Part I

1.

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \xrightarrow{-R_1 \to R_3} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_2 \to R_3} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_4} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of **S** is 2.

2. Let $r_4 = \alpha$ and $r_3 = \beta$, then

$$\begin{aligned} r_2 &= -r_3 + r_4 \\ &= \alpha - \beta \\ r_1 &= r_2 \\ &= \alpha - \beta \end{aligned}$$

3. Let $\alpha = 3$ and $\beta = 1$, then

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \\ r_3 &= 1 \\ r_4 &= 3 \end{aligned}$$

4. By conservation of mass, whatever flows into a junction (metabolite) must flow out. Thus r_1 is always equal to r_2 and $r_4 = r_2 + r_3$. So, if we specify any two reactions (except r_1 and r_2), we know the values of all the other reactions. This implies that the system has only two independent pieces of information, so the rank should be two.

Part II

1. (a) $f''(x) = 216x^2 + 672x + 408$

$$\begin{aligned} x = -3: &\quad f''(x) = 336 > 0 \Rightarrow \text{local minimum} \\ x = -1: &\quad f''(x) = -48 < 0 \Rightarrow \text{local maximum} \\ x = -2/3: &\quad f''(x) = 56 > 0 \Rightarrow \text{local minimum} \end{aligned}$$

   (b) This is an unconstrained optimization problem, so all extrema occur when $f'(x) = 0$. Since f' is a third-order polynomial, our local maximum and minima are the only extreme. Also, $f \to \infty$ has $x \to \infty$ or $x \to -\infty$, so there is no global maximum but potentially a global minimum. Testing our local minima:

$$\begin{aligned} f(-3) &= -179 \\ f(-2/3) &\approx -52 \end{aligned}$$

   Since $f(-3) < f(-2/3)$ we know that $x = -3$ is the global argmin and $f(-3) = -179$ is the global minimum.

2. (a) Normal form: $\hat{\mathbf{a}} \cdot \mathbf{x} = d$

$$y = (3/2)x + 2 \Rightarrow -(3/2)x + y = 2$$

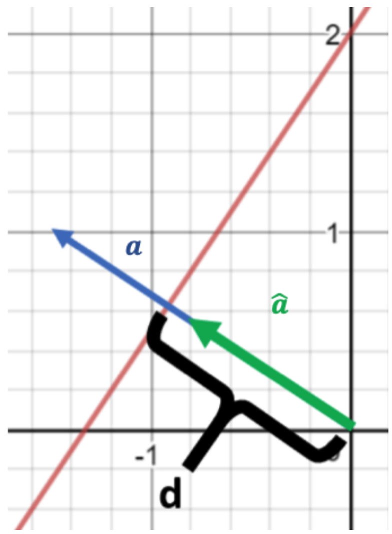
$$\mathbf{a} = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \|\mathbf{a}\| = \sqrt{13}/2 &\Rightarrow \hat{\mathbf{a}} = \frac{2}{\sqrt{13}} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \\ &\Rightarrow d = b/\|\mathbf{a}\| = 4/\sqrt{13} \end{aligned}$$

$$\frac{2}{\sqrt{13}} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \frac{4}{\sqrt{13}}$$

   (b)



(c)

$$\begin{pmatrix} -3/2 & 1 & 2 \\ -3 & -2 & -1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 9 & -1/2 \\ 0 & 1 & 5/4 \end{pmatrix}$$

The intersection is a single point at $x = -1/2$ and $y = 5/4$.

Part III

1.

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -x_2 \sin x_1 & \cos x_1 \\ 0 & 4x_2 \end{pmatrix}$$

2.

$$\mathbf{f}(\mathbf{x}_0) = \begin{pmatrix} 1 \cos 1 \\ 2(1)^2 - 1 \end{pmatrix} = \begin{pmatrix} \cos 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3.

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 - \mathbf{J}^{-1}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0) \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -\sin 1 & \cos 1 \\ 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} \cos 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1.48 \\ 0.75 \end{pmatrix} \\ \mathbf{f}(\mathbf{x}_1) &= \begin{pmatrix} 0.75 \cos 1.48 \\ 2(0.75)^2 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.0668 \\ 0.1250 \end{pmatrix} \end{aligned}$$

4.

$$\begin{aligned} \mathbf{x}_2 &= \mathbf{x}_1 - \mathbf{J}^{-1}(\mathbf{x}_1)\mathbf{f}(\mathbf{x}_1) \\ &= \begin{pmatrix} 1.48 \\ 0.75 \end{pmatrix} - \begin{pmatrix} -0.75 \sin 1.48 & \cos 1.48 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0.0668 \\ 0.1250 \end{pmatrix} \\ &= \begin{pmatrix} 1.566 \\ 0.708 \end{pmatrix} \\ \mathbf{f}(\mathbf{x}_2) &= \begin{pmatrix} 0.708 \cos 1.566 \\ 2(.708)^2 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.0034 \\ 0.0035 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
\mathbf{x}_3 &= \mathbf{x}_2 - \mathbf{J}^{-1}(\mathbf{x}_2)\mathbf{f}(\mathbf{x}_2) \\
&= \begin{pmatrix} 1.566 \\ 0.708 \end{pmatrix} - \begin{pmatrix} -0.708 \sin 1.566 & \cos 1.566 \\ 0 & 2.832 \end{pmatrix}^{-1} \begin{pmatrix} 0.0034 \\ 0.0035 \end{pmatrix} \\
&= \begin{pmatrix} 1.571 \\ 0.707 \end{pmatrix} \\
\mathbf{f}(\mathbf{x}_3) &= \begin{pmatrix} 0.707 \cos 1.571 \\ 2(0.707)^2 - 1 \end{pmatrix} \\
&= \begin{pmatrix} 5.76 \times 10^{-6} \\ 3.00 \times 10^{-6} \end{pmatrix}
\end{aligned}
$$

We are converging to a root since the norms of the values $\mathbf{f}(\mathbf{x})$ are getting closer to zero: $\|\mathbf{f}(\mathbf{x}_1)\| > \|\mathbf{f}(\mathbf{x}_2)\| > \|\mathbf{f}(\mathbf{x}_3)\|$.