3 Exercises for the third lecture day of the course on quantum field theory

3.1 Exercise 1

Given the Dirac hamiltonian,

$$\hat{H} = -i\hbar c \sum_{j=1}^{3} \gamma^0 \gamma^j \partial_j + mc^2 \gamma^0 \,, \tag{1}$$

show that $\hat{\vec{L}} = -i\hbar\, \vec{r} \times \nabla$ does not commute with the hamiltonian while,

$$\hat{\vec{L}} = -i\hbar\, \vec{r} \times \nabla + \frac{\hbar}{2}\, \vec{\sigma} \,, \tag{2}$$

does commute with the hamiltonian. We used the notation $\vec{\sigma}$ for $(\sigma_{23}, \sigma_{31}, \sigma_{12})$ or equivalently $(\sigma^1, \sigma^2, \sigma^3)$ where $\sigma^j = \gamma^5 \gamma^0 \gamma^j$. If time permits, convince yourself that the components of both forms of $\hat{\vec{L}}$ satisfy the same algebra under the commutation (*i.e.* $[\hat{L}_1, \hat{L}_2] = i\hbar\, \hat{L}_3$ and cyclic).

3.2 Exercise 2

Work out the relation between $\gamma^5\, v_r(\vec{p})$ and $\sigma_{\vec{p}}\, v_r(\vec{p})$ in the ultra-relativistic limit ($|\vec{p}\,| \gg mc$).

3.3 Exercise 3

We normalized the spinors in momentum space by,

$$u_r^\dagger(\vec{p}) u_r(\vec{p}) = v_r^\dagger(\vec{p}) v_r(\vec{p}) = \frac{E_{\vec{p}}}{mc^2} \,. \tag{3}$$

Show that besides the various relations we explicitly derived during the lecture we also get that (3) implies that,

$$\begin{aligned} \bar{v}_r(\vec{p}) v_s(\vec{p}) &= -\delta_{rs} \,, \\ \bar{u}_r(\vec{p}) v_s(\vec{p}) &= \bar{v}_r(\vec{p}) u_s(\vec{p}) = 0 \,. \end{aligned} \tag{4}$$

3.4 Exercise 4

Given a particle with mass m, charge q and position vector $\vec{r}$. It moves in an electromagnetic field characterized by $\vec{E}$ and $\vec{B}$. Newton's second law gives,

$$m\, \ddot{\vec{r}} = \vec{F}_L \,, \tag{5}$$

where $\vec{F}_L$ is the Lorentz force,

$$\vec{F}_L = q\Big(\vec{E} + \frac{1}{c}\,\dot{\vec{r}} \times \vec{B}\Big)\,. \tag{6}$$

We solve the homogeneous Maxwell equations by introducing a scalar and a vector potential ϕ and $\vec{A}$,

$$\vec{B} = \nabla \times \vec{A}\,, \qquad \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,. \tag{7}$$

Show that the components of the Lorentz force can be written as,

$$F_x = -\frac{\partial V}{\partial x} + \frac{d}{dt}\frac{\partial V}{\partial \dot{x}}\,, \tag{8}$$

and similarly for F_y and F_z where V is given by,

$$V = q\big(\phi - \frac{1}{c}\,\dot{\vec{r}} \cdot \vec{A}\big)\,. \tag{9}$$

3.5 Exercise 5

Having covered the Dirac equation and the Dirac field during the lecture, carefully (re)read/study appendix A and chapter 4 in Mandl and Shaw. You can skip those parts which discuss *Majorana fermions*. This will be studied in great detail (in the context of neutrino physics) and in a much clearer way in the course *Electroweak and Strong Interactions* next semester.

3.6 Exercise 6

Carefully study section 6.1 in Mandl and Shaw. Determine yourself the dimensions of a scalar, the Maxwell and the Dirac field in both the SI system and in the natural system.

3.7 Exercise 7

While we covered all of it in the lecture, study section 1.5 and 6.2 in Mandl and Shaw. In order to prepare for next lecture, read section 6.3 as well.