Gravitation without a Principle of Equivalence

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THE previous article has considered the observational and experimental facts and has concluded that there is no substantial evidence to support the belief that the coupling constants of the weak interactions are independent of time or place. Consequently, it is possible that the principle of equivalence may be satisfied, if at all, only when the contributions to the binding energy of a system having their origin in the weak interactions are neglected. This article considers a form which a theory of gravitation may take when the principle of equivalence is satisfied in a weakened form only.

Jordan has previously considered a similar problem,* and Fierz has made a critical analysis of Jordan's theory.†

The great difficulty with constructing a theory of gravitation is the paucity of experimental evidence. After 40 years there are still only the four famous observational checks of the theory of relativity. Of these only two have any real accuracy. With so few experimental facts to guide one, any number of *ad hoc* theories can be constructed. To choose between them, standards going beyond the observational evidence must be introduced. The danger of judging a theory on the basis of elegance, simplicity, or perfection is obvious.

While "elegance" may not be a valid criterion for judging a theory, there are a few rules for the construction of a formalism which if followed should improve the prospects for later agreement with observation. First, it should be noted that there is much experimental evidence on the validity of the Lorentz invariance of the strong interactions and a theory of gravity should reduce to the usual Lorentz invariant theory in the absence of the gravitational field. Second, the theory should introduce a minimum of new elements. Third, on the assumption that nature is basically simple, the simplest of several alternatives should be chosen. The theory to be described accepts Mach's principle, the cosmological principle, and is generally covariant. Also as much of the principle of equivalence as is supported by the Eötvös experiment is accepted.

The general features of a theory of gravitation without a principle of equivalence are easily outlined. The motivation for introducing a Riemannian metric into the geometry of space and time is now largely absent, as there is no single universal gravitational acceleration at a given space-time point. Simply by redefining units of length and time as functions of space-time coordinates, the curvatures of a space are modified. With a proper redefinition of units making them dependent upon coordinates and orientation of an infinitesimal interval a curved space can be converted into a flat one and vice versa. Rosen[1] has shown how to formulate general relativity within the framework of a flat metric.

To illustrate the arbitrary character of the choice of metric tensor, consider the following physical example. Twelve identical rods can be normally assembled into a hexagonal pattern with 6 rods joining at the center. If this assembly is carried out in a suitable gravitational field, the 6 rods no longer join at the center. There are at least two geometrical explanations for this result. The conventional one is that the rods have not changed but are now in a curved space which "causes" a gap to open at the center of the geometrical figure. Another possible explanation is that the gravitational field has shortened the radial rods relative to the circumferential ones. The two explanations are equivalent in the sense that they both agree as to the existence of the gap in the geometrical figure constructed out of real atoms.

It has been argued[2] that space is "really" curved and that the rods do not change their "real" length. Without splitting hairs over the meaning of the word "really" this argument is based on the assumption that such changes are presumably independent of the material out of which the rods are constructed. However, all rods are constructed from electrons, protons, and neutrons held together almost completely by the strong interactions. They have a common structure and could vary in length in a common way.

Two theories which differ only in the definition of the units employed are equivalent. Nonetheless, there are advantages to be obtained from the use of a flat metric. In the conventional theory the Riemannian metric is used to transform the gravitational force away. In a flat metric the gravitational effects are to be regarded as associated with a force field just as electromagnetic or meson forces are related to a field. Because of this common basis it becomes possible to make use of analogy in constructing a theory of gravity.

By analogy with electromagnetic or meson force fields it is reasonable to expect that when viewed sufficiently closely the gravitational effects would be quantum in nature. Stated more exactly, it might be expected that the gravitational force acting on an elementary particle would have its origin in a local interaction of the particle with virtual particles present in the vacuum.

* P. Jordan, *Schwerkraft und Weltall* (Vieweg, Braunschweig, 1955), second edition.
† M. Fierz, Helv. Phys. Acta **29**, 128 (1956).

[1] N. Rosen, Phys. Rev. **57**, 147 (1940).
[2] H. P. Robertson, *Albert Einstein*, edited by Paul A. Schilpp (Tudor, New York, 1951).

Remembering the virtual electron-positron pairs reputed to be present in the vacuum as a result of zero point fluctuations, it becomes interesting to inquire whether the gravitational effect can be linked to these particles already present. If so, it should be possible eventually to construct a theory of particles and obtain the gravitational interaction as a weak effect connected with more primitive strong interactions. A less ambitious approach would be to start in the middle of the problem, to ignore the quantum aspects of the interaction of a particle with a bath of virtual particles, and to treat this interaction as a classical field.

The most striking effect of the presence of virtual pairs in the vacuum is the polarizability of the vacuum. This property suffers from divergence difficulties which are usually ameliorated by "renormalization." By defining the velocity of light in empty space as c and "renormalizing," the vacuum polarization effects are made to disappear for a weak electromagnetic wave in free space whereas they still contribute to the space charge about a charged particle. This, however, is arbitrary. The velocity of light in a "bare" space could be greatly different from c or even meaningless.

With a "cutoff" theory there are no divergences and the vacuum polarization can be treated as physically meaningful. It may be significant that a wavelength cut off at the gravitational radius of a particle would not be detectable in any presently known experiment. It should be noted that with such a cutoff theory, the velocity of light in "bare" space is not c. It is c only after including vacuum polarization effects.

It is helpful to remember that before Lorentz no distinction was made between the polarization of a vacuum and the polarization of a dielectric medium. Lorentz first clearly saw that the $\dot{D}$ term in Maxwell's first equation should, on physical grounds, be separated into two parts with the $4\pi\dot{P}$ term representing polarization currents in the dielectric medium. These polarization currents were to be treated like all other currents in computing the magnetic field. The $\dot{E}$ term, on the other hand, was commonly thought to be physically quite different. In the early 20th Century, the vacuum by definition contained no charges and currents. In recent years vacuum polarization effects have been recognized as existing and having physical importance. It would appear to be reasonable to assume that the $\dot{E}$ term represents the flow of virtual charged particles in the vacuum and that the whole of $\dot{D}$ is to be interpreted as a "displacement current."

With the neglect of quantum effects the polarizability of the vacuum can be described by classical field quantities ϵ and μ.

From this point of view the vacuum is to be treated like a dielectric medium. In free space far from particles, it is a medium without dispersion as there is no mechanism which would lead to the absorption of a single photon and the production of a pair.

If the vacuum is thus considered to have a structure, to be a dielectric medium which can be polarized by electromagnetic fields, it becomes important to inquire about the effect of motion relative to such a medium. Could such a motion be measurable by electromagnetic means? The motion of an ordinary dielectric medium leads to measurable effects (Fizeau). That such motional effects should be absent for the vacuum is reasonable on the basis of the following physical argument.

Consider an electron moving through a normal dielectric medium such as helium gas. The electric field about the electron can be expanded as a Fourier integral in time. Up to a frequency of about 10^{15} cycles/sec the helium is equivalent to a moderately continuous medium of constant dielectric constant ϵ. There is a Lorentz contraction in this part of the electric field by a Lorentz factor computed by taking the velocity of light to be the wave velocity in the medium. On the other hand, the parts of the field associated with frequencies which are very much higher (above the strong absorption frequencies) are not greatly affected by the helium gas and the field is contracted by a factor computed for the velocity in free space. The energy stored in the low-frequency part of the electromagnetic field contributes in an anomalous way to the effective mass of the electron but this contribution is small compared with the total mass.

Consider the situation when there is no dispersion in the medium. All frequency components are contracted by a Lorentz factor computed with the velocity of light equal to the wave velocity in the medium. It would be reasonable to expect that the theory would be Lorentz covariant using the wave velocity in the medium as the velocity of light. It should not be possible to use an electromagnetic effect to determine a velocity relative to such a nondispersive medium.

GRAVITATION AN ELECTROMAGNETIC EFFECT?

The fact that many of the properties of gravitation can be accounted for in terms of an interaction with a polarizable medium is an old idea which has recurred from time to time.[3] The physical idea is simply that a space variation in the polarizabilities of the vacuum will lead to a number of results familiar as typical gravitation effects. For example, an increase of the index of refraction of the vacuum in the vicinity of the sun will cause a bending of light toward the sun.

The gravitational force on a charged particle is interpreted as resulting from a change in its electromagnetic self-energy with position as a result of a variation in the polarizability of the vacuum. A gradient in the polarizability results in a force acting on the charged particle. This force results in part from the polarization charges induced in the vacuum by the charged particle. For a medium having a nonvanishing

[3] H. A. Wilson, Phys. Rev. 17, 54 (1921) introduced many of the physical concepts employed in this paper.

gradient in its polarizabilities there is more induced charge on one side of the particle than on the other and the electrostatic interaction with the induced charges leads to a force acting on the particle in the direction of increasing gradient.

For a complex system such as a hydrogen atom there is an added force having its origin in a variation of the binding energy of the atom with position. The neutron is to be regarded as a compound system containing charged particles.

As a result of the change of binding energy with position it would be expected that the energy or frequency of a photon emitted by the atom would depend upon the location of the atom in the field. This could conceivably lead to the phenomenon of the gravitational red shift.

Also, it would be expected that as a result of the variation in the dielectric constant of the vacuum the Bohr radius and other atomic lengths would be a function of position. This would lead to a shortening and bending of meter sticks depending upon their location. If such meter sticks are defined as unchanged, the resulting metric of space-like surfaces is Riemannian.

Thus there is a possibility of accounting for all the observed gravitational effects within the physical framework outlined above. That gravitation should be electromagnetic in origin is not unreasonable. In the previous paper it was shown that the strength of the gravitational interaction appears to be related to the strength of the electrical interaction and to the size of the universe.

In order to find a functional relation between the dielectric constant ϵ and μ, the permeability of the vacuum, we consider quantitatively the effect of a polarizable medium on the hydrogen atom. The binding energy can be written as

$$\tfrac{1}{2}M_e\alpha^2, \tag{1}$$

where M_e is the electron's rest energy and

$$\alpha=\frac{e^2}{\epsilon\hbar c} \tag{2}$$

is the fine structure constant. e is the "true" charge of the electron and assuming the validity of Maxwell's equations (charge conservation), it is a constant. The dielectric constant ϵ is present in (2) because of the effect of the dielectric medium in reducing the strength of the electrostatic interaction between the electron and proton. $\hbar$ is also a constant if angular momentum is to be conserved. This can be seen by considering a circularly polarized photon carrying an angular momentum $\hbar$. As it propagates through space $\hbar$ remains unchanged if angular momentum is to be conserved.

The velocity of light in a polarizable medium can be written with a suitable choice of units as

$$c=(\epsilon\mu)^{-\frac{1}{2}}, \tag{3}$$

where μ is the permeability of the medium. In the previous paper it was shown that there are experimental and observational reasons for believing that the fine structure constant (2) varies, if at all, only slowly with position and time. Hence, we shall assume that α is a constant. In view of the fact that we are attempting to associate the gravitational field with a gradient in ϵ, from (2) and (3), α can be constant only if

$$\epsilon=\mu. \tag{4}$$

The variation in index of refraction about the sun that is required to obtain a deflection of light of the amount expected and observed (roughly) may be calculated using Maxwell's equations. The result is

$$\epsilon=\mu\cong 1+\frac{2GM}{r}, \tag{5}$$

where G is the gravitational constant based on an energy measure of mass, M is the sun's mass in energy units, and r is the distance from the sun. The second term on the right-hand side of (5) is clearly associated with the presence of the sun.

What about the first term? Does it have its origin in the remainder of the matter in the universe? To investigate this possibility we form the integral

$$2G\int_0^R \frac{4\pi\rho r^2}{r}dr=4\pi G\rho R^2, \tag{6}$$

where ρ is the density of matter in the universe and R is the Hubble radius. Inserting the appropriate value for G and assuming $R=5.4\times10^{27}$ cm, $\rho/c^2=4\times10^{-29}$ g/cm^3, (6) is equal to unity. This density of matter is compatible with astronomical observations which give a value of $\rho/c^2=10^{-30}$ g/cm^3 for galactic matter only.

From the standpoint of Mach's principle this is a highly satisfactory result. In this interpretation, the polarizability of the vacuum at any point depends upon the distribution of the distant galactic matter. As the inertial properties of matter are associated with these polarizabilities, the inertial properties are, as suggested by Mach, determined by the remainder of the matter of the universe.

COORDINATES AND UNITS

Of necessity the preceding discussion has been somewhat inexact, with the physical quantities not carefully defined. By basing the discussion upon a flat space-time, the question of how physical units are to be defined has been left somewhat vague. Atomic units of length and time are no longer suitable local standards as these lead to a Riemannian metric. While it is conceivable that units of length and time might be introduced simply by requiring that the resulting space be flat, the resulting definition would be generally not unique. For example, a stereographic projection may be used to map a Cartesian coordinate system of a plane upon a sphere and with a suitable definition of a local unit of length the metric of the sphere (with the singular point excluded) is flat, but the choice of the singular point on the sphere is arbitrary and the resulting definition of the unit of length is not unique.

What is needed is a prescription for correcting local atomic units of length, time, etc., so that they give a "true" measure. Most of the problem is already solved. The assumptions that e and $\hbar$ are constant means that these two quantities can serve as "true" local units. However, a third unit is needed. In view of the assumption that ϵ affects atomic lengths and times, what is needed is a local measure of ϵ. It will appear as a derived result of the theory that ϵ is in principle locally measurable by determining locally the ratio of electrical to gravitational forces. Thus, a flat metric is obtained if local units of length and time are assumed to be given by local atomic units corrected by a function of ϵ.

The system of units used with a flat metric will be called "Newtonian." Local atomic units will be called "atomic" or sometimes "proper." As explained later a characteristic atomic length expressed in Newtonian units varies with ϵ as

$$L=L_0/\epsilon^{\frac{1}{2}}, \tag{7}$$

also a characteristic atomic frequency varies as

$$\omega=\omega_0/\epsilon^{\frac{1}{2}}, \tag{8}$$

with L_0 and ω_0 constants.

In addition to the problem of defining local measures of physical quantities it is necessary to discuss coordinate systems. In view of the flat space-time it is always possible to choose Cartesian systems of coordinates. Furthermore with the assumption of the Cosmological principle it is possible to define for every point a unique time direction (cosmic time). It is assumed that this time direction is everywhere parallel to the time axis of a particular Cartesian coordinate system. This Cartesian coordinate system, which for obvious reasons will be called "Newtonian," can be characterized by saying that along the world line of any fixed position point the universe appears uniform.

Because of this high degree of symmetry, the vacuum should appear isotropic in this coordinate system and the vacuum polarizabilities should be scalars. Local inhomogeneities would destroy this symmetry and it is conceivable that the polarizability about a body such as the sun would be a tensor quantity. On the other hand, the simplest assumption to make is that the polarizability is a scalar even in the presence of local irregularities in the mass distribution, and this assumption will be made here.

Although a special coordinate system has been introduced, it has been done within the framework of Mach's principle. Namely this special coordinate system is determined by the distribution of matter in the universe. Furthermore, the equations of motion can be written in a generally covariant manner. The equations do, however, take on a particularly simple physically understandable form for the Newtonian coordinate system.

Units are so chosen that the metric tensor of the Newtonian coordinate system with Newtonian units is γ_{ij}

$$\gamma_{11}=\gamma_{22}=\gamma_{33}=-1,\quad \gamma_{44}=1,\quad \gamma_{ij}=0,\quad i\neq j. \tag{9}$$

Although the physical significance of the equations is more transparent when expressed in Newtonian units it is always possible to choose proper or local atomic units if desired. In this case assuming for the moment the validity of (7) and (8) the metric tensor for this particular coordinate frame is g_{ij}

$$g_{11}=g_{22}=g_{33}=-\epsilon,\quad g_{44}=1/\epsilon,\quad g_{ij}=0,\quad i\neq j. \tag{10}$$

Whether g_{ij} is taken to be an ordinary tensor or the metric tensor, the infinitesimal invariant

$$ds^2=g_{ij}dx^i dx^j, \tag{11}$$

is a useful quantity representing an infinitesimal interval measured in local atomic units.

POSTULATE OF LOCAL LORENTZ COVARIANCE

Before proceeding further we shall summarize the assumptions which are being made.

1. The theory is based upon a flat metric.
2. Mach's principle is assumed.
3. The Cosmological principle is assumed.
4. A special Cartesian coordinate system ("Newtonian") determined by the distribution of matter in the universe is introduced such that the universe as a whole appears isotropic from any fixed position point. While generally covariant equations can and will be written, the Newtonian coordinate system because of the cosmological principle has a special physical significance.
5. The vacuum is polarizable and Maxwell's equations for a polarizable medium are valid in the Newtonian coordinate system. The electric and magnetic specific inductive capacities of the vacuum ϵ, μ are scalars.
6. A variation in ϵ, μ affect the self-energy of a particle. Characteristic atomic lengths, frequencies and energies are affected by a variation in ϵ and μ.
7. e and $\hbar$ are constant. (Conservation of charge and angular momentum.)
8. Neglecting the small contribution to the binding energy of a complex system having an origin in the weak interactions, the self-energy of a particle, whether elementary or complex, varies with ϵ as

$$M=M_0\epsilon^{-\frac{1}{2}},\quad M_0=\text{constant}.$$

It is shown later that assumption 8 can be obtained from a limited principle of equivalence. Also the fact that all atoms obey the same functional relation is justified directly by the Eötvös experiment (see the preceding paper).

From the above assumptions several important results are obtained.

9. The fine structure constant

$$\alpha=\frac{e^2}{\epsilon\hbar c}$$

is constant (independent of ϵ and μ). Also all other development parameters of strong interactions are constant. This follows if both the self-energies of the elementary particles and complex atoms are to obey No. 8.

10. From 9 it follows that $\epsilon=\mu$.

11. From 8, 9, and 10 it follows that all characteristic atomic lengths vary together as

$$L=L_0\epsilon^{-\frac{1}{2}},\quad L_0=\text{constant}.$$

[For example compute the Bohr radius as

$$a_0=\frac{e^2}{\epsilon M_e}\alpha^{-2}.]$$

12. From 8 and 7 all atomic frequencies vary as

$$\omega=\omega_0\epsilon^{-\frac{1}{2}}$$

(since $\hbar$ is constant and $M\sim\hbar\omega$).

The above list of assumptions by itself would constitute a weak scaffolding upon which to construct a theory of gravity. Presumably a large number of *ad hoc* theories could be constructed within this framework to account for the very few observational facts. However, there is a strong assumption still to be made which greatly limits the number of possibilities.

Physical arguments were given earlier to support the assumption that an atom moving relative to the Newtonian frame would experience a time dilatation and Lorentz contraction in accordance with the local velocity of light. This suggests that equations be written in a way which we define as local-Lorentz covariant. Stated exactly the Lagrangian or Lagrangian density of a field theory should be written in its usual Lorentz invariant form but with c no longer constant but equal to the reciprocal of ϵ.

Since this Lagrangian density reduces for constant ϵ to the usual Lorentz invariant density, all the observational experience concerning special relativity can be brought to bear on the problem of choosing the correct Lagrangian density.

EQUATIONS OF MOTION

Rather than consider immediately the formulation of the general equations we first limit ourselves to a simple problem: Find the motion of a test particle in a given gravitational field. The scalar ϵ is assumed known and given. Following the prescription outlined above the Lagrangian is written for the Newtonian coordinate system as a local Lorentz invariant.

$$L=-M(1-\epsilon^2v^2)^{\frac{1}{2}}=-M_0\epsilon^{-\frac{1}{2}}(1-\epsilon^2v^2)^{\frac{1}{2}}. \qquad (12)$$

Remember that the velocity of light $c=\epsilon^{-1}$. M is the rest energy of the particle. The equations of motion are obtained from the variational principle

$$\delta\int_{t_1}^{t_2}Ldt=0,\quad dt=dx^4. \qquad (13)$$

This gives for Euler equations

$$\frac{d}{dt}\left[\frac{M\epsilon^2\mathbf{v}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\right]=\tfrac{1}{2}M\left[\frac{1+\epsilon^2v^2}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\right]\frac{1}{\epsilon}\nabla\epsilon. \qquad (14)$$

The momentum of the particle is

$$\mathbf{p}=\frac{\partial L}{\partial\mathbf{v}}=\frac{M\epsilon^2\mathbf{v}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}, \qquad (15)$$

and the Hamiltonian is

$$H=\mathbf{p}\cdot\mathbf{v}-L=\frac{M}{(1-\epsilon^2v^2)^{\frac{1}{2}}}. \qquad (16)$$

It is a constant of the motion, (the energy of the particle) if ϵ is time independent. For a time dependent ϵ the rate of change of the energy of the particle is

$$\frac{dH}{dt}=-\frac{\partial L}{\partial t}=-\frac{1}{2}\frac{M}{(1-\epsilon^2v^2)^{\frac{1}{2}}}(1+\epsilon^2v^2)\frac{1}{\epsilon}\frac{\partial\epsilon}{\partial t}. \qquad (16\text{a})$$

Note that from Eq. (23) the gravitational force $\mathbf{F}$ is

$$\mathbf{F}=\frac{1}{2}\frac{M}{(1-\epsilon^2v^2)^{\frac{1}{2}}}(1+\epsilon^2v^2)\frac{1}{\epsilon}\nabla\epsilon. \qquad (17)$$

The force acting on the particle is proportional to the particle's energy (ϵ time independent), but it contains the added factor $(1+\epsilon^2v^2)$. This added factor serves to double the gravitational force for a rapidly moving particle. This leads to double the transverse gravitational acceleration for a photon and twice the Newtonian deflection of light by the sun. This result is in agreement with observations.

Although a rapidly moving particle has double the normal weight, this does not contribute anomalously to the weight of a bound system. Consider a photon confined to a box with perfectly reflecting walls. The photon is too heavy by a factor of 2. However, a small displacement of the box upward results in a change in volume of the box because of expansion of the walls. The photon gas does work on the walls as they expand and the resulting work done by the photon is just sufficient to reduce its effective weight to the normal value.

Equation (12) can be written in generally covariant form by introducing the tensor g_{ij} (not the metric tensor) defined by (10) and the infinitesimal invariant of (11). In this notation (12) becomes

$$L=-M_0\frac{ds}{dt}. \tag{18}$$

The variational principle takes on the invariant form

$$\delta\int_{s_1}^{s_2} ds=0, \tag{19}$$

since M_0 is a constant.

Equation (19) is identical with the corresponding equation from the Einstein formalism where the orbit appears as a geodesic. Thus, the only differences between the two theories concern the form of the tensor g_{ij} and not the equations of motion with a given g_{ij}.

HOMOGENEOUS FIELD EQUATION

From the physical picture which has been constructed it is reasonable to expect that the scalar field ϵ should exhibit retardation effects with gravitational waves traveling with the local velocity of light. Consequently, it is assumed that an invariant Lagrangian density for the gravitational field alone can be constructed by writing the usual Lagrangian density for a scalar field, an invariant quadratic in gradients of ϵ

$$L=\frac{1}{2k}h^{ij}\epsilon_{,i}\epsilon_{,j}. \tag{20}$$

Here h^{ij} is the reciprocal of the tensor g_{ij}, i.e.,

$$g_{ik}h^{kj}=\delta_i^j, \tag{20a}$$

and

$$\epsilon_{,i}=\frac{\partial\epsilon}{\partial x^i}, \tag{21}$$

k is a constant. This is the same form as the standard Lorentz invariant Lagrangian density of a zero mass scalar field and appears to be the simplest invariant Lagrangian density which will yield gravitational waves of the appropriate velocity. While (20) could be multiplied by an arbitrary function of the scalar ϵ without destroying its invariance, this is not done, and the physical reason for omitting such a factor is discussed later.

The gravitational wave equation for matter free space is obtained from the variational principle

$$0=\delta\int L\sqrt{-\gamma}\,d^4x. \tag{22}$$

Here as usual $\sqrt{-\gamma}\,d^4x$ is the invariant volume element.

The wave equation is obtained as the Euler equation

$$\frac{\partial}{\partial x^i}\left((-\gamma)^{\frac{1}{2}}\frac{\partial L}{\partial\epsilon_{,i}}\right)-\sqrt{-\gamma}\frac{\partial L}{\partial\epsilon}=0. \tag{23}$$

Conservation laws for the gravitational field alone are obtained by defining the energy momentum tensor $G_j{}^i$ as

$$G_j{}^i=\epsilon_{,j}\frac{\partial L}{\partial\epsilon_{,i}}-\delta_j{}^iL. \tag{24}$$

The covariant divergence (γ is the metric tensor) of the energy momentum tensor vanishes. The proof of this is simple. For the Newtonian-Cartesian coordinate system the Christoffel symbols all vanish and the covariant divergence and ordinary divergence are equal

$$G_{j,\,i}{}^i=G_{j;\,i}{}^i. \tag{25}$$

However, for this coordinate system $\sqrt{-\gamma}=1$ and making use of (23) it is found that

$$G_{j,\,i}{}^i=0. \tag{26}$$

The vanishing of the covariant divergence for this coordinate system implies its vanishing in all coordinate systems since

$$G_{j;\,i}{}^i=0 \tag{27}$$

is a tensor equation.

The vanishing of the ordinary divergence (26) for the Newtonian coordinate system can be interpreted in the usual way as representing the conservation of energy and momentum of the field. Equation (27) represents the same conservation laws expressed in general curvilinear coordinates.

In the Einstein theory the vanishing of the covariant divergence of an energy-momentum tensor does not represent a conservation relation as (26) does not exist in this formalism. In the present theory, however, the existence of at least one coordinate system for which the covariant divergence equals the ordinary divergence and vanishes is sufficient to define conservation relations which can be expressed in any arbitrary coordinate system.

The components of the tensor $G_j{}^k$ are obtained from (24) as

$$G_j{}^i=\frac{1}{k}h^{ik}\epsilon_{,k}\epsilon_{,j}-\delta_j{}^i\frac{1}{2k}h^{kl}\epsilon_{,k}\epsilon_{,l}. \tag{27a}$$

Written out for the Newtonian coordinate system the energy density (positive definite) is

$$G_4{}^4=\frac{1}{2k}\left[\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)^2+\frac{1}{\epsilon}(\nabla\epsilon)^2\right]. \tag{27b}$$

The three components of momentum density are

$$-G_\alpha{}^4=-\frac{1}{k}\epsilon\epsilon_{,4}\epsilon_{,\alpha}=-\frac{1}{k}\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)\left(\frac{\partial\epsilon}{\partial x^\alpha}\right), \tag{27c}$$

$\alpha=1, 2, 3.$

The three components of the energy flux density are

$$G_4{}^\alpha=-\frac{1}{k\epsilon}\left(\frac{\partial\epsilon}{\partial t}\right)\left(\frac{\partial\epsilon}{\partial x^\alpha}\right)=-\frac{1}{\epsilon^2}G_\alpha{}^4. \quad (27d)$$

The remaining elements of $G_j{}^i$ constitute the stress tensor of the field.

THE INHOMOGENEOUS FIELD EQUATION

By adding to the Lagrangian density (20) terms of the type of (12) a Lagrangian density representing particles, the gravitational field and their interaction is obtained. This may be written for the Newtonian frame as

$$L=-\sum_i M_i(1-\epsilon^2 v_i{}^2)^{\frac{1}{2}}\delta(\mathbf{r}-\mathbf{r}_i)+\frac{1}{2k}h^{ij}\epsilon_{,j}\epsilon_{,j}. \quad (28)$$

Here $\mathbf{r}$ represents the three position components of the field point. $\mathbf{r}_i$ refers to the position of the ith particle and $\mathbf{v}_i$ to its velocity.

The variational principle (22) gives as the Euler equation for ϵ

$$\frac{\partial}{\partial x^i}\frac{\partial L}{\partial\epsilon_{,i}}-\frac{\partial L}{\partial\epsilon}=0, \quad (29)$$

which becomes, when written out in detail for the Newtonian coordinate system,

$$\nabla^2\epsilon-\epsilon^2\frac{\partial^2\epsilon}{\partial t^2}=\frac{1}{2}\left[\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)^2+\frac{1}{\epsilon}(\nabla\epsilon)^2\right]$$
$$-\frac{k}{2}\sum_i\frac{M_i(1+\epsilon^2 v_i{}^2)}{(1-\epsilon^2 v_i{}^2)^{\frac{1}{2}}}\delta(\mathbf{r}-\mathbf{r}_i). \quad (30)$$

The first term on the right is easily seen from (27b) to be $kG_4{}^4$ and represents k times the gravitational energy density. The material particles and gravitational energy serve as sources of gravitational waves.

That a particle should be a source of a gradient of ϵ is understandable when it is recalled that a gradient of ϵ leads to a force acting on the particle. It would be expected on grounds of Newton's third law that the particle would be a source of the field which acts upon it.

MECHANICS OF CONTINUOUS MATTER

The dynamics of point particles moving under the influence of gravitational forces as described by (14) and (30) is not completely satisfactory. It is necessary to subtract the self-gravitational field of a particle before computing the force on a particle or the total gravitational energy of the field. If this is not done, divergent results are obtained. The difficulty is of essentially the same type as the self-field problem encountered in the Lorentz-Maxwell theory. There is one important difference, however; the gravitational field (30) is nonlinear and the subtraction cannot be carried out in a consistent fashion. On the other hand the nonlinearities become important only at a distance from a particle of the order of the gravitational radius.

From a formal viewpoint matter is best considered to be a continuous medium, as the self-field problem then disappears. This of course is nothing but a way of ignoring the particle-structure problem which cannot be solved in any case.

The field equations for a continuous medium moving under the influence of gravity only are conveniently written in terms of energy momentum tensors. Inasmuch as the equations of motion of a particle in a given gravitational field (g_{ij}) are identical with the equations of motion of the Einstein theory, these equations can be obtained from the latter theory in the form of the vanishing of the covariant divergence of the energy momentum tensor of matter with zero stresses. Owing to the difference in the units employed with the metric tensors γ_{ij} and g_{ij}, the matter tensor of the Einstein theory must, for the Newtonian frame, be multiplied by $\sqrt{-g}$ to obtain the energy momentum tensor for matter in the electromagnetic theory $M_i{}^k$. Hence, the Einstein equations of motion are

$$\left(\frac{1}{\sqrt{-g}}M_i{}^k\right)_{;k}=0, \quad (31)$$

where the covariant divergence is computed with g_{ij} as the metric tensor.

$$M_i{}^k=\rho g_{ij}u^j u^k, \quad (32)$$

where ρ is the matter energy density in a coordinate frame for which the matter is locally at rest.[4] u^i is the four velocity dx^i/ds. Making use of a standard formula the covariant divergence (31) can be written as

$$0=\sqrt{-g}\left(\frac{1}{\sqrt{-g}}M_i{}^k\right)_{;k}=M_{i,k}{}^k-\frac{1}{2}\frac{\partial g_{lm}}{\partial x^i}h^{ln}M_n{}^m. \quad (33)$$

Making use of (10), (11), and (20a) the last term in (42) can be evaluated for the Newtonian frame to give

$$-\frac{1}{2}\frac{\partial g_{lm}}{\partial x^i}h^{ln}M_n{}^m=\frac{1}{2}\frac{\rho}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{1+\epsilon^2v^2}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{1}{\epsilon}\epsilon_{,i}. \quad (34)$$

Because of the Lorentz contraction the matter density is increased by a Lorentz contraction factor. Comparing (34) with (17) it is apparent that (34) represents the gravitational force density for $i=1, 2, 3$. For $i=4$ it represents the negative of the rate per unit volume that work is being done on matter by gravitational forces [see (16a)].

The first term on the right of (33) represents, for the Newtonian frame, the total derivative (i.e., co-moving)

[4] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), p. 296.

with respect to time of the density of matter momentum and energy. To see this (32) is written as

$$M_i{}^k=\rho g_{ij}\frac{\epsilon v_j v_k}{(1-\epsilon^2v^2)}, \tag{35}$$

with v_α representing an ordinary velocity component $\alpha=1, 2, 3$ and $v_4=1$

$$-M_{\alpha,k}{}^k=\frac{\rho\sqrt{\epsilon}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{d}{dt}\left[\frac{\epsilon^{\frac{3}{2}}v_\alpha}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\right]+\frac{\epsilon^{\frac{3}{2}}v_\alpha}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\left[\frac{\rho\sqrt{\epsilon}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}v_k\right]_{,k}. \tag{36}$$

Also the condition of matter continuity is

$$\left[\frac{\rho\sqrt{\epsilon}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}v_k\right]_{,k}=0. \tag{36a}$$

Equation (36a) can be understood from a particle model for the continuous medium. The coefficient of v_k is proportional to the particle number density and this divergence vanishes if numbers of particles are conserved. Hence

$$-M_{\alpha,k}{}^k=\frac{\rho\sqrt{\epsilon}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{d}{dt}\left[\frac{\epsilon^{\frac{3}{2}}v_\alpha}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\right], \quad \alpha=1,2,3. \tag{37}$$

In similar fashion

$$M_{4,k}{}^k=\frac{\rho\sqrt{\epsilon}}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{d}{dt}\left[\frac{1}{\sqrt{\epsilon}(1-\epsilon^2v^2)^{\frac{1}{2}}}\right]. \tag{37a}$$

These equations represent total time derivatives of the matter densities of momentum and energy respectively.

Comparing (33), (34), (37), and (37a) with (14) and (25a) it is seen that (42) is equivalent to the particle equations (14) and (16a).

With γ_{ij} as the metric tensor and for the Newtonian coordinate system ordinary derivatives and covariant derivatives are equivalent. Hence, the covariant relation equivalent to (33) is

$$0=M_{i;k}{}^k-\tfrac{1}{2}g_{lm;i}h^{ln}M_n{}^m, \tag{38}$$

where the covariant derivatives are now defined with γ_{ij} as the metric tensor. Equation (38) constitutes 4 field equations for the 4 field quantities $\mathbf{v}$, ρ.

For a continuous medium the field equation, (30), for ϵ becomes for the Newtonian coordinate system

$$\nabla^2\epsilon-\epsilon^2\frac{\partial^2\epsilon}{\partial t^2}=\frac{1}{2}\left[\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)^2+\frac{1}{\epsilon}(\nabla\epsilon)^2\right]-\frac{k}{2}\frac{\rho(1+\epsilon^2v^2)}{(1-\epsilon^2v^2)}. \tag{39}$$

This field equation can be expressed in terms of the energy momentum tensor of the gravitational field (27a). Multiply (39) by $-\epsilon_{,i}/k\epsilon$ which form (27a) can then be written for the Newtonian coordinate system as

$$0=G_{i,k}{}^k-\frac{1}{2}\frac{\rho}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{1+\epsilon^2v^2}{(1-\epsilon^2v^2)^{\frac{1}{2}}}\frac{1}{\epsilon}\epsilon_{,i}. \tag{40}$$

From (34) this can be written in generalized coordinates as the tensor equation.

$$0=G_{i;k}{}^k+\tfrac{1}{2}g_{lm;i}h^{ln}M_n{}^m. \tag{41}$$

The covariant derivatives are of course based upon γ_{ij} as the metric tensor. The four field equations, (41), are equivalent as (40) divided by $\epsilon_{,i}$ is independent of the index i.

Conservation relations for the two fields in interaction are obtained by adding together (38) and (41) to obtain

$$0=(G_i{}^k+M_i{}^k)_{;k}. \tag{42}$$

For the Newtonian coordinate system this reduces to an ordinary divergence and defines conservation relations for momentum and energy in a manner similar to that described earlier for the gravitational field alone. For the Newtonian frame the integration of the ordinary divergence of the total energy momentum tensor over the volume bounded by two surfaces of constant time gives, after converting to a surface integral,

$$\int_{t_1}(G_i{}^4+M_i{}^4)d^3x=\int_{t_2}(G_i{}^4+M_i{}^4)d^3x. \tag{43}$$

Hence the total energy and momentum of the system is conserved.

Angular momentum and torque stress densities may be defined for the Newtonian coordinate system by defining the quantities (not tensor components)

$$L_{\alpha\beta}{}^j=x_\alpha(G_\beta{}^j+M_\beta{}^j)-x_\beta(G_\alpha{}^j+M_\alpha{}^j), \quad \alpha,\beta\neq 4. \tag{44}$$

Conservation of angular momentum takes the form of the divergence relation

$$0=L_{\alpha\beta,j}{}^j, \tag{45}$$

which in a manner similar to the derivation of (43) gives

$$\int_t L_{\alpha\beta}{}^4d^3x=\text{const}, \tag{46}$$

where $L_{\alpha\beta}{}^4$ are the three components of angular momentum density of the field.

CENTRAL FORCES

In order to evaluate the constant k, the field equation (39) will be solved for the case of static central symmetry. The resulting solution can be compared directly with the Schwarzschild solution of the corresponding problem of general relativity.

We consider a central spherically symmetrical source of gravitational field. For points outside the source the scalar ϵ satisfies the equation,

$$\nabla^2\epsilon-\frac{1}{2\epsilon}(\nabla\epsilon)^2=0. \tag{47}$$

The solution to this equation for which $\epsilon\rightarrow 1$ asymptotically is

$$\epsilon=\left(1+\frac{a}{r}\right)^2, \tag{48}$$

where a is a positive constant which must be adjusted to join properly with the interior solution of (39).

The tensor g_{ij} is obtained by substituting (48) in (10). This may be compared directly with the Schwarzschild solution[5] in isotropic coordinates for which

$$\begin{aligned} g_{11}=g_{22}=g_{33}&=-\left(1+\frac{G_0M_0}{2r}\right)^4,\\ g_{44}&=\left(\frac{1-G_0M_0/2r}{1+G_0M_0/2r}\right)^2. \end{aligned} \tag{49}$$

Neglecting $(G_0M_0/2r)^2$ compared with unity the two expressions are equal. At the surface of the sun $(G_0M_0/2r)^2$ has a value of 10^{-12}. It is doubtful that it will ever be possible to detect observationally the effect of a term this small. Thus, the two theories give essentially the same result for the planet and light trajectories in the solar system.

They also give essentially the same result for the gravitational red shift. An atom at the surface of the sun emits a photon of energy

$$\hbar\omega=\hbar\omega_0\epsilon^{-\frac{1}{2}}, \tag{50}$$

where $\hbar\omega_0$ is the energy of the photon emitted by a similar atom at great distance where $\epsilon=1$. The photon travels without shift in either energy or frequency (Newtonian units). Compared with the photon of a distant atom it appears to be shifted in frequency by an amount

$$\Delta\omega=\omega_0-\omega=\omega_0(1-\epsilon^{-\frac{1}{2}}), \tag{51}$$

$$\frac{\Delta\omega}{\omega_0}\cong\frac{kM_0}{16\pi r}\cong\frac{G_0M_0}{r}, \tag{52}$$

which is the usual approximate expression from general relativity for the red shift.

One of the fundamental assumptions of the theory introduced earlier concerned the ϵ dependence of the rest energy of a particle. The assumption $M=M_0\epsilon^{-\frac{1}{2}}$ can be justified on the grounds of the applicability of a limited principle of equivalence. If it be assumed that the gravitational red shift disappears for an experiment performed wholly within a freely falling elevator, it is found that the above relation for the rest energy of a particle is the only one compatible with this assumption. The gravitational red shift (52) is compensated by a Doppler effect resulting from the change in velocity of fall during the propagation time of the light from one atom to the other. It is also found that light travels in "straight" paths in the elevator, meter sticks being defined as "straight" but being for Newtonian units curved by just the right amount to make the light beam appear to be straight.

To evaluate the constant k, we consider first a general static solution to (39) in the weak field approximation. Matter is at rest, gravitational forces being balanced by other stresses not specified. These other stresses would also be sources of gravitational fields but may be neglected in the weak field limit. We assume that $\epsilon\rightarrow\epsilon_0$ asymptotically. For static solutions (39) becomes

$$\nabla^2\epsilon-\frac{1}{\epsilon}(\nabla\epsilon)^2=2\epsilon^{\frac{1}{2}}\nabla^2\epsilon^{\frac{1}{2}}=-\frac{k}{2}\rho. \tag{53}$$

This can be written as

$$\nabla^2\epsilon^{\frac{1}{2}}=-\frac{k}{4}\frac{\rho_0}{\epsilon}, \tag{54}$$

where ρ_0 is the matter energy density with $\epsilon=1$. In the weak field approximation ϵ can be replaced by the constant ϵ_0 on the right side of (54). Then the general solution with the proper asymptotic dependence is

$$\epsilon^{\frac{1}{2}}=\epsilon_0^{\frac{1}{2}}+\frac{k}{16\pi\epsilon_0}\int\frac{\rho_0(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3x'. \tag{55}$$

For a spherically symmetric source (55) becomes

$$\epsilon^{\frac{1}{2}}=\epsilon_0^{\frac{1}{2}}+\frac{kM_0}{16\pi\epsilon_0 r}, \tag{56}$$

where M_0 is the volume integral of ρ_0 over the region interior to the field point r. $M_0/\epsilon_0^{\frac{1}{2}}$ is the weak field approximation for the internal matter energy. If (56) is squared and compared with (48), (10), and (49) for $\epsilon_0=1$, k can be evaluated as

$$k=16\pi G_0. \tag{57}$$

One interesting question concerns the dependence upon ϵ_0 of the ratio of gravitational to electrical forces between matter in the weak field approximation. Consider two identical electrically charged static spherical mass distributions. The gravitational force acting on one of the masses due to the other is from (56) and (17)

$$F_g=\frac{M}{2\epsilon}\frac{\partial\epsilon}{\partial r}=-\frac{kM_0^2}{16\pi\epsilon_0^2r^2}. \tag{58}$$

The electrostatic interaction is

$$F_e=-\frac{q^2}{\epsilon_0 r^2}. \tag{59}$$

[5] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, London, 1939), p. 205.

The ratio of these two forces is

$$\frac{F_e}{F_g}=\frac{G_0\epsilon_0 M_0^2}{q^2}. \tag{60}$$

For fixed total matter M_0 and charge q, the ratio varies as ϵ_0. By measuring this ratio, one can in principle experimentally determine local $\epsilon=\epsilon_0$.

ELECTROMAGNETIC INTERACTIONS

To include the Maxwell field in the formalism in a proper way, the procedure outlined earlier is followed again. The Lagrangian density of the Lorentz invariant theory is written with c replaced by ϵ^{-1}. To this is added (29) the density of the gravitational field. The total is, for the Newtonian frame

$$L=-\sum_i M_i(1-\epsilon^2 v_i^2)^{\frac{1}{2}}\delta(\mathbf{r}-\mathbf{r}_i)+\sum_i e_i(\mathbf{A}\cdot\mathbf{v}_i-\varphi)\delta(\mathbf{r}-\mathbf{r}_i)$$
$$+\frac{1}{8\pi}\left(\epsilon E^2-\frac{1}{\epsilon}B^2\right)+\frac{1}{2k}\left[\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)^2-\frac{1}{\epsilon}(\nabla\epsilon)^2\right], \tag{61}$$

where

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}-\nabla\varphi,$$
$$\mathbf{B}=\nabla\times\mathbf{A}, \tag{62}$$

and the Lorentz condition

$$\nabla\cdot\mathbf{A}+\frac{\partial\varphi}{\partial t}=0, \tag{63}$$

is assumed to be satisfied.

The variation principle

$$\delta\int L d^4x=0, \tag{64}$$

gives as the equation of motion of the ith particle

$$\frac{d}{dt}\left[\frac{M_i\mathbf{v}_i\epsilon^2}{(1-\epsilon^2v_i^2)^{\frac{1}{2}}}\right]=e_i(\mathbf{E}+\mathbf{v}_i\times\mathbf{B})$$
$$+\frac{1}{2}\frac{1+\epsilon^2v_i^2}{(1-\epsilon^2v_i^2)^{\frac{1}{2}}}M_i\frac{1}{\epsilon}\nabla\epsilon. \tag{65}$$

Here the first term on the right is the Lorentz force and the second term is the previously obtained gravitational force. The variation principle also gives the two Maxwell equations that are not identities.

$$\nabla\times\left(\frac{1}{\epsilon}\mathbf{B}\right)-\frac{\partial}{\partial t}(\epsilon\mathbf{E})=4\pi\sum_i e_i\mathbf{v}_i\delta(\mathbf{r}-\mathbf{r}_i),$$
$$\nabla\cdot(\epsilon\mathbf{E})=4\pi\sum_i e_i\delta(\mathbf{r}-\mathbf{r}_i). \tag{66}$$

It also gives the field equation for the gravitational field as

$$\nabla^2\epsilon-\epsilon^2\frac{\partial^2\epsilon}{\partial t^2}=-k\left\{\sum_i\frac{M_i(1+\epsilon^2v_i^2)}{2(1-\epsilon^2v_i^2)^{\frac{1}{2}}}\delta(\mathbf{r}-\mathbf{r}_i)\right.$$
$$\left.+\frac{1}{8\pi}\left(\epsilon E^2+\frac{1}{\epsilon}B^2\right)-\frac{1}{2k}\left[\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)^2+\frac{1}{\epsilon}(\nabla\epsilon)^2\right]\right\}. \tag{67}$$

In this coordinate system the electromagnetic energy density and the gravitational energy density serve as source terms for the generation of gravitational waves but they couple with opposite signs. Per unit energy, the electromagnetic field couples with twice the strength of a slowly moving particle. This is in agreement with the extra factor of two for the gravitational deflection of light.

On the right side of (67) the first term represents the total effect of the particle (at a distance) and the particle self-fields are to be omitted from the 2nd and 3rd terms. Presumably with a proper classical field theory of particles the self-fields of the particle (2nd and 3rd terms) would be included as sources of gravitational field. Because of the difference in sign of these two terms, the self-electric and gravitational energies could be each large compared with the total particle energy and still generate the proper gravitational field. There is, hence, a possibility within this framework of a charged particle of very small characteristic radius (gravitational radius) held together by gravitational forces. On the other hand, one has little reason to believe that a theory having its origin in macroscopic phenomena only should be valid at such small distances.

There is one piece of unfinished business regarding the form of (20). It was shown that without violating invariance arguments this term could be multiplied by an arbitrary function of ϵ. However, on physical grounds it would be expected that for the Newtonian coordinate system the third term on the right of 73 would, like the 2nd term, be an energy density. For this to be true the function of ϵ must be omitted.

The Lagrangian density (61) is easily written in generally covariant form by introducing the covariant four potential $A_i=(\mathbf{A},-\varphi)$ and the usual antisymmetric field tensor

$$F_{ik}=A_{i,k}-A_{k,i}. \tag{68}$$

In this notation, for the Newtonian frame

$$L=-\sum_i M_{i0}\frac{ds}{dt}\delta(\mathbf{r}-\mathbf{r}_i)+\sum_i e_iA_j\frac{dx^j}{dt}\delta(\mathbf{r}-\mathbf{r}_i)$$
$$-\frac{\epsilon}{16\pi}F_{ik}F_{rs}h^{ir}h^{ks}+\frac{1}{2k}h^{ij}\epsilon_{,i}\epsilon_{,j}. \tag{69}$$

As $(ds/dt)\delta(\mathbf{r}-\mathbf{r}_i)$ is assumed to transform like a scalar, this Lagrangian density is a scalar. The invariant form of (64) is

$$\delta\int L\sqrt{-\gamma}d^4x=0. \tag{70}$$

As previously, divergence difficulties with self-fields can be avoided by using a macroscopic description of matter. In the usual way an energy-momentum tensor for the electromagnetic field can be defined as

$$E_i{}^j=-\frac{\epsilon}{4\pi}F_{ik}F_{nm}h^{km}h^{nj}-\delta_i{}^j\frac{\epsilon}{16\pi}F_{mn}F_{rs}h^{mr}h^{ns}. \tag{71}$$

In the Newtonian coordinate system the energy density is

$$E_4{}^4=\frac{1}{8\pi}\left(\epsilon E^2+\frac{1}{\epsilon}B^2\right). \tag{72}$$

The momentum density is

$$-E_\alpha{}^4=\frac{\epsilon}{4\pi}(\mathbf{E}\times\mathbf{B})_\alpha,\quad \alpha=1,2,3. \tag{73}$$

The Poynting vector is

$$E_4{}^\alpha=\frac{1}{4\pi\epsilon}(\mathbf{E}\times\mathbf{B})_\alpha=-\frac{1}{\epsilon^2}E_\alpha{}^4. \tag{74}$$

The remainder of the terms constitute the Maxwell stress tensor.

For a continuous charge-current distribution the charge and current densities can be combined in a four vector which in the Newtonian coordinate system is

$$j^k=(\mathbf{j},\rho). \tag{75}$$

Making use of (68), (62), and (75) Maxwell's equations (66) can be written for a general coordinate system as

$$(\epsilon F_{sr}h^{is}h^{rj})_{;j}=-4\pi j^i. \tag{76}$$

For the Newtonian coordinate system the covariant divergence reduces to an ordinary divergence.

For the Newtonian coordinate system the ordinary divergence of the energy-momentum tensor of the electromagnetic field is from (71) and (76)

$$E_{i,j}{}^j=-F_{ik}j^k-E_4{}^4\frac{1}{\epsilon}\epsilon_{,i}, \tag{77}$$

$$E_{\alpha,j}{}^j=\rho E_\alpha+(\mathbf{j}\times\mathbf{B})_\alpha-\frac{1}{8\pi}\left(\epsilon E^2+\frac{1}{\epsilon}B^2\right)\frac{1}{\epsilon}\epsilon_{,\alpha},\qquad \alpha=1,2,3, \tag{78}$$

$$E_{4,j}{}^j=-\mathbf{j}\cdot\mathbf{E}-\frac{1}{8\pi}\left(\epsilon E^2+\frac{1}{\epsilon}B^2\right)\frac{1}{\epsilon}\epsilon_{,4}. \tag{79}$$

Equation (78) represents the force per unit volume exerted by the electromagnetic field. Equation (79) is the negative of the rate of energy transfer from electromagnetic to other forms of energy.

Equation (77) can be written in general covariant form by first rewriting the last term as a derivative of the Lagrangian density of the electromagnetic field. In generally covariant form (77) becomes

$$E_{i;j}{}^j=-F_{ik}j^k+\frac{\partial}{\partial\epsilon}\left(\frac{\epsilon}{16\pi}F_{ik}F_{rs}h^{ir}h^{ks}\right)\epsilon_{,i}. \tag{80}$$

For a continuous matter distribution [see (39)], (67) becomes

$$\nabla^2\epsilon-\epsilon^2\frac{\partial^2\epsilon}{\partial t^2}=-k\left\{\frac{1}{2}\frac{\rho(1+\epsilon^2v^2)}{(1-\epsilon^2v^2)}+\frac{1}{8\pi}\left(\epsilon E^2+\frac{1}{\epsilon}B^2\right)-\frac{1}{2k}\left[\epsilon\left(\frac{\partial\epsilon}{\partial t}\right)^2+\frac{1}{\epsilon}(\nabla\epsilon)^2\right]\right\}. \tag{81}$$

If this equation is multiplied by $-\epsilon_{,i}/k\epsilon$ and compared with (27a) and (40) it can be written as

$$0=G_{i,k}{}^k-\tfrac{1}{2}\rho\frac{1+\epsilon^2v^2}{1-\epsilon^2v^2}\frac{1}{\epsilon}\epsilon_{,i}-\frac{1}{8\pi}\left(\epsilon E^2+\frac{1}{\epsilon}B^2\right)\frac{\epsilon_{,i}}{\epsilon}. \tag{82}$$

Equation (82) can be written in generally covariant form by making use of (34)

$$0=G_{i;k}{}^k+\tfrac{1}{2}g_{lm;i}h^{lm}M_n{}^m+\frac{\partial}{\partial\epsilon}\left(\frac{\epsilon}{16\pi}F_{ik}F_{rs}h^{ir}h^{ks}\right)\epsilon_{,i}. \tag{83}$$

For a continuous medium (65) can be written as

$$\frac{d}{dt}\left(\frac{\rho\epsilon^2\mathbf{v}}{1-\epsilon^2v^2}\right)=\rho_e\mathbf{E}+\mathbf{j}\times\mathbf{B}+\tfrac{1}{2}\rho\frac{1+\epsilon^2v^2}{1-\epsilon^2v^2}\frac{1}{\epsilon}\nabla\epsilon. \tag{84}$$

Here ρ again represents the matter energy density in a co-moving coordinate system. Making use of (32), (33), (34), (37), (62), (68), and (75), (84) can be written

$$0=M_{i;k}{}^k-\tfrac{1}{2}g_{lm;i}h^{ln}M_n{}^m-F_{ik}j^k. \tag{85}$$

This is obviously generally covariant.

Defining the total energy-momentum tensor as the sum of the tensors for the electromagnetic field, gravitational field and matter

$$T_i{}^k=E_i{}^k+G_i{}^k+M_i{}^k, \tag{86}$$

the conservation of energy and momentum is expressed by

$$T_{i;k}{}^k=0. \tag{87}$$

The vanishing of the covariant divergence of $T_i{}^k$ follows from (85), (80), and (83).

The fact that the covariant divergence reduces to an ordinary divergence for the Newtonian coordinate system suffices to define conservation laws. Also angular momentum and torque densities can be defined for the

Newtonian coordinate system as

$$L_{\alpha\beta}{}^{j}=x_\alpha T_\beta{}^{j}-x_\beta T_\alpha{}^{j},\quad \alpha,\beta=1,2,3. \tag{88}$$

The angular momentum of a closed system is conserved.

Although the physical picture of polarizable space was used as a guide in constructing the above theory, this interpretation is not fundamental to the formalism. The formalism has a structure of its own, that is independent of the philosophic interpretations. In this more general sense the theory is simply a scalar field theory of gravity based upon a flat space-time and local Lorentz invariance.

COSMOLOGY

The cosmological principle was taken to be a fundamental assumption of the theory. Namely, from any fixed position point of a Newtonian frame the universe is assumed to be on the average uniform. This implies that matter is on the average fixed in position relative to the Newtonian coordinate frame, for motion would introduce a lack of uniformity as seen by an observer located where the matter would be moving. In like manner the scalar field variable ϵ and matter density must be position independent.

If the lack of uniformity of matter and its random motion are ignored, the time dependence of ϵ can be computed from energy conservation. Equation (87) becomes for the Newtonian frame, remembering the position independence of the densities

$$0=T_{4,k}{}^{k}=T_{4,4}{}^{4}, \tag{89}$$

or

$$T_4{}^4=U\quad\text{(const)}. \tag{90}$$

Assuming that electromagnetic energy can be ignored, from (86), (35), and (27), (90) can be written

$$U=\frac{1}{2k}\epsilon\left(\frac{d\epsilon}{dt}\right)^2+\frac{\rho_0}{\sqrt{\epsilon}}, \tag{91}$$

with ρ_0 constant.

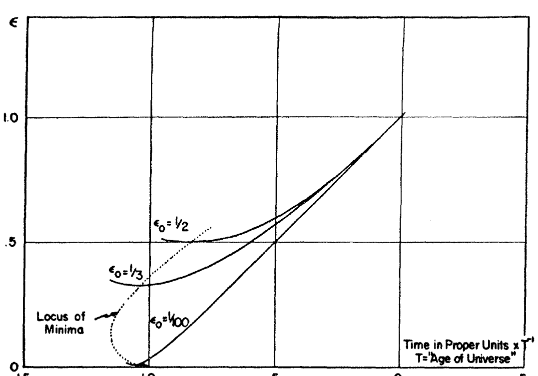

FIG. 1. Variation with time of vacuum polarizability for various energy densities.

Equation (91) can be integrated to give

$$\pm(2kU)^{\frac{1}{2}}t=2\epsilon^{\frac{1}{4}}(\epsilon^{\frac{1}{2}}-\epsilon_0{}^{\frac{1}{2}})^{\frac{1}{2}}[\tfrac{1}{3}\epsilon+(5/12)\epsilon_0{}^{\frac{1}{2}}\epsilon^{\frac{1}{2}}+\tfrac{5}{8}\epsilon_0]$$
$$+(5/4)\epsilon_0{}^{\frac{3}{2}}\log\left\{\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{1}{4}}+\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}\right\}, \tag{92}$$

with

$$\epsilon_0=\left(\frac{\rho_0}{U}\right)^2. \tag{93}$$

The constant of integration is chosen to cause the minimum of ϵ, (ϵ_0) to occur at $t=0$.

For some purposes it is desirable to obtain ϵ as a function of τ, the "proper" or atomic time. Multiplying (91) by

$$\left(\frac{dt}{d\tau}\right)^2=\epsilon, \tag{94}$$

and integrating gives

$$\pm(2kU)^{\frac{1}{2}}\tau=\epsilon^{\frac{1}{4}}(\epsilon^{\frac{1}{2}}-\epsilon_0{}^{\frac{1}{2}})^{\frac{1}{2}}(\epsilon^{\frac{1}{2}}+\tfrac{3}{2}\epsilon_0{}^{\frac{1}{2}})$$
$$+\tfrac{3}{2}\epsilon_0\log\left\{\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{1}{4}}+\left[\left(\frac{\epsilon}{\epsilon_0}\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}\right\}. \tag{95}$$

Equation (95) with a shift in time zero is plotted in Fig. 1. U is chosen to give an asymptotic slope of unity and the time zero is adjusted to make $\epsilon=1$ at $t=0$.

Although all matter is at rest in this model there is a galactic red shift. With increasing ϵ, the photon emitted in the past has more energy than its present counterpart. This might be thought to cause a "blue shift." However, a photon loses energy with increasing ϵ at twice the rate of loss characteristic of an atom, hence there is a net shift toward the red. The energy lost by a photon becomes converted to gravitational field energy. Combining (16) and (16a), it is apparent that the energy of a photon varies as ϵ^{-1} whereas that of a stationary atom varies as $\epsilon^{-\frac{1}{2}}$. The red shift is consequently given by

$$\frac{\lambda_1}{\lambda_2}=\epsilon^{\frac{1}{2}}. \tag{96}$$

Here the ratio refers to the wavelength of light of distant origin λ_2 and the laboratory light λ_1. ϵ is the value of the dielectric constant at the time the light was emitted and ϵ is assumed to be unity now.

Although photon concepts were used to obtain the galactic red shift these particle ideas are not necessary. It is easily seen from Maxwell's equations that for time dependent but space independent ϵ, an electromagnetic wave propagates without a change in wavelength but with its frequency varying as ϵ^{-1}.

For small red shifts (96) can be written, assuming $\epsilon\gg\epsilon_0$

$$\frac{1}{\lambda}\frac{d\lambda}{d\tau}=\tfrac{1}{2}\epsilon^{-\frac{1}{2}}\frac{d\epsilon}{d\tau}=\tfrac{1}{2}(2kU)^{\frac{1}{2}}. \tag{97}$$

Using the present value of the Hubble constant of 6.5×10^9 years for the reciprocal of the left side of (97) gives an energy density of space

$$U=9\times10^{-29}\ \mathrm{g/cm^3}. \tag{98}$$

Defining the "age of the universe" as the negative time at which ϵ extrapolates to zero (Fig. 1), gives an age of the universe of 3.25×10^9 years with this choice of U. Remembering the changes in the Hubble constant in the past years, another factor two change is perhaps not excluded. The evidence from isotope abundance and radioactive dating favors an age of the galaxy of at least 6.5×10^9 years.[6] If this is taken as the "age" of the universe, one obtains $U=2.25\times10^{-29}$ g/cm^3. With a minimum matter density of the universe equal to that of galactic matter (10^{-30} g/cm^3), there is a lower bound on ϵ_0 from (93) of

$$\epsilon_0>0.002. \tag{99}$$

It is interesting to consider the effect of a changing ϵ on the random motion in the universe. Consider a test particle moving through the idealized uniform space. Its momentum is a constant of the motion. Hence, from (15)

$$\epsilon^{\frac{1}{2}}\frac{\epsilon v}{(1-\epsilon^2v^2)^{\frac{1}{2}}}=\text{const.} \tag{100}$$

With increasing ϵ the ratio of particle to light velocity (ϵv) decreases. Thus, the random kinetic energy measured in atomic units falls with time and the kinetic temperature of galactic matter decreases with time. This energy is converted into gravitational energy [see (82)].

Much of the formalism developed in the past 40 years by the cosmologists is concerned with kinematics and is equally valid in the present theory. The tensor g_{ij} (10) is easily transformed into the standard form of Robertson and Lemaitre[7] through the substitution of proper time for coordinate time using (94). This gives as an expression for interval measured in proper units

$$ds^2=-\epsilon[(dx^1)^2+(dx^2)^2+(dx^3)^2]+d\tau^2, \tag{101}$$

with ϵ given by (104).

This is the type of universe characterized as open and flat. Since in the present theory the equations of motion of a particle are identical with those of general relativity, the kinematical description of the universe based upon (101) is identical with that of general relativity.

Equation (91) and its solution (92) are based upon the assumption that (37a) is valid or that the total matter is conserved. Only under this condition does the last term in (91) have this form. On the other hand, without violating conservation of energy, momentum, or angular momentum the condition (37a) could be relaxed. Namely, a mechanism for the production of particles by the conversion of gravitational energy into matter energy could be postulated. This would allow a variety of continuous creation universes.

A cataclysmic production of particles could also be obtained within the framework of the theory. The universe might be visualized as initially free of particles ($t<0$) and containing only gravitational energy. The time dependence of ϵ would be given by (92) with the negative sign and with $\epsilon_0=0$. ϵ would be decreasing with time, varying as $(-t)^{\frac{2}{3}}$. One might postulate in a completely *ad hoc* fashion the creation of heavy neutral bosons at a rate varying as $\epsilon^{-n}(d\epsilon/dt)^2$ with $n>\frac{1}{3}$. This would result in a cataclysmic production of bosons at $t=0$. The heavy bosons would then quickly decay into protons and electrons. The production of particles is accompanied by a rapid increase of ϵ from a low value to ϵ_0 after which production effectively ceases. From here on ϵ varies with time in accordance with (92).

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From (95) it is apparent that asymptotically ϵ is proportional to the age of the universe on an atomic time scale. From (53) and (95), the ratio of the gravitational to electrical interaction between two elementary particles varies asymptotically inversely as the age of the universe. This agrees with Dirac's hypothesis,[8] and suggests that the time dependence be examined of the remainder of the physical and astrophysical constants of Fig. 1 of the preceding paper.

The theory has been constructed to make the numbers in the first column constants, hence time independent. The present theory says nothing about the Fermi interactions except that their strength would suggest that they might be time dependent.

The only other number requiring discussion is in the last column, the number of particles in the universe out to the Hubble radius. Basing the discussion upon the asymptotic dependence expressed in atomic units, the volume inside the Hubble radius is proportional to $\tau^3\sim\epsilon^3$ and the average distance between particles varies as $\epsilon^{\frac{1}{2}}$. This gives a total number of particles which varies as $\tau^{\frac{3}{2}}$ instead of τ^2 as suggested by Dirac's considerations.

There is a simple explanation for this difference. The present theory contains a small dimensionless number ϵ/ϵ_0 which is time dependent. In the absence of a theory, any of the large dimensionless numbers could contain a factor in the form of a power of this number without changing its order of magnitude. Consequently, the time dependence of a number cannot be inferred from its magnitude alone.

From this point of view there is a single large dimensionless number which is statistical in origin. This is the number of particles in the universe. The age of the universe, "now," is not random but is conditioned by biological factors. The radiation rate of a star varies as

[6] F. Hoyle *et al.*, Science **124**, 611 (1956).

[7] See reference 5, p. 369 [Eq. (148.20)].

[8] P. A. M. Dirac, Proc. Roy. Soc. (London) **A165**, 199 (1938).

ϵ^{-7} [9] and for very much larger values of ϵ than the present value, all stars would be cold. This would preclude the existence of man to consider this problem. On the other hand, if ϵ/ϵ_0 were presently very much larger, the very rapid production of radiation at earlier times would have converted all hydrogen into heavier elements, again precluding the existence of man. This suggests that ϵ/ϵ_0 is presently a relatively small number, perhaps under ten. The universe can be characterized as young.

Some insight into the puzzle of the constants of Table I of the previous paper is gained by noting that for any solution to the field and orbit equations (39) and (14), a family of solutions can be obtained through the use of a simple scaling transformation.

Let n represent the density of particles of rest energy m. Then a macroscopic treatment of matter employs the matter density function ρ which is related to the particle density, according to (36a), as

$$nm=\frac{\rho}{(1-\epsilon^2v^2)^{\frac{1}{2}}}=\frac{\rho_0}{\sqrt{\epsilon}(1-\epsilon^2v^2)^{\frac{1}{2}}}. \tag{102}$$

For any solution to (39), (14), and (102) another is obtained through the transformation

$$\begin{aligned}\epsilon(\mathbf{r},t)&\to\gamma^2\epsilon(\gamma^{-1}\mathbf{r},\gamma^{-3}t),\\ \mathbf{v}(\mathbf{r},t)&\to\gamma^{-2}v(\gamma^{-1}\mathbf{r},\gamma^{-3}t),\\ n(\mathbf{r},t)&\to\gamma n(\gamma^{-1}\mathbf{r},\gamma^{-3}t),\\ \rho_0(\mathbf{r},t)&\to\gamma\rho_0(\gamma^{-1}\mathbf{r},\gamma^{-3}t),\end{aligned} \tag{103}$$

with γ a constant.

If this transformation is applied to the cosmological problem [and (92)] it is found that the number of particles in the universe scales as γ^4, the age and radius of the universe in atomic units scale as γ^2, and the ratio of electrical to gravitational forces scales as γ^2. The energy density U is independent of γ. Consequently this trans formation preserves the internal relations exhibited by Fig. 1 of the previous paper. Apparently, to make some sense out of the regularity of these numbers in the framework of the present theory, it is necessary to assume two things, that the universe is young (i.e., $\epsilon/\epsilon_0\sim 1$) and that it has a characteristic energy density $\sim 10^{-8}$ erg/cm^3 when expressed in Newtonian units.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the help of several of the author's colleagues, particularly Professor V. Bargmann who read the manuscript and made several helpful suggestions.

[9] E. Teller, Phys. Rev. **73**, 801 (1948).