Newton-Raphson Method

Consider $\dfrac{dy}{dx} = f(x,y)$

The solution can be numerically approximated

with $w_{n+1} = w_n + hf(x_n, w_n)$

$$\Rightarrow w_{n+1} - hf(x_n, w_n) - w_n = 0$$

$$F(w_{n+1}; x_n, w_n) = 0$$

solve for w_{n+1} in terms of x_n, w_n

In general, this is a root finding problem, which also arises when trying to find fixed points. Start with an initial guess x_0, and iterate to find x_n. By Taylor series expansion:

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n) + O(\Delta^2)$$

where $\Delta = x_{n+1} - x_n$

$$\Rightarrow \underbrace{f(x_{n+1})}_{\text{replace with } 0} \approx f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

$$0 = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

$$(x_{n+1} - x_n) f'(x_n) = -f(x_n)$$

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Newton-Raphson root finding method

This method transforms $f(x)=0$ into a 1^{st} order difference equation. What are the fixed points of $x_{n+1} = F(x_n)$ where

$$F(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \quad ?$$

<u>Aside:</u> See notes in 6-1 and 6-2 for linear stability analysis of difference equations

Solve $x^* = F(x^*)$, since x_n must be equal to x_{n+1}

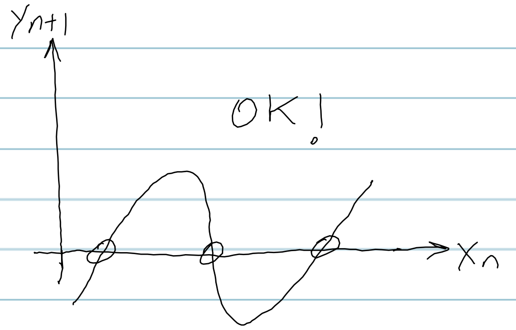
$$\Rightarrow \quad x^* = x^* - \frac{f(x^*)}{f'(x^*)}$$

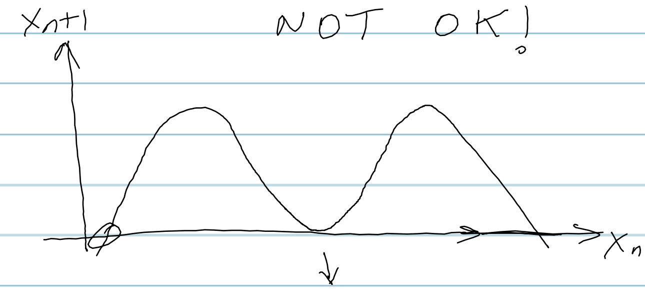
$$\Rightarrow f(x^*) = 0$$

Then, the roots of $f(x)=0$ are the fixed points of its discretized version

$$x_{n+1} = F(x_n)$$

provided that $f'(x^*) \neq 0$.





Problematic since $f'(x^*) = 0$

The derivative of the RHS must be < 1

$$|F'(x^*)| < 1$$

If the points were unstable, N-R method would not converge.

Every fixed point is locally stable as long as the first derivative is different from zero.

It should be possible to find every root of f using N-R.

<u>Example:</u> Use N-R to find roots of

$$x^2 - x = 0$$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^2 - x_n}{2x_n - 1}$$

We know that the roots are $x_1^* = 0$, $x_1^* = 1$

Try $x_0 = 2$. $x_1 = 2 - \frac{4-2}{2-1} = 1.33$

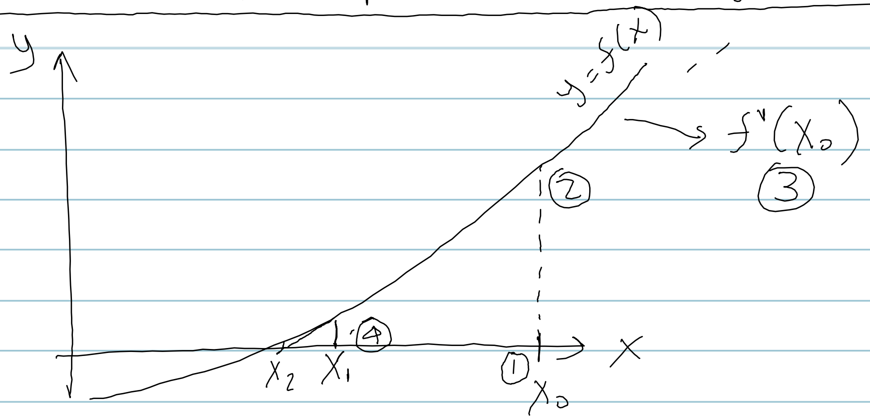
$x_2 = 1.07$; $x_3 = 1.004$ ...

Now, try $x_0 = 0.2$

Then, $x_1 = 0.2 - \frac{0.04 - 0.2}{0.4 - 1} = -0.067$

$x_2 = -0.004 \ldots$

Geometric interpretation of N-R



Convergence to a root is quadratic with N-R provided x_0 is sufficiently close to x^*.