24300 HW5

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Problem 1: Eigenvalues of Important Transformations

(1)

Given the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To find the eigenvalues λ and corresponding eigenvectors $\mathbf{v}$ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Setting the determinant of $A - \lambda I$ to zero:

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{pmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (-\lambda)(-\lambda) - (1)(1) = \lambda^2 - 1$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

Eigenvalues: $\lambda = 1$ and $\lambda = -1$.

For $\lambda = 1$:

Solve $(A - I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This leads to the equation:

$$-v_1 + v_2 = 0 \implies v_2 = v_1$$

Eigenvector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -1$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This leads to the equation:

$$v_1 + v_2 = 0 \implies v_2 = -v_1$$

Eigenvector: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Interpretation

- The eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ remains unchanged after multiplication by A, meaning $A\mathbf{v} = \mathbf{v}$.
- The eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is reflected by A, meaning $A\mathbf{v} = -\mathbf{v}$.

(2)

Given the rotation matrix:

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The characteristic equation:

$$\det(B - \lambda I) = 0$$

$$B - \lambda I = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

$$\det(B - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$
$$(\cos\theta - \lambda)^2 = -\sin^2\theta$$
$$\cos\theta - \lambda = \pm i\sin\theta$$

$$\lambda = \cos\theta + i\sin\theta$$

or

$$\lambda = \cos\theta - i\sin\theta$$

Eigenvalues: $\lambda = \cos\theta + i\sin\theta \quad \text{or} \quad \cos\theta - i\sin\theta$

Or: $\lambda = e^{\pm i\theta} = \cos\theta \mp i\sin\theta$.

(3)

To find the eigenvalues λ of C, solve the characteristic equation:

$$\det(C - \lambda I) = 0$$

Compute $C - \lambda I$:

$$C - \lambda I = \frac{1}{6}\begin{pmatrix} 5-6\lambda & -1 & 2 \\ -1 & 5-6\lambda & 2 \\ 2 & 2 & 2-6\lambda \end{pmatrix}$$

Let $M = \begin{pmatrix} 5-6\lambda & -1 & 2 \\ -1 & 5-6\lambda & 2 \\ 2 & 2 & 2-6\lambda \end{pmatrix}$.

$$\det(M) = (5-6\lambda)\begin{vmatrix} 5-6\lambda & 2 \\ 2 & 2-6\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 2 \\ 2 & 2-6\lambda \end{vmatrix} + 2\begin{vmatrix} -1 & 5-6\lambda \\ 2 & 2 \end{vmatrix}$$

$$\begin{aligned} M_1 &= (5-6\lambda)(2-6\lambda) - (2)(2) \\ &= (10 - 30\lambda - 12\lambda + 36\lambda^2) - 4 \\ &= 6 - 42\lambda + 36\lambda^2 \end{aligned}$$

$$\begin{aligned} M_2 &= -1(2-6\lambda) - (2)(2) \\ &= (-2+6\lambda) - 4 \\ &= 6\lambda - 6 \end{aligned}$$

$$\begin{aligned} M_3 &= -1(2) - (5-6\lambda)(2) \\ &= (-2) - (10-12\lambda) \\ &= -12 + 12\lambda \end{aligned}$$

$$\begin{aligned} \det(M) &= (5-6\lambda)(6-42\lambda+36\lambda^2) + (6\lambda-6) + 2(-12+12\lambda) \\ &= [30 - 210\lambda + 180\lambda^2 - 36\lambda + 252\lambda^2 - 216\lambda^3] + (6\lambda - 6) + (-24 + 24\lambda) \\ &= [30 - 246\lambda + 432\lambda^2 - 216\lambda^3] + (30\lambda - 30) \\ &= -216\lambda + 432\lambda^2 - 216\lambda^3 \end{aligned}$$

$$\det(M) = -216\lambda(1 - 2\lambda + \lambda^2)$$

Set to 0:

$$\det(M) = -216\lambda(\lambda-1)^2 = 0$$
$$\lambda(\lambda-1)^2 = 0$$

Eigenvalues: $\lambda = 0$ and $\lambda = 1$ (with multiplicity 2).

For $\lambda = 1$:

Solve $(C - I)\mathbf{v} = \mathbf{0}$.

Compute $C - I$:

$$C - I = \frac{1}{6}\begin{pmatrix} 5-6 & -1 & 2 \\ -1 & 5-6 & 2 \\ 2 & 2 & 2-6 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & -4 \end{pmatrix}$$

Multiply both sides by 6:

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Set up the system of equations:

$$\begin{aligned} -v_1 - v_2 + 2v_3 &= 0 \quad (1) \\ -v_1 - v_2 + 2v_3 &= 0 \quad (2) \\ 2v_1 + 2v_2 - 4v_3 &= 0 \quad (3) \end{aligned}$$

Equation (1) and (2):

$$-v_1 - v_2 + 2v_3 = 0 \implies v_1 + v_2 = 2v_3$$

Equation (3):

$$2v_1 + 2v_2 - 4v_3 = 0 \implies v_1 + v_2 = 2v_3$$

$$v_1 + v_2 = 2v_3$$

Let $v_2 = s$ and $v_3 = t$, where s and t are free variables.

$$v_1 = 2v_3 - v_2 = 2t - s$$

First Eigenvector ($\mathbf{v}^{(1)}$):

Let $s = 0$ and $t = 1$:

$$\begin{aligned} v_1 &= 2(1) - 0 = 2 \\ v_2 &= 0 \\ v_3 &= 1 \end{aligned}$$

Thus,

$$\mathbf{v}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Second Eigenvector ($\mathbf{v}^{(2)}$):

Let $s = 1$ and $t = 0$:

$$\begin{aligned} v_1 &= 2(0) - 1 = -1 \\ v_2 &= 1 \\ v_3 &= 0 \end{aligned}$$

Thus,

$$\mathbf{v}^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

These two eigenvectors are linearly independent and correspond to the eigenvalue $\lambda = 1$.

For $\lambda = 0$:

To solve $C\mathbf{v} = \mathbf{0}$.

Multiply both sides by 6:

$$\begin{pmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Set up the system of equations:

$$\begin{aligned} 5v_1 - v_2 + 2v_3 &= 0 \quad (1) \\ -v_1 + 5v_2 + 2v_3 &= 0 \quad (2) \\ 2v_1 + 2v_2 + 2v_3 &= 0 \quad (3) \end{aligned}$$

From equation (3):

$$v_1 + v_2 + v_3 = 0 \implies v_3 = -v_1 - v_2$$

Substitute v_3 into equations (1) and (2):

Equation (1):

$$5v_1 - v_2 + 2(-v_1 - v_2) = 0 \implies 5v_1 - v_2 - 2v_1 - 2v_2 = 0 \implies 3v_1 - 3v_2 = 0$$

$$v_1 = v_2$$

Equation (2):

$$-v_1 + 5v_2 + 2(-v_1 - v_2) = 0 \implies -v_1 + 5v_2 - 2v_1 - 2v_2 = 0 \implies -3v_1 + 3v_2 = 0$$

$$v_1 = v_2$$

$$v_3 = -v_1 - v_2 = -2v_1$$

Therefore, the eigenvector corresponding to $\lambda = 0$ is:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Eigenvector: $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Therefore: Eigenvalues of C are $\lambda = 0$ and $\lambda = 1$ (with multiplicity 2).

Eigenvectors for $\lambda = 1$: $\mathbf{v}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $\mathbf{v}^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Eigenvector for $\lambda = 0$: $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

(4)

Given that P is a projector ($P^2 = P$), and $\mathbf{v}$ is an eigenvector with eigenvalue λ:

$$P\mathbf{v} = \lambda\mathbf{v}$$

$$P^2\mathbf{v} = P(P\mathbf{v}) = P(\lambda\mathbf{v}) = \lambda P\mathbf{v} = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$$

Since $P^2 = P$:

$$P^2\mathbf{v} = P\mathbf{v} \implies \lambda^2\mathbf{v} = \lambda\mathbf{v}$$

So we have:

$$\lambda^2\mathbf{v} = \lambda\mathbf{v}$$

$$\lambda^2\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \implies \lambda(\lambda - 1)\mathbf{v} = \mathbf{0}$$

Since $\mathbf{v} \neq \mathbf{0}$, it holds that:

$$\lambda(\lambda - 1) = 0$$

Therefore, the eigenvalues of P are $\lambda = 0$ or $\lambda = 1$.

Question 2: Eigenvalues of 2×2 Matrices

(1)

Consider the generic 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The trace of A is:

$$\text{trace}(A) = a + d.$$

The determinant of A is:

$$\det(A) = ad - bc.$$

(2)

The characteristic polynomial $p(\lambda)$ of A is given by:

$$p(\lambda) = \det(A - \lambda I),$$

where I is the identity matrix.

$$A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc$$

$$
\begin{aligned}
&= ad - a\lambda - d\lambda + \lambda^2 - bc \\
&= \lambda^2 - (a+d)\lambda + (ad - bc) \\
&= \lambda^2 - \text{trace}(A) \cdot \lambda + \det(A)
\end{aligned}
$$

This shows that:

$$\det(A - \lambda I) = \lambda^2 - \text{trace}(A) \cdot \lambda + \det(A).$$

(3)

The eigenvalues are the roots of the characteristic equation:

$$\lambda^2 - \text{trace}(A) \cdot \lambda + \det(A) = 0.$$

Applying the quadratic formula:

$$\lambda = \frac{\text{trace}(A) \pm \sqrt{[\text{trace}(A)]^2 - 4\det(A)}}{2}.$$

Thus, the eigenvalues of A in terms of its trace and determinant are:

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}.$$

(4)

The discriminant D of the quadratic equation determines the nature of the eigenvalues:

$$D = [\text{trace}(A)]^2 - 4\det(A).$$

Eigenvalues are real if $D \geq 0$:

$$
\begin{aligned}
&[\text{trace}(A)]^2 - 4\det(A) \geq 0 \\
&(a+d)^2 - 4(ad - bc) \geq 0.
\end{aligned}
$$

Eigenvalues are identical (repeated root) if $D = 0$:

$$[\text{trace}(A)]^2 - 4\det(A) = 0$$

$$(a+d)^2 - 4(ad - bc) = 0.$$

Eigenvalues are complex conjugates if $D < 0$:

$$[\text{trace}(A)]^2 - 4\det(A) < 0$$

$$(a+d)^2 - 4(ad - bc) < 0.$$

Question 3: Determinants of Jacobians and Integration after Change of Coordinates

(1)

The Jacobian matrix $J(f(y))$ is defined as:

$$J(f(y)) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} \end{pmatrix}.$$

Each partial derivative:

$$\begin{aligned} \frac{\partial f_1}{\partial y_1} &= 2y_1, \\ \frac{\partial f_1}{\partial y_2} &= y_3, \\ \frac{\partial f_1}{\partial y_3} &= y_2, \\ \frac{\partial f_2}{\partial y_1} &= 0, \\ \frac{\partial f_2}{\partial y_2} &= 1, \\ \frac{\partial f_2}{\partial y_3} &= 1, \\ \frac{\partial f_3}{\partial y_1} &= 0, \\ \frac{\partial f_3}{\partial y_2} &= 0, \\ \frac{\partial f_3}{\partial y_3} &= 3y_3^2. \end{aligned}$$

Therefore, the Jacobian matrix is:

$$J(f(y)) = \begin{pmatrix} 2y_1 & y_3 & y_2 \\ 0 & 1 & 1 \\ 0 & 0 & 3y_3^2 \end{pmatrix}.$$

(2)

$$\det(J(f(y))) = \det \begin{pmatrix} 2y_1 & y_3 & y_2 \\ 0 & 1 & 1 \\ 0 & 0 & 3y_3^2 \end{pmatrix}.$$

Since the matrix is upper triangular, the determinant is the product of the diagonal elements:

$$\det(J(f(y))) = (2y_1) \times 1 \times (3y_3^2) = 6y_1y_3^2.$$

Thus,

$$\det(J(f(y))) = 6y_1y_3^2.$$

(3)

The transform is:

$$\begin{cases} x_1 = f_1(y) = y_1 \cos(y_2), \\ x_2 = f_2(y) = y_1 \sin(y_2). \end{cases}$$

The Jacobian matrix $J(f(y))$ is:

$$J(f(y)) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}.$$

Each partial derivatives:

$$\begin{aligned} \frac{\partial f_1}{\partial y_1} &= \cos(y_2), \\ \frac{\partial f_1}{\partial y_2} &= -y_1 \sin(y_2), \\ \frac{\partial f_2}{\partial y_1} &= \sin(y_2), \\ \frac{\partial f_2}{\partial y_2} &= y_1 \cos(y_2). \end{aligned}$$

Thus, the Jacobian matrix is:

$$J(f(y)) = \begin{pmatrix} \cos(y_2) & -y_1 \sin(y_2) \\ \sin(y_2) & y_1 \cos(y_2) \end{pmatrix}.$$

Find the determinant:

$$\begin{aligned} \det\left(J(f(y))\right) &= \cos(y_2) \cdot (y_1 \cos(y_2)) - (-y_1 \sin(y_2)) \cdot \sin(y_2) \\ &= y_1 \cos^2(y_2) + y_1 \sin^2(y_2) \\ &= y_1 \left(\cos^2(y_2) + \sin^2(y_2)\right) \\ &= y_1. \end{aligned}$$

Therefore, the determinant of the Jacobian is equal to the radius r:

$$\det\left(J(f(y))\right) = y_1 = r.$$

Question 4: Diagonalization

(1)

$$A - \lambda I = \begin{pmatrix} -5-\lambda & 6 & 9 \\ 0 & -2-\lambda & 0 \\ -3 & 6 & 7-\lambda \end{pmatrix}.$$

$$
\begin{aligned}
p(\lambda) &= \det \begin{pmatrix} -5-\lambda & 6 & 9 \\ 0 & -2-\lambda & 0 \\ -3 & 6 & 7-\lambda \end{pmatrix} \\
&= 0 \cdot \det \begin{pmatrix} 6 & 9 \\ 6 & 7-\lambda \end{pmatrix} + (-2-\lambda) \cdot \det \begin{pmatrix} -5-\lambda & 9 \\ -3 & 7-\lambda \end{pmatrix} + 0 \cdot \det \begin{pmatrix} -5-\lambda & 6 \\ -3 & 6 \end{pmatrix} \\
&= (-2-\lambda)\left[(-5-\lambda)(7-\lambda) - (-3)(9)\right] \\
&= (-2-\lambda)\left[(-5-\lambda)(7-\lambda) + 27\right].
\end{aligned}
$$

Therefore,

$$p(\lambda) = (-2-\lambda)(\lambda^2 - 2\lambda - 8).$$

$$p(\lambda) = -(\lambda+2)^2(\lambda-4).$$

Setting $p(\lambda) = 0$, the eigenvalues are:

$$\lambda = -2 \quad \text{(algebraic multiplicity 2)}, \quad \lambda = 4.$$

Therefore, A has eigenvalues $\lambda = -2$ (repeated) and $\lambda = 4$.

(2)

The vectors are:

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For v_1:

$$
\begin{aligned}
Av_1 &= \begin{pmatrix} -5 & 6 & 9 \\ 0 & -2 & 0 \\ -3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \cdot 2 + 6 \cdot 1 + 9 \cdot 0 \\ 0 \cdot 2 + (-2) \cdot 1 + 0 \cdot 0 \\ -3 \cdot 2 + 6 \cdot 1 + 7 \cdot 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix} \\
&= -2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -2v_1.
\end{aligned}
$$

Thus, v_1 is an eigenvector corresponding to $\lambda = -2$.

For v_2:

$$
\begin{aligned}
Av_2 &= \begin{pmatrix} -5 & 6 & 9 \\ 0 & -2 & 0 \\ -3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \cdot 3 + 6 \cdot 0 + 9 \cdot 1 \\ 0 \cdot 3 + (-2) \cdot 0 + 0 \cdot 1 \\ -3 \cdot 3 + 6 \cdot 0 + 7 \cdot 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -2 \end{pmatrix} \\
&= -2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = -2v_2.
\end{aligned}
$$

Therefore, v_2 is also an eigenvector corresponding to $\lambda = -2$.

For v_3:

$$
\begin{aligned}
Av_3 &= \begin{pmatrix} -5 & 6 & 9 \\ 0 & -2 & 0 \\ -3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \cdot 1 + 6 \cdot 0 + 9 \cdot 1 \\ 0 \cdot 1 + (-2) \cdot 0 + 0 \cdot 1 \\ -3 \cdot 1 + 6 \cdot 0 + 7 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} \\
&= 4 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4v_3.
\end{aligned}
$$

Thus, v_3 is an eigenvector corresponding to $\lambda = 4$.

The eigenvalue $\lambda = -2$ has two linearly independent eigenvectors v_1 and v_2. Therefore, the geometric multiplicity of $\lambda = -2$ is 2.

The nullity of $A + 2I$ is equal to the geometric multiplicity of $\lambda = -2$. Since we have two independent eigenvectors for $\lambda = -2$, which can be derived by solving $A + 2I = 0$, the dimension of the null space of $A + 2I$ is 2, and the nullity is 2.

(3)

Matrix V:

$$V = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Swap Row 1 and Row 2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

R2 = R2 - 2 R1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Simplify Row 2:

$$R2 = \begin{pmatrix} 0 & 3 & 1 & 1 & -2 & 0 \end{pmatrix}$$

$$R2 = \frac{1}{3} R2$$

R3 = R3 - R2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$$R3 = \frac{3}{2} R3$$

$R2 = R2 - \frac{1}{3} R3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{3}{2} \end{array}\right]$$

Thus,

$$V^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}$$

Since we have V^{-1} using Gauss-Jordan elimination, V is invertible.

(4)

Given

$$V = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{1}{2} & -1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & 1 & \dfrac{3}{2} \end{pmatrix}$$

$$V\Lambda = V \cdot \Lambda = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -4 & -6 & 4 \\ -2 & 0 & 0 \\ 0 & -2 & 4 \end{pmatrix}$$

$$A = (V\Lambda)V^{-1} = \begin{pmatrix} -4 & -6 & 4 \\ -2 & 0 & 0 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{1}{2} & -1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & 1 & \dfrac{3}{2} \end{pmatrix}$$

First row of A:

$$\begin{aligned} a_{11} &= (-4)(0) + (-6)\left(\frac{1}{2}\right) + (4)\left(-\frac{1}{2}\right) = 0 - 3 - 2 = -5 \\ a_{12} &= (-4)(1) + (-6)(-1) + (4)(1) = -4 + 6 + 4 = 6 \\ a_{13} &= (-4)(0) + (-6)\left(-\frac{1}{2}\right) + (4)\left(\frac{3}{2}\right) = 0 + 3 + 6 = 9 \end{aligned}$$

Second row of A:

$$\begin{aligned} a_{21} &= (-2)(0) + (0)\left(\frac{1}{2}\right) + (0)\left(-\frac{1}{2}\right) = 0 + 0 + 0 = 0 \\ a_{22} &= (-2)(1) + (0)(-1) + (0)(1) = -2 + 0 + 0 = -2 \\ a_{23} &= (-2)(0) + (0)\left(-\frac{1}{2}\right) + (0)\left(\frac{3}{2}\right) = 0 + 0 + 0 = 0 \end{aligned}$$

Third row of A:

$$\begin{aligned} a_{31} &= (0)(0) + (-2)\left(\frac{1}{2}\right) + (4)\left(-\frac{1}{2}\right) = 0 - 1 - 2 = -3 \\ a_{32} &= (0)(1) + (-2)(-1) + (4)(1) = 0 + 2 + 4 = 6 \\ a_{33} &= (0)(0) + (-2)\left(-\frac{1}{2}\right) + (4)\left(\frac{3}{2}\right) = 0 + 1 + 6 = 7 \end{aligned}$$

Thus, the matrix A is:

$$A = \begin{pmatrix} -5 & 6 & 9 \\ 0 & -2 & 0 \\ -3 & 6 & 7 \end{pmatrix}$$

We have verified that:

$$A = V\Lambda V^{-1}$$

Therefore, A is diagonalizable with the given V and Λ.