A Brief Introduction to Bayesian Statistics

Frameworks of Statistical Inference

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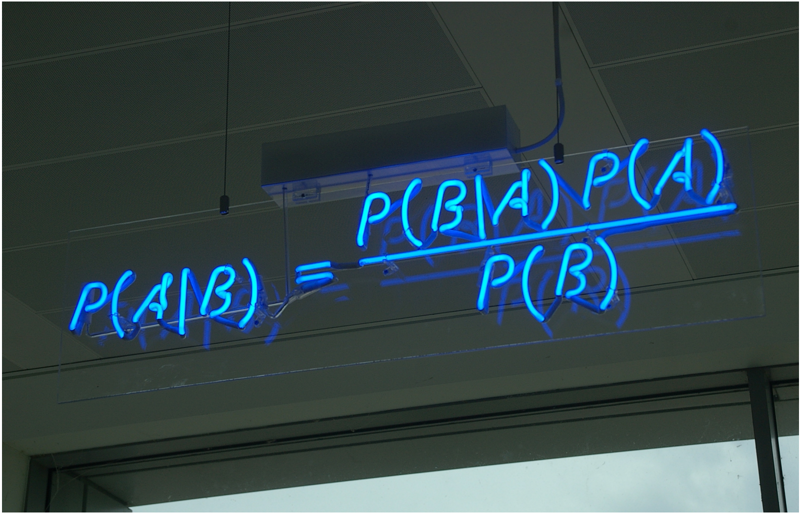
- Bayesian: Using prior information and data, evaluate the degree of belief in specific hypotheses, recognizing that data is one realization of some distribution of a parameter.

Bayes Theorem



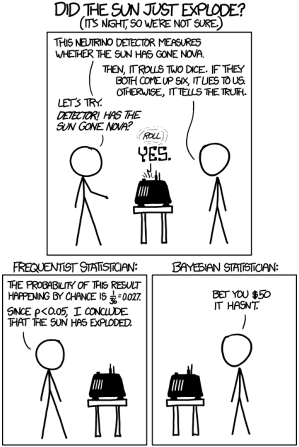
Bayes Theorem

$$p(Hypothesis|Data) = \frac{P(Data|Hypothesis)p(Hypothesis)}{p(Data)}$$

Bayes Theorem

$$p(\theta|X) = \frac{p(X|\theta)P(\theta)}{p(X)}$$

Bayes Theorem in Action


DID THE SUN JUST EXPLODE?
(IT'S NIGHT, SO WE'RE NOT SURE.)
THIS NEUTRINO DETECTOR MEASURES WHETHER THE SUN HAS GONE NOVA.
THEN, IT ROLLS TWO DICE. IF THEY BOTH COME UP SIX, IT LIES TO US. OTHERWISE, IT TELLS THE TRUTH.
LET'S TRY.
DETECTOR! HAS THE SUN GONE NOVA?
ROLL
YES.
FREQUENTIST STATISTICIAN:
THE PROBABILITY OF THIS RESULT HAPPENING BY CHANCE IS $\frac{1}{36}=0.027$. SINCE $p<0.05$, I CONCLUDE THAT THE SUN HAS EXPLODED.
BAYESIAN STATISTICIAN:
BET YOU $50 IT HASN'T.

http://xkcd.com/1132/

Bayes Theorem in Action

$$p(SunExplodes|Yes) = \frac{p(Yes|SunExplodes)p(SunExplodes)}{p(Yes)}$$

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We know/assume:
p(Sun Explodes) = 0.0001, P(Yes | Sun Explodes) = 35/36

Bayes Theorem in Action

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We know/assume:
p(Sun Explodes) = 0.0001, P(Yes | Sun Explodes) = 35/36

We can calculate:
p(Yes) = P(Yes | Sun Explodes)p(Sun Explodes) + P(Yes | Sun Doesn't Explode)p(Sun Doesn't Explodes)

$$= 35/36 * 0.0001 + 1/36 * 0.9999 = 0.0277775$$

credit: Amelia Hoover

Bayes Theorem in Action

$$p(SunExplodes|Yes) = \frac{p(Yes|SunExplodes)p(SunExplodes)}{p(Yes)}$$

$$p(SunExplodes|Yes) = \frac{0.0001 * 35/36}{0.028} = 0.0035$$

Incorporating Prior Information about the Sun Exploding gives us a very different answer

Bayes Theorem in Action

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$$p(SunExplodes|Yes) = \frac{0.0001 * 35/36}{0.028} = 0.0035$$

Incorporating Prior Information about the Sun Exploding gives us a very different answer

Note, we can also explicitly evaluate the probability of an alternate hypothesis - p(Sun Doesn't Explode | Yes)

The Marginal Distribution in the Denominator

$$p(\theta|X) = \frac{p(X|\theta)P(\theta)}{\sum_{i=0}^{j} p(X|\theta_i)p(\theta_i)}$$

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What are alternate parameter values but alternate hypotheses?

Denominator - marginal distribution - becomes an integral of likelihoods if θ is continuous. It normalizes the equation to be between 0 and 1.

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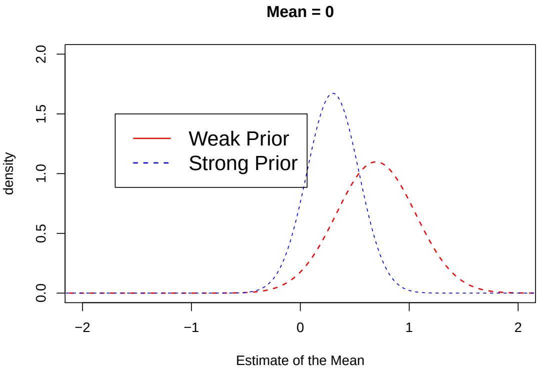
- Conjugate priors make some forms of Bayes Theorem analytically solveable
- If we have objective prior information - from pilot studies or the literature - we can use it to obtain a more informative posterior distribution
- If we do not, we can use a weak or flat prior (e.g., N(0,1000)). Note: constraining the range of possible values can still be weakly informative - and in some cases beneficial

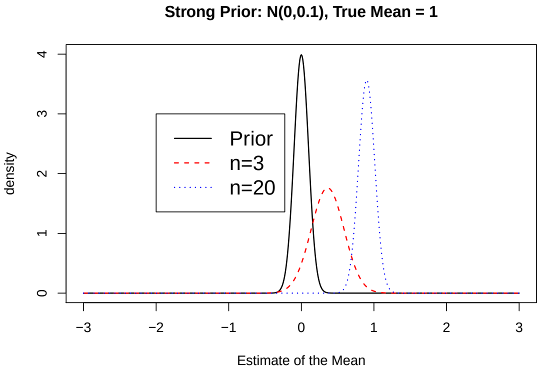
The Influence of Priors

Here’s the posterior distribution drawn using the same sample - but in one case with a weak prior, and one a strong prior.



Priors and Sample Size

The influence of priors decreases with same size. A large sample size 'overwhelms' the prior.



Evaluation of a Posterior: Frequentist Confidence Intervals

In Frequentist analyses, the **95% Confidence Interval** of a parameter is the region in which, were we to repeat the experiment an infinite number of times, the *true value* would occur 95% of the time. For normal distributions of parameters:

$$\hat{\beta} - t(\alpha, df)SE_{\beta} \leq \beta \leq \hat{\beta} + t(\alpha, df)SE_{\beta}$$

Evaluation of a Posterior: Bayesian Credible Intervals

In Bayesian analyses, the **95% Credible Interval** is the region in which we find 95% of the possible parameter values. The observed parameter is drawn from this distribution. For normally distributed parameters:

$$\hat{\beta} - 2 * \hat{SD} \leq \hat{\beta} \leq \hat{\beta} + 2 * \hat{SD}$$

where $\hat{SD}$ is the SD of the posterior distribution of the parameter β. Note, for other types of parameters, the distribution may be different.

Bayes Theorem Expanded

$$p(\theta|X) = \frac{p(X|\theta)P(\theta)}{\sum_{i=0}^{j} p(X|\theta_i)p(\theta_i)}$$ - Algebraically Solvable

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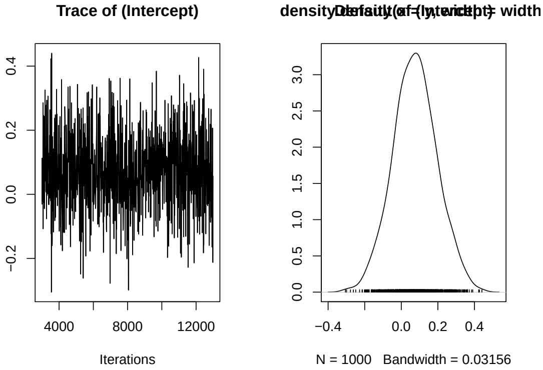
$p(\theta|X) = \frac{\int p(X|\theta)P(\theta|\eta)p(\eta)d\eta}{\int\int p(X|\theta)p(\theta)d\theta d\eta}$ - Hierarchical Model: need numerical integration approach with random hyperparameters

Markov Chain Monte Carlo Sampling (MCMC)



Markov Chain Monte Carlo Sampling (MCMC)

If we cannot analytically solve a distribution, we can still simulate from it:

- Chose a set of starting values X at t=0
- Chose a random set of parameters, Y, from the distribution parameterized by X
- Select a uniorm random number between 0 and 1, U
- If U $\leq$ f(X,Y), X(t+1) = Y. Otherwise, X(t+1) = X.
- Rinse and repeat

Markov Chain Monte Carlo Sampling (MCMC)

This is a time series. To use it for inference to sample from the final stationary distribution:

- Discard a 'burn in' set of samples
- 'Thin' your chain to reduce temporal autocorrelation
- Examine chain for convergence on your posterior distribution
- Evaluate multiple chains to ensure convergence to a single distribution

Many different samplers using different decision rules for f. We use the Gibbs Sampler commonly.

Software Options for MCMC

- WinBUGS `http://www.mrc-bsu.cam.ac.uk/bugs/`
- OpenBUGS `http://www.openbugs.info/w/`
- JAGS `http://mcmc-jags.sourceforge.net/`
- STAN `http://mc-stan.org/`
- MCMCglmm in R
- MCMCpack in R

BUGS code for a Simple Linear Regression

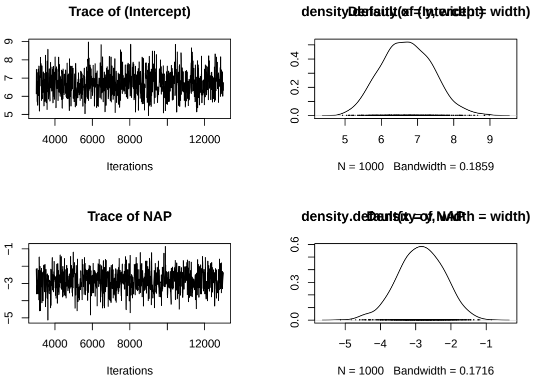
```
model {
# Prior
    alpha ~ dnorm(0,0.001)
    beta ~ dnorm(0,0.001)
    sigma ~ dunif(0,100)
# Likelihood
for (i in 1:n){
    y[i] ~ dnorm(mu[i],tau)
    mu[i] <- alpha + beta*x[i]
  }
}
```

Example: The RIKZ Beaches and Tide Height

```r
rikz <- read.csv("./data/rikz.csv")
rikz$Beach <- factor(rikz$Beach)
#
library(MCMCglmm)
NAPMod <- MCMCglmm(Richness ~ NAP, data=rikz, verbose=F)
```

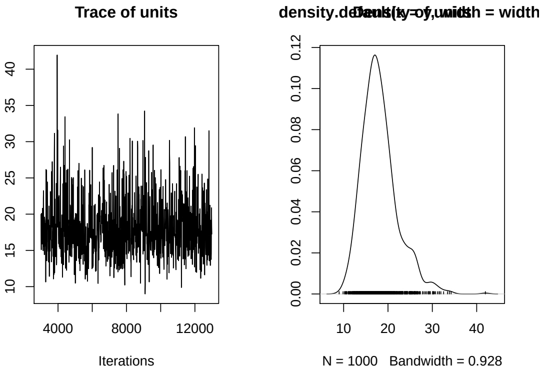
Plots of Chains

```r
plot(NAPMod$Sol)
```

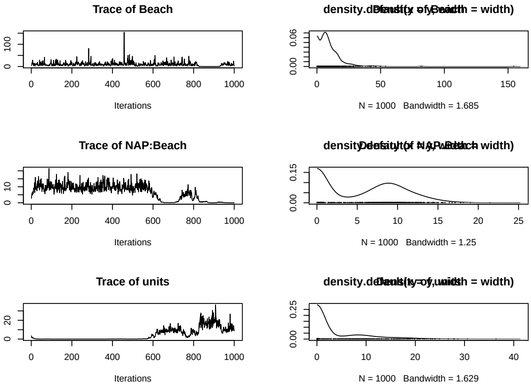


Plots of Chains

```r
plot(NAPMod$VCV)
```



Sometimes Problems are Obvious



Did you Thin Enough?

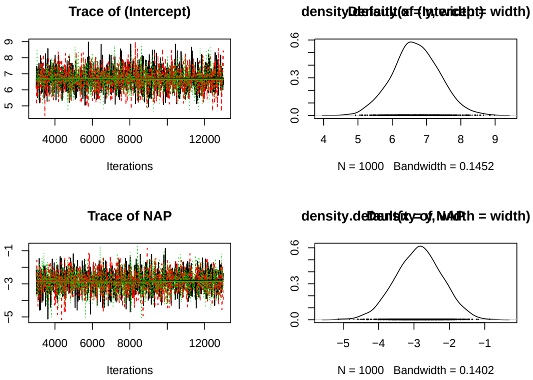
```
autocorr(NAPMod$Sol)

# , , (Intercept)
#
#          (Intercept)       NAP
# Lag 0       1.000000 -0.338722
# Lag 10      0.030775 -0.009913
# Lag 50      0.003102 -0.015534
# Lag 100    -0.050866  0.031194
# Lag 500    -0.077621  0.031906
#
# , , NAP
#
#          (Intercept)      NAP
# Lag 0      -0.338722  1.00000
# Lag 10     -0.008204  0.01056
# Lag 50     -0.008502  0.03985
# Lag 100     0.038907 -0.01880
# Lag 500     0.046233  0.01864
```

Did You Converge: Assessing with Multiple Chains

```r
NAPMod2 <- MCMCglmm(Richness ~ NAP, data=rikz, verbose=F)
NAPMod3 <- MCMCglmm(Richness ~ NAP, data=rikz, verbose=F)
#
library(coda)
chainList <- mcmc.list(NAPMod$Sol, NAPMod2$Sol, NAPMod3$Sol)
```

Did You Converge: Assessing with Multiple Chains

```r
plot(chainList)
```



The Gelman-Rubin Diagnostic

Diagnostic should be close to 1.

```
gelman.diag(chainList)

# Potential scale reduction factors:
#
#             Point est. Upper C.I.
# (Intercept)          1       1.01
# NAP                  1       1.00
#
# Multivariate psrf
#
# 1
```

Evaluating Results

```
summary(NAPMod)

#
#  Iterations = 3001:12991
#  Thinning interval  = 10
#  Sample size  = 1000
....
```

Evaluating Results

```
summary(NAPMod)


....


#  R-structure:  ~units
#
#        post.mean l-95% CI u-95% CI eff.samp
# units      18.2     11.4     26.7      867
#
....
```

Evaluating Results

```
summary(NAPMod)


....

#  Location effects: Richness ~ NAP
#
#             post.mean l-95% CI u-95% CI eff.samp  pMCMC
# (Intercept)      6.71     5.34     8.00     1000 <0.001
# NAP             -2.85    -4.02    -1.54     1000 <0.001
```

Your 95% Credible Interval

```
HPDinterval(NAPMod$Sol)

#                 lower  upper
# (Intercept)  5.345  7.997
# NAP         -4.019 -1.536
# attr(,"Probability")
# [1] 0.95
```

The Bayesian Approach to MMI: The DIC

$$DIC = D\bar(\theta) + pD$$

from Spiegelhalter et al 2002

$D\bar(\theta)$ is the average deviance and pD = Effective # of parameters

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The Bayesian Approach to MMI: The DIC

$$DIC = D\bar(\theta) + pD$$

from Spiegelhalter et al 2002

$D\bar(\theta)$ is the average deviance and pD $=$ Effective $\#$ of parameters

$pD = D\bar(\theta) - D(\bar{\theta})$

```
NAPMod$DIC

# [1] 260
```

Setting Priors

```
prior<-list(B=list(mu=c(0,-3),V=diag(c(1e+10, 1))))
#
NAPMod_Prior <- MCMCglmm(Richness ~ NAP,
                         data=rikz, verbose=F, prior=prior)
```

Strong Priors Can Alter Parameters

```
summary(NAPMod)$solutions

#             post.mean l-95% CI u-95% CI eff.samp pMCMC
# (Intercept)     6.710    5.345    7.997     1000 0.001
# NAP            -2.849   -4.019   -1.536     1000 0.001

summary(NAPMod_Prior)$solutions

#             post.mean l-95% CI u-95% CI eff.samp pMCMC
# (Intercept)     6.696    5.355    7.863     1000 0.001
# NAP            -2.884   -3.838   -1.917     1000 0.001
```

Random Effects

MCMCglmm allows random effects & family much like nlme

```
MCMCglmm(y ~ x, random = z + x:z)
```

Implies that the intercept varies randomly by z and the slope of x varies by z. Equivalent to (1+x | z)

Exercise: Off the MCMC Shorline

- Fit a model with a NAP*angle1 interaction and random effect of beach
- Evaluate the model and whether it is fit well
- Compare the coefficients to a model with a strong prior that the interaction is -5.