30100: Mathematical Statistics I **Winter 2022**

Homework 1

Lecturer: Chao Gao

The homework is due on Jan 20.

1. Read the general proof of factorization theorem in Lehmann and Romano (Page 45).

2. Consider i.i.d. $X_1, \cdots, X_n \sim \text{Uniform}(0, \theta)$ and the statistic $T = \max_{1 \leq i \leq n} X_i$. Derive the conditional distribution $(X_1, \cdots, X_n)|T$ and show it is independent of θ. Hint: You may first guess the answer and check it is correct according to the definition of conditional distribution. Or you can first solve Problem Problem 6.2 of Chapter 1 in Lehmann and Casella.

3. Is it true that a one-dimensional sufficient statistic must be minimal? Either prove it or construct a counter example.

4. Suppose there exists a one-dimensional minimal sufficient statistic. Does that imply any two-dimensional sufficient statistic to be non-minimal? Either prove it or construct a counter example.

5. Chapter 1, Problem 6.4 in Lehmann and Casella.

6. Chapter 1, Problem 6.7 in Lehmann and Casella.

7. For an exponential family of canonical form $\exp(\sum_{j=1}^{d} \eta_j T_j(x) - A(\eta))h(x)$, show $A(\eta)$ is a convex function and the natural parameter space $H = \{\eta : -\infty < A(\eta) < \infty\}$ is a convex set.

8. Consider a linear model $y|X \sim N(X\beta, I_n)$, where $X \in \mathbb{R}^{n\times p}$ is a random design matrix whose marginal distribution is arbitrary. Show it is an exponential family and write down a $p + p^2$ dimensional sufficient statistic for $\beta \in \mathbb{R}^p$.

9. Chapter 1, Problem 5.6 in Lehmann and Casella.

10. Chapter 1, Problem 5.25 (a) (b) in Lehmann and Casella.