Linear Stability Analysis of Delay equations

Equilibrium occurs when $x(t) = x^*$ is a solution for all t. Therefore, at the equilibrium point,

$$f(x^*, x^*, x^*, \ldots, x^*) = 0$$

Recall the simple case of ODE perturbation.

$$\eta = x - x^*, \quad \dot{\eta} = \dot{x} = f(x^*) + J(x - x^*) - O(\eta^2)$$

Introduce the notation

$$\underset{\sim}{x} \equiv x(t), \quad x_\tau \equiv x(t-\tau)$$

Let x^* be an equilibrium. Let $\delta x(t)$ be the displacement from equilibrium with a perturbation that lasts from $t = t_0 - \tau_{max}$ to t_0.

Then, $x = x^* + \delta x$

$\Rightarrow \dot{x} = \delta\dot{x} = f(x^* + \delta x, x^* + \delta x_{\tau_1}, \ldots, x^* + \delta x_{\tau_k})$

Since all quantities are small, linearize. Then,

$$\delta\dot{x} = J_0\, \delta \underset{\sim}{x} + J_{\tau_1} \delta x_{\tau_1} + J_{\tau_2} \delta x_{\tau_2} + \ldots$$

Use the ansatz (educated guess verified later by results) $\delta\dot{x}(t) = Ae^{\lambda t}$.

Substituting this ansatz in the previous equation:

$$\lambda e^{\lambda t} A = \left(J_0 e^{\lambda t} + e^{\lambda(t-\tau_1)} J_{\tau_1} + e^{\lambda(t-\tau_2)} J_{\tau_2} \ldots\right) A$$

Both sides can be divided by $e^{\lambda t}$.

This leaves the product of two terms that have to be zero:

$$\underbrace{\left(J_0 + e^{-\lambda\tau_1} J_{\tau_1} + e^{-\lambda\tau_2} J_{\tau_2} + \ldots\right)}_{\text{has to be zero}} A = 0$$

cannot be zero... otherwise the zero solution is used.

Since J and A are matrices (and the equality $\delta x(t) = e^{\lambda t} A$ has to be understood on the left-hand side as $\delta I x$, where I is the identity matrix), then

$$\det\left(J_0 + e^{-\lambda \tau_1} J_{\tau_1} + e^{-\lambda \tau_2} J_{\tau_2} + \dots \right) = 0$$

This equation looks like an ordinary eigenvalue problem, except for the exponential terms. Once we expand out the determinant, we obtain a quasi polynomial. In the case of ODEs, the characteristic equation is a polynomial of degree n with exactly n complex roots.

In the case of DDEs, quasipolynomials have infinite number of roots in the complex plane. There is no universal strategy to deal with this problem.