Stat 301

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, I_p)$ $\theta \in \mathbb{R}^p$ $\|\hat{\theta} - \theta\|^2$

$p \geq 3$ $\bar{X}$ is inadmissible.

$$\hat{\theta}_{JS} = \left(1 - \frac{p-2}{n\|\bar{X}\|^2}\right)\bar{X}.$$

an empirical Bayes view (Efron & Morris).

empirical Bayes framework. (Robbins).

$$p(x|\theta), \quad \pi(\theta|\tau^2)$$

$\tau^2 \to$ hyperparameter.

can estimate τ^2 using $m(x|\tau^2) = \int p(x|\theta)\pi(\theta|\tau^2)\,d\theta$.

particularly useful when $\theta \in \mathbb{R}^p$ for large p

and $\theta_1, \cdots, \theta_p \mid \tau^2$ are iid.

Compound decision theory $\quad \frac{1}{p}\mathbb{E}_\theta \|\hat{\theta}-\theta\|^2 = \mathbb{E}_\theta \frac{1}{p}\sum_{j=1}^{p}(\hat{\theta}_j - \theta_j)^2.$

$$\theta \sim \frac{1}{p}\sum_{j=1}^{p}\delta_{\theta_j}$$

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derivation of $\hat{\theta}_{JS}$ from empirical Bayes.

$$X_1, \cdots, X_n \mid \theta \overset{iid}{\sim} N(\theta, I_p). \quad \theta \mid \tau^2 \sim N(0, \tau^2 I_p).$$

$$\hat{\theta} = \mathbb{E}(\theta \mid X_1, \cdots, X_n) = \frac{n}{n + 1/\tau^2}\bar{X} = \left(1 - \frac{1/n}{1/n + \tau^2}\right)\bar{X}$$

$$m(X \mid \tau^2) = \int p(X\mid\theta)\pi(\theta\mid\tau^2)\, d\theta.$$

$$X_i = \theta + Z_i, \quad Z_i \overset{iid}{\sim} N(0, I_p). \qquad \theta = \tau W, \quad W \sim N(0, I_p)$$

$$\Rightarrow \quad X_i = \tau W + Z_i, \quad i = 1, \cdots, n$$

$$\Rightarrow \quad \bar{X} = \tau W + \bar{Z} \sim N\left(0, \left(\tau^2 + \frac{1}{n}\right) I_p\right).$$

$$\frac{\|\bar{X}\|^2}{\tau^2 + \frac{1}{n}} \sim \chi^2_p \Rightarrow \frac{\tau^2 + \frac{1}{n}}{\|\bar{X}\|^2} \sim \text{inv-}\chi^2_p.$$

$$\mathbb{E}\left(\frac{\tau^2 + \frac{1}{n}}{\|\bar{X}\|^2}\right) = \frac{1}{p-2}. \Leftrightarrow \mathbb{E}\left(\frac{p-2}{\|\bar{X}\|^2}\right) = \frac{1}{\tau^2 + \frac{1}{n}}.$$

method of moment: $\frac{p-2}{\|\bar{X}\|^2}$ is an unbiased estimator of $\frac{1}{\tau^2 + \frac{1}{n}}$.

$$\hat{\theta}_{JS} = \left(1 - \frac{p-2}{n\|\bar{X}\|^2}\right) \bar{X}.$$

Theorem: $p \geq 3$. $\mathbb{E}_\theta \|\hat{\theta}_{JS} - \theta\|^2 < \frac{p}{n} = \mathbb{E}_\theta \|\bar{X} - \theta\|^2 \quad \forall \theta \in \mathbb{R}^p$.

Lemma (Stein's identity): $Z \sim N(0,1)$

$$\mathbb{E} Z g(Z) = \mathbb{E} g'(Z).$$

Proof: $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $\quad \phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(-x) = -x\phi(x)$.

$$\mathbb{E} Z g(Z) = \int x g(x) \phi(x) dx = -\int \phi'(x) g(x) dx.$$

$$= -\left(g(x)\phi(x) \Big|_{-\infty}^{\infty} - \int g'(x) \phi(x) dx \right)$$

$$= \mathbb{E} g'(Z).$$

$Z \sim N(0, I_p)$ $\quad g: \mathbb{R}^p \to \mathbb{R}^p$

$$\mathbb{E}\langle Z, g(Z)\rangle = \mathbb{E}\langle \nabla, g(Z)\rangle = \sum_{j=1}^{p} \mathbb{E}\frac{\partial}{\partial z_j} g_j(Z)$$

Proof of Thm: $\mathbb{E}_\theta \|\hat{\theta}_{JS} - \theta\|^2 = \mathbb{E}_\theta \left\| \left(1 - \frac{p-2}{n\|\bar{X}\|^2}\right)\bar{X} - \theta \right\|^2.$

$$= \mathbb{E}_\theta \left\| \bar{X} - \theta - \frac{p-2}{n\|\bar{X}\|^2}\bar{X} \right\|^2$$

$\bar{X} \sim N(\theta, \frac{1}{n} I_p).$

$\Leftrightarrow \bar{X} = \theta + \frac{1}{\sqrt{n}} Z \quad Z \sim N(0, I_p)$

$= \frac{1}{\sqrt{n}}(\sqrt{n}\theta + Z)$

$= \frac{1}{\sqrt{n}}(\mu + Z)$

$\mu = \sqrt{n}\theta$

$$= \mathbb{E}\left\| \frac{1}{\sqrt{n}} Z - \frac{p-2}{\|\mu+Z\|^2} \frac{1}{\sqrt{n}}(\mu+Z) \right\|^2.$$

$$= \frac{1}{n}\mathbb{E}\left\| Z - \frac{p-2}{\|\mu+Z\|^2}(\mu+Z) \right\|^2.$$

$$= \frac{1}{n}\left(\mathbb{E}\|Z\|^2 + \mathbb{E}\frac{(p-2)^2}{\|\mu+Z\|^2} - 2(p-2)\mathbb{E}\left\langle Z, \frac{\mu+Z}{\|\mu+Z\|^2}\right\rangle \right).$$

analysis of $\mathbb{E}\left\langle z, \frac{\mu+z}{\|\mu+z\|^2}\right\rangle$

$$g(z) = \frac{\mu+z}{\|\mu+z\|^2} = \begin{pmatrix} g_1(z) \\ \vdots \\ g_p(z) \end{pmatrix}.$$

$$g_j(z) = \frac{\mu_j+z_j}{\|\mu+z\|^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad \frac{\partial}{\partial z_j} g_j(z) = \frac{\|\mu+z\|^2 - 2(\mu_j+z_j)^2}{\|\mu+z\|^4}$$

$$\mathbb{E}\left\langle z, \frac{\mu+z}{\|\mu+z\|^2}\right\rangle = \sum_{j=1}^{p} \mathbb{E}\frac{\partial}{\partial z_j} g_j(z)$$

$$= \mathbb{E}\sum_{j=1}^{p} \frac{\|\mu+z\|^2 - 2(\mu_j+z_j)^2}{\|\mu+z\|^4} = \mathbb{E}\frac{p\|\mu+z\|^2 - 2\|\mu+z\|^2}{\|\mu+z\|^4}$$

$$= \mathbb{E}\frac{p-2}{\|\mu+z\|^2}$$

$$\mathbb{E}_\theta \|\hat{\theta}_{JS} - \theta\|^2 = \frac{1}{n}\left(p + \mathbb{E}\frac{(p-2)^2}{\|\mu+Z\|^2} - 2(p-2)\,\mathbb{E}\,\frac{p-2}{\|\mu+Z\|^2} \right).$$

$$= \frac{1}{n}\left(p - (p-2)^2\,\mathbb{E}\,\frac{1}{\|\mu+Z\|^2} \right). \qquad Z \sim N(0, I_p) \quad \mu = \sqrt{n}\,\theta.$$

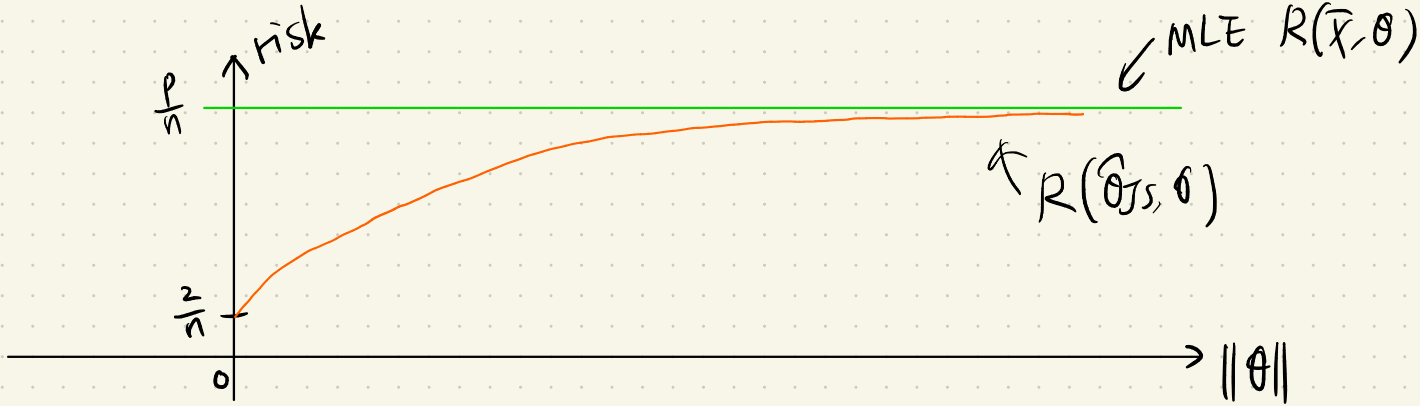
$$< \frac{p}{n}. \qquad \square.$$

$$R(\hat{\theta}_{JS}, \theta) = \frac{1}{n}\left(p - (p-2)^2\,\mathbb{E}\,\frac{1}{\|\sqrt{n}\theta + Z\|^2} \right).$$

$$R(\hat{\theta}_{JS}, 0) = \frac{1}{n}\left(p - (p-2)^2\,\mathbb{E}\,\frac{1}{\|Z\|^2} \right) = \frac{2}{n}.$$

by symmetry $R(\hat{\theta}_{JS}, \theta)$ depends on θ through $\|\theta\|$.

as $\|\theta\| \to \infty$ $\quad R(\hat{\theta}_{JS}, \theta) \to \frac{p}{n}.$



$$\sup_{\theta\in\mathbb{R}^p} R(\hat{\theta}_{JS},\theta) = \sup_{\theta\in\mathbb{R}^p} R(\bar{X},\theta) = \frac{p}{n}$$

Shrinkage estimation $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, I_p)$. $\hat{\theta}_c = c\bar{X}$.

$$R(\hat{\theta}_c,\theta) = \mathbb{E}_\theta\|c\bar{X}-c\theta\|^2 + \|c\theta-\theta\|^2.$$

$$= c^2\frac{p}{n} + (c-1)^2\|\theta\|^2 = f(c).$$

$$f'(c) = 2c\frac{p}{n} + 2(c-1)\|\theta\|^2 = 0$$

$$\Rightarrow c^* = \frac{\|\theta\|^2}{\frac{p}{n} + \|\theta\|^2}$$

$$\hat{\theta}_{c^*} = \frac{\|\theta\|^2}{\frac{p}{n} + \|\theta\|^2}\overline{X} = \left(1 - \frac{p}{p + n\|\theta\|^2}\right)\overline{X}$$

$$R(\hat{\theta}_{c^*}, \theta) = \left(\frac{b}{a+b}\right)^2 a + \left(\frac{a}{a+b}\right)^2 b \qquad \begin{cases} a = \frac{p}{n} \\ b = \|\theta\|^2 \end{cases}$$

oracle estimator

$$= \frac{b^2 a + a^2 b}{(a+b)^2} = \frac{ab}{a+b} \leq \min(a, b)$$

$$= \min\left(\frac{p}{n}, \|\theta\|^2\right).$$

$$R(\hat{\theta}_{JS}, \theta) = \frac{1}{n}\left(p - (p-2)^2\, \mathbb{E}\frac{1}{\|Z+\mu\|^2}\right) \qquad \begin{cases} Z \sim N(0, I_p) \\ \mu = \sqrt{n}\,\theta \end{cases}$$

$$\|Z+\mu\|^2 \sim \chi^2_{p,\|\mu\|^2} \overset{d}{=} \chi^2_{p+2N}, \quad N \sim \text{Poisson}\left(\frac{\|\mu\|^2}{2}\right)$$

$$\mathbb{E}\frac{1}{\|Z+\mu\|^2} = \mathbb{E}\frac{1}{\chi^2_{p+2N}} = \mathbb{E}\left(\mathbb{E}\left(\frac{1}{\chi^2_{p+2N}} \,\Big|\, N\right)\right)$$

$$= \mathbb{E}\frac{1}{p+2N-2} \geq \underset{\text{Jensen}}{\frac{1}{\mathbb{E}(p+2N-2)}}$$

$$= \frac{1}{p+\|\mu\|^2-2}$$

$$R(\hat{\theta}_{JS}, \theta) \leq \frac{1}{n}\left(p - (p-2)^2 \frac{1}{p+\|u\|^2-2}\right)$$

$$= \frac{1}{n} \frac{p^2+p\|u\|^2-2p-p^2-4+4p}{p+\|u\|^2-2} = \frac{1}{n} \frac{p\|u\|^2+2(p-2)}{p+\|u\|^2-2}$$

$$= \frac{1}{n}\left(2 + \frac{(p-2)\|u\|^2}{p-2+\|u\|^2}\right). \qquad u = \sqrt{n}\,\theta$$

$$= \frac{2}{n} + \frac{\frac{p-2}{n}\|\theta\|^2}{\frac{p-2}{n}+\|\theta\|^2} \leq \frac{2}{n} + \frac{\frac{p}{n}\|\theta\|^2}{\frac{p}{n}+\|\theta\|^2}.$$

$$= \frac{2}{n} + R(\hat{\theta}_{c^*}, \theta).$$

Theorem (oracle inequality):

$p \geq 3$

$$R(\hat{\theta}_{JS}, \theta) \leq \inf_c R(\hat{\theta}_c, \theta) + \frac{2}{n} \rightarrow \text{dimension-free}$$