Linear Models: Interactions

BIOE 498/598 PJ

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What is an interaction?

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What is there is another effect that depends on both x_1 and x_2? This is an **interaction** between x_1 and x_2.

How do we model interactions?

We model the interaction of x_1 and x_2 using the product of these variables.

$$y = \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

The coefficient β_{12} is the effect size of the interaction.

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Why do we multiply x_1 and x_2? There are at least two ways to interpret this term.

The coded factor interpretation

Often we set up design matrices using **coded variables**. If we're testing the variable at two levels, we code the variable as "on/off" ($\{0, 1\}$) or "low/high" ($\{-1, +1\}$).

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x_1	x_2	$x_1 x_2$
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0	1	0
1	0	0
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x_1	x_2	$x_1 x_2$
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1	0	0
1	1	1

high/low $\rightarrow$ interaction when both "high" or both "low"

x_1	x_2	$x_1 x_2$
-1	-1	$+1$
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

The augmented slope interpretation

We can also interpret the interaction as one variable changing the effect of the other variable.

$$\begin{aligned} y &= \beta_1 x_1 + \beta_2(x_1) x_2 + \epsilon \\ &= \beta_1 x_1 + (\beta_2 + \beta_{12} x_1) x_2 + \epsilon \\ &= \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon \end{aligned}$$

Interactions with `lm`

Recall the data frame from out blood pressure clinical trial:

```
## # A tibble: 6 x 3
##   BPchange treated male
##      <dbl> <lgl>   <lgl>
## 1   -0.525 TRUE    FALSE
## 2    4.17  TRUE    FALSE
## 3    6.03  TRUE    TRUE
## 4   -1.40  TRUE    FALSE
## 5    0.493 TRUE    FALSE
## 6   12.9   FALSE   TRUE
```

Adding an interaction term to our model

```
##
## Call:
## lm(formula = BPchange ~ treated + male + treated:male, data =
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -20.465  -4.407   2.309   5.887  18.738
##
## Coefficients:
##                       Estimate Std. Error t value Pr(>|t|)
## (Intercept)             12.319      7.895   1.560    0.145
## treatedTRUE            -10.090      9.116  -1.107    0.290
## maleTRUE                14.898      9.116   1.634    0.128
## treatedTRUE:maleTRUE     1.049     12.893   0.081    0.936
##
## Residual standard error: 11.17 on 12 degrees of freedom
## Multiple R-squared:  0.5606, Adjusted R-squared:  0.4507
## F-statistic: 5.102 on 3 and 12 DF,  p-value: 0.01666
```

A shortcut for adding interactions and main effects

```
##
## Call:
## lm(formula = BPchange ~ treated * male, data = bp_data)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -20.465  -4.407   2.309   5.887  18.738
##
## Coefficients:
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A shortcut for adding interactions and main effects

```
##
## Call:
## lm(formula = y ~ x1 * x2 * x3)
##
## Residuals:
##         1         2         3         4         5         6
##  0.053936  0.121901 -0.188750 -0.081015  0.261358  0.292505 -
##         8         9        10
## -0.326646 -0.078732 -0.052560
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   -4.729     30.580  -0.155    0.891
## x1            17.778     95.430   0.186    0.869
## x2             9.121     46.268   0.197    0.862
## x3             6.258     34.767   0.180    0.874
## x1:x2        -28.809    143.813  -0.200    0.860
## x1:x3        -21.140    110.656  -0.191    0.866
## x2:x3        -11.096     53.031  -0.209    0.854
## x1:x2:x3      32.847    163.304   0.201    0.859
##
```

How many interactions are there?

term	x_1	x_2	x_3
β_0	0	0	0
$\beta_1 x_1$	1	0	0
$\beta_2 x_2$	0	1	0
$\beta_3 x_3$	0	0	1
$\beta_{12} x_1 x_2$	1	1	0
$\beta_{13} x_1 x_3$	1	0	1
$\beta_{23} x_2 x_3$	0	1	1
$\beta_{123} x_1 x_2 x_3$	1	1	1

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$\beta_{123} x_1 x_2 x_3$	1	1	1

A model with n factors has 2^n possible terms; $2^n - n - 1$ of these are interactions.

Hierarchical ordering to the rescue

Hierarchical Ordering principle

- Lower order effects are more likely to be important than higher order effects.
- Effects of the same order are equally likely to be important.

How many interactions are there?

n	intercept	main effects	TWI	higher-order
1	1	1	0	0
2	1	2	1	0
3	1	3	3	1
4	1	4	6	5
5	1	5	10	16
6	1	6	15	42
7	1	7	21	99
8	1	8	28	219
9	1	9	36	466
10	1	10	45	968

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7	1	7	21	99
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10	1	10	45	968

We will design experiments that focus on main effects and two-way interactions.

Hierarchical ordering to the rescue

Hierarchical Ordering Principle

- Lower-order effects are more likely to be important than higher-order effects.
- Effects of the same order are equally likely to be important.

If we neglect an important higher-order term, the effects can appear anywhere in our model!

We can design the experiment to constrain where higher-order effects appear.

Things to remember about interactions

- Interaction are modeled as the product of variables.
- The interaction effect is “above and beyond" the independent effects (synergy/super-additivity, antagonism/sub-additivity).
- Higher-order interactions are possible (e.g. $x_1 x_2 x_3$), but these are rare.
- Proper experiment design is needed when "ignoring" higher-order interactions.