Linearity:

Proportional

$$f(kx) = kf(x)$$

$$f(0x) = 0f(x) = 0 \qquad y = mx + b$$

Additive

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

After spending years studying algebra, you might think that there are many byzantine rules that govern fields. In fact, there are only five. The five axioms describe only two operations (addition and multiplication) and define two special elements that must be in every field (0 and 1).

1.2 The Field Axioms

Given elements a, b, and c in a field:

1. **Associativity**.

$$a + b + c = (a + b) + c = a + (b + c)$$
$$abc = (ab)c = a(bc)$$

2. **Commutativity**.

$$a + b = b + a$$
$$ab = ba$$

3. **Distribution** of multiplication over addition.

$$a(b + c) = ab + ac$$

4. **Identity**. There exist elements 0 and 1, both in the field, such that

$$a + 0 = a$$
$$1 \times a = a$$

5. **Inverses**.

   - For all a, there exists an element $(-a)$ in the field such that $a + (-a) = 0$. The element $-a$ is called the *additive inverse* of a.
   - For all $a \neq 0$, there exists an element (a^{-1}) in the field such that $a \times a^{-1} = 1$. The element a^{-1} is called the *multiplicative inverse* of a. $1/a$

It might surprise you that only five axioms are sufficient to recreate everything you know about algebra. For example, nowhere do we state the special property of zero that $a \times 0 = 0$ for any number a. We don't need to state this property, as it follows from the field axioms.

Theorem. $a \times 0 = 0$.

$$0 \times a = 0$$

$$(1 + -1) \times a = 0$$

$$(a + -a) = 0$$

$$ab = 0 \iff a = 0 \text{ or } b = 0$$

Suppose $a \neq 0$

$$a^{-1} a b = a^{-1} 0$$

$$1b = 0$$

$$b = 0$$

$$x^2 - 7x + 12 = 0$$

$$(x-3)(x-4) = 0$$

$$x = 3, x = 4$$

Vector.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Scalars: x, y, α, k

Vectors: $\underline{x}, \underline{y}, \underline{\alpha}$

Matrices: $\underline{A}, \underline{X}, \ldots$

Vector Addition: Elementwise

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3-1 \\ -1+4 \\ 2+6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

Zero

$$\underline{x} + \underline{0} = \underline{x}$$

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3+0 \\ 1+0 \\ 2+0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$\underline{x}$, what is $-\underline{x}$?

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplication

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

} Nonsense

Scalar Multiplication is elementwise

$$\underset{\text{scalar}}{k}\ \underset{\text{vector}}{\underline{x}}$$

$$-6 \times \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -6\times 2 \\ -6\times 0 \\ -6\times 4 \end{pmatrix} = \begin{pmatrix} -12 \\ 0 \\ -24 \end{pmatrix}$$

$$k\underline{x} = \underline{0} = \begin{pmatrix} kx_1 \\ \vdots \\ kx_n \end{pmatrix}$$

Vectors are not well-ordered.

$$\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 5 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

NORM. $\approx$ magnitude

$\mathbb{R}^n \longmapsto \mathbb{R}$

Properties of NORMS.

$\|\underline{x}\| \geq 0$ non-negativity

$\|\underline{x}\| = 0 \iff \underline{x} = \underline{0}$

2-norm, Pythagorean norm, Euclidean norm

$$\|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\|\underline{x}\|_2 = \left(x_1^2 + \cdots x_n^2\right)^{1/2}$$

k-norm

$$\|\underline{x}\|_k = \left(\sum_{i=1}^{n} x_i^k\right)^{1/k}$$

$$\underline{x} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad \|\underline{x}\| = \sqrt{3^2 + (-4)^2}$$
$$= 5$$

<u>Unit Vectors</u> have norm = 1

$\underline{x} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$ is a unit vector

$$\|\underline{x}\| = \left(\left(\tfrac{3}{5}\right)^2 + \left(-\tfrac{4}{5}\right)^2\right)^{1/2}$$

$$= \left(\tfrac{9}{25} + \tfrac{16}{25}\right)^{1/2}$$

$$= \sqrt{\tfrac{25}{25}} = 1$$

If $\underline{x}$ is not a unit vector, $\frac{1}{\|\underline{x}\|}\underline{x}$ is.

Dot product (inner product)

$$\underline{x} \cdot \underline{y} = \|\underline{x}\| \, \|\underline{y}\| \cos\theta$$