FAQs: Vector Spaces, Span, and Basis

A set of orthogonal vectors are always linearly independent. Are a set of linearly independent vectors always orthogonal?

No. We proved in §11.5 that orthogonal vectors are always linearly independent. However, linear independence does not imply orthogonality. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These vectors are linearly independent since there are no coefficients c_1 and c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

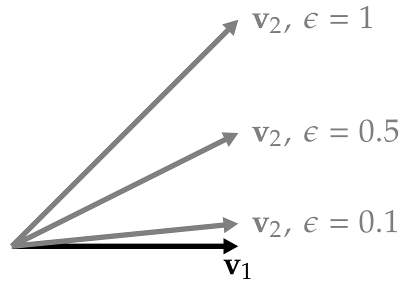
In particular, we cannot cancel out the second dimension of $\mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, but they are not orthogonal:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \times 1 + 0 \times 1 = 1 \neq 0$$

Going further, let's consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}$$

for some small number $\epsilon > 0$. These vectors are always linearly independent since we cannot cancel out the ϵ in the second dimension of vector $\mathbf{v}_2$. However, for small values of ϵ the two vectors are very close together as shown below.



Even vectors that are arbitrarily close together can be linearly independent.