Show that $f(0) = 0$ for any linear system.

For any linear system

$$f(kx) = kf(x)$$

$$f(\underbrace{k\,0}) = kf(0)$$

$$f(0) = kf(0)$$

$$f(0) = 0$$

What is the angle between the vectors

$$\underline{x} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad \underline{y} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{x} \cdot \underline{y} = \|\underline{x}\| \, \|\underline{y}\| \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \, \|\underline{y}\|} \right)$$

$$\|\underline{x}\| = \sqrt{3^2 + 0^2 + (-1)^2} = \sqrt{10}$$

$$\|\underline{y}\| = \sqrt{2^2 - 1^2 + 1^2} = \sqrt{6}$$

$$\underline{x} \cdot \underline{y} = 3 \times 2 + 0 \times (-1) - 1 \times 1 = 5$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{10} \cdot \sqrt{6}} \right) = 0.87 \simeq 50^\circ$$

Is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ a unit vector?

$$\left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2+0^2+1^2} = \sqrt{2} \qquad \text{NO.}$$

Normalize it:

$$\begin{pmatrix} 1/\sqrt{2} \\ 0/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\left\| \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\| = \sqrt{\left(1/\sqrt{2}\right)^2+0^2+\left(1/\sqrt{2}\right)^2} = \sqrt{1/2+1/2} = 1$$

Is $f(x) = \frac{dx}{dt}$ linear?

$$f(k_1 x_1 + k_2 x_2) = \frac{d}{dt}\left(k_1 x_1 + k_2 x_2\right)$$

$$= \frac{d}{dt}(k_1 x_1) + \frac{d}{dt}(k_2 x_2)$$

$$= k_1 \frac{dx_1}{dt} + k_2 \frac{dx_2}{dt}$$

$$= k_1 f(x_1) + k_2 f(x_2)$$

yes; $\frac{dx}{dt}$ is linear

What is the angle between

$$\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 4 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 8 \\ 0 \\ -6 \\ 0 \\ 12 \end{pmatrix} \qquad \theta = \cos^{-1}\left(\frac{\underline{x} \cdot \underline{y}}{\|x\| \, \|y\|}\right)$$

$$\underline{x} \cdot \underline{y} = 1 \times 0 + 0 \times 8 + 3 \times 0 + 0 \times -6 + 4 \times 0 + 0 \times 12$$

$$= 0$$

$\Rightarrow$ $\underline{x}$ + $\underline{y}$ are orthogonal.

$\Rightarrow \theta = 90^\circ$

What is $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$

$$= 3 \times 2 + 1 \times 2 - 2 \times 4 = 0$$

$\therefore$ these vectors are orthogonal.

$$\underline{A} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \quad \underline{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \underline{C} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$(\underline{A}\,\underline{B})\underline{C} = \underline{A}(\underline{B}\,\underline{C})$$

$$\underline{A}\,\underline{B} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}$$

$$\underline{B}\,\underline{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

$$(\underline{A}\,\underline{B})\underline{C} = \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 6 & 0 \end{pmatrix}$$

$$\underline{A}(\underline{B}\,\underline{C}) = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 6 & 0 \end{pmatrix}$$

$\underline{A} \in \mathbb{R}^{3\times4}$ $\quad \underline{B} \in \mathbb{R}^{4\times3}$ $\quad \underline{C} \in \mathbb{R}^{3\times3}$

What permutations of $\underline{A}\,\underline{B}\,\underline{C}$ are conformable?
What are the resulting dimensions?

$\underline{A}\,\underline{B}\,\underline{C}$	$(3\times4)(4\times3)(3\times3)$	3×3
$\underline{A}\,\underline{C}\,\underline{B}$	$(3\times4)(3\times3)(4\times3)$	$\times$
$\underline{B}\,\underline{A}\,\underline{C}$	$(4\times3)(3\times4)(3\times3)$	$\times$
$\underline{B}\,\underline{C}\,\underline{A}$	$(4\times3)(3\times3)(3\times4)$	4×4
$\underline{C}\,\underline{A}\,\underline{B}$	$(3\times3)(3\times4)(4\times3)$	3×3
$\underline{C}\,\underline{B}\,\underline{A}$	$(3\times3)(4\times3)(3\times4)$	$\times$

Show that $(\underline{A}\,\underline{B})^T = \underline{B}^T \underline{A}^T$

$\underline{A} \in \mathbb{R}^{m \times n}$, $\underline{B} \in \mathbb{R}^{n \times p}$ $\Rightarrow$ $\underline{A}\,\underline{B} \in \mathbb{R}^{m \times p} \Rightarrow (\underline{A}\,\underline{B})^T \in \mathbb{R}^{p \times m}$

$\underline{B}^T \in \mathbb{R}^{p \times n}$, $\underline{A}^T \in \mathbb{R}^{n \times m}$ $\Rightarrow$ $\underline{B}^T \underline{A}^T \in \mathbb{R}^{p \times m}$

$(\underline{A}\,\underline{B})^T$ for $\underline{A}\,\underline{B}$, $c_{ij} = \underline{A}(i,:) \cdot \underline{B}(:,j)$

$(\underline{A}\,\underline{B})^T$, $c_{ij} = \underline{A}(j,:) \cdot \underline{B}(:,i)$

$\underline{B}^T \underline{A}^T$, $c_{ij} = \underline{B}^T(i,:) \cdot \underline{A}^T(:,j)$

$= \underline{B}(:,i) \cdot \underline{A}(j,:) = \underline{A}(j,:) \cdot \underline{B}(:,i)$

Is $\underline{A}^T\underline{A}$ always conformable?

$$\underline{A} \in \mathbb{R}^{m \times n} \quad , \quad \underline{A}^T \in \mathbb{R}^{n \times m}$$

$$\underset{(n \times m)(m \times n)}{\underline{A}^T \underline{A}} = \underline{C} \in \mathbb{R}^{n \times n}$$

How about $\underline{A}\underline{A}^T$?

$$\underset{(m \times n)(n \times m)}{\underline{A}\,\underline{A}^T} = \underline{C}' \in \mathbb{R}^{m \times m}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{Q} = \begin{pmatrix} q_{11} & q_{12} \\ 0 & q_{22} \end{pmatrix}$$

What is $\underline{x}^T \underline{Q}\, \underline{x}$?

$$\underline{x}^T \underline{Q}\, \underline{x}, \quad \underline{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ 0 & q_{22} & q_{23} \\ 0 & 0 & q_{33} \end{pmatrix}$$

$$= q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{12}x_1x_2 + q_{13}x_1x_3 + q_{23}x_2x_3$$

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ 0 & q_{22} \end{pmatrix} = \begin{pmatrix} q_{11}x_1 & q_{12}x_1 + q_{22}x_2 \end{pmatrix}$$

$$\underbrace{(1\times 2)(2\times 2)}_{1\times 2}$$

$$(\underline{x}^T \underline{Q})\underline{x} = \begin{pmatrix} q_{11}x_1 & q_{12}x_1 + q_{22}x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= q_{11}x_1^2 + q_{12}x_1x_2 + q_{22}x_2^2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 3 & 4 & 8 \\ 9 & 3 & 4 & 7 \end{pmatrix}$$

$$\xrightarrow{R_2 - 4R_1} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 9 & 3 & 4 & 7 \end{pmatrix}$$

$$\xrightarrow{R_3 - 9R_1} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & -5 & -20 \end{pmatrix}$$

$$\xrightarrow{-R_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & -6 & -5 & -20 \end{pmatrix}$$

$$\xrightarrow{R_3 + 6R_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -5 & 4 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5} R_3} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$\boxed{\begin{aligned} x_1 &= -1/5 \\ x_2 &= 4 \\ x_3 &= -4/5 \end{aligned}}$$

$$\underline{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\underline{I}}_{1 \leftrightarrow 2}$$

$$\underline{\underline{I}}_{1 \leftrightarrow 2}\,\underline{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\underline{\underline{I}}_{2R_2}\,\underline{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\underline{A}\,\underline{\underline{I}}_{1 \leftrightarrow 2} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{pmatrix}$$

Is it Linear?

$$\frac{du}{dx} + \sin(3x)u = 0 \qquad \text{in } u \qquad \text{yes}$$

$$\underline{\underline{u}} + \sin(3x)\underline{\underline{u}} = 0$$

$$\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x \partial t} + \sin t \qquad \text{in } T \qquad \text{yes}$$

✓ ✓ ✓ ✓

$$y\frac{d^2y}{dx} = 4 \qquad \text{in } y \qquad \text{No}$$

$$\underline{\underline{y\,y = 4}}$$

Dirichlet B.C.: $u(0)=\alpha, \ u(1)=\beta$

Neumann B.C.: $u'(0)=\alpha, \ u'(1)=\beta$

Mixed:

$$\frac{du}{dx}+3u=0 \ , \quad u'(0)=1$$

Solve on $[0,1]$
4 nodes

Backward F.D.

$x=$	0	$1/3$	$2/3$	1
node	0	1	2	3

$\Delta x = 1/3$

@ $x=1/3$ (node 1)

$$\frac{u^{(1)}-u^{(0)}}{1/3}+3u^{(1)}=0$$

@ $x=2/3$ (node 2)

$$\frac{u^{(2)}-u^{(1)}}{1/3}+3u^{(2)}=0$$

@ $x=1$ (node 3)

$$\frac{u^{(3)}-u^{(2)}}{1/3}+3u^{(3)}=0$$

@ $x=0$ (node 0)

$$\frac{u^{(1)}-u^{(0)}}{1/3}=1 \quad \text{from. B.C.}$$

$$\underline{A} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}, \quad \underline{A}^{-1}?$$

$$(\underline{A} \;\; \underline{\underline{I}}) \longrightarrow (\underline{\underline{I}} \;\; \underline{A}^{-1})$$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \underline{A}^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\left(\underline{A}^{-1}\right)^{-1} = \underline{A}$$

$\underline{A} = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}$; find $\underline{A}^{-1}$ using a product of elementary matrices.

$$\underline{A} = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_1} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \qquad \underline{E}_{-\frac{1}{2}R_1} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{E}_{R_1+2R_2} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\underline{A}^{-1} = \underline{E}_{R_1+2R_2}\underline{E}_{-\frac{1}{2}R_1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 2 \\ 0 & 1 \end{pmatrix}$$

$$\underline{A}\,\underline{A}^{-1} = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$$

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$$\xrightarrow{-\frac{1}{5} R_3} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

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$$\underline{\underline{I}}_{1\leftrightarrow 2}\,\underline{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\underline{\underline{I}}_{2R_2}\,\underline{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\underline{A}\,\underline{\underline{I}}_{1\leftrightarrow 2} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{pmatrix}$$