24300 HW1

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Question 1: Gaussian Elimination and Back-Substitution

Consider the linear system:

$$\begin{aligned} 5x_1 + 3x_2 + 4x_3 - x_4 &= 0 \\ -10x_1 - 2x_2 - 6x_3 + 2x_4 &= 4 \\ -2x_2 + 2x_3 - 3x_4 &= 1 \\ 5x_1 + 3x_2 - 2x_3 &= 4 \end{aligned}$$

solve this system using Gaussian elimination.

(1)

The augmented matrix:

$$\left[\begin{array}{cccc|c} 5 & 3 & 4 & -1 & 0 \\ -10 & -2 & -6 & 2 & 4 \\ 0 & -2 & 2 & -3 & 1 \\ 5 & 3 & -2 & 0 & 4 \end{array}\right]$$

(2)

1. row 2 = 2 row 1 + row 2:

$$\left[\begin{array}{cccc|c} 5 & 3 & 4 & -1 & 0 \\ 0 & 4 & 2 & 0 & 4 \\ 0 & -2 & 2 & -3 & 1 \\ 5 & 3 & -2 & 0 & 4 \end{array}\right]$$

2. row 4 = row 1 - row 4

$$\left[\begin{array}{cccc|c} 5 & 3 & 4 & -1 & 0 \\ 0 & 4 & 2 & 0 & 4 \\ 0 & -2 & 2 & -3 & 1 \\ 0 & 0 & -6 & -1 & -4 \end{array}\right]$$

3. row 3 = 2 row 3 + row 2:

$$\left[\begin{array}{cccc|c} 5 & 3 & 4 & -1 & 0 \\ 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & -6 & -6 & 6 \\ 0 & 0 & -6 & -1 & -4 \end{array}\right]$$

4. row 4 = row 4 - row 3:

$$\left[\begin{array}{cccc|c} 5 & 3 & 4 & -1 & 0 \\ 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 6 & -6 & 6 \\ 0 & 0 & 0 & 5 & -10 \end{array}\right]$$

(3)

solve by back-substitution:

From row 4:

$$5x_4 = -10 \implies x_4 = -2$$

From row 3:

$$6x_3 - 6x_4 = 6 \implies 6x_3 - 6(-2) = 6 \implies 6x_3 + 12 = 6 \implies x_3 = -1$$

From row 2:

$$4x_2 + 2x_3 = 4 \implies 4x_2 + 2(-1) = 4 \implies 4x_2 - 2 = 4 \implies x_2 = \frac{3}{2}$$

From row 1:

$$5x_1 + 3x_2 + 4x_3 - x_4 = 0 \implies 5x_1 + 3\left(\frac{3}{2}\right) + 4(-1) - (-2) = 0$$

$$5x_1 + \frac{9}{2} - 4 + 2 = 0 \implies 5x_1 + \frac{5}{2} = 0 \implies 5x_1 = -\frac{5}{2} \implies x_1 = -\frac{1}{2}$$

Thus, the solution to the system is:

$$\begin{aligned} x_1 &= -\frac{1}{2} \\ x_2 &= \frac{3}{2} \\ x_3 &= -1 \\ x_4 &= -2 \end{aligned}$$

(4)

compute Ax and check if it equals b.

Given:

$$A = \begin{bmatrix} 5 & 3 & 4 & -1 \\ -10 & -2 & -6 & 2 \\ 0 & -2 & 2 & -3 \\ 5 & 3 & -2 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ -1 \\ -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 4 \end{bmatrix}$$

Ax:

$$Ax = \begin{bmatrix} 5 & 3 & 4 & -1 \\ -10 & -2 & -6 & 2 \\ 0 & -2 & 2 & -3 \\ 5 & 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ -1 \\ -2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 5\left(-\frac{1}{2}\right) + 3\left(\frac{3}{2}\right) + 4(-1) + (-1)(-2) \\ -10\left(-\frac{1}{2}\right) + (-2)\left(\frac{3}{2}\right) + (-6)(-1) + 2(-2) \\ 0\left(-\frac{1}{2}\right) + (-2)\left(\frac{3}{2}\right) + 2(-1) + (-3)(-2) \\ 5\left(-\frac{1}{2}\right) + 3\left(\frac{3}{2}\right) + (-2)(-1) + 0(-2) \end{bmatrix}$$

$$Ax = \begin{bmatrix} -\frac{5}{2} + \frac{9}{2} - 4 + 2 \\ 5 - 3 + 6 - 4 \\ -3 - 2 + 6 \\ -\frac{5}{2} + \frac{9}{2} + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 4 \end{bmatrix}$$

$Ax = b$, so the solution is verified, Thus, the solution to the system is:

$$x_1 = -\frac{1}{2}$$
$$x_2 = \frac{3}{2}$$
$$x_3 = -1$$
$$x_4 = -2$$

Question 2: Properties of the Products

(1)

For any compatible vectors $\mathbf{v}$ and $\mathbf{w}$, show that the inner product $\mathbf{v}^\top \mathbf{w}$ commutes, i.e.,

$$\mathbf{v}^\top \mathbf{w} = \mathbf{w}^\top \mathbf{v}.$$

Proof:

Let $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$. The inner product $\mathbf{v}^\top \mathbf{w}$ is defined as:

$$\mathbf{v}^\top \mathbf{w} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Similarly, the inner product $\mathbf{w}^\top \mathbf{v}$ is:

$$\mathbf{w}^\top \mathbf{v} = y_1 x_1 + y_2 x_2 + \cdots + y_n x_n = \sum_{i=1}^{n} y_i x_i.$$

Since scalar multiplication is commutative ($x_i y_i = y_i x_i$ for all i), each corresponding term in the sums is equal. Therefore:

$$\mathbf{v}^\top \mathbf{w} = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \mathbf{w}^\top \mathbf{v}.$$

Thus, the inner product commutes:

$$\mathbf{v}^\top \mathbf{w} = \mathbf{w}^\top \mathbf{v}.$$

(2)

Statement 1: For any compatible vectors $\mathbf{v}, \mathbf{w}, \mathbf{z}$, show that:

$$\mathbf{v}^\top(\mathbf{w} + \mathbf{z}) = \mathbf{v}^\top\mathbf{w} + \mathbf{v}^\top\mathbf{z}.$$

Proof:

Let the vectors be defined as:

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

Thus,

$$\begin{aligned} \mathbf{v}^\top(\mathbf{w} + \mathbf{z}) &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 + z_1 \\ y_2 + z_2 \\ \vdots \\ y_n + z_n \end{pmatrix} \\ &= x_1(y_1 + z_1) + x_2(y_2 + z_2) + \cdots + x_n(y_n + z_n) \\ &= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + \cdots + x_ny_n + x_nz_n. \end{aligned}$$

on the right-hand of the equation:

$$\begin{aligned} \mathbf{v}^\top\mathbf{w} + \mathbf{v}^\top\mathbf{z} &= \left(\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right) + \left(\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \right) \\ &= (x_1y_1 + x_2y_2 + \cdots + x_ny_n) + (x_1z_1 + x_2z_2 + \cdots + x_nz_n) \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n + x_1z_1 + x_2z_2 + \cdots + x_nz_n. \end{aligned}$$

Therefore,

$$\mathbf{v}^\top(\mathbf{w} + \mathbf{z}) = \mathbf{v}^\top\mathbf{w} + \mathbf{v}^\top\mathbf{z}.$$

Statement 2: For any compatible matrix A and vectors $\mathbf{v}, \mathbf{w}$, show that:

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}.$$

Proof:

Let A be an $m \times n$ matrix with elements a_{ij}, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let the vectors $\mathbf{v}$ and $\mathbf{w}$ be n-dimensional column vectors:

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

$A(\mathbf{v} + \mathbf{w})$ results in an m-dimensional vector. The i-th component of $A(\mathbf{v} + \mathbf{w})$ is:

$$\begin{aligned} [A(\mathbf{v} + \mathbf{w})]_i &= a_{i1}(x_1 + y_1) + a_{i2}(x_2 + y_2) + \cdots + a_{in}(x_n + y_n) \\ &= a_{i1}x_1 + a_{i1}y_1 + a_{i2}x_2 + a_{i2}y_2 + \cdots + a_{in}x_n + a_{in}y_n. \end{aligned}$$

On the right-hand side of the equation, compute $A\mathbf{v}$ and $A\mathbf{w}$ separately:

The i-th component of $A\mathbf{v}$ is:

$$[A\mathbf{v}]_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n.$$

The i-th component of $A\mathbf{w}$ is:

$$[A\mathbf{w}]_i = a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n.$$

Sum them:

$$\begin{aligned} [A\mathbf{v}]_i + [A\mathbf{w}]_i &= (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) + (a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n) \\ &= a_{i1}x_1 + a_{i1}y_1 + a_{i2}x_2 + a_{i2}y_2 + \cdots + a_{in}x_n + a_{in}y_n. \end{aligned}$$

Thus,

$$[A(\mathbf{v} + \mathbf{w})]_i = [A\mathbf{v}]_i + [A\mathbf{w}]_i.$$

Since this equality holds for each $i = 1, 2, \ldots, m$:

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}.$$

(3)

Statement 1: For any scalar λ and compatible vectors $\mathbf{v}, \mathbf{w}$, show that:

$$(\lambda\mathbf{v})^\top \mathbf{w} = \mathbf{v}^\top(\lambda\mathbf{w}) = \lambda(\mathbf{v}^\top\mathbf{w}).$$

Proof:

Let the vectors $\mathbf{v}$ and $\mathbf{w}$ be n-dimensional column vectors:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.$$

taking the scalar multiplication of $\mathbf{v}$ by λ and then taking the inner product with $\mathbf{w}$:

$$\begin{aligned}
(\lambda\mathbf{v})^\top\mathbf{w} &= (\lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix})^\top \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \\
&= (\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix})^\top \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \\
&= (\lambda v_1, \lambda v_2, \dots, \lambda v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \\
&= \lambda v_1 w_1 + \lambda v_2 w_2 + \cdots + \lambda v_n w_n \\
&= \lambda(v_1 w_1 + v_2 w_2 + \cdots + v_n w_n) \\
&= \lambda(\mathbf{v}^\top\mathbf{w}).
\end{aligned}$$

Similarly, taking the scalar multiplication of $\mathbf{w}$ by λ and then taking the inner product with $\mathbf{v}$:

$$\begin{aligned}
\mathbf{v}^\top(\lambda\mathbf{w}) &= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^\top (\lambda \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}) \\
&= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^\top \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \\ \vdots \\ \lambda w_n \end{pmatrix} \\
&= (v_1, v_2, \dots, v_n) \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \\ \vdots \\ \lambda w_n \end{pmatrix} \\
&= \lambda v_1 w_1 + \lambda v_2 w_2 + \cdots + \lambda v_n w_n \\
&= \lambda(v_1 w_1 + v_2 w_2 + \cdots + v_n w_n) \\
&= \lambda(\mathbf{v}^\top\mathbf{w}).
\end{aligned}$$

Therefore:

$$(\lambda\mathbf{v})^\top\mathbf{w} = \mathbf{v}^\top(\lambda\mathbf{w}) = \lambda(\mathbf{v}^\top\mathbf{w}).$$

Statement 2: For any scalar λ and compatible matrix A, show that:

$$(\lambda A)\mathbf{v} = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}).$$

Proof:

Let A be an $m \times n$ matrix with elements a_{ij}, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let the vector $\mathbf{v}$ be an n-dimensional column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}.$$

The multiplication $(\lambda A)\mathbf{v}$ results in an m-dimensional vector. The i-th component of $(\lambda A)\mathbf{v}$ is:

$$\begin{aligned} [(\lambda A)\mathbf{v}]_i &= \sum_{j=1}^{n} (\lambda a_{ij}) v_j \\ &= (\lambda a_{i1}) v_1 + (\lambda a_{i2}) v_2 + \cdots + (\lambda a_{in}) v_n \\ &= \lambda (a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{in} v_n) \\ &= \lambda \left(\sum_{j=1}^{n} a_{ij} v_j \right) \\ &= \lambda [A\mathbf{v}]_i. \end{aligned}$$

Similarly, for the $\lambda\mathbf{v}$:

$$\lambda\mathbf{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}.$$

Then, the i-th component of $A(\lambda\mathbf{v})$ is:

$$\begin{aligned} [A(\lambda\mathbf{v})]_i &= \sum_{j=1}^{n} a_{ij} (\lambda v_j) \\ &= a_{i1}(\lambda v_1) + a_{i2}(\lambda v_2) + \cdots + a_{in}(\lambda v_n) \\ &= \lambda (a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{in} v_n) \\ &= \lambda \left(\sum_{j=1}^{n} a_{ij} v_j \right) \\ &= \lambda [A\mathbf{v}]_i. \end{aligned}$$

Therefore, for each component i:

$$[(\lambda A)\mathbf{v}]_i = [A(\lambda\mathbf{v})]_i = \lambda [A\mathbf{v}]_i.$$

Since this holds for all $i = 1, 2, \ldots, m$:

$$(\lambda A)\mathbf{v} = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}).$$

Question 3: Applications of Linear Systems (Polynomial Interpolation)

(1)

Since:

$$g(x_i) = f(x_i) \quad \text{for} \quad i = 1, 2, 3, 4.$$

we have:

$$\begin{cases} g(x_1) = c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3 = f(x_1) \\ g(x_2) = c_0 + c_1 x_2 + c_2 x_2^2 + c_3 x_2^3 = f(x_2) \\ g(x_3) = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 = f(x_3) \\ g(x_4) = c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 = f(x_4) \end{cases}$$

Substituting the given values:

$$\begin{cases} c_0 + c_1(-1) + c_2(-1)^2 + c_3(-1)^3 = -5 \\ c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 = 1 \\ c_0 + c_1(1) + c_2(1)^2 + c_3(1)^3 = 5 \\ c_0 + c_1(2) + c_2(2)^2 + c_3(2)^3 = 25 \end{cases}$$

Simplifying:

$$\begin{cases} c_0 - c_1 + c_2 - c_3 = -5 \\ c_0 = 1 \\ c_0 + c_1 + c_2 + c_3 = 5 \\ c_0 + 2c_1 + 4c_2 + 8c_3 = 25 \end{cases}$$

The linear system can be represented in matrix form as:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \\ 25 \end{bmatrix}$$

(2)

augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -5 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & 4 & 8 & 25 \end{array}\right]$$

row 2 = row 2 - row 1
row 3 = row 3 - row 1

row 4 = row 4 - row 1

the matrix becomes:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -5 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 2 & 0 & 2 & 10 \\ 0 & 3 & 3 & 9 & 30 \end{array}\right]$$

then:
row 3 = row 3 - 2 row 2
row 4 = row 4 - 3 row 2

The augmented matrix becomes:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -5 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 6 & 6 & 12 \end{array}\right]$$

row 4 = row 4 - 3 row 3

The augmented matrix becomes:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -5 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 6 & 18 \end{array}\right]$$

From the row 4:

$$6c_3 = 18 \implies c_3 = 3$$

From the row 3:

$$2c_2 = -2 \implies c_2 = -1$$

From the row 2:

$$c_1 - c_2 + c_3 = 6 \implies c_1 - (-1) + 3 = 6 \implies c_1 + 1 + 3 = 6 \implies c_1 = 2$$

From the row 1:

$$c_0 - c_1 + c_2 - c_3 = -5 \implies c_0 - 2 + (-1) - 3 = -5 \implies c_0 - 6 = -5 \implies c_0 = 1$$

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Thus, the function is:

$$g(x) = 1 + 2x - x^2 + 3x^3$$

(3)

To verify that $g(x_i) = f(x_i)$ for each $i = 1, 2, 3, 4$:

For $x_1 = -1$:
$$g(-1) = 1 + 2(-1) - (-1)^2 + 3(-1)^3 = 1 - 2 - 1 - 3 = -5 = f(-1)$$

For $x_2 = 0$:
$$g(0) = 1 + 2(0) - (0)^2 + 3(0)^3 = 1 = f(0)$$

For $x_3 = 1$:
$$g(1) = 1 + 2(1) - (1)^2 + 3(1)^3 = 1 + 2 - 1 + 3 = 5 = f(1)$$

For $x_4 = 2$:
$$g(2) = 1 + 2(2) - (2)^2 + 3(2)^3 = 1 + 4 - 4 + 24 = 25 = f(2)$$

Therefore, $g(x) = 1 + 2x - x^2 + 3x^3$ is the interpolating cubic polynomial for the given data points.

Question 4: Singular Systems from the Row Perspective

(1)

Steps to Determine Singularity

1. Use elementary row operations to reduce A to an upper triangular matrix.

2. If during the elimination process we obtain rows of all zeros (i.e., all coefficients in that row become zero), this indicates that the rank of A is less than $\min\{m, n\}$, which means that A is not in full rank, A is **singular**.

3. Otherwise, if A is in full rank, which means that the simplified upper triangular matrix of A does not contain any row of all zeros, A is **nonsingular**.

Testing for Solutions When A is Singular:

1. When A is singular, the system $A\mathbf{x} = \mathbf{b}$ may have either no solutions or infinitely many solutions.

2. First we should have the augmented matrix A with $\mathbf{b}$.

3. Then perform same row reduction on $[A \mid \mathbf{b}]$: as we did to A.

4. If $\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) < \min\{m, n\}$, That is, if rows in $[A \mid \mathbf{b}]$ reduces to $[0\ 0\ \ldots\ 0 \mid 0]$, the system has infinitely many solutions.

5. If $\text{rank}(A) < \text{rank}([A \mid \mathbf{b}])$, that is, if a row in $[A \mid \mathbf{b}]$ reduces to $[0\ 0\ \ldots\ 0 \mid c]$ where $c \neq 0$, representing an impossible equation $0 = c$, the system has no solutions.

Why singularity only depends on A
1. Singularity is a property of the matrix A because it is determined only by the relationships among its rows (or columns). The vector $\mathbf{b}$ does not influence the linear dependence or independence of A's rows.

2. $\mathbf{b}$ will only decide whether a solution exists, it does not alter the singularity of A.

(2)

solve for x such that $A\text{x} = \mathbf{b}$

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ -1 \\ 3 \end{pmatrix}$$

The augmented matrix $[A|\mathbf{b}]$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 1 & 4 \\ -1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 2 & 0 & 3 \end{array}\right]$$

row 2 = row 2 - 2 row 1
row 3 = row 3 + row 1
row 4 = row 4 - row 1

The augmented matrix becomes:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 3 \\ 0 & -1 & -1 & 1 & -2 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

row 3 = row 3 + row 2
The augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 3 \\ 0 & -1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rewrite the matrix to following system of equations:

$$\begin{cases} x_1 + x_2 + 2x_3 = 3 \\ -x_2 - x_3 + x_4 = -2 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Equation 2:

$$-x_2 - x_3 + x_4 = -2 \implies x_2 = -x_3 + x_4 + 2$$

Equation 1:

$$x_1 + x_2 + 2x_3 = 3$$

Substitute x_2 from Equation 2:

$$x_1 + (-x_3 + x_4 + 2) + 2x_3 = 3 \implies x_1 + x_3 + x_4 + 2 = 3 \implies x_1 = -x_3 - x_4 + 1$$

Let $x_3 = t$ and $x_4 = s$, where $t, s \in R$ are free variables.

$$\begin{cases} x_1 = 1 - t - s \\ x_2 = 2 - t + s \\ x_3 = t \\ x_4 = s \end{cases}$$

The solution can be written as:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in R.$$

Question 5: Nullspace

(1)

Given the matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 & -2 \\ -2 & 1 & 0 & 2 \\ 1 & 1 & 9 & -4 \\ -1 & 1 & 3 & 0 \end{pmatrix}$$

To solve $A\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the 4-entry zero vector, and find the dimension of the nullspace of A:

The augmented matrix $[A \mid \mathbf{0}]$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ -2 & 1 & 0 & 2 & 0 \\ 1 & 1 & 9 & -4 & 0 \\ -1 & 1 & 3 & 0 & 0 \end{array}\right]$$

row 2 = 2 row 1 + row 2
row 3 = row 1 - row 3
row 4 = row 1 + row 4

This gives:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & -1 & -6 & 2 & 0 \\ 0 & 1 & 6 & -2 & 0 \end{array}\right]$$

row 3 = row 2 + row 3
row 4 = row 2 - row 4

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we derive the following system of equations:

$$\begin{cases} x_1 + 3x_3 - 2x_4 = 0 \\ x_2 + 6x_3 - 2x_4 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Let $x_3 = t$ and $x_4 = s$, where $t, s \in R$ are free variables.

$$\begin{aligned} x_1 &= -3t + 2s \\ x_2 &= -6t + 2s \\ x_3 &= t \\ x_4 &= s \end{aligned}$$

The general solution to the system is:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in R$$

(2)

The nullspace of A is spanned by two free variables t and s.

The basis for the nullspace is:

$$\left\{ \begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Thus,

$$\text{Nullity}(A) = 2$$

(3)

Given the matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 & -2 \\ -2 & 1 & 0 & 2 \\ 1 & 1 & 9 & -4 \\ -1 & 1 & 3 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -6 \\ 2 \\ -16 \\ -4 \end{pmatrix}$$

to solve the system $A\mathbf{x} = \mathbf{b}$:

The augmented matrix is:

Then,

row 2 = 2 row 1 + row 2
row 3 = row 1 - row 3
row 4 = row 1 + row 4

This gives:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & -6 \\ 0 & 1 & 6 & -2 & -10 \\ 0 & -1 & -6 & 2 & 10 \\ 0 & 1 & 6 & -2 & -10 \end{array}\right]$$

row 3 = row 2 + row 3
row 4 = row 2 - row 3

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & -6 \\ 0 & 1 & 6 & -2 & -10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The row-reduced matrix derive the system:

$$\begin{cases} x_1 + 3x_3 - 2x_4 & = -6 \\ x_2 + 6x_3 - 2x_4 & = -10 \\ 0 & = 0 \\ 0 & = 0 \end{cases}$$

Let $x_3 = t$ and $x_4 = s$, where $t, s \in R$ are free variables.

$$\begin{aligned} x_1 &= -6 - 3t + 2s \\ x_2 &= -10 - 6t + 2s \\ x_3 &= t \\ x_4 &= s \end{aligned}$$

The particular solution is

$$\begin{pmatrix} -6 \\ -10 \\ 0 \\ 0 \end{pmatrix}$$

when t, s = 0.

The nullspace vector is:

$$t \begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in R$$

Thus, the final solution to the system $A\mathbf{x} = \mathbf{b}$ is:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6 \\ -10 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in R$$

(4)

proof of $(A\mathbf{x})^\top = \mathbf{x}^\top A^\top$

Let A be an $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Let $\mathbf{x}$ be a column vector in R^n:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Then, the matrix-vector product $A\mathbf{x}$ is:

$$A\mathbf{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

the transpose of $A\mathbf{x}$ is:

$$(A\mathbf{x})^\top = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, & \cdots, & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

And, the transpose of the vector $\mathbf{x}$ is:

$$\mathbf{x}^\top = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

The transpose of A, denoted as $A^\top$, is the $n \times m$ matrix:

$$A^\top = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

the matrix-vector product $\mathbf{x}^\top A^\top$ is:

$$\mathbf{x}^\top A^\top = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Performing the multiplication gives:

$$\mathbf{x}^\top A^\top = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, & \cdots, & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Therefore,

$$(A\mathbf{x})^\top = \mathbf{x}^\top A^\top$$

Let A be an $m \times n$ matrix, and
$\mathbf{x} \in \text{Null}(A)$, meaning $A\mathbf{x} = \mathbf{0}$
$\mathbf{v} \in \text{Range}(A^\top)$, meaning there exists some vector $\mathbf{b} \in R^m$ such that $\mathbf{v} = A^\top \mathbf{b}$.

We need to prove:

$$\mathbf{x}^\top \mathbf{v} = 0$$

Since $\mathbf{v} = A^\top \mathbf{b}$:

$$\mathbf{x}^\top \mathbf{v} = \mathbf{x}^\top A^\top \mathbf{b}$$

Using the equation $(A\mathbf{x})^\top = \mathbf{x}^\top A^\top$ proved above, we have:

$$\mathbf{x}^\top A^\top \mathbf{b} = (A\mathbf{x})^\top \mathbf{b}$$

Since $\mathbf{x} \in \text{Null}(A)$, $A\mathbf{x} = \mathbf{0}$. Thus:

$$(A\mathbf{x})^\top \mathbf{b} = \mathbf{0}^\top \mathbf{b} = 0$$

Therefore:

$$\mathbf{x}^\top \mathbf{v} = 0$$

Therefore, any vector $\mathbf{x} \in \mathrm{Null}(A)$ is orthogonal to any vector $\mathbf{v} \in \mathrm{Range}(A^\top)$.