Numerical Solutions of ODEs

So far: Qualitative information and asymptotic behavior.

Need: Quantitative information

Use numerical methods to approximate trajectories.

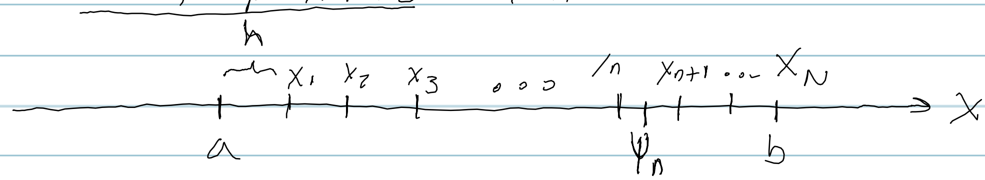
General case: $\frac{dy}{dx} = f(x,y)$, $a \leq x \leq b$

(possibly non-autonomous)

$y(a) = \alpha$, initial condition.

The simplest numerical method is Euler's method: Partition the interval $[a,b]$ into N equal subintervals of length $h = \frac{b-a}{N}$ where h is called "step size".

The goal is to find approximate solutions at the mesh points x_n



At $y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(\psi_n)$

by Taylor expansion

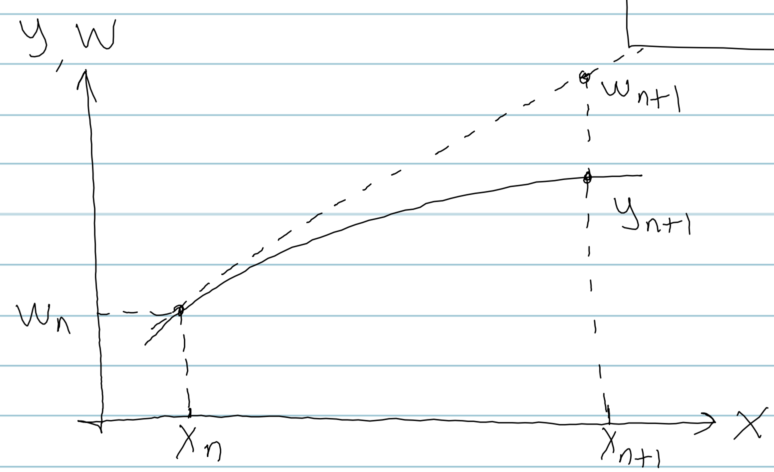
$$= y(x_n) + h f(x_n, y_n) + \frac{h^2}{2} y''(\psi_n)$$

where $\psi_n \in [x_n, x_{n+1}]$

To get Euler's method, just keep linear terms. Let $\boxed{w_n}$ be the ~~approximate~~ approximate value for y_n using Euler's approximation. Then,

$$\boxed{\begin{aligned} w_{n+1} &= w_n + h f(x_n, w_n) \\ w_0 &= \alpha \end{aligned}}$$

Euler's Method

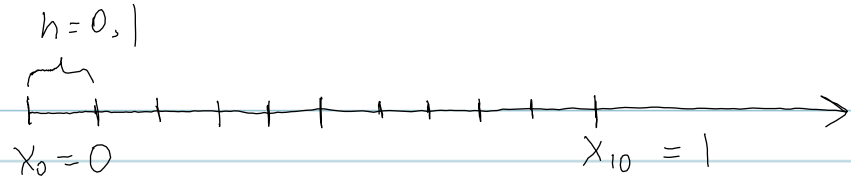


The first order differential equation is approximated by a 1st order difference equation.

This raises two issues: (i) truncation error
(ii) stability

Example: $\dfrac{dy}{dx} = x^2 + y$ on $[0,1]$

Discretize with $N=10$ subintervals:



Euler approximation:

$$w_{n+1} = w_n + 0.1\,(x_n^2 + w_n)\,, \quad n = 0 \dots 9$$

$$w_0 = 2$$

Build a table:

n	$w_n + 0.1(x_n^2 + w_n)$	=
0	$2 + 0.1(0 + 2)$	2.2
1	$2.2 + 0.1(0.1 + 2.2)$	2.43
⋮		

HW: complete the table and compare with exact solution.

Local truncation errors measure the amount by which the exact solution (y) fails to satisfy the difference equation for the approximation.

Euler:
$$\frac{w_{n+1} - w_n}{h} = f(x_n, w_n)$$

$$\rightarrow \frac{w_{n+1} - w_n}{h} - f(x_n, w_n) = 0$$

y_n will not satisfy this equation in general.

Deviation from 0 is the truncation error.

Euler truncation error at x_n

is: $\tau_n = \dfrac{y_{n+1} - y_n}{h} - f(x_n, y_n)$

Note: If $\tau_n = 0$ $\forall$ $n = 0, \ldots, N-1$, then exact and approximate solutions are the same for $[a, b]$.

Need to know something about τ_n without knowing the y_n's. From Taylor:

$$y_{n+1} = y_n + h f(x_n, y_n) + \frac{h^2}{2} y''(\psi_n) \quad \text{where } \psi_n \in [x_n, x_{n+1}]$$

Replace in τ_n:

$$\tau_n = \frac{y_n + h f(x_n, y_n) + \frac{h^2}{2} y''(\psi_n) - y_n}{h} - f(x_n, y_n)$$

$$\tau_n = f(x_n, y_n) + \frac{h}{2} y''(\psi_n) - f(x_n, y_n)$$

$$\tau_n = \frac{h}{2} y''(\psi_n)$$

So, τ_n is proportional to h, or, in other words: $\tau_n = O(h)$, i.e. Euler's method is an $O(h)$ approximation.

It would be better to have an $O(h^p)$ method, with $p > 1$, because τ_n would decrease more rapidly as mesh points are increased.