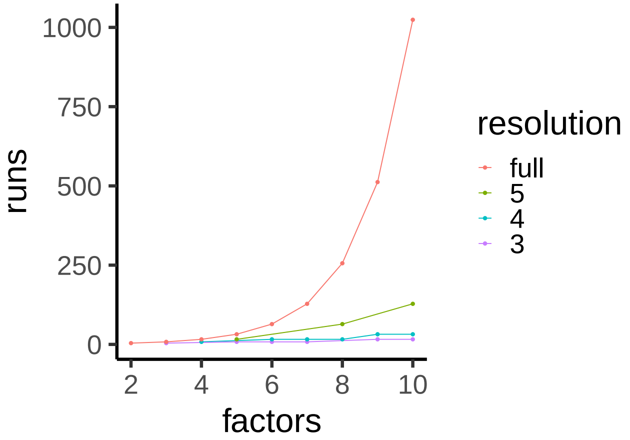
Alternative Fractional Factorial Designs

BIOE 498/598 PJ

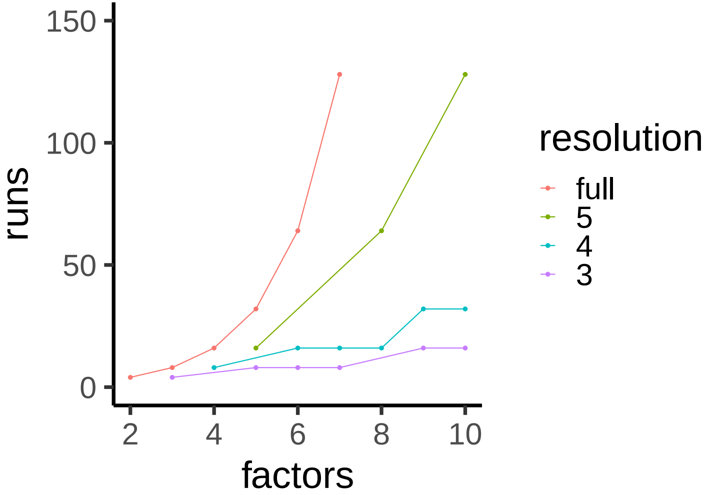
Spring 2021

How low can we go?

The efficiency of fractional factorial designs offsets the exponential increase in runs for factorial designs.



How low can we go? (zoomed in)



number of runs

number of factors	8	16	32	64	128	256	512	1024	2048	4096
						only the MA design				
3	full									
4	IV	full								
5	III	V	full							
6	III	IV	VI	full						
7	III	IV	IV	VII	full					
8		IV	IV	V	VIII	full				
9		III	IV	IV	VI	IX	full			
10		III	IV	IV	V	VI	X	full		
11		III	IV	IV	V	VI	VII	XI	full	
12		III	IV	IV	IV	VI	VI	VIII	XII	full
13		III	IV	IV	IV	V	VI	VII	VIII	XIII
14		III	IV	IV	IV	V	VI	VII	VIII	IX
15		III	IV	IV	IV	V	VI	VII	VIII	VIII
16			IV	IV	IV	V	VI	VI	VIII	VIII
17			III	IV	IV	V	VI	VI	VII	VIII
18			III	IV	IV	IV	VI	VI	VII	VIII
19			III	IV	IV	IV	V	VI	VII	VIII
20			III	IV	IV	IV	V	VI	VII	VIII
21			III	IV	IV	IV	V	VI	VII	VIII
22			III	IV	IV	IV	V	VI	VII	VIII
23			III	IV	IV	IV	V	VI	VII	VIII
24			III	IV	IV	IV	IV	VI	VI	VIII
Resolution III up to			31	63	127			factors.		
Resolution IV up to				32	64	80	160	factors.		
Resolution V up to number of factors:								33	47	65
Resolution VI up to number of factors:								24	34	48
First design is MA up to number of factors:			31	63	127	36	29	28	32	26

Gromping, 2014

J. Stat. Software

Foldover Designs

Imagine a 2^{6-3}_{III} design with

$$D = AB, \quad E = AC, \quad F = BC$$

$$\begin{aligned} I &= ABD = ACE = BCF = DEF \\ &= BCDE = ACDF = ABEF \end{aligned}$$

After analysis, we find that both B and D are significant.

Since $D = AB$, the significance of D might be due to B and AB.

We can *augment* the design by doubling the runs *with D flipped*. This clears D and its interactions.

Run	A	B	C	D	E	F
1	$-$	$-$	$-$	$+$	$+$	$+$
2	$+$	$-$	$-$	$-$	$-$	$+$
3	$-$	$+$	$-$	$-$	$+$	$-$
4	$+$	$+$	$-$	$+$	$-$	$-$
5	$-$	$-$	$+$	$+$	$-$	$-$
6	$+$	$-$	$+$	$-$	$+$	$-$
7	$-$	$+$	$+$	$-$	$-$	$+$
8	$+$	$+$	$+$	$+$	$+$	$+$

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Run	A	B	C	D	E	F
1	−	−	−	+	+	+
2	+	−	−	−	−	+
3	−	+	−	−	+	−
4	+	+	−	+	−	−
5	−	−	+	+	−	−
6	+	−	+	−	+	−
7	−	+	+	−	−	+
8	+	+	+	+	+	+
9	−	−	−	−	+	+
10	+	−	−	+	−	+
11	−	+	−	+	+	−
12	+	+	−	−	−	−
13	−	−	+	−	−	−
14	+	−	+	+	+	−
15	−	+	+	+	−	+
16	+	+	+	−	+	+

Mirror image designs

If we combine a Resolution III design with its mirror image (all factors flipped), we have a Resolution IV design with all main effects clear.

If we add a blocking factor we can perform the experimental batches sequentially.

As with foldover designs, mirror image designs are only necessary if more than one main effect is significant.

Plackett-Burman (PB) Designs

The number of runs in a fractional factorial design is always a power of two (8, 16, 32, ...).

Plackett-Burman designs allow run sizes in multiples of four regardless of the number of factors.

PB designs have no generators or defining relation (pro & con).

Creating a PB design (up to 23 factors)

1. Start with the first run from the following table.

Runs	Factor Levels
12	$++-+++---+-$
20	$++--++++-+-+----++-$
24	$+++++-+-++--++--+-+----$

2. Cycle the factor levels by one to get run #2. Repeat for 11, 19, or 23 runs.
3. Set the final run to all low ($-$).
4. If the number of factors k is less than the number of runs, select the first k columns.

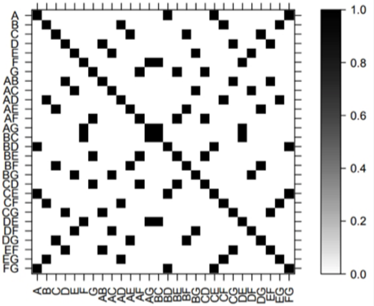
A 12-run PB design

Run	A	B	C	D	E	F	G	H	J	K	L
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

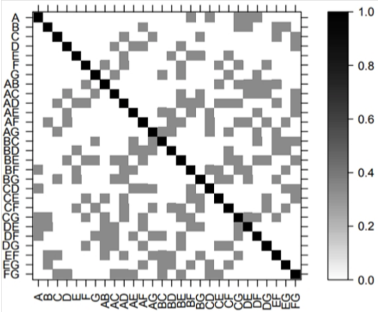
Note: We skip "I" when naming factors as this symbol is used for the intercept.

Confounding in PB designs

- Factors in **FF** designs are *confounded* (perfectly correlated).
- Factors in **PB** designs are *partially correlated* (complex aliasing).



(b) 2^{7-4}_{III} design

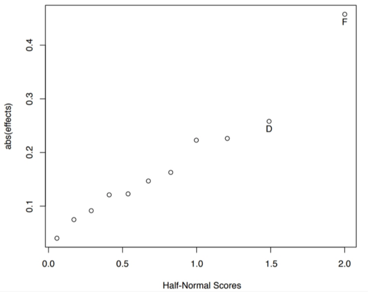


(a) Plackett-Burman Design

Example PB design: Cast fatigue

Table 6.11 *Design Matrix and Lifetime Data for Cast Fatigue Experiment*

Run	A	B	C	D	E	F	G	c8	c9	c10	c11	
1	+	−	+	+	+	−	−	−	+	−	+	4.733
2	−	+	+	+	−	−	−	+	−	+	+	4.625
3	+	+	+	−	−	−	+	−	+	+	−	5.899
4	+	+	−	−	−	+	−	+	+	−	+	7.000
5	+	−	−	−	+	−	+	+	−	+	+	5.752
6	−	−	−	+	−	+	+	−	+	+	+	5.682
7	−	−	+	−	+	+	−	+	+	+	−	6.607
8	−	+	−	+	+	−	+	+	+	−	−	5.818
9	+	−	+	+	−	+	+	+	−	−	−	5.917
10	−	+	+	−	+	+	+	−	−	−	+	5.863
11	+	+	−	+	+	+	−	+	−	+	−	6.058
12	−	−	−	−	−	−	−	−	−	−	−	4.809



This design includes 7 factors; however, effects are estimated for all columns. The last 4 “factors” are interactions with complex aliasing.

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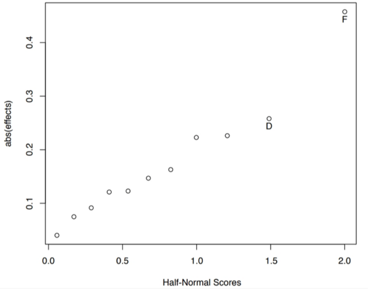

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The complex aliasing of PB designs allow us to fit models with main and TWI terms **provided the number of terms is small**. This feature is called the *hidden projection property*.

What effects should I include?

Many small models can be built from the 7 main effects and 21 TWIs.
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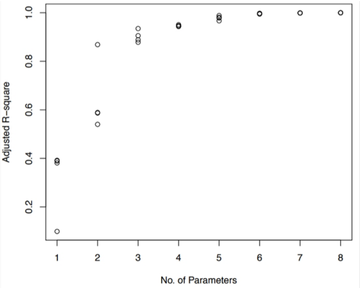
We use *subset selection* to find good models with few terms.

```
> castfr <- castf[ , c(1:7, 12)]
> library(leaps)
> modpbr<-regsubsets(y ~ (.)^2, data=castfr,
+ method="exhaustive",nvmax=4,nbest=4)
> rs <- summary(modpbr)
> plot(c(rep(1:4,each=4)), rs$adjr2, xlab="No.
+ ylab="Adjusted R-square")
> plot(modpbr,scale="r2")
```



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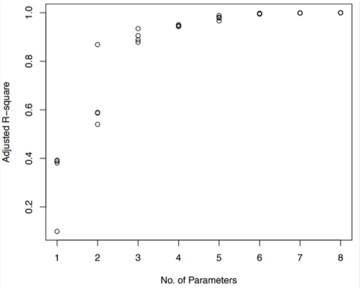


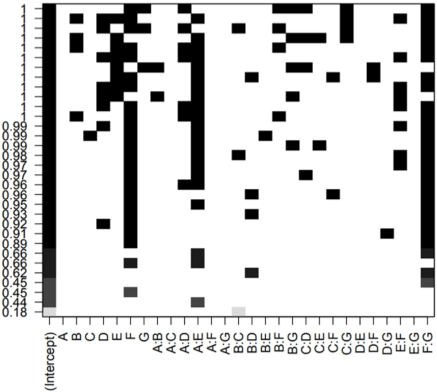
We stop adding effects when the model improvement diminishes.

Here 3 parameters is a good cutoff.

What parameters should be selected?

Number of Terms in Model	Adjusted R-Square	Variables in Model
1	0.3921	FG
1	0.3896	F
1	0.3814	AE
1	0.0993	BC
2	0.8686	F FG
2	0.5891	AE FG
2	0.5870	F AE
2	0.5403	BD FG
3	0.9348	F AE FG
3	0.9056	F BD FG
3	0.8886	D F FG
3	0.8785	F DG FG
4	0.9507	F AE EF FG
4	0.9465	F AE CD FG
4	0.9439	F AD AE FG
4	0.9438	F BD CF FG

Figure 6.15 *Plot of Model Terms by R^2*



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3	0.9056	F BD FG
3	0.8886	D F FG
3	0.8785	F DG FG
4	0.9507	F AE EF FG
4	0.9465	F AE CD FG
4	0.9439	F AD AE FG
4	0.9438	F BD CF FG

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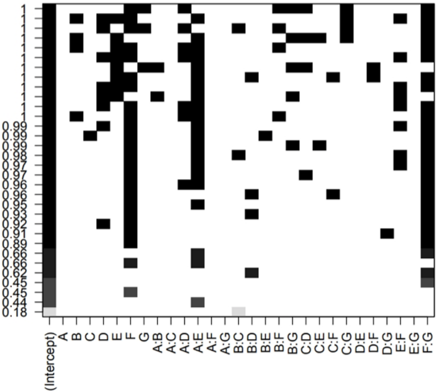

Be mindful of the *heredity effect*: A model that includes an interaction should also include the corresponding main effects.

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- Fractional factorial and PB designs use two level factors $(+/-)$.
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  - Software packages choose OA designs from catalogs.

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- One solution is Orthogonal Array Designs (OAs).
  - OAs are "hand-crafted" for mixtures of 2- and 3-level factors.
  - Software packages choose OA designs from catalogs.
- Analysis of OAs is similar to PB designs
  - Resolution III, no defining relation
  - Complex aliasing, hidden projection
  - Models with few parameters can be fit directly to the data.

(Fractional) Factorial Summary

- Fractional designs are the **most** efficient method to screen large numbers of factors.
- Factors are confounded, but the alias structure is known.
- PB designs are an alternative if
  1. a specific # of runs is needed, or
  2. you don't want a secondary experiment.
- Factors with >2 levels require OA designs.