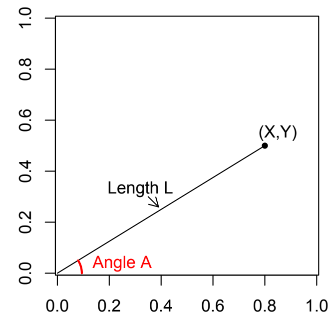
Joint distributions (part 4)

Lecture 7a (STAT 24400 F24)

Example: Joint distribution, functions of r.v. pair

Suppose that (X, Y) is drawn uniformly from the unit square.
Drawing the segment from $(0, 0)$ to (X, Y), let A = angle & L = length



Joint distribution example (cont)

1. Express A and L in terms of X and Y.

   $L = \sqrt{X^2 + Y^2}$

   $\tan(A) = \frac{Y}{X} \quad \Rightarrow \quad A = \arctan\frac{Y}{X}$

2. Are A and L independent?

   No. Possible values of L depend on the value of A.

   — e.g., if $A = 0$ then we must have $L \leq 1$,
   but if $A = \frac{\pi}{4}$ then we can have $L > 1$.

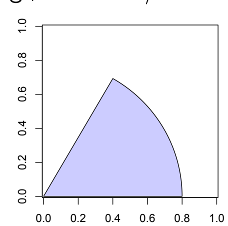
   (Note this approach is simpler than trying to prove via joint density.)

Joint distribution example (cont.)

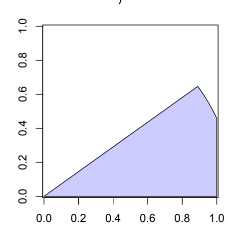
3. What is the joint distribution of A and L?

$$\begin{aligned} F_{A,L}(s,t) &= \mathbb{P}(A \leq s, L \leq t) \\ &= \mathbb{P}(\arctan(Y/X) \leq s, \sqrt{X^2+Y^2} \leq t) \\ &= \text{Area}\big(\{(x,y) \in [0,1]^2 : \arctan(y/x) \leq s, \sqrt{x^2+y^2} \leq t\}\big) \end{aligned}$$

E.g., for $s = \pi/3$ and $t = 0.8$:



for $s = \pi/5$ and $t = 1.1$:



Review: Independence

- If events A and B are independent, the probability of their intersection

$$\mathbb{P}(AB) = \mathbb{P}(A)\,\mathbb{P}(B)$$

- If two r.v.'s X and Y are independent, their joint CDF factors:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

In addition,

  - If X and Y are discrete and independent, their joint PMF

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

  - If X and Y are continuous and independent, their joint density

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Joint distribution for i.i.d. r.v.'s

Suppose $X_1, \ldots, X_n$ are independent and identically distributed ("i.i.d.") r.v.'s drawn from a distribution with CDF F.

- Independence $\Rightarrow$ $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^n F(x_i)$
- If F is discrete (or continuous),
  then the joint PMF (or joint density) is a product also:

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^n p(x_i)$$

or

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^n f(x_i)$$

Order statistics

For an n-tuple $X_1, \ldots, X_n$, the **order statistics** are the ranked values:

- $X_{(1)}$ denotes the smallest value, i.e., $X_{(1)} = \min\{X_1, \ldots, X_n\}$
- $X_{(2)}$ denotes the next-smallest value
- ...
- $X_{(n)}$ denotes the largest value, i.e., $X_{(n)} = \max\{X_1, \ldots, X_n\}$

<u>Note</u>: if there are ties, then the same value appears multiple times
(e.g., if we observe 3, 5, 3, then $X_{(1)} = X_{(2)} = 3$ and $X_{(3)} = 5$)

Order statistics

Order statistics is an important tool in statistics.

The probability laws of order statistics can be used for questions such as:

- What is the probability that the largest draw among n draws will exceed some value?
  (This will help us to quantify and control extreme events.)
- How accurately does the sample median estimate the true median of the distribution?
- ... and many more

Order statistics for i.i.d. data — distribution of $\min_i X_i$

Suppose $X_1, \ldots, X_n$ are i.i.d. draws from a distribution with CDF F.

What is the distribution of $X_{(1)} = \min_i X_i$?

$$\begin{aligned} F_{X_{(1)}}(x) = \mathbb{P}(X_{(1)} \le x) &= 1 - \mathbb{P}(X_{(1)} > x) \\ &= 1 - \mathbb{P}(\min\{X_1, \ldots, X_n\} > x) \\ &= 1 - \mathbb{P}(X_i > x \text{ for all } i = 1, \ldots, n) \\ &= 1 - \prod_{i=1}^{n} \mathbb{P}(X_i > x) \;=\; 1 - (1 - F(x))^n \end{aligned}$$

If the original distribution is continuous with density $f = F'$, then

$$f_{X_{(1)}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X_{(1)}}(x) = n(1 - F(x))^{n-1} \cdot \frac{\mathrm{d}}{\mathrm{d}x} F(x) = n(1 - F(x))^{n-1} \cdot f(x).$$

Order statistics for i.i.d. data — distribution of $\max_i X_i$

Suppose $X_1, \ldots, X_n$ are i.i.d. draws from a distribution with CDF F

What is the distribution of $X_{(n)} = \max_i X_i$?

$$\begin{aligned} F_{X_{(n)}}(x) &= \mathbb{P}(X_{(n)} \le x) \\ &= \mathbb{P}(X_i \le x \text{ for all } i = 1, \ldots, n) \\ &= \prod_{i=1}^{n} \mathbb{P}(X_i \le x) = F(x)^n \end{aligned}$$

If the original distribution is continuous with density $f = F'$:

$$f_{X_{(n)}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X_{(n)}}(x) = nF(x)^{n-1} \cdot \frac{\mathrm{d}}{\mathrm{d}x} F(x) = nF(x)^{n-1} \cdot f(x).$$

Joint distribution of $(\min_i X_i, \max_i X_i)$ for i.i.d. data

Suppose $X_1, \ldots, X_n$ are i.i.d. draws from a distribution with CDF F

What is the joint distribution of $X_{(1)}$ and $X_{(n)}$?

$$\begin{aligned} F_{X_{(1)}, X_{(n)}}(x, y) &= \mathbb{P}(X_{(1)} \le x, X_{(n)} \le y) \\ &= \mathbb{P}(X_{(n)} \le y) - \mathbb{P}(X_{(1)} > x, X_{(n)} \le y) \\ &= \mathbb{P}(X_{(n)} \le y) - \mathbb{P}(x < X_i \le y \text{ for all } i = 1, \ldots, n) \\ &= F(y)^n - (F(y) - F(x))^n \end{aligned}$$

Remarks Distributions for other order statistics (e.g. $\mathbb{P}(X_{(3)} < x)$) and their joint distributions (e.g. for $X_{(2)}$ and $X_{(3)}$) can also be obtained analogously.
The derivations involve detailed counting this are more technical (omitted).

Order statistics: Exponential distribution

Example

Suppose $X_1, \ldots, X_n$ are i.i.d. Exponential(λ).

What are the CDF and density of $X_{(n)}$?

$$\begin{aligned} F_{X_{(n)}}(x) &= F(x)^n = (1 - e^{-\lambda x})^n \\ f_{X_{(n)}}(x) &= nF(x)^{n-1} \cdot f(x) = n(1 - e^{-\lambda x})^{n-1} \cdot \lambda e^{-\lambda x} \end{aligned}$$

What are the CDF and density of $X_{(1)}$?

$$\begin{aligned} F_{X_{(1)}}(x) &= 1 - (1 - F(x))^n = 1 - e^{n\lambda x} \\ f_{X_{(1)}}(x) &= n(1 - F(x))^{n-1} \cdot f(x) \\ &= n\big(1 - (1 - e^{-\lambda x})\big)^{n-1} \cdot \lambda e^{-\lambda x} = (n\lambda) e^{-(n\lambda)x} \end{aligned}$$

$\Rightarrow \quad X_{(1)} \sim \text{Exponential}(n\lambda)$