STAT 245 HW3 Solution

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January 2023

Q1

Using the facts that $Z_1 \stackrel{d}{=} -Z_1$ if $Z_1 \sim N(0,1)$ and Z_1, Z_2 are independent, $\forall t$,

$$\begin{aligned}
P(Z_1/|Z_2| \leq t) &= P(Z_1/Z_2 \leq t|Z_2 > 0)P(Z_2 > 0) + P(-Z_1/Z_2 \leq t|Z_2 < 0)P(Z_2 < 0) \\
&= P(Z_1/Z_2 \leq t|Z_2 > 0)P(Z_2 > 0) + P(Z_1/Z_2 \leq t|Z_2 < 0)P(Z_2 < 0) \\
&= P(Z_1/Z_2 \leq t),
\end{aligned}$$

so $Z_1/|Z_2|$ is also Cauchy.

Q2

Using the fact that expectation is a linear, i.e. $E[AX] = AE[X]$ for any matrix A and random vector X,

$$\begin{aligned}
\mathrm{Cov(AX, BY)} &- E(AX - E[AX])(BY - E[BY])^T \\
&= E(AX)(BY)^T - E[AX]E[BY]^T \\
&= AE[XY^T]B^T - AE[X]E[Y]^T B^T \\
&= A(E[XY^T] - E[X]E[Y]^T)B^T \\
&= A\mathrm{Cov(X, Y)B^T}.
\end{aligned}$$

Q3

$$\begin{aligned}
f_{X_1,\cdots,X_n}(x_1, \cdots, x_n) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma^2}^n} e^{-\sum_{i=1}^n -\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi \mathrm{det}(\sigma^2 I_n)}} e^{-\frac{1}{2}(\vec{x}-\mu 1_n)^T(\sigma^2 I_n)(\vec{x}-\mu 1_n)},
\end{aligned}$$

so $(X_1, \cdots, X_n) \sim N_n(\mu 1_n, \sigma^2 I_n)$

Q4

(a)

$$P^2 = \frac{1}{n^2} 1_n (1_n^T 1_n) 1_n^T = \frac{1}{n} 1_n (n) 1_n^T = \frac{1}{n} 1_n 1_n^T = P.$$

Using this identity, we have

$$(I_n - P)^2 = I_n - 2P + P^2 = I_n - P = I_n - P,$$

and

$$(I_n - P)P = P - P^2 = 0.$$

(b) $\sum_{i=1}^n (Z_i - \bar{Z}) = \|Z - 1_n \bar{Z}\|^2 = \|Z - 1_n \frac{1}{n} 1_n^T Z\|^2 = \|Z - PZ\|^2$.
$\bar{Z} = \frac{1}{n} 1_n^T Z = \frac{1}{n} 1_n^T PZ + \frac{1}{n} 1_n^T (I_n - P)Z$, but $1_n^T (I_n - P) = 1_n^T - 1_n^T \frac{1}{n} 1_n 1_n^T = 1_n^T - 1_n^T = 0$.

(c) Notice that, $P^T = P$,

$$\begin{aligned} \mathrm{Cov}((I_n - P)Z, PZ) &= (I_n - P)\mathrm{Cov}(Z, Z)P^T \\ &= (I_n - P)I_n P^T \\ &= (I_n - P)P \\ &= 0. \end{aligned}$$

Q5

(a)

$$\begin{aligned} P(Z^2 \le t) &= P(-\sqrt{t} \le Z \le \sqrt{t}) \\ &= P(Z \le \sqrt{t}) - P(Z \le -\sqrt{t}) \\ &= P(Z \le \sqrt{t}) - P(Z \ge \sqrt{t}) \\ &= 2P(Z \le \sqrt{t}) - 1. \end{aligned}$$

(b)

$$\begin{aligned} \frac{d}{dt} P(Z^2 \le t) &= 2 \frac{d}{dt} P(Z \le \sqrt{t}) \\ &= t^{-1/2} \phi(\sqrt{t}) \\ &= \frac{1}{\sqrt{2\pi t}} e^{-t/2}. \end{aligned}$$

Q6

Method 1

Direct calculation through inverting block matrix, check this:
https://statproofbook.github.io/P/mvn-cond.

Method 2

Let $Z = X + AY$, if we can find A such that $\mathrm{Cov}(Z, Y) = 0$, then Z is independent of Y because they are normal.

$$\mathrm{Cov}(X + AY, Y) = \mathrm{Cov}(X, Y) + A\mathrm{Cov}(Y, Y) = \Sigma_{xy} + A\Sigma_{yy}.$$

Therefore $A = -\Sigma_{xy}\Sigma_{yy}^{-1}$, and we also have

$$\begin{aligned}
E[Z|Y] &= E[Z] \\
&= E[X] + AE[Y] \\
&= \mu_x - \Sigma_{xy}\Sigma_{yy}^{-1}\mu_y. \\
\mathrm{Cov(Z|Y)} &= \mathrm{Cov(Z)} \\
&= \mathrm{Cov(X, X) + ACov(Y, X) + ACov(X, Y)A^T + ACov(Y, Y)A^T} \\
&= \Sigma_x - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} + \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} \\
&= \Sigma_x - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}.
\end{aligned}$$

Then using the fact that

$$E[X|Y] = AY + E[Z|Y], \quad \mathrm{Cov}(X|Y) = \mathrm{Cov}(Z|Y),$$

we can easily get $X|Y \sim N(\mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(Y - \mu_y), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})$.

Q7

Method 1

Using symmetry, we know that $E[X_i|\bar{X}] = E[X_1|\bar{X}]$ for all $i = 1, \cdots, n$, so it's easy to show $E[X_i|\bar{X}] = \frac{1}{n}nE[X_i|\bar{X}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i|\bar{X}] = E[\bar{X}|\bar{X}] = \bar{X}$, then the rest of the questions are trivial to us.

Method 2

We know $E[X_1] = E[\bar{X}] = \mu$ and $\mathrm{Var}(X_1) = \sigma^2$ and $\mathrm{Var}(\bar{X}) = 1/n$. Moreover,

$$\mathrm{Cov}(X_1, \bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Cov}(X_1, X_i) = \frac{1}{n}\mathrm{Cov}(X_1, X_1) = 1/n.$$

Therefore,

$$\begin{pmatrix} X_1 \\ \bar{X} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} 1 & 1/n \\ 1/n & 1/n \end{pmatrix}\right).$$

Using the formula from Question 6 will give us $E[X_1|\bar{X}] = \mu + \frac{1}{n}n(\bar{X} - \mu) = \bar{X}$. It is obvious that we can get $E[X_i|\bar{X}] = \bar{X}$ for other i using the same method.

If we condition on the sample mean, the expected value for the average of any subset of the sample is the sample mean.