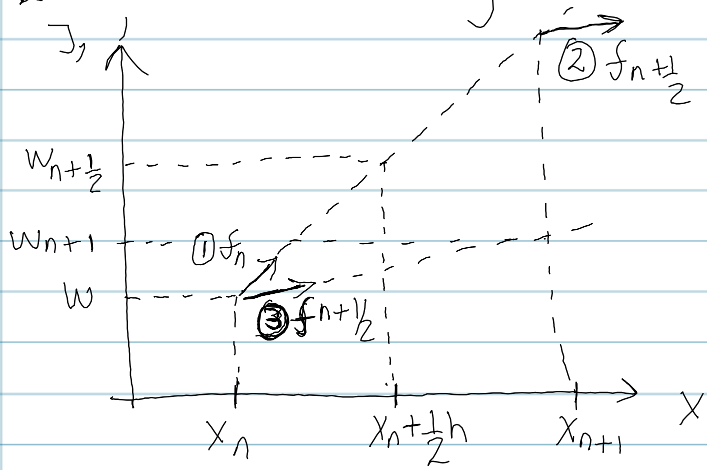
2^{ND} Order Runge-Kutta Method



$K_1 = h\,\underbrace{f(x_n, w_n)}_{\text{slope at } x_n}$

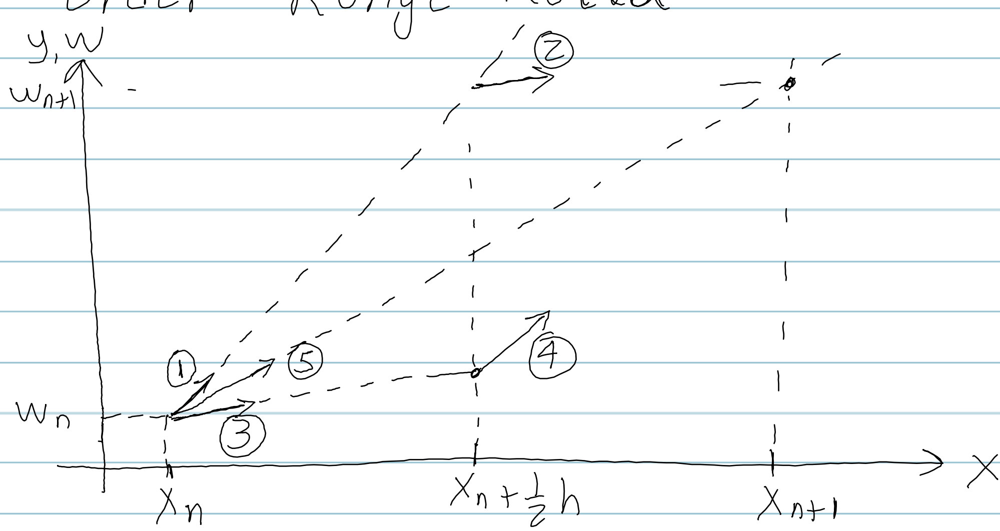
$K_2 = h\,\underbrace{f(x_n + 1/2\,h,\ w_n + 1/2\,K_1)}_{\text{slope at midpoint}}$

$w_{n+1} = w_n + K_2$

2^{ND} order Runge-Kutta

$\tau = O(h^2)$

4^{th} Order Runge-Kutta



$K_1 = hf(x_n, w_n)$

$K_2 = hf(x_n + 1/2\,h,\ w_n + 1/2\,K_1)$

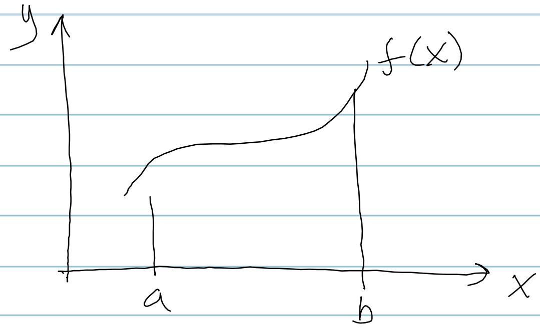
$K_3 = hf(x_n + 1/2\,h,\ w_n + 1/2\,K_2)$

$K_4 = hf(x_n + h,\ w_n + K_3)$

$w_{n+1} = w_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\tau = O(h^4)$

Runge-Kutta 4^{TH} order uses the weighted average of the K's to project to the next approximate w_{n+1}. 4^{TH} order R-K can be derived from Simpson's Rule for numerical quadrature.



$$\int_a^b f(x)\,dx \approx \frac{b-a}{3N}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n)\Big]$$

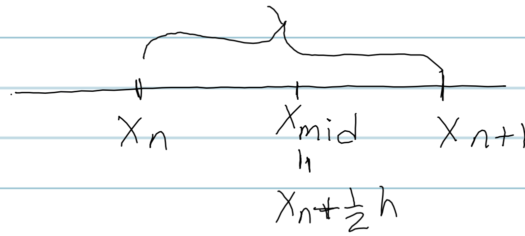
Let $N=2$. Then,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\Big[f(x_0) + 4(x_1) + f(x_2)\Big]$$

Back to $\frac{dy}{dx} = f(x,y)$, suppose that f depends on x only, so $\frac{dy}{dx} = f(x)$. Integrate over $[x_n, x_{n+1}]$:

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x)\,dx$$

Approximate with Simpson's rule $(N=2)$



$$\Rightarrow \int_{x_n}^{x_{n+1}} f(x)\,dx \approx \frac{h}{6}\left[f(x_n) + 4f(x_{mid}) + f(x_{n+1})\right]$$

Using K definitions from 4^{TH} order R-K method described before:

$$= \frac{h}{6}\left[\frac{K_1}{h} + 4\left(\frac{K_2}{h}\right) + \frac{K_4}{h}\right]$$

$$= \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

So, $y_{n+1} - y_n \approx \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$, or

$$y_{n+1} \approx y_n + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

which is the same as R-K 4^{TH} order.

Note: Simpson rule is a 2^{ND} order method.

Note: An adaptive step size is used when the solution to the ODE has a wide range of slopes.