STAT 24300 - Numerical Linear Algebra
Assignment 6: Spectral Linear Algebra

Question 1: LU decomposition

We've seen in class that reduction can produce useful matrix decompositions. In particular, the RREF produced by performing row reduction induced the CR decomposition. Like other factorizations, CR expresses a matrix A as a product of simpler factors. We have also shown that each step of the reduction process (row combination, permutation, scaling) can all be represented by representation with the correct matrix. Thus, we can encode the standard row reduction process as iterative multiplication by a sequence of matrices. When invertible, this process produces an alternative to CR which speeds solving linear systems. Namely, $A = LU$ where L is lower triangular and U is upper triangular. Then, given L and U, we can solve $Ax = b$ efficiently ($\mathcal{O}(n^2)$ operations instead of $\mathcal{O}(n^3)$). Let's work through an example which illustrates the LU decomposition.

Let:

$$A = \begin{bmatrix} 3 & 3 & 2 \\ -6 & -2 & -1 \\ 6 & 18 & 14 \end{bmatrix}. \tag{1}$$

Use the following algorithm to compute an LU factorization of A such that $A = LU$.

1. Set $L = I, U = A$ where I is the 3×3 identity matrix.
2. for $i = 1 : n - 1$
   - for $j = i + 1 : n$
     - $l_{ji} = u_{ji}/u_{ii}$
     - Set U_j to $U_j - l_{ji}U_i$ where U_m represents the m-th row of U.

Use the LU decomposition to compute the determinant of the matrix A.

Question 2: Solving for Eigenvalues and Eigenvectors

Consider the matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

1. Compute the determinants of A and B directly.
2. Write down the characteristic polynomials $\det(A - \lambda I)$ and $\det(B - \lambda I)$. (Note: The characteristic polynomial of a matrix A is a polynomial in λ given by $p(\lambda) = \det(A - \lambda I)$.)
3. Solve for the eigenvalues λ by finding the roots of the characteristic polynomials.
4. Confirm that the determinant of A and B equal the products of their eigenvalues.
5. Find the eigenvectors associated with each eigenvalue by solving $(A - \lambda I)v = 0$ and $(B - \lambda I)v = 0$ for each λ. Remember that $v \neq 0$. Normalize each eigenvector.

Question 3: Matrix Powers and Matrix Functions

Suppose that A is diagonalizable. Then we can write $A = V\Lambda V^{-1}$.

1. Show that $A^2 = V\Lambda^2 V^{-1}$ and $A^3 = V\Lambda^3 V^{-1}$. Show that $A^k = V\Lambda^k V^{-1}$ for any natural number k where Λ^k is the diagonal matrix with entries $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. (Hint: Use induction)
2. Suppose that $|\lambda_j| > 1$ then what happens to $|\lambda_j^k|$ as k diverges? What if $|\lambda_j| < 1$?

These conclusions allow us to compute and express matrix powers in a simple way. They also lead naturally to matrix functions. If we can evaluate A^2 and A^3 etc, then what about $A^{1/2}$ or $A^{1/3}$? What about A^0 or A^{-1}?

Well, suppose we have a scalar valued function $f(x)$ which has a convergent power series expansion $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$. Then we can extend it to matrices by defining:

$$f(A) = \sum_{k=0}^{\infty} \alpha_k A^k. \tag{3}$$

Substituting in $A = V\Lambda V^{-1}$ gives:

$$f(A) = \sum_{k=0}^{\infty} \alpha_k V\Lambda^k V^{-1} = V\left[\sum_{k=0} \alpha_k \Lambda^k\right] V^{-1} = Vf(\Lambda)V^{-1} \tag{4}$$

where $f(\Lambda) = \text{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))$ is the diagonal matrix with entries equal to f evaluated at the eigenvalues. Note that, in general $f(A)$ is not given by plugging each entry of A into f, just like the inverse of a matrix is not given by reciprocating every entry. Instead, $f(A)$ is given by replacing every eigenvalue of A with f evaluated at each eigenvalue.

This is a pretty neat trick. It is very useful in differential equations, where solutions are often expressed in terms of $\exp(At)$. In particular, if $\frac{d}{dt}x(t) = Ax(t)$ then $x(t) = e^{At}x(0)$. It is also useful if we want to factor matrices, for example, if we want to find the square root of a matrix.

Suppose that:

$$A = \frac{1}{3}\begin{bmatrix} 1 & 8 \\ 4 & 5 \end{bmatrix}. \tag{5}$$

1. Compute the eigenvalue decomposition (if it exists) of A.
2. Compute e^A. Justify your computations.
3. What is x^0 for any scalar x? Use your answer to show that $A^0 = I$ for any diagonalizable matrix A.

Question 4: Computing the Singular Value Decomposition (SVD) using Eigenvalues

All matrices admit an SVD, $A = U\Sigma V^\intercal$. If A is $m \times n$ then U is $m \times m$ and orthonormal, V is $n \times n$ and orthonormal, and Σ is $m \times n$ and diagonal. Notice that if we compute:

$$\begin{aligned} AA^\intercal &= (U\Sigma V^\intercal)(V\Sigma^\intercal U^\intercal) = U(\Sigma\Sigma^\intercal)U^\intercal \\ A^\intercal A &= (V\Sigma^\intercal U^\intercal)U\Sigma V^\intercal = V(\Sigma^\intercal\Sigma)V^\intercal \end{aligned} \tag{6}$$

where the products $V^\intercal V$ and $U^\intercal U$ both vanish since U and V are orthonormal matrices so $V^\intercal V = U^\intercal U = I$. But then $U(\Sigma\Sigma^\intercal)U^\intercal$ is the eigenvalue decomposition of $AA^\intercal$ since $U^\intercal = U^{-1}$, and $\Sigma\Sigma^\intercal$ is a diagonal matrix (product of two diagonal matrices). The diagonal entries are $\sigma_1^2, \sigma_2^2, \ldots$ so the eigenvalues of $AA^\intercal$ are the singular values of A squared. Then the columns of U are the (normalized) eigenvectors of $AA^\intercal$. The same logic applies to $A^\intercal A$, only with V instead of U.

It follows that we can find the singular values and vectors by finding the eigenvalue decomposition of $AA^\intercal$ and $A^\intercal A$. It is always possible to find such a decomposition since $AA^\intercal$ and $A^\intercal A$ are real symmetric matrices, so have real eigenvalues, and are unitarily diagonalizable (can be diagonalized by orthonormal matrices).

Let:

$$A = \begin{bmatrix} \frac{9}{5}\frac{1}{\sqrt{2}} & 1 & 1 & \frac{12}{5}\frac{1}{\sqrt{2}} \\ \frac{9}{5}\frac{1}{\sqrt{2}} & -1 & -1 & \frac{12}{5}\frac{1}{\sqrt{2}} \end{bmatrix} \tag{7}$$

1. Compute $AA^\intercal$.
2. Compute the eigenvalues of $AA^\intercal$ and confirm that these are $9, 4$. Compute the (positive) square roots of the eigenvalues. These are your singular values σ_1, σ_2.
3. Find the corresponding eigenvectors u_1, u_2 of $AA^\top$ in such a way that $U^\intercal U = I$, where U is the matrix with columns given by u_1, u_2.
4. We could compute the columns of V in the same way, but $A^\intercal A$ is 4 by 4, so that would be a mess. If we solved for the columns of V in the same way we would find:

   $$V = \begin{bmatrix} \frac{3}{5} & 0 & 0 & \frac{4}{5} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{4}{5} & 0 & 0 & -\frac{3}{5} \end{bmatrix} \tag{8}$$

   Confirm that the columns of V are orthogonal to one another, and are normalized (have length equal to one).
5. Make the diagonal matrix:

   $$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{bmatrix} \tag{9}$$

   Confirm that $A = U\Sigma V^\intercal$.