COSC 341 – Tutorial 3, Solutions

1. Show that the set of even natural numbers is countable.

   Let $EN = \{n | n \in \mathbb{N}, n \text{ is even}\}$ denote the set of even natural numbers and let $f : \mathbb{N} \to EN$ be a function from $\mathbb{N}$ to EN with $f(n) = 2n$. For proving that EN is countable we will prove that f is bijective:

   (a) injectivity:

   $$\begin{aligned} & \text{Let } f(n) = f(m) \in EN \\ & \Rightarrow 2n = f(n) = f(m) = 2m \\ & \Rightarrow n = m \\ & \Rightarrow \text{f is injective} \end{aligned}$$

   (b) surjectivity:

   $$\begin{aligned} & \text{Let } m \in EN \text{ be an arbitrary element of } EN \text{ and let } n = \frac{m}{2} \in \mathbb{N} \\ & \Rightarrow f(n) = f(\frac{m}{2}) = m \\ & \Rightarrow \text{f is surjective} \end{aligned}$$

2. Show that the set of even integers is countable.

   Let EZ denote the set of even integers and let $f : \mathbb{N} \to EZ$ with

   $$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n-1 & \text{if } n \text{ is odd} \end{cases}$$

   For proving that EZ is countable we will prove that f is bijective:

   (a) injectivity:

   $$\begin{aligned} & \text{Let } f(n) = f(m) \in EZ \text{ be an arbitrary element of } EZ \\ & \Rightarrow \text{It is either } f(n) = f(m) \geq 0 \text{ or } f(n) = f(m) < 0 \\ & \text{If } f(n) = f(m) \geq 0 \\ & \Rightarrow n = f(n) = f(m) = m \\ & \Rightarrow n = m \\ & \text{If } f(n) = f(m) < 0 \\ & \Rightarrow -n-1 = f(n) = f(m) = -m-1 \\ & \Rightarrow n = m \\ & \Rightarrow \text{f is injective} \end{aligned}$$

   (b) surjectivity:

   $$\begin{aligned} & \text{Let } z \in EZ \\ & \text{If } z \geq 0 \Rightarrow \text{For } x = z \in \mathbb{N} \text{ it holds } f(x) = f(z) = z \\ & \text{If } z < 0 \Rightarrow \text{For } x = -z-1 \in \mathbb{N} \text{ it holds } f(x) = f(-z-1) = z \\ & \Rightarrow \text{f is surjective} \end{aligned}$$

3. Show that the set $\{f|f : \mathbb{N} \to \mathbb{N}\}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

   Suppose to the contrary that $\{f|f : \mathbb{N} \to \mathbb{N}\}$ is countable. List each function as $f_0, f_1, ...$

$$\begin{array}{c|lllll} & 0 & 1 & 2 & 3 & \cdots \\ \hline f_0 & f_0(0) & f_0(1) & f_0(2) & f_0(3) & \cdots \\ f_1 & f_1(0) & f_1(1) & f_1(2) & f_1(3) & \cdots \\ f_2 & f_2(0) & f_2(1) & f_2(2) & f_2(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

   Now define the function $f : \mathbb{N} \to \mathbb{N}$ by:

$$f(n) = f_n(n) + 1$$

   By the definition of f it is different to every function in our list. Therefore our list does not include all possible functions which contradicts our assumption that $\{f|f : \mathbb{N} \to \mathbb{N}\}$ is countable. Therefore $\{f|f : \mathbb{N} \to \mathbb{N}\}$ must be uncountable.

4. Show that the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ is uncountable.

   As with all diagonal arguments, we argue by contradiction. Suppose, contrary to what we want to prove, that $\mathcal{P}(\mathbb{N})$ were countable. In that case we could list all subsets of $\mathbb{N}$ as:

$$A_0, A_1, A_4, \ldots$$

   Now we define a set $A = \{i|i \in \mathbb{N}, i \notin A_i\} \subseteq \mathbb{N}$. There is no k for which $A = A_k$. If there was such a k, it would either be

$$\begin{aligned} k \in A \Rightarrow k \notin A_k = A \text{ (by the definition of } A) &\Rightarrow \text{ contradiction} \\ \text{or } k \notin A \Rightarrow k \in A = A_k \text{ (by the definition of } A) &\Rightarrow \text{ contradiction} \end{aligned}$$

   So our list of subsets was *not* complete as we claimed it was, and hence no such list can exist.

5. Show that, for any set A, $|A| < |\mathcal{P}(A)|$.

   $|A| < |\mathcal{P}(A)|$ means that there is an injective, but no surjective function form A to $\mathcal{P}(A)$.
   An example for an injective function from A to $\mathcal{P}(A)$ is $g : A \to \mathcal{P}(A)$ with $g(a) = \{a\}$.
   Now we need to prove that there is no surjective function from A to $\mathcal{P}(A)$. We do this by contradiction: Suppose that we had a surjective function $f : A \to \mathcal{P}(A)$. We reach a contradiction by defining a set $B \subseteq A$ not in the range of f. Define:

$$B = \{a \in A \,:\, a \notin f(a)\} \subseteq A$$

   Now we need to show that it is a contradiction for B to be in the range of f. To see this, suppose that $B = f(b)$. Now we ask "Is b an element of B"? If so, then by the definition of B, $b \notin f(b)$ i.e. $b \notin B$. A contradiction. If not, then since $b \notin f(b)$, we should have $b \in B$. Another contradiction. Both possibilities lead to contradictions, so no such function can exist.

Homework

1. Show that the set of total functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable.

   As with all diagonal arguments, we argue by contradiction. Suppose, contrary to what we want to prove, that the set of total functions from $\mathbb{N}$ to $\{0, 1\}$ were countable. In that case we could list all these functions as:

   $$f_0, f_1, f_2, \ldots$$

   Now we define a function $g : \mathbb{N} \to \{0, 1\}$ that does not appear in the list (achieving our contradiction). We simply set, for each $i \in \mathbb{N}$:

   $$g(i) = 1 - f_i(i)$$

   (so if $f_i(i) = 1$, $g(i) = 0$ and vice versa). Thus, for any i, g disagrees with f_i at at least one point (namely i) and possibly many others, i.e. $g \neq f_i$ for any i. So our list of functions was *not* complete as we claimed it was, and hence no such list can exist.

2. We can define the set $\mathbb{N}$ of natural numbers as:

   $$\begin{array}{l} 0 \in \mathbb{N} \\ \text{If } n \in \mathbb{N}\text{, then } n + 1 \in \mathbb{N} \end{array}$$

   We call this a *recursive* definition.
   Give recursive definitions of:

   (a) The set of even natural numbers $EN = \{2n | n \in \mathbb{N}\}$

   $$\begin{array}{l} 0 \in EN \\ \text{If } n \in EN\text{, then } n + 2 \in EN \end{array}$$

   (b) The set $P = \{1, 2, 4, 8, 16, \ldots\}$ of powers of 2 within $\mathbb{N}$

   $$\begin{array}{l} 1 \in P \\ \text{If } n \in P\text{, then } n + n \in P \end{array}$$