Applications in Ecology

<u>Competition model:</u> Two species competing for the same resource. Lokta-Volterra model is the most well known.

x = rabbit population density

y = sheep density.

$\Rightarrow$ $\dot{x}$ = (logistic growth) − (competition with y)

$\dot{y}$ = (logistic growth) − (competition with x)

<u>Logistic growth:</u> $ax - x^2$, where $a > 0$

For rabbits: $3x - x^2$

For sheeps: $2y - y^2$

<u>Competition:</u> bxy, where $b > 0$

For rabbits: $2xy$

For sheep: xy

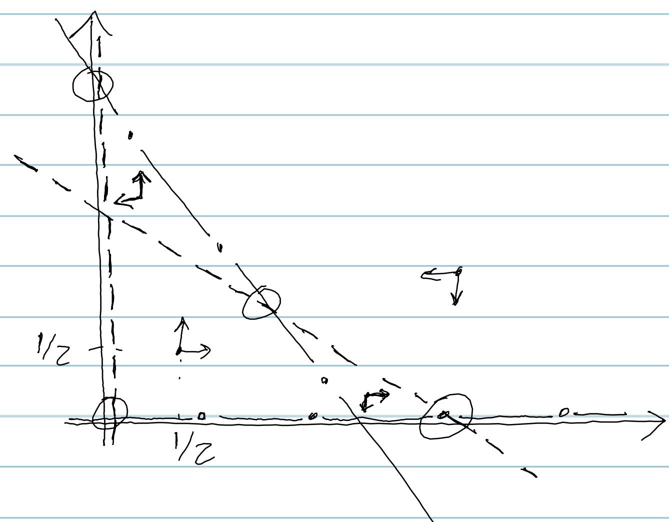
$$\Rightarrow \begin{cases} \dot{x} = 3x - x^2 - 2xy, & x \geq 0 \\ \dot{y} = 2y - y^2 - xy, & y \geq 0 \end{cases}$$

x-nullcline:

$$\dot{x} = 0$$
$$3x - x^2 - 2xy = 0$$
$$2xy = 3x - x^2$$
$$\Rightarrow x = 0, \quad y = \frac{3}{2} - \frac{1}{2}x \quad \rightarrow 2 \text{ branches}$$

y-nullcline:

$$\dot{y} = 0$$
$$2y - y^2 - xy = 0$$
$$y = 0, \quad 2 - y = x \quad \rightarrow 2 \text{ branches}$$



--- x-nullcline
-·-· y-nullcline
O fixed points
-o-o-o- stable manifold of the separatrix given by $\underset{\sim}{x}_4$

Steady-states: where nullclines intersect

$$\underset{\sim}{x}_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \underset{\sim}{x}_2^* = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \underset{\sim}{x}_3^* = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \underset{\sim}{x}_4^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Stability analysis: Find Jacobian

$$J = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - 2y - x \end{bmatrix}$$

$$J(\underset{\sim}{x}_1^*) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 3, \lambda_2 = 2 \quad \underline{\text{unstable}}$$

$$J(\underset{\sim}{x}_2^*) = \begin{bmatrix} 3-4 & 0 \\ -2 & 2-4 \end{bmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = -2 \quad \underline{\text{stable}}$$

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$$J(\underset{\sim}{x_3}) = \begin{bmatrix} 3-6 & -6 \\ 0 & 2-3 \end{bmatrix} \Rightarrow \lambda_1 = -1, \ \lambda_2 = -3$$ stable

$$J(\underset{\sim}{x_4^*}) = \begin{bmatrix} 3-4 & -2 \\ -1 & 2-2-1 \end{bmatrix} \Rightarrow \lambda_1 = -1+\sqrt{2}, \lambda_2 = -1-\sqrt{2}$$ saddle

Phase plane trajectories: Choose a point, e.g $(1/2, 1/2)$. Evaluate it in $\mathring{x}, \mathring{y}$:

$$\mathring{x} = \frac{3}{2} - \frac{1}{4} - \frac{2}{4}$$
$$= 3/4 > 0$$
$$\mathring{y} = 2(1/2) - 1/4 - 1/4$$
$$= 1/2 > 0$$

see graphic at this point.

Basin of attraction: Find eigenvectors for $\underset{\sim}{x_4}$ because it is the saddle point.

$$\underset{\sim}{v_1} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \qquad \underset{\sim}{v_2} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Both species rule each other out: principle of competitive exclusion: two species competing for the same resource cannot coexist.