Eigenvectors and Eigenvalues

BIOE 210

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or, let $\mathbf{x}_2 = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$:

$$\mathbf{A}\mathbf{x}_2 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \end{pmatrix} = \mathbf{x}_2$$

The matrix **A** isn't special.

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Let $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\mathbf{A}\mathbf{x}_3 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix} \neq \lambda \mathbf{x}_3$$

The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ aren't special.

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$$

$$\mathbf{B}\mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \neq \lambda \mathbf{x}_1$$

$$\mathbf{B}\mathbf{x}_2 = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -13 \\ 21 \end{pmatrix} \neq \lambda \mathbf{x}_2$$

Eigenvectors and eigenvalues

The relationship between the matrix $\mathbf{A}$ and the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ is special. These vectors are *eigenvectors* of the matrix.

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

The associated scalar λ is called the *eigenvalue*.

Properties of eigenvectors and eigenvalues

- Only square matrices can have eigenvectors and eigenvalues.
- All eigenvectors for a matrix are linearly independent.
- An $n \times n$ matrix can have up to n eigenvectors.
- All eigenvectors for a matrix are unique, but the eigenvalues can repeat.

The eigenbasis

A matrix is *perfect* if it has a complete set of eigenvectors. These eigenvectors form a basis called the *eigenbasis*.

We can decompose vectors over the eigenbasis.

$$\mathbf{x} = a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n$$

This simplifies matrix multiplication.

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{A}\left(a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n\right) \\ &= a_1 \mathbf{A}\mathbf{v}_1 + \cdots + a_n \mathbf{A}\mathbf{v}_n \\ &= a_1 \lambda_1 \mathbf{v}_1 + \cdots + a_n \lambda_n \mathbf{v}_n \end{aligned}$$

Finding eigenvectors by the Power Method

$$\mathbf{Ax} = a_1\lambda_1\mathbf{v}_1 + \cdots + a_n\lambda_n\mathbf{v}_n$$

$$\mathbf{A}^2\mathbf{x} = a_1\lambda_1^2\mathbf{v}_1 + \cdots + a_n\lambda_n^2\mathbf{v}_n$$

$$\mathbf{A}^k\mathbf{x} = a_1\lambda_1^k\mathbf{v}_1 + \cdots + a_n\lambda_n^k\mathbf{v}_n$$

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As k increases, the term with the largest λ will overtake the sum. Thus

$$\lim_{k\to\infty} \mathbf{A}^k\mathbf{x} \propto \mathbf{v}_{\mathrm{max}}$$

Applications

Eigenvectors and eigenvalues have many applications in science and engineering.

For this class, skip the sections on *Solving Systems of ODEs* and *Stability of Linear ODEs*. These topics overlap with other BIOE courses.

Please read the sections on *Positive Definite Matrices* and *Network Centrality*.

The geometry of eigenvectors

If we decompose a vector over an eigenbasis, matrix multiplication is akin to scaling each dimension by a constant (the eigenvalue).

$$
\begin{aligned}
\mathbf{x} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \\
\mathbf{A}\mathbf{x} &= \mathbf{A}(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) \\
&= \lambda_1 a_1\mathbf{v}_1 + \lambda_2 a_2\mathbf{v}_2
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The total “volume" change after multiplying by a matrix is the product of the eigenvalues.

$$\frac{\mathrm{volume}(\mathbf{Ax})}{\mathrm{volume}(\mathbf{x})} = \frac{\lambda_1 a_1 \lambda_2 a_2 \lambda_3 a_3}{a_1 a_2 a_3} = \lambda_1\lambda_2\lambda_3$$

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$$\frac{\mathrm{volume}(\mathbf{A}\mathbf{x})}{\mathrm{volume}(\mathbf{x})} = \frac{\lambda_1 a_1 \lambda_2 a_2 \lambda_3 a_3}{a_1 a_2 a_3} = \lambda_1\lambda_2\lambda_3$$

This volume change is a characteristic of the matrix called the *determinant*.

$$\det(\mathbf{A}) = \lambda_1\lambda_2\cdots\lambda_n$$

Properties of the determinant

Although we won't prove it in this course, **a matrix has an inverse if and only if the determinant of the matrix is nonzero**.

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The following statements are equivalent for a square matrix $\mathbf{A}$:

- $\mathbf{A}$ can be transformed into the identity matrix by elementary row operations.
- The system $\mathbf{Ax} = \mathbf{y}$ is solvable and has a unique solution.
- $\mathbf{A}$ is full rank.
- $\mathbf{A}^{-1}$ exists.
- $\det(\mathbf{A}) \neq 0$.

Wrapping up the Field Axioms

Using the determinant we can concisely state our last field axiom. Recall that for scalars we required a multiplicative inverse exist for any nonzero member of the field, i.e.

For all scalars $a \neq 0$ there exists a^{-1} such that $aa^{-1} = 1$.

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For vector spaces, we have the following:

For all square matrices $\mathbf{A}$ where $\det(\mathbf{A}) \neq 0$,
there exists $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.