Linear Models

BIOE 210

Is there a better method?

We can derive estimates for any type of linear model. But the amount of math scales cubicly with the number of parameters (why?).

Instead, let's turn to a matrix formalism for linear models.

A matrix formalism for linear models

Let's write out one equation for each observation of the model $y = \beta_0 + \beta_1 x$.

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\begin{aligned}
-0.05 &= \beta_0 + 0.07\beta_1 + \epsilon_1 \\
0.40 &= \beta_0 + 0.16\beta_1 + \epsilon_2 \\
0.66 &= \beta_0 + 0.48\beta_1 + \epsilon_3 \\
0.65 &= \beta_0 + 0.68\beta_1 + \epsilon_4 \\
1.12 &= \beta_0 + 0.83\beta_1 + \epsilon_5
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\begin{pmatrix} -0.05 \\ 0.40 \\ 0.66 \\ 0.65 \\ 1.12 \end{pmatrix} = \begin{pmatrix} 1 & 0.07 \\ 1 & 0.16 \\ 1 & 0.48 \\ 1 & 0.68 \\ 1 & 0.83 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix}
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Solving the linear system

A few points about $\mathbf{y} = \mathbf{X}\beta + \epsilon$:

- The unknowns are β, not $\mathbf{X}$.
- The coefficient matrix $\mathbf{X}$ is called the *model matrix*.
- The design matrix $\mathbf{X}$ is rarely square.

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The solution to this system that minimizes the errors in ϵ is

$$\beta = \mathbf{X}^{+}\mathbf{y}$$

where $\mathbf{X}^{+}$ is the *pseudoinverse* of $\mathbf{X}$.

For next time

- We will see how to calculate the pseudoinverse in Part III.
- Next time we will demonstrate how to formulate and solve more complex linear models.