Delay Differential Equations

D.D.E. with constant lags $\tau_i > 0$ have the general form

$$\dot{x}(t) = f\left(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_k)\right)$$

Let's focus on single delay:

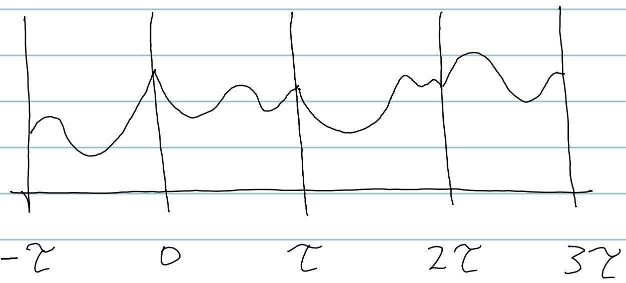
$$\dot{x} = f(x(t), x(t-\tau))$$

The solution is a mapping from the interval $[t-\tau, t]$ to the interval $[t, t+\tau]$:

$$x : f([t-\tau, t]) \longrightarrow f([t, t+\tau])$$

i.e. the solution can be thought of as a sequence of functions $f_0(t), f_1(t), \ldots$ defined over a set of contiguous time intervals of length τ.

The points $t = 0, \tau, 2\tau, \dots$ are called knots. The solution often has discontinuities at the knots.



Example:

$$\dot{x} = (t-1)\cdot x$$

Suppose we have $x(t) = f_{i-1}(t)$ for some interval $[t_{i-1}, t_i]$. Then, over the interval $[t_i, t_{i+1}]$ we have

$$\int_{f_{i-1}(t_i)}^{x(t)} dx' = \int_{t_i}^{t} f_{i-1}(t'-1)\,dt'$$

Aside: Recall: $\frac{dx}{dt} = f_{i-1}$, which is a separable equation. Then $\int dx = \int f_{i-1}\,dt$

$$\Rightarrow x(t) = f_i(t) = f_{i-1}(t) - \int_{t_i}^{t} f_{i-1}(t-1)\,dt$$

This example illustrates the method of steps

Assume that I. C. is $x(t) = 1$ for $t \in [-1, 0]$.

In the interval $[0,1]$ we have

$$x(t) = 1 - \int_0^t 1\,dt = 1 - t$$

In the interval $[1,2]$ we have

$$x(t) = 0 - \int_1^t [1 - (t'-1)]\,dt'$$

$$= -\left[2t' - \frac{1}{2}t'^2\right]_1^t$$

$$= -2t + \frac{1}{2}t^2 + \frac{3}{2}$$

In the interval $[2,3]$, the solution is

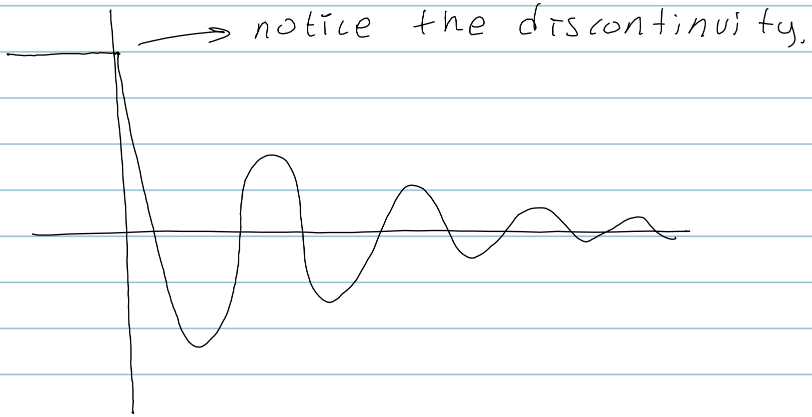
$$x(t) = -\frac{1}{2} - \int_2^t \left[-2(t'-1) + \frac{1}{2}(t'-1)^2 + \frac{3}{2}\right] dt'$$

$$= -\frac{1}{2} - \left[-(t'-2)^2 + \frac{1}{6}(t'-1)^3 + \frac{2}{3}t'\right]_2^t$$

$$= 5/3$$

and so on.

The solution can be computed, and looks like:



Numerical methods can yield wrong results if the assume continuous derivatives at the knots.