Homework 1 Solution.

1. Moments of Poisson distribution (20 pts).

$X \sim \text{Poisson}(\lambda)$ $E(X) = \lambda$ $Var(X) = \lambda$.

$E(X^2) = \lambda^2 + \lambda$.

$$EX^3 - 3EX^2 + 2EX = E[X(X-1)(X-2)]$$

$$= \sum_{k=0}^{\infty} k(k-1)(k-2) e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3$$

$$\Rightarrow EX^3 = \lambda^3 + 3\lambda^2 + \lambda.$$

$$EX^4 - 6EX^3 + 11EX^2 - 6EX = E[X(X-1)(X-2)(X-3)]$$

$$= \lambda^4$$

$$\Rightarrow EX^4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

Or. Use Moment Generating Function.

2. Poisson & χ^2-tails. (20).

①. $P(0 \le Y_{2(c+1)} \le 2\lambda) = \int_0^{2\lambda} f_{\chi^2_{2(c+1)}}(y)\, dy$. trivial.

②. $P(X_\lambda > c+1) = P(0 \le Y_{2(c+1)} \le 2\lambda)$.

$$P(X_\lambda > c+1) = 1 - P(X_\lambda \le c) = 1 - \sum_{i=0}^{c} \frac{\lambda^i e^{-\lambda}}{i!}$$

$$\frac{d}{d\lambda} P(X_\lambda > c+1) = e^{-\lambda} - \sum_{i=1}^{c} e^{-\lambda} \lambda^{i-1} \frac{i-\lambda}{i!}$$

$$= e^{-\lambda} - e^{-\lambda}\left(1 - \frac{\lambda^c}{c!}\right) = e^{-\lambda} \frac{\lambda^c}{c!}$$

$$= 2 f_{\chi^2_{2(c+1)}}(2\lambda) = \frac{d}{d\lambda} \int_0^{2\lambda} f_{\chi^2_{2(c+1)}}(y)\, dy.$$

3. Approx. to Binomial. prob. (20).

(6pts) (a). • $P(X=3) = \binom{7}{3} 0.3^3 0.7^4 = 0.2269$.

• Normal Approx:

$$z_1 = \frac{2.5 - 7\times 0.3}{\sqrt{7\times 0.3\times 0.7}} = 0.3299$$

$$z_2 = \frac{3.5 - 7\times 0.3}{\sqrt{7\times 0.3\times 0.7}} = 1.1547$$

$$P(z_1 \le z \le z_2) = 0.2466.$$

• Poisson Approx: $P(X=k) \approx \frac{e^{-np}(np)^k}{k!}$

$$P(X=3) \approx \frac{e^{-7\times 0.3}(7\times 0.3)^3}{3!} = 0.1890.$$

• Normal slightly better.

For Normal approx.
n not large enough.

For Poisson approx.
p not small enough.

(7pts) (b). • $P(X=11) = 0.03573$

• Normal: $z_1 = -1.7751$ $\quad z_2 = -1.4523$

$$P(z_1 \le z \le z_2) = 0.0353.$$

• Poisson: $P(X=11) \approx 0.0496$

• Normal works well.
n large & p close to 0.5

(7 pts) (c) • $P(X=2) = 0.1842$.

• Normal: $z_1 = 0.5006$ $\quad z_2 = 1.5019$

$$P(z_1 \le Z \le z_2) = 0.2418$$

• Poisson: $P(X=2) = 0.1839$

• Poisson works better, p small.

4. Conditional dist'n in Poisson Process. (20 pts).

(10) (a) $$P(X_s = k \mid X_t = n) = \frac{P(X_t = n \mid X_s = k)\, P(X_s = k)}{P(X_t = n)}$$

$$P(X_t = n \mid X_s = k) = P(X_{t-s} = n-k) = \frac{e^{-(t-s)\lambda}\,[(t-s)\lambda]^{n-k}}{(n-k)!}$$

$$P(X_s = k) = \frac{e^{-s\lambda}(s\lambda)^k}{k!}$$

$$P(X_t = n) = \frac{e^{-t\lambda}(t\lambda)^n}{n!}$$

$$\Rightarrow \quad P(X_s = k \mid X_t = n) = \binom{n}{k}\frac{(t-s)^{n-k}s^k}{t^n}$$

(10) (b) $$P(T_1 \le s \mid X_t = 1) = P(X_s = 1, X_t - X_s = 0 \mid X_t = 1)$$

$0 < s < t$

$$= \frac{P(X_s = 1, X_t - X_s = 0)}{P(X_t = 1)}$$

$$= \frac{P(X_s = 1)\, P(X_t - X_s = 0)}{P(X_t = 1)}$$

$$= \frac{s e^{-\lambda s} e^{-(t-s)\lambda}}{t e^{-\lambda t}} = \frac{s}{t}$$

5. Data from Poisson Process. (20 pts)

(10) (a).
$$L(\lambda) = \prod_{i=1}^{180} \frac{e^{-10\lambda}(10\lambda)^{x_i}}{x_i!} \prod_{j=1}^{20} \frac{e^{-20\lambda}(20\lambda)^{y_j}}{y_j!}$$

$$l(\lambda) = -180(10\lambda) + \sum_{i=1}^{180} x_i \ln(10\lambda) - \sum_{i=1}^{180} \ln(x_i!) - 20(20\lambda) + \sum_{j=1}^{20} y_j \ln(20\lambda) - \sum_{j=1}^{20} \ln(y_j!)$$

$$l'(\lambda) = -2200 + \frac{1}{\lambda}\sum_{i=1}^{180} x_i + \frac{1}{\lambda}\sum_{j=1}^{20} y_j = 0$$

$$\hat{\lambda} = \frac{\sum_{i=1}^{180} x_i + \sum_{j=1}^{20} y_j}{2200}$$

Using the data
$$\hat{\lambda} = \frac{347}{2200} = 0.1577.$$

(10) (b)
$$x_i \sim \text{Poisson}(10\lambda) \quad i=1,\ldots,180$$
$$y_j \sim \text{Poisson}(20\lambda) \quad j=1,\ldots,20$$
x_i, y_i are indep.

$$\Rightarrow \sum_{i=1}^{180} x_i + \sum_{j=1}^{20} y_j \sim \text{Poisson}(2200\lambda).$$
$$\hat{\lambda} \sim \frac{\text{Poisson}(2200\lambda)}{2200}.$$

By CLT, $\hat{\lambda} \sim N(\lambda, \frac{\lambda}{2200})$.