STAT 24300 - Numerical Linear Algebra
Assignment 2: Fundamental subspaces and least squares

Question 1: Linear Dependence and Independence

Test whether the following matrices have linearly independent columns. (Hint: try to solve the homogeneous equations $Ax = 0$ and $Bx = 0$ where 0 is a vector of all zeroes.)

$$A = \begin{bmatrix} -2 & 3 & -3 \\ -6 & 9 & -11 \\ -4 & 6 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 2 \\ 1 & 0 & 3 \\ -1 & 0 & 0 \end{bmatrix}$$

Question 2: Row rank and column rank

(i) Find a value for q (if possible) so that the matrix $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$ has rank (a) 1 (b) 2 (c) 3.

(ii) Repeat the same exercise as in part (i) with the matrix $B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$.

Question 3: Fundamental subspaces

Let

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \\ 3 & -7 & 8 & -5 \\ -1 & 3 & -4 & 3 \end{pmatrix}.$$

1. Find the span of the null space of A. Use this to find the rank of A.
2. Suppose

$$b = \begin{pmatrix} 3 \\ 7 \\ 9 \\ -7/3 \end{pmatrix}.$$

   Can you find a solution to the linear system $Ax = b$? If so, what is the dimension of the solution space?
3. Now suppose

$$b = \begin{pmatrix} 3 \\ 7 \\ 0 \\ 0 \end{pmatrix}.$$

   Can you find a solution to the linear system $Ax = b$? Explain your answer in terms of fundamental subspaces of A.

Question 4: Least Squares with Calculus

Consider the least squares problem:

$$\text{find } x \text{ that minimizes } \|Ax - b\|^2 = (Ax - b)^\mathsf{T}(Ax - b).$$

In class we introduced the normal equations with geometric arguments. We argued that, at the solution x_*, the discrepancy $Ax_* - b$ must be orthogonal to the range of A, so lies in the nullspace of $A^\intercal$. Let's try and prove the same thing directly by computing the gradient of the objective function $f(x) = \|Ax - b\|^2$ with respect to x.

1. Let $f(x) = \|Ax - b\|^2$. Show that $f(x) = x^\intercal A^\intercal Ax - 2x^\intercal A^\intercal b + b^\intercal b$. (Hint: remember that inner products are distributive, the transpose of a product is the product of the transposes - in reverse order, and that any scalar equals itself when transposed since a scalar is both a row and column vector).

2. Show that the gradient of the inner product $v^\intercal x$ with respect to x is just the vector v by showing that $\partial_{x_k} v^\intercal x = v_k$.

3. In general, any expression of the form $x^\intercal Mx$ is called a quadratic form. You may assume that M is symmetric, i.e, $M^\top = M$. Show that $x^\intercal Mx = \sum_{i,j=1}^{n} m_{ij}x_ix_j$ by first expanding Mx as a sum, then by expanding the inner product $x^\intercal(Mx)$. Show that the gradient $\nabla x^\intercal Mx = (M^\intercal + M)x = 2Mx$ by computing $\partial_{x_k} x^\intercal Mx$.

4. Parts 2 and 3 tell you how to compute $\nabla f(x)$ (substitute $A^\intercal A$ for M, and $A^\intercal b$ for v). Show that:

$$\nabla f(x) = 2(A^\intercal Ax - A^\intercal b) \tag{1}$$

   The objective function $f(x)$ is minimized when its gradient is zero. Show that $f(x)$ is minimized (the least squares problem is solved) if x_* solves the linear system:

$$A^\intercal Ax_* = A^\intercal b. \tag{2}$$

   That is, show that normal equations must hold at the minimizer.