# 24300 HW7

Bin Yu

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## Question 1

Prove: if $Q$ is an orthonormal (orthogonal) matrix, then $\|Qx\| = \|x\|$ for any $x$.

Since $Q$ is orthonormal, we have $Q^TQ = I$. Hence, for any vector $x$,

$$\|Qx\|^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|^2.$$

Taking square roots gives $\|Qx\| = \|x\|$.

By the singular value decomposition, any matrix $A \in R^{m\times n}$ of rank $r$ can be written as

$$A = U\Sigma V^T,$$

where $U \in R^{m\times m}$ and $V \in R^{n\times n}$ are orthonormal matrices, and

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix}_{m\times n},$$

with $r = \text{rank}(A)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. The remaining singular values are zero.

For any nonzero vector $x \in R^n$,

$$\frac{\|Ax\|}{\|x\|} = \frac{\|U\Sigma V^Tx\|}{\|x\|}.$$

Set $y = V^Tx$. Since $V$ is orthonormal, $\|y\| = \|x\|$. Thus,

$$\frac{\|Ax\|}{\|x\|} = \frac{\|U\Sigma y\|}{\|y\|}.$$

$\|U\Sigma y\| = \|\Sigma y\|$ because $U$ is orthonormal:

$$\frac{\|Ax\|}{\|x\|} = \frac{\|\Sigma y\|}{\|y\|}.$$

If $y = (y_1, y_2, \ldots, y_n)^T$, then

$$\Sigma y = (\sigma_1 y_1, \sigma_2 y_2, \ldots, \sigma_r y_r, 0, \ldots, 0)^T,$$

Thus,

$$\|\Sigma y\|^2 = \sum_{i=1}^{r} \sigma_i^2 y_i^2.$$

Since $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, and given $r \leq \min(m, n)$:

$$\|\Sigma y\|^2 \leq \sigma_1^2 \sum_{i=1}^{r} y_i^2 \leq \sigma_1^2 \|y\|^2.$$

Taking square roots,

$$\|\Sigma y\| \leq \sigma_1 \|y\|.$$

$$\frac{\|\Sigma y\|}{\|y\|} \leq \sigma_1.$$

Since this holds for all $y \neq 0$:

$$\max_{\|y\| \neq 0} \frac{\|\Sigma y\|}{\|y\|} = \sigma_1.$$

Therefore,

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_1.$$

## Question 2

By definition, the Frobenius norm of $A$ is

$$\|A\|_{\text{Fro}}^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2.$$

For m by n matrix A:

$$A^\top A \in R^{n \times n},$$

and the $(j, j)$-th entry of $A^\top A$ is

$$(A^\top A)_{jj} = \sum_{i=1}^{m} A_{ij}^2.$$

Therefore, summing over $j$,

$$\text{Trace}(A^\top A) = \sum_{j=1}^{n} (A^\top A)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij}^2 = \|A\|_{\text{Fro}}^2.$$

$$\|A\|_{\text{Fro}}^2 = \text{Trace}(A^\top A).$$

And,

$$\text{Trace}(AB) = \sum_{i=1}^{m} (AB)_{ii}.$$

the $(i, i)$-th element of $AB$ is given by

$$(AB)_{ii} = \sum_{k=1}^{n} A_{ik} B_{ki},$$

Therefore,

$$\text{Trace}(AB) = \sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} B_{ki}.$$

The matrix $BA$ is $n \times n$, and its $(j, j)$-th element is

$$(BA)_{jj} = \sum_{l=1}^{m} B_{jl} A_{lj}.$$

$$\text{Trace}(BA) = \sum_{j=1}^{n} (BA)_{jj} = \sum_{j=1}^{n} \sum_{l=1}^{m} B_{jl} A_{lj}.$$

$$\text{Trace}(AB) = \sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} B_{ki},$$
$$\text{Trace}(BA) = \sum_{j=1}^{n} \sum_{l=1}^{m} B_{jl} A_{lj}.$$

let $i = l$ and $k = j$:

$$\text{Trace}(BA) = \sum_{k=1}^{n} \sum_{i=1}^{m} B_{ki} A_{ik}.$$

Since multiplication of scaler is commutative, $A_{ik} B_{ki} = B_{ki} A_{ik}$. Thus,

$$\text{Trace}(AB) = \text{Trace}(BA).$$

By the SVD:

$$A = U \Sigma V^\top,$$

where $U \in R^{m \times m}$ and $V \in R^{n \times n}$ are orthonormal matrices, and

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix} \in R^{m \times n},$$

with $r = \text{rank}(A)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Starts from:

$$\|A\|_{\text{Fro}}^2 = \text{Trace}(A^\top A).$$

Substitute $A = U \Sigma V^\top$:

$$A^\top A = (U \Sigma V^\top)^\top (U \Sigma V^\top) = V \Sigma^\top U^\top U \Sigma V^\top.$$

Since $U$ is orthonormal, $U^\top U = I$. Thus

$$A^\top A = V \Sigma^\top \Sigma V^\top.$$

Therefore,

$$\|A\|_{\text{Fro}}^2 = \text{Trace}(A^\top A) = \text{Trace}(V\Sigma^\top \Sigma V^\top).$$

Using the cyclic property of the trace,

$$\text{Trace}(V\Sigma^\top \Sigma V^\top) = \text{Trace}(\Sigma^\top \Sigma V^\top V).$$

Since $V^\top V = I$, we have

$$\text{Trace}(V\Sigma^\top \Sigma V^\top) = \text{Trace}(\Sigma^\top \Sigma).$$

The matrix $\Sigma^\top \Sigma$ is a diagonal matrix whose diagonal entries are the squared singular values $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ (and zeros for any remaining diagonal elements if $r < \min(m, n)$). Thus,

$$\text{Trace}(\Sigma^\top \Sigma) = \sum_{j=1}^{r} \sigma_j^2.$$

we have shown that

$$\|A\|_{\text{Fro}}^2 = \sum_{j=1}^{r} \sigma_j^2.$$

# Question 3

## (1)

First, consider $A^\top = (U\Sigma V^\top)^\top = V\Sigma^\top U^\top$. Since $\Sigma$ has the form:

$$\Sigma = \begin{pmatrix} D \\ 0 \end{pmatrix} \quad \text{where } D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r).$$

Thus,

$$A^\top = V\Sigma^\top U^\top = V \begin{pmatrix} D & 0 \end{pmatrix} U^\top.$$

Consider a vector $x \in R^m$. Then

$$A^\top x = V \begin{pmatrix} D & 0 \end{pmatrix} U^\top x.$$

Let $y = U^\top x$. Since $U$ is orthonormal, $y$ is just the coordinates of $x$ in the basis formed by the columns of $U$:

$$U^\top x = y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

$$A^\top x = V \begin{pmatrix} D & 0 \end{pmatrix} y = V \begin{pmatrix} Dy_{1:r} \\ 0 \end{pmatrix} = V \begin{pmatrix} \sigma_1 y_1 \\ \sigma_2 y_2 \\ \vdots \\ \sigma_r y_r \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $y_{1:r} = (y_1, y_2, \ldots, y_r)^\top$ are the first $r$ components of $y$.

Since $V = [v_1 \ v_2 \ \cdots \ v_n]$:

$$A^\top x = \sigma_1 y_1 v_1 + \sigma_2 y_2 v_2 + \cdots + \sigma_r y_r v_r.$$

Here no vectors $v_{r+1}, \ldots, v_n$ appear in this linear combination. As $x$ varies over all of $R^m$, the vector $(y_1, \ldots, y_r)$ can produce any vector in $R^r$. Therefore, $A^\top x$ can produce any linear combination of $v_1, \ldots, v_r$, which shows:

$$\text{range}(A^\top) \subseteq \text{span}(v_1, \ldots, v_r).$$

For reverse inclusion, for any vector $w \in \text{span}(v_1, \ldots, v_r)$. We can write $w = \sum_{i=1}^r c_i v_i$ for some scalars $c_i$. If we let $y_{1:r} = (c_1/\sigma_1, \ldots, c_r/\sigma_r)$ and $y_{r+1:m} = 0$, and set $x = Uy$ (since $U$ is orthonormal), then by the same calculation $A^\top x = w$. Therefore, every vector in $\text{span}(v_1, \ldots, v_r)$ can be attained by $A^\top x$ for some $x$. Thus

$$\text{range}(A^\top) = \text{span}(v_1, \ldots, v_r).$$

## (2)

the null space of $A^\top$:

$$\text{null}(A^\top) = \{x \in R^m : A^\top x = 0\}.$$

From the previous part,

$$A^\top = V \begin{pmatrix} D & 0 \end{pmatrix} U^\top.$$

If $A^\top x = 0$, then

$$V(D, 0)U^\top x = 0.$$

Set $y = U^\top x$:

$$(D, 0)y = 0 \implies Dy_{1:r} = 0.$$

Since $D$ is invertible (all $\sigma_i > 0$), $Dy_{1:r} = 0$ implies $y_1 = \cdots = y_r = 0$.

Therefore,

$$y = \begin{pmatrix} 0 \\ y_{r+1:m} \end{pmatrix}$$

for some $y_{r+1:m} \in R^{m-r}$.

From $x = Uy$ (since $U$ is orthonormal),we have

$$x = U \begin{pmatrix} 0 \\ y_{r+1:m} \end{pmatrix} = y_{r+1} u_{r+1} + \cdots + y_m u_m.$$

Thus, any vector in $\text{null}(A^\top)$ is a linear combination of $u_{r+1}, \ldots, u_m$. Conversely, if $x$ is in the span of $u_{r+1}, \ldots, u_m$, then $U^\top x$ has zeros in the first $r$ components, ensuring $(D, 0)y = 0$ and thus $A^\top x = 0$.

Therefore:

$$\text{null}(A^\top) = \text{span}(u_{r+1}, \ldots, u_m).$$

## Question 4

### (1)

**Given:**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The singular values of $A$ are the square roots of the eigenvalues of $A^\top A$. First, compute $A^\top$:

$$A^\top = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, compute $A^\top A$:

$$A^\top A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$A^\top A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

=

Let $\lambda$ be an eigenvalue. solve $\det(A^\top A - \lambda I) = 0$:

$$A^\top A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}.$$

Solve:

$$-\lambda^3 + 5\lambda^2 - 4\lambda = -\lambda(\lambda^2 - 5\lambda + 4) = -\lambda(\lambda - 4)(\lambda - 1) = 0$$

The eigenvalues are $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 0$.

The singular values $\sigma_i$ of $A$ are the square roots of these eigenvalues:

$$\sigma_1 = \sqrt{4} = 2, \quad \sigma_2 = \sqrt{1} = 1, \quad \sigma_3 = \sqrt{0} = 0.$$

Thus, the nonzero singular values of $A$ are 2 and 1, and the rank of $A$ is $r = 2$.

An SVD of $A$ is $A = U\Sigma V^\top$, where

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find $V$ from the eigenvectors of $A^\top A$.

The matrix $V$ is formed from the orthonormal eigenvectors of $A^\top A$.

For $\lambda_1 = 4$: Solve $(A^\top A - 4I)v = 0$.

$$A^\top A - 4I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$

An eigenvector is $(2, 1, 1)$. Normalize it:

$$\|(2, 1, 1)\| = \sqrt{4 + 1 + 1} = \sqrt{6}.$$

So

$$v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$: Solve $(A^\top A - I)v = 0$.

$$A^\top A - I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

An eigenvector is $(-1, 1, 1)$. Normalize it:

$$\|(-1, 1, 1)\| = \sqrt{1 + 1 + 1} = \sqrt{3}.$$

So

$$v_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = 0$: Solve $A^\top A v = 0$. An eigenvector is $(0, 1, -1)$.

$$\|(0, 1, -1)\| = \sqrt{0 + 1 + 1} = \sqrt{2}.$$

$$v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Thus,

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

<u>For $u_1$:</u>

$$Av_1 = A\left(\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \cdot 2 + 0 \cdot 1 + 0 \cdot 1 \\ 1 \cdot 2 + 0 \cdot 1 + 0 \cdot 1 \\ 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Divide by $\sigma_1 = 2$:

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

For $u_2$:

$$Av_2 = A\left(\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}\right) = \frac{1}{\sqrt{3}}\begin{pmatrix}1\cdot(-1)+0\cdot 1+0\cdot 1\\1\cdot(-1)+0\cdot 1+0\cdot 1\\1\cdot(-1)+1\cdot 1+1\cdot 1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}.$$

Divide by $\sigma_2 = 1$ (no change):

$$u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}.$$

Since $U$ is orthonormal and must have 3 orthonormal columns, we find $u_3$ orthogonal to $u_1$ and $u_2$. A suitable choice is:

$$u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Check that $u_3$ is orthogonal to both $u_1$ and $u_2$. It is. Thus:

$$U = \begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0\end{pmatrix}.$$

We have:

$$\Sigma = \begin{pmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}, \quad \Sigma^+ = \begin{pmatrix}1/2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$

The pseudo inverse is:

$$A^+ = V\Sigma^+U^\top.$$

$$A^+ = \begin{pmatrix}\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0\end{pmatrix}.$$

$$A^+ = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2}\end{pmatrix}.$$

## (2)

The minimum norm solution to

$$\min_{x\in R^3}\|Ax-b\|$$

is given by:

$$x_* = A^+b.$$

$$b = \begin{pmatrix}0\\2\\2\end{pmatrix}.$$

Thus:

$$A^{+}b = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/4 & -1/4 & 1/2 \\ -1/4 & -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

$$(1/2) * 0 + (1/2) * 2 + 0 * 2 = 1.$$
$$(-1/4) * 0 + (-1/4) * 2 + (1/2) * 2 = (-1/2) + 1 = 1/2.$$
$$(-1/4) * 0 + (-1/4) * 2 + (1/2) * 2 = (-1/2) + 1 = 1/2.$$

So:

$$x* = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

## Question 5: Stability and Conditioning

**Given:** For $\beta > 1$, let

$$A = \frac{1}{25} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & 1/\beta \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

$$U = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

since,

$$U^T U = \frac{1}{25} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$U$ is an orthonormal matrix.

In SVD, we need $V^T$ and $U$ are orthonormal matrix, so we can choose $V^T = U$, where

$$V^\top = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

The singular values of $A$ are determined by the diagonal matrix with entries $\beta$ and $1/\beta$. Thus, we have:

$$\Sigma = \begin{pmatrix} \beta & 0 \\ 0 & 1/\beta \end{pmatrix}.$$

SVD of $A$:

$$A = U\Sigma V^\top = \left( \frac{1}{5} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \begin{pmatrix} \beta & 0 \\ 0 & 1/\beta \end{pmatrix} \left( \frac{1}{5} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right).$$
$$A = U\Sigma V^\top.$$

### (2)

We know

$$x = A^{-1}b = V\Sigma^{-1}U^\top b$$

Here,

$$\Sigma^{-1} = \begin{pmatrix} 1/\beta & 0 \\ 0 & \beta \end{pmatrix}.$$

$$U^\top = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}, \quad b = \frac{1}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

$$U^\top b = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\frac{1}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{1}{125}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{1}{125}\begin{pmatrix} 25 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}$$

Next:

$$\Sigma^{-1}U^\top b = \begin{pmatrix} 1/\beta & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1/5 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5\beta} \\ 0 \end{pmatrix}.$$

multiply by $V$

$$x_1 = V\begin{pmatrix} 1/(5\beta) \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1/(5\beta) \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3/(5\beta) \\ 4/(5\beta) \end{pmatrix} = \begin{pmatrix} 3/(25\beta) \\ 4/(25\beta) \end{pmatrix}.$$

$$x_1 = \begin{pmatrix} \frac{3}{25\beta} \\ \frac{4}{25\beta} \end{pmatrix}.$$

### (3)

For $x_2$:

$$U^\top b = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\frac{1}{25}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{125}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

$$U^\top b = \frac{1}{125}\begin{pmatrix} 0 \\ 25 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/5 \end{pmatrix}.$$

Apply $\Sigma^{-1}$:

$$\Sigma^{-1}U^\top b = \begin{pmatrix} 1/\beta & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 0 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta/5 \end{pmatrix}.$$

Multiply by $V$:

$$x_2 = V\begin{pmatrix} 0 \\ \beta/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ \beta/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -(4\beta)/5 \\ (3\beta)/5 \end{pmatrix} = \begin{pmatrix} -\frac{4\beta}{25} \\ \frac{3\beta}{25} \end{pmatrix}.$$

### (4)

We have:

$$x_1 = \begin{pmatrix} \frac{3}{25\beta} \\ \frac{4}{25\beta} \end{pmatrix}, \quad x_2 = \begin{pmatrix} -\frac{4\beta}{25} \\ \frac{3\beta}{25} \end{pmatrix}.$$

As $\beta \to \infty$, $x_1 \to (0, 0)$ and $x_2$ grows without bound. The two $b$ vectors differ only slightly in direction, yet their solutions differ dramatically for large $\beta$. This shows that $A$ is ill-conditioned.

The condition number of $A$ is approximately $\beta^2$, which grows large with $\beta$, making the solution highly sensitive to changes in $b$.