Chapter 11: Vector Spaces, Span, and Basis (Part I)

1. **Are the vectors**
$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$
linearly independent?

2. **Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis in $\mathbb{R}^3$?**

3. **Decompose the vector**
$$\mathbf{u} = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}$$
onto the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

4. **Are the coefficients you found in Problem 3 the only ones that decompose $\mathbf{u}$ over the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?**

Solutions

1. Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

linearly independent?

Let's form a matrix $\mathbf{V}$ using the vectors as columns.

$$\mathbf{V} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix}$$

This matrix has three rows and three columns. If all three columns are linearly independent, the matrix will have rank three.

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix}$$
$$\xrightarrow{-2R_1 \to R_3} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 4 & 1 \end{pmatrix}$$
$$\xrightarrow{-4R_2 \to R_3} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 9 \end{pmatrix}$$
$$\xrightarrow{(1/9)R_3} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

The reduced form of $\mathbf{V}$ shows that $\text{rank}(\mathbf{V}) = 3$, so the vectors are linearly independent.

2. Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis in $\mathbb{R}^3$?

Yes. The space $\mathbb{R}^3$ has dimension three, and we have three vectors. We know from the previous question that these three vectors are linearly independent. Combined, these facts prove that the vectors are a basis for $\mathbb{R}^3$. Since they are a basis, we also know that the vectors span $\mathbb{R}^3$.

3. Decompose the vector

$$\mathbf{u} = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}$$

onto the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

We are looking for scalars a_1, a_2, and a_3 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{u}$$

which is equivalent to solving the linear system

$$\mathbf{V}\mathbf{a} = \mathbf{u}$$

where $\mathbf{V}$ is a matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$; and $\mathbf{a}$ is a vector of the scalars a_1, a_2, and a_3:

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}$$

We can solve this by Gaussian elimination to find that $a_1 = -2$, $a_2 = 1$, and $a_3 = 3$.

4. Are the coefficients you found in Problem 3 the only ones that decompose $\mathbf{u}$ over the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?

Yes. Since the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are a basis, the decomposition of any vector onto them is unique.