Representations of the Lorentz and the Poincaré group in $d = 3 + 1$

Alexander Sevrin

0 Disclaimer

These notes were quickly put together in order to address some of the questions the author was receiving from students. There is no intention whatsoever to be either exhaustive or mathematically rigorous. The topics discussed below should normally be part of a thorough course on group theory (*e.g.* *Groups and Symmetries* (B-KUL-G0S96A) taught in Leuven during the first semester by Prof. Dr. Antoine Van Proeyen). There are numerous (good) books on group theory. Two of them which I like are [1] and [2]. The representation theory of the Poincaré group was originally worked out by Eugene Wigner in [3]. A very thorough text, though completely written in French, on the Lorentz and the Poincaré group and their representations can be found in the lecture notes of Prof. Dr. Marc Henneaux: http://www.solvayinstitutes.be/pdf/Marc/SO3SO3_1_18112014.pdf. Finally, as these are notes intended to complement the introductory *Quantum Field Theory* course, we restrict ourselves to the case where the dimensionality of space-time is $d = 1+3$. The representation theory of the Poincaré group varies wildly when going from one dimensionality to another. We use natural units and put $c = 1$ and $\hbar = 1$.

1 Finite dimensional representations of $SU(2)$

The rotation group in $\mathbb{R}^3$ (the mathematical notation for the group of proper rotations in $\mathbb{R}^3$ is $SO(3)$) plays a central role in physics as does its universal covering group $SU(2)$ (see appendix A). Infinitesimal rotations are generated by the components of the angular momentum which in quantum mechanics reads,

$$\hat{\vec{L}} = -i\,\vec{r} \times \nabla\,. \tag{1}$$

Denoting $t_a = \hat{L}^a$, $t_a = t_a^\dagger$, $a \in \{1, 2, 3\}$, we find that they satisfy the commutator algebra[1],

$$[t_a, t_b] = i\,\varepsilon_{abc}\,t_c\,, \tag{2}$$

which is the *Lie algebra* of $SU(2)$ and $SO(3)$ (sometimes denoted by $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ or A_1 and B_1).

Introduce a new basis,

$$h \equiv 2t_3\,, \quad e \equiv t_1 + it_2\,, \quad f = e^\dagger \equiv t_1 - it_2\,. \tag{3}$$

From eq. (2) we get the commutator algebra in this basis,

$$[h, e] = +2e\,, \quad [h, f] = -2f\,, \quad [e, f] = h\,. \tag{4}$$

We now introduce an infinite dimensional representation of this algebra. The basis vectors of this representation are defined by a real number $n \in \mathbb{R}$ and they are given by,

$$f^r|n\rangle\,, \qquad r \in \mathbb{N}\,, \tag{5}$$

where,

$$e|n\rangle = 0\,, \qquad h|n\rangle = n|n\rangle\,. \tag{6}$$

The basis vector $|n\rangle$ is called the *highest weight vector*. Using this and the commutation relations given in eq. (4), we immediately get the action of the generators on an arbitrary basis vector of the representation,

$$\begin{aligned} e\,f^r|n\rangle &= r(n-r+1)f^{r-1}|n\rangle\,, \\ h\,f^r|n\rangle &= (n-2r)f^r|n\rangle\,, \\ f\,f^r|n\rangle &= f^{r+1}|n\rangle\,. \end{aligned} \tag{7}$$

Now note that if $n \in \mathbb{N}$ we get that,

$$e\,f^{n+1}|n\rangle = 0\,. \tag{8}$$

Put differently, $f^{n+1}|n\rangle$ is another highest weight vector. This immediately implies that $f^r|n\rangle$ with $r \geqslant n+1$ span an *invariant subspace*. They can be put consistently to zero. Indeed the action of either h, e or f on one of these vectors gives again a vector which has been put to zero.

[1] We introduced the completely antisymmetric ε-symbol, ε_{abc}, in 3 dimensions and $\varepsilon_{123} = +1$.

So we end up with a finite dimensional unitary irreducible representation characterized by a highest weight vector $|n\rangle$, with $n \in \mathbb{N}$ which is defined by,

$$e|n\rangle = 0\,, \qquad h|n\rangle = n|n\rangle\,, \qquad f^{n+1}|n\rangle = 0\,. \tag{9}$$

This gives the irreducible $n+1$ dimensional unitary representation of $SU(2)$ whose basis vectors are proportional to $f^r|n\rangle$ with $r \in \mathbb{N}$ and $0 \leqslant r \leqslant n$. The proportionality constant is defined by normalizing the vectors to one. Assuming the highest weight vector has norm one,

$$\langle n|n\rangle = 1\,, \tag{10}$$

we introduce the notation,

$$|n,r\rangle \equiv \alpha(n,r) f^r |n\rangle\,, \tag{11}$$

where $\alpha(n,r)$ is determined by the requirement that the vector $|n,r\rangle$ has norm one[2],

$$\langle n,r|n,r\rangle = 1\,. \tag{12}$$

Using the hermiticity properties of the generators, $h^\dagger = h$, $e^\dagger = f$, and eq. (7), we find the action of the generators on the normalized basis vectors,

$$\begin{aligned} h|n,r\rangle &= (n-2r)|n,r\rangle\,, \\ e|n,r\rangle &= \sqrt{r(n-r+1)}\,|n,r+1\rangle\,, \\ f|n,r\rangle &= \sqrt{(r+1)(n-r)}\,|n,r-1\rangle\,. \end{aligned} \tag{13}$$

Till now we have used a notation common in the mathematical literature. In order to connect to a language more familiar to physicists, we introduce the angular momentum $j \in \frac{1}{2}\mathbb{N}$ which is related to n introduced above, by $j = n/2$. We denote now the basis vectors by,

$$|j,m\rangle \quad \text{with} \quad m \in \{-j, -j+1, -j+2, \cdots, +j\}\,, \tag{14}$$

where m can be viewed as the projection of the angular momentum on the 3-axis (in units $\hbar$). Using eq. (13), we have,

$$\begin{aligned} h|j,m\rangle &= 2m|j,m\rangle\,, \\ e|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle\,, \\ f|j,m\rangle &= \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle\,, \end{aligned} \tag{15}$$

[2] This determines $\alpha(n,r)$ modulo a multiplicative phase which we choose to be one.

which should be familiar from your course on quantum mechanics.

For a given value of j this gives us immediately the representation. *E.g.* taking $j = 1/2$, one finds,

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{16}$$

or in the basis eq. (2) ,

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

In a similar way we find for $j = 1$,

$$h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad e = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{18}$$

which in the basis eq. (2) becomes,

$$t_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

Finally, let us end by introducing the (quadratic) Casimir operator[3] of $SU(2)$, C_2,

$$\begin{aligned} C_2 &= \sum_{a=1}^{3} t_a t_a \\ &= \frac{1}{4}\Big(h(h+1) + 4fe\Big). \end{aligned} \tag{20}$$

One verifies that C_2 commutes with all generators,

$$[C_2, t_a] = 0, \qquad \forall a \in \{1, 2, 3\}. \tag{21}$$

As a consequence it has the same eigenvalue when acting on any of the basis vectors of a given irreducible representation,

$$C_2|j, m\rangle = j(j+1)|j, m\rangle. \tag{22}$$

[3] A Casimir operator commutes with all the generators of the algebra.

2 The Lorentz algebra and its representations

We take the mostly minus convention for the metric, *i.e.* $\eta = \mathrm{diag}(+1 -1 -1 -1)$. We also introduce the completely antisymmetric ε-symbol $\varepsilon_{\mu\nu\rho\sigma}$ and put $\varepsilon_{0123} = +1$.

In quantum mechanics the generators of infinitesimal Lorentz transformations, $\hat{M}_{\mu\nu}$ are given by,

$$\hat{M}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu} \,, \tag{23}$$

where $S_{\mu\nu}$ is the "intrinsic" or "spin" part. They satisfy the commutator algebra,

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] \;=\; i\,\eta_{\nu\rho}\hat{M}_{\mu\sigma} + i\,\eta_{\mu\sigma}\hat{M}_{\nu\rho} - i\,\eta_{\mu\rho}\hat{M}_{\nu\sigma} - i\,\eta_{\nu\sigma}\hat{M}_{\mu\rho} \,. \tag{24}$$

Introducing a new basis J_a and K_a, $a \in \{1, 2, 3\}$, for the algebra,

$$\begin{aligned} J_a &= \frac{1}{2}\varepsilon_{abc}\hat{M}^{bc} \,, \\ K_a &= \hat{M}^{0a} \,, \end{aligned} \tag{25}$$

we find that eq. (24) translates into,

$$\begin{aligned} [J_a, J_b] &= i\varepsilon_{abc}J_c \,, \\ [J_a, K_b] &= i\varepsilon_{abc}K_c \,, \\ [K_a, K_b] &= -i\varepsilon_{abc}J_c \,. \end{aligned} \tag{26}$$

Finally recombine these generators into L_a and R_a,

$$\begin{aligned} L_a &= \frac{1}{2}\left(J_a + i\,K_a\right) , \\ R_a &= \frac{1}{2}\left(J_a - i\,K_a\right) , \end{aligned} \tag{27}$$

which satisfy two commuting copies of the $SU(2)$ Lie algebra,

$$\begin{aligned} [L_a, L_b] &= i\varepsilon_{abc}L_c \,, \\ [R_a, R_b] &= i\varepsilon_{abc}R_c \,, \\ [L_a, R_b] &= 0 \,. \end{aligned} \tag{28}$$

We can use the representation theory developed in the previous section and characterize a representation of the Lorentz algebra by two half-integers

(j_1, j_2), j_1, $j_2 \in \frac{1}{2}\mathbb{N}$, j_1 referring to the representation of the $SU(2)$ generated by the L's and j_2 to the representation of the $SU(2)$ generated by the R's..

Let us illustrate this for spinors. We introduce the Dirac γ-matrices, γ_μ which satisfy,

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2\eta_{\mu\nu} \mathbb{1}_{4\times 4}\,, \\ \gamma_\mu^\dagger &= \gamma_0\gamma_\mu\gamma_0\,. \end{aligned} \tag{29}$$

We also introduce $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, satisfying,

$$\{\gamma_5, \gamma_\mu\} = 0\,, \qquad \gamma_5^\dagger = \gamma_5\,. \tag{30}$$

Under an infinitesimal Lorentz transformation parameterized by $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$, a spinor $\psi(x)$ transforms as (see your QFT course),

$$\delta\psi(x) = \psi'(x') - \psi(x) = -\frac{i}{2}\,\varepsilon^{\mu\nu} S_{\mu\nu}\,\psi(x)\,, \tag{31}$$

with $S_{\mu\nu}$ given by ,

$$S_{\mu\nu} = \frac{i}{4}\,[\gamma_\mu, \gamma_\nu]. \tag{32}$$

Using this one finds,

$$\begin{aligned} L_a &= \frac{1}{2}\,(\frac{1}{2}\,\varepsilon_{abc}S^{bc} + i\,S^{0a}) = -\frac{1}{2}\,\gamma^0\gamma^a P_L\,, \\ R_a &= \frac{1}{2}\,(\frac{1}{2}\,\varepsilon_{abc}S^{bc} - i\,S^{0a}) = +\frac{1}{2}\,\gamma^0\gamma^a P_R\,, \end{aligned} \tag{33}$$

where P_L and P_R resp. are projection operators[4]

$$P_L = \frac{1}{2}\,(\mathbb{1}_{4\times 4} - \gamma_5)\,, \qquad P_L = \frac{1}{2}\,(\mathbb{1}_{4\times 4} + \gamma_5)\,, \tag{34}$$

projecting out left- and right-handed spinors resp.

Calculating the Casimir operator for each of the $SU(2)$'s, one finds,

$$\sum_{a=1}^{3} L_a L_a = \frac{3}{4}\,P_L\,, \qquad \sum_{a=1}^{3} R_a R_a = \frac{3}{4}\,P_R\,, \tag{35}$$

which should be compared to eq. (22). So we conclude that the left-handed spinor transform in the $(j_1 = 1/2, j_2 = 0)$ while a right-handed spinor transforms in the $(j_1 = 0, j_2 = 1/2)$ representation. A Dirac spinor $\psi = \psi_L + \psi_R$ transforms then in the $(j_1 = 1/2, j_2 = 0) \oplus (j_1 = 0, j_2 = 1/2)$ representation.

Continuing like this one gets,

[4] $P_L + P_R = \mathbb{1}_{4\times 4}$, $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$.

- $(0,0)$: the scalar representation,
- $(\frac{1}{2},0)$: the left-handed spinor representation ($\psi_L = \frac{1}{2}(\mathbb{1} - \gamma_5)\psi$),
- $(0,\frac{1}{2})$: the right-handed spinor representation ($\psi_R = \frac{1}{2}(\mathbb{1} + \gamma_5)\psi$),
- $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$: the Dirac spinor representation ($\psi = \psi_L + \psi_r$),
- $(\frac{1}{2},\frac{1}{2})$: the vector representation (A_μ),
- $(1,0)$: the selfdual 2-form ($F_{\mu\nu} = -F_{\nu\mu}$ and $F_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$),
- $(0,1)$: the anti-selfdual 2-form ($F_{\mu\nu} = -F_{\nu\mu}$ and $F_{\mu\nu} = -\frac{1}{2}\,\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$),
- $(1,0) \oplus (0,1)$: the 2-form ($F_{\mu\nu} = -F_{\nu\mu}$),
- $\cdots$

3 The Poincaré algebra and its representations

The Poincaré group consists of translations,

$$x^\mu \to x'^\mu = x^\mu + a^\mu\,, \tag{36}$$

and Lorentz transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,, \tag{37}$$

where,

$$\Lambda^\mu{}_\rho\, \eta^{\rho\sigma}\Lambda^\nu{}_\sigma = \eta^{\mu\nu}\,. \tag{38}$$

For infinitesimal Poincaré transformations we get,

$$\delta x^\mu = \varepsilon^\mu + \varepsilon^\mu{}_\nu\, x^\nu\,, \tag{39}$$

where ε^μ and $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$ real, constant and infinitesimal.

We can rewrite eq. (39) as,

$$\delta x^\mu = -i\,\varepsilon^\rho P_\rho\, x^\mu + \frac{i}{2}\,\varepsilon^{\rho\sigma} M_{\rho\sigma} x^\mu\,, \tag{40}$$

where,

$$P_\mu = i\partial_\mu \quad \text{and} \quad M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\,. \tag{41}$$

The operators P_μ and $M_{\mu\nu}$ generate infinitesimal Poincaré transformations and as to be expected they close under commutation,

$$\begin{aligned}
[P_\mu, P_\nu] &= 0\,, \\
[M_{\mu\nu}, P_\rho] &= i\,\eta_{\nu\rho}P_\mu - i\,\eta_{\mu\rho}P_\nu\,, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\,\eta_{\nu\rho}M_{\mu\sigma} + i\,\eta_{\mu\sigma}M_{\nu\rho} - i\,\eta_{\mu\rho}M_{\nu\sigma} - i\,\eta_{\nu\sigma}M_{\mu\rho}\,. \qquad (42)
\end{aligned}$$

In quantum mechanics the situation is slightly more general as a particle can have spin. The expression for the Lorentz generators is then modified to,

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}\,, \qquad (43)$$

where $S_{\mu\nu}$ are (constant) matrices which are such that the M's defined in eq. (43) still satisfy eq. (42). The operator $S_{\mu\nu}$ is called the spin[5].

One Casimir operator is readily found, it is simply $P^2 = P_\mu P^\mu$. Another one is obtained from the Pauli-Lubanski (pseudo-)vector[6], W^μ,

$$W_\mu \equiv \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}P^\nu S^{\rho\sigma}\,, \qquad (44)$$

which obviously satisfies $W \cdot P = 0$. One readily verifies,

$$\begin{aligned}
[P_\mu, W_\nu] &= 0\,, \\
[M_{\mu\nu}, W_\rho] &= i\,\eta_{\nu\rho}W_\mu - i\,\eta_{\mu\rho}W_\nu\,. \qquad (45)
\end{aligned}$$

From this we also get that,

$$[W_\mu, W_\nu] = i\,\varepsilon_{\mu\nu}{}^{\rho\sigma}P_\rho W_\sigma\,. \qquad (46)$$

Eq. (45) implies that $W^2 = W_\mu W^\mu$ commutes with all generators of the Poincaré algebra, so it is a Casimir operator as well. Its explicit form is given by,

$$W^2 = -\frac{1}{2}\,(P^2 M_{\mu\nu}M^{\mu\nu} - 2P^\rho M_{\rho\mu}P_\sigma M^{\sigma\mu})\,. \qquad (47)$$

The eigenvalues of P^2 and W^2 will characterize the representations of the Poincaré group. Let us start with P^2, its eigenvalue can either be strictly

[5] Stricto senso the spin is the intrinsic part of the angular momentum alone, *i.e.* $\vec{L}_{spin} = (S_{23}, S_{31}, S_{12})$.

[6] In a non-covariant notation the components of the Pauli-Lubanski vector can be written as $W^0 = \vec{P}\cdot\vec{L}_{spin}$ and $\vec{W} = P^0\,\vec{L}_{spin} - \vec{P}\times\vec{K}$, where $\vec{K} = (S^{01}, S^{02}, S^{03})$.

positive, $+M^2 > 0$, strictly negative, $-M^2 < 0$, or zero. When the eigenvalue is strictly negative we are dealing with a tachyon, a particle which is (fortunately) not observed in nature. So we will not consider this case any further. We are left with two cases, either the eigenvalue of P^2 is $M^2 > 0$ or it is $M^2 = 0$. The former are called massive representations (or states or particles) and the latter are the massless representations.

1. $\underline{M^2 > 0}$
   Calling the eigenvalues of P^μ (they mutually commute, so they can be simultaneously diagonalized), p^μ, we get that $p_\mu p^\mu = M^2 > 0$. We can always perform a Lorentz transformation so that the 4-momentum assumes the form $(p^0, \vec{p}) = (M, \vec{0}\,)$. Having done this, we can still perform an arbitrary rotation in $\mathbb{R}^3$ without altering the 4-momentum. So an $SO(3)$, or better its $SU(2)$ cover, subgroup of the Lorentz group survives and an additional feature of a massive state is how it transforms under this subgroup. This was worked out in previous section.

   Concluding: a massive representation is characterized by its mass M, its 3-momentum $\vec{p}$ (its energy E is then given by $E = \sqrt{M^2 + \vec{p} \cdot \vec{p}}$) and a representation of $SU(2)$ characterized by the spin $j \in \frac{1}{2}\mathbb{N}$. As we saw in the first section, such a state has $2j + 1$ degrees of freedom.

   Turning to the Pauli-Lubanski vector we find that in this particular Lorentz frame $W^0 = 0$ and $\vec{W} = M\,\vec{L}_{spin}$, which implies that the eigenvalue of the Casimir operator W^2 is proportional to the $SU(2)$ Casimir: $-M^2 j(j + 1)$, which is of course valid in any Lorentz frame.

2. $\underline{M^2 = 0}$
   Using a Lorentz transformation we can always bring the 4-momentum in the form $(p^0, \vec{p}) = (E, 0, 0, \pm E)$, where $E > 0$. With this we find that the Casimir W^2 is given by,

   $$W^2 = -E^2((L^1 \pm K^2)^2 + (L^2 \mp K^1)^2)\,, \tag{48}$$

   i.e. $W^2 \leqslant 0$. The case $W^2 < 0$ is not realized in nature and will not be considered any further.

   So we have that $P^2 = W^2 = P \cdot W = 0$, implying that W and P are proportional to each other,

   $$W^\mu = h\,P^\mu\,, \tag{49}$$

   where the proportionality constant h is called the helicity. As the name indicates, it is the projection of the spin on the direction of movement.

Indeed combining $\vec{W} = h\,\vec{P}$ with the explicit expression given for $\vec{W}$ in footnote 6, on finds,

$$h = \frac{\vec{p}\cdot\vec{L}_{spin}}{|\vec{p}\,|}\,. \tag{50}$$

Quantum mechanics further restricts the helicity to be half integer,

$$h \in \frac{1}{2}\,\mathbb{Z}\,. \tag{51}$$

Concluding: a massless representation is characterized by its 3-momentum $\vec{p}$ (its energy E is then given by $E = |\vec{p}\,|$) and its helicity $h = \pm j$ where $j \in \frac{1}{2}\mathbb{N}$.

4 An example

Let us illustrate the difference between massless and massive states with an example: the vector field $A_\mu(x)$. Under an infinitesimal Lorentz transformation parameterized by $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ it transforms as,

$$\delta A_\rho(x) = A'_\mu(x') - A_\mu(x) = \varepsilon_\rho{}^\sigma A_\sigma(x) \ = -\frac{i}{2}\,\varepsilon^{\mu\nu}(S_{\mu\nu})_\rho{}^\sigma A_\sigma(x)\,, \tag{52}$$

with,

$$(S_{\mu\nu})_\rho{}^\sigma = i(\eta_{\mu\rho}\delta^\sigma_\nu - \eta_{\nu\rho}\delta^\sigma_\mu)\,. \tag{53}$$

For future reference we also give the components of the spin $\vec{L}_{spin}$,

$$(L^a_{spin})_\rho{}^\sigma \ = \frac{1}{2}\,\varepsilon_{abc}(S^{bc})_\rho{}^\sigma = i\,\varepsilon_{abc}\,\delta^b_\rho\,\eta^{c\sigma}\,. \tag{54}$$

We consider the Lagrange density given by,

$$\mathcal{L} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \frac{M^2}{2}\,A^\mu A_\mu\,, \tag{55}$$

where the field strength $F_{\mu\nu}$ is given by,

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu\,. \tag{56}$$

The equations of motion which follow from eq. (55) are,

$$\Box A^\mu - \partial^\mu\partial_\nu A^\nu + M^2 A^\mu = 0\,. \tag{57}$$

We have to distinguish two cases: $M^2 = 0$ and $M^2 > 0$:

1. $\underline{M^2 = 0}$
   This is just the free Maxwell field. The theory has a gauge invariance,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu f(x)\,, \tag{58}$$

   allowing us to impose the Lorenz gauge,

$$\partial_\mu A^\mu = 0\,, \tag{59}$$

   which does not fully fix the gauge symmetry. Gauge transformations with harmonic functions, $\Box f = 0$, are compatible with the Lorenz gauge. Combining the equation of motion with the gauge choice, we find an equivalent set of equations satisfied by the vector,

$$\begin{aligned} \Box A^\mu &= 0\,, \\ \partial_\mu A^\mu &= 0. \end{aligned} \tag{60}$$

   A complete set of solutions to the first of these equations are plane waves. The second one, the Lorenz condition, eliminates one of the 4 polarization degrees of freedom. A residual gauge transformation allows for eliminating an additional polarization degree of freedom leaving the two transversal polarization degrees of freedom as the physical degrees of freedom (see the first QFT lecture).

   Let us now choose a wave moving in the 3-direction, $p^\mu : (E, 0, 0, E)$ and take[7] for the two physical transversal polarization vectors $\varepsilon^\mu : (0, 1, i, 0)/\sqrt{2}$ and $\bar{\varepsilon}^\mu : (0, 1, -i, 0)/\sqrt{2}$. One immediately verifies that $\vec{p} \cdot \vec{L}_{spin}/|\vec{p}|$ acts diagonally on ε and $\bar{\varepsilon}$ resp. with eigenvalues $+1$ and -1 resp. So we are dealing with a massless spin 1 field with two helicity degrees of freedom: $h = \pm 1$.

2. $\underline{M^2 > 0}$
   The theory has *no* gauge symmetry. Taking the 4-divergence of the equation of motion, eq. (57), we get $\partial_\mu A^\mu = 0$ and we find that eq. (57) is equivalent to,

$$\begin{aligned} \left(\Box + M^2\right) A^\mu &= 0\,, \\ \partial_\mu A^\mu &= 0\,. \end{aligned} \tag{61}$$

   Once again the full set of solutions of the first equation are given by plane waves while the second equation eliminates one polarization degree of freedom leaving 3 physical polarization degrees of freedom. The

[7] Compared to the QFT lectures, we choose here complex transversal polarization vectors so that the the action of $\vec{p} \cdot \vec{L}_{spin}/|\vec{p}|$ is indeed diagonal.

theory describes a massive vector, *i.e.* a spin 1 ($j = 1$) with $2j + 1 = 3$ degrees of freedom.

Let us just verify this for a wave moving in the 3-direction $p^\mu : (E = \sqrt{p^2 + M^2}, 0, 0, p)$ with $p > 0$. Again we choose the two transversal polarization vectors as $\varepsilon^\mu : (0, 1, i, 0)/\sqrt{2}$ and $\bar{\varepsilon}^\mu : (0, 1, -i, 0)/\sqrt{2}$. For the physical longitudinal polarization we choose, $\varepsilon_3^\mu : (p, 0, 0, E)/M$. The eigenvalues of $\vec{p} \cdot \vec{L}_{spin}/|\vec{p}|$ are now $+1$ on ε, 0 on ε_3 and -1 on $\bar{\varepsilon}$, indeed exactly what we expect for a field describing massive spin 1 particles.

A $SU(2)$ versus $SO(3)$

The elements of the group $SO(3)$ are proper rotations in $\mathbb{R}^3$. Euler angles (see your course on analytical mechanics) provide an adequate way to parameterize rotations. One starts by rotating the frame over an angle $\phi \in [0, 2\pi]$ (the "precession angle") around the 3-axis, which is followed by a rotation over an angle $\theta \in [0, \pi]$ (the "nutation angle") around the new 2-axis and one ends by making a rotation over an angle $\psi \in [0, 2\pi]$ (the "proper rotation angle") around the new 3-axis. In this way we obtain a general element $R(\psi, \theta, \phi)$ of the group $SO(3)$,

$$
\begin{aligned}
&R(\psi,\theta,\phi) = \\
&\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}
\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\
&\begin{pmatrix} \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi & \cos\theta\cos\psi\sin\phi + \sin\psi\cos\phi & -\sin\theta\cos\psi \\ -\cos\theta\sin\psi\cos\phi - \cos\psi\sin\phi & \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \sin\theta\sin\psi \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix},
\end{aligned}
\tag{62}
$$

where ψ, $\phi \in \mathbb{R}$ mod 2π and $\theta \in [0, \pi]$.

We now turn to the group $SU(2)$. In the defining representation, this is the group of all 2×2 unitary matrices with determinant equal to $+1$. An arbitrary element U of this group can be written as,

$$
U = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \tag{63}
$$

where u, $v \in \mathbb{C}$ and $|u|^2 + |v|^2 = 1$. Writing $u = y_1 + iy_2$ and $v = y_3 + iy_4$, with $y_a \in \mathbb{R}$, $a \in \{1, \cdots, 4\}$, we can rewrite the latter condition as,

$$\sum_{a=1}^{4} y_a^2 = 1\,, \tag{64}$$

which defines a 3-dimensional sphere S^3 with radius 1. An explicit parameterization of U is the Hopf parameterization,

$$\begin{aligned} U(\psi,\theta,\phi) &= e^{i\frac{\psi}{2}\sigma_3} e^{i\frac{\theta}{2}\sigma_2} e^{i\frac{\phi}{2}\sigma_3} \\ &= \begin{pmatrix} e^{\frac{i}{2}(\psi+\phi)}\cos\frac{\theta}{2} & e^{\frac{i}{2}(\psi-\phi)}\sin\frac{\theta}{2} \\ -e^{-\frac{i}{2}(\psi-\phi)}\sin\frac{\theta}{2} & e^{-\frac{i}{2}(\psi+\phi)}\cos\frac{\theta}{2} \end{pmatrix}, \end{aligned} \tag{65}$$

where $\theta \in [0, \pi]$, $\psi \in \mathbb{R} \bmod 2\pi$ and $\phi \in \mathbb{R} \bmod 4\pi$. The matrices σ_a, $a \in \{1, 2, 3\}$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{66}$$

We will now establish a 2 to 1 relation between $SU(2)$ and $SO(3)$. Introduce a generic 2×2 traceless hermitian matrix $\mathbb{X} = \sum_{a=1}^{3} x^a \sigma_a$, with $x^a \in \mathbb{R}$ and the Pauli matrices σ_a were given in eq. (66). Note that $\det \mathbb{X} = -\sum_{a=1}^{3} (x^a)^2$. Consider now the map,

$$\mathbb{X} \to \mathbb{X}' = U(\psi,\theta,\phi)\, \mathbb{X}\, U(\psi,\theta,\phi)^\dagger\,. \tag{67}$$

As $\mathbb{X}'$ is again hermitian and traceless, it can be written as $\mathbb{X}' = \sum_{a=1}^{3} x'^a \sigma_a$. From the fact that $\det \mathbb{X}' = \det \mathbb{X}$, or $\sum (x'^a)^2 = \sum (x^a)^2$, we get that x' and x are related by a rotation (*i.e.* an element of $SO(3)$). An explicit calculation gives,

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = R(\psi,\theta,\phi) \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{68}$$

where $R(\psi,\theta,\phi)$ was given in eq. (62). However note that for the $SO(3)$ element in eq. (62) $\phi \in \mathbb{R} \bmod 2\pi$ while for the $SU(2)$ element in eq. (65) $\phi \in \mathbb{R} \bmod 4\pi$. So the previous construction mapped two elements of $SU(2)$ to a single element of $SO(3)$:

$$U(\psi,\theta,\phi),\, U(\psi,\theta,\phi+2\pi) \to R(\psi,\theta,\phi)\,. \tag{69}$$

From eq. (65) we get that $U(\psi,\theta,\phi+2\pi) = -U(\psi,\theta,\phi)$. Comparing this to eqs. (63) and (64), one sees that in order to obtain $SO(3)$ from $SU(2)$

one identifies the points (y_1, y_2, y_3, y_4) and $-(y_1, y_2, y_3, y_4)$ on the 3-sphere. Concluding: $SU(2)$ is quivalent to a 3-sphere while $SO(3)$ corresponds to a 3-sphere where the antipodal points are identified.

Finally let us end with some thoughts about infinitesimal $SO(3)$ and $SU(2)$ transformations. As the difference between $SO(3)$ and $SU(2)$ is a global issue, we expect the infinitesimal transformations to be the same. Let us start with an infinitesimal $SU(2)$ transformation. Writing a 2×2 unitary matrix as $U = e^{iH}$ with H a hermitian matrix, we find when requiring that $\det U = 1$ that H has to be traceless[8]. So a 2×2 unitary matrix with determinant 1 which is infinitesimally close to the 2 matrix can be written as,

$$U = \mathbb{1}_{2\times 2} + i \sum_{a=1}^{3} \varepsilon^a t_a \,, \tag{70}$$

where $t_a = \sigma_a/2$ and the Pauli matrices were given in eq. (66). One checks that t_a, $a \in \{1, 2, 3\}$, satisfies eq. (2).

Turning now to infinitesimal rotations in $\mathbb{R}^3$. Rotating a vector $\vec{r} = x^1, x^2, x^3)$ over infintesimal angles $\delta\theta^a$ around the a-axes, $a \in \{1, 2, 3\}$, gives,

$$\begin{pmatrix} \delta x^1 \\ \delta x^2 \\ \delta x^3 \end{pmatrix} = \begin{pmatrix} 0 & \delta\theta^3 & -\delta\theta^2 \\ -\delta\theta^3 & 0 & \delta\theta^1 \\ \delta\theta^2 & -\delta\theta^1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = -i \sum_{a=1}^{3} \delta\theta^a t_a \,, \tag{71}$$

and one verifies that t_a satisfies eq. (2) as well.

So, as expected, we find that infinitesimal $SU(2)$ and $SO(3)$ transformations satisfy the same commutator algebra. Let me end by stating that the representations introduced in eq. (14) are ok for $SU(2)$ however if we restrict ourselves to $SO(3)$, then only $j \in \mathbb{N}$ is allowed.

[8] We used that $\det e^A = e^{\operatorname{tr} A}$.

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