E}(2.56Y) + \text{Var}(0.8Y) \\ &= 2.56\mathbb{E}(Y) + 0.8^2\text{Var}(Y) \\ &= 2.56 \cdot 1.6 + 0.8^2 \cdot 5.12 = 7.3728 \end{aligned}$$