Homework 1 Solutions

1. We compute the moment generating function (MGF) of $X \sim \text{Poisson}(\lambda)$ and take the derivative. Recall that the MGF of Poisson(λ) is

$$M_X(t) := \mathbb{E}(e^{tX}) = e^{\lambda(e^t-1)} \quad \forall t \in \mathbb{R}.$$

Then, the fourth moment is obtained by, after doing some tedious calculations,

$$\mathbb{E}(X^4) = \frac{d^4}{dt^4} M_X(t)\bigg|_{t=0} = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda.$$

2. The expectation is obtained as follows:

$$\mathbb{E}(X) = \int_0^\infty x p(x)\,dx = \int_0^\infty \lambda x e^{-\lambda x}\,dx = -xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\bigg|_0^\infty = \frac{1}{\lambda}.$$

3. The exact binomial probability of $X \sim \text{Binomial}(n, p)$ is

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

which can be computed by `dbinmom(k, n, p)` in R or `binom.pmf(k, n, p)` after declaring `from scipy.stats import binom` in Python with the SciPy package.

The normal approximation relies on $X \approx np + \sqrt{np(1-p)} \cdot Z$ for $Z \sim N(0,1)$. In this case, noticing $\mathbb{P}(X = k) = \mathbb{P}(k - 0.5 < X \leq k + 0.5)$, we have

$$\begin{aligned}
\mathbb{P}(X = k) &\approx \mathbb{P}\left(\frac{k - 0.5 - np}{\sqrt{np(1-p)}} < Z \leq \frac{k + 0.5 - np}{\sqrt{np(1-p)}}\right) \\
&= \Phi\left(\frac{k + 0.5 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k - 0.5 - np}{\sqrt{np(1-p)}}\right),
\end{aligned}$$

where Φ is the cumulative distribution function of $Z \sim N(0,1)$. To compute $\Phi(z)$, use `pnorm(z)` in R or `norm.cdf(z)` after declaring `from scipy.stats import norm` in Python with the SciPy package.

Lastly, the Poisson approximation relies on $X \approx \text{Poisson}(np)$ so that

$$\mathbb{P}(X = k) \approx e^{-np} \frac{(np)^k}{k!},$$

which can be computed by `dpois(k, n*p)` in R or `poisson.pmf(k, n*p)` after declaring `from scipy.stats import poisson` in Python with the SciPy package. The results are shown in Table 1.

For (a), neither approximation scheme is good. For (b), the normal approximation is good because both np and $n(1-p)$ are greater than 10, which makes the shape of X close to that of the corresponding normal distribution. For (c), the Poisson approximation is good because n is large while p is small.

	exact	normal	Poisson
(a) $n = 7$, $p = 0.3$, $k = 3$	0.2269	0.2466	0.1890
(b) $n = 40$, $p = 0.4$, $k = 11$	0.0357	0.0353	0.0496
(c) $n = 400$, $p = 0.0025$, $k = 2$	0.1842	0.2418	0.1839

Table 1: Results for Q3.

4-(a). The log-likelihood is

$$\begin{aligned}\ell(p) &= \log \prod_{i=1}^{n} (1-p)^{1-X_i} \\ &= \sum_{i=1}^{n} \left(X_i \log(p) + (1-X_i)\log(1-p)\right) \\ &= n(\bar{X}\log(p) + (1-\bar{X})\log(1-p)),\end{aligned}$$

where $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ is the sample mean. Taking the derivative, we have

$$\ell'(p) = n\left(\frac{\bar{X}}{p} - \frac{1-\bar{X}}{1-p}\right),$$

from which we can deduce that ℓ is maximized at

$$\widehat{p}_{\mathsf{MLE}} := \bar{X}.$$

4-(b). The mean, variance, and MSE of $\widehat{p}_{\mathsf{MLE}}$ are

$$\begin{aligned}\mathbb{E}(\widehat{p}_{\mathsf{MLE}}) &= \mathbb{E}(\bar{X}) = p, \\ \mathrm{Var}(\widehat{p}_{\mathsf{MLE}}) &= \mathrm{Var}(\bar{X}) = \frac{p(1-p)}{n}, \\ \mathrm{MSE}(\widehat{p}_{\mathsf{MLE}}) &= \mathbb{E}(\widehat{p}_{\mathsf{MLE}} - p)^2 = \mathrm{Var}(\widehat{p}_{\mathsf{MLE}}) + \underbrace{(\mathbb{E}(\widehat{p}_{\mathsf{MLE}}) - p)^2}_{(\mathrm{Bias}(\widehat{p}_{\mathsf{MLE}}))^2} = \frac{p(1-p)}{n}.\end{aligned}$$

4-(c). By the central limit theorem,

$$\frac{\sqrt{n}(\widehat{p}_{\mathsf{MLE}} - p)}{\sqrt{p(1-p)}} = \frac{\sqrt{n}(\bar{X} - p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1).$$

4-(d). Based on the above asymptotic distribution, Wald's confidence interval with 95% is

$$\widehat{p}_{\mathsf{MLE}} \pm z_{0.975}\sqrt{\frac{\widehat{p}_{\mathsf{MLE}}(1-\widehat{p}_{\mathsf{MLE}})}{n}},$$

where z_q is the q-th quantile of $N(0,1)$.

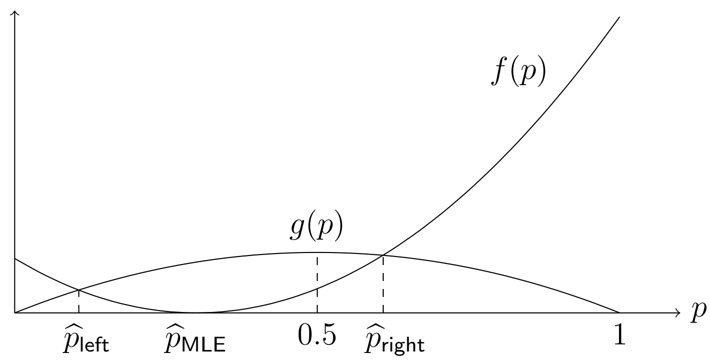

Figure 1: Visualization of Wilson's method.

4-(e). For Wilson's method, we want

$$\frac{n(\widehat{p}_{\mathsf{MLE}} - p)^2}{p(1-p)} \leq z_{0.975}^2,$$

and we solve the following quadratic equation of p:

$$n(\widehat{p}_{\mathsf{MLE}} - p)^2 = z_{0.975}^2 p(1-p).$$

The two solutions are

$$\widehat{p}_{\mathsf{left}} = \frac{2n\widehat{p}_{\mathsf{MLE}} + z_{0.975}^2 - z_{0.975}\sqrt{4n\widehat{p}_{\mathsf{MLE}}(1-\widehat{p}_{\mathsf{MLE}}) + z_{0.975}^2}}{2(n + z_{0.975}^2)},$$
$$\widehat{p}_{\mathsf{right}} = \frac{2n\widehat{p}_{\mathsf{MLE}} + z_{0.975}^2 + z_{0.975}\sqrt{4n\widehat{p}_{\mathsf{MLE}}(1-\widehat{p}_{\mathsf{MLE}}) + z_{0.975}^2}}{2(n + z_{0.975}^2)}.$$

4-(f). We draw two quadratic functions of p, say, $f(p) = (\widehat{p}_{\mathsf{MLE}} - p)^2$ and $g(p) = \frac{z_{0.975}^2 p(1-p)}{n}$. See Figure 1. The two functions f and g intersect at $\widehat{p}_{\mathsf{left}}$ and $\widehat{p}_{\mathsf{right}}$.

4-(g). Notice that g has symmetry about the vertical line $p = 0.5$, while f has symmetry about the vertical line $p = \widehat{p}_{\mathsf{MLE}}$. Therefore, one can deduce that

$$\begin{aligned}
\widehat{p}_{\mathsf{MLE}} - \widehat{p}_{\mathsf{left}} < \widehat{p}_{\mathsf{right}} - \widehat{p}_{\mathsf{MLE}} \quad &\text{if } \widehat{p}_{\mathsf{MLE}} < 0.5, \\
\widehat{p}_{\mathsf{MLE}} - \widehat{p}_{\mathsf{left}} > \widehat{p}_{\mathsf{right}} - \widehat{p}_{\mathsf{MLE}} \quad &\text{if } \widehat{p}_{\mathsf{MLE}} > 0.5, \\
\widehat{p}_{\mathsf{MLE}} - \widehat{p}_{\mathsf{left}} = \widehat{p}_{\mathsf{right}} - \widehat{p}_{\mathsf{MLE}} \quad &\text{if } \widehat{p}_{\mathsf{MLE}} = 0.5.
\end{aligned}$$

Alternatively, one can analytically derive this by noticing that

$$\begin{aligned}
\widehat{p}_{\mathsf{MLE}} - \widehat{p}_{\mathsf{left}} < \widehat{p}_{\mathsf{right}} - \widehat{p}_{\mathsf{MLE}} &\Leftrightarrow 0 < \widehat{p}_{\mathsf{left}} + \widehat{p}_{\mathsf{right}} - 2\widehat{p}_{\mathsf{MLE}} \\
&\Leftrightarrow 0 < \frac{2n\widehat{p}_{\mathsf{MLE}} + z_{0.975}^2}{n + z_{0.975}^2} - 2\widehat{p}_{\mathsf{MLE}} \\
&\Leftrightarrow \widehat{p}_{\mathsf{MLE}} < 0.5.
\end{aligned}$$