BIOE 210, SPRING 2019

EXAM 1

You have 80 minutes to complete this exam.
You may use notes or printouts from the course website,
but **no electronic resources**.
A standalone scientific or graphing calculator is allowed.

Circle your final answer for each question.

Name

KEY

PART I (14 POINTS; 2 POINTS EACH)

(1) TRUE or FALSE. If $\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ \theta \end{pmatrix}$, there exists a scalar θ such that $\|\mathbf{x}\| = -4$.

$\|\underline{x}\| \geq 0 \;\; \forall \; \underline{x}$

(2) If f is a linear system with $f(2) = 5$ and $f(a) = 3$, what is $f(2a + 2)$?

$$
\begin{aligned}
f(2a+2) &= f(2a) + f(2) \\
&= 2f(a) + f(2) \\
&= 2(3) + 5 \\
&= 11
\end{aligned}
$$

(3) What is the inverse of the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$? (Hint: this is a special type of matrix that you've used on your homework.)

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \underline{I}(2,3)$$

$$\underline{I}^{-1}(2,3) = \underline{I}(-2,-3) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

(4) If the vectors $\begin{pmatrix} \theta \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ \theta \\ 3 \end{pmatrix}$ are orthogonal, what is θ?

$$\begin{pmatrix} \theta \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ \theta \\ 3 \end{pmatrix} = 3\theta + 2\theta - 3 = 0$$

$$\theta = 3/5$$

(5) Given the following matrices and their inverses,

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

what is the matrix $\left((\mathbf{BA})^{-1}\mathbf{BAB}\right)^{-1}$?

$$\left((\underline{B}\underline{A})^{-1}\underline{B}\underline{A}\underline{B}\right)^{-1}$$

$$= \left(\underline{I}\,\underline{B}\right)^{-1}$$

$$= \left(\underline{B}\right)^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

(6) Let the function $f = \alpha\beta x^2 + \alpha x + \alpha\beta$. Circle all that are correct:

(a) The function f is linear with respect to α.
(b) The function f is linear with respect to β.
(c) The function f is linear with respect to x.

f is affine in β
and quadratic in x.

(7) Rank the following from 1 = most efficient to 3 = least efficient for solving a linear system $\mathbf{Ax} = \mathbf{y}$.

(2) Find the inverse $\mathbf{A}^{-1}$ using the side-by-side method and multiply $\mathbf{A}^{-1}\mathbf{y}$.
(3) Find the inverse $\mathbf{A}^{-1}$ using elementary matrices and multiply $\mathbf{A}^{-1}\mathbf{y}$.
(1) Find the row echelon form of the augmented matrix $[\mathbf{A}\ \mathbf{y}]$ and use back substitution.

Part II (4 points)

Compute the product

$$\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -b \\ 2a & 2+3b \end{pmatrix}$$

PART III (8 POINTS)

Consider the following system of equations:

$$2x_3 = 4$$
$$x_1 + 2x_2 - x_3 = 6$$
$$-x_2 + x_3 = 2$$

(1) Set up an augmented matrix and convert it to row echelon form using row operations.

$$\begin{pmatrix} 0 & 0 & 1 & 4 \\ 1 & 2 & -1 & 6 \\ 0 & -1 & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -1 & 6 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{-R_2} \begin{pmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

(2) Solve the system of equations using back substitution.

$$x_3 = 4$$

$$x_2 - x_3 = -2 \Rightarrow x_2 = 2$$

$$x_1 + 2x_2 - x_3 = 6$$

$$\Rightarrow x_1 = 0$$

Part IV (4 points)

Find the inverse of the matrix $\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ using the side-by-side method.

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{-R_1} \begin{pmatrix} 1 & -3 & 0 & -1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

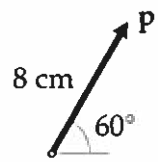
$$\xrightarrow{1/2 R_2} \begin{pmatrix} 1 & -3 & 0 & -1 \\ 0 & 1 & 1/2 & 0 \end{pmatrix}$$

$$\xrightarrow{3R_2 \to R_1} \begin{pmatrix} 1 & 0 & 3/2 & -1 \\ 0 & 1 & 1/2 & 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3/2 & -1 \\ 1/2 & 0 \end{pmatrix}$$

PART V (4 POINTS)

Using translation and rotation matrices, write an expression for $\mathbf{p}$, the location of the end of the arm below. You do not need to compute the product of the matrices.



$$\underline{p} = \underline{R}(60°)\,\underline{T}(8,0)\,\underline{o}_d \qquad [cm]$$

$$= \begin{pmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

PART VI (16 POINTS)

We want to approximate the solution of the differential equation

$$\frac{d^2u}{dx^2} + x\frac{du}{dx} = 0, \quad u(0) = 3, \quad u(1) = 0$$

using four nodes spanning the interval $[0, 1]$.

(1) Discretize the interval $[0, 1]$ with four nodes. Write the node number and the value of x at each node.

node	0	1	2	3
$x =$	0	$1/3$	$2/3$	1

(2) Write finite difference approximations for the interior nodes.

@ node 1: $$\frac{u^{(2)} - 2u^{(1)} + u^{(0)}}{(1/3)^2} + \frac{1}{3}\,\frac{u^{(2)} - u^{(1)}}{1/3} = 0$$

@ node 2: $$\frac{u^{(3)} - 2u^{(2)} + u^{(1)}}{(1/3)^2} + \frac{2}{3}\,\frac{u^{(3)} - u^{(2)}}{1/3} = 0$$

(3) Write finite difference approximations for the boundary nodes.

@ node 0: $u^{(0)} = 3$

@ node 3: $u^{(3)} = 0$

(4) Rewrite the equations as a matrix equation of the form **Ax** = **y**. (You do not need to solve the system of equations.)

Simplify node 1:

$$\frac{1}{9}u^{(2)} - \frac{2}{9}u^{(1)} + \frac{1}{9}u^{(0)} + \frac{1}{9}u^{(2)} - \frac{1}{9}u^{(1)} = 0$$

$$2u^{(2)} - 3u^{(1)} + u^{(0)} = 0$$

Simplify node 2:

$$\frac{1}{9}u^{(3)} - \frac{2}{9}u^{(2)} + \frac{1}{9}u^{(1)} + \frac{2}{9}u^{(3)} - \frac{2}{9}u^{(2)} = 0$$

$$3u^{(3)} - 4u^{(2)} + u^{(1)} = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$