24400 HW1

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1 Question 1

(a)

Total ways to select 13 cards from 52 cards:

$$\binom{52}{13}$$

Total ways to have exactly 1 Heart in 13 cards:

$$\binom{13}{1} \times \binom{39}{12}$$

Probability of exactly 1 Heart:

$$P(\text{Exactly 1 Heart}) = \frac{\binom{13}{1} \times \binom{39}{12}}{\binom{52}{13}} = \frac{13 \times \frac{39!}{12!(39-12)!}}{\frac{52!}{13!(52-13)!}} = \frac{13 \times \frac{39!}{12!27!}}{\frac{52!}{13!39!}}$$

$$\approx 0.08 \approx 8\%$$

(b)

Total ways to select 5 cards out of 52 cards:

$$\binom{52}{5}$$

Total ways to select 4 cards from different suits and 1 card from any suit:

$$4 \times \binom{13}{2} \times \binom{13}{1}^3$$

Probability of 4 cards from each suits:

$$P(\text{4 cards from each suits:}) = \frac{4 \times \binom{13}{2} \times \binom{13}{1}^3}{\binom{52}{5}}$$

$$= \frac{4 \times \frac{13\times12}{2} \times 13 \times 13 \times 13}{2,598,960} = \frac{4 \times 78 \times 2197}{2,598,960} \approx 0.2635 \approx 26.35\%$$

(c)

Ways to order 4 kings:

$$4!$$

Ways to order 48 other cards:

$$48!$$

Total ways to order 52 cards:

$$52!$$

Probability of getting 4 consecutive kings:

$$P(\text{4 consecutive kings:}) = \frac{48! \times 4!}{52!} \approx 0.00000369$$

2 Question 2

(a)

A: At least one coin landed Tails.
B: Both coins landed Tails.
Using conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

A^c: No coin landed with Tails, we have:

$$P(A^c) = \frac{1}{2} \times \frac{1}{2}$$

$$P(A) = 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

and:

$$P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

since when B occurs, A must occur:

$$P(A \cap B) = P(B) = \frac{1}{4}$$

Thus:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(b)

There are 4 states whose name begins with the letter "I".
The total number of ways to select two coins from the 50-state collection is:

$$\binom{50}{2} = \frac{50 \times 49}{2} = 1225$$

The number of ways to select two coins from the 4 "I" states is:

$$\binom{4}{2} = \frac{4 \times 3}{2} = 6$$

Thus, the probability is:

$$P = \frac{6}{1225} \approx 0.0049$$

(c)

A: Event that do not see the name Illinois.
A^c: Event that the name Illinois is seen.
B : One of the coins is the Illinois quarter.
Now we can calculate $P(A^c)$, the probability of seeing "Illinois", to have this event, we have to select Illinois and 1 coin out of other 49 coins, and Illinois coin comes up tails:

$$P(A^c) = \frac{\binom{49}{1}\binom{1}{1}}{\binom{50}{2}} \times \frac{1}{2} = \frac{1}{50}$$

$$P(A) = 1 - P(A^c) = \frac{49}{50}$$

Next, we calculate $P(B)$, the probability that one of the coins is the Illinois quarter:

$$P(B) = \frac{\binom{49}{1}\binom{1}{1}}{\binom{50}{2}} = \frac{1}{25}$$

Use the Bayes' rule to find $P(B|A)$:

$$P(B|A) = \frac{P(A|B) \times P(B)}{P(A)}$$

Since when B occurs, that one coin must be Illinois, to make A occur, this coin needs to come up tails:

$$P(A|B) = \frac{1}{2}$$

Thus:

$$P(B|A) = \frac{\frac{1}{25} \times \frac{1}{2}}{\frac{49}{50}} = \frac{1}{49}$$

(d)

Let random variable X be the total number of G's in the state names of the two coins tossed. The possible values of X are 0, 1, 2, and 3.

All Possible Outcomes

- $X = 0$: No G states.
- $X = 1$: One G state and one non-G state.
- $X = 2$: One Georgia and one non-G state or two G states.
- $X = 3$: One Georgia and one G state.

$$P(X = 0) = \frac{\binom{43}{2}}{\binom{50}{2}} = \frac{43 \times 42}{50 \times 49} = \frac{903}{1225} = \frac{129}{175}$$

$$P(X = 1) = \frac{\binom{6}{1} \times \binom{43}{1}}{\binom{50}{2}} = \frac{6 \times 43}{50 \times 49} = \frac{258}{1225}$$

$$P(X = 3) = \frac{\binom{1}{1} \times \binom{6}{1}}{\binom{50}{2}} = \frac{6}{1225}$$

$$P(X = 2) = 1 - P(X = 0) - P(X = 1) - P(X = 3) = 1 - \frac{129}{175} - \frac{258}{1225} - \frac{6}{1225} = \frac{58}{1225}$$

PMF:

x	$P(X = x)$
0	$\frac{129}{175}$
1	$\frac{258}{1225}$
2	$\frac{58}{1225}$
3	$\frac{6}{1225}$

3 Question 3

(a)

	Disease	No Disease
Positive	$P(\text{Positive}\|\text{Disease}) = 0.95$	$P(\text{Positive}\|\text{No Disease}) = 0.20$
Negative	$P(\text{Negative}\|\text{Disease}) = 0.05$	$P(\text{Negative}\|\text{No Disease}) = 0.80$

We have known:

$$P(\text{Disease}) = 0.05$$

$$P(\text{No Disease}) = 1 - 0.05 = 0.95$$

Using total probability rule:

$$P(\text{Positive}) = P(\text{Disease}) \times P(\text{Positive}|\text{Disease}) + P(\text{No Disease}) \times P(\text{Positive}|\text{No Disease})$$

$$= 0.05 \times 0.95 + 0.95 \times 0.20$$

$$= 0.0475 + 0.19 = 0.2375 = 23.75\%$$

(b)

We need to calculate $P(\text{Disease}|\text{Positive})$, using Bayes' Theorem:

$$P(\text{Disease}|\text{Positive}) = \frac{P(\text{Positive}|\text{Disease}) \times P(\text{Disease})}{P(\text{Positive})}$$

From (a), we have

$$P(\text{Positive}) = 0.2375$$

Substituting the known values:

$$P(\text{Disease}|\text{Positive}) = \frac{0.95 \times 0.05}{0.2375}$$

$$= \frac{0.0475}{0.2375} = 0.2 = 20\%$$

(c)

$d = P(\text{Disease})$.
We have:

$$P(\text{No Disease}) = 1 - d$$

The probability of a positive test result for the first screening is:

$$P(\text{First Positive}) = P(\text{Positive}|\text{Disease}) \times P(\text{Disease}) + P(\text{Positive}|\text{No Disease}) \times P(\text{No Disease})$$

$$= 0.95d + 0.2(1-d)$$

For the second test, since the sample in the second test comes from the people having disease and don't have disease given the first test is positive, therefore:

$$P(\text{Second Positive}) = P(\text{Second Positive}|\text{Disease}) \times P(\text{Disease}|\text{First Positive})$$

$$+P(\text{Second Positive}|\text{No Disease}) \times P(\text{No Disease}|\text{First Positive})$$

Then use Bayes' rule for $P(\text{Disease}|\text{First Positive})$:

$$P(\text{Disease}|\text{First Positive}) = \frac{P(\text{First Positive}|\text{Disease}) \times P(\text{Disease})}{P(\text{First Positive})}$$

$$P(\text{Disease}|\text{First Positive}) = \frac{0.95 \times d}{0.95d + 0.2(1-d)}$$

Similarly, for $P(\text{No Disease}|\text{First Positive})$:

$$P(\text{No Disease}|\text{First Positive}) = \frac{0.2(1-d)}{0.95d + 0.2(1-d)}$$

Thus,

$$P(\text{Second Positive}) = \frac{0.95d}{0.95d + 0.2(1-d)} \times 0.95 + \frac{0.2(1-d)}{0.95d + 0.2(1-d)} \times 0.2$$

$$= 0.25$$

solving for d:

$$d = \frac{10}{675} \approx 0.0148 \approx 1.48\%$$

(d)

The probability that no one in the group has the disease is:

$$P(\text{No one has Disease}) = (1-d)^k$$

Therefore, the probability that at least one person has the disease is:

$$P(\text{At least one person has Disease}) = 1 - (1-d)^k$$

The probability of a positive test result can be calculated as follow, using total probability rule:

$$P(\text{Positive}) = P(\text{Positive}|\text{ At least one person has Disease}) \times P(\text{At least one person has Disease})$$

$$+P(\text{Positive}|\text{ No one has Disease}) \times P(\text{No one has Disease})$$

$$= 0.95 \times [1 - (1-d)^k] + 0.2 \times (1-d)^k$$

The probability that at least one person in the group has the disease given a positive test result is:

$$P(\text{At least one person has Disease}|\text{Positive})$$

$$= \frac{P(\text{At least one person has Disease}) \times P(\text{Positive}|\text{At least one person has Disease})}{P(\text{Positive})}$$

$$= \frac{0.95 \times [1-(1-d)^k]}{0.95 \times [1-(1-d)^k] + 0.2 \times (1-d)^k} \quad (0 \leq d \leq 1)$$

4 Question 4

(a)

The total number of ways to select and order l cards is:

$$\frac{25!}{(25-l)!}$$

The total ways to have favorable results, that is, to arrange A or B in l^{th} place and another A or B in the $l-1$ places in front of the last card, and then order other $(l-2)$ cards out of 23 cards can be expressed as:

$$\binom{2}{1} \times (l-1) \times \frac{23!}{[23-(l-2)]!}$$

Thus, the probability mass function of L is:

$$P(L=l) = \frac{\binom{2}{1} \times (l-1) \times \frac{23!}{[23-(l-2)]!}}{\frac{25!}{(25-l)!}} = \frac{2(l-1)}{25 \times 24} = \frac{l-1}{300} \quad (l = 2, 3, 4, \ldots, 25)$$

(b)

The total number of ways to arrange the remaining $(25-k)$ cards is:

$$(25-k)!$$

In the favorable results, we treat the combination of A and B as a whole unit, meaning we only need to arrange $(24-k)$ cards, and then arrange A and B (either AB or BA), therefore, the total number of ways to arrange the favorable results is:

$$(24-k)! \times 2!$$

Thus, the probability that A and B are drawn consecutively after k draws is:

$$P(\text{probability that A and B are drawn consecutively after k draws})$$

$$= \frac{2 \times (24-k)!}{(25-k)!} = \frac{2}{25-k} \quad (k = 0, 1, 2, 3, \ldots, 23)$$