Eigenvalues and Eigenvectors

1. Which of the following are eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$$

and what are the associated eigenvalues?

a. $\begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

b. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

c. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

has eigenvectors and eigenvalues

$$\lambda_1 = 1.382, \quad \mathbf{v}_1 = \begin{pmatrix} -0.8507 \\ 0.5257 \end{pmatrix}$$

$$\lambda_2 = 3.618, \quad \mathbf{v}_2 = \begin{pmatrix} -0.5257 \\ 0.8507 \end{pmatrix}$$

2. Decompose the vector $\mathbf{u} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ onto the eigenvectors of X.

3. Using your solution to Question #2, compute the product Xu.

4. Is the matrix X positive definite? What does this tell you about the function $f(x_1, x_2) = 2x_1^2 + 2x_1x_2 + 3x_2^2$?

5. Using MATLAB, calculate the eigenvalues for the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & 2 & 5 \\ -6 & 4 & 5 \end{pmatrix}$$

Using the eigenvalues, compute the determinant of the matrix A. Verify your answer using the `det` function in MATLAB.

Solutions

1. Which of the following are eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$$

and what are the associated eigenvalues?

a. $\begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

b. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

c. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

a. $\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} -3\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{pmatrix} = 3 \times \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

This is an eigenvector with eigenvalue 3.

b. $\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This is not an eigenvector.

c. $\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This is an eigenvector with eigenvalue 1.

2. Decompose the vector $\mathbf{u} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ onto the eigenvectors of X.

We can decompose $\mathbf{u}$ by solving the system $\mathbf{V}\mathbf{a} = \mathbf{u}$ where $\mathbf{V}$ is a matrix with the two eigenvectors as columns.

$$\begin{pmatrix} -0.8507 & 0.5257 \\ 0.5257 & 0.8507 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Solving this systems in MATLAB we find that $a_1 = 3.078$ and $a_2 = -0.727$. Thus

$$\begin{aligned} \mathbf{u} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \\ \begin{pmatrix} -3 \\ 1 \end{pmatrix} &= 3.078 \begin{pmatrix} -0.8507 \\ 0.5257 \end{pmatrix} - 0.727 \begin{pmatrix} 0.5257 \\ 0.8507 \end{pmatrix} \end{aligned}$$

3. Using your solution to Question #2, compute the product Xu.

We can write the product $\mathbf{X}\mathbf{u}$ using our decomposition of $\mathbf{u}$ from Question #2.

$$\mathbf{X}\mathbf{u} = \mathbf{X}(a_1\mathbf{v}_1 + a_2\mathbf{v}_2)$$

Now we can distribute $\mathbf{X}$ and simplify because $\mathbf{X}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for any eigenvector $\mathbf{v}_i$.

$$\begin{aligned} \mathbf{X}\mathbf{u} &= \mathbf{X}(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) \\ &= a_1\mathbf{X}\mathbf{v}_1 + a_2\mathbf{X}\mathbf{v}_2 \\ &= a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2 \end{aligned}$$

Now let's substitute in numerical values.

$$\begin{aligned} \mathbf{X}\mathbf{u} &= a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2 \\ &= 3.078 \times 1.382 \times \begin{pmatrix} -0.8507 \\ 0.5257 \end{pmatrix} - 0.727 \times 3.618 \times \begin{pmatrix} 0.5257 \\ 0.8507 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 0 \end{pmatrix} \end{aligned}$$

4. Is the matrix X positive definite? What does this tell you about the function $f(x_1, x_2) = 2x_1^2 + 2x_1x_2 + 3x_2^2$?

Both of the eigenvalues for **X** are positive, so the matrix **X** is positive definite.

The function $f(x_1, x_2) = 2x_1^2 + 2x_1x_2 + 3x_2^2$ can be written as

$$f(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ is positive definite, we know the function f is convex.

5. Using MATLAB, calculate the eigenvalues for the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & 2 & 5 \\ -6 & 4 & 5 \end{pmatrix}$$

Using the eigenvalues, compute the determinant of the matrix A. Verify your answer using the `det` function in MATLAB.

```
>> A = [-3 1 0; 0 2 5; -6 4 5];
>> [V,D] = eig(A)

V =

    0.0590   -0.5760    0.2957
    0.6444    0.5277    0.8870
    0.7624   -0.6243   -0.3548


D =

    7.9161         0         0
         0   -3.9161         0
         0         0   -0.0000
```

The eigenvectors are the columns of `V`, and the eigenvalues are the diagonal elements of `D`.

The determinant of **A** is the product of the eigenvalues.

$$\begin{aligned}\det(\mathbf{A}) &= \lambda_1\lambda_2\lambda_3 \\ &= 7.9161 \times -3.9161 \times 0.0 \\ &= 0\end{aligned}$$

The matrix **A** has a determinant of zero, so it must be rank deficient. Using MATLAB's rank function we see that the rank is only two.