Quantum Field Theory 3

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1 Exercise 1

The commutator with the Hamiltonian with a component of the angular momentum $\hat{\vec{L}} = -i\hbar\vec{r} \times \nabla$ is

$$
\begin{aligned}
[\hat{H}, \hat{L}_i] &= [\hat{H}, \varepsilon_{ijk} r_j p_k] && (1)\\
&= [-i\hbar c\gamma^0\gamma^l\partial_l + mc^2\gamma^0, \varepsilon_{ijk} r_j p_k] && (2)\\
&= c\gamma^0\gamma^l[p_l, \varepsilon_{ijk} r_j p_k] && (3)\\
&= \varepsilon_{ijk} c\gamma^0\gamma^l[p_l, r_j]p_k && (4)\\
&= -i\hbar\varepsilon_{ijk} c\gamma^0\gamma^l\delta_{lj}p_k && (5)\\
&= -i\hbar c\gamma^0(\varepsilon_{ilk}\gamma^l p_k) && (6)\\
&= -i\hbar c\gamma^0(\vec{\gamma} \times \vec{p})_i, && (7)
\end{aligned}
$$

so that

$$[\hat{H}, \hat{\vec{L}}] = -i\hbar c\gamma^0(\vec{\gamma} \times \vec{p}). \quad (8)$$

Next

$$
\begin{aligned}
[\hat{H}, \frac{\hbar}{2}\sigma^j] &= \frac{\hbar}{2}[\hat{H}, \gamma^5\gamma^0\gamma^j] && (9)\\
&= \frac{\hbar}{2}[-i\hbar c\gamma^0\gamma^l\partial_l + mc^2\gamma^0, \gamma^5\gamma^0\gamma^j] && (10)\\
&= \frac{\hbar}{2}c[\gamma^0\gamma^l, \gamma^5\gamma^0\gamma^j]p_l + \frac{\hbar}{2}mc^2[\gamma^0, \gamma^5\gamma^0\gamma^j]. && (11)\\
& && (12)
\end{aligned}
$$

Now

$$\sum_l [\gamma^0\gamma^l, \gamma^5\gamma^0\gamma^j] = \sum_l (\gamma^0\gamma^l\gamma^5\gamma^0\gamma^j - \gamma^5\gamma^0\gamma^j\gamma^0\gamma^l) \tag{13}$$
$$= \sum_l (\gamma^l\gamma^5\gamma^j + \gamma^5\gamma^j\gamma^l) \tag{14}$$
$$= \gamma_5 \sum_l ([\gamma^j, \gamma^l]) \tag{15}$$
$$= 2\gamma_5 \sum_{l\neq j} \gamma^j\gamma^l, \tag{16}$$

and

$$[\gamma^0, \gamma^5\gamma^0\gamma^j] = \gamma^0\gamma^5\gamma^0\gamma^j - \gamma^5\gamma^0\gamma^j\gamma^0 \tag{17}$$
$$= -\gamma^5\gamma^j + \gamma^5\gamma^j = 0. \tag{18}$$

Therefore

$$[\hat{H}, \frac{\hbar}{2}\sigma^j] = \hbar c\gamma^5\gamma^j \sum_{l\neq j} \gamma^l p_l. \tag{19}$$

When f.e. $j = 1$, then

$$[\hat{H}, \frac{\hbar}{2}\sigma^j] = \hbar c\gamma^5\gamma^1 \sum_{l\neq j} \gamma^l p_l \tag{20}$$
$$= i\hbar c\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1(\gamma^2 p_2 + \gamma^3 p_3) \tag{21}$$
$$= -i\hbar c\gamma^0(\gamma^2\gamma^3\gamma^2 p_2 + \gamma^2\gamma^3\gamma^3 p_3) \tag{22}$$
$$= -i\hbar c\gamma^0(\gamma^3 p_2 - \gamma^2 p_3) \tag{23}$$
$$= i\hbar c\gamma^0(\vec{\gamma} \times \vec{p})_1. \tag{24}$$

Doing an analogue calculation for the other components, we obtain

$$[\hat{H}, \frac{\hbar}{2}\vec{\sigma}] = i\hbar c\gamma^0(\vec{\gamma} \times \vec{p}). \tag{25}$$

As a result

$$[\hat{H}, -i\hbar\vec{r} \times \nabla + \frac{\hbar}{2}\vec{\sigma}] = 0. \tag{26}$$

2 Exercise 2

The Dirac equation for the spinor $v_r(\vec{p})$, gives

$$(\not{p} + mc)v_r(\vec{p}) = 0.$$

This means

$$p_0\gamma^0 v_r(\vec{p}) = -p_j\gamma^j v_r(\vec{p}) - mc\, v_r(\vec{p})$$
$$\gamma^0 v_r(\vec{p}) = \frac{p^j\gamma^j}{p^0} v_r(\vec{p}) - \frac{mc}{p^0} v_r(\vec{p})$$

where in the last step we divided by p_0 and raised all the indices.
Premultiplying the previous equation by $\gamma^5\gamma^0$ gives

$$\gamma^5 v_r(\vec{p}) = \frac{p^j\gamma^5\gamma^0\gamma^j}{p^0} - \frac{mc}{p_0}\gamma^5\gamma^0 v_r(\vec{p})$$

Note that for a relativistic particle the following relation holds

$$\frac{1}{p_0} = \frac{1}{\sqrt{\vec{p}^{\,2} + m^2c^2}} = \frac{1}{|\vec{p}|}\Big(1 + \mathcal{O}\Big(\frac{m^2c^2}{\vec{p}^{\,2}}\Big)\Big)$$

Further using the relation $\sigma^j = \gamma^5\gamma^0\gamma^j$ we find

$$\gamma^5 v_r(\vec{p}) = \sigma_{\vec{p}}\, v_r(\vec{p}) + \mathcal{O}\Big(\frac{mc}{|\vec{p}|}\Big). \tag{27}$$

In the ultra-relativistic limit where $|\vec{p}| \gg mc$ or when the particle is massless equation (27) becomes

$$\gamma^5 v_r(\vec{p}) = \sigma_{\vec{p}}\, v_r(\vec{p}).$$

3 Exercise 3

We first note that

$$\overline{v}_r(\vec{p})\sigma_{\vec{p}} v_s(\vec{p}) = \overline{v}_r(\vec{p})(\sigma_{\vec{p}} v_s(\vec{p})) = (-1)^s \overline{v}_r(\vec{p}) v_s(\vec{p}), \tag{28}$$

$$\overline{v}_r(\vec{p})\sigma_{\vec{p}} v_s(\vec{p}) = \overline{(\sigma_{\vec{p}} v_r(\vec{p}))} v_s(\vec{p}) = (-1)^r \overline{v}_r(\vec{p}) v_s(\vec{p}), \tag{29}$$

since $\sigma_{\vec{p}}$ is Hermitian, so that

$$\overline{v}_r(\vec{p}) v_s(\vec{p}) \sim \delta_{rs}. \tag{30}$$

Further

$$\not{p} v_r(\vec{p}) = -mc v_r(\vec{p}), \tag{31}$$

from which it follows that

$$\gamma^0 p_0 v_r(\vec{p}) = -\gamma^j p_j v_r(\vec{p}) - mc v_r(\vec{p}), \tag{32}$$

$$\begin{aligned}\overline{v_r(\vec{p})}v_r(\vec{p}) = v_r(\vec{p})^\dagger\gamma^0 v_r(\vec{p}) &= -v_r(\vec{p})^\dagger\gamma^j v_r(\vec{p})\frac{p_j}{p_0} - \frac{mc}{p_0}v_r(\vec{p})^\dagger v_r(\vec{p}) && (33)\\ &= -v_r(\vec{p})^\dagger\gamma^j v_r(\vec{p})\frac{p_j}{p_0} - \frac{mc}{p_0}\frac{E_{\vec{p}}}{mc^2} && (34)\\ &= -v_r(\vec{p})^\dagger\gamma^j v_r(\vec{p})\frac{p_j}{p_0} - 1. && (35)\end{aligned}$$

The left hand side of the equation is real

$$\begin{aligned}[v_r(\vec{p})^\dagger\gamma^0 v_r(\vec{p})]^\dagger &= v_r(\vec{p})^\dagger\gamma^{0\dagger} v_r(\vec{p}) && (36)\\ &= v_r(\vec{p})^\dagger\gamma^0 v_r(\vec{p}), && (37)\end{aligned}$$

while the first term at the right hand side is imaginary

$$\begin{aligned}[v_r(\vec{p})^\dagger\gamma^j v_r(\vec{p})]^\dagger &= v_r(\vec{p})^\dagger\gamma^{j\dagger} v_r(\vec{p}) && (38)\\ &= -v_r(\vec{p})^\dagger\gamma^j v_r(\vec{p}), && (39)\end{aligned}$$

because

$$\gamma^{j\dagger} = \gamma^0\gamma^j\gamma^0 = -\gamma^j\gamma^0\gamma^0 = -\gamma^j. \tag{40}$$

Therefore

$$\overline{v_r(\vec{p})}v_r(\vec{p}) = -1, \tag{41}$$

and

$$\overline{v_r(\vec{p})}v_s(\vec{p}) = -\delta_{rs}. \tag{42}$$

Equations A.25, A.26 in Mandl and Shaw lead to

$$\overline{u}_r(\vec{p}) = \overline{u}_r(\vec{p})\frac{\not{p}}{mc}, \tag{43}$$

$$v_s(\vec{p}) = -\frac{\not{p}}{mc}v_s(\vec{p}). \tag{44}$$

These lead to

$$\overline{u}_r(\vec{p})v_s(\vec{p}) = -\overline{u}_r(\vec{p})\frac{\not{p}^2}{m^2c^2}v_s(\vec{p}) = -\overline{u}_r(\vec{p})v_s(\vec{p}), \tag{45}$$

because $\not{p}^2 = m^2c^2$ using A.19b. As a result

$$\overline{u}_r(\vec{p})v_s(\vec{p}) = 0, \tag{46}$$

and

$$\overline{v}_r(\vec{p})u_s(\vec{p}) = (\overline{u}_s(\vec{p})v_r(\vec{p}))^\dagger = 0. \tag{47}$$

4 Exercise 4

Given is the description of the Lorentz force,

$$\vec{F} = q\Big(\vec{E} + \frac{1}{c}\dot{\vec{r}} \times \vec{B}\Big).$$

Now substitute the scalar and vector potential ϕ and $\vec{A}$ into the previous equation using

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

We find

$$\vec{F} = q\Big(-\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\dot{\vec{r}} \times \big(\nabla \times \vec{A}\big)\Big).$$

Further using the triple product identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

the previous equation becomes

$$\vec{F} = q\Big(-\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\nabla\big(\dot{\vec{r}} \cdot \vec{A}\big) - \frac{1}{c}\big(\dot{\vec{r}} \cdot \nabla\big)\vec{A}\Big).$$

The total derivative of $\vec{A}$ is given by

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \big(\dot{\vec{r}} \cdot \nabla\big)\vec{A}$$

Therefore

$$\vec{F} = q\Big(-\nabla(\phi - \frac{1}{c}\dot{\vec{r}} \cdot \vec{A}) - \frac{1}{c}\frac{d\vec{A}}{dt}\Big)$$

From this we find the x-component is

$$\begin{aligned} F_x &= -\frac{\partial\, q\big(\phi - \frac{1}{c}\dot{\vec{r}} \cdot \vec{A}\big)}{\partial x} - \frac{q}{c}\frac{dA_x}{dt} \\ &= -\frac{\partial\, q\big(\phi - \frac{1}{c}\dot{\vec{r}} \cdot \vec{A}\big)}{\partial x} + \frac{d}{dt}\frac{\partial\, q\big(\phi - \frac{1}{c}\dot{\vec{r}} \cdot \vec{A}\big)}{\partial \dot{x}} \end{aligned}$$

where in the last step we used the fact that the scalar potential is only a function of x and t.