Chapter 1

Slope and Rate of Change

Chapter Summary and Goal

This chapter will start with a discussion of slopes and the tangent line. This will rapidly lead to heuristic developments of limits and the derivative.

Student Learning Objectives

The student will:

1. Be able to distinguish between the slopes of secant lines and tangent lines.
2. Understand the concept of a limit.
3. Understand the relationship between slopes of a function and the derivative.
4. Learn the definition of a derivative.
5. Learn the relationship between tangent lines and the velocity problem.

The Mathematics of Change

Calculus is about change. More specifically, if gives us ways to explain, using mathematics, how a variable y might change when a variable x changes. Why do we care about this? Because change is a fact of life. We can use calculus to figure out how fast our local reservoir will empty during the next drought; to plan a flight path between Atlanta and London that uses the least fuel; to figure out how long it will take to pay off a mortgage; to figure out the dimensions of the largest rectangular tree-house you can build with a fixed amount of plywood; or to predict how fast a penny dropped off the top of the CN Tower in Toronto would be falling when it hit the ground. All of these problems involve figuring out how one variable changes in comparison to, or as mathematicians often like to say, ***with respect to***, another variables.

You've probably already come across this concept in an algebra or pre-calculus class through the use of the **slope** of a line. Suppose we draw the plot of a non-vertical line

$$y = mx + b \tag{1.1}$$

where m is the slope and b is the y intercept (figure 1.1). If we pick *any* two points (x_1, y_1) and (x_2, y_2) on that line, then the slope is defined as

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.2}$$

Calculus

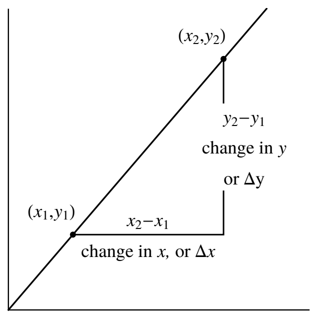
Calculus is the **quantitative description** of how things change in comparison to one another. It has two branches:

a) **Differential Calculus**: calculation and application of the derivative to find rates of change.

b) **Integral Calculus**: calculate of areas under curves and inversion of the process of differentiation.

We have introduced the terms **rise** and **run** in equation 1.2 to refer to the **change in** x and **change in** y, respectively. These terms are commonly used throughout algebra and analytic geometry. Sometimes we also use the expressions Δy and Δx to refer to the rise and the run,

Figure 1.1: Slope of a line.



$$\Delta y = y_2 - y_1 \quad \text{read this as “Delta y”} \tag{1.3}$$
$$\Delta x = x_2 - x_1 \quad \text{read this as “Delta x”} \tag{1.4}$$

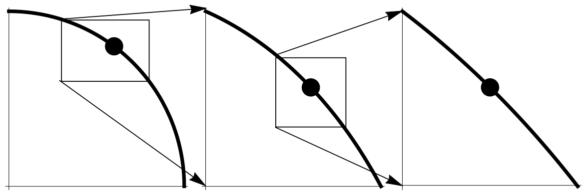
Here the Greek letter Δ is not a variable that is multiplied by the x or the y but part of the variable name, so that Δx is single complete variable name, and Δy is a single complete variable name.[1]

Tangent Lines

We want to extend the concept of the slope of a line to curves. Since curves can bend in any which-way they might choose, the idea of a single slope that applies to the entire curve doesn't make much sense. Instead, given any particular curve, and a particular point P, on that curve, we observe that if we look at it through a powerful enough magnifying glass, it looks more and more like a straight line as we get closer and closer to that point (see figure 1.2). Imagine, then, that we can figure out the slope m of this almost-nearly-straight line, and construct the line through P. This new line, which we have just drawn, has the *same slope as the curve at* P. We call this line the **tangent line to the curve at** P. The word tangent is derived from the latin verb *tangere*, to touch, as in, to touch, but not to cross or intersect. A tangent line (as we have described it above) just touches the curve at a single point, but does not cross it

[1] Technically, Δ is an operator that means “change in” so Δx means “change in x”. It comes from the Liebniz notation that we will study in chapter 8.

Figure 1.2: Illustration of a point on a smooth curve, and two successive blow-ups under the magnifying glass, as you move from the left to the root, showing how the points in the neighborhood of the smooth curve look very much like a straight line if you look at the point under a magnifying glass and ignore the rest of the figure.

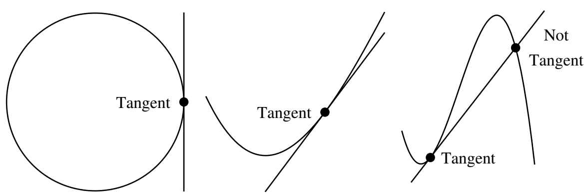


or intersect it (figure 1.3). The ***slope of a curve at the point P*** is defined to be the slope of the tangent line at P.

Definition 1.1. Slope of a Curve

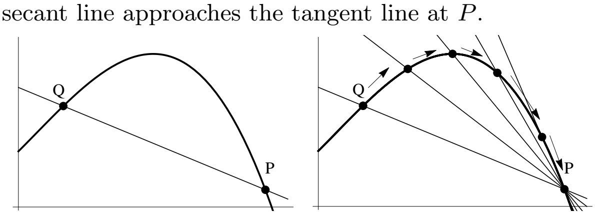
The slope of a curve at a point P is the slope m of a line that is tangent to the curve at P.

Figure 1.3: Examples of tangent lines. The tangent line is just touching, but not intersecting, at the point of tangency.



Let $f(x)$ be a smooth function. We want to develop an easy, methodical process for calculating the slope of the tangent line through a point P on the plot of $f(x)$. Let the coordinates of P be (x_1, y_1). Let $Q = (x_2, y_2)$ be another point on the curve, and construct the secant line through P and Q (see the curve in the left frame of figure 1.4.) Next, we imagine marching point Q

Figure 1.4: Using a sequence of secant lines to define the tangent line. The image on the left shows a single secant line between the points Q and P. Then, on the right, as the point Q approaches P, the secant line approaches the tangent line at P.



towards P, as indicated by the arrows in the right frame of figure 1.4. As Q gets **closer and closer** to P, the secant line gets closer and closer to the tangent line through P. We call this process of one point getting closer and closer to another point **taking a limit** or a **limiting**

process. We denote the limiting process by $Q \to P$, which we read as "Q goes to P," or "the limit as Q goes to P."

As Q is marching closer and closer to P, Q is trying to get as close as possible. Q may actually get all the way to P; but there are some cases where it may not actually get there. For example, we may have chosen P to be a hole in the function (a point $x = c$ where $f(x)$ is not defined, surrounded by an open neighborhood in which $f(x)$ is defined). It is often still possible to define the slope at a hole, even though the function is not defined there.

In order to do calculations, it is more convenient to talk about the coordinates (x, y) instead of P and Q. Since both P and Q lie on the function $f(x)$, we have

$$P = (x_1, y_1) = (x_1, f(x_1)) \tag{1.5}$$
$$Q = (x_2, y_2) = (x_2, f(x_2)) \tag{1.6}$$

Since x is the **independent variable** and y is the **dependent variable**, instead of saying Q approaches P, we say that x_2 approaches x_1, which we write as $x_2 \to x_1$. Instead of saying "the limit as Q goes to P" we say "the limit as x_2 goes to x_1." At the same time, we observe that $y_2 \to y_1$ because $Q \to P$.

Since the slope of the secant is

$$m_{\text{secant}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1.7}$$

we can then define the **slope of the tangent line at** x_1 as the limit of m_{secant} as $x_2 \to x_1$. We will call this slope $f'(x_1)$.

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1.8}$$

If we define the fixed point $x_1 = a$ and let $x = x_2$ then (1.8) becomes

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1.9}$$

Equation 1.9 gives the slope of the tangent line at the point $x = a$. It is a number, and is called the derivative of $f(x)$ at the point $x = a$.

If we instead define $h = x_2 - x_1$ in (1.8) then $x_2 = x_1 + h$. So when $x_2 \to x_1$, h must go to zero; it is the horizontal distance between the coordinates of P and Q. Substituting,

$$f'(x_1) = \lim_{x_1 + h \to x_1} \frac{f(x_1 + h) - f(x_1)}{h} \tag{1.10}$$

Since there is only one x coordinate (that of P) the index is no longer needed, and the equation for the **slope of the tangent line** at x becomes

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{1.11}$$

We call the slope of the tangent line at x the **derivative** of $f(x)$ and denote it by $f'(x)$. When the limit (1.11) exists we say that $f(x)$ is **differentiable at** $\boldsymbol{x}$. We emphasize here that while the slope of a line is a number, and the slope of a curve at a particular point, the derivative of a function is another function. ***The derivative*** $f'(x)$ ***gives the slope of*** $f(x)$ ***as a continually changing function of*** x.

Definition 1.2. The Derivative

The **derivative of a function** $f(x)$ is the **function** $f'(x)$ defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1.12}$$

The **derivative at the point** $x = a$ is a **number** $f'(a)$ that may be calculated either by by setting $x = a$ in the formula for $f'(x)$ or by calculating a limit such as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{1.13}$$

or

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{1.14}$$

Example 1.1. Find the slope of the tangent line to $y = x^2$ at the point $(1, 1)$ by simulating the process of $Q \to P$ empirically, and use this slope to calculate the equation of the tangent line at $(1, 1)$.

Solution. By an *empirical* calculation we mean we want to experimentally calculate the values of the slop as Q marches toward P. Let $Q = (x, y) = (x, x^2)$ be any other point on the curve of $y = x^2$. Then the slope is

$$m = \lim_{x_2\to x_1} \frac{y_2 - y_1}{x_2 - x_1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x - 1} = x + 1 \tag{1.15a}$$

We will arbitrarily pick a sequence of values of x that approach 1 from the left, and see what the slope appears to be approaching:

x	$m = 1 + x$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.999
0.9999999999	1.9999999999

as $x \to 1$ from the left, it would appear that $m \to 2$. What about if $x \to 1$ from the right? We can repeat the empircal calculation:

x	$m = 1 + x$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001
1.0000000001	2.0000000001

It would appear that $m \to 2$ as $x \to 1$ from the right as well. Thus the tangent line through $(1, 1)$ is (using the point-slope form of the equation of a line),

$$y - 1 = 2(x - 1) = 2x - 2 \tag{1.15b}$$

Bringing the 1 to the right hand side gives the equation in the more standard slope-intercept form of $y = 2x - 1$. □

As it turns it, we could have simply plugged $x = 1$ into the last step of equation (1.15a)

$$m = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} (x + 1) = 2 \tag{1.16}$$

We will see why we can do this when we discuss the limit laws (such as theorem 3.9) in chapter 2. In this case, since $f(x) = x + 1$ is a polynomial, theorem 3.9 allows us to simply substitute the value of x into the formula for $f(x)$. We *could not* have plugged $x = 1$ into $g(x) = \dfrac{x^2-1}{x-1}$, even though $g(x) = f(x)$ for all $x \neq 1$, to calculate the limit because that would have led to the rather perplexing conundrum of fraction equal to 0/0. We will also discus 0/0 limits in chapter 2.

Average and Instantaneous Rate of Change

Suppose that the amount or quantity of something is a function of time: position, altitude of an airplane, the odometer on your car, amount of flour in a cannister on your kitchen counter, amount of gasoline in your car. We want to be able to describe that quantity $y = f(t)$ changes with time. Suppose there is an amount $y_1 = f(t_1)$ at time t_1, and an amount $y_2 = f(t_2)$ at time t_2.

Let $\Delta t = t_2 - t_1$ be the change in t, and $\Delta y = y_2 - y_1$ be the corresponding change in y over the timespan Δt starting at t_1.

Then the **average rate of change** of y is the total change in y divided by Δt, and we will denote this by v_{average}:

$$v_{\text{average}} = \frac{\Delta y}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} \tag{1.17}$$

If y is the *position*, for example, then v is the *average velocity* or *speed*, and is measured in km/sec. If y is the amount of gasoline in your car, it may be measured in gallons/day. If y is the number of gallons in a tank of water, then dy/dt is rate at which you use the water.

We observe that **the average rate of change is the slope of the secant line** to $f(t)$ from time $t = t_1$ to time $t = t_2$.

We use the term **instantaneous rate of change** to refer to the limit (see figure 1.5)

$$v_{\text{instantenous}}(t_1) = \lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \tag{1.18}$$

which is precisely the derivative at t_1. Hence

$$v_{\text{instantaneous}}(t) = f'(t) = \lim_{a \to t} \frac{f(a) - f(t)}{a - t} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \tag{1.19}$$

The instantaneous rate of change at any time $t = a$ is the slope of the tangent line at $t = a$, and is equal to the value of the derivative $f'(a)$.

Example 1.2. In the absence of air resistance, the distance an object falls when it is dropped is related to the time since it has fallen by the equation

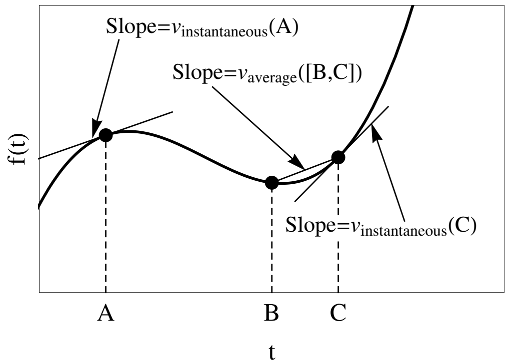
$$y = \frac{1}{2} g t^2 \tag{1.20}$$

where $g = 9.80665 \text{ m/sec}^2$ is the Earth standard acceleration due to gravity. Estimate the velocity that a US penny has when it hits the ground if it is dropped off the top of the tallest building in Los Angeles (the US Bank tower, 310 Meters).

Solution. We first solve for the time it takes to hit the ground; solving (1.20) for t,

$$t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 310}{9.80665}} = 7.95125 \text{ sec} \tag{1.21a}$$

Figure 1.5: Average and instantaneous rate of change. The function $f(t)$ is plotted on the dependent axis and time on the independent axis. The instantaneous rate of change at any particular time, such as at the time points A and C (illustrated) is the slope of the tangent line at that point. The average rate of change is measured over and interval between two points, and is the slope of the secant line connecting the values of the function at the two endpoints of the interval $[A, B]$.



Next, we need to figure out the instantaneous velocity at $t = 7.95125$. The average velocity between t_1 and t_2 is

$$v_{\text{average}} = \frac{gt_2^2/2 - gt_1^2/2}{t_2 - t_1} = \frac{g}{2} \cdot \frac{(t_2 - t_1)(t_2 + t_1)}{t_2 - t_1} = \frac{g(t_2 + t_1)}{2} \tag{1.21b}$$

The instantaneous velocity is

$$v(t_1) = \lim_{t_2 \to t_1} \frac{g(t_2 + t_1)}{2} \tag{1.21c}$$

The instantaneous velocity at $t = 7.95125$ is then

$$v(7.95125) = \lim_{t \to 7.95125} \frac{g(t + 7.95125)}{2} \tag{1.21d}$$

Calculating this limit empirically from the left we obtain the following values:

t	$g(t + 7.95125)/2$
7.9	77.7238
7.95	77.9690
7.951	77.9739
7.9512	77.9749
7.95125	77.9751

Here is a table as we approach from the right

t	$g(t + 7.95125)/2$
8	78.2142
7.96	78.018
7.952	77.9788
7.9513	77.9754
7.95126	77.9752

Truncating to three decimals we estimate that the instantaneous velocity is 77.975 meters/second.

In fact, due to air resistance, the speed would probably only be about half that fast (because the equation we started with is actually incorrect). □

Exercises

Empirically estimate the slope of each of the following functions at the specified point.

1. $y = x^3 - 4$ at $x = 2$ ans: 12
2. $y = \sin x$ at $x = \dfrac{\pi}{3}$ ans: 0.5
3. $y = x(x-1)$ at $x = 0.5$ ans: 0
4. $y = x^4 - x^2$ at $x = 1$ ans: 2
5. $y = \sqrt{x}$ at $x = 2$ ans: 0.353553
6. $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$ ans: -0.0625

Use an empirical estimate of the slope to find the equation of the tangent line to each of the following functions at the specified point.

7. $y = x^4 - x^2$ at $x = 1$ ans:$y = 2x - 2$
8. $y = x - x^3$ at $x = 2$ ans: $y = 16 - 11x$
9. $y = \tan x$ at $x = \dfrac{\pi}{4}$ ans: $y = 2x - 0.571$
10. $y = x \sin\left(\dfrac{1}{x}\right)$ at $x = \dfrac{1}{x}$
    ans: $y = \pi x - 1$
11. Suppose that a ball is thrown straight up into the air with an initial velocity of 50 ft/sec so that its height in feet after t seconds is given by $y = 50t - 16t^2$.
    (a) Find the average velocity over the period $1 \le t \le 1.5$
    (b) Find the average velocity over the period $1 \le t \le 1.1$
    (c) Find the average velocity over the period $1 \le t \le 1.01$
    (d) Estimate the instantaneous velocity at $t = 1$
12. The position of a particle moving in a straight line is given by $t^2 - 4t + 8$ meters, where t is measured in seconds.
    (a) Estimate the average velocity over the interval $[3, 4]$
    (b) Estimate the average velocity over the interval $[3.5, 4]$
    (c) Estimate the average velocity over the interval $[4, 5]$
    (d) Estimate the average velocity over the interval $[4, 4.5]$
    (e) Estimate the instantaneous velocity at $t = 4$.