Homework 5 Solutions

1. We know that $y_1 - \hat\beta_0 - \hat\beta_1 x_1, \ldots, y_n - \hat\beta_0 - \hat\beta_1 x_n, \hat\beta_0, \hat\beta_1$ are jointly normal. Hence, we conclude that $\sum_{i=1}^n (y_i - \hat\beta_0 - \hat\beta_1 x_i)^2$ is independent of $(\hat\beta_0, \hat\beta_1)$ if the covariance of the vectors $(y_1 - \hat\beta_0 - \hat\beta_1 x_1, \ldots, y_n - \hat\beta_0 - \hat\beta_1 x_n) \in \mathbb{R}^n$ and $(\hat\beta_0, \hat\beta_1) \in \mathbb{R}^2$ is zero. To see this, we show that for any $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
\mathrm{Cov}(\hat\beta_0, y_i - \hat\beta_0 - \hat\beta_1 x_i) &= 0, \\
\mathrm{Cov}(\hat\beta_1, y_i - \hat\beta_0 - \hat\beta_1 x_i) &= 0.
\end{aligned}$$

We first show the latter. To this end, we observe that for any $i \in \{1, \ldots, n\}$,

$$\mathrm{Cov}(y_i - \bar y, \bar y) = \mathrm{Cov}(y_i, \bar y) - \mathrm{Var}(\bar y) = \frac{\mathrm{Var}(y_i)}{n} - \frac{\sigma^2}{n} = 0.$$

Now, recall that $\hat\beta_1 = \frac{\sum_{i=1}^n (x_i - \bar x)(y_i - \bar y)}{s_x^2}$, where $s_x^2 = \sum_{i=1}^n (x_i - \bar x)^2$. Therefore,

$$\mathrm{Cov}(\hat\beta_1, \bar y) = \frac{\sum_{i=1}^n (x_i - \bar x)\mathrm{Cov}(y_i - \bar y, \bar y)}{s_x^2} = 0.$$

Meanwhile, we have for any $i \in \{1, \ldots, n\}$,

$$\mathrm{Var}(y_i - \bar y) = \mathrm{Var}(y_i) + \mathrm{Var}(\bar y) - 2\mathrm{Cov}(y_i, \bar y) = \frac{(n-1)\sigma^2}{n}.$$

For $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, we have

$$\mathrm{Cov}(y_i - \bar y, y_j - \bar y) = -\mathrm{Cov}(\bar y, y_j) - \mathrm{Cov}(y_i, \bar y) + \mathrm{Var}(\bar y) = -\frac{\sigma^2}{n}.$$

Therefore,

$$\mathrm{Cov}(\hat\beta_1, y_i - \bar y) = \frac{\sigma^2}{n s_x^2}(n(x_i - \bar x) - \sum_{j=1}^n (x_j - \bar x)) = \frac{\sigma^2 (x_i - \bar x)}{s_x^2}.$$

Accordingly,

$$\mathrm{Cov}(\hat\beta_1, \hat\beta_1) = \sum_{i=1}^n \frac{x_i - \bar x}{s_x^2}\mathrm{Cov}(\hat\beta_1, y_i - \bar y) = \frac{\sigma^2}{s_x^2}.$$

Now, recall that $\hat\beta_0 = \bar y - \hat\beta_1 \bar x$. Then,

$$\begin{aligned}
\mathrm{Cov}(\hat\beta_1, y_i - \hat\beta_0 - \hat\beta_1 x_i) &= \mathrm{Cov}(\hat\beta_1, y_i - \bar y - \hat\beta_1(x_i - \bar x)) \\
&= \mathrm{Cov}(\hat\beta_1, y_i - \bar y) - (x_i - \bar x)\mathrm{Cov}(\hat\beta_1, \hat\beta_1) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{Cov}(\hat\beta_0, y_i - \hat\beta_0 - \hat\beta_1 x_i) &= \mathrm{Cov}(\bar y, y_i - \hat\beta_0 - \hat\beta_1 x_i) - \bar x \mathrm{Cov}(\hat\beta_1, y_i - \hat\beta_0 - \hat\beta_1 x_i) \\
&= \mathrm{Cov}(\bar y, y_i - \bar y - \hat\beta_1(x_i - \bar x)) \\
&= \mathrm{Cov}(\bar y, y_i - \bar y) - (x_i - \bar x)\mathrm{Cov}(\bar y, \hat\beta_1) \\
&= 0.
\end{aligned}$$

2-(a). Let $n = n_1 + n_2$, $Y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\beta = (\beta_0, \beta_1) \in \mathbb{R}^2$,

$$X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \in \mathbb{R}^{n\times 2}, \quad \text{and} \quad \Sigma = \sigma^2 \begin{pmatrix} I_{n_1} & 0 \\ 0 & 2I_{n_2} \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Then, $Y \sim N(X\beta, \Sigma)$. The log-likelihood of $\beta \in \mathbb{R}^2$ is

$$\ell(\beta) = -\frac{(Y - X\beta)^\top \Sigma^{-1} (Y - X\beta)}{2} + \text{ term independent of } \beta.$$

By taking the derivatives with respect to β_0 and β_0, we can see that

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X^\top \Sigma^{-1} X)^{-1} X^\top \Sigma^{-1} Y. \tag{1}$$

2-(b). Accordingly, we have

$$\mathbb{E} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X^\top \Sigma^{-1} X)^{-1} X^\top \Sigma^{-1} \mathbb{E}(Y) = \beta.$$

2-(c). We have

$$\mathrm{Cov}(\hat{\beta}) = (X^\top \Sigma^{-1} X)^{-1} X^\top \Sigma^{-1} \mathrm{Cov}(Y) \Sigma^{-1} X (X^\top \Sigma^{-1} X)^{-1} = (X^\top \Sigma^{-1} X)^{-1}.$$

2-(d). By (1), we can deduce that the joint distribution of $\hat{\beta}_0, \hat{\beta}_1$ is normal. Hence,

$$\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} \sim N(\beta, (X^\top \Sigma^{-1} X)^{-1}).$$