24400 HW3

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Question 1

(a)

To find $E(X)$:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_0^1 x \cdot 4x^3 dx \\
&= \int_0^1 4x^4 dx \\
&= 4 \cdot \frac{x^5}{5}\Big|_0^1 \\
&= \frac{4}{5}.
\end{aligned}$$

(b)

To Find $E(X^2)$ and $\mathrm{Var}(X)$:

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_0^1 x^2 \cdot 4x^3 dx \\
&= \int_0^1 4x^5 dx \\
&= 4 \cdot \frac{x^6}{6}\Big|_0^1 \\
&= \frac{2}{3}.
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{2}{3} - \left(\frac{4}{5}\right)^2 \\
&= \frac{2}{3} - \frac{16}{25} \\
&= \frac{50}{75} - \frac{48}{75} \\
&= \frac{2}{75}.
\end{aligned}
$$

(c)

To find the PDF for $Y = X^4$:

$$
\begin{gathered}
Y = X^4, \quad X \in [0,1], Y \in [0,1], \\
X = g^{-1}(y) = y^{1/4}, \\
\frac{dX}{dY} = \frac{1}{4} y^{-3/4}.
\end{gathered}
$$

Thus, using the change of variables formula:

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|, \\
f_Y(y) &= 4(y^{1/4})^3 \cdot \frac{1}{4} y^{-3/4}, \\
&= y^0 = 1.
\end{aligned}
$$

Therefore, $f_Y(y) = 1$ for $0 \leq y \leq 1$.

Question 2

(a)

To avoid the sequence HT, the possible sequences must satisfy the following conditions:

1. **All T's sequence**: There is only one such sequence, consisting of 10 T's (TTTTTTTTTT).

2. **Sequences with H's but no HT**: Once an H appears, all subsequent tosses must also be H to avoid forming HT. This means:

- H can first appear at any position from 1 to 10.
- All tosses after the first H are H.
- Before the first H, there can be any number of T's.

For a sequence of length 10, H can first appear at positions 1 through 10, resulting in 10 different sequences.

Thus,

$$\text{Total sequences} = 1 + 10 = 11$$

The total number of possible sequences is $2^{10} = 1024$.

Therefore:

$$P(X = 0) = \frac{\text{Number of sequences that avoid HT}}{\text{Total number of sequences}} = \frac{11}{1024}$$

$$P(X = 0) = \frac{11}{1024}$$

(b)

To find $E(X)$, consider X as the sum of indicator random variables.

For $n = 2$ to 10, define:

$$I_n = \begin{cases} 1, & \text{if toss } n-1 \text{ is H and toss } n \text{ is T} \\ 0, & \text{otherwise} \end{cases}$$

Thus:

$$X = \sum_{n=2}^{10} I_n$$

Since the coin is fair and tosses are independent:

$$P(I_n = 1) = P(\text{toss } n-1 \text{ is H}) \times P(\text{toss } n \text{ is T}) = \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \frac{1}{4}$$

Therefore:

$$E(I_n) = \frac{1}{4}$$

There are 9 indicator variables (from $n = 2$ to $n = 10$), so:

$$E(X) = \sum_{n=2}^{10} E(I_n) = 9 \times \frac{1}{4} = \frac{9}{4}$$

$$E(X) = \frac{9}{4}$$

Question 3

Given the joint probability distribution $p(x, y) = P(X = x, Y = y)$:

	$Y=2$	$Y=3$	$Y=4$	$Y=5$
$X=1$	0.1	0.1	0.0	0.0
$X=2$	0.0	0.2	0.2	0.1
$X=3$	0.0	0.0	0.1	0.2

(a)

To calculate marginal distribution of X and Y:

use the formula:

$$P_X(x) = P(X = x) = \sum_y P(X = x, Y = y) = \sum_y p(x, y)$$

For $X = 1$:

$$\begin{aligned} P_X(1) &= \sum_y p(1, y) = p(1, 2) + p(1, 3) + p(1, 4) + p(1, 5) \\ &= 0.1 + 0.1 + 0.0 + 0.0 = 0.2 \end{aligned}$$

For $X = 2$:

$$\begin{aligned} P_X(2) &= \sum_y p(2, y) = p(2, 2) + p(2, 3) + p(2, 4) + p(2, 5) \\ &= 0.0 + 0.2 + 0.2 + 0.1 = 0.5 \end{aligned}$$

For $X = 3$:

$$\begin{aligned} P_X(3) &= \sum_y p(3, y) = p(3, 2) + p(3, 3) + p(3, 4) + p(3, 5) \\ &= 0.0 + 0.0 + 0.1 + 0.2 = 0.3 \end{aligned}$$

So the marginal distribution of X is:

$$P_X(x) = \begin{array}{c|ccc} x & 1 & 2 & 3 \\ \hline P(X = x) & 0.2 & 0.5 & 0.3 \end{array}$$

Similarly, we use:

$$P_Y(y) = P(Y = y) = \sum_x P(X = x, Y = y) = \sum_x p(x, y)$$

For $Y = 2$:

$$\begin{aligned} P_Y(2) &= \sum_x p(x,2) = p(1,2) + p(2,2) + p(3,2) \\ &= 0.1 + 0.0 + 0.0 = 0.1 \end{aligned}$$

- For $Y = 3$:

$$\begin{aligned} P_Y(3) &= \sum_x p(x,3) = p(1,3) + p(2,3) + p(3,3) \\ &= 0.1 + 0.2 + 0.0 = 0.3 \end{aligned}$$

- For $Y = 4$:

$$\begin{aligned} P_Y(4) &= \sum_x p(x,4) = p(1,4) + p(2,4) + p(3,4) \\ &= 0.0 + 0.2 + 0.1 = 0.3 \end{aligned}$$

- For $Y = 5$:

$$\begin{aligned} P_Y(5) &= \sum_x p(x,5) = p(1,5) + p(2,5) + p(3,5) \\ &= 0.0 + 0.1 + 0.2 = 0.3 \end{aligned}$$

So the marginal distribution of Y is:

$$P_Y(y) = \begin{array}{c|cccc} y & 2 & 3 & 4 & 5 \\ \hline P(Y=y) & 0.1 & 0.3 & 0.3 & 0.3 \end{array}$$

(b)

To find the conditional distribution of Y given $X = 1$ and the conditional distribution of Y given $X = 2$ use the formula:

$$P_{Y|X}(y|x) = P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P_X(x)} = \frac{p(x,y)}{P_X(x)}$$

Conditional distribution of Y given $X = 1$:

Since $P_X(1) = 0.2$, to get $P(Y = y \mid X = 1)$:

For $Y = 2$:

$$P(Y = 2 \mid X = 1) = \frac{p(1,2)}{P_X(1)} = \frac{0.1}{0.2} = 0.5$$

For $Y = 3$:

$$P(Y = 3 \mid X = 1) = \frac{p(1,3)}{0.2} = \frac{0.1}{0.2} = 0.5$$

For $Y = 4$:

$$P(Y = 4 \mid X = 1) = \frac{p(1,4)}{0.2} = \frac{0.0}{0.2} = 0$$

For $Y = 5$:

$$P(Y = 5 \mid X = 1) = \frac{p(1,5)}{0.2} = \frac{0.0}{0.2} = 0$$

So the conditional distribution of Y given $X = 1$ is:

$$P(Y = y \mid X = 1) = \begin{array}{c|cccc} y & 2 & 3 & 4 & 5 \\ \hline P(Y = y \mid X = 1) & 0.5 & 0.5 & 0 & 0 \end{array}$$

To find conditional distribution of Y given $X = 2$:

Since that $P_X(2) = 0.5$, to get $P(Y = y \mid X = 2)$:

For $Y = 2$:

$$P(Y = 2 \mid X = 2) = \frac{p(2,2)}{P_X(2)} = \frac{0.0}{0.5} = 0$$

For $Y = 3$:

$$P(Y = 3 \mid X = 2) = \frac{p(2,3)}{0.5} = \frac{0.2}{0.5} = 0.4$$

For $Y = 4$:

$$P(Y = 4 \mid X = 2) = \frac{p(2,4)}{0.5} = \frac{0.2}{0.5} = 0.4$$

For $Y = 5$:

$$P(Y = 5 \mid X = 2) = \frac{p(2,5)}{0.5} = \frac{0.1}{0.5} = 0.2$$

So the conditional distribution of Y given $X = 2$ is:

$$P(Y = y \mid X = 2) = \begin{array}{c|cccc} y & 2 & 3 & 4 & 5 \\ \hline P(Y = y \mid X = 2) & 0 & 0.4 & 0.4 & 0.2 \end{array}$$

(c)

$$E(X) = \sum_x x \cdot P_X(x)$$

$$E(Y) = \sum_y y \cdot P_Y(y)$$

Thus,

$$\begin{aligned}
E(X) &= \sum_x x \cdot P_X(x) \\
&= 1 \cdot P_X(1) + 2 \cdot P_X(2) + 3 \cdot P_X(3) \\
&= 1 \cdot 0.2 + 2 \cdot 0.5 + 3 \cdot 0.3 \\
&= 0.2 + 1.0 + 0.9 = 2.1
\end{aligned}$$

and,

$$\begin{aligned}
E(Y) &= \sum_y y \cdot P_Y(y) \\
&= 2 \cdot P_Y(2) + 3 \cdot P_Y(3) + 4 \cdot P_Y(4) + 5 \cdot P_Y(5) \\
&= 2 \cdot 0.1 + 3 \cdot 0.3 + 4 \cdot 0.3 + 5 \cdot 0.3 \\
&= 0.2 + 0.9 + 1.2 + 1.5 = 3.8
\end{aligned}$$

Therefore,

$$E(X) = 2.1$$
$$E(Y) = 3.8$$

(d)

Since we have:

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

where,

$$E(X^2) = \sum_x x^2 \cdot P_X(x)$$

Similarly for Y:

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$$

Where:

$$E(Y^2) = \sum_y y^2 \cdot P_Y(y)$$

Therefore,

$$\begin{aligned}
E(X^2) &= \sum_x x^2 \cdot P_X(x) \\
&= 1^2 \cdot P_X(1) + 2^2 \cdot P_X(2) + 3^2 \cdot P_X(3) \\
&= 1 \cdot 0.2 + 4 \cdot 0.5 + 9 \cdot 0.3 \\
&= 0.2 + 2.0 + 2.7 = 4.9
\end{aligned}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 4.9 - (2.1)^2 = 4.9 - 4.41 = 0.49$$

For $E(Y^2)$:

$$\begin{aligned}
E(Y^2) &= \sum_y y^2 \cdot P_Y(y) \\
&= 2^2 \cdot P_Y(2) + 3^2 \cdot P_Y(3) + 4^2 \cdot P_Y(4) + 5^2 \cdot P_Y(5) \\
&= 4 \cdot 0.1 + 9 \cdot 0.3 + 16 \cdot 0.3 + 25 \cdot 0.3 \\
&= 0.4 + 2.7 + 4.8 + 7.5 = 15.4
\end{aligned}$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 15.4 - (3.8)^2 = 15.4 - 14.44 = 0.96$$

Therefore,

$$Var(X) = 0.49$$
$$Var(Y) = 0.96$$

(e)

To check if X and Y are independent is to check whether for all x and y:

$$P(X = x, Y = y) = P_X(x) \cdot P_Y(y)$$

Check for $X = 1$ and $Y = 2$:

$$P(X = 1, Y = 2) = p(1, 2) = 0.1$$
$$P_X(1) \cdot P_Y(2) = 0.2 \times 0.1 = 0.02$$

Since $0.1 \neq 0.02$, $P(X = 1, Y = 2) \neq P_X(1) \cdot P_Y(2)$.

Therefore, X and Y are **dependent**.

Question 4

(a)

Given that:

$$f(x, y) = C(x^2 - y^2)e^{-x}, \quad \text{for } x \in [0, \infty),\ y \in [-x, x].$$

Since $f(x, y)$ is a valid joint probability density function (PDF), which means:

$$\iint f(x, y)\, dy\, dx = 1.$$

$$\iint f(x, y)\, dy\, dx = \int_{x=0}^{\infty} \left(\int_{y=-x}^{x} C(x^2 - y^2) e^{-x}\, dy \right) dx = 1.$$

The inner integral:

$$\begin{aligned}
\int_{-x}^{x} (x^2 - y^2)\, dy &= \int_{-x}^{x} x^2\, dy - \int_{-x}^{x} y^2\, dy \\
&= x^2 \int_{-x}^{x} dy - \int_{-x}^{x} y^2\, dy \\
&= x^2\, [y]_{-x}^{x} - \left[\frac{y^3}{3} \right]_{-x}^{x} \\
&= x^2 (x - (-x)) - \left(\frac{x^3}{3} - \frac{(-x)^3}{3} \right) \\
&= 2x^3 - \left(\frac{x^3}{3} + \frac{x^3}{3} \right) \\
&= 2x^3 - \left(\frac{2x^3}{3} \right) \\
&= \frac{6x^3}{3} - \frac{2x^3}{3} \\
&= \frac{4x^3}{3}.
\end{aligned}$$

Therefore, the inner integral is:

$$e^{-x} \int_{-x}^{x} (x^2 - y^2)\, dy = e^{-x} \cdot \frac{4x^3}{3} = \frac{4x^3 e^{-x}}{3}.$$

The outer integral:

$$\int_{x=0}^{\infty} C \left(\frac{4x^3 e^{-x}}{3} \right) dx = 1.$$

$$C \cdot \frac{4}{3} \int_0^{\infty} x^3 e^{-x}\, dx = 1.$$

Since we have the Gamma function:

$$\int_0^{\infty} x^n e^{-x}\, dx = \Gamma(n + 1) = n!, \quad \text{for } n > -1.$$

Therefore,

$$\int_0^{\infty} x^3 e^{-x}\, dx = \Gamma(4) = 3! = 6.$$

Thus,

$$C \cdot \frac{4}{3} \int_0^{\infty} x^3 e^{-x}\, dx.$$

$$C \cdot \frac{4}{3} \times 6 = 1$$

$$C = \frac{1}{8}$$

(b)

The marginal density $f_X(x)$ is given by:

$$f_X(x) = \int_{-x}^{x} f(x,y)\,dy,\ \ y \in [-x, x].$$

$$f_X(x) = \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)e^{-x}\,dy = \frac{1}{8}e^{-x} \int_{-x}^{x} (x^2 - y^2)\,dy.$$

From part (a), we already have:

$$\int_{-x}^{x} (x^2 - y^2)\,dy = \frac{4x^3}{3}.$$

Thus,

$$f_X(x) = \frac{1}{8}e^{-x} \cdot \frac{4x^3}{3} = \frac{x^3 e^{-x}}{6}, \quad \text{for } x \geq 0.$$

Therefore,

$$f_X(x) = \frac{x^3 e^{-x}}{6}, \quad x \in [0, \infty).$$

(c)

The conditional density $f_{Y|X}(y|x)$ for $x \in [0, \infty)$, $y \in [-x, x]$ is given by:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}.$$

$$f_{Y|X}(y|x) = \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{x^3 e^{-x}}{6}} = \frac{1}{8}(x^2 - y^2)e^{-x} \times \frac{6}{x^3 e^{-x}}.$$

$$f_{Y|X}(y|x) = \frac{6(x^2 - y^2)}{8x^3} = \frac{3(x^2 - y^2)}{4x^3}.$$

Thus,

$$f_{Y|X}(y|x) = \frac{3(x^2 - y^2)}{4x^3}, \quad y \in [-x, x],\ x \geq 0.$$

(d)

To determine if X and Y are independent, we need to check if:

$$f(x, y) = f_X(x) f_Y(y).$$

The joint density is:

$$f(x, y) = \frac{1}{8}(x^2 - y^2)e^{-x}.$$

Since the term $(x^2 - y^2)$ involves both x and y, and the support of Y depends on X, since $y \in [-x, x]$, the joint density $f(x, y)$ cannot be written as the product $f_X(x) f_Y(y)$ (which is product of some function of X and some function of Y)

Moreover, from part (c):

$$f_{Y|X}(y|x) = \frac{3(x^2 - y^2)}{4x^3}.$$

If $f_{Y|X}(y|x) = f_Y(y)$, X and Y are independent. However, this conditional density depends on x, so it can not be equal to $f_Y(y)$.

Conclusion:

Since the joint density $f(x, y)$ cannot be written as the product $f_X(x) f_Y(y)$, and the conditional density $f_{Y|X}(y|x)$ depends on x, the variables X and Y are **dependent**.

Question 5

(a)

Given Triangle formed by the points $(1, 0)$, $(0, 1)$, and $(0, -1)$. The sides of the triangle are defined by the equations:

- $y = 1 - x$
- $y = x - 1$
- $x = 0$

The area of this triangle is 1.

Since the point (X, Y) is chosen uniformly at random from the triangle, the joint probability density function $f(x, y)$ is constant within the triangle and zero outside. Given that the area is 1:

$$f(x, y) = 1 \quad \text{for } x \in [0, 1], \quad y \in [x - 1, 1 - x]$$

The conditional probability is given by:

$$P(X > 0.1 \mid Y > 0.1) = \frac{P(X > 0.1 \text{ and } Y > 0.1)}{P(Y > 0.1)}$$

To calculate $P(Y > 0.1)$, we can integrate $f(x, y)$ from 0.1 to 1 in y, and for a fixed y value, we can see the possible x values lie between 0 and $1 - y$ (here $y > 0.1$). Hence, we first integrate x from 0 to 1 - y, and then integrate y from 0 to 1:

$$
\begin{aligned}
P(Y > 0.1) &= \int_{y=0.1}^{1} \int_{x=0}^{-y+1} f(x, y)\,dx\,dy \\
&= \int_{y=0.1}^{1} \int_{x=0}^{-y+1} 1\,dx\,dy \\
&= \int_{y=0.1}^{1} (-y + 1)\,dy \\
&= \left[-\frac{y^2}{2} + y\right]_{0.1}^{1} \\
&= \left(-\frac{1^2}{2} + 1\right) - \left(-\frac{(0.1)^2}{2} + 0.1\right) \\
&= \left(-\frac{1}{2} + 1\right) - \left(-\frac{0.01}{2} + 0.1\right) \\
&= \left(\frac{1}{2}\right) - (-0.005 + 0.1) \\
&= \frac{1}{2} - 0.095 \\
&= 0.405
\end{aligned}
$$

To Compute $P(X > 0.1 \text{ and } Y > 0.1)$, we can integrate $f(x, y)$ from 0.1 to 0.9 in y, since when x = 0.1, y = 0.9 ($y > 0.1$). Also for a fixed y value, the possible x values are between 0.1 and $1 - y$ (here $y > 0.1$). Hence, we first integrate x from 0.1 to 1 - y, and then integrate y from 0.1 to 0.9:

$$
\begin{aligned}
P(X > 0.1 \text{ and } Y > 0.1) &= \int_{y=0.1}^{0.9} \int_{x=0.1}^{-y+1} f(x, y)\,dx\,dy \\
&= \int_{y=0.1}^{0.9} \int_{x=0.1}^{-y+1} 1\,dx\,dy \\
&= \int_{y=0.1}^{0.9} (-y + 1 - 0.1)\,dy \\
&= \int_{y=0.1}^{0.9} (0.9 - y)\,dy \\
&= \left[0.9y - \frac{y^2}{2}\right]_{0.1}^{0.9} \\
&= \left(0.9 \times 0.9 - \frac{0.9^2}{2}\right) - \left(0.9 \times 0.1 - \frac{(0.1)^2}{2}\right) \\
&= (0.81 - 0.405) - (0.09 - 0.005) \\
&= 0.405 - 0.085 \\
&= 0.32
\end{aligned}
$$

$$
P(X > 0.1 \mid Y > 0.1) = \frac{P(X > 0.1 \text{ and } Y > 0.1)}{P(Y > 0.1)}
$$

$$
P(X > 0.1 \mid Y > 0.1) = \frac{0.32}{0.405} = \frac{64}{81}
$$

Thus,

$$P(X > 0.1 \mid Y > 0.1) = \frac{64}{81}$$

(b)

For $x \in [0,1]$, the limits of y are from $y = x - 1$ to $y = -x + 1$. Therefore, the marginal density is:

$$\begin{aligned} f_X(x) &= \int_{y=x-1}^{-x+1} f(x,y)\,dy \\ &= \int_{y=x-1}^{-x+1} 1\,dy \\ &= (-x+1) - (x-1) \\ &= 2(1-x) \end{aligned}$$

The CDF of X for $0 \le x \le 1$ is given by:

$$\begin{aligned} F_X(x) &= \int_{t=0}^{x} f_X(t)\,dt \\ &= \int_{t=0}^{x} 2(1-t)\,dt \\ &= 2\left(t - \frac{t^2}{2}\right)\Big|_0^x \\ &= 2\left(x - \frac{x^2}{2}\right) \\ &= 2x - x^2 \end{aligned}$$

Therefore,

- For $x < 0$:

$$F_X(x) = 0$$

- For $0 \le x \le 1$:

$$F_X(x) = 2x - x^2$$

- For $x > 1$:

$$F_X(x) = 1$$

The cumulative distribution function of X is:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0; \\ 2x - x^2, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } x > 1. \end{cases}$$

Question 6

(a)

Since $T = \min(T_1, T_2, T_3)$, and T_1, T_2, T_3 are independent, we have:

Since:

$$
\begin{aligned}
P(T_i > t) &= 1 - F_i(T_i) \\
&= 1 - (1 - e^{-\lambda_i t}) = e^{-\lambda_i t}, \quad t \geq 0
\end{aligned}
$$

We have

$$
\begin{aligned}
P(T > t) &= P(T_1 > t, T_2 > t, T_3 > t) = P(T_1 > t) \cdot P(T_2 > t) \cdot P(T_3 > t) \\
&= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 t} = e^{-(\lambda_1 + \lambda_2 + \lambda_3) t}
\end{aligned}
$$

Therefore, the CDF is:

$$
F_T(t) = 1 - P(T > t) = 1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3) t}, \quad t \geq 0
$$

(b)

To calculate $P(S = s)$, since we check the circuit every second, so if $S = s$, then T must fulfill $s - 1 < T \leq s$, so we have:

$$
P(S = s) = P(s - 1 < T \leq s) = F_T(s) - F_T(s - 1)
$$

Therefore,

$$
\begin{aligned}
P(S = s) &= \left(1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3) s}\right) - \left(1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3)(s-1)}\right) \\
&= e^{-(\lambda_1 + \lambda_2 + \lambda_3)(s-1)} - e^{-(\lambda_1 + \lambda_2 + \lambda_3) s}, \quad s = 1, 2, 3, \ldots
\end{aligned}
$$

(c)

(i)

$$
P(N = n, S = s) = P(S = s) \cdot P(N = n | S = s)
$$

Since when we are given $(S = s)$, that is, in s -1 seconds, we check the circuit n times during these times (i.e. $s - 1$ times of trials in total). Therefore, the probability of checking exactly n times in $s - 1$ opportunities follows a Binomial distribution:

$$
P(N = n | S = s) = \binom{s-1}{n} p^n (1 - p)^{s-1-n}
$$

Therefore,

$$
P(N = n, S = s) = (e^{-(\lambda_1 + \lambda_2 + \lambda_3)(s-1)} - e^{-(\lambda_1 + \lambda_2 + \lambda_3) s}) \cdot \binom{s-1}{n} p^n (1 - p)^{s-n-1}, \quad s = 1, 2, 3, \ldots,, \quad n = 0, 1, 2, \ldots
$$

(ii)

To find the PMF of N, we can sum the joint PMF over all possible values of S, since $n \leq s-1$, $s \geq n+1$

$$P(N=n) = \sum_{s=n+1}^{\infty} P(N=n, S=s)$$

Thus,

$$P(N=n) = \sum_{s=n+1}^{\infty} \left[\left(e^{-(\lambda_1+\lambda_2+\lambda_3)(s-1)} - e^{-(\lambda_1+\lambda_2+\lambda_3)s} \right) \cdot \binom{s-1}{n} p^n (1-p)^{s-n-1} \right], \quad s = 1, 2, 3, \ldots, \quad n = 0, 1, 2, \ldots$$