Assignment 7 (two pages plus two tables)

Statistics 24400 (Autumn 2024)

Due on Gradescope, Tuesday, November 19 by 9 am

Requirements

Your answers should be typed or clearly written, started with your name, Assignment 7, STAT 24400;
saved as `Lastname-Firstname-hw7.pdf`, uploaded to Gradescope under P-set7, tag the pages for each question.
You may discuss approaches with others. However the assignment should be devised and written by yourself.
To get full credit, you must provide the reasoning of main steps of the derivation leading to your answer.

Problem assignments (related sections in the text: 6.2-6.3, 7.3, 8.3-8.5)

1. (*Common sampling distributions*)

   Suppose X_i *i.i.d.* $\sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$, and Z_j *i.i.d.* $\sim N(0, 1)$ for $j = 1, \ldots, k$, and all variables X_i, Z_j's are independent. Define $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and $\bar{Z} = \frac{1}{k}\sum_{j=1}^{k} Z_j$. Identify the distribution of each of the following variables, including the values of the parameters (e.g., degrees of freedom for χ^2, t distributions, mean&variance for normal distribution).

   (a) $\bar{X} + \bar{Z}$

   (b) $k\bar{Z}^2$

   (c) $Z_1/\sqrt{Z_2^2}$

   (d) $Z_1^2 + Z_2^2$

   (e) $\displaystyle\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} + \sum_{j=1}^{k}(Z_i - \bar{Z})^2$

   (f) $\dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma\sqrt{\sum_{j=1}^{k} Z_j^2/k}}$

2. (*Confidence interval for the mean*)

   Suppose that $X_1, \ldots, X_{20}$ are *i.i.d.* samples from a $N(\mu, \sigma^2)$ distribution, with μ and σ^2 unknown. Let $\bar{X} = 14.1$ and $S^2 = 9.9$ be the sample mean and sample variance of our data. Calculate a 95% confidence interval for μ.
   (The t-distribution table in Rice's textbook is attached at the end of this assignment.)

   Notes: To obtain probabilities and cutoff values for the t distribution, you can use the attached t-table or search online for "t distribution table" — be aware that the notation $t_{.95}$ in this table is *not* the same as the critical t value for 95% confidence (you should look at the picture in the table to see what is meant by $t_{.95}$). Or use R commands `pt` & `qt` instead.

3. (*Properties of sample estimators*)

   (a) Suppose $\hat{\theta}_1$ and $\hat{\theta}_2$ are uncorrelated and both are unbiased estimators of parameter θ, and that $\mathrm{Var}(\hat{\theta}_1) = 2 \cdot \mathrm{Var}(\hat{\theta}_2)$.

      i. Show that for any constant c, the wighted average $\hat{\theta}_3 = c\hat{\theta}_1 + (1 - c)\hat{\theta}_2$ is an unbiased estimator.

      ii. Find the c for which $\hat{\theta}_3$ has the smallest mean squared error MSE $= \mathbb{E}[(\hat{\theta}_3 - \theta)^2]$.

      iii. Are there any values of $c \in [0, 1]$ for which $\hat{\theta}_3$ is better (in the sense of MSE) than both $\hat{\theta}_1$ and $\hat{\theta}_2$? Which values?

   (b) Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ where $0 < p < 1$. Calculate $\mathrm{Cov}(\bar{X}, S^2)$.

      i. Is the covariance zero or nonzero (your answer might depend on p)?

      ii. Are $\bar{X}$ and S^2 independent or not independent (your answer might depend on p)?

   (Hint: It might be easier to make a table of all the possibilities and calculate $\bar{X}$ and S^2 for each case.)

4. (*Method of maximum likelihood*)

   Suppose that we are working with a family of densities

   $$f(x \mid \theta) = (\theta + 1)x^\theta$$

   supported on $x \in [0, 1]$. The parameter θ must satisfy $\theta > -1$ for this to be a valid density.

   (a) Suppose we draw i.i.d. samples $X_1, \ldots, X_n$ from the density $f(x \mid \theta)$.
   Calculate the maximum likelihood estimator (MLE), $\hat{\theta}$.

   (b) Calculate the Fisher information $I(\theta)$.

   (c) Calculate the approximate normal distribution of the MLE $\hat{\theta}$, if the true parameter is θ_0 and the sample size n is large. (The mean and variance of the normal may depend on n and/or θ_0, but should not depend on any other quantities.)

5. (*Method of moments*)

   Consider the same parametric family as in the problem above,

   $$f(x \mid \theta) = (\theta + 1)x^\theta$$

   supported on $x \in [0, 1]$, for some parameter $\theta > -1$. Above we estimated θ with its MLE $\hat{\theta}$. In this problem we will use the method of moments (MoM) for estimating θ, instead of the MLE.

   (a) Assuming a known value of the parameter θ, calculate $\mu = \mathbb{E}(X)$ where X is a single draw from the density $f(x \mid \theta)$.

   (b) Now solve for the MoM estimator $\hat{\theta}$ as a function of the data $X_1, \ldots, X_n$.

6. (*Estimating variance for normal data*)

   Suppose that we observe data $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathrm{N}(0, \theta)$, where the variance θ is unknown.

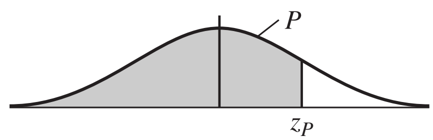
   (Note: We are writing variance $= \theta$, instead of σ^2 to make it clear that we are working with the variance rather than with the standard deviation as our parameter.)

   (a) Calculate the MLE estimate for θ as a function of the data $X_1, \ldots, X_n$.

   (b) Calculate the Fisher information $I(\theta)$.

   (c) Calculate the approximate normal distribution of the MLE $\hat{\theta}$, if the true variance is θ_0 and the sample size n is large. (The mean and variance of the normal may depend on n and/or θ_0, but should not depend on any other quantities.)

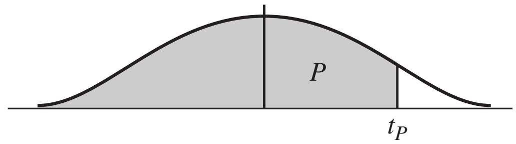
TABLE 2 Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for –1.62 equals 1 – .9474 = .0526.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

TABLE 4 Percentiles of the *t* Distribution



df	$t_{.60}$	$t_{.70}$	$t_{.80}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$
1	.325	.727	1.376	3.078	6.314	12.706	31.821	63.657
2	.289	.617	1.061	1.886	2.920	4.303	6.965	9.925
3	.277	.584	.978	1.638	2.353	3.182	4.541	5.841
4	.271	.569	.941	1.533	2.132	2.776	3.747	4.604
5	.267	.559	.920	1.476	2.015	2.571	3.365	4.032
6	.265	.553	.906	1.440	1.943	2.447	3.143	3.707
7	.263	.549	.896	1.415	1.895	2.365	2.998	3.499
8	.262	.546	.889	1.397	1.860	2.306	2.896	3.355
9	.261	.543	.883	1.383	1.833	2.262	2.821	3.250
10	.260	.542	.879	1.372	1.812	2.228	2.764	3.169
11	.260	.540	.876	1.363	1.796	2.201	2.718	3.106
12	.259	.539	.873	1.356	1.782	2.179	2.681	3.055
13	.259	.538	.870	1.350	1.771	2.160	2.650	3.012
14	.258	.537	.868	1.345	1.761	2.145	2.624	2.977
15	.258	.536	.866	1.341	1.753	2.131	2.602	2.947
16	.258	.535	.865	1.337	1.746	2.120	2.583	2.921
17	.257	.534	.863	1.333	1.740	2.110	2.567	2.898
18	.257	.534	.862	1.330	1.734	2.101	2.552	2.878
19	.257	.533	.861	1.328	1.729	2.093	2.539	2.861
20	.257	.533	.860	1.325	1.725	2.086	2.528	2.845
21	.257	.532	.859	1.323	1.721	2.080	2.518	2.831
22	.256	.532	.858	1.321	1.717	2.074	2.508	2.819
23	.256	.532	.858	1.319	1.714	2.069	2.500	2.807
24	.256	.531	.857	1.318	1.711	2.064	2.492	2.797
25	.256	.531	.856	1.316	1.708	2.060	2.485	2.787
26	.256	.531	.856	1.315	1.706	2.056	2.479	2.779
27	.256	.531	.855	1.314	1.703	2.052	2.473	2.771
28	.256	.530	.855	1.313	1.701	2.048	2.467	2.763
29	.256	.530	.854	1.311	1.699	2.045	2.462	2.756
30	.256	.530	.854	1.310	1.697	2.042	2.457	2.750
40	.255	.529	.851	1.303	1.684	2.021	2.423	2.704
60	.254	.527	.848	1.296	1.671	2.000	2.390	2.660
120	.254	.526	.845	1.289	1.658	1.980	2.358	2.617
∞	.253	.524	.842	1.282	1.645	1.960	2.326	2.576