$P_{\alpha,n}$ on $\mathcal{T}_n$ for $n \leqslant 8$

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1. Introduction

In this document we compute the probabilities of all cladograms in every $\mathcal{T}_n$ with $n = 2, \ldots, 8$. Although Ford gives in [1, §7] these probabilities for $n = 2, \ldots, 6$, for the convenience of the reader we also include them. We shall denote the shapes of dendrograms by means of their Newick format [2], representing their leaves with a symbol $*$. We use the following formula for the probability of a cladogram $T \in \mathcal{T}_n$:

$$P_{\alpha,n}(T) = \frac{2^{n-1}}{n! \cdot \Gamma_\alpha(n)} \prod_{(a,b)\in NS(T)} \varphi_\alpha(a,b)$$

where, for every $a, b \in \mathbb{N} \setminus \{0\}$,

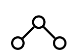
$$\varphi_\alpha(a,b) = \frac{\alpha}{2}\binom{a+b}{a} + (1-2\alpha)\binom{a+b-2}{a-1}.$$

We gather in Table 1 below the values of $\varphi_\alpha(a,b)$ with $a \leqslant b$ and $a + b \leqslant 8$, which are used in the explicit computations given in this document.

In the sections below we classify the dendrograms in each $\mathcal{T}_n$ according to their shape in $\mathcal{T}_n^*$ and then, for each one of these shapes, we give the number of dendrograms with that shape (which is equal to $n!$ divided by 2 to the power of its number of symmetric branch points; see [1, Lem. 31]) and the probability of each one of these dendrograms; hence, the probability of every tree in $\mathcal{T}_n^*$ can be easily computed as the product of these two numbers.

2. $\mathcal{T}_2$

There is only one tree in $\mathcal{T}_2$, with shape

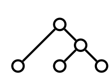


$(*, *);$

Thus, of course, it has probability 1.

3. $\mathcal{T}_3$

All trees in $\mathcal{T}_2$ have shape



$(*, (*, *));$

There are 3 of them, and each one has probability

$$\frac{2^2}{3! \cdot \Gamma_\alpha(3)} \varphi_\alpha(1,2)\varphi_\alpha(1,1) = \frac{1}{3}$$

as it should be, since the α-model is shape invariant.

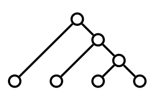
(a,b)	$\varphi_\alpha(a,b)$
$(1,1)$	$1-\alpha$
$(1,2)$	$(2-\alpha)/2$
$(1,3)$	1
$(2,2)$	$2-\alpha$
$(1,4)$	$(2+\alpha)/2$
$(2,3)$	$3-\alpha$
$(1,5)$	$1+\alpha$
$(2,4)$	$(8-\alpha)/2$
$(3,3)$	$2(3-\alpha)$
$(1,6)$	$(2+3\alpha)/2$
$(2,5)$	$(10+\alpha)/2$
$(3,4)$	$5(4-\alpha)/2$
$(1,7)$	$1+2\alpha$
$(2,6)$	$2(3+\alpha)$
$(3,5)$	$15-2\alpha$
$(4,4)$	$5(4-\alpha)$

Table 1: Values of $\varphi_\alpha(a,b)$ for $a \leqslant b$ and $a+b \leqslant 8$

4. $\mathcal{T}_4$

In $\mathcal{T}_4$ there are:

(4.1) Trees with shape

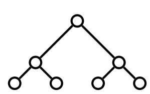


$(*, (*, (*, *)));$

There are $4!/2 = 12$ of them, and each one has probability

$$\frac{2^3}{4! \cdot \Gamma_\alpha(4)} \varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1) = \frac{1}{6(3-\alpha)}$$

(4.2) Trees with shape



$((*, *), (*, *));$

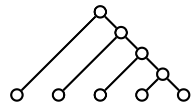
There are $4!/2^3 = 3$ of them, and each one has probability

$$\frac{2^3}{4! \cdot \Gamma_\alpha(4)} \varphi_\alpha(2,2)\varphi_\alpha(1,1)^2 = \frac{1-\alpha}{3(3-\alpha)}$$

5. $\mathcal{T}_5$

In $\mathcal{T}_5$ there are:

(5.1) Trees with shape

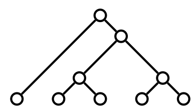


$(*, (*, (*, (*, *))));$

There are $5!/2 = 60$ of them, and each one has probability

$$\frac{2^4}{5! \cdot \Gamma_\alpha(5)} \varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1) = \frac{2+\alpha}{30(4-\alpha)(3-\alpha)}$$

(5.2) Trees with shape

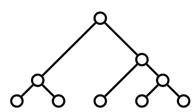


$(*, ((*, *), (*, *)));$

There are $5!/2^3 = 15$ of them, and each one has probability

$$\frac{2^4}{5! \cdot \Gamma_\alpha(5)} \varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2+\alpha)}{15(4-\alpha)(3-\alpha)}$$

(5.3) Trees with shape



$((*, *), (*, (*, *)));$

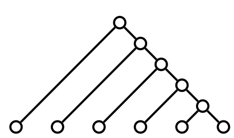
There are $5!/2^2 = 30$ of them, and each one has probability

$$\frac{2^4}{5! \cdot \Gamma_\alpha(5)} \varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)}{15(4-\alpha)}$$

6. $\mathcal{T}_6$

In $\mathcal{T}_6$ there are:

(6.1) Trees with shape

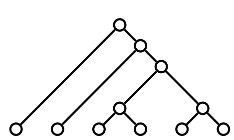


$(*, (*, (*, (*, (*, *)))));$

There are $6!/2 = 360$ of them, and each one has probability

$$\frac{2^5}{6! \cdot \Gamma_\alpha(6)} \varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1) = \frac{(1+\alpha)(2+\alpha)}{90(5-\alpha)(4-\alpha)(3-\alpha)}$$

(6.2) Trees with shape

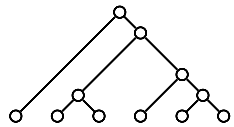


$(*, (*, ((*, *), (*, *))));$

There are $6!/2^3 = 90$ of them, and each one has probability

$$\frac{2^5}{6! \cdot \Gamma_\alpha(6)} \varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+\alpha)(2+\alpha)}{45(5-\alpha)(4-\alpha)(3-\alpha)}$$

(6.3) Trees with shape

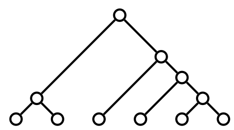


$(*, ((*, *), (*, (*, *))));$

There are $6!/2^2 = 180$ of them, and each one has probability

$$\frac{2^5}{6! \cdot \Gamma_\alpha(6)} \varphi_\alpha(1,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+\alpha)}{45(5-\alpha)(4-\alpha)}$$

(6.4) Trees with shape

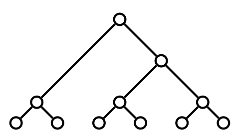


$((*, *), (*, (*, (*, *))));$

There are $6!/2^2 = 180$ of them, and each one has probability

$$\frac{2^5}{6! \cdot \Gamma_\alpha(6)} \varphi_\alpha(2,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(8-\alpha)}{90(5-\alpha)(4-\alpha)(3-\alpha)}$$

(6.5) Trees with shape

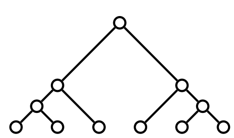


$((*, *), ((*, *), (*, *)));$

There are $6!/2^4 = 45$ of them, and each one has probability

$$\frac{2^5}{6! \cdot \Gamma_\alpha(6)} \varphi_\alpha(2,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(8-\alpha)}{45(5-\alpha)(4-\alpha)(3-\alpha)}$$

(6.6) Trees with shape



$((*, (*, *)), (*, (*, *)));$

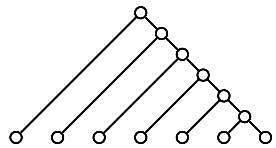
There are $6!/2^3 = 90$ of them, and each one has probability

$$\frac{2^5}{6! \cdot \Gamma_\alpha(6)} \varphi_\alpha(3,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2-\alpha)}{45(5-\alpha)(4-\alpha)}$$

7. $\mathcal{T}_7$

In $\mathcal{T}_7$ there are:

(7.1) Trees with shape

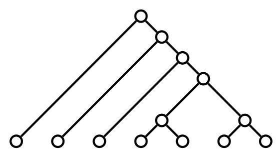


$(*,(*,(*,(*,(*,(*,*))))));$

There are $7!/2 = 2520$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(1,6)\varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1) = \frac{(1+\alpha)(2+\alpha)(2+3\alpha)}{630(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(7.2) Trees with shape

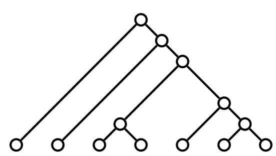


$(*,(*,(*,((*,*),(*,*)))));$

There are $7!/2^3 = 630$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(1,6)\varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+\alpha)(2+\alpha)(2+3\alpha)}{315(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(7.3) Trees with shape

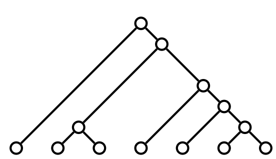


$(*,(*,((*,*),(*,(*,*)))));$

There are $7!/2^2 = 1260$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(1,6)\varphi_\alpha(1,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+\alpha)(2+3\alpha)}{315(6-\alpha)(5-\alpha)(4-\alpha)}$$

(7.4) Trees with shape

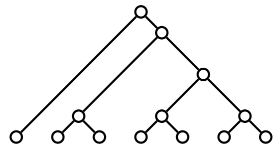


$(*,((*,*),(*,(*,(*,*)))));$

There are $7!/2^2 = 1260$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(1,6)\varphi_\alpha(2,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(8-\alpha)(2+3\alpha)}{630(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(7.5) Trees with shape

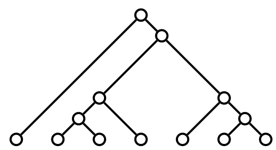


$(*, ((*, *), ((*, *), (*, *))));$

There are $7!/2^4 = 315$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(1,6)\varphi_\alpha(2,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(8-\alpha)(2+3\alpha)}{315(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(7.6) Trees with shape

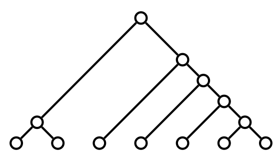


$(*, ((*, (*, *)), (*, (*, *))));$

There are $7!/2^3 = 630$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(1,6)\varphi_\alpha(3,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2-\alpha)(2+3\alpha)}{315(6-\alpha)(5-\alpha)(4-\alpha)}$$

(7.7) Trees with shape

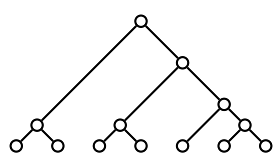


$((*, *), (*, (*, (*, (*, *)))));$

There are $7!/2^2 = 1260$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(2,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2+\alpha)(10+\alpha)}{630(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(7.8) Trees with shape

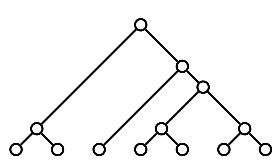


$((*, *), ((*, *), (*, (*, *))));$

There are $7!/2^3 = 630$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(2,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(10+\alpha)}{315(6-\alpha)(5-\alpha)(4-\alpha)}$$

(7.9) Trees with shape

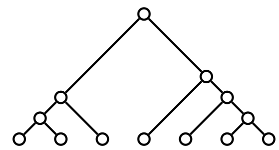


$((*, *), (*, ((*, *), (*, *))));$

There are $7!/2^4 = 315$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(2,5)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(2+\alpha)(10+\alpha)}{315(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

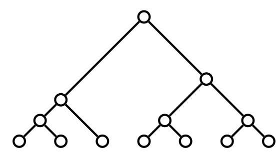
(7.10) Trees with shape



There are $7!/2^2 = 1260$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(3,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2-\alpha)}{126(6-\alpha)(5-\alpha)(3-\alpha)}$$

(7.11) Trees with shape



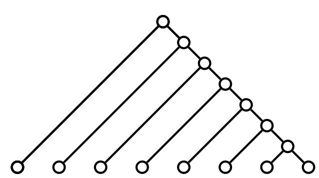
There are $7!/2^4 = 315$ of them, and each one has probability

$$\frac{2^6}{7! \cdot \Gamma_\alpha(7)} \varphi_\alpha(3,4)\varphi_\alpha(1,2)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(2-\alpha)}{63(6-\alpha)(5-\alpha)(3-\alpha)}$$

8. $\mathcal{T}_8$

In $\mathcal{T}_8$ there are:

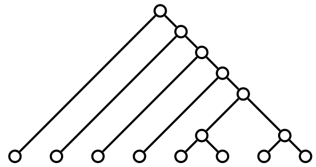
(8.1) Trees with shape



There are $8!/2 = 20160$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(1,6)\varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)$$
$$= \frac{(1+\alpha)(1+2\alpha)(2+\alpha)(2+3\alpha)}{2520(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.2) Trees with shape

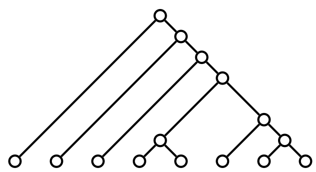


$(*,(*,(*,(*,((*,*),(*,*))))));$

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(1,6)\varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^2$$
$$= \frac{(1-\alpha)(1+\alpha)(1+2\alpha)(2+\alpha)(2+3\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.3) Trees with shape

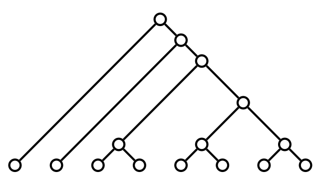


$(*,(*,(*,((*,*),(*,(*,*))))));$

There are $8!/2^2 = 10080$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(1,6)\varphi_\alpha(1,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+\alpha)(1+2\alpha)(2+3\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)}$$

(8.4) Trees with shape

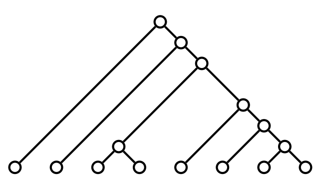


$(*,(*,((*,*),((*,*),(*,*)))));$

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(1,6)\varphi_\alpha(2,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(1+2\alpha)(2+3\alpha)(8-\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.5) Trees with shape

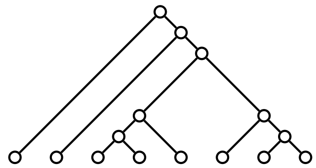


$(*,(*,((*,*),(*,(*,(*,*))))));$

There are $8!/2^2 = 10080$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(1,6)\varphi_\alpha(2,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+2\alpha)(2+3\alpha)(8-\alpha)}{2520(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.6) Trees with shape

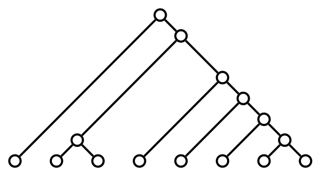


$(*, (*, ((*, (*, *)), (*, (*, *)))));$

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(1,6)\varphi_\alpha(3,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+2\alpha)(2-\alpha)(2+3\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)}$$

(8.7) Trees with shape

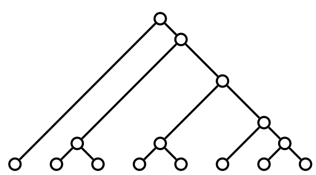


$(*, ((*, *), (*, (*, (*, (*, *))))));$

There are $8!/2^2 = 10080$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(2,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+2\alpha)(2+\alpha)(10+\alpha)}{2520(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.8) Trees with shape

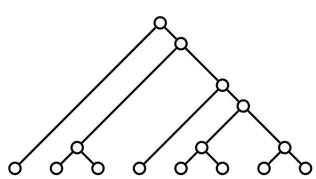


$(*, ((*, *), ((*, *), (*, (*, *)))));$

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(2,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(1+2\alpha)(10+\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)}$$

(8.9) Trees with shape

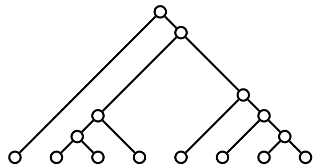


$(*, ((*, *), (*, ((*, *), (*, *)))));$

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(2,5)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(1+2\alpha)(2+\alpha)(10+\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.10) Trees with shape

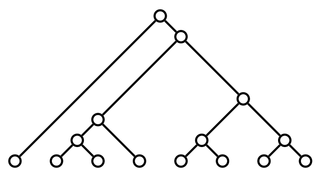


$(*, ((*, (*, *)), (*, (*, (*, *)))));$

There are $8!/2^2 = 10080$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(3,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+2\alpha)(2-\alpha)}{504(7-\alpha)(6-\alpha)(5-\alpha)(3-\alpha)}$$

(8.11) Trees with shape

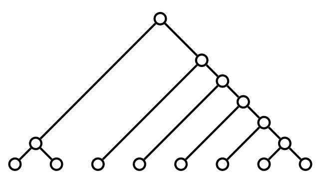


$(*, ((*, (*, *)), ((*, *), (*, *))));$

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(1,7)\varphi_\alpha(3,4)\varphi_\alpha(1,2)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(1+2\alpha)(2-\alpha)}{252(7-\alpha)(6-\alpha)(5-\alpha)(3-\alpha)}$$

(8.12) Trees with shape

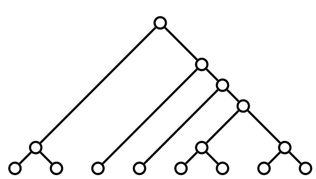


$((*, *), (*, (*, (*, (*, (*, *))))));$

There are $8!/2^2 = 10080$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(2,6)\varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(1+\alpha)(2+\alpha)(3+\alpha)}{630(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.13) Trees with shape

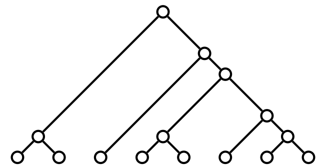


$((*, *), (*, (*, ((*, *), (*, *)))));$

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(2,6)\varphi_\alpha(1,5)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(1+\alpha)(2+\alpha)(3+\alpha)}{315(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.14) Trees with shape

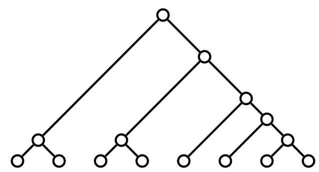


((∗, ∗), (∗, ((∗, ∗), (∗, (∗, ∗)))));

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(2,6)\varphi_\alpha(1,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(1+\alpha)(3+\alpha)}{315(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)}$$

(8.15) Trees with shape

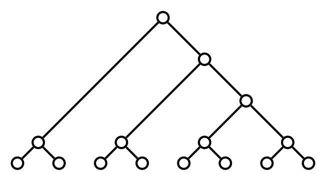


((∗, ∗), ((∗, ∗), (∗, (∗, (∗, ∗)))));

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(2,6)\varphi_\alpha(2,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(3+\alpha)(8-\alpha)}{630(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.16) Trees with shape

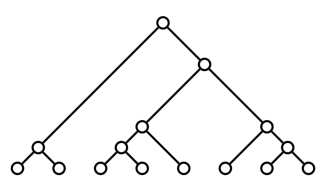


((∗, ∗), ((∗, ∗), ((∗, ∗), (∗, ∗))));

There are $8!/2^5 = 1260$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(2,6)\varphi_\alpha(2,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^4 = \frac{(1-\alpha)^3(3+\alpha)(8-\alpha)}{315(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.17) Trees with shape

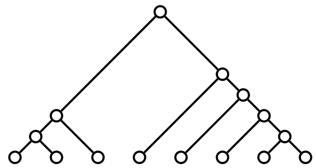


((∗, ∗), ((∗, (∗, ∗)), (∗, (∗, ∗))));

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(2,6)\varphi_\alpha(3,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(2-\alpha)(3+\alpha)}{315(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)}$$

(8.18) Trees with shape

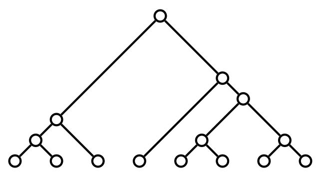


$((*,(*,*)),(*,(*,(*,(*,*)))));$

There are $8!/2^2 = 10080$ of them, and each one has probability

$$\frac{2^7}{8!\cdot\Gamma_\alpha(8)}\varphi_\alpha(3,5)\varphi_\alpha(1,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2-\alpha)(2+\alpha)(15-2\alpha)}{2520(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.19) Trees with shape

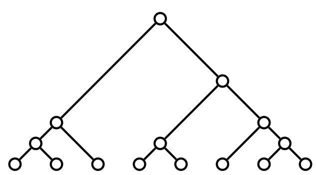


$((*,(*,*)),(*,((*,*),(*,*))));$

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8!\cdot\Gamma_\alpha(8)}\varphi_\alpha(3,5)\varphi_\alpha(1,2)\varphi_\alpha(1,4)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(2-\alpha)(2+\alpha)(15-2\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)}$$

(8.20) Trees with shape

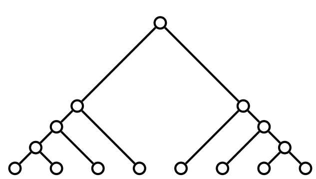


$((*,(*,*)),((*,*),(*,(*,*))));$

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8!\cdot\Gamma_\alpha(8)}\varphi_\alpha(3,5)\varphi_\alpha(2,3)\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(2-\alpha)(15-2\alpha)}{1260(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)}$$

(8.21) Trees with shape

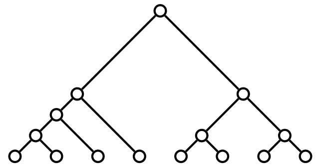


$((*,(*,(*,*))),(*,(*,(*,*))));$

There are $8!/2^3 = 5040$ of them, and each one has probability

$$\frac{2^7}{8!\cdot\Gamma_\alpha(8)}\varphi_\alpha(4,4)\varphi_\alpha(1,3)^2\varphi_\alpha(1,2)^2\varphi_\alpha(1,1)^2 = \frac{(1-\alpha)(2-\alpha)}{252(7-\alpha)(6-\alpha)(5-\alpha)(3-\alpha)}$$

(8.22) Trees with shape

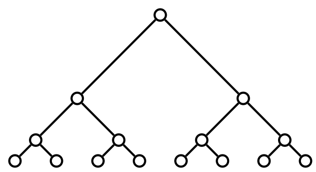


$((*, (*, (*, *))), ((*, *), (*, *)));$

There are $8!/2^4 = 2520$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(4,4)\varphi_\alpha(1,3)\varphi_\alpha(1,2)\varphi_\alpha(2,2)\varphi_\alpha(1,1)^3 = \frac{(1-\alpha)^2(2-\alpha)}{126(7-\alpha)(6-\alpha)(5-\alpha)(3-\alpha)}$$

(8.23) Trees with shape



$(((*, *), (*, *)), ((*, *), (*, *)));$

There are $8!/2^7 = 315$ of them, and each one has probability

$$\frac{2^7}{8! \cdot \Gamma_\alpha(8)} \varphi_\alpha(4,4)\varphi_\alpha(2,2)^2\varphi_\alpha(1,1)^4 = \frac{(1-\alpha)^3(2-\alpha)}{63(7-\alpha)(6-\alpha)(5-\alpha)(3-\alpha)}$$

References

[1] D. Ford. Probabilities on cladograms: Introduction to the alpha model. PhD Thesis (Stanford University). arXiv preprint arXiv:math/0511246 [math.PR] (2005).

[2] The Newick tree format: `http://evolution.genetics.washington.edu/phylip/newicktree.html` (last visited, 01/05/2017).