Matrix Decompositions

BIOE 210

Matrix multiplication using the eigenbasis

Let's consider a square $n \times n$ matrix $\mathbf{A}$ with a complete set of eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$.

The eigenvectors form a basis (the *eigenbasis*), so we can decompose another vector $\mathbf{x}$ onto them.

$$\mathbf{x} = a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n$$

We can use the decomposition of $\mathbf{x}$ to perform matrix multiplication by $\mathbf{A}$.

$$\mathbf{A}\mathbf{x} = a_1 \lambda_1 \mathbf{v}_1 + \cdots + a_n \lambda_n \mathbf{v}_n$$

Matrix multiplication as a multistep transformation

Using the eigenbasis we can break down matrix multiplication into three steps:

1. Decompose $\mathbf{x}$ onto the eigenbasis.
2. Scale each term by the respective eigenvalue.
3. Reassemble the decomposed vectors into the output vector.

Step 1: Decompose $\mathbf{x}$ onto the eigenbasis

We already know how to decompose a vector onto a basis. First we assemble a matrix using the basis vectors as columns.

$$\mathbf{V} = (\mathbf{v}_1 \, \mathbf{v}_2 \, \cdots \, \mathbf{v}_n)$$

Then we find the coefficients $\mathbf{a}$ by solving the linear system

$$\mathbf{V}\mathbf{a} = \mathbf{x}$$

This has a unique solution

$$\mathbf{a} = \mathbf{V}^{-1}\mathbf{x}$$

Step 2: Scale the coefficients by the eigenvalues

Multiplying by the matrix **A** scales each coefficient in our decomposition by the corresponding eigenvalue:

$$
\begin{aligned}
a_1 &\longrightarrow \lambda_1 a_1 \\
a_2 &\longrightarrow \lambda_2 a_2 \\
&\vdots \\
a_n &\longrightarrow \lambda_n a_n
\end{aligned}
$$

We can scale all of the coefficients at once using a matrix with the eigenvalues along the diagonal:

$$
\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad \mathbf{\Lambda a} = \begin{pmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ \vdots \\ \lambda_n a_n \end{pmatrix}
$$

Step 3: Reassembling the output vector

We can also use matrix multiplication to reassemble a decomposed vector into the output vector.

Notice that the inverse matrix $\mathbf{V}^{-1}$ decomposes a vector.

$$\mathbf{a} = \mathbf{V}^{-1}\mathbf{x}$$

The original matrix undoes this operation since it is the inverse of the inverse.

$$\begin{aligned} \mathbf{Va} &= \mathbf{V}\mathbf{V}^{-1}\mathbf{x} \\ &= \mathbf{x} \end{aligned}$$

Reassembling the decomposed vector is the same as multiplying by the matrix $\mathbf{V}$.

Putting it all together

We have broken down matrix multiplication into three sequential matrix operations:

$$
\begin{aligned}
\mathbf{Ax} &= \underbrace{(\text{reassembly})}_{\mathbf{V}}\underbrace{(\text{scaling})}_{\mathbf{\Lambda}}\underbrace{(\text{decomposition})}_{\mathbf{V}^{-1}}\mathbf{x} \\
&= \mathbf{V\Lambda V}^{-1}\mathbf{x}
\end{aligned}
$$

The matrix $\mathbf{A}$ can therefore be decomposed into the product of three matrices, $\mathbf{V\Lambda V}^{-1}$.

This matrix decomposition is called the *eigendecomposition* of $\mathbf{A}$.

Limitations of the eigendecomposition

The eigendecomposition $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$ only works when the matrix $\mathbf{A}$ meets certain requirements:

- $\mathbf{A}$ must be square.
- $\mathbf{A}$ must have a complete set of eigenvectors.
- $\mathbf{V}^{-1}$ must exist.[1]

There is a more general form of matrix decomposition that relaxes these requirements.

[1] The inverse $\mathbf{V}^{-1}$ will always exist when $\mathbf{A}$ has a complete set of eigenvectors since the eigenvectors are linearly independent and therefore $\mathbf{V}$ is full rank.

The Singular Value Decomposition (SVD)

Any $m \times n$ matrix $\mathbf{A}$ can be decomposed into the product of three matrices

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{T}}$$

where

- $\mathbf{U}$ is an orthogonal $m \times m$ matrix.
- $\mathbf{\Sigma}$ is a diagonal $m \times n$ matrix with nonzero entries.
- $\mathbf{V}$ is an orthogonal $n \times n$ matrix.

Parts of the SVD

The matrices $\mathbf{U}$ and $\mathbf{V}$ are *orthogonal* matrices, meaning their columns are an orthonormal set of basis vectors. This also means that

$$\mathbf{U}^{-1} = \mathbf{U}^{\mathsf{T}}, \quad \mathbf{V}^{-1} = \mathbf{V}^{\mathsf{T}}$$

The columns in $\mathbf{U}$ and $\mathbf{V}$ are called the left and right *singular vectors*. The singular vectors have similar properties to eigenvectors:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$$

and

$$\mathbf{A}^{\mathsf{T}}\mathbf{u}_i = \sigma_i \mathbf{v}_i$$

where the scalars σ_i are the *singular values*.

Parts of the SVD

The matrix $\mathbf{\Sigma}$ is a diagonal $m \times n$ matrix. The entries along the diagonal are the singular values.

Singular values are always non-negative. They are similar to eigenvalues for square matrices, but they are not the same.[2]

If we arrange $\mathbf{\Sigma}$ such that the singular values are in descending order, the SVD of a matrix is unique.

[2]The singular values of a square matrix are the squares of the eigenvalues of the matrix.

Applications of the SVD

We will discuss several examples of the SVD, including:

- Calculating the rank of a matrix.
- Finding a “pseudoinverse" of non-square matrices.
- Approximating matrices with simpler matrices.
- Image compression.