# A few Bayesian lectures for the Uninitiated

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# Part 3: MCMC


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# Bayes?!

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Likelihood and prior are fine now, but what is that strange denominator, $P(\text{data})$?

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This means:
We compute the probability of the data by integrating over all values of $\theta$ (aka "integrating out" or "marginalising over" $\theta$).
(The $\Omega$ is the parameter space, i.e. all possible value combinations for multidimensional $\theta$.)

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Problem: How to sample $X$ proportional to $P(X)$?

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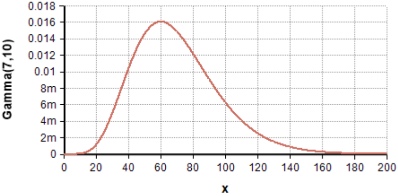

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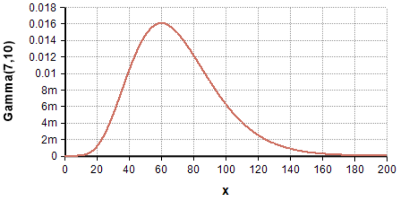


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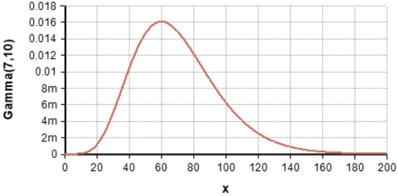


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Enter the "Metropolis" algorithm.

In line with Stigler's Law of Eponymy, the "Metropolis" algorithm was invented by physicist Marshall Rosenbluth, after being posed the problem by Ed Teller, and first implemented by his wife Arianna Rosenbluth, but written up by Metropolis and these three, plus Ed Teller's wife Augusta, in 1953.

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Ad 4. Acceptance is thus more likely the more similar the likelihood of $x_1$ and $x_2$ are.

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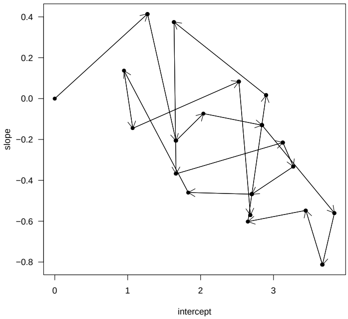
The above MCMC is an *autocorrelated* random walk in parameter space, spending more time at higher probabilities.



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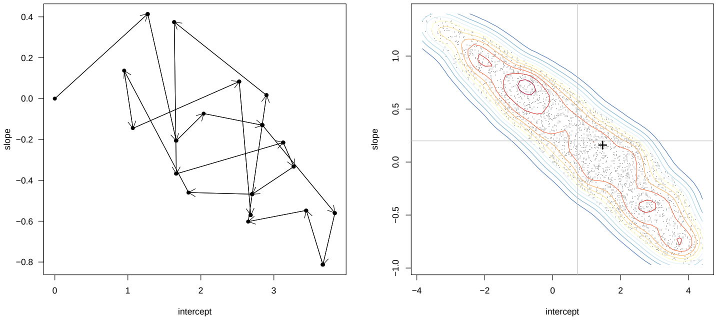

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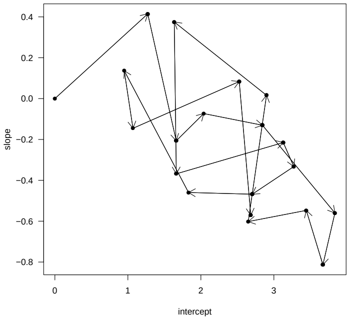
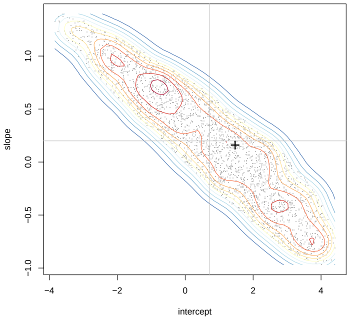
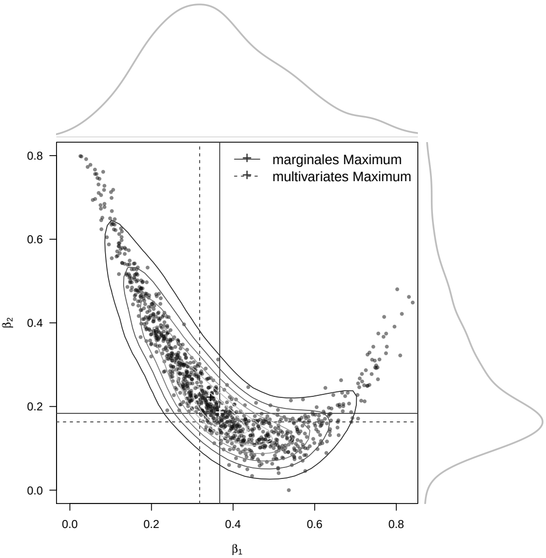



For many parameters, proposals are often drawn conditionally on other dimensions ("Gibbs sampling", hence JAGS).

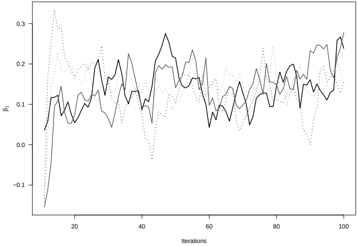
# MCMC: the “banana” problem

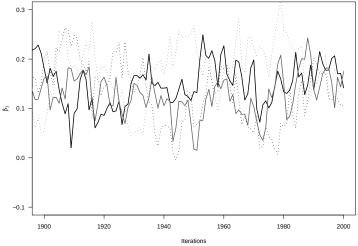



multivariate MAP $\neq$ marginal MAP

# MCMC: mixing and convergence

How many samples do we need to work out the shape of the posterior?



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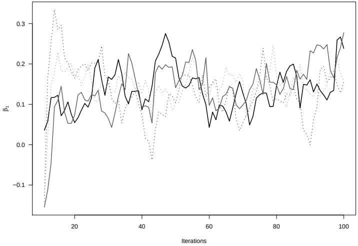


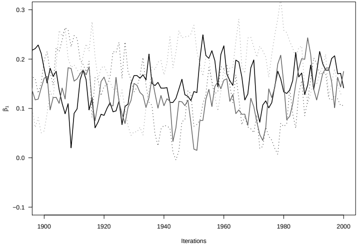


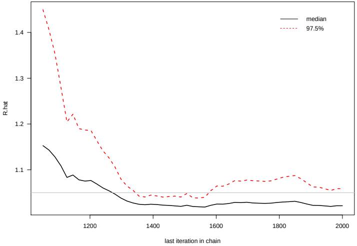

# MCMC: take-home-message

- Markov chain Monte Carlo is an ingenious way to sample proportional to an unknown distribution (e.g. the model's likelihood).
- It is slow and requires attention after fitting.
- MCMC is a must-know algorithm/concept in statistics, far beyond Bayes.
- If you want, you can use MCMC as robust-but-slow optimiser (if so, check modern extensions, such as DREAM, DEzs, SMC as implemented in **BayesianTools**).

**Next time:**
**Crazy little thing called *prior*?**