$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -4 \\ -2 & 0 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 2 & -4 & -2 \\ -2 & 0 & -6 & -4 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \xrightarrow{2R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow 0=0$$

$x_3 = \alpha$

$x_2 - 2x_3 = -1$

$x_2 - 2\alpha = -1$

$x_2 = 2\alpha - 1$

$x_1 + 3x_3 = 2$

$x_1 + 3\alpha = 2$

$x_1 = 2 - 3\alpha$

$$\left.\begin{aligned} x_1 &= 2 - 3\alpha \\ x_2 &= 2\alpha - 1 \\ x_3 &= \alpha \end{aligned}\right\} \text{Parameterized Sol}^{\underline{n}}$$

Let $\alpha = 0$

$x_1 = 2$

$x_2 = -1$

$x_3 = 0$

Let $\alpha = -3$

$$\left.\begin{aligned} x_1 &= 11 \\ x_2 &= -7 \\ x_3 &= -3 \end{aligned}\right\} \text{Particular or Specific Sol}^{\underline{n}}$$

RANK of a matrix is the # of linearly independent rows in the matrix.

A set of vectors are lin. ind. iff

$$\underline{x}_i = c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_n\underline{x}_n$$

$$c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_n\underline{x}_n = \underline{0}$$

$$\Rightarrow c_1, c_2, \cdots c_n = 0$$

Properties of Rank

$\text{Rank}(\underline{A}) \leq \min\{m, n\}$ when $\underline{A} \in \mathbb{R}^{m \times n}$

$\text{RANK}(\underline{A}) = \text{Rank}(\underline{A}^T)$

$\Rightarrow$ Row Rank = column Rank

$\text{RANK}(\underline{A}) = 0 \Leftrightarrow \underline{A} = \underline{0}$

$\underline{A}\,\underline{x} = \underline{y}$, $\underline{x} \in \mathbb{R}^n$ is solvable iff

$\text{RANK}(\underline{A}) = \text{Rank}([\underline{A}\ \underline{y}])$. If

$\text{RANK}(\underline{A}) = n$, then the sol$^{\underline{n}}$ is unique.

OTW, there are infinitely many sol$^{\underline{n}}$s.

Homogeneous: $\underline{A}\,\underline{x} = \underline{0}$

$\underline{0}$ is always a sol'n. (trivial)

$\text{RANK}(\underline{A}) = \text{Rank}([\underline{A}\ \underline{0}])$

Non-homogeneous $\underline{A}\,\underline{x} = \underline{y}$, $\underline{y} \neq \underline{0}$

1. Full RANK: $\text{RANK}(\underline{A}) = n$

There is always a solution.

$$\underbrace{\text{RANK}(\underline{A})}_{\substack{4\times 4 \\ \text{Rank}=4}} = \underbrace{\text{RANK}([\underline{A}\ \underline{y}])}_{\substack{4\times 5 \\ \text{Rank}=4}}$$

UNIQUE:

$$c_1 \underline{A}(:,1) + c_2 \underline{A}(:,2) + \cdots + c_n \underline{A}(:,n) = \underline{y}$$

$$\underline{A}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \underline{y} \Rightarrow \underline{x} = \underline{c}$$

3. Rank-deficient, non-homogeneous.

$$\text{RANK}(\underline{A}) = \text{RANK}([A \ \underline{y}])$$

$$\text{RANK}\,\underline{A} < n$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -4 \\ -2 & 0 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -4 & -2 \\ -2 & 0 & -6 & -4 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \\ -2 & 0 & -6 & -4 \end{pmatrix}$$

$$\xrightarrow{2R_1 \to R_3} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad \text{Rank} = 3$$

$$\underset{=2}{\text{RANK}(\underline{A})} \neq \underset{=3}{\text{Rank}([\underline{A}\ \underline{y}])}$$

No solution!