General Relativity 2019-20

Homework 2

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Problem 1

1.

First start by considering the Einstein's equation in vacuum with a non-zero cosmological constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{1}$$

Taking the trace of this equation gives us

$$R - \frac{1}{2}R\,n + \Lambda\,n = 0 \tag{2}$$

where n is the dimension of the space-time. Here we used the identity $g^{\mu}_{\mu} = n$.
Now multiplying both sides of Eq. 2 by $\frac{g_{\mu\nu}}{2-n}$ gives us

$$\frac{1}{2}R\,g_{\mu\nu} - \frac{\Lambda n}{n-2}g_{\mu\nu} = 0.$$

If we add this to Eq. 1 the result is found to be

$$R_{\mu\nu} - \frac{\Lambda}{\frac{n}{2}-1}g_{\mu\nu} = 0. \tag{3}$$

This is exaclty what we were looking for, we conclude that the constant c is given by

$$c = \left(\frac{n}{2} - 1\right)^{-1},$$

note that this constant is reel since the dimension of the universe is also a real scalar.
For a space-time with dimension 4 we get the relation

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

2.

The aim of this exercise is to find the most general spherically symmetric solution to the Einstein's equation in vacuum with a non-zero cosmological constant Λ. This solution has to also reduce to the ordinary Schwarzschild solution when $\Lambda \to 0$.
Following the same reasoning as in chapter 5.2 of Carrol we find that the most general static and spherical symmetric metric is given by Eq. 5.38,

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + d\Omega^2.$$

The Ricci tensor for this general metric has been calculated in Carrol and is given by Eqs. 5.41,

$$\begin{aligned}
R_{tt} &= \left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] + e^{2(\beta-\alpha)}\left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] \\
R_{tr} &= \frac{2}{r}\partial_t\beta \\
R_{\theta\theta} &= e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}
\end{aligned}$$

Now our calculations will not follow the book anymore since the Ricci tensor does not equal zero but rather obeys Eq.3.
From the fact that $R_{tr} = 0$ we get

$$\partial_t\beta = 0$$

Using this equality we find that by taking the derivative of $R_{\theta\theta} = c\Lambda g_{\theta\theta}$ we get,

$$\partial_t\partial_r\alpha = 0$$

We therefore find the same result as in the case where $\Lambda = 0$, namely

$$\begin{aligned}
\beta &= \beta(r) \\
\alpha &= f(r) + g(t).
\end{aligned}$$

This enables us to use a coordinate transformation and redifine our metric as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + d\Omega^2.$$

This result can be seen as a verification of the fact that any spherical symmetric vacuum metric posses an timelike Killing vector.
The resulting components of the Ricci tensor are

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}
\end{aligned}$$

To find some restriction on α and β we calculate

$$e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta)$$

Now substituting Eq. 3 into this expression, we get

$$\begin{aligned} -e^{2(\beta-\alpha)}\,c\Lambda e^{2\alpha} + c\Lambda e^{2\beta} &= \frac{2}{r}(\partial_r\alpha + \partial_r\beta) \\ 0 &= \frac{2}{r}(\partial_r\alpha + \partial_r\beta) \end{aligned}$$

A non-zero cosmological constant thus does not alter the result of this equation. We conclude that $\alpha = -\beta$ also must hold.
Now we look at $R_{\theta\theta}$. Again from $R_{\theta\theta} = c\Lambda g_{\theta\theta}$ we find

$$\begin{aligned} c\Lambda r^2 &= e^{2\alpha}(-r2\partial_r\alpha - 1) + 1 \\ 1 - c\Lambda r^2 &= \partial_r(re^{2\alpha}) \end{aligned}$$

Solving this equation gives us

$$\begin{aligned} R_s + r - \frac{c\Lambda r^3}{3} &= re^{2\alpha} \\ 1 + \frac{R_s}{r} - \frac{c\Lambda r^2}{3} &= e^{2\alpha} \end{aligned}$$

here R_s is some undetermined constant. Using this result our metric becomes

$$ds^2 = -\left(1 + \frac{R_s}{r} - \frac{c\Lambda r^2}{3}\right)dt^2 + \left(1 + \frac{R_s}{r} - \frac{c\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Since our metric should reduce to the ordinary Schwarzschild metric when $\Lambda \to 0$, we can interpret this constant R_s as the Schwarschild radius

$$R_s = -2GM$$

3.

First start by noticing that this metric has two killing vectors leading to conservation of energy and angular momentum, the conservation of angular momentum enables us to choose

$$\theta = \frac{\pi}{2}.$$

The Killing vectors become,

$$\begin{aligned} K_\mu &= \left[-\left(1 - \frac{2GM}{r} - \frac{c\Lambda r^2}{3}\right), 0, 0, 0\right] \\ R_\mu &= (0, 0, 0, r^2) \end{aligned}$$

These lead to the conserved quantities

$$\begin{aligned} E = -K_\mu \frac{dx^\mu}{d\lambda} &= \left(1 - \frac{2GM}{r} - \frac{c\Lambda r^2}{3}\right)\frac{dt}{d\lambda} \\ L = R_\mu \frac{dx^\mu}{d\lambda} &= r^2 \frac{d\phi}{d\lambda} \end{aligned}$$

Using the normalisation of the four momentum we get the following equation

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{4}$$

$$\epsilon = \left(1 - \frac{2GM}{r} - \frac{c\Lambda r^2}{3}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r} - \frac{c\Lambda r^2}{3}\right)^{-1}\left(\frac{dt}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right) \tag{5}$$

Here $\epsilon = 1$ for massive particles and $\epsilon = 0$ for massless particles.
Multiplying both sides of Eq. 5 by $-g_{tt}$ we get,

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r} - \frac{c\Lambda r^2}{3}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0$$

This equation can be rewritten in a more familiar form,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2,$$

where,

$$V(r) = \frac{1}{2}\epsilon - \frac{GM}{r}\epsilon - \frac{c\Lambda r^2}{6}\epsilon + \frac{L^2}{2r^2} - \frac{GML^2}{2r^3} - \frac{c\Lambda L^2}{6}$$

This potential is plotted in Fig. 1. In this figure the potential with $\Lambda = 0$ is presented in blue and $\Lambda \neq 0$ in orange. We see that close to the origin there is not a substantial difference between the two potentials but at large distances the potential with $\Lambda \neq 0$ keeps decreasing. This is due to extra factor of $-r^2$, which will result in the potential dropping to $-\infty$ when $r \to \infty$

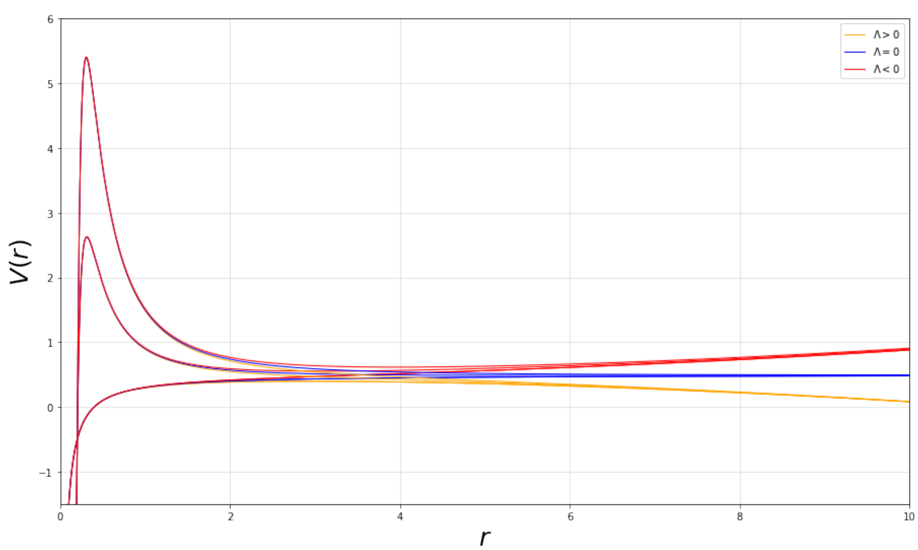

Figure 1: Plot of the Potential with $\Lambda = 0$(blue), $\Lambda > 0$(orange) and $\Lambda < 0$(red)

Problem 2

We consider the kerr metric given by

$$
\begin{aligned}
ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMr}{\rho^2} a \sin^2\theta (dt d\varphi + d\varphi dt) \\
&+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta\right] d\varphi^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(r) &= r^2 - 2GMr + a^2, \\
\rho^2(r, \theta) &= r^2 + a^2 \cos^2\theta.
\end{aligned}
$$

The aim of the exercise is now to calculate the Komar integral given by

$$
J = -\frac{1}{8\pi G} \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} n_\mu \sigma_\nu \nabla^\mu \mathcal{R}^\nu.
$$

Here $\mathcal{R}$ is given by

$$
\mathcal{R}^\mu = (\partial_\varphi)^\mu = \delta^\mu_\varphi = (0, 0, 0, 1)
$$

The vectors n and σ are normal to Σ and $\partial\Sigma$, respectively, and $\sqrt{\gamma^{(2)}}$ is the determinant of the induced metric on $\partial\Sigma$. The induced metric on $\partial\Sigma$ and it's determinant are given by

$$
ds^2 = r^2[d\theta^2 + \sin\theta d\varphi^2] \to \sqrt{\gamma^{(2)}} = r^2 \sin^2\theta
$$

From the fact that n and σ are the normal vectors we derive that the only non-zero components are n_t and σ_r. Now using the relations $n_\mu n^\mu = -1$ and $\sigma^\mu \sigma_\mu = 1$, the normal vectors are given by

$$
\begin{aligned}
g^{\mu\nu} n_\mu n_\nu = -1 &\to g^{tt} n_t n_t = -1 \\
g^{\mu\nu} \sigma_\mu \sigma_\nu = 1 &\to g^{rr} \sigma_r \sigma_r = 1
\end{aligned}
$$

To now calculate these components so that the vectors a properly normalised, the inverse metric $g^{\mu\nu}$ is needed. This is a non trivial calculation due to the cross terms $g_{t\varphi}$ being non zero. Since the inverse of the $r\theta$ part is trivial we only need to compute the inverse of the $t\phi$ block

$$
\tilde{g}_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{t\varphi} \\ g_{\varphi t} & g_{\varphi} \end{pmatrix}
$$

and its determinant is given by

$$
\begin{aligned}
g &= g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2 \\
&= -\left(1 - \frac{2GMr}{\rho^2}\right) \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta\right] - \frac{4G^2M^2r^2}{\rho^4} a^2 \sin^4\theta \\
&= -\left(1 - \frac{2GMr}{\rho^2}\right) \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - [r^2 - 2GMr + a^2] a^2 \sin^2\theta\right] - \frac{4G^2M^2r^2}{\rho^4} a^2 \sin^4\theta \\
&= -\frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - [r^2 - 2GMr + a^2] a^2 \sin^2\theta\right] + \frac{2GMr}{\rho^2} \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - [r^2 + a^2] a^2 \sin^2\theta\right] \\
&= -\Delta \sin^2\theta
\end{aligned}
$$

From this we find

$$\tilde{g}^{\mu\nu} = -\frac{1}{\Delta \sin^2\theta}\begin{pmatrix} g_{\varphi\varphi} & -g_{t\varphi} \\ -g_{t\varphi} & g_{tt} \end{pmatrix}.$$

Using this we easily find the intire inverse metric,

$$g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & 0 & g^{t\varphi} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ g^{t\varphi} & 0 & 0 & g^{\varphi\varphi} \end{pmatrix}$$

where

$$\begin{aligned} g^{tt} &= -\frac{1}{\Delta}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right], \\ g^{t\varphi} &= -\frac{2GMr}{\Delta\rho^2}a, \\ g^{\varphi\varphi} &= \frac{1}{\Delta\sin^2\theta}\left(1-\frac{2GMr}{\rho^2}\right) = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}. \end{aligned}$$

Now we have determined the normal vectors as

$$\begin{aligned} \sigma &= \left(0, \left(\frac{\rho^2}{\Delta}\right)^{\frac{1}{2}}, 0, 0\right), \\ n &= \left(\frac{\sqrt{\Delta}}{\sqrt{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}}, 0, 0, 0\right). \end{aligned}$$

Now we can start to compute the integral. First calcutlate

$$\begin{aligned} n_\mu\sigma_\nu\nabla^\mu\mathcal{R}^\nu &= \frac{\rho}{\sqrt{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}}\nabla^t\mathcal{R}^r \\ &= \frac{\rho}{\sqrt{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}}(g^{tt}\nabla_t\mathcal{R}^r + g^{t\varphi}\nabla_\varphi\mathcal{R}^r) \\ &= \frac{\rho}{\sqrt{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}}\left(g^{tt}\Gamma^r_{t\lambda}\mathcal{R}^\lambda + g^{t\varphi}\Gamma^r_{\varphi\lambda}\mathcal{R}^\lambda\right) \\ &= \frac{\rho}{\sqrt{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}}\left(g^{tt}\Gamma^r_{t\varphi}\mathcal{R}^\varphi + g^{t\varphi}\Gamma^r_{\varphi\varphi}\mathcal{R}^\varphi\right). \end{aligned}$$

Here the Christoffel symbols need to be calculated using

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}).$$

They are found to be

$$\begin{aligned} \Gamma^r_{t\varphi} &= -\frac{2GM\Delta a\sin^2\theta(r^2 - a^2\cos^2\theta)}{2\rho^6}, \\ \Gamma^r_{\varphi\varphi} &= \frac{\Delta\sin^2\theta}{2\rho^6}[-2r\rho^4 + 2GMa^2\sin^2\theta(r^2-a^2\cos^2\theta)]. \end{aligned}$$

Plug this all into the integral and in the same time take the limit of $r\to\infty$ we find

$$\begin{aligned} J &= -\frac{1}{8\pi G}\int_0^\pi d\theta\int_0^{2\pi} d\varphi\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu\mathcal{R}^\nu \\ &= Ma? \end{aligned}$$