8.2: You are building a linear model $y = \beta_0 + \beta_1 x + \beta_2 x^2$ using the following data:

x	y
1	4
2	5
-1	2

Compute the total quadratic loss for your model when $\beta_0 = 1$, $\beta_1 = 0.2$, and $\beta_2 = -0.1$.

x	y^{true}	y^{pred}
1	4	1.1
2	5	1
-1	2	0.7

$$L = \sum (y_i^{true} - y_i^{pred})^2$$

$$= (4-1.1)^2 + (5-1)^2 + (2-0.7)^2$$

$$= 8.41 + 16 + 1.69$$

$$= 26.1$$

8.4: You are building a linear model $y = \beta_0 + \beta_1 x + \beta_2 x^2$ using the following data:

x	y
1	4
2	5
-1	2

Write out a system of equations in a matrix formalism using your model and the above data.

$$\underline{y} = \underline{X}\,\underline{\beta} + \underline{\varepsilon}$$

$$\begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & -1 & (-1)^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

8.5 Without using the `pinv` function in Matlab, calculate the pseudoinverse of the design matrix and the least-squares estimates of the parameters β_0, β_1, and β_2.

$$\underline{X}^{+} = (\underline{X}^T \underline{X})^{-1} \underline{X}^T$$

$$= \begin{pmatrix} 1.11 & -0.33 & 0.22 \\ 0.33 & 0 & -0.33 \\ -0.44 & 0.33 & 0.11 \end{pmatrix}$$

Also, since $\underline{X}$ is square for this problem, $\underline{X}^{+} = \underline{X}^{-1}$, so we could use the 'inv' function instead.

9.2.1: In §8.3.2 we fit the model $y = \beta_0 + \beta_1 x$ to the data in the following table.

x	y_{true}	y_{pred}	Residual
0.07	-0.05	0.1047	-0.1547
0.16	0.40	0.2136	0.1864
0.48	0.66	0.6008	0.0592
0.68	0.65	0.8428	-0.1928
0.83	1.12	1.0243	0.0957

The best parameter estimates were $y = 0.020 + 1.21x$. What is the RMSE for this model?

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum(\text{residual})^2}$$

$$= 0.147$$

9.5: Linearize the model $y = \sqrt{\beta_0 + \beta_1 x}$.

$$y^2 = \beta_0 + \beta_1 x$$

Write out a system of matrix equations using your linearized model and the following data.

x	y
1	4
2	5
-1	2

$$\begin{pmatrix} 4^2 \\ 5^2 \\ 2^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

10.5: Calculate the Jacobian matrix of the function

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} \cos x_2 \\ \log x_1 \\ x_1 x_2 x_3 \end{pmatrix}$$

$$\underline{J}(\underline{x}) = \begin{pmatrix} 0 & -\sin x_2 & 0 \\ 1/x_1 & 0 & 0 \\ x_2 x_3 & x_1 x_3 & x_1 x_2 \end{pmatrix}$$

12.1: Start at $x = 0$, calculate two iterations of gradient descent for the function $y = (x-3)^2 + 4$ using a step size of $\alpha = 0.1$.

$$g(x) = \frac{dy}{dx} = 2(x-3)$$

$$x^{(0)} = 0$$

$$x^{(1)} = x^{(0)} - \alpha g(x^{(0)})$$

$$= 0 - 0.1\left(2(0-3)\right)$$

$$= 0.6$$

$$x^{(2)} = x^{(1)} - \alpha g(x^{(1)})$$

$$= 0.6 - 0.1\left(2(0.6-3)\right)$$

$$= 1.08$$

13.1: A bag contains five red balls and three green balls. What are the odds that a randomly selected ball will be red?

$$P(\text{Red}) = 5/_{5+3} = 5/8$$

$$\text{odds}(\text{Red}) = \frac{P(\text{red})}{1-P(\text{red})} = \frac{5/8}{1-5/8} = 5/3$$

13.4: You fit a logistic regression model that predicts the probability of Illinois' basketball team winning given the number of fouls committed by the opposing team. The best fit model is

$$P(\text{win}) = 0.3 + 0.2[\text{fouls}]$$

What is the odds ratio of the number of fouls? What is the interpretation of the odds ratio?

odds ratio (fouls) = $e^{0.2} \approx 1.22$

one more foul by the opposing team increases the odds of winning by a factor of 1.22.

15.1: How far is the plane $x_1 - 2x_2 + 0.3x_3 = 5$ from the origin?

$$\underline{a} \cdot \underline{x} = b$$

$$\begin{pmatrix} 1 \\ -2 \\ 0.3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 5$$

$$\|\underline{a}\| = \sqrt{1^2 + (-2)^2 + 0.3^2} \approx 2.37$$

$$d = b / \|\underline{a}\| = 5 / 2.37 = 2.11$$

15.2: What is the intersection of the lines $2x_1 + x_2 = 4$ and $x_1 - x_2 = 3$?

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow 2} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{-2R_1 \to R_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \end{bmatrix}$$

$$x_2 = -2/3$$

$$x_1 - \left(-2/3\right) = 3$$

$$x_1 = 7/3$$

The intersection is at $\left(7/3, -2/3\right)$.

15.2: What is the intersection of the planes $x_1 - x_2 + 3x_3 = 1$, $3x_1 + x_2 - 4x_3 = 2$, and $-2x_1 + 2x_2 - 6x_3 = -2$?

$$\begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & -4 \\ -2 & 2 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 1 \\ 3 & 1 & -4 & 2 \\ -2 & 2 & -6 & -2 \end{array}\right] \xrightarrow{-3R_1 \to R_2} \left[\begin{array}{cccc} 1 & -1 & 3 & 1 \\ 0 & 4 & -13 & -1 \\ -2 & 2 & -6 & -2 \end{array}\right]$$

$$\xrightarrow{2R_1 \to R_2} \left[\begin{array}{cccc} 1 & -1 & 3 & 1 \\ 0 & 4 & -13 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = \alpha$.

$4x_2 - 13\alpha = -1$

$\Rightarrow x_2 = \dfrac{13\alpha - 1}{4}$

$x_1 - \dfrac{13\alpha - 1}{4} + 3\alpha = 1$

$\Rightarrow x_1 = \dfrac{3 - \alpha}{4}$

The intersection is the line with points

$$\begin{pmatrix} \frac{3-\alpha}{4} \\ \frac{13\alpha - 1}{4} \\ \alpha \end{pmatrix}$$

for all α.

16.2: You want to build an SVM classifier that predicts if a cell line will respond to a drug based on the expression of three genes. Write a quadratic program based on the following four observations.

Gene 1	Gene 2	Gene 3	Response	
1.6	2.4	0.1	yes	+1
2.3	1.4	0.6	no	−1
1.0	0.8	0.2	yes	+1
1.9	2.1	0.4	no	−1

$$\min_{a_1, a_2, a_3, b} \quad a_1^2 + a_2^2 + a_3^2$$

s.t.

$$1.6a_1 + 2.4a_2 + 0.1a_3 \geq b + 1$$

$$2.3a_1 + 1.4a_2 + 0.6a_3 \leq b - 1$$

$$1.0a_1 + 0.8a_2 + 0.2a_3 \geq b + 1$$

$$1.9a_1 + 2.1a_2 + 0.4a_3 \leq b - 1$$