# Lecture 5 - Bifurcations on the line

Introduce the concept of bifurcation with an example:

Ex: Analyze the dynamics of the following ODE: $\dot{x} = r - x - e^{-x}$

That is to say, find fixed points, and draw a phase portrait.

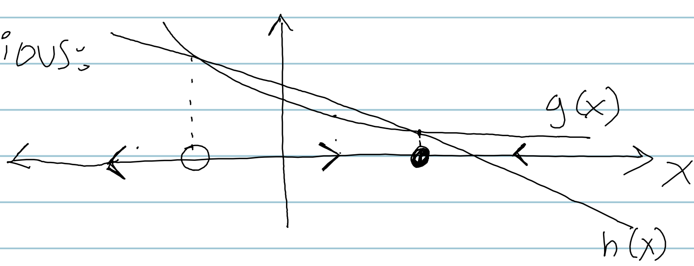
Sol: Separate the function $\dot{x}$ into two components: $\dot{x} = f(x)$

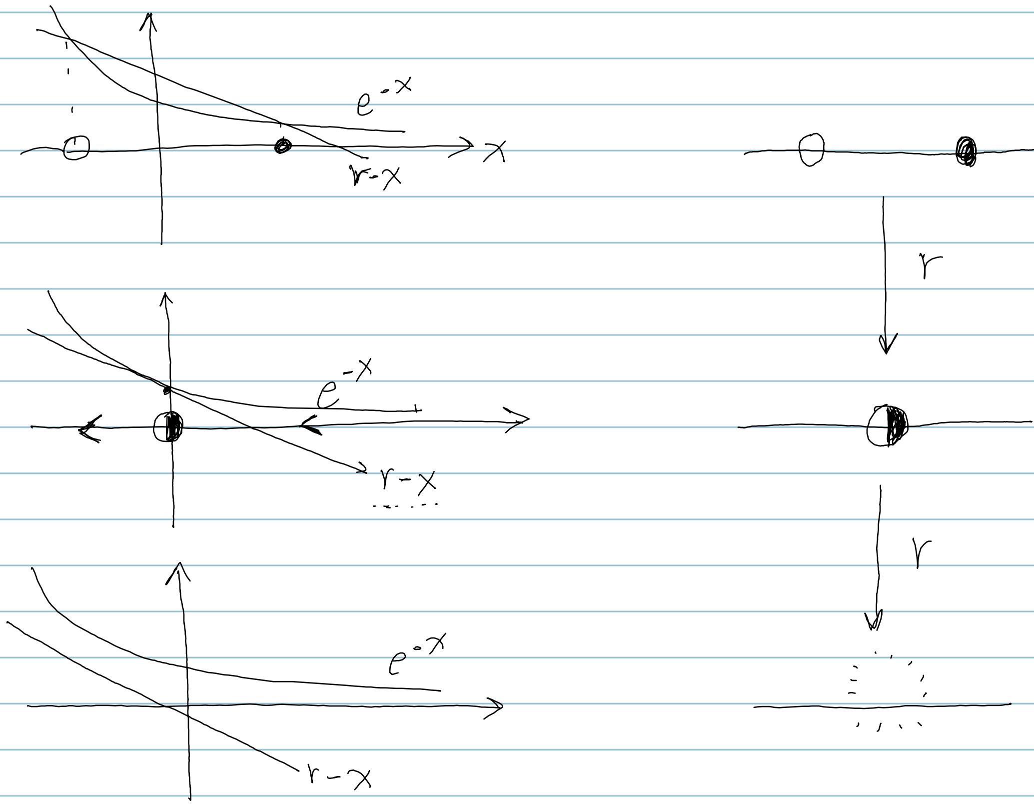
$$= h(x) - g(x),$$

where $h(x) = r - x$, $g(x) = e^{-x}$

The function $f(x)$ is equivalent to the substraction of $h(x)$ and $g(x)$.

This is obvious:



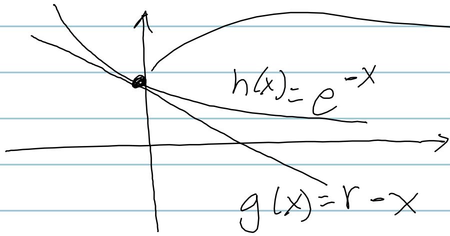



The fixed point occurs at $x^*$

$$f(x^*) = 0 \Rightarrow r - x^* - e^{-x^*} = 0$$

At the bifurcation point,

$$\frac{d}{dx}(g(x^*)) = \frac{d}{dx}(h(x^*))$$




the slope of $g(x)$ equals the slope of $h(x)$

$$-1 = -e^{-x}$$

Aside

Why? Because $\frac{d}{dx}(r-x) = -1$

$$\frac{d}{dx}(e^{-x}) = -e^{-x}$$

$$f(x) = \cancel{g/h}\ \underbrace{h(x)}_{r-x} - \underbrace{g(x)}_{e^{-x}}$$

Since $-1 = -e^{-x} \Rightarrow 1 = e^{-x}$

$x = 0$, this is the point in which $h(x)$ and $g(x)$ touch at one single point.

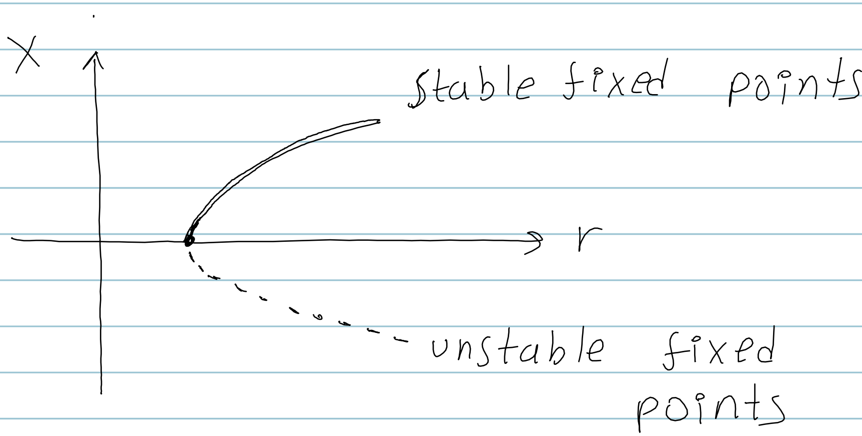
$\boxed{\lambda_c^* = 0}$ → there is one half-stable fixed point.

Since we know $x_c^*$, then we can find $r_c$ $\rightarrow$ which is the value of $r$ in which the system changes behavior, i.e. it goes from two fixed points to no fixed points at all.

So, replacing:

$$r_c - x_c^* = e^{-x_c^*}$$

$$\boxed{r_c = 1}$$



To understand the previous plot, do a Taylor series expansion of $f(x)$ about zero, since $x_c = 0$.

$$f(x) = r - x - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right]$$

$$= r - 1 - \frac{x^2}{2!} + \dots$$

$$f(x) \approx r - 1 - \frac{x^2}{2}$$ $\rightarrow$ a parabola close to the point of bifurcation.

<u>Def:</u> <u>Saddle node bifurcation:</u>

After linearization, the system takes the form

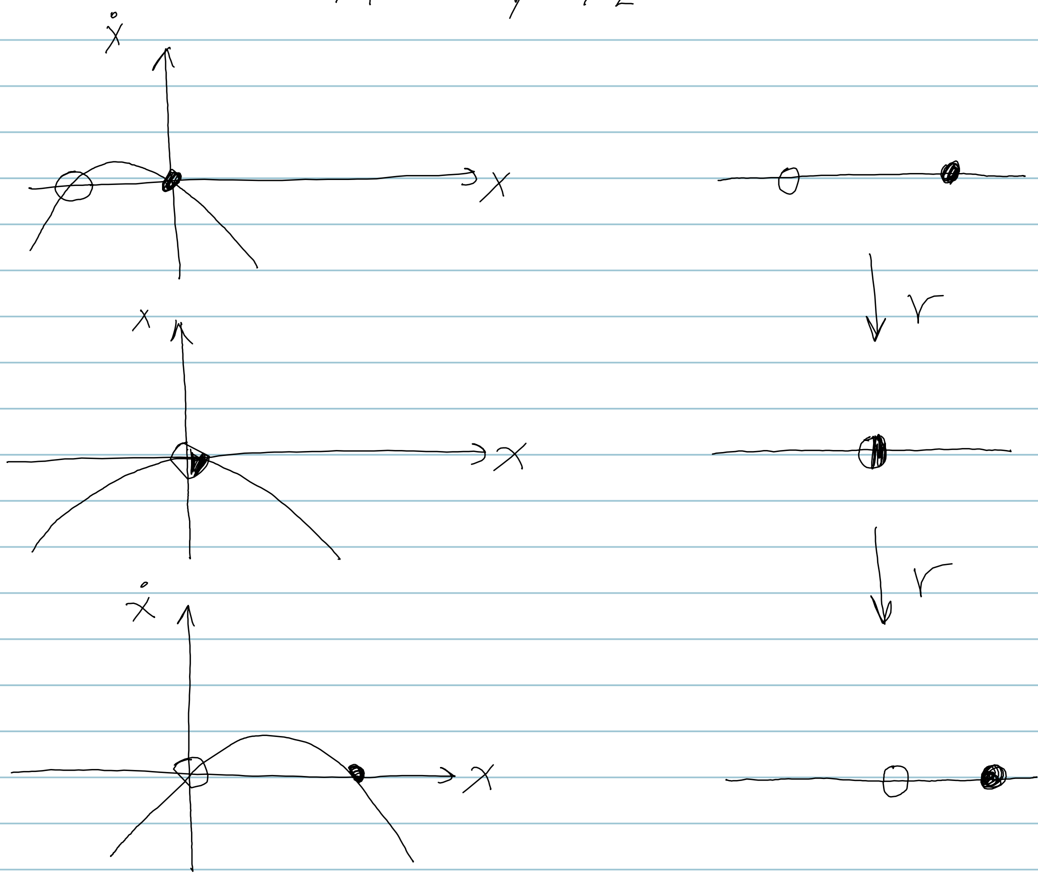
$$\dot{x} = a + bx^2$$

<u>Note:</u> This gives rise to the bifurcation plot in the previous example.

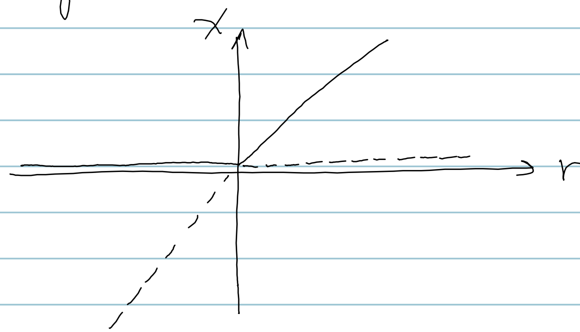
Def: Transcritical bifurcation:

After linearization, the system becomes $\dot{x} = rx - x^2$

F. P. $x_1^* = 0$, $x_2^* = r$



The transcritical bifurcation diagram is:

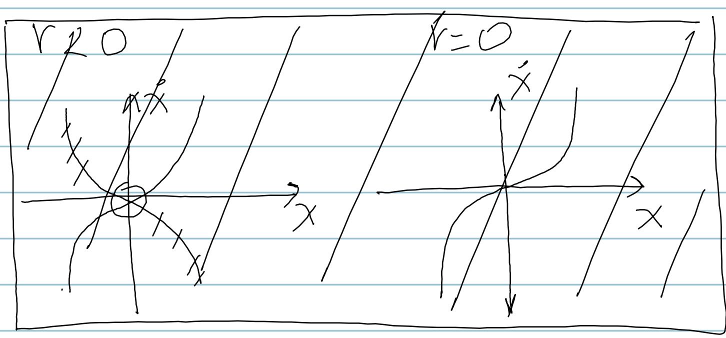



<u>Def:</u> Pitchfork bifurcation: After linearization:

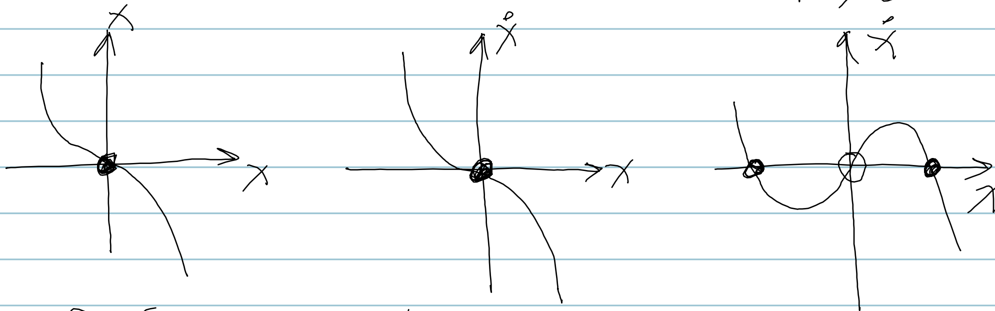
a) <u>Supercritical</u>: $\dot{x} = rx - x^3$

F.P. $rx - x^3 = 0 \Rightarrow x(r - x^2) = 0$

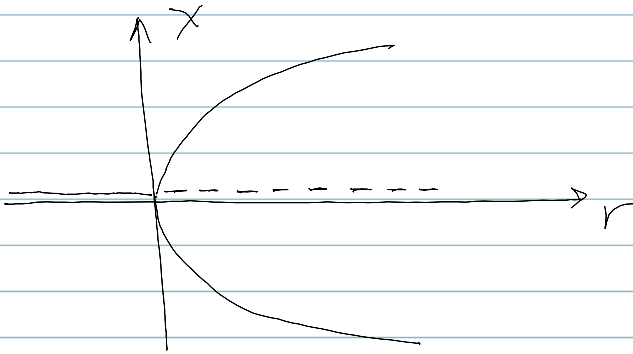
$x_1^* = 0, \quad x_2^* = \sqrt{r}, \quad x_3^* = -\sqrt{r}$



$r<0$ $\qquad$ $r=0$ $\qquad$ $r>0$



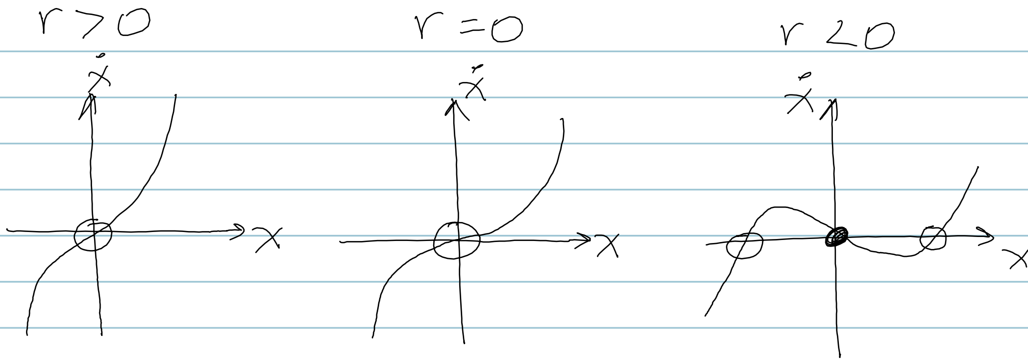

Bifurcation diagram



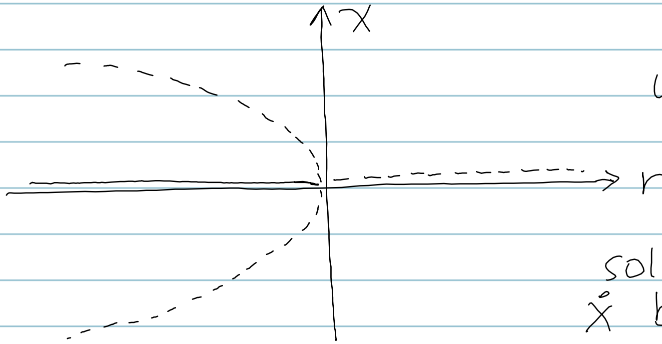

b) Subcritical : After linearization, the system becomes $\dot{x} = rx + x^3$

F.P. $rx + x^3 = 0$, $x_1^* = 0$, $x_2^* = \sqrt{-r}$

$x_3^* = -\sqrt{-r}$




Bifurcation diagram:



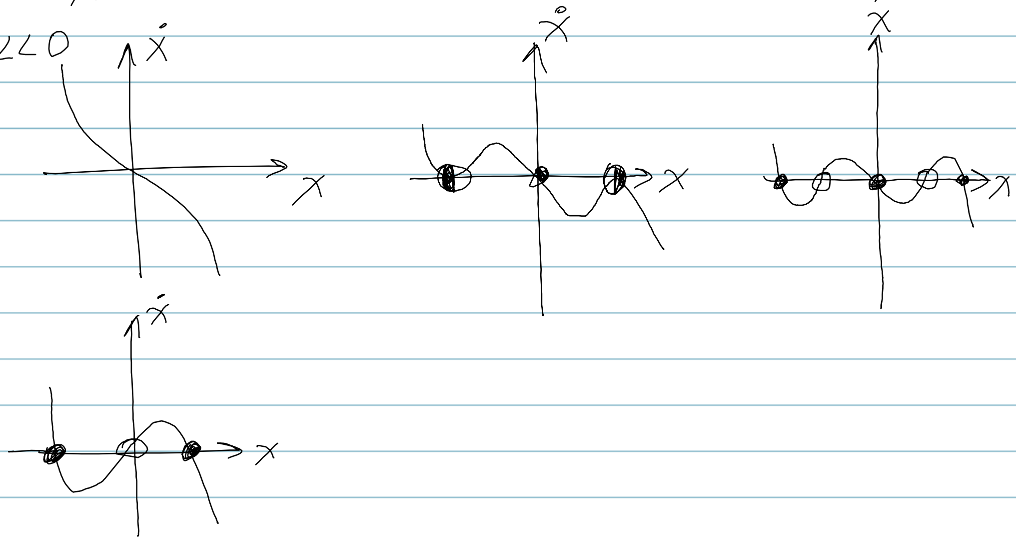

This is an undesirable situation, because: solutions of $\dot{x}$ blow up to infinity.

Biological systems are bounded, and in most cases robust. Therefore a subcritical pitchfork bifurcation is unrealistic. In this case, take into consideration additional terms of the Taylor series.

**Def:** Subcritical pitchfork bifurcation with restoring term: After ~~linear~~ Taylor series expansion,

$$\dot{x} = rx + x^3 - x^5$$

$r << 0$




Bifurcation diagram