STAT 32950 Assignment 0

Bin Yu

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Question 1

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}},$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^n y_i.$$

Question 2

Use:

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> x = runif(30); y = x^3 + rnorm(30)/3
> plot(x, y); abline(lm(y ~ x))
> title("Scatter Plot with Least Square Regression Line")
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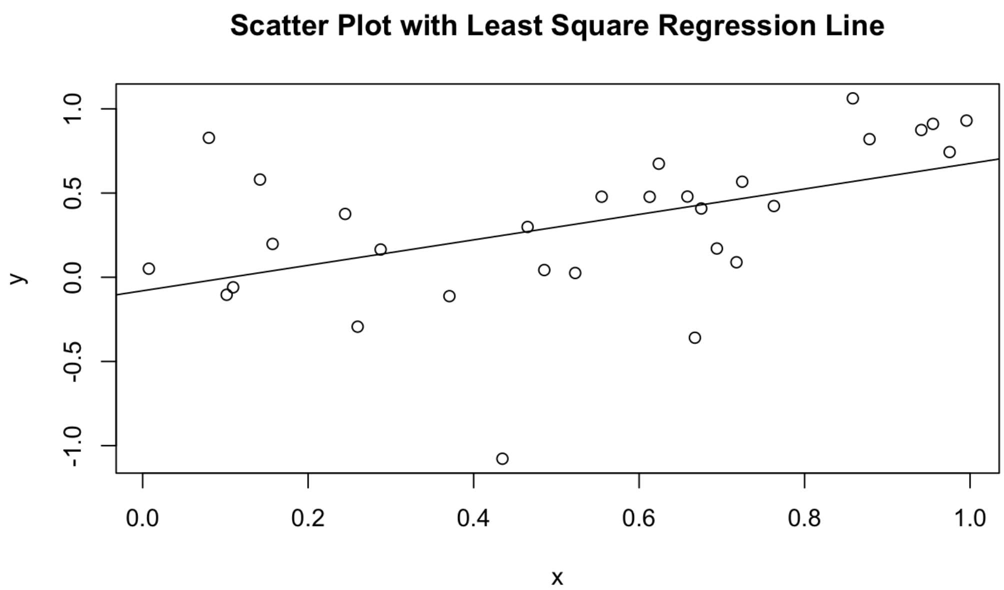

Figure 1: Scatter Plot with Least Square Regression Line

Question 3

(a)

(i) Rows of BA in terms of rows of A

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} -1 & -4 & 3 \\ 5 & 0 & -6 \\ 7 & 5 & 4 \end{pmatrix}.$$

Since B is diagonal, it multiplies each row of A by the corresponding diagonal entry of B:

- The first row of BA is the same as the first row of A.
- The second row of BA is 3 times the second row of A.
- The third row of BA is -2 times the third row of A.

(ii) Columns of AB in terms of columns of A

$$AB = \begin{pmatrix} -1 & -4 & 3 \\ 5 & 0 & -6 \\ 7 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Right-multiplying by a diagonal matrix B scales each column of A:

- The first column of AB is the same as the first column of A, which is a_1.
- The second column of AB is 3 times the second column of A, which is $3a_2$.
- The third column of AB is -2 times the third column of A, which is $-2a_3$.

(iii) Rows of EA in terms of rows of A

Consider

$$EA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & -4 & 3 \\ 5 & 0 & -6 \\ 7 & 5 & 4 \end{pmatrix}.$$

Each row of EA is formed by the corresponding row of E multiplying A. Label the rows of A by r_1, r_2, r_3. Then:

$$\mathrm{Row}_1(E) = (0\ 1\ 0), \quad \mathrm{Row}_2(E) = (0\ 0\ 1), \quad \mathrm{Row}_3(E) = (1\ 0\ 0).$$

Hence:

- The first row of EA is $r_2(A)$ (the second row of A).
- The second row of EA is $r_3(A)$ (the third row of A).
- The third row of EA is $r_1(A)$ (the first row of A).

(iv) Columns of AE in terms of columns of A

Now consider

$$AE = \begin{pmatrix} -1 & -4 & 3 \\ 5 & 0 & -6 \\ 7 & 5 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let the columns of A be a_1, a_2, a_3. The columns of E are

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus each column of AE is A times one of these e_j:

- The first column of AE is $A\,e_1 = a_3$.
- The second column of AE is $A\,e_2 = a_1$.
- The third column of AE is $A\,e_3 = a_2$.

(b)

Let

$$v = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}.$$

,

$$A = \begin{pmatrix} -1 & -4 & 3 \\ 5 & 0 & -6 \\ 7 & 5 & 4 \end{pmatrix} \longrightarrow a_1 = \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix}, a_2 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}, a_3 = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \longrightarrow b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

1) Writing v as a linear combination of the a_i's

We want to solve

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}.$$

Equivalently,

$$x\,a_1 + y\,a_2 + z\,a_3 = v.$$

Solving the system

$$\begin{pmatrix} -1 & -4 & 3 \\ 5 & 0 & -6 \\ 7 & 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}$$

gives

$$x = 3, \quad y = -1, \quad z = 2.$$

Hence

$$v = 3\,a_1 - a_2 + 2\,a_3.$$

2) Writing v as a linear combination of the b_i's

We want to solve:

$$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}.$$

where

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

Hence

$$xb_1 + yb_2 + zb_3 = \begin{pmatrix} x \\ 3y \\ -2z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}.$$

Matching components:

$$x = 7, \quad 3y = 3 \Rightarrow y = 1, \quad -2z = 24 \Rightarrow z = -12.$$

Thus

$$v = 7\,b_1 + 1\,b_2 - 12\,b_3.$$

3) Writing v as a linear combination of the e_i's

We want to solve

$$E\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}.$$

where,

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

we compute:

$$x\,e_1 + y\,e_2 + z\,e_3 = \begin{pmatrix} y \\ z \\ x \end{pmatrix} = v = \begin{pmatrix} 7 \\ 3 \\ 24 \end{pmatrix}.$$

Therefore:

$$y = 7, \quad z = 3, \quad x = 24.$$

So:

$$v = 24\,e_1 + 7\,e_2 + 3\,e_3.$$

Question 4

(a) Proof by Mathematical Induction

Proof:

Base Case: Let $k = 1$. Then

$$\int_0^\infty xe^{-x}\,dx.$$

Use integration by parts. Let

$$u = x \quad \text{and} \quad dv = e^{-x}\,dx.$$

Then

$$du = dx, \quad v = -e^{-x}.$$

Hence,

$$\int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx.$$

As $x \to \infty$, $xe^{-x} \to 0$, and at $x = 0$, $xe^{-x} = 0$. Therefore,

$$\int_0^\infty xe^{-x}\,dx = 0 + \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = (0) - (-1) = 1.$$

Thus, for $k = 1$,

$$\int_0^\infty xe^{-x}\,dx = 1! = 1,$$

which completes the base case.

Inductive Step: Assume for some integer $k \geq 1$ that

$$\int_0^\infty x^k e^{-x}\,dx = k!.$$

We need to show that

$$\int_0^\infty x^{k+1}e^{-x}\,dx = (k+1)!.$$

Again, use integration by parts. Let

$$u = x^{k+1}, \quad dv = e^{-x}\,dx.$$

Then

$$du = (k+1)x^k\,dx, \quad v = -e^{-x}.$$

So

$$\int_0^\infty x^{k+1}e^{-x}\,dx = -x^{k+1}e^{-x}\Big|_0^\infty + \int_0^\infty (k+1)x^k e^{-x}\,dx.$$

As $x \to \infty$, $x^{k+1}e^{-x} \to 0$, and at $x = 0$, $x^{k+1}e^{-x} = 0$. Therefore,

$$\int_0^\infty x^{k+1}e^{-x}\,dx = (k+1)\int_0^\infty x^k e^{-x}\,dx.$$

By the inductive hypothesis, this is

$$(k+1)\cdot k! = (k+1)!.$$

This completes the inductive step.

Therefore, by the principle of mathematical induction, for all integers $k \geq 1$,

$$\int_0^\infty x^k e^{-x}\,dx = k!.$$

(b) Commutativity of 4×4 Matrices

We wish to show that for 4×4 non-diagonal matrices A and B, the product AB need not equal BA. We prove this by give a counter example:

Consider the 4×4 matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which are clearly not equal. Hence, in general, $AB \neq BA$ for non-diagonal matrices A and B.

Question 5

(a) Derive the PDF of X

Since U is uniform on $[-1, 1]$, its PDF is

$$f_U(u) = \begin{cases} \frac{1}{2}, & -1 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We define $X = U^2$. Then $X \in [0, 1]$. To find the distribution of X, we first compute its CDF, $F_X(x) = P(X \leq x)$.

- If $x < 0$, then $X \leq x$ is impossible because $X \geq 0$. Hence $F_X(x) = 0$ for $x < 0$.
- If $x \geq 1$, then $X \leq x$ is always true ($X \in [0, 1]$). Hence $F_X(x) = 1$ for $x \geq 1$.
- For $0 \leq x < 1$, we have

  $$F_X(x) = P(U^2 \leq x) = P\big(-\sqrt{x} \leq U \leq \sqrt{x}\big) = P(U \leq \sqrt{x}) - P(U \leq -\sqrt{x}).$$

  Hence:

  $$F_X(x) = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{2}\,du = \frac{1}{2} \times \big(2\sqrt{x}\big) = \sqrt{x}.$$

Therefore, the CDF of X is

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \sqrt{x}, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Differentiating $F_X(x)$ with respect to x on $(0,1)$ gives the PDF of X:

$$f_X(x) = \frac{d}{dx}\big(\sqrt{x}\big) = \frac{1}{2\sqrt{x}}, \quad 0 < x < 1.$$

Hence, the PDF is

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Calculate the Mean $E[X]$

$$E[X] = \int_0^1 x\, f_X(x)\, dx = \int_0^1 x \cdot \frac{1}{2\sqrt{x}}\, dx = \int_0^1 \frac{\sqrt{x}}{2}\, dx = \frac{1}{2}\int_0^1 \sqrt{x}\, dx = \frac{1}{2} \cdot \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}.$$

Hence,

$$E[X] = \frac{1}{3}.$$