Hamiltonian Path from Hamiltonian Circuit

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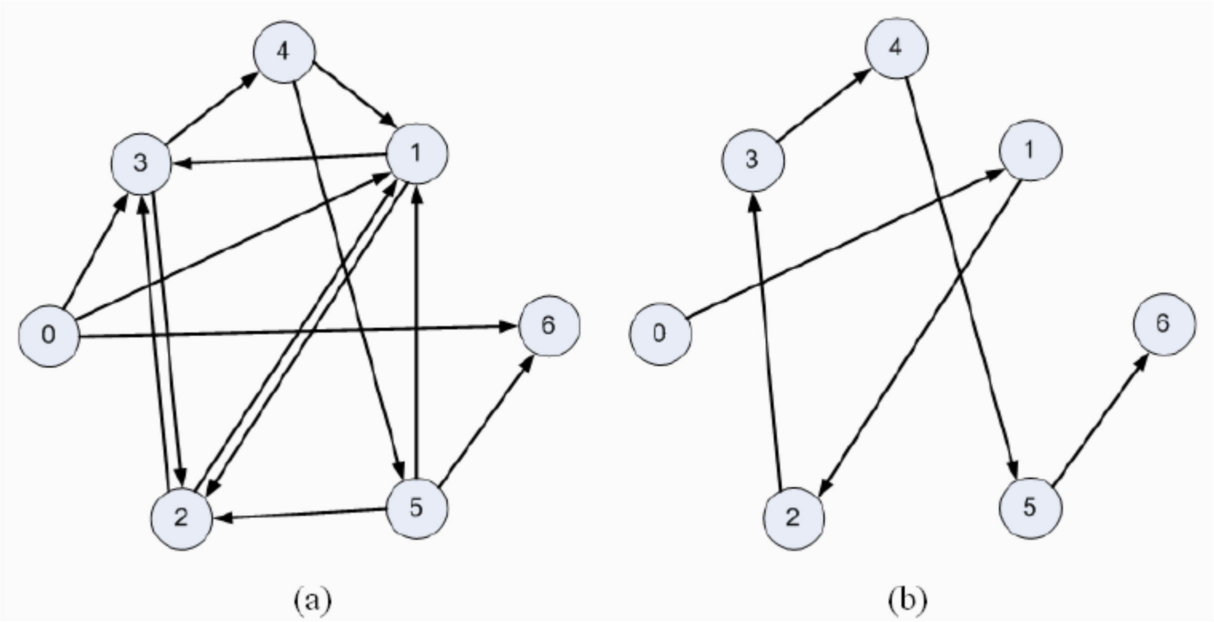
Hamiltonian Path(HP) – Need prove NP-Complete (NP-C)

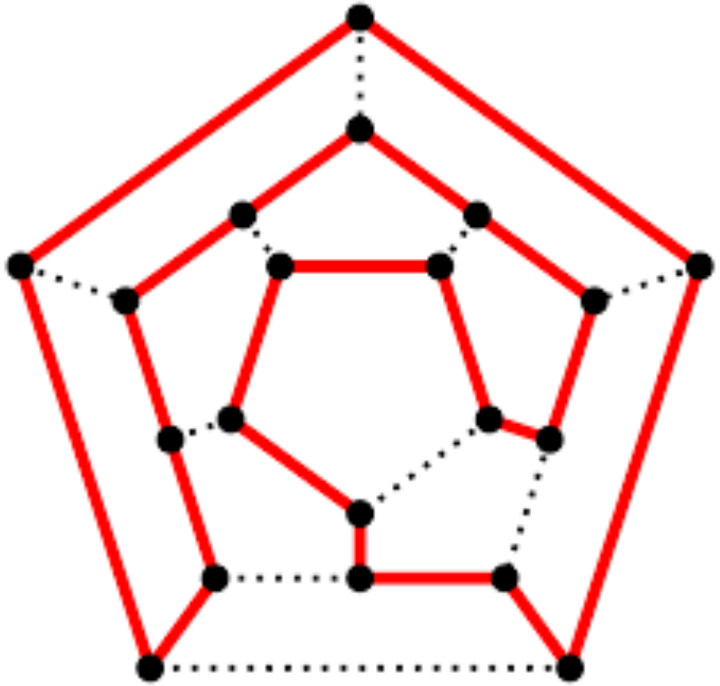
- Input: A graph G with some edges and vertices. G = (V,E)
- Question: Is there a path that visits every vertex in graph only once.



https://www.researchgate.net/publication/242393153_CONCENTRATION-CONTROLLED_LENGTH-BASED_DNA_COMPUTING_FOR_WEIGHTED_GRAPH_PROBLEMS_WITH_NOVEL_READOUT_APPROACH_USING_REAL-TIME_PCR/figures?lo=1

Hamiltonian Circuit(HC) – known NP-Complete

- Input: A graph G with some edges and vertices. G = (V,E)
- Question: Is there a circuit that visits every vertex in graph only once.



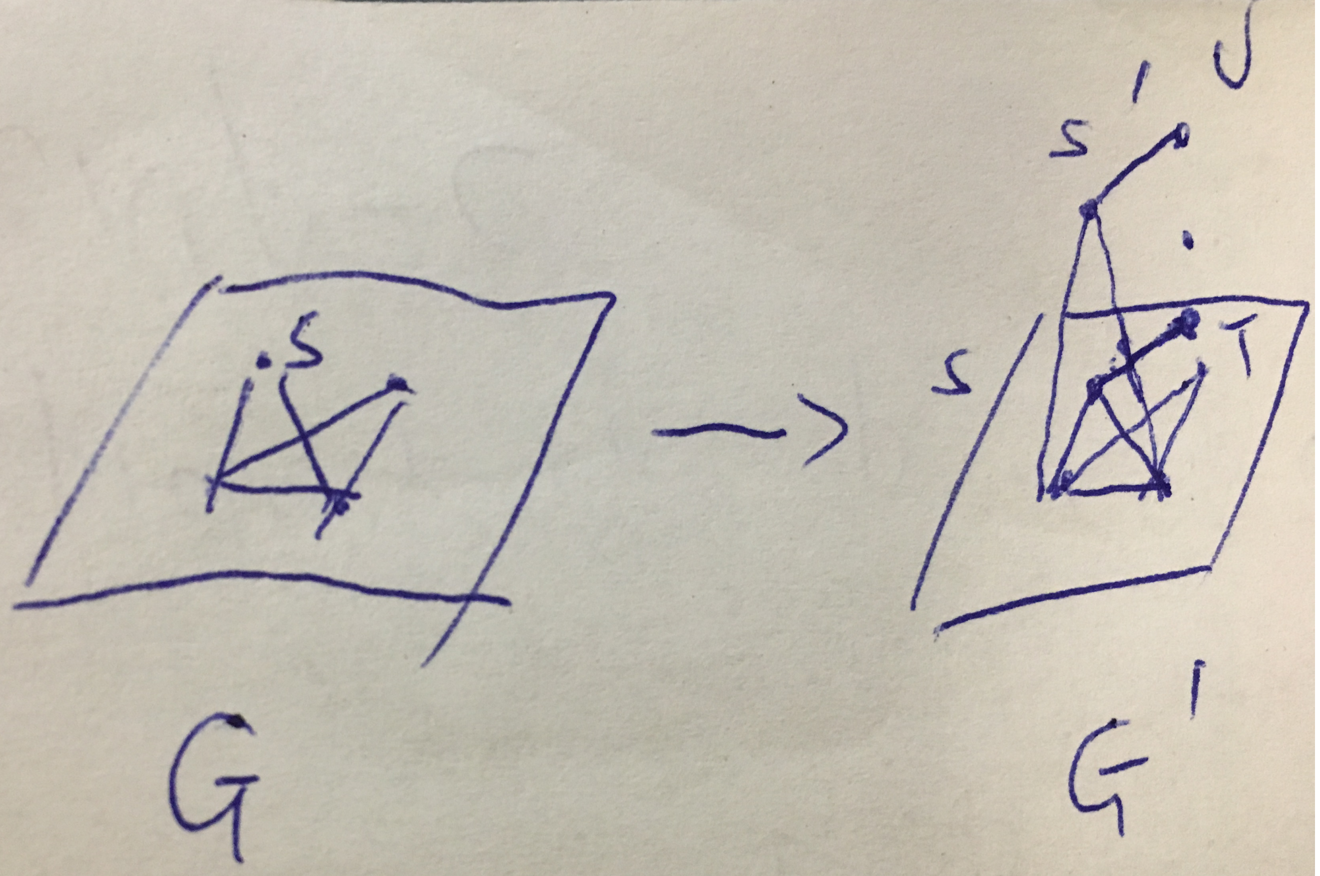
https://simple.wikipedia.org/wiki/Hamiltonian_path

Proof of HP belonging to NP

- Give a graph G = (V,E) and solution S, we can exam the correctness of one solution S by go through the path in solution in polynomial time (P-time). If the path go through every vertex only once, we know it is correct. otherwise, It’s wrong. So, HP is definitely a NP problem.

Proof of HP is NP-C by the Karp reduction

- Polynomial transformation of any instance I' of HC into an instance I of HP.
  - I’ is Graph G = (V,E), question is there existing an HC?
  - Build I: Build a new graph G’ by adding a new vertex S’ into Graph G, this vertex S’ is a copy of any vertex S in U and let S’ has same edge as S. Add 2 new vertices, U, T such that U is only connected to S’ and T is only connected to S. For this new Graph G’ = (V’,E’), the constructed problem becomes: does there exist an HP?
- Justify
  - We can do this in P-time, since we add 3 vertices and n + 2 edges(n is number of edge contain vertex S).


S
G
S'
U
S
T
G'

Proof of HP is NP-C by the Karp reduction

- Polynomial algorithm that transformation of a valid solution I of HP into a solution I’ of HC

  A valid solution for problem I would be of the form (T, S, $a_0 \dots a_n$, S’, U) , since T and U are vertices with degree 1, they must be endpoints. We then transform the solution by removing vertices T, U and the respective edges, and connect the edge connecting S’ to vertex S and remove vertex S’ and it’s remaining edges to get solution I’ = (S, $a_0 \dots a_n$, S), which must be a HC.

- Justify

  1. we can construct the I’ from I in polynomial time. Since we just remove 3 vertices , remove 3 edges and add 1 new edge.
  2. The vertex connect to S’, can also connect to the vertex S. Since S’ is a copy of S.
  3. The solution I (HP), must start and end with T and U. Since T and U are of degree 1 thus the HP will have T and U as endpoints.

Proof that that I' is positive implies I is positive

Let I’ = (S, a_0 … a_n, S) be a valid solution for HC, since any point can be a start point on a HC. We can infer I = (T, S, a_0 … a_n, S’, U) is valid solution for HP. Since I’ is a circuit go though all the vertex S, a_0 … a_n for once, except S, T, U. We just copy S’ from S. So if edge a_nS exist, a_nS’ must exist, replace a_nS with a_nS’ ,thus I has path go through all vertex including S’ , T, U. If you reverse the direction of I, it still a HP.

Let’s reverse it

- HC – need prove

  Input: A graph G with some edges and vertices. G = (V,E)

  Question: Is there a circuit that visits every vertex in graph only once.

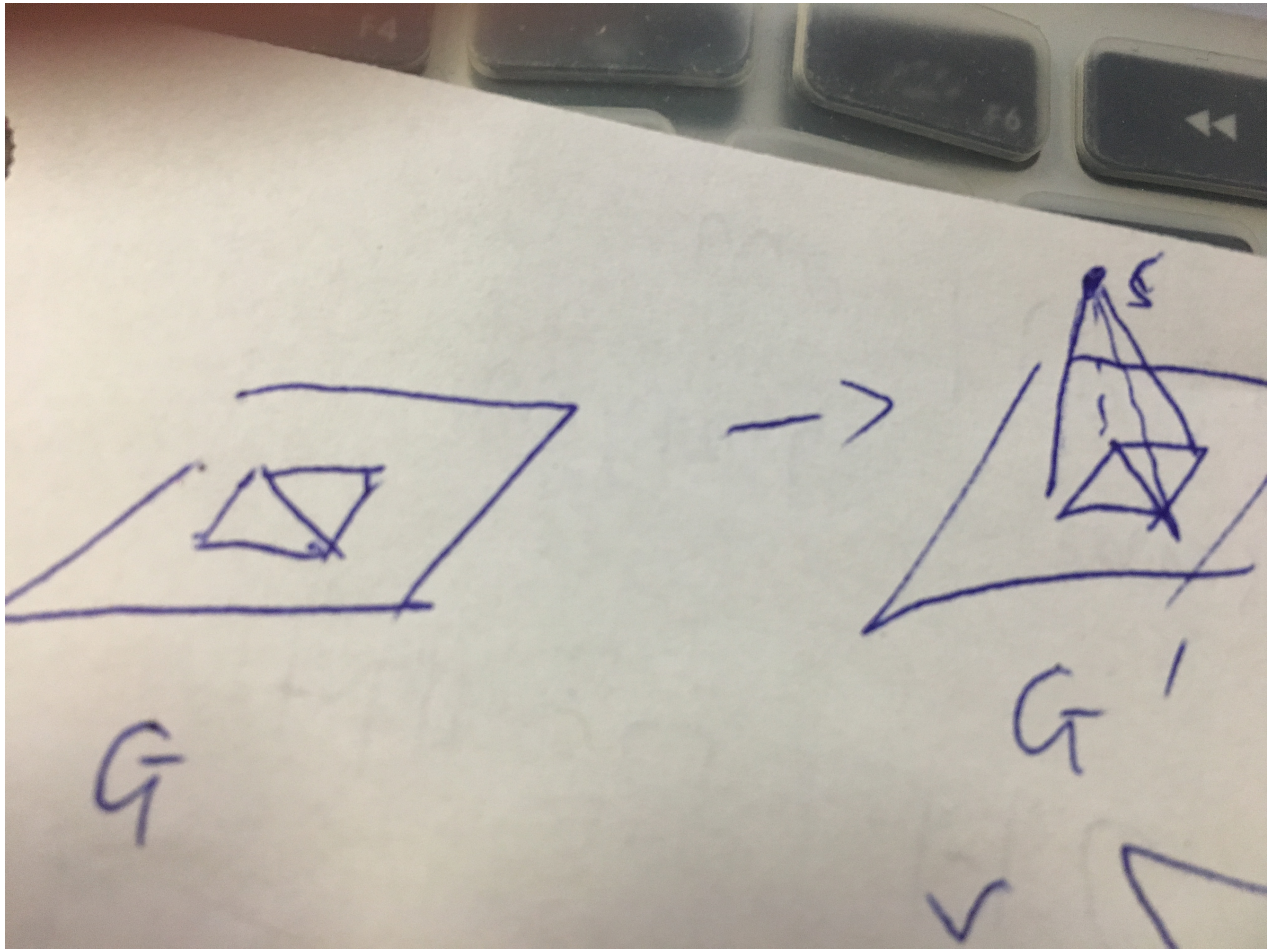
- HP - known NP-C
  - Input: A graph G with some edges and vertices. G = (V,E)
  - Question: Is there a path that visits every vertex in graph only once .

Proof of HC belonging to NP

Give a graph G = (V,E) and and solution S, we can exam the correctness of this solution S by go through the circuit in solution in polynomial time (P-time). If the circuit go through every vertex only once, we know it is correct. otherwise, It’s wrong. So, HC is definitely a NP problem.

Proof of HC is NP-C by the Karp reduction

- Polynomial transformation of any instance I' of HP into an instance I of HC.
  - I’ is Graph G = (V,E), question is there existing an HP?
  - Build I: Add a new vertex S into Graph G and connect S with every vertex in G to form new edges. The new Graph G’ = (V’,E’), question does there exist an HC?
- Justify
  - We can do this in P-time, since we add one vertex and n edges(n is number of vertex in G).


S
G
G'

Proof of HP is NP-C by the Karp reduction

- Polynomial algorithm that transformation of a valid solution I of HC into a solution I’ of HP

  For a valid solution I , we just remove the edges contain vertex S along with vertex S, to get solution I’.

- Justify

  1. we can construct the I’ from I in polynomial time. Since we just remove two edges contain vertex S.
  2. We can guarantee S in the cycle since S connected to all another vertex.

Proof that that I' is positive implies I is positive

Let I’ = (a_0 … a_n,) is a valid solution for HP. We can infer I = (S, a_0 … a_n, S) is valid solution for HC. Since I’ is a path go though all the vertex, a_0 … a_n go through all vertex except S. Since S connected to all vertex, so . edge Sa_n,S a_0 must exist, and I is a circuit go through all vertex include S.

The problem pi that will be proven to be NP-Complete

Hamiltonian Path

Input: A graph $G = (V, E)$

Question: Is there an Hamiltonian Path in G; that is to say, a path that visits every vertex in the graph without visiting the same vertex twice.

Input

$G=(V,E)$

The problem pi' that is known to be NP-Complete

Hamiltonian Circuit

Input: A graph $G = (V, E)$

Question: Is there an Hamiltonian Circuit in G; that is to say, a circuit that visits every vertex in the graph without visiting the same vertex twice.

Input

$G=(V,E)$

Proof of belonging to NP

$$HP(v^*, V^*, E^*) = \max_{|E|}\left(\begin{matrix}\forall v \in V^* \ni \\ \exists e = (v^*, v) \in E\end{matrix}\left(HP(v, V^* - v, E^* \cup e)\right)\right)$$

$$HP(v^*, V^* = \{\}, E^*) = E^*$$

$$HP(v^*, V^* \ni \forall v \in V^*, (v^*, v) \notin E, E^*) = E^*$$

$$HP(v^*, V^*, E^* \ni (|E^*| = \max(|E^*|) = |V| - 1) \wedge ((a,b) \in E)) = E^*$$

$e = (S_0, v) \in E,$

$|HP(v, V - S_0 - v, (e))| = |V| - 1$

Alternative

$\wedge \exists (a, b) \in E \ni a \in V, b \in V$

Guess S_0 arbitrarily

Probabilistic Solution, select $S_0 \ni \forall e \in E, S_0 \in e \rightarrow e \in E' \wedge |E'| = \max(|E'|)$

NP because the correct solution requires selecting the correct S_0.

polynomial transformation of any instance I' of pi' into an instance I of pi [proven]

$e=(S_o, v) \in E,$
$|HP(v, V-S_o-v, (e))| = |V|-1$

$e=(S_o, v) \in E,$
$|HP(v, V-S_o-v, (e))| = |V|-1$

$\wedge (S_i, S_o) \in E$

Alternative

$\wedge \exists (a, b) \in E \ni a \in V, b \in V$

$\wedge \exists (a, b) \in E \ni a \in V, b \in V$

$a = S_i \; , b = S_o$

proof that that I' is positive implies I is positive

$$(|HP(v, V - S_0 - v, (e))| = |V| - 1) \wedge (S_i, S_0) \in E \rightarrow$$
$$(|HP(v, V - S_0 - v, (e))| = |V| - 1)$$

polynomial algorithm that transforms a solution of I into a solution of I' [proven]

$$\frac{e=(S_0, v)\in E,\quad |HP(v, V-S_0-v, (e))| = |V|-1}{}$$

$$\frac{e=(S_0, v)\in E,\quad |HP(v, V-S_0-v, (e))| = |V|-1}{\wedge\ (S_i, S_0)\in E}$$

Alternative

$\wedge\ \exists (a, b)\in E\ \exists\, a\in V, b\in V$

$\wedge\ \exists (a, b)\in E\ \exists\, a\in V, b\in V$

$a=S_i\ , b=S_0$

Sources

1: Advanced Algorithms: NP-Completeness, Spring 2020, Erik Saule

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