Regression - The Linear Model

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Review concepts learned

- Multilevel (Hierarchical) Modeling
- Mixture Models

Motivating examples for today

- Regression models

Linear model

Linear regression

Linear regression is one of the most widely used statstical tools.

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Learning objective: To set up the relevant Bayesian models and draw samples from posterior distributions for parameters θ and future observables $\tilde{y}$.

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- Sometimes, a single variable x_j may be of primary interest and consider it the *treatment* variable, labeling the other components of x as the *control* variables

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- Assumed to be uncorrelated and to have common variance σ^2, which becomes an additional parameter

The Linear Model

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Answer: $\theta = \left(\beta_1, \ldots, \beta_k, \sigma^2\right)$

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Statistical inference problem is to estimate the parameters θ, conditional on $\mathbb{X}$ and $\mathbb{y}$.

Notation for a basic normal linear model

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The likelihood becomes

$$f\left(\mathrm{y}|\beta, \sigma^2\right) = \left(2\pi\sigma^2\right)^{\frac{-n}{2}} exp\left[\left(\mathrm{y} - \mathbb{X}\beta\right)'\left(\mathrm{y} - \mathbb{X}\beta\right) / \left(2\sigma^2\right)\right]$$

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$\left(\mathrm{y} - \mathbb{X}\beta\right)'\left(\mathrm{y} - \mathbb{X}\beta\right)$ is referred to as the *quadratic form*

Least squares estimator of β

$$\hat{\beta} = \left(\mathbb{X}^T\mathbb{X}\right)^{-1}\mathbb{X}^T\mathrm{y}$$

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Residual sum of squares

$$S = \left(\mathrm{y} - \mathbb{X}\hat{\beta}\right)^T \left(\mathrm{y} - \mathbb{X}\hat{\beta}\right)$$

The posterior distribution

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Question: We are interested in the posterior distribution of...

The posterior distribution

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We factor the join posterior distribution for β and σ^2 as

$$p\left(\beta, \sigma^2 | y\right) = p\left(\beta | \sigma^2, y\right) p\left(\sigma^2 | y\right)$$

The posterior distribution

Here, we are interested in determining first, the posterior distribution for β, conditioning on σ^2, and then the marginal posterior distribution for σ^2.

The conditional posterior distribution of β, given σ^2

$$\beta|\sigma^2, \mathbb{y} \sim \mathcal{N}\left(\hat{\beta}, \mathbf{V}_\beta \sigma^2\right),$$

Least squares estimator of β

$$\hat{\beta} = \left(\mathbb{X}^T\mathbb{X}\right)^{-1}\mathbb{X}^T\mathrm{y}$$

$\mathbf{V}_\beta$

$$\mathbf{V}_\beta = \left(\mathbb{X}^T\mathbb{X}\right)^{-1}$$

Marginal posterior distribution of σ^2

$$\sigma^2|\mathrm{y} \sim IG\left(\frac{n-k}{2}, s^2\right),$$

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What you usually do in practice is to draw inferences by simulation

1. Draw simulations of σ^2
2. Then, draw simulations of $\beta|\sigma^2$

non-Bayesian estimates of β and σ^2

- $\hat{\beta}$
- s^2

We obtain the classical standard error estimate for β by setting σ^2 to s^2

Case study : Trees

Plotting Outcome (Volume) as a function of Covariates (Girth, Height)

Source: http://www.columbia.edu/~cjd11/charles_dimaggio/DIRE/styled-4/styled-11/code-8/

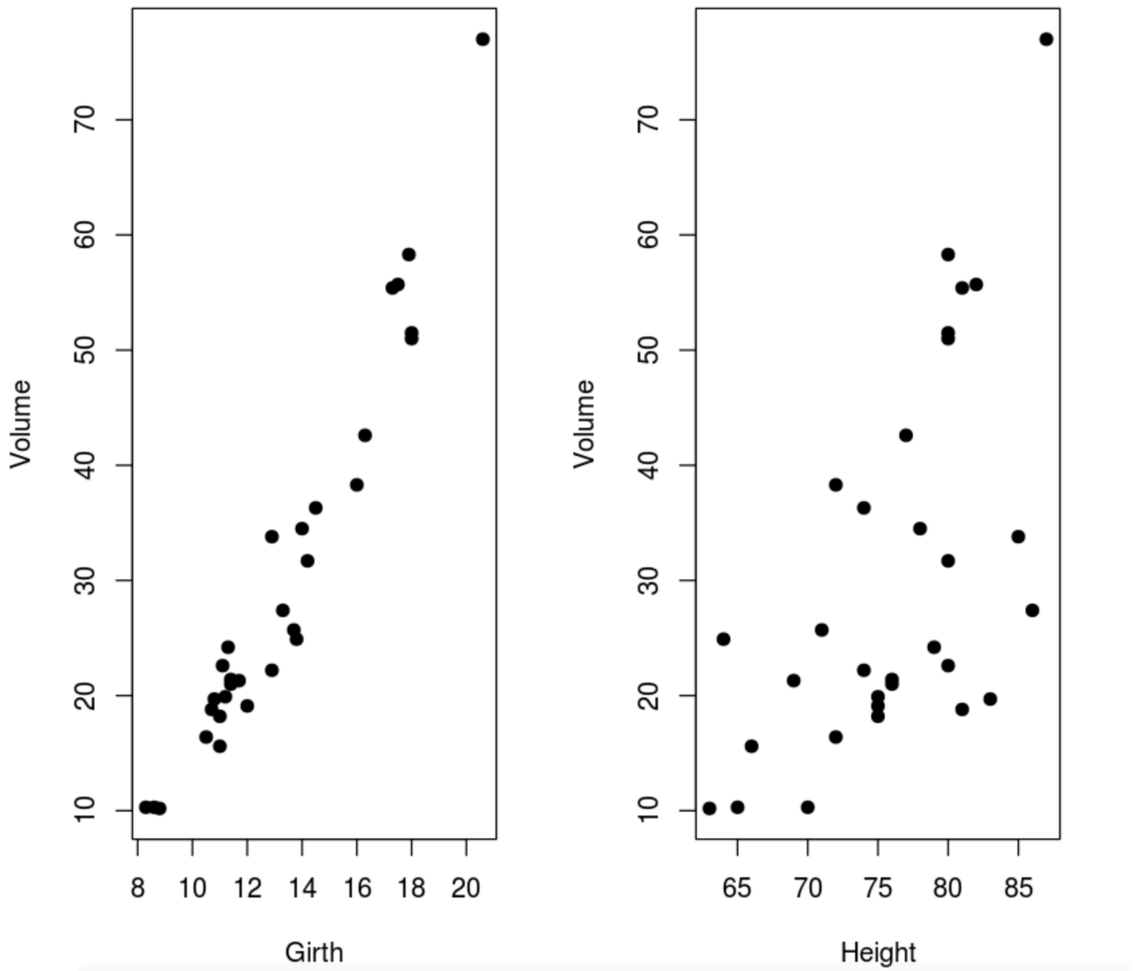
```
data(trees)
attach(trees)
dim(trees)
trees
```

Case study : Trees

31 3

Girth	Height	Volume
8.3	70	10.3
8.6	65	10.3
8.8	63	10.2
10.5	72	16.4
10.7	81	18.8
10.8	83	19.7
11.0	66	15.6
11.0	75	18.2
11.1	80	22.6
11.2	75	19.9
11.3	79	24.2
11.4	76	21.0
11.4	76	21.4
11.7	69	21.3
12.0	75	19.1
12.9	74	22.2

Case study : Trees



Case study : Trees

```
## MLE fit of Regression Model
model <- lm(Volume~Girth+Height)
summary(model)
```

Case study : Trees

```
Call:
lm(formula = Volume ~ Girth + Height)

Residuals:
    Min      1Q  Median      3Q     Max
-6.4065 -2.6493 -0.2876  2.2003  8.4847

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -57.9877     8.6382  -6.713 2.75e-07 ***
Girth         4.7082     0.2643  17.816  < 2e-16 ***
Height        0.3393     0.1302   2.607   0.0145 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.882 on 28 degrees of freedom
Multiple R-squared:  0.948,     Adjusted R-squared:  0.9442
F-statistic:   255 on 2 and 28 DF,  p-value: < 2.2e-16
```

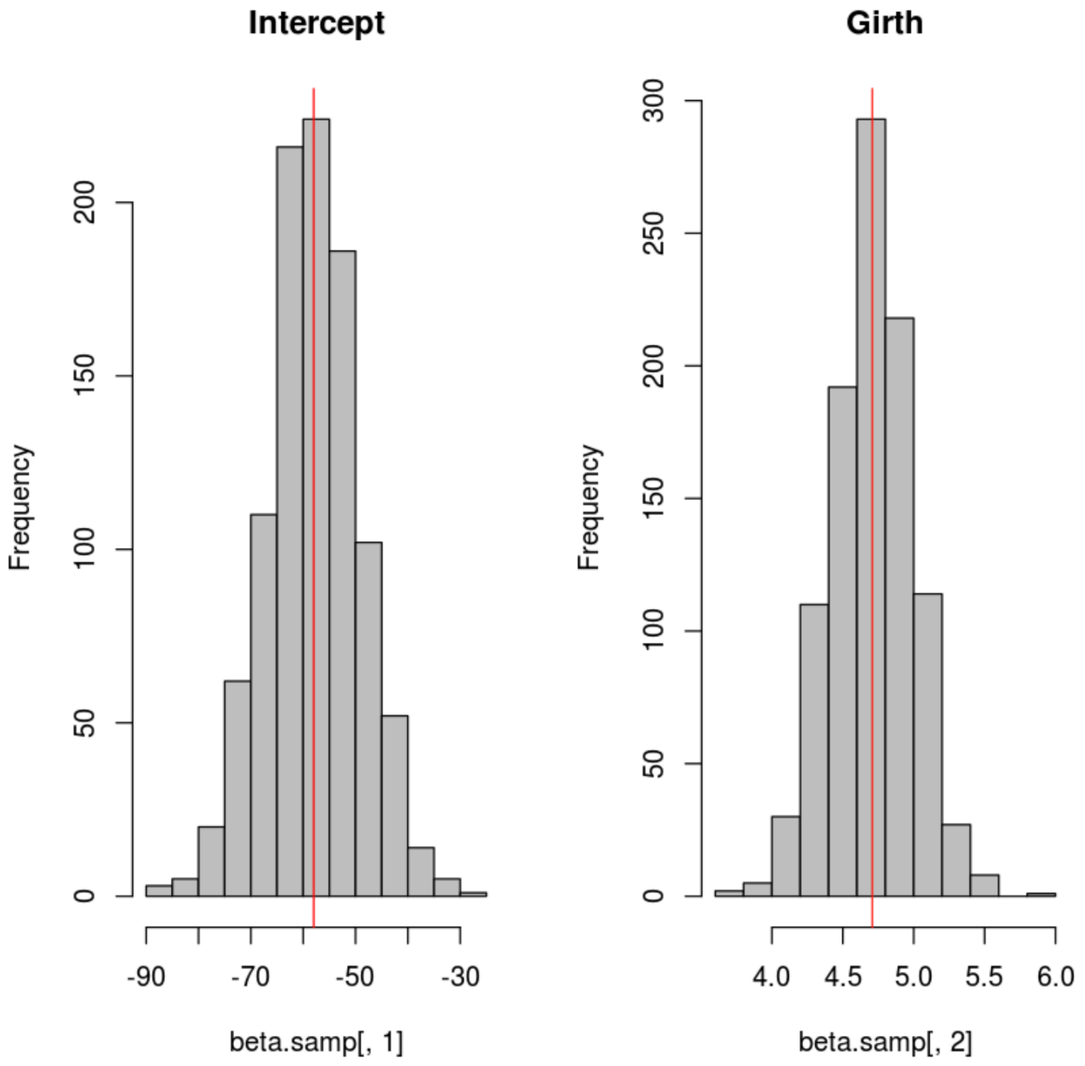
Sampling from posterior distribution of coefficients beta and variance sigsq

```
beta.hat <- model$coef
  n <- length(Volume)
  k <- length(beta.hat)
  s2 <- (n-k)*summary(model)$sigma^2
  V.beta <- summary(model)$cov.unscaled
```

Sampling from posterior distribution of coefficients beta and variance sigsq

```
numsamp <- 1000
beta.samp <- matrix(NA,nrow=numsamp,ncol=k)
sigsq.samp <- rep(NA,numsamp)
for (i in 1:numsamp){
    temp <- rgamma(1,shape=(n-k)/2,rate=s2/2)
    cursigsq <- 1/temp # sampling from IG
    curvarbeta <- cursigsq*V.beta # sigma^2 *V_{beta}
    curvarbeta.chol <- t(chol(curvarbeta))
    z <- rnorm(k,0,1)
    curbeta <- beta.hat+curvarbeta.chol%*%z # sampling from normal
    sigsq.samp[i] <- cursigsq
    beta.samp[i,] <- curbeta
  }
```

Sampling from posterior distribution of coefficients beta and variance sigsq



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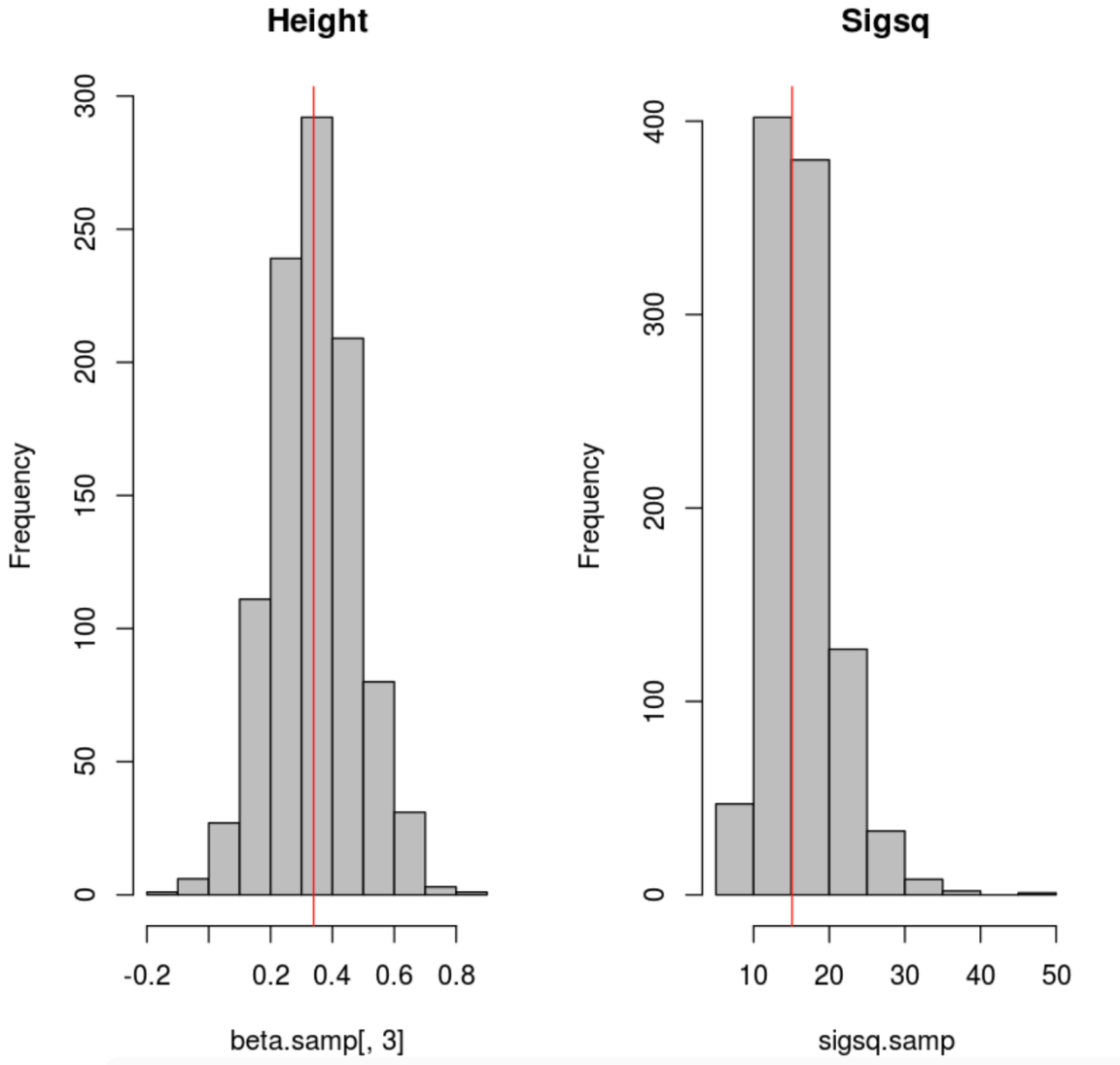

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Posterior predictive distribution for new data

Suppose we apply the regression model to a new set of data, for which we have observed the matrix $\tilde{\mathbb{X}}$ of explanatory variables, and we wish to predict the outcomes, $\tilde{\mathbb{y}}$.

- If β and σ^2 were known exactly, the vector $\tilde{\mathbb{y}}$ would have a normal distribution with mean $\tilde{\mathbb{X}}\beta$ and variance matrix $\sigma^2\mathbf{I}$.
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To draw a random sample $\tilde{\mathbb{y}}$ from its posterior predictive distribution

- We first draw $\left(\beta, \sigma^2\right)$ from the joint posterior distribution
- Then, we draw $\tilde{\mathbb{y}} \sim \mathcal{N}\left(\tilde{\mathbb{X}}\beta, \sigma^2\mathbf{I}\right)$

Posterior Predictive Sampling for new tree with girth = 18 and height = 80

```
Xstar <- c(1,18,80)  # new tree with girth = 18 and height = 80

  Ystar.samp <- rep(NA,numsamp)
  for (i in 1:numsamp){
     Ystar.hat <- sum(beta.samp[i,]*Xstar)
     Ystar.samp[i] <- rnorm(1,mean=Ystar.hat,sd=sqrt(sigsq.samp[i]))
  }
```

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```
Xstar <- c(1,18,

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```
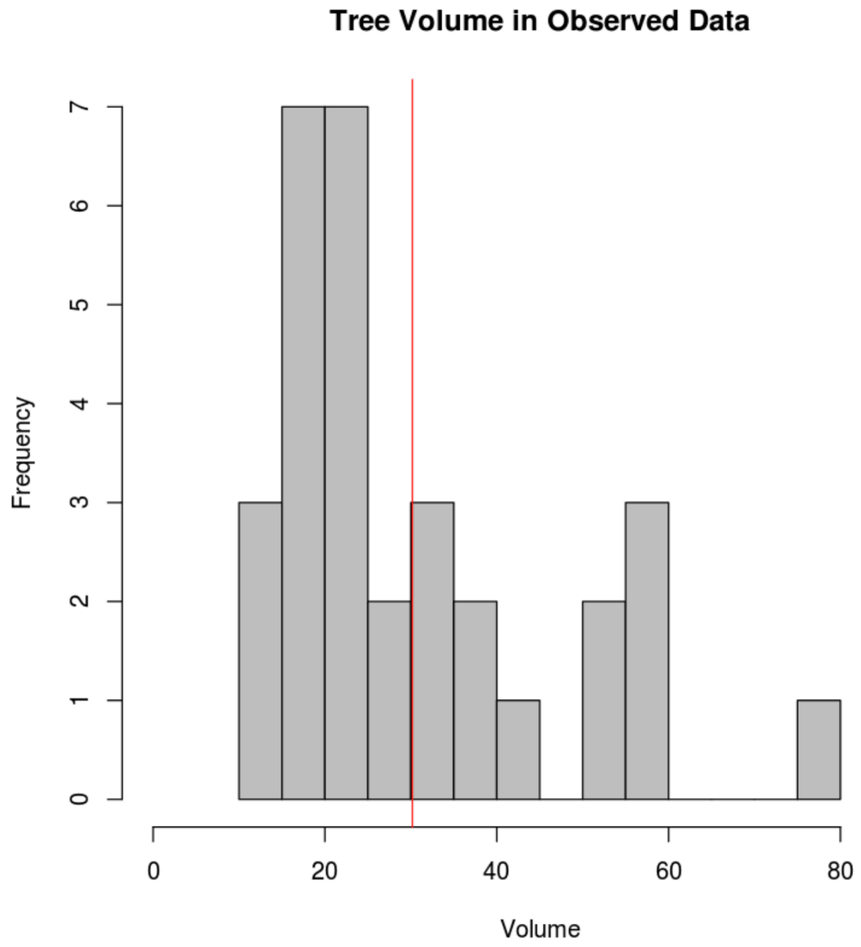

```
i]))
```



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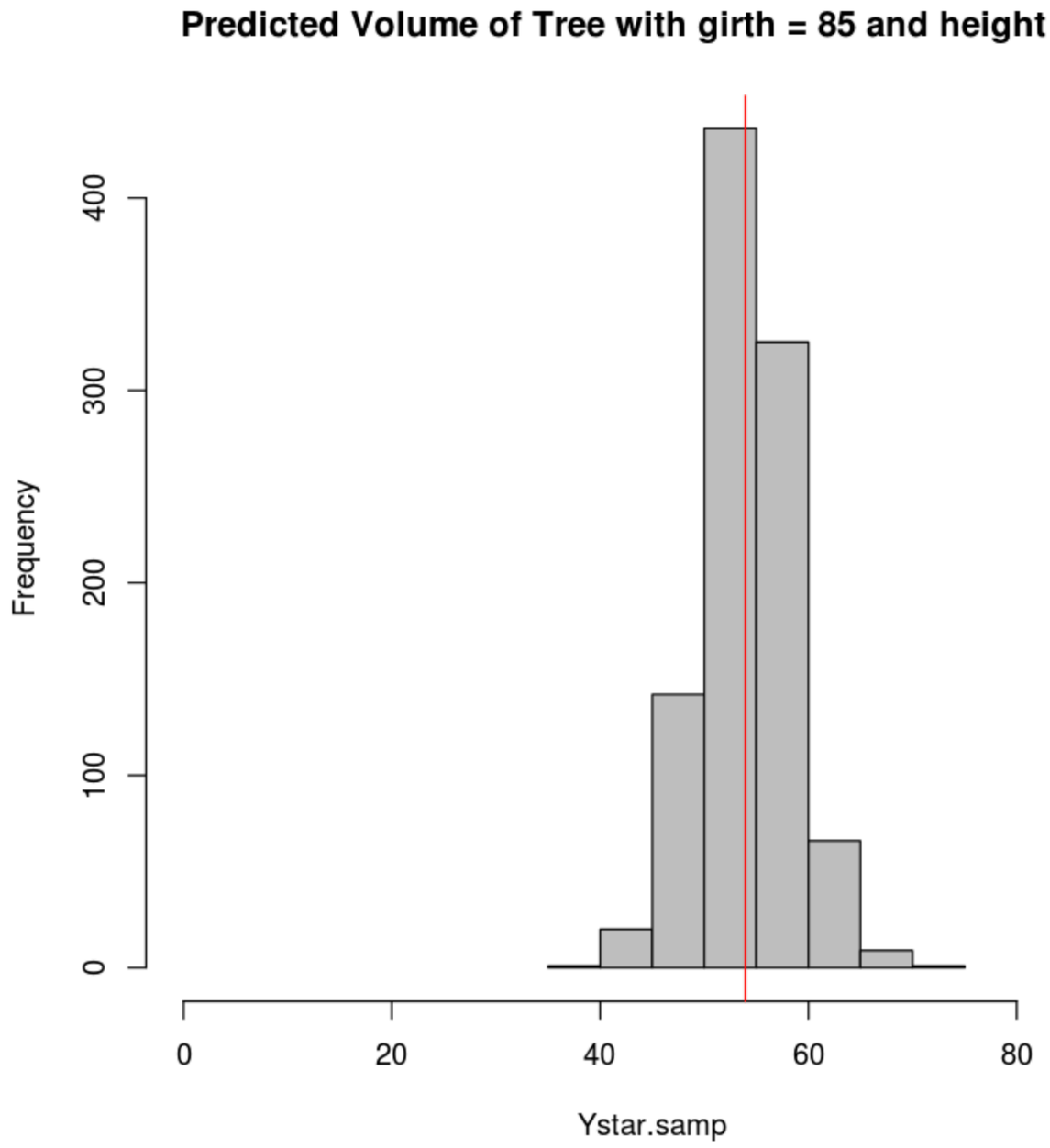
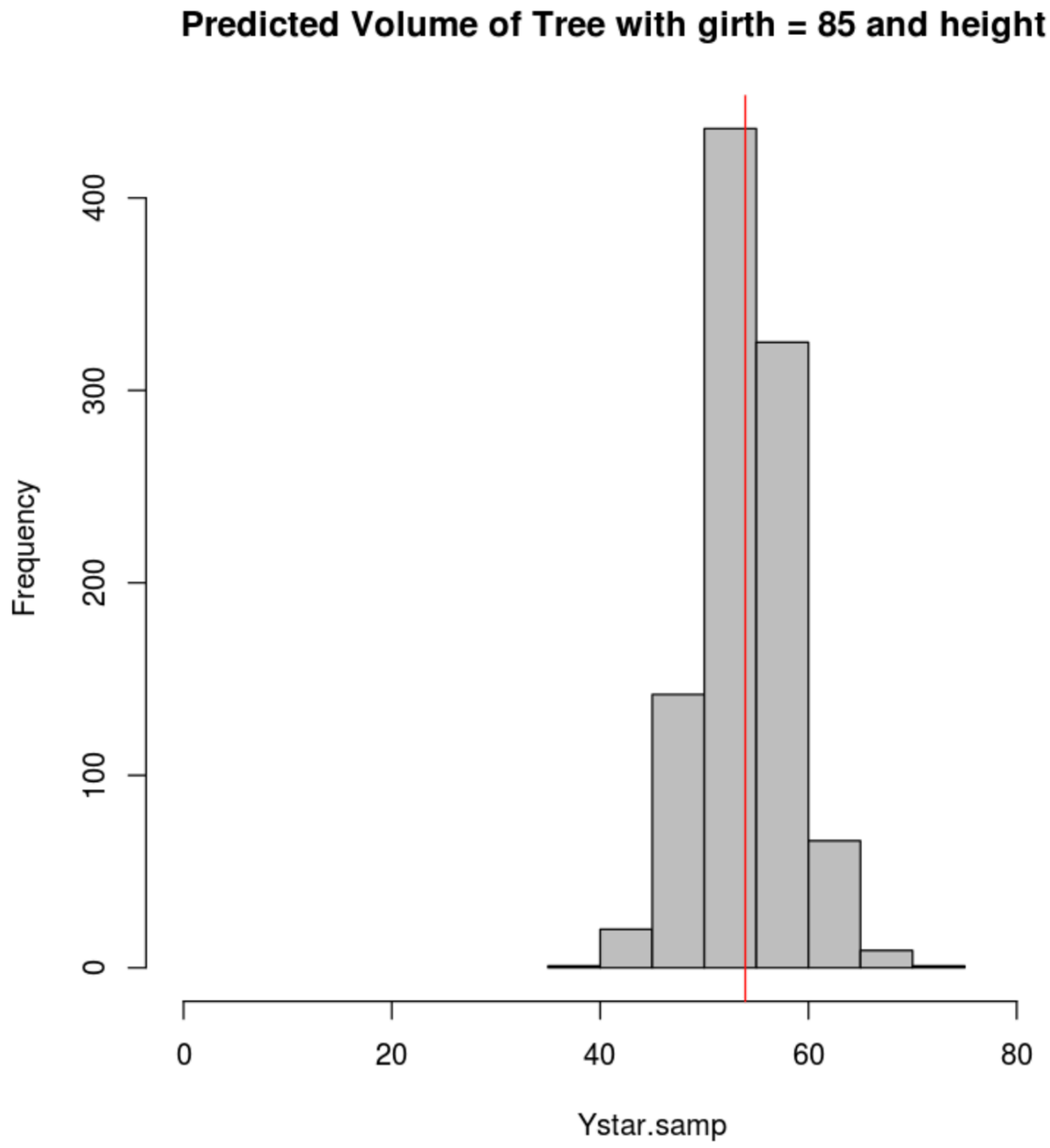

Predicted Volume of Tree with girth = 85 and height = 80
Frequency
0
100
200
300
400
0
20
40
60
80
Ystar.samp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