Orthogonality

BIOE 210

Vector Decomposition

We decompose a vector $\mathbf{u}$ over the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ by finding scalars $a_1, a_2, \ldots, a_n$ such that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

Usually the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are easier to interpret than the vector $\mathbf{u}$. Decomposition helps us understand how vectors like $\mathbf{u}$ can be constructed from simpler parts.

Basis Vectors

A set of n vectors is a basis for a vector space if and only if

1. The number of vectors (n) equals the dimension of the space.
2. The vectors span the space.
3. The vectors are linearly independent.

Basis vectors are ideal sets for decomposing vectors since *there is always a unique set of scalars that decompose any vector onto a basis.*

Decomposing Vectors

How do we decompose the vector $\mathbf{u}$ onto a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$?

1. Assemble a matrix $\mathbf{V}$ using the basis vectors as columns:

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$$

2. Solve the linear system $\mathbf{Va} = \mathbf{u}$ to find the scalars $a_1, a_2, \ldots, a_n$.

Decomposing a vector onto a basis requires solving a linear system. Is there a simpler way?

Orthonormal Basis Vectors

Decomposing onto a basis is much easier if our basis is *orthonormal*. An orthonormal basis has two additional requirements:

1. The vectors are *mutually orthogonal*, i.e. every vector in the basis is orthogonal to every other vector in the basis.
2. Every vector in the basis is a unit vector.

Decomposing onto an Orthonormal Basis

Given an orthonormal basis $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \ldots, \hat{\mathbf{v}}_n$, the decomposition of the vector $\mathbf{u}$,

$$\mathbf{u} = a_1\hat{\mathbf{v}}_1 + a_2\hat{\mathbf{v}}_2 + \cdots + a_n\hat{\mathbf{v}}_n$$

has coefficients

$$\begin{aligned} a_1 &= \mathbf{u} \cdot \hat{\mathbf{v}}_1 \\ a_2 &= \mathbf{u} \cdot \hat{\mathbf{v}}_2 \\ &\vdots \\ a_n &= \mathbf{u} \cdot \hat{\mathbf{v}}_n \end{aligned}$$

Decomposing onto an orthonormal basis requires only dot products, not solving a linear system!

Finding Orthonormal Basis Vectors

Orthonormal basis vectors are great, but how do we find them? In the last part of the chapter we introduce the Gram-Schmidt algorithm for building an orthonormal set of vectors starting from a non-orthonormal set.

Realize that the Gram-Schmidt algorithm produces a new set of vectors. If you want an orthonormal set that is “close" to a starting set, use Gram-Schmidt. If you want to decompose a vector onto a non-orthonormal basis, you would still need to solve the linear system $\mathbf{Va} = \mathbf{u}$.

In the coming chapters we will show how orthonormal sets of vectors can be generated directly from matrices.